Homework 1

Yi Yang (yy3089)

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1 E1

For n = 1, let $Y = X_1^2$

$$\begin{aligned} P(Y \leqslant y) &= P(X_1^2 \leqslant y) \\ &= P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(\sqrt{-y}) \\ &= 2\Phi(\sqrt{y}) - 1 \end{aligned} \tag{1.1}$$

Thus:

$$\begin{aligned} f(y) &= 2(\Phi(\sqrt{y}))' * (\sqrt{y})' \\ &= 2 * \frac{1}{\sqrt{2\pi}} e^{-y/2} * \frac{1}{2} \frac{1}{\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y/2} * \frac{1}{\sqrt{y}} \end{aligned} \tag{1.2}$$

When n = 1, there is no possibility that $X^2 < 0$, so:

$$\begin{aligned} \frac{1}{\Gamma(n/2)2^{n/2}} y^{n/2-1} e^{-y/2} &= \frac{1}{\Gamma(1/2)2^{1/2}} x^{1/2-1} e^{-y/2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y/2} * \frac{1}{\sqrt{y}} \end{aligned} \tag{1.3}$$

We assume that when n = k, this equation is valid, then for n = k + 1:

$$\sum_{i=1}^{K+1} X_i^2 = \chi_K^2 + \chi_1^2 \tag{1.4}$$

Thus:

$$
\begin{aligned}
f(x) &= \int_0^x \frac{1}{\Gamma(k/2)2^{k/2}} \tau^{k/2-1} e^{-\tau/2} \frac{1}{\Gamma(k/2)2^{k/2}} (x-\tau)^{k/2-1} e^{1(x-\tau)/2} dx \\
&= \frac{e^{-x/2}}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} \int_0^x t^{\frac{k}{2}-1}(x-t)^{\frac{1}{2}-1} dt \\
&= \frac{e^{-x/2}}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} \int_0^1 (ux)^{\frac{k}{2}-1}(x-ux)^{-\frac{1}{2}} dux \\
&= \frac{e^{-x/2}}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} x^{(k-1)/2} \int_0^1 u^{\frac{k}{2}-1}(1-u)^{\frac{1}{2}-1} du \\
&= \frac{e^{-x/2}\beta(k/2,1/2)x^{(k-1)/2}}{2^{(k+1)/2}\Gamma(k/2)\Gamma(1/2)} \\
&= \frac{1}{\Gamma((k+1)/2)2^{(k+1)/2}} x^{(k+1)/2-1} e^{-x/2}
\end{aligned} \tag{1.5}
$$

For $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $u = t/x$

2 E2

Let:

$$
A = \begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{2\cdot 1}} & -\frac{1}{\sqrt{2\cdot 1}} & 0 & \cdots & 0 \\
\frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & -\frac{2}{\sqrt{3\cdot 2}} & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{n-1}{\sqrt{n(n-1)}}
\end{pmatrix} \tag{2.1}
$$

We can easily compute that $A^T \cdot A = E$
Let $X = (X_1, X_2, \cdots, X_n)^T$ and $Y = A \cdot X$. We can compute $Y_1 = \frac{1}{\sqrt{n}} n\bar{X} = \sqrt{n}\bar{X}$ so that we can get $\bar{X} = \frac{1}{\sqrt{n}} Y_1$ and $\sum_{i=1}^n Y_i^2 = Y^T Y = X^T A^T A X = \sum_{i=1}^n X_i^2$
The joint PDF for $Y_1, Y_2, \cdots, Y_n$ is :

$$
\begin{aligned}
p(Y_1, \cdots, Y_n) &= \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\frac{\sum_{i=1}^n Y_i^2 - 2\sqrt{n} Y_1 \mu + n\mu^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}^n} e^{-\frac{\sum_{i=2}^n Y_i^2 + (Y_1 - \sqrt{n}\mu)^2}{2\sigma^2}}
\end{aligned} \tag{2.2}
$$

$Y_2, \cdots, Y_n \sim N(0, \sigma^2)$, $Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$From the result of Problem 1.a we know that:

$$
\begin{aligned}
(n-1)S^2 &= \sum_{i=1}^n X_i^2 - n\bar{X}^2 \\
&= \sum_{i=1}^n X_i^2 - (\sqrt{n}\bar{X})^2 \\
&= \sum_{i=1}^n Y_i^2 - Y_1^2 \\
&= \sum_{n=2}^n Y_i^2
\end{aligned} \tag{2.3}
$$

Thus:

$$
\frac{(n-1)S^2}{\sigma^2} \overset{d}{=} \chi^2_{n-1} \tag{2.4}
$$