Snapshot of four benchmark models

For Macro Modeling Class Term Project

Dr. Guanyi Yang
Assitant Professor
Economics Department
St. Lawrence University
gyang@stlawu.edu

January 23, 2019

Contents

1 Neoclassical RBC model

A simple and original version of the RBC model (or optimal growth model can be written as a form of competitive market economy with a representative household and a representative firm in a general equilibrium, or a planner's economy.

I will start with a version of competitive market economy, and then a version of a planner's economy.

1.1 Competitive market economy version

1.1.1 HH

There is a representative HH who is living infinitely. The HH maximizes lifetime utility as her value at any given point in time. Every period, in order to maximize lifetime utility, she chooses current labor supply N_t, current consumption C_t, and savings $K_{t+!}$. This same problem repeats every period.

How to write an infinite horizon problem recursively:
Let V be the lifetime value of the HH at any given period, where $V_t = \sum_t^\infty \beta^t u(c_t, l_t)$. While this is one way to write/describe V_t, by summing up all the utility values every period from period t on until infinity; we can also think in another way. Since this HH is living forever, it is less meaningful to keep track time period by period. For this HH, everyday is a repeat of previous day. In this case, we only need to differentiate the difference of today (t) and tomorrow ($t+1$). At time t, $V_t = \sum_t^\infty \beta^t u(c_t, l_t)$; while at time $t+1$, $V_t + 1 = \sum_{t+1}^\infty \beta^t u(c_{t+1}, l_{t+1})$. And $V_t = u(c_t, l_t) + \sum_{t+1}^\infty \beta^t u(c_{t+1}, l_{t+1})$. Then you can simply write it $V_t = u(c_t, l_t) + \beta V_{t+1}$, for any t in her infinity lifetime. We can further abstract away from subscript t and $t+1$, and use an apostrophe $'$ to represent $t+1$:

$$\begin{aligned} V(K) &= \max_{C,N,K'} \{u(C, L) + \beta V(K')\} \\ &\text{s.t.} \\ C + K' &= wN + (1+r)K + \Pi - I \\ L + N &= 1 \\ K' &= (1-d)K + I \end{aligned} \tag{1}$$

State variable:
This is the best example of showing what state variable looks like. In the HH's problem, we have several quantity variables C, N, K', K, I, among which C, N, I are the endogenous

choices the HH makes every period. However, K is something that's made in the previous period, passing down to the current period; while leaving this period with an update of K' for the next period. In other words, K carries information from the history to the current state. Entering current state with given K (exogenous) to the current period. One makes decisions of how much capital to accumulate, which endogenously generate K' from the current period.

When solving this problem, the first thing is to solve the state variable K', given each K. Hence we call K' a policy function, or decision rule, or law of motion. We can give it another name: $K' = g(K)$ to describe this state variable essence (i.e. given previous value K, how much is K' if this HH optimizes lifetime value). Then, the rest of the float variables (C, N, I) are merely a function of K'.

1.1.2 Firm

The firm is a much simpler entity. Instead of maximizing lifetime value, it only needs to decide how much labor N_t to hire and how much capital K_t to rent from HH so to make the most profit every period. The period profit of the firm comes from total revenue from output Y, subtracting labor cost at wage rate w, capital rental cost at rate r, and adding back un-depreciated capital at the end of the period $(1-d)K$.

$$\Pi = \max_{N,K}\{Y - wN - rK + (1-d)K\} \tag{2}$$

We can assign output Y according to the production function:

$$Y = F(K, N)$$

1.1.3 General equilibrium

A recursive competitive equilibrium is a set of functions: quantities $G(K), g(K)$, Value $V(K)$, Prices $r(K), w(K)$, such that:

1. $V(K)$ solves HH's problem and $g(K) = K'$ is the associated policy function.

2. Prices are competitively determined

$$r(K) = D_1F(K, N) + 1 - d$$

$$w(K) = D_2F(K, N)$$

3. Individual decisions are consistent with aggregates.

Notice that in the definition of general equilibrium, I made skipped several steps. In price setting, I already assumed that firms make optimal choices. In a competitive equilibrium, $MRPN = w$ and $MRPK = 1 + R$, where $R = r + d$. This is coming from the first order condition of firm's profit function. Taking first order derivative *w.r.t.* labor N, you get the wage function; and taking first order derivative *w.r.t.* capital K, you get the interest rate function. In addition, I already assumed directly, all HH's savings goes to firm's capital. In other words, HH's lend her additional resources to firm as productive capital and receive rent payment $R = r + d$ (interest payment plus depreciation payment.

