Assignment 3 — Travelling Waves

2019862s

Monday 28th March, 2016

1 Travelling waves

1.1 A single species reaction-diffusion equation

Consider the reaction-diffusion equation with linear population growth,

$$u_t = Du_{xx} + ru, \tag{1}$$

where D, r are positive constants. Suppose that at time $t = 0$, all the population is concentrated at $x = 0$, i.e. $u(x, 0) = \delta(x)$, the Dirac delta distribution.

Question 1.1. *Show that $n(x,t) = u(x,t)\exp(-rt)$ satisfies the diffusion equation*

$$n_t = Dn_{xx}. \tag{2}$$

Solution 1.1. We know that $u(x,t)$ satisfies Equation (1). Thus, we rearrange the expression for $n(x,t)$ to obtain an expression for $u(x,t)$ in terms of $n(x,t)$ as follows

$$u(x,t) = n(x,t)e^{rt}.$$

Substituting the obtained expression for u into Equation (1) yields

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= D\frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t) \\ \implies \frac{\partial}{\partial t}\left(n(x,t)e^{rt}\right) &= D\frac{\partial^2}{\partial x^2}\left(n(x,t)e^{rt}\right) + rn(x,t)e^{rt}. \end{aligned}$$

Using the Chain and Product rules, we get

$$\begin{aligned} \frac{\partial n(x,t)}{\partial t}e^{rt} + rn(x,t)e^{rt} &= D\left(\frac{\partial^2 n(x,t)}{\partial x^2}e^{rt} + \cancelto{0}{n(x,t)\frac{\partial^2 e^{rt}}{\partial x^2}}\right) + rn(x,t)e^{rt} \\ \implies n_t e^{rt} + \cancel{rne^{rt}} &= Dn_{xx}e^{rt} + \cancel{rne^{rt}}. \end{aligned}$$

The two $n(x,t)$ terms cancel on both sides of the equality. Finally, we divide throughout by e^{rt} to obtain

$$n_t = Dn_{xx}$$

as required. Hence, we conclude that $n(x,t) = u(x,t)\exp(-rt)$ satisfies the simple diffusion equation, shown in Equation (2). $\square$

Question 1.2. *Equation (2) with the initial condition $n(0,x) = \delta(x)$ has the fundamental solution*

$$n(x,t) = \hat{n}(x,t) \equiv \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right).$$

Hence derive the corresponding fundamental solution $\hat{u}(x,t)$, for Equation (1) with the Dirac delta initial condition.

Solution 1.2. From Question 1.1 we know that the solution $n(x,t) = u(x,t)\exp(-rt)$ satisfies Equation (2). Imposing the Dirac delta initial conditions, Equation (1) has a solution $\hat{u}(x,t)$ of the form

$$\hat{u}(x,t) = \hat{n}(x,t)e^{rt}.$$

Using the fundamental solution of Equation (2), we can find $\hat{u}(x,t)$ by substituting the given expression for $\hat{n}(x,t)$ above, namely

$$\begin{aligned}
\hat{u}(x,t) =& \hat{n}(x,t)e^{rt} \\
\implies \hat{u}(x,t) =& \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right)\exp(rt) \\
\implies & \boxed{\hat{u}(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2 + 4Drt^2}{4Dt}\right)},
\end{aligned}$$

which is the fundamental solution $\hat{u}(x,t)$ for Equation (1). □

Question 1.3. *Let $u_c > 0$ be a detection threshold. Solve $\hat{u}(x,t) = u_c$ to show the position of the leading edge $x_c(t)$ of the population is given by*

$$x_c(t) = \pm 2t\sqrt{Dr - \frac{D\ln\left(2u_c\sqrt{\pi Dt}\right)}{t}}. \tag{3}$$

Show that the leading edge moves with speed

$$c = \pm 2\sqrt{rD} \tag{4}$$

as time gets large.

Solution 1.3. We have to solve $\hat{u}(x,t) = u_c$, or in other words

$$u_c = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x_c^2 + 4Drt^2}{4Dt}\right).$$

First, we multiply the LHS by the factor $\sqrt{4\pi Dt}$, yielding

$$u_c\sqrt{4\pi Dt} = \exp\left(\frac{-x_c^2 + 4Drt^2}{4Dt}\right).$$

Taking the natural logarithm ($\log_e = \ln$) on both sides of the equation gives

$$\ln\left(u_c\sqrt{4\pi Dt}\right) = \ln\left(\exp\left(\frac{-x_c^2 + 4Drt^2}{4Dt}\right)\right)$$

$$\implies \ln\left(u_c\sqrt{4\pi Dt}\right) = \left(\frac{-x_c^2 + 4Drt^2}{4Dt}\right).$$

Note that since u_c, D, and t are positive constants, we do not need to worry about the absolute value of the expressions in the logarithms. Also, the logarithm on the RHS cancels with the exponential. Now rearranging the equation for x_c^2, we obtain

$$-x_c^2 + 4Drt^2 = 4Dt \ln\left(u_c\sqrt{4\pi Dt}\right) \qquad \text{common denominator,}$$

$$\implies -x_c^2 = 4Dt \ln\left(u_c\sqrt{4\pi Dt}\right) - 4Drt^2 \qquad \text{all terms except } x_c^2 \text{ at LHS,}$$

$$\implies x_c^2 = 4Drt^2 - 4Dt \ln\left(u_c\sqrt{4\pi Dt}\right) \qquad \text{multiply by } -1,$$

$$\implies x_c^2 = 4t^2\left(Dr - \frac{\ln\left(2u_c\sqrt{\pi Dt}\right)}{t}\right) \qquad \text{common factor } 4t^2,$$

$$\implies \sqrt{x_c^2} = \sqrt{4t^2\left(Dr - \frac{\ln\left(2u_c\sqrt{\pi Dt}\right)}{t}\right)} \qquad \text{take square root,}$$

$$\therefore \quad \boxed{x_c = \pm 2t\sqrt{Dr - \frac{2\ln\left(u_c\sqrt{\pi Dt}\right)}{t}}}.$$

We are looking for travelling wave solutions, which maintain a constant shape and move with constant speed. Such solutions satisfy $u(x,t) = \tilde{u}(x-ct)$ for all t, where c is the wave speed. We would like to examine the waves in a frame moving at speed c. We introduce the travelling wave variable z. Let $z = x - ct$, such that $u(x,t) = \tilde{u}(x-ct) = \tilde{u}(z)$. If z is constant, then so is $\tilde{u}(z)$ [5]. In order to determine the travelling wave solutions of Equation (1), we need to map the solutions of the PDE to the travelling wave frame, and hence, transform it to an ODE. Thus, we compute the partial derivatives using the Chain rule as follows.

$$\frac{\partial \tilde{u}}{\partial t} = \frac{d\tilde{u}}{dz}\frac{\partial z}{\partial t} = -c\frac{d\tilde{u}}{dz}, \qquad \frac{\partial \tilde{u}}{\partial x} = \frac{d\tilde{u}}{dz}\frac{\partial z}{\partial x} = \frac{d\tilde{u}}{dz}.$$

Similrly, $\frac{\partial^2 \tilde{u}}{\partial x^2} = \frac{d^2\tilde{u}}{dz^2}$. Substituting the obtained expressions in Equation (1), we obtain

$$-c\frac{d\tilde{u}}{dz} = D\frac{d^2\tilde{u}}{dz^2} + r\tilde{u},$$

which is indeed an ODE. For simplicity, we drop the tilde-s. Let $u' = \frac{du}{dz}$. Then

$$-c\frac{d\tilde{u}}{dz} = D\frac{d^2\tilde{u}}{dz^2} + r\tilde{u} \quad \longmapsto \quad -cu' = Du'' + ru.$$

