Note on Error Mitigation

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1 Type of error

Qubit operations are susceptible to various types of errors due to imperfect control pulses, qubit-qubit couplings (crosstalk), and environmental noise. In order to improve qubit performance, it is necessary to identify the types and magnitudes of these errors and reduce them.

1. State preparation and measurement (SPAM)
   (a) intrinsic
   (b) extrinsic
2. Gate
   (a) 1 qubit operation
   (b) 2 qubit operation

It will be useful to classify SPAM errors into two different types, which we will call *intrinsic* and *extrinsic*. Intrinsic SPAM errors are those that are inherent in the state preparation and measurement process. One example is an error initializing the $|0\rangle$ state due to thermal populations of excited states. Another is dark counts when attempting to measure, say, the $|1\rangle$ state. Extrinsic SPAM errors are those due to errors in the gates used to transform the initial state to the starting state (or set of states) for the experiment to be performed.

Intrinsic SPAM errors are of particular relevance to fault-tolerant quantum computing, since it turns out that quantum error correction (QEC) requirements are much more stringent on gates than on SPAM.

1. Dephasing
2. Amplitude and phasing damping
3. Homogeneous depolarizing

1. Localized Markovian
2. Unbiased statistical fluctuation

Below are some concrete examples of quantum noise and quantum operations. They are also important in understanding the practical effects of noise on quantum systems, and how noise can be controlled by techniques such as error-correction.

Quantum operations can be represented in the operator-sum representation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \tag{1}$$

where the operators $\{E_k\}$ are known as operation elements.

1.1 Depolarizing

Imagine we take a single qubit, and with probability p that qubit is depolarized. That is, it is replaced by the completely mixed state, $I/2$. With probability $1-p$ the qubit is left untouched. The state of the quantum system after this noise is:

$$\mathcal{E}(\rho) = \frac{pI}{2} + (1-p)\rho \tag{2}$$

In the operator-sum representation,

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho X) \tag{3}$$

1.2 Amplitude damping

the description of energy dissipation-effects due to loss of energy from a quantum system.

$$\mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger \tag{4}$$

where

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \tag{5}$$

$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}. \tag{6}$$

γ can be thought of as the probability of losing energy. The E_1 operation changes a $|1\rangle$ state into a $|0\rangle$ state, corresponding to the physical process of losing a quantum of energy to the environment. E_0 leaves $|0\rangle$ unchanged, but reduces the amplitude of a $|1\rangle$ state; physically, this happens because a quantum of energy was not lost to the environment, and thus the environment now perceives it to be more likely that the system is in the $|0\rangle$ state, rather than the $|1\rangle$ state.

$\mathcal{E}_{\mathrm{GAD}}$, called generalized amplitude damping, is defined for single qubits by the operation elements

$$E_0 = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{bmatrix}, \tag{7}$$

$$E_1 = \sqrt{p}\begin{bmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{bmatrix}, \tag{8}$$

$$E_2 = \sqrt{1-p}\begin{bmatrix}\sqrt{1-\gamma} & 0\\ 0 & 1\end{bmatrix}, \tag{9}$$

$$E_3 = \sqrt{1-p}\begin{bmatrix}0 & 0\\ \sqrt{\gamma} & 0\end{bmatrix}. \tag{10}$$

where the stationary state ρ_∞, which statisfies $\mathcal{E}_{\mathrm{GAD}}(\rho_\infty) = \rho_\infty$ is,

$$\rho_\infty = \begin{bmatrix}p & 0\\ 0 & 1-p\end{bmatrix}. \tag{11}$$

When γ is replaced with a time-varying function like $1 - e^{t/T_1}$, you can visualize the effects of amplitude damping as a flow on the Bloch sphere, which moves every point in the unit ball towards a fixed point at $|0\rangle$.

1.3 Phase damping

A noise process that is uniquely quantum mechanical, which describes the loss of quantum information without loss of energy, is phase damping. The energy eigenstates of a quantum system do not change as a function of time, but do accumulate a phase which is proportional to the eigenvalue. When a system evolves for an amount of time which is not precisely known, partial information about this quantum phase – the relative phases between the energy eigenstates – is lost. A phase kick, the angle of rotation θ is random. The randomness could originate, for example, from a deterministic interaction with an environment, which never again interacts with the system and thus is implicitly measured. Let us assume that the phase kick angle θ is well represented as a random variable which has a Gaussian distribution $e^{-\theta^2/4\lambda}$ with mean 0 and variance 2λ.

The random phase kicking causes the expected value of the off-diagonal elements of the density matrix to decay exponentially to zero with time, $e^{-\lambda}$. That is a characteristic result of phase damping.

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \tag{12}$$

where

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \tag{13}$$

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}. \tag{14}$$

By applying the unitary freedom of quantum operations, we find that a unitary recombination of E_0 and E_1 gives a new set of operation elements for phase damping;

$$E_0' = \sqrt{\alpha} I \tag{15}$$

$$E_1' = \sqrt{1-\alpha} Z, \tag{16}$$

where $\alpha = (1 + \sqrt{1-\lambda})/2$. Thus the phase damping quantum operation is exactly the same as the phase flip channel.

Phase damping is often referred to as a T_2 relaxation process, for historical reasons, where dephasing time is related to γ as $e^{-t/2T_2} = \sqrt{1-\gamma}$. As a function of time, the amount of damping increases, corresponding to an inwards flow towards σ_z-axis.