1.2 RBC model in a planner's economy

For our purpose, this may be a much simplified way to look at the issue. There is a benevolent planner who already knows the optimal allocation of resources, and cares about the maximum welfare of the society (measured by everyone's utility).

$$\begin{aligned} V(K) &= \max_{C,N,K'} \{u(C,L) + \beta V(K')\} \\ &\text{s.t.} \\ C + K' &= F(K,N) + (1-d)K \\ L + N &= 1 \end{aligned} \tag{3}$$

Notice that the physical resource constraint has already skipped a few steps. We can write it in a more patient way:

$$\begin{aligned} C + I = Y &= F(K,N) \\ K' &= I + (1-d)K \end{aligned}$$

1.3 Bewley-Hansen/Rogerson Indivisible labor

Rogerson et al. (1988) and Hansen (1985) use an employment lottery concept to representative HH's labor supply choices. Each household is like a communal living that has many members. Each member draw a lottery and decides whether to work or not. Working member of the HH brings back money to share with the entire HH. Each member doesn't care whether she is working or not; only the HH cares the share of the members who gets the leisure. You can think of it in the sense that members taking turns to work and rest decided by a fair lottery with outcome matches the work/leisure ratio within the household.

This is a theoretical extension to the original RBC model, which gives a utility function that makes a lot of the theoretical results tractable. This abstract also comes from the observation in the U.S. labor market where majority of individuals work full time, or 1/3 of a day with minor fluctuations.

This theoretical extension gives a nice utility function to the RBC model:

$$U(c, l) = logc - B(1 - l) \tag{4}$$

where $n = 1 - l$, and B is this disutility parameter that serves as a lottery, determining the share of members in a HH working.

Some commonly used parameter values for simple version of RBC models (If your time period is one quarter):

- subjective discount rate $\beta = 0.99$;
- capital depreciation rate $d = 0.0176$;
- disutility parameter of working $B = 2.786$;
- production function $F(K, N) = K^\alpha N^{1-\alpha}$, where capital share of output $\alpha = 0.261$

1.4 Further readings and topic exploration

The model introduced here are the foundation of the neoclassical optimal growth and real business cycle framework. The original groundwork of this framework comes from Solow (1956), Cass (1965), Ramsey (1928), Koopmans et al. (1963), King, Plosser, and Rebelo (1988), Lucas Jr et al. (1976), Lucas (1977), etc.

As the most widely used workhorse models in modern real macro studies, this framework can help you describe a lot of aggregate dynamics and answer a lot of policy questions. Here I list a few potentially interesting and doable extensions for the term project:

1.4.1 Business Cycle Fluctuations

You can introduce a series of aggregate fluctuations to the model. The model framework listed above as no uncertainty at all. One way to extend it to study aggregate real business cycle fluctuation of the economy is to introduce a aggregate exogenous state variable, a TFP shock Z to the model. Often times, the shock in studies follow what econometricians called AR(1) process:

$$logZ' = \rho logZ + \epsilon$$

where ϵ is a random *iid* shock that comes from a normal distribution $\epsilon \in N(\mu, \sigma)$. Some possible topics to explore in this direction can be examining and understanding the business cycle quality of the model. You may want to find data on the mean, variance, and auto-correlation of GDP with other major macro variables, and find the corresponding parameter values in the shock process that allow the model to replicate what data has.

1.4.2 Fiscal policies

The framework described above has no government, hence no tax at all. In reality, we have all kinds of tax implemented by a government. In a stationary steady state equilibrium, this is also a good starting framework to examine what impact certain tax policy would do to the economy.

Between the relationship of tax and income, we can put tax in three categories: proportionary tax, progressive tax, and regressive tax. Proportionary tax means that regardless of income level, one pays τ portion of your income to tax. Progressive tax means that the higher the income you get, the more tax you pay. Regressive tax means that the higher the income, the less tax you pay.

There are many ways of implementing the tax policy. You may have distortionary tax, as a tax rate on some parts of the budget constraint, or lump-sum tax. One most common example of distortionary tax is a distortionary income tax: $w(1-\tau)$. One most simple lump-sum tax is just adding a $-T$ to the end of the budget constraint. You can certainly branch out to other forms of tax, or tax on other features we have in the model (including firms too).

You also need to think about what the government is going to do with the tax. One easy way is just to say, using it in government spending G. A more trickier way is to think of tax as a transfer back to the economy. Then you need to think who to receive the transfer and in what way (lump-sum or proportion, and to what feature).

2 Search and match model

Search-and-match model is widely applied in various scenarios. Consumers search for sellers, and adding a commonly agreed and acceptable good, money, to reduce search friction, we can model how money play a role in an otherwise classical dichotomy world. Unemployed workers seek jobs and firms with unfilled positions seek unemployed workers. Given the labor market condition, not everyone finds a job immediately. Marriage market, single persons look for partners; housing market pairs buyers and sellers; etc.

In this section, I focus on the application of search-and-match framework in a labor market. Compared to the RBC framework (often referred to as friction-less neoclassical model too), a search-and-match model is the most widely used framework that studies 1-1 relationship between each job seeker and a job, and the most widely used frictional economy framework.