Not let $u' = v$. Then we have $-cv' = Dv + ru$. Thus, we have to solve the following system of ODEs.

$$\begin{aligned} u' &= v, \\ v' &= -\frac{c}{D}v - \frac{c}{r}u. \end{aligned}$$

We can now analyze the system in the phase plane. Hence, we need to find the eigenvalues and corresponding eigenvectors of the Jacobian matrix of the system, namely

$$J = \begin{bmatrix} 0 & 1 \\ -\dfrac{r}{D} & -\dfrac{c}{D} \end{bmatrix}.$$

Now, let us find the characteristic polynomial and thus, the eigenvalues of the Jacobian.

$$\det \begin{vmatrix} 0-\lambda & 1 \\ -\dfrac{r}{D} & -\dfrac{c}{D} - \lambda \end{vmatrix} = (-\lambda)(-\frac{c}{D} - \lambda) - (-\frac{r}{D}) = 0$$

$$\implies \lambda^2 + \frac{c}{D}\lambda + \frac{r}{D} = 0.$$

The roots of the characteristic polynomial are the eigenvalues we need, namely

$$\lambda_{\pm} = \frac{-\dfrac{c}{D} \pm \sqrt{\dfrac{c^2}{D^2} - \dfrac{4r}{D}}}{2}.$$

Indeed, the real part of the eigenvalue is negative, i.e. $\Re(\lambda) < 0$. In order to ensure the stability of the trivial steady state, we require real eigenvalues, thus we want the expression in the square root to be non-negative, that is

$$\frac{c^2}{D^2} - \frac{4r}{D} \geq 0 \implies c^2 \geq 4r\frac{D^{\cancel{2}}}{\cancel{D}}.$$

However, note that the speed must be a constant. Thus, we neglect the inequality, and we look for the minimum constant speed. Taking the square root, we obtain

$$c = \pm\sqrt{4rD} \implies c = \pm 2\sqrt{rD},$$

as required. □

2 A Lotka–Volterra model of competition

Travelling waves connect two steady states of the underlying spatially-homogeneous one-species systems, so we will look for travelling waves, which make similar connections in a two-species system.

The (scaled) Lotka–Volterra model of competition between species $u(x,t)$ and $v(x,t)$ with diffusion is given by

$$u_t = u_{xx} + u(1-u-av), \tag{5}$$
$$v_t = Dv_{xx} + rv(1-v-bu). \tag{6}$$

Here D is the relative diffusion coefficient, r is the ratio of the growth rates, and a and b are the interspecies competition coefficients. These four parameters are all positive constants. We know from Ex. Sheet 1, Q. 4, the interaction dynamics alone (i.e. in the absence of space) can generate four different outcomes:

- If $a, b < 1$, then both semi-trivial steady states are unstable and the positive coexistence is stable.
- If $a > 1 > b$, then $(0, v^*)$ is stable and $(u^*, 0)$ is unstable. There is no positive coexistence steady state.
- If $b > 1 > a$, we have the previous case in reverse.
- If $a, b > 1$, then both semi-trivial steady states are stable and the positive coexistence steady state is unstable.

The trivial steady state is always unstable. Initially $u = 1$ throughout the spatial domain. Its competitor v is then introduced and we examine if v can invade.

Question 2.1. *Find the linearized PDEs at the* $(u,v) = (1,0)$ *steady state and show that the equation for* v *decouples, and is given by*

$$v_t = Dv_{xx} + rv(1-b). \tag{7}$$

Solution 2.1. Let $f(u,v) = u(1-u-av)$ and let $g(u,v) = rv(1-v-bu)$. Let us first compute the partial derivatives with respect to u and v of each reaction term.

$$\begin{aligned} f_u &= 1-2u-av, & f_v &= -au \\ g_u &= -brv, & g_v &= r-2rv-bru. \end{aligned}$$

Then we linearize f and g around the point $(u,v) = (1,0)$ as follows:

$$\begin{aligned} f(u,v) &= f(1,0) + uf_u\big|_{(1,0)} + vf_v\big|_{(1,0)} + H.O.T. \\ g(u,v) &= g(1,0) + ug_u\big|_{(1,0)} + vg_v\big|_{(1,0)} + H.O.T. \end{aligned}$$

$$\implies \quad \begin{aligned} f(u,v) &= 0 + u(-1) + v(-a) + H.O.T. \\ g(u,v) &= 0 + 0u + v(r-br) + H.O.T. \end{aligned}$$

$$\implies \quad \begin{aligned} f(u,v) &= -u-av \\ g(u,v) &= r(1-b)v. \end{aligned}$$

Therefore, the linearized PDEs at the point $(u, v) = (1, 0)$ are

$$\begin{aligned} u_t &= u_{xx} - u - av, \\ v_t &= Dv_{xx} + r(1-b)v. \end{aligned}$$

Observe that the PDE for v does *not* include a u term, but depends solely on v. Thus, the second PDE decouples, and indeed is

$$\boxed{v_t = Dv_{xx} + r(1-b)v},$$

as required. Hence, we first seek a solution of the v equation, and then derive the u solution from the obtained result. □

Question 2.2. *Use the results from Section 1 above to show that the minimum travelling wave speed for v is*

$$c_v = 2\sqrt{Dr(1-b)}, \tag{8}$$

provided $b < 1$. Explain why v cannot invade when $b > 1$.

Solution 2.2. Equation (8) differs from Equation (1) by a factor of $(1-b)$ in the homogeneous term. Thus, assuming we have the same initial conditions, using the results from Section 1, we can deduce that the leading edge $\tilde{x}_c(t)$ of the v population is given by

$$\tilde{x}_c(t) = \pm 2t\sqrt{Dr(1-b) - \frac{D\ln\left(2v_c\sqrt{\pi Dt}\right)}{t}}.$$

Again, using the same analysis as in Question 1.3, we are looking for travelling wave solutions.

Let $z = x - ct$. Thus, $\dfrac{\partial v}{\partial t} = -c\dfrac{dv}{dz}$ and $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{d^2 v}{dx^2}$. Then Equation (8) becomes

$$-cv' = Dv'' + rv(1-b),$$

where $v' = \dfrac{dv}{dz}$. This gives the following system of ODEs.

$$\begin{aligned} v' &= w, \\ w' &= -\frac{c}{D}w - \frac{rv(1-b)}{D}. \end{aligned}$$

As before, we compute the Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ -\dfrac{r(1-b)}{D} & -\dfrac{c}{D} \end{bmatrix}.$$

And the characteristic polynomial is

$$\det\begin{vmatrix} 0-\mu & 1 \\ -\dfrac{r(1-b)}{D} & -\dfrac{c}{D} - \mu \end{vmatrix} = (-\mu)(-\frac{c}{D} - \mu) - (-\frac{r(1-b)}{D}) = 0$$

$$\implies \mu^2 + \frac{c}{D}\mu + \frac{r(1-b)}{D} = 0.$$

Thus, the two eigenvalues are

$$\mu\pm = \frac{-\dfrac{c}{D} \pm \sqrt{\dfrac{c^2}{D^2} - \dfrac{4r(1-b)}{D}}}{2}.$$

As before, we require the expression under the square root to be positive, in order to have real eigenvalues. Furthermore, we seek the *minimum* speed, which gives the condition

$$\begin{aligned} & c_{\min}^2 = 4rD(1-b), \\ \implies \quad & c_{\min} = \pm 2\sqrt{rD(1-b)}. \end{aligned}$$

Note that $r > 0$, $D > 0$, and $b > 0$. However, we want to make sure that the expression under the radical is positive to avoid a complex valued speed. Thus, we impose the condition that $0 < b < 1$.