1.4 Phase flip

$$\mathcal{E}(\rho) = p\rho + (1-p) Z \rho Z \tag{17}$$

1.5 Bit flip

The bit flip channel flips the state of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1-p$. It has operation elements

$$\mathcal{E}(\rho) = \sqrt{p} I \rho \sqrt{p} I + \sqrt{1-p} X \rho \sqrt{1-p} X \tag{18}$$

1.6 Bit-phase flip

$$\mathcal{E}(\rho) = \sqrt{p} I \rho \sqrt{p} I + \sqrt{1-p} X \rho \sqrt{1-p} Y \tag{19}$$

2 Error models

The decay time, (T_1) and dephasing time, (T_2)

3 Qubit characterization methods

Several methods of qubit characterization are currently available[1]. In chronological order of their development, the main techniques are:

1. quantum state tomography (QST)
2. quantum process tomography (QPT)
3. randomized benchmarking (RB)
4. quantum gate set tomography (GST)

3.1 Quantum state tomography (QST)

3.2 Quantum process tomography (QPT)

A quantum operation on a d-dimensional quantum system can be completely determined by experimentally measuring the output density matrices produced from d^2 pure state inputs.

3.3 Randomized benchmarking (RB)

3.4 Quantum gate set tomography (GST)

GST arose from the observation that QPT is inaccurate in the presence of SPAM errors. In QPT, the starting states must form an informationally complete basis of the Hilbert-Schmidt space on which the gate being estimated acts. These are typically created by applying gates to a given initial state, usually the $|0\rangle$ state, and these gates themselves may be faulty.

According to recent results from IBM, a 50-fold increase in intrinsic SPAM error reduces the surface code threshold by only a factor of 3-4. Therefore QPT – the accuracy of which degrades with increasing SPAM – would not be able to determine if a qubit meets threshold requirements when the ratio of intrinsic SPAM to gate error is large.

This is not an issue for extrinsic SPAM errors, which go to zero as the errors on the gates go to zero. Nevertheless, extrinisic SPAM error interferes with diagnostics: as an example, QPT cannot distinguish an over-rotation error on a single gate from the same error on all gates. In addition, Merkel, et al. have found that, for a broad range of gate error – including the thresholds of leading QEC code candidates – the ratio of QPT estimation error to gate error increases as the gate error itself decreases. This makes QPT less reliable as gate quality improves.

Extrinsic SPAM error is also unsatisfactory from a theoretical point of view: QPT assumes the ability to perfectly prepare a complete set of states and measurements. In

[1] Introduction to Quantum Gate Set Tomography, D. Greenbaum, arXiv:1509.02921

method	assumption	advantage	disadvantage	scalability
quantum state tomography				
quantum process tomography				
randomized benchmarking				
quantum gate set tomography				

reality, these states and measurements are prepared using the same faulty gates that QPT attempts to characterize. One would like to have a characterization technique that takes account of SPAM gates self-consistently. We shall see that GST is able to resolve all of these issues.

Another approach to dealing with SPAM errors is provided by randomized benchmarking. RB is based on the idea of twirling – the gate being characterized is averaged in a such a way that the resulting process is depolarizing with the same average fidelity as the original gate. The depolarizing parameter of the averaged process is measured experimentally, and the result is related back to the average fidelity of the original gate. This technique is independent of the particular starting state of the experiment, and therefore is not affected by SPAM errors. However, RB has several shortcomings which make it unsatisfactory as a sole characterization technique for fault-tolerant QIP. For one thing, it is limited to Clifford gates, and so cannot be used to characterize a universal gate set for quantum computing. For another, RB provides only a single metric of gate quality, the average fidelity. This can be insufficient for determining the correct qubit error model to use for evaluating compatibility with QEC. Several groups have shown that qualitatively different errors can produce the same average gate fidelity, and in the case of coherent errors the depolarizing channel inferred from the RB gate fidelity underestimates the effect of the error. Finally, RB assumes the errors on subsequent gates are independent. This assumption fails in the presence of non-Markovian, or time-dependent noise. GST suffers from this assumption as well, but the long sequences used in RB make this a more pressing issue.

Despite these apparent shortcomings, RB has been used with great success by several groups to measure gate fidelities and to diagnose and correct errors. RB also has the advantage of scalability – the resources required to implement RB (number of experiments, processing time) scale polynomially with the number of qubits being characterized. QPT and GST, on the other hand, scale exponentially with the number of qubits. As a result, these techniques will foreseeably be limited to addressing no more than 2-3 qubits at a time.

GST and RB may end up complementing each other as elements of a larger characterization protocol for any future multi-qubit quantum computer.

TODO: summarize them in table.

4 Type of error mitigation technics

1. Error extrapolation
2. Quasiprobability decomposition
3. Quantum subspace expansion. quantum channels
4. Process tomography protocols
   (a) Gate set tomography

4.1 Quasiprobability decomposition

4.1.1 Motivation

4.1.2 Theory

Utility of "twirling" operations in minimizing the cost[2]. For the extrapolation method, their optimisation is to observe that typically for the classes of noise most common in experiments it is appropriate to assume that the expected values of the observation will decay exponentially with he severity of the circuit noise, rather than polynomial.

4.1.3 Experiment

4.1.4 Summary

4.2 Process tomography protocols

Localized Markovian errors

4.2.1 Motivation

4.2.2 Theory

Single-qubit Clifford gates and measurements are universal in computing expectation values. Any quantum operation is a linear map. Single qubit Clifford gates and measurements yield a complete set of linear independent maps. Any error can be simulated or subtracted by decomposition of the error using complete operation set.

By combining GST and the complete set decomposition, any localized Markovian errors in the QC can be systematically mitigated, so that the error in the final computational output is due to unbiased statistical fluctuation.

[2]Error Mitigation for Short-Depth Quantum Circuits, K. Temme, S. Bravyi, and J. M. Gambetta, Phys. Rev. Lett. 119, 180509

4.2.3 Experiment

4.2.4 Summary

5 Summary

1. Error mitigation method
2. Applicable error type
3. Efficiency
4. Cost (per qubit)