2.0.1 Overview of the model

There is a representative worker looking for a job to maximize lifetime value, and there is a firm deciding to post vacancies. Job seeker and firm meet through a matching function that transforms unemployed workers and unfilled jobs into matched productive pairs. Paired worker and firm then bargain over wage based on a "surplus" sharing rule.

The model can be written in a market/decentralized economy or in a planner's economy. I only present the decentralized version of the economy here.

2.0.2 Individual job seekers

For the sake of simplicity, we assume all workers are "hand-to-mouth" workers. They eat up all what they have and there is no savings.

Individuals can either be matched (employed), or unmatched (unemployed). We need to separately describe the value of matched vs unmatched workers.

Unmatched worker has the following lifetime value V^u, where one receives a simple linear utility function $u(c) = c$ (the utility is 1-1 linked to the value of consumption). Since the worker has no labor income, we assume there is some unemployment benefit b, which is all the budget she can use. In the next period, she have probability p_1 that she will find a job and become a matched worker V^e, and otherwise (probability $1 - p_1$) remaining as unemployed V^u.

$$\begin{aligned} V^u &= c + \beta(p_1 V^e + (1 - p_1)V^u) \\ &\text{s.t.} \\ c &= b \end{aligned} \tag{5}$$

Likewise, a matched/employed worker has the following lifetime value V^e. The worker receives wage income w. Matched workers also face some exogenous probability of losing the job (job separation shock) δ, and become unemployed V^u. Hence, with probability $1 - \delta$, she still remain employed.

$$\begin{aligned} V^e &= c + \beta((1 - \delta)V^e + \delta V^u) \\ &\text{s.t.} \\ c &= w \end{aligned} \tag{6}$$

Adding aggregate uncertainty:

The original version of the search-and-match model in labor market was invented to study the business cycle fluctuations of the labor market. In the spirit of that, let's introduce an aggregate shock to the economy z. The shock enters the economy to the production function of the firm (will formally introduce right after), as a TFP shock. The shock follows a Markov Chain process. Markov Chain is just a fancier way of saying the shock today depends on the shock yesterday (hence, a chain). To formally describe it, we let z_i of any date (subscript i describes the shock today) comes from a grid of choices $z_i \in Z = \{z_1, z_2, z_3 ..., z_{N_z}\}$. The probability of us landing in a particular realization of state z from the $Z = \{z_1, z_2, z_3 ..., z_{N_z}\}$ is: $\pi_{ij} = Pr\{z' = z_j | z = z_i\}$. What this is saying is that: the probability of getting a particular value for z_j (tomorrow), given where we are currently at z_i with some particular value is π_{ij}. I will teach you how to incorporate that in class using Tauchen (1986) method [1].

Incorporating such uncertainty, we can transform the worker's values to:

$$
\begin{aligned}
V^u(z_i) &= c + \beta(p_1(z_i)\sum_{j=1}^{N_z}\pi_{ij}V^e(z_j) + (1-p_1(z_i))\sum_{j=1}^{N_z}\pi_{ij}V^u(z_j)) \\
\text{s.t.} \\
c &= b
\end{aligned}
\tag{7}
$$

What I did here, is that given economy being good or bad (different values of z_i), the value of this worker/job seeker may differ. Hence $V(\cdot)$ is a function of z_i. We also know from intuition, that if the economy is good, it is easier for workers to find jobs, otherwise it is difficult. Hence the probability of finding job is a function of aggregate state too $p_1(z_i)$. And lastly, we don't really kno what future is going to be like. We can only guess where future is, given our current state. So we assign a probability pi_{ij}, given where we are now at i, the probability of going to state j for each possible value tomorrow. We sum up the value of each of these possible realizations for tomorrow and get our expected value of tomorrow, when we are at today's standpoint. This gives us $\sum_{j=1}^{N_z}\pi_{ij}V^e(z_j)$, where we sum up the expected value of good or bad condition of tomorrow being employed, and $\sum_{j=1}^{N_z}\pi_{ij}V^u(z_j))$ where we sum up the expected value of good or bad condition of tomorrow being unemployed.

Likewise, we have the value function of an employed worker:

$$
\begin{aligned}
V^e(z_i) &= c + \beta((1-\delta)\sum_{j=1}^{N_z}\pi_{ij}V^e(z_j) + \delta\sum_{j=1}^{N_z}\pi_{ij}V^u(z_j)) \\
\text{s.t.} \\
c &= w(z_i)
\end{aligned}
\tag{8}
$$

[1]We assume the aggregate productivity shock follows: $logz' = \rho logz + \epsilon$, where $\epsilon \sim N(0, \sigma_e^2)$

2.0.3 Firms

The firm has a slightly different concept as what you have seen before in RBC style models. There is one firm owned by the HH. The firm can post unlimited number of jobs based on the profitability of each job posting (or market condition). Once firm decides to post a job, the job is on the market looking for employees. Similar to individual workers, we have two types of jobs: matched/employed job J^e, and unmatched/unfulfilled job J^u.