We chose the travelling wave variable to be $z = x - ct$, which means that we assume that the wave is travelling in the positive x-direction. So it is natural to seek a speed $c > 0$. Thus,

$$\boxed{c_{\min} = 2\sqrt{rD(1-b)}}.$$

□

Question 2.3. MATLAB *script, using* **`pdepe`** *PDE solver to solve Equations (5) and (6).*

Solution 2.3. The following script contains the PDE solver, BCs, ICs. Note that we specify Neumann boundary conditions

Listing 1: Script used to solve the system of PDEs with Neumann boundary conditions.

```
% Lab 3: 2019862s.

% PDE solver.
function pdeFileNeumannBCs()
m=0;                % Symmetry parameter.
x=0:0.1:400;        % Spatial coordinates.
t=0:10:110;         % Time span.

% Solve the system of PDEs by invoking the PDE function,
% Initial Conditions, and Boundary Conditions.
sol=pdepe(m,@diffusionPDE,@diffusionIC,@diffusionBC,x,t);

% Approximate the u component solution for all t and all x.
u1=sol(:,:,1);
% Approximate the v component solution for all t and all x.
u2=sol(:,:,2);

% Calculating the wave speed from the numerical solutions.
front_locu=zeros;
for i=9:10
```

```
    % Calculate v for t=80 and t=90 for all x. Store if v>0.001.
    front_vecu=u2(i,:)>0.001;
    % Multiply the v vector by the spatial position x
    % to obtain the location of the leading edge (before v=0).
    front_locu(i-8)=max(front_vecu.*x);
end
% Calculate the speed via a finite difference approximation.
speedu=diff(front_locu);
% Find the average wave speed of v by dividing through
% the time interval.
cv=sum(speedu)/10;
disp(cv);

% Plot of u and v as functions of x for all t.
figure
for j=1:12
    plot(x,u1(j,:),'k');
    hold on
    plot(x,u2(j,:),'k--');
end
title('Travelling waves of u and v. Neumann BCs, a=0.5');
xlabel('Distance x');
ylabel('u(x,t) and v(x,t)');
axis([0, 400, -0.1, 1.1]); legend('u(x,t)','v(x,t)');
% -----------------------------------------------------------------
% PDE function - specify parameters, flux, and source terms.
function [c,flux,source] = diffusionPDE(~,~,u,DuDx)
% Parameters:
c=[1;1];    % Coefficients of partial u and v w.r.t. time.
Du=1;     % Diffusion coefficient in u equation.
Dv=2;       % Diffusion coefficient in v equation.
r=1.5;      % Ratio of growth rates

% Interspecies coefficients:
a=0.5;      % u decreases due to interaction with v.
b=0.7;      % v decreases due to interaction with u.

% Define the flux (diffusion term) of the PDEs.
flux=[Du;Dv].*DuDx;

% Define the source (reaction term) of the PDEs.
source=[u(1)*(1-u(1)-a*u(2)); r*u(2)*(1-u(2)-b*u(1))];
% -----------------------------------------------------------------
% Initial conditions
function u0 = diffusionIC(x)
% Initiate a vector for the ICs.
u0=[1;1];
% If statement for the value of x.
if x >= 0 && x <= 10     % For x=[0,10],
```

```
70     u0(2) = 1;              % v = 1.
71 else                        % For x=(10,400],
72     u0(2) = 0;              % v = 0.
73 end
74 % -----------------------------------------------------------------
75 % Boundary conditions
76 function [pl,ql,pr,qr] = diffusionBC(~,~,~,ur,~)
77 % Neumann boundary conditions.
78 % No specifications about
79 % the values of the functions on the left, this is
80 % already taken care of by the ICs.
81 pl=[0;0];
82 % Both derivatives w.r.t. x are zero on the left. Neumann.
83 ql=[1;1];
84 % The values of u and v on the right are calculated
85 % numerically by the pdepe sovler, no need to specify BCs.
86 pr=[0;0];
87 % Both derivatives w.r.t. x are zero on the right. Neumann.
88 qr=[1;1];
89
90 % % Mixed boundary conditions.
91 % % Both u and v are already described by ICs on the left.
92 % pl = [0;0];
93 % % Both derivatives w.r.t. x are zero on the left. Neumann.
94 % ql = [1;1];
95 % % Set u=1 and v=0 on the right boundary. Dirichlet BC.
96 % pr = [ur(1)-1;ur(2)];
97 % % Both derivatives w.r.t. x are zero on the right.
98 % qr = [0;0];
```

In particular, the script above has the following functionalities:

- Lines 4–16: The PDE solver.
  - Lines 5–7: Specify the spatial and time coordinates.
  - Line 11: The variable `sol` contains the numerical solutions for u and v, obtained by invoking the `pdepe` command, which calls the system of PDEs (`@diffusionPDE`), the initial conditions (`@diffusionIC`), and the boundary conditions (`@diffusionBC`).
  - Lines 14 and 16: `u1` is the variable, storing the u solutions; `u2` is the variable, storing the v solutions.
- Lines 19–32: Calculating the wave speed (code is copied from `WaveSpeed.m` program on Moodle).
  - Line 19: Initiate a vector for the `front_locu` variable (pre-allocate for faster speed).
  - Line 20: A `for` loop, which iterates through $t = 80$ and $t = 90$.

- Line 22: `front_vecu` is a row vector with entries 1 or 0, consisting of 4001 columns (which is the number of spatial intervals). We are interested in the last column entry with value 1 — this is where $v > 0.001$. Think of the binary values stored in `front_vecu` as the following rule — when $v > 0.001$, we hold a value of 1 for true; when $v < 0.001$, we hold a value of 0, indicating the rule is false. Then $v = 0$ for the rest of x. When $t = 80$, the last column entry where $v = 1$ is with index 1661, and when $t = 90s$, the index is 1849.
- Line 25: Here, we seek the location of the front of the wave at $t = 10$ and $t = 20$. We take the pointwise product of the row vector `front_vecu` with the row vector x. Thus, we obtain the spatial coordinates, where $v = 1$, right before it drops down to $v = 0$. We seek the `max` entry, that is, the final location where $v = 1$. This occurs at the 1661 spatial interval for $t = 80$, which is when $x = 166$, and at the 1849 interval for $t = 90$, which is when $x = 184.8$.
- Lines 28–32: We use a finite difference formula for approximating the speed — take the difference, and divide by the length of the time interval, that is $184.8 - 166 = 18.8$ and then $18.8/10 = 1.88$. **N.B.** *Note that we iterate through* $t = 80$ *and* $t = 90$ *as this is when the wave is finally stabilized and thus, has constant speed.*

• Lines 35–44: Plotting the solutions of u and v as functions of x at time $t = [0 : 10 : 110]$ on the same graph.

• Lines 47–62: Specifying the system of PDEs to be solved.

- Lines 49–56: Specifying the parameters of the system — coefficients of partial derivatives with respect to time, of diffusivity, of growth rates, and of interspecies interactions.
- Line 59: Define the flux term of the PDEs, that is, the diffusion coefficients in front of u_{xx} and v_{xx}.
- Line 62: Define the source term, or in other words, the homogeneous terms of the two PDEs.

• Lines 65–73: Specifying the initial conditions of the PDEs.

- Line 67: Initialize a column vector $u0$.
- Lines 69–73: An `if` statement for the values of v. If $x \in [0, 10]$, then $v = 1$. Otherwise (`else`), $v = 0$.

• Lines 77–88: Specifying the boundary conditions of the PDEs.

- Lines 81 and 83: Neumann boundary conditions on the left. We consider only the derivatives of u and v with respect to time and require them to be equal to zero. That is, we require smooth, continuous, and constant values of u and v in order to avoid any 'jumps' / 'peaks' / 'drops'.
- Lines 86 and 88: Neumann boundary conditions on the right as well. Same as above.

It is very important to discuss the choice of boundary conditions. There are four possible combinations for the boundary conditions.

- Neumann boundary conditions on the left — `pl=[0;0]` and `ql=[1;1]`, and Neumann BCs on the right — `pr=[0;0]` and `qr=[1;1]`.
- Neumann BCs on the left — `pl=[0;0]` and `ql=[1;1]`, and Dirichlet BCs on the right — `pr=[ur(1)-1;ur(2)]` and `qr=[0;0]`.
- Dirichlet BCs on the left — `pl=[ul(1)-1; ul(2)-1]` and `ql=[0;0]`, and Neumann on the right — `pr=[0;0]` and `qr=[1;1]`.
- Dirichlet BCs on the left — `pl=[ul(1)-1; ul(2)-1]` and Dirichlet BCs on the right — `pr=[ur(1)-1;ur(2)]` and `qr=[0;0]`.

Initially, it seemed like all four would work properly. However, after implementing all four combinations, unexpected behaviour was observed for the travelling waves at $t = 10$, when Dirichlet BCs were implemented for the left boundary. This is illustrated later on in the report. *Thus, it is necessary to impose Neumann BCs on the left boundary.* The right boundary, on the other hand, did not cause any problems whatsoever. Hence, both Neumann and Dirichlet BCs work perfectly fine for the right boundary. This is why the script in Listing 1 contains the commented code in Lines 92–98. There is no difference between running the script with the boundary conditions in Lines 81–88, or with the ones specified in Lines 92–98. □

Question 2.4. *Plot the solutions for u and v as functions of x at time $t = [0 : 10 : 110]$ on the same graph. Now let $a = 0.5$. What are the differences and similarities between the travelling wave solutions in the two cases? Explain why this difference occurs.*

Solution 2.4. Let us first explore what happens to the travelling waves with Dirichlet BCs on the left. Figure 1 uses Dirichlet BCs on both sides, and Figure 2 uses Dirichlet on the left and Neumann on the right. The parameters are $a = 1.5$, $b = 0.7$, $r = 1.5$, and $D = 2$.

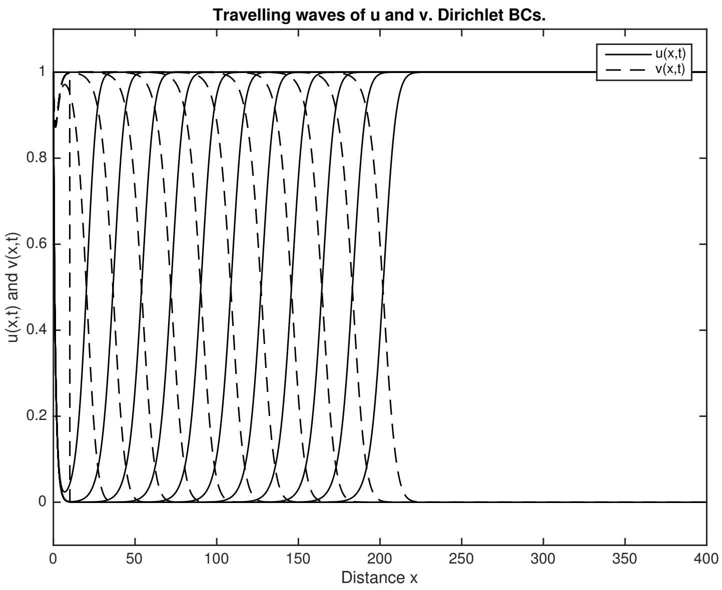


Figure 1: Plot of u and v as functions of x for all t with Neumann BCs.

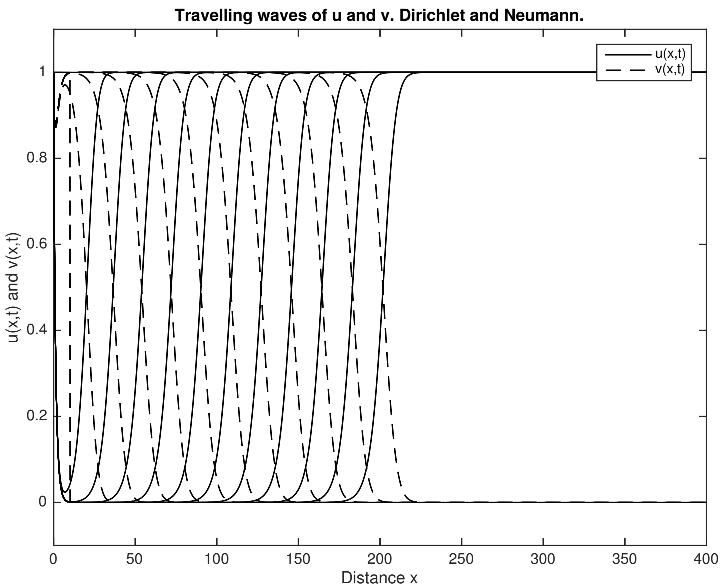


Figure 2: Plot of u and v as functions of x for all t with mixed BCs.

Observe that there is a 'drop' in both figures at $t \approx 10$, which is exactly the behavior we discussed in the previous question. A zoomed in version of the drop is provided below in Figure 3.

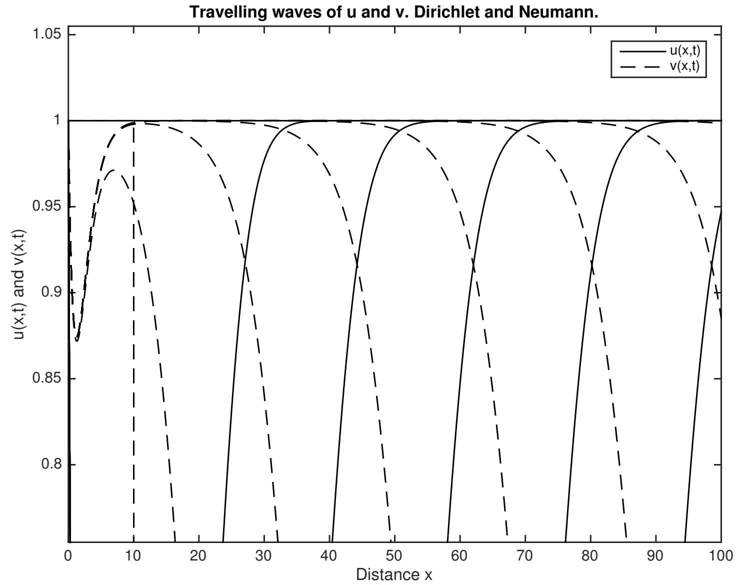


Figure 3: A close up of the drop at $t = 10$ when we use Dirichlet BCs on the left.

Having illustrated the strange behaviour of the waves when we apply Dirichlet BCs on the left, we now use the appropriate Neumann conditions on the left. We run the script to plot the solutions of u and v using Neumann BCs. Figure 4 is created using Neumann BCs on both sides, and Figure 5 implements Neumann on the left and Dirichlet on the right. From now on, we refer to the combination of Neumann on the left and Dirichlet on the right as *the proper mixed BCs*. The parameters are again $a = 1.5$, $b = 0.7$, $r = 1.5$, and $D = 2$.