$$J^u(z_i) = -\kappa + \beta(p_2(z_i)\sum_{j=1}^{N_z}\pi_{ij}J^e(z_j) + (1-p_2(z_i))\sum_{j=1}^{N_z}\pi_{ij}J^u(z_j)) \tag{9}$$

Once firm decides to post a new vacancy, it needs to pay vacancy posting fee κ. If the job is unfulfilled, the position doesn't produce any revenue. Hence the current period value of a new unfulfilled posting is: $-\kappa$. In the next period, the job has probability $p_2(z_i)$ of finding a worker and become a productive filled position J^e. Alternatively, $1-p_2(z_i)$ is the probability the job is left unfilled.

$$J^e(z_i) = z - w + \beta((1-\delta)\sum_{j=1}^{N_z}\pi_{ij}J^e(z_j) + \delta\sum_{j=1}^{N_z}\pi_{ij}J^u(z_j)) \tag{10}$$

For a filled position, the firm is productive. I assume the production function $y = z_iF(N) = z_i$ (i.e. each worker generate output 1, multiplied by TFP z_i).Firm has to pay the worker wage w. Same as worker's scenario, any matched pair has a job destruction probability δ. If the job is unlucky, it will get separated, and become an unfilled position J^u next period with probability δ. Otherwise, it stays productive and matched with probability $1-\delta$.

All the aggregate notation and interpretation for the firm follows directly from individual worker's case.

Free entry condition:
Observe the value of unfilled position J^u. If the position is not filled, it needs to pay a fixed vacancy posting cost. In a competitive *labor* market, firm will continually entering the market and posting new vacancies if it sees a positive vacancy value $J^u > 0$. This $J^u > 0$ may happen if the future expected value of a vacancy posting is larger than the initial posting cost κ. And the future positive value comes from the value of being matched $J^e(z_j)$.

However, the more jobs posted, the harder it is for firms to recruit workers, since total number of population is fixed. So the more the firm needs to pay to attract workers. This drives down the profit value of a potentially matched position $J^(z_j)$. Therefore, it drives down the value of a vacancy posting in the current period. In other words, the more jobs posted, the harder and less profitable it is for each job.

If, on the other hand, the value of unfilled job is negative $J^u < 0$, firm will not enter the market to hire new workers at all. The posting cost outweighs the future expected profit.

In equilibrium, therefore, we will have a balance, in which $J^u = 0$, as what we call **free entry condition**. There is just right number of jobs posting vacancies where the future expected profit just cancels out the posting cost κ. This condition should hold in all time as long as we are in equilibrium. And it makes everything so much easier, i.e. you can substitute $J^u = 0$ into any where of the above equations where there is J^u.

Let's see how it simplifies everything:

First from the unfilled position value, we get:

$$0 = -\kappa + \beta p_2(z_i) \sum_{j=1}^{N_z} \pi_{ij} J^e(z_j) \tag{11}$$

Since Equation 11 is produced by our condition when $J^u = 0$ for J^u from Equation 9, let's officially refer the **free entry condition** as Equation 11.

Then, substitute $J^u = 0$ to Equation 10, the matched position value, we get:

$$J^e(z_i) = z - w + \beta(1-\delta) \sum_{j=1}^{N_z} \pi_{ij} J^e(z_j) \tag{12}$$

2.0.4 Matching technology

One part of the model I left unexplained, the probability of job seekers matched with a job p_1 and the probability a vacancy matched with a employee p_2. In other words, how do vacancies and job seekers get matched?

In this framework, we use this matching technology, $m(u, v)$.

$$m(u, v) = A u^{\alpha} v^{1-\alpha} \tag{13}$$

In the above Equation 13, u represents the number of job seekers (unemployed number), and v represents the number of vacancy postings. $m(u, v)$ is the magic market, where job seekers and vacancies meet and generate number of matched pairs. Giving number of job seekers u, number of vacancies v, we feed in the function and get $m(u, v)$, which is the exact number of new matches every period.

Using u and v, we can form an index: **Labor market tightness** $\theta = v/u$. If v is large, θ is large, or the labor market is tight, meaning the economy is good, and it is hard for firm to hire a worker; vice versa.

The function is the right hand side of Equation 13, with parameter $A > 0$ describing the efficiency of the labor market matching process. $0 < \alpha < 1$ describes elasticity of matching w.r.t u, or how elastic/responsive it is for a firm to find an employee given each additional job posting.