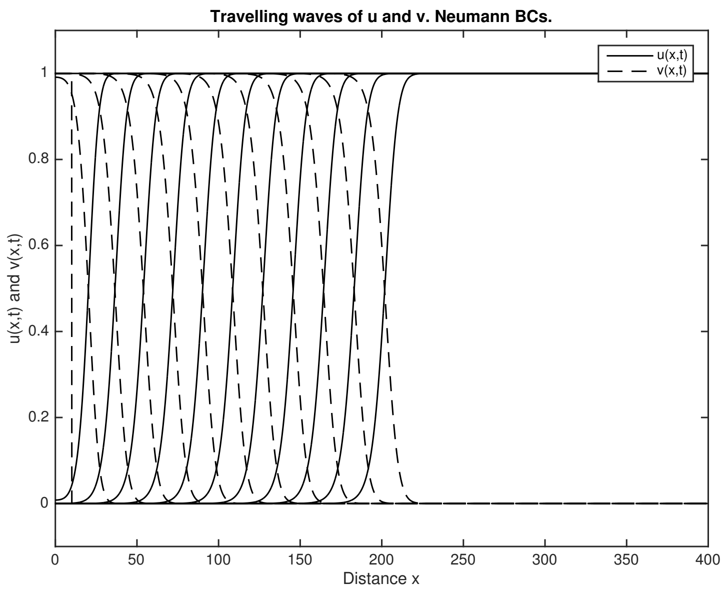


Figure 4: Plot of u and v as functions of x for all t with Neumann BCs.

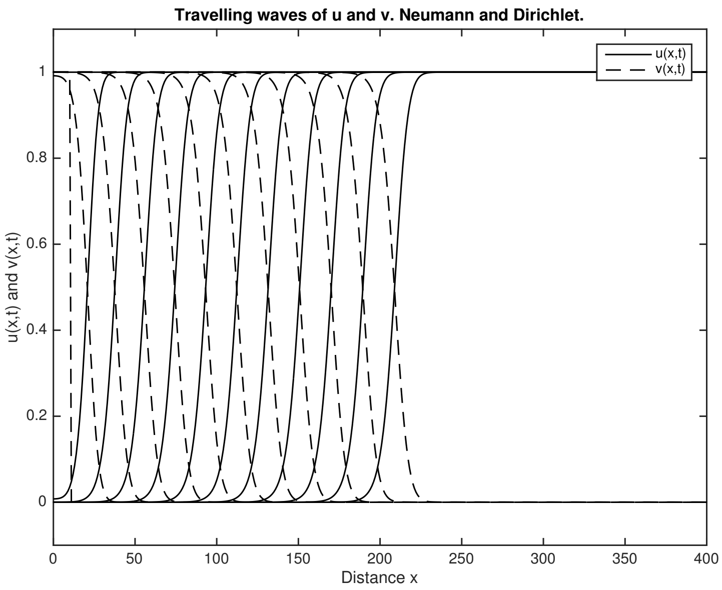


Figure 5: Plot of u and v as functions of x for all t with proper mixed BCs.

It is evident from the figures above that both the Neumann and the proper mixed BCs yield the same graphs of the waves. Furthermore, we note that the parameters used for the figures above are $a = 1.5$, $b = 0.7$, $r = 1.5$, and $D = 2$. Now, we set $a = 0.5$ and using Neumann and proper mixed BCs, we plot the solutions of u and v as functions of x.

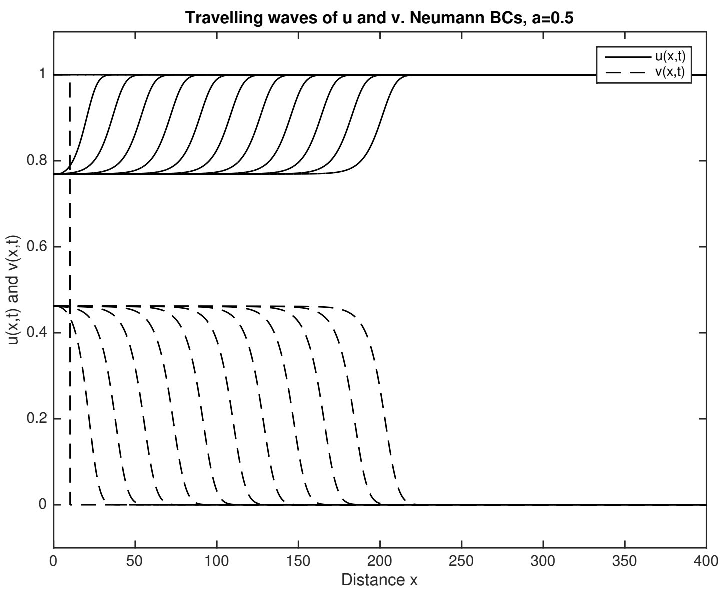


Figure 6: Plot of u and v as functions of x for all t with Neumann BCs and $a = 0.5$.

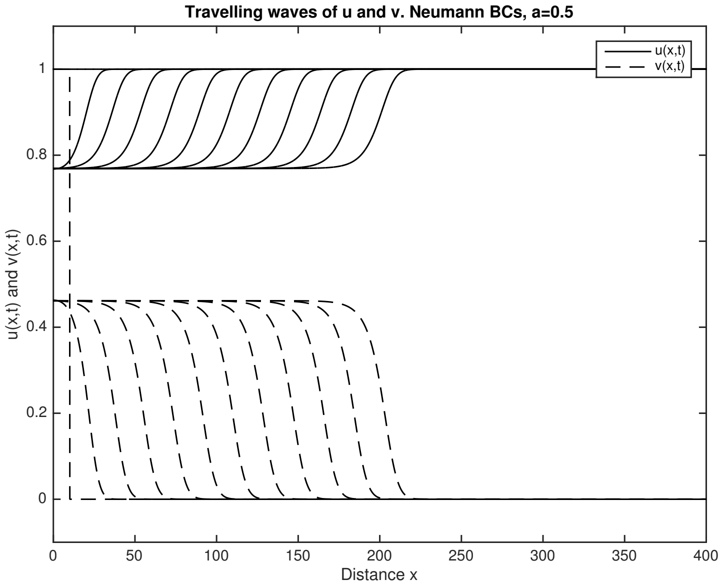
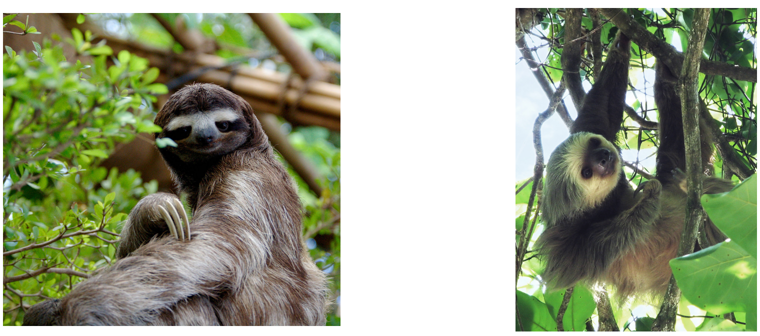


Figure 7: Plot of u and v as functions of x for all t with proper mixed BCs and $a = 0.5$.

Again, we observe that the graphs in both Figures 6 and 7 are the same. Thus, from now on, we simply use Neumann BCs on both the left and right, as we have justified why both Neumann on the left and right, and Neumann on the left and Dirichlet on the right work.

Note that both in Figure 4 and in Figure 6 there is a dashed vertical line at $t = 10$, which is formed due to the initial conditions that $v = 1$ for $t = [0, 10]$. **N.B.** For illustrating purposes, we assume that species u is the three-toed sloth *Bradypus* (see Figure 8a), species v is the two-toed sloth *C. hoffmanni* (see Figure 8b), the space $x = [0, 400]$ is a 1-dimensional island measured in meters on the west of Quepos, Costa Rica, and the time scale $t = [0, 110]$ is measured in days [6]. The two sloth species are competing over plant resources and they do not predate on each other. The island is isolated, and we assume there are no other species interfering with the existence of the sloths.

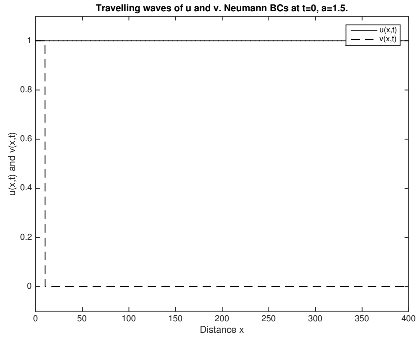


(a) The three-toed sloth *Bradypus* Link .

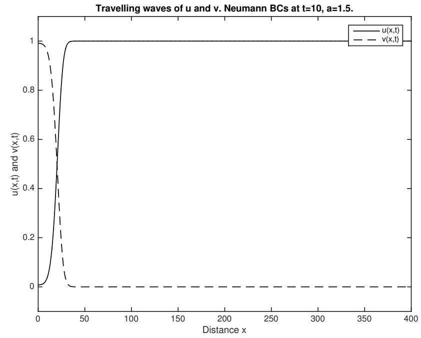
(b) The two-toed sloth *C. hoffmanni* Link.

Figure 8: Sloths. Source: Wikipedia [6].

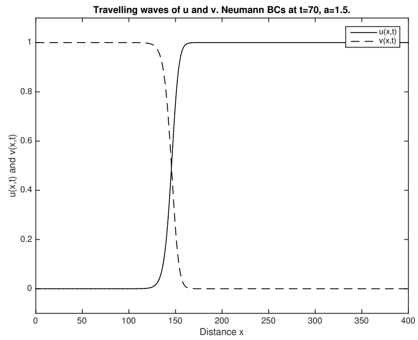
In order to be able to analyze the graphs in more detail, below we plot the travelling waves of u and v with $a = 1.5$ at three particular time instances, $t = 0$, $t = 10$ and $t = 70$.



(a) Plot of u and v as functions of x at $t = 0$ and $a = 1.5$.



(b) Plot of u and v as functions of x at $t = 10$ and $a = 1.5$.

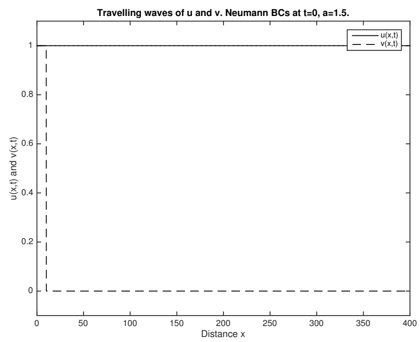


(c) Plot of u and v as functions of x at $t = 70$ and $a = 1.5$.

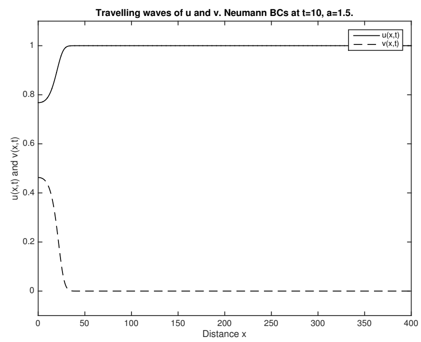
Figure 9: Plots of u and v at $t = 0$, $t = 10$ and $t = 70$ and $a = 1.5$.

Let us now analyze the produced graphs. Figure 9a shows the initial conditions. Observe that $u = 1$ for all x, and $v = 1$ for $x = [0, 10]$. This means that our 1-dimensional island is initially inhabited by the three-toed sloth u. Then we introduce the two-toed sloth v in the first 10 meters of the island. As time increases, the sloths v (slowly) invade the island and start feeding on the plants. At $t = 10$, sloths v have already propagated to $x \approx 35$ (c.f. Figure 9b). This causes sloths u to move forward in space, i.e. they move to the inner parts of the island. As time progresses, sloths v successfully invade the island, and cause the native sloths u to be dispersed in a much smaller area. This leads to a much greater competition for resources both between and within the species. Thus, we conclude that when the interspecies coefficients $b < 1$ and $a > 1$, the sloths v successfully invade the territory of the u species. Thus, the steady state $(0, v^*)$ is stable, the steady state $(u^*, 0)$ is unstable, and there is no positive coexistence, as sloths v are the more dominant species and outcompete the u sloths. Notice that this is expected as a shows the rate at which the u sloths decrease due to the interaction with the v sloths, and b shows the reverse. Hence, we conclude that when $a = 1.5$ and $b = 0.7$, the two-toed sloths v invade successfully the three-toed sloths u, and outcompete them.

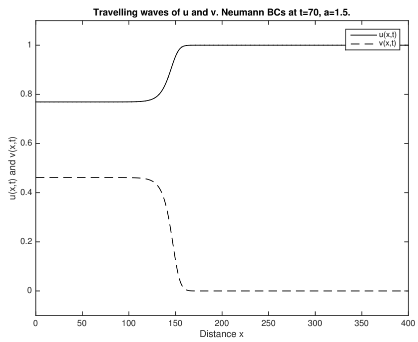
Next we set $a = 0.5$ and examine the travelling wave solutions.



(a) Plot of u and v as functions of x at $t = 0$ and $a = 0.5$.



(b) Plot of u and v as functions of x at $t = 10$ and $a = 0.5$.



(c) Plot of u and v as functions of x at $t = 70$ and $a = 0.5$.

Figure 10: Plots of u and v at $t = 0$, $t = 10$ and $t = 70$ and $a = 0.5$.

On the day we introduce sloths v to the island inhabited by sloths u, we observe the same setting as before. On the tenth day, $t = 10$, the v sloths have settled on one part of the island, say the eastern part, and sloths u are concentrated in the western (see Figure 10b). As time progresses, both species continue propagating on their respective parts of the island. There is no competition for plant resources between the species, and we observe that sloths u and v happily coexist. Hence, when the interspecies coefficient $a = 0.5$ there is very little interspecific competition and the two species exhibit positive coexistence. Thus, indeed we confirm the assumption that when both $a, b < 1$, there is positive coexistence and this is the stable steady state. In terms of similarities, in the two cases ($a = 1.5$ and $a = 0.5$), both the u and v travelling waves propagate with approximately the same speeds, in the same direction. However, when $a = 1.5$, both u and v range between 0 and 1, in contrast with when $a = 0.5$ — u starts at ≈ 0.78, and v at ≈ 0.44, meaning that the density of the u sloths is higher than the density of the v sloths . This occurs because the positive coexistence steady state is with coordinates (u_c, v_c), which can be found from the system of PDEs in Equation 5 and 6. Also, when $a = 1.5$, the speed wave $c_v = 1.8800$, and when $a = 0.5$, $c_v = 1.8700$. We see that the difference in the speeds of the leading edges of the v travelling waves is not significantly high. However, we expect a large difference if we vary b. Hence, we proceed to the next question to examine the effect of b on the wave speed c_v. □

Question 2.5. *Run the* MatLab *code and vary b in order to record the values of the wave speed, obtained for each value of b. You will find the theoretical prediction accurately predicts the travelling wave speed. What effect does b have on the wave speed? By considering the interaction between u and v in the PDE explain why you might expect b to have this effect on the wave speed.*

Solution 2.5. We use the parameters as above $a = 1.5$, $r = 1.5$, and $D = 2$, and vary $b = \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Table 1 below shows the values obtained numerically and theoretically for the wave speed c_v upon varying b.