Given $m(u, v)$ being the number of new matches, and u being number of job seekers, we can form the probability of job seekers getting hired: $p_1 = m(u, v)/u$. If you plug in the explicit format of $m(u, v)$ and definition of θ, we will have the following closed form of job finding probability:

$$p_1 = \frac{m(u, v)}{u} = A\theta^{1-\alpha}$$

And similarly, we can define the probability of each vacancy posting finding an employee as: $p_2 = m(u, v)/v$, and explicitly:

$$p_2 = \frac{m(u, v)}{v} = A\theta^{-\alpha}$$

Notice, previously in defining values for individuals and job postings, we have aggregate state z_i entering the probability $p_1(z_i)$ and $p_2(z_i)$. This is because v is endogenously generated by the economy from the free entry condition, hence market tightness $\theta = \frac{v}{u}$ is endogenously generated because of v. When the economy is good, we have $v(z)$ high, more job openings, and when the economy is bad, we have $v(z)$ low, few openings. Transferring this concept into market tightness, we have: $\theta(z)$ fluctuating with the aggregate shock z. Using the previous functional format, we know, now, that z is the aggregate state variable of the economy (even though it is exogenous to the model); and $p_1(z), p_2(z)$ are endogenous probability depending on the aggregate state of the economy.

2.0.5 Wage sharing rule

Lastly, we want to know how the wage rate is set in the economy. Since you probably have noticed, the model is quite different in the sense that it is not a competitive labor market per period. Free entry condition only determines the profitability of job vacancy posting. When free entry condition is met, the matched jobs will have a positive profit as in Equation 11.

Another way to look at it, in an RBC model where we have competitive market economy, it is when a large number of firms and a large number of workers coming to one market for one job, hence competitive, and wage is set by the market to clear labor supply and demand; everyone is a price taker. In search-and-match framework, only one worker and one job is met, none-competitive. We need to give the model an arbitrary wage setting. One commonly used wage setting rule is through Nash Bargaining. The job seeker and the job bargain for a wage (really a surplus sharing rule), so that it reaches some optimal sharing decision. And this is completely none-market force, and only arbitrary.

$$w(z) = arg\max_{w}\{(V^e(w;z) - V^u(z))^{\phi} J^e(w;z)^{1-\phi}\} \tag{14}$$

Equation 14 describes such arbitrary bargaining rule. For both sides, the wage is only acceptable if the matched value is higher than unmatched value. For a worker, she would like to accept a wage that makes her employed value $V^e(w;z)$ higher than (or at least equal to) her stay-home unmatched value $V^u(z)$. Otherwise, why would she work? And here, an official name for it, $(V^e(w;z) - V^u(z))$, is matched surplus of the worker.

Same for the job post. The matched value for a job $J^e(w;z)$ has to be greater than (at least equal) to the unfilled value J^u, which is 0 given free entry condition. And the official name here $J^e(w;z) - 0$ is the matched surplus of the job post.

Parameter $0 < \phi < 1$ is the relative bargaining power of worker. If *phi* is high, the worker has high bargaining power in finding the wage rate; otherwise the firm has high bargaining power. $w(z)$ is set to make this surplus sharing objective (right hand side of Equation 14) optimal.

You can assign parameters values following from Hagedorn and Manovskii (2008):

- b Value of unemployment insurance 0.955;
- ϕ Workers bargaining power 0.052;
- α Matching elasticity 0.66 (Nakajima, 2012);
- A Labor market efficiency 0.6246 (Nakajima, 2012);
- κ Cost of vacancy posting 0.584;
- β Discount rate 0.991 for annual and $0.991^{\frac{1}{12}}$ for monthly
- δ Separation rate 0.0081;
- ρ Persistence of productivity shock 0.9895;
- δ_e^2 Variance of iid shock in productivity process 0.0034;

2.1 Further readings and topic exploration

While many people have built on the search-and-match frameowork, Diamond (1982), Mortensen (1982), and Pissarides (1985) jointly developed the original form of the framework and receive Nobel price in 2010. Many important studies contribute to our understanding of DMP framework, including Shimer (2005), Hall (2005), Hagedorn and Manovskii (2008), and so on.

As introduced at the beginning of search-and-match section, many directions of applications can be explored using the search-and-match framework.

2.1.1 Business Cycle Fluctuations

The original introduction of search-and-match framework in the labor market is to study the business cycle qualities of labor market. Since the model framework introduced above has already included aggregate fluctuations, you only need to simulate the economy and examine how the model describe the business cycle qualities of labor market fluctuations. In particular, we can look at the standard deviations and auto-correlation of job posting, unemployment, labor market tightness, job finding rate, and job separation rate, and their relationship to the aggregate fluctuation. It is a great project if you can compare your model generated values to the real data moments; and possibly changing the parameter values I assigned above to find the model generated values as close to data as possible.

2.1.2 Business cycle labor market policies

In this domain, you may also try some policy counterfactuals after solving the model. We already have unemployment insurance b as a great policy to examine. Increasing or decreasing the unemployment insurance, how does labor market fluctuate in the business cycle? It sheds light to any unemployment protection and its impact on business cycle dynamics.