Table 1: The effect of b on the wave speed c_v

Coefficient b	Numerical wave speed c_v	Theoretical wave speed c_v
0.1	3.2800	3.2863
0.3	2.8800	2.8983
0.5	2.4300	2.4495
0.7	1.8800	1.8974
0.9	1.0900	1.0954

Further, we plot the obtained numerical and theoretical values for the wave speed c_v against b.

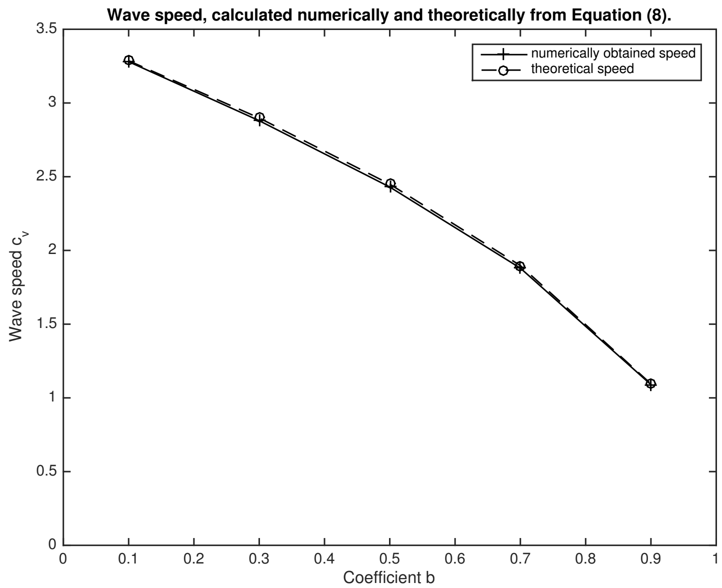


Figure 11: Plot of the wave speed c_v against b.

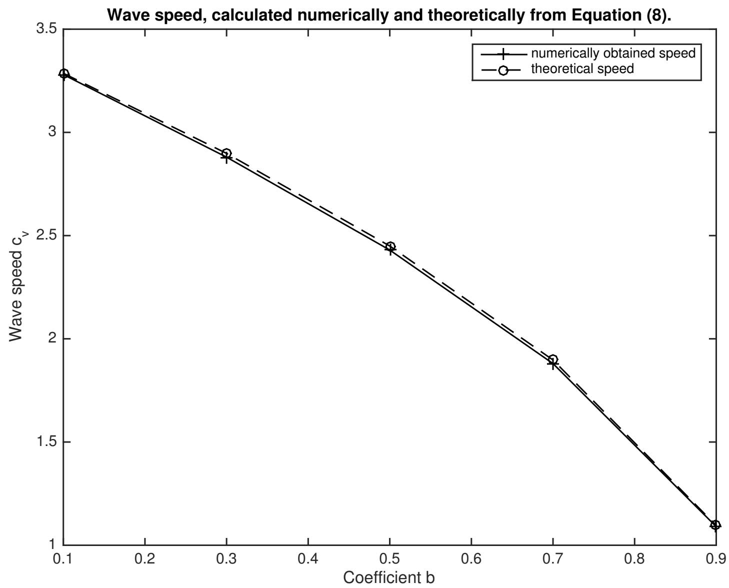


Figure 12: Zoomed version of the plot of the wave speed c_v against b.

We can see from Figure 12 that indeed the theoretical prediction accurately predicts the travelling wave speed c_v, which is expected as $c_v \sim \sqrt{1-b}$. Even though there are some small differences, as shown in Table 1, the overall prediction is quite accurate. Hence, we observe that an increase in the value of b leads to a decrease of the wave speed c_v. This makes sense as b is the interspecies coefficient, which describes the rate at which the sloths v decrease due to interaction with the sloths u. Thus, as b increases, the competition between the two species increases, and the invading species v are unable to outcompete the native species u, in other words, the impact of u on v increases. Hence, the travelling wave of the v sloths is unable to propagate in space, which is mathematically expressed by a very slow or non-existent wave speed at all. Note that b is theoretically strictly bounded above by 1 (as $c_v \notin \mathbb{C}$). However, when running the MATLAB code with values of b greater than 1, we still obtain some travelling wave solutions, but the wave speed approaches 0. For instance, if $b = 1.1$, then $c_v = 0.47$, if $b = 1.2$, then $c_v = 0.32$; if $b = 1.3$, then $c_v = 0.1900$, etc. Thus, we conclude that the theoretical approximation is quite accurate, and we can neglect the numerical values of c_v for $b > 1$, as $c_v \to 0$. □

Question 2.6. *Now examine what happens to the travelling wave when $D = 0.01$, which is equivalent to u having a much larger diffusion coefficient than v. As u is travelling faster than v we can see from the plot that the population density of u at the front of the wave is less than 1. Why would this mean the theoretical wave prediction underestimates the wave speed? Would you expect the theoretical wave speed prediction to be accurate if $D >> 1$? Explain your answer.*

Solution 2.6. Now we plot the solutions for u and v for $x = [0 : 0.05 : 30]$ and $t = [0 : 10 : 110]$. The plot is presented below in Figure 13.

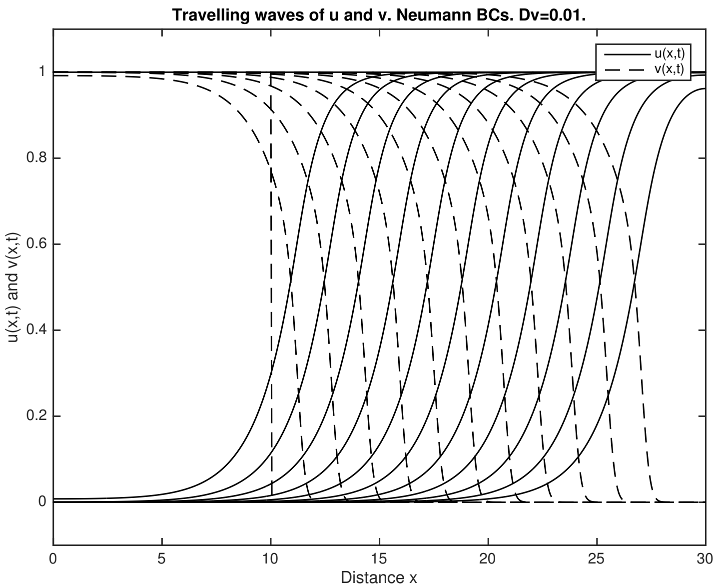


Figure 13: Plot of u and v as functions of x, where $x = [0 : 0.05 : 30]$, for all t.

The numerically computed value for the wave speed is $c_{v,num} = 0.1600$. On the other hand, the value of c_v predicted by Equation (8) is $c_{v,th} = 0.1342$. A screenshot of the console is shown below in Figure 14.

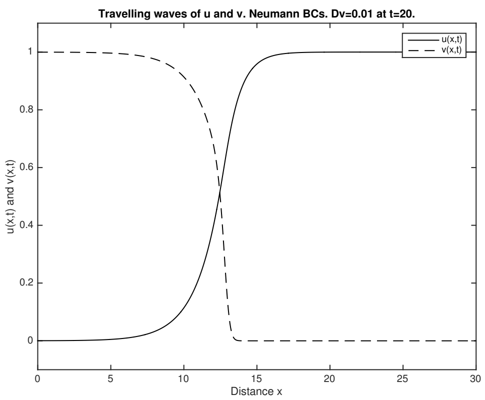
```
cvNum =

    0.1600


cvTh =

    0.1342
```

Figure 14: Console output for the numerical and theoretical values of c_v when $D = 0.01$.

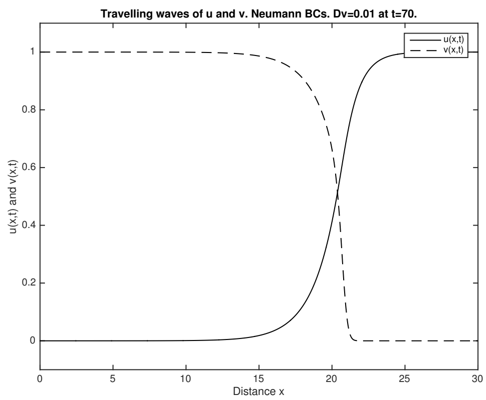
Thus, we see that the theoretical prediction for the wave speed underestimates the numerically computed value, which may be caused by both of either of the following:

- The numerically computed value is obtained by a finite difference method. Thus, as long as MATLAB registers a difference between the locations of the leading edges, a numerical value is computed.
- The theoretical prediction is calculated using Equation (8), which is $c_v = 2\sqrt{Dr(1-b)}$. There is no explicit dependence on the interspecies coefficient a, on the diffusivity of species u ($D_u = 1$), or simply on the ratio of the two diffusivities. Thus, we can safely assume that the theoretical prediction does not yield precise results, if we vary the value of D.

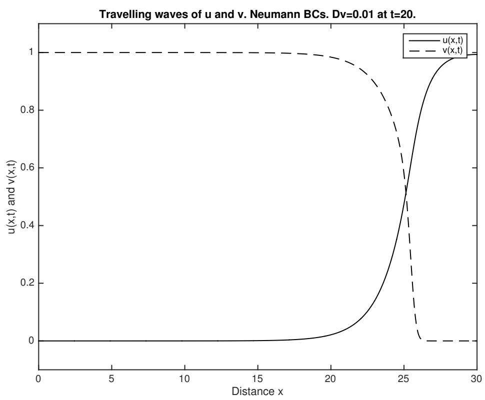
Now let us analyze the plots in Figure 13. We observe a similar trend to the one presented in Figure 4 — there is no positive coexistence. Now this time the sloths u have much larger diffusivity, meaning that the rate at which they propagate in space is higher, compared to the diffusivity of the v sloths. The ratio between the diffusivities (if we assume that the diffusivity of u is 1, and that the diffusivity of v is 0.01), is $\dfrac{D_u}{D_v} = \dfrac{1}{0.01} = 100$, which means that the u sloths are travelling 100 times faster than the v sloths. This is somewhat reflected on the plot in Figure 13, but in order to see it more clearly, we plot three particular instances below.



(a) Plot of u and v at $t = 20$ and $D = 0.01$.



(b) Plot of u and v at $t = 70$ and $D = 0.01$.



(c) Plot of u and v at $t = 110$ and $D = 0.01$.

Figure 15: Plots of u and v at $t = 20$ and $t = 70$ and $D = 0.01$.

Now we can clearly see how the sloths propagate in time. At $t = 20$, we see that sloths u have propagated to ≈ 15 meters in the island, while the invading sloths v are behind at ≈ 13 meters. The same trend is observed at $t = 70$ and $t = 110$. Notice, however, that in this setting the leading edge of the travelling wave of the u sloths is slightly below 1. Also, the slopes of the u and v waves are different, making them non-symmetric. Thus we see that the invasion of the v sloths, even though much slower, is still successful, and causes a disturbance in the density of the u sloths. This means that the native u sloths are harmed by the v sloths, which may result in a decrease of the u sloth population due to possibly low amounts of plant resources. This occurs since indeed the u sloths propagate faster in space, but there exists some interspecies competition with the invading sloths

v, which leads to an uneven distribution of the sloths u, and hence, a density less than 1. Again, this is clearly reflected in the numerical solution, but underestimated by the theoretical one, as the latter does not take into account any of the coefficients, which describe the reaction-diffusion behavior of the u species. So far we can conclude that D is not the diffusivity of the v sloths, but rather represents a ratio between the diffusivities D_u and D_v. Most likely, during the non-dimensionalization of the system of PDEs, a substitution was made, and D was defined as $D = \dfrac{D_u}{D_v}$. Hence, the numerical value for the wave speed, which uses the finite difference method, is much more precise as it relies on computing the values for the wave fronts of v by solving the whole system of PDEs and considering all coefficients, including a, which is the interspecific competition coefficient in the equation for u. On the contrary, the theoretical prediction is solely based on the values of r, D, and b, which is unacceptable as we are solving a system of PDEs, not a single PDE for v. Even though theoretically v decouples, as shown analytically above, the numerical solution of the system of PDEs takes into account both equations and gives more precise results.

In general, if indeed D is defined as the ratio between the diffusivities, we can expect the theoretical approach to be yielding accurate predictions for a specific range of values. We saw in the previous question that the theoretical prediction correctly predicts the wave speed for $D = 2$. Then, as we changed $D = 0.01$, the theoretical prediction definitely underestimated the value for the wave speed. Thus we can expect that the theoretical prediction accurately predicts the wave speed c_v for values close to 2. A small numerical experiments was conducted. Below in Table 2 we summarize the obtained results.

Table 2: Varying the values of D to predict the wave speed c_v accurately

Diffusivity ratio D	Numerical wave speed c_v	Theoretical wave speed c_v
0.01	0.1600	0.1342
0.1	0.4300	0.4243
0.3	0.7300	0.7348
0.5	0.9400	0.9487
4	2.6500	2.6833
5	2.9700	3.0000
6	3.2400	3.2863

From the results obtained in Table 2 we see that the numerical and theoretical wave speeds behave nicely (meaning that the two values are approximately the same) for values of the diffusivity $D = [0.3, 5]$. Now, the reasons for choosing this range:

- For the infimum, we choose the last value before the theoretical prediction underestimates the numerical one, i.e. $c_{v,th} < c_{v,num}$, which is at $D = 0.1$, hence, we select $D = 0.3$.

- For the supremum, we allow a tolerance of 1%. Hence, we choose the value where the theoretical exceeds the numerical value by less than or equal to 1%, which occurs at $D = 5$. The next step at $D = 6$ we have an increase of 0.0143%, hence, we choose the previous one.

Of course, this approximation is very crude, and the experiment can be further extended by calculating the values of the wave speed for much more values of the diffusivity. However, it serves as a first order approximation to understand the behavior of the theoretical prediction, and to find sensible bounds for D, where the theoretical and numerical predictions agree. Thus, if the ratio between D_u and D_v, that is, D, is between 0.3 and 5, or in other words, if $D_u = 0.3D_v$ and $D_u = 5D_v$, the theoretical prediction yields a sensible value for the value of the wave speed, similar to the numerical approximation using the finite difference method. If, however, D is beyond those bounds, $D >> 1$, then we should use a numerical solver, as we cannot neglect the effect of the equation describing the reaction-diffusion of the species u. $\square$

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