In addition, you may also consider changing the parameter value of worker's bargaining power α. This can be related to legal protections to workers. Government raising employee protection litigation increases bargaining power of the workers. What does that do to the labor market fluctuations?

3 Lucas fruit tree asset pricing model

The models above describe aggregate goods market and labor market. The credit market, a major market in aggregate economy, is left un-examined. Lucas fruit tree asset pricing model lays the foundation in describing risky asset ownership decisions and how risky premium is priced in asset. Rather than the completely unique framework as the previous two, this one is mainly playing a change at the budget constraint by introducing a risky asset.

3.1 Overview of the model

Lucas Jr (1978) describes his model by assuming that each household initially owns a fruit tree (share of risky asset/stock). These trees yield fruit (dividends) with uncertain amount, which households consume but cannot store. Therefore, the name of the model as *Lucas fruit tree asset pricing.*

3.2 Firm

There is one representative firm with *decreasing returns to scale* technology ($0 < \alpha < 1$) and with only labor input:

$$y = F(n) = zn^{\alpha}$$

. Aggregate TFP z follows a Markov chain as described in previous models (mostly in search-and-match framework). To formally describe it, we let z_i of any date (subscript i describes the shock today) comes from a grid of choices $z_i \in Z = \{z_1, z_2, z_3..., z_{N_z}\}$. We assume the aggregate productivity shock follows: $logz' = \rho logz + \epsilon$, where $\epsilon \sim N(0, \sigma_e^2)$. The probability of us landing in a particular realization of state z from the $Z = \{z_1, z_2, z_3..., z_{N_z}\}$ is: $\pi_{ij} = Pr\{z' = z_j | z = z_i\}$. What this is saying is that: the probability of getting a particular value for z_j (tomorrow), given where we are currently at z_i with some particular value is π_{ij}.

The firm's problem is static:

$$d(z) = \max_{n} \{zn^{\alpha} - wn\} \tag{15}$$

Notice here I use d to represent dividend rather than π as what we frequently seen in an one-period model in class. In a competitive market, wage is paid at marginal product of labor, and all dividends are rebated to households.

3.3 HH

There is one representative household. Each household holds a fruit tree (i.e. in an economy of one firm and one HH, each HH just own this one firm). We assume HH doesn't value leisure, and only values consumption, so she will spend all her time working (i.e. $n = 1$). Hence, her wage income is $w(z_i)$.

$$V(s, z_i) = \max_{c \geq 0, s' \geq 0} \{u(c) + \beta \sum_{j=1}^{N_z} \pi_{ij} V(s', z_j)\}$$
$$\text{s.t.} \tag{16}$$
$$c + p(z_i)s' = (d(z_i) + p(z_i))s + w(z_i)$$

HH chooses how much share of stocks to own next period s', given current period share of stock s. Here, share of stock is the fruit tree. The stock is priced (asset pricing) at price $p(z_i)$. The stock price fluctuates in the market depending on the aggregate state. In addition, the "fruit tree" also yields fruits or dividend $d(z_i)$.

Equity return:

One purchase each share of stock at price $p(z_i)$, and receives the return of each share of stock (or risky asset) as $d(z_j) + p(z_j)$. The equity return is formally defined as:

$$e(z_t; z_{t+1}) = \frac{d(z_{t+1}) + p(z_{t+1})}{p(z_t)}$$

3.4 General equilibrium

A Recursive Competitive Equilibrium in our stock market production economy is a set of functions for value $V(s, z_i)$, quantities $g(s, z_i)$ and $n(z_i)$, and prices $w(z_i)$, $d(z_i)$ and $p(z_i)$ such that:

1. $V(s, z_i)$ solves HH's problem in Equation 16, and $g(s, z_i)$ is the associated decision rule for share of equity $s' = g(s, z_i)$.

2. Prices are competitively determines:

$$w(z_i) = az_i$$

$$d(z_i) = (1 - a)z_i$$

.

3. Individual decisions are consistent with aggregates.

$$g(1, z_i) = 1$$

For this model framework, let's try the benchmark parameterization as having utility function: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, and parameter values:

- α Labor share of output 0.66;
- γ Risk aversion parameter 1.5;
- β Subjective discounting factor 0.96 (for annual frequency, you may adjust it to the corresponding values for other frequencies);
- ρ Persistence of aggregate shock 0.909 (Khan and Thomas, 2013);
- σ_e^2 variance of *iid* part of aggregate shock 0.014 (Khan and Thomas, 2013).

3.5 Further readings and topic exploration

The foundation of asset pricing models is laid by Lucas Jr (1978). Later on, the most important advancement comes from Mehra and Prescott (1985), a discovery of equity premium puzzle. Nada (2013) has a rather good complication of history of studies on this line of research.

3.5.1 Equity premium puzzle

One pf the biggest questions in the literature of asset pricing is centered at equity premium puzzle. What it is saying is that given a standard CRRA utility function, the risk aversion ratio γ value that is often used in everywhere else can not generate the equity premium observed in the data. If this is some direction of exploration, the original Mehra and Prescott (1985) and the literature review by Nada (2013) are definitely two papers to start reading.

What you can do for this topic is that you may want to find empirical evidence that describe the equity returns over some long period of time, finding the mean and variance of equity return. You may also want to repeat the analysis by finding the mean and variance of some benchmark risk free asset over the same period of time. Equity premium is the mean of risky asset return minus the mean of risk-less asset return.

Then, using the benchmark model framework, you can change the value for γ and find out what value would it need to generate the level of risky equity return you observed in data.

4 Overlapping generation model

Rather than a stand alone model framework like the previous ones, overlapping generation model is a concept that introduces aging in any otherwise infinite horizon framework. It creates a description of lifecycle (and even intergenerational relationship) into any model framework.

Instead of making the benchmark setting overly complicated, I present the overlapping generation concept in a partial equilibrium under the otherwise standard RBC framework.

4.1 Lifecycle HH

Instead of an infinitely living household, the household in benchmark framework lives a set years of life T, after which, this person "exits" the model and disappears; otherwise, everything is the same as the neoclassical RBC model.

Once we know that at age $T+1$, the hh doesn't exist any more. At age T, the last existing age, this person is not going to save anything at all, and eat up everything she has. Hence the value function becomes:

For age $t = T$:

$$\begin{aligned} V_T(s, z_i) &= \max_{c \geq 0, 0 \leq l \leq 1} \{u(c, l)\} \\ &\text{s.t.} \\ c &= s + w(z_i)n \\ n + l &= 1 \end{aligned} \tag{17}$$

Once we know our condition in the last age, we build the model backwards, age by age. Writing it recursively, we have:

For age $t < T$:

$$\begin{aligned} V_t(s, z_i) &= \max_{c \geq 0, s' \geq 0, 0 \leq l \leq 1} \{u(c, l) + \beta \sum_{j=1}^{N_z} \pi_{ij} V_{t+1}(s', z_j)\} \\ &\text{s.t.} \\ c + s' &= (1+r)s + w(z_i)n \\ n + l &= 1 \end{aligned} \tag{18}$$

In this setting, instead of aggregate TFP shock that z represented in previous ones, I let z to be an idiosyncratic shock. Or putting it even more straightforward, z is a stream for wage rate that varies from time to time, i.e. $w_t = z_t$. The value of wage received at age t is the shock realization z_t at age t.

Since it is a partial equilibrium framework, we can call it a day now. There is no firm, nor any need to declare a general equilibrium definition. There is also no endogenous price generation, since wage rate w is given by the random shock process, exogenously.

4.2 Overlapping generations

The setup described above merely talks about one generation. How does it have overlapping generation?

Overlapping generation means that in any given period of time t, you would have people with all ages present in the economy. Each age represents a generation, hence overlapping generations. Table 1 describes this concept. At every date t, there is an old generation reaching age T exits the model and disappears. In the meantime, there is a new generation entering the model starting at age 1 to replace the existing generation. Hence every generation overlaps for some periods.

Table 1: Overlapping generation at time t

t	1	2	3	...	T-1	T	T+1	...
generation 1	age 1	age 2	age 3	...	age T-1	age T		
generation 2		age 1	age 2	...	age T-2	age T-1	age T	
generation 3			age 1	...	age T-3	age T-2	age T-1	...
.				...				...
.				...				...
.				...				...
generation T-1				...	age 1	age 2	age 3	...
generation T				...		age 1	age 2	...
generation T+1				...			age 1	...
.				...				...
.				...				...
.				...				...

This overlapping concept doesn't impact the solving algorithm of the model in the stationary equilibrium. In simulating the model, however, that's when we display the "overlapping" idea.

We can parameterize the utility function as: $u(c,l) = \frac{c^{1=\gamma}}{1-\gamma} + \psi\frac{(1-n)^{1-\sigma}}{1-\sigma}$. We can assign parameter values as:

- ρ Persistence of wage shock 0.986;
- σ_e^2 Variance of *iid* part of wage stream 0.036;
- *gamma* Risk aversion 2;
- *sigma* parameter for Frisch elasticity 4.3;
- β Subjective discounting 0.96 (for annual level);

- r Interest rate 0.04 (set since there is no equilibrium concept to generate interest rate);
- T total years of life 50.

4.3 Further readings and topic exploration

The overlapping generation concept was pioneered by Allais (1947), Samuelson (1958) and Diamond (1965). It has ever since expanded to incorprate into almost every models in modern macroeconomics. Auerbach, Kotlikoff, et al. (1987) and Attanasio (1999) have a more recent collection of reviews of overlapping generations (OLG) models.

4.3.1 Risk and lifecycle consumption profile

One of the most widely used purpose of OLG framework is to model the lifecycle profiles of HHs. As the foundation of macroeconomic dynamics, only we learn how the consumption of each individual behave, do we learn how aggregate consumption fluctuation is formed. In this topic, you need to look for data that describe the life-cycle consumption profiles of households, solve the model and find if your model can generate the lifecycle consumption stream that is similar to data pattern.

You can play more in this direction. While finding the consumption pattern over the lifecycle, you may also record the savings pattern over the lifecycle, as well as labor supply choices.

Now, play with the risk part of the model, the z shock. If you increase the variance of the shock σ_e^2, say, doubling it, what is going to happen to the simulated data patterns (consumption, savings, and labor supply) over the lifecycle? Why do you think they change the way they are?

4.3.2 Social security and retirement policy experiment

You may modify the model a little to describe not just working age, but also retirement age. For example, let's say people work for $T = 40$ years, and then retire, and live under retirement plan for another 15 years. The retired individuals don't earn labor income. Instead, they have retirement income.

For the sake of simplicity, let's just make the retired individuals having budget constraint:

$$c = b$$

, where b is retirement social security benefit. Since it is a partial equilibrium framework, you don't necessarily need to worry about where this b comes from. You may play with the value of b, and see how people change their lifetime labor supply, savings, and consumption choices.

How about increase the retirement age? You may change the working age from T years to $T+5$, then people retire. Since we fixed the total age each individual can live, previously retirement has 15 years, now you only have 10 years in retirement. What would people do in their consumption, savings, and labor supply?

References

Allais, M., 1947. Economie et intérêt. Imprimerie National, Paris .

Attanasio, O. P., 1999. Consumption. Handbook of macroeconomics 1, 741–812.

Auerbach, A. J., Kotlikoff, L. J., et al., 1987. Dynamic fiscal policy. Cambridge University Press.

Cass, D., 1965. Optimum growth in an aggregative model of capital accumulation. The Review of economic studies 32, 233–240.

Diamond, P. A., 1965. National debt in a neoclassical growth model. The American Economic Review 55, 1126–1150.

Diamond, P. A., 1982. Wage determination and efficiency in search equilibrium. The Review of Economic Studies 49, 217–227.

Hagedorn, M., Manovskii, I., 2008. The cyclical behavior of equilibrium unemployment and vacancies revisited. American Economic Review 98, 1692–1706.

Hall, R. E., 2005. Employment fluctuations with equilibrium wage stickiness. American economic review 95, 50–65.

Hansen, G. D., 1985. Indivisible labor and the business cycle. Journal of monetary Economics 16, 309–327.

Khan, A., Thomas, J. K., 2013. Credit shocks and aggregate fluctuations in an economy with production heterogeneity. Journal of Political Economy 121, 1055–1107.

King, R. G., Plosser, C. I., Rebelo, S. T., 1988. Production, growth and business cycles: I. the basic neoclassical model. Journal of monetary Economics 21, 195–232.

Koopmans, T. C., et al., 1963. On the concept of optimal economic growth. Tech. rep., Cowles Foundation for Research in Economics, Yale University.

Lucas, R., 1977. Understanding business cycles. In: *Carnegie-Rochester Conference Series on Public Policy*, Elsevier, vol. 5, pp. 7–29.

Lucas Jr, R., et al., 1976. Econometric policy evaluation: A critique. In: *Carnegie-Rochester Conference Series on Public Policy*, Elsevier, vol. 1, pp. 19–46.

Lucas Jr, R. E., 1978. Asset prices in an exchange economy. Econometrica 46, 1429–1445.

Mehra, R., Prescott, E. C., 1985. The equity premium: A puzzle. Journal of monetary Economics 15, 145–161.

Mortensen, D. T., 1982. The matching process as a noncooperative bargaining game. In: *The economics of information and uncertainty*, University of Chicago Press, pp. 233–258.

Nada, S., 2013. Equity premium puzzle: Not solved yet. BUSINESS ADMINISTRATION 48.

Nakajima, M., 2012. Business cycles in the equilibrium model of labor market search and self-insurance. International Economic Review 53, 399–432.

Pissarides, C. A., 1985. Short-run equilibrium dynamics of unemployment, vacancies, and real wages. The American Economic Review 75, 676–690.

Ramsey, F. P., 1928. A mathematical theory of saving. The economic journal 38, 543–559.

Rogerson, R., et al., 1988. Indivisible labor, lotteries and equilibrium. Journal of Monetary Economics 21, 3–16.

Samuelson, P. A., 1958. An exact consumption-loan model of interest with or without the social contrivance of money. Journal of political economy 66, 467–482.

Shimer, R., 2005. The cyclical behavior of equilibrium unemployment and vacancies. American economic review 95, 25–49.

Solow, R. M., 1956. A contribution to the theory of economic growth. The quarterly journal of economics 70, 65–94.

Tauchen, G., 1986. Finite state markov-chain approximations to univariate and vector autoregressions. Economics letters 20, 177–181.