Infinite Horizon Optimal Growth Model

Dr. Guanyi Yang

St. Lawrence University

February 28, 2019

Introduction

- Now we are going to extend the previously introduced finite horizon models to an infinite horizon.
- i.e. a full-fledged optimal growth model
- We are also going to introduce uncertainty to the model, i.e. stochastic quality.
- By the end of this lecture, you will have seen the basic features of any DSGE model (Dynamic Stochastic General Equilibrium).

Plan

Start with

We will start with **infinite horizon** optimal growth model

End with

Infinite horizon optimal growth model with stochastic TFP (total factor productivity), i.e. **aggregate risk**

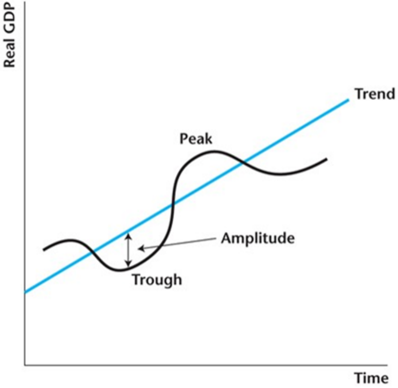
Review of a Two-Period Optimal Growth Model

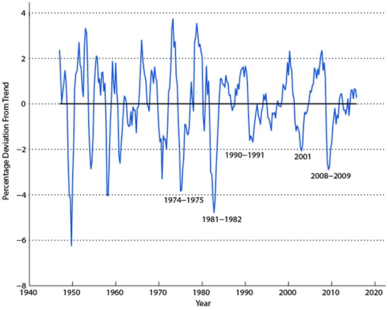
$$
\begin{aligned}
V &= \max_{c,c',K'} \{u(c) + \beta u(c')\} \\
&\text{s.t.} \\
c + K' &= F(K,1) + (1-d)K \\
c' &= F(K',1) + (1-d)K' \\
&\text{K given}
\end{aligned}
\tag{1}
$$

side-note: For those of you working on lifecycle model (i.e. finite horizon model), the algorithm solving it is essentially the same as the one for two-period optimal growth model that we talked about (with sample code in Sakai).

Power of infinite horizon

In reality, no one lives forever. However, history repeats. When we look at an aggregate economy in a long horizon, especially a *developed economy*, it often display the *steady state* quality.





There is a constant trend, and the economy fluctuates along the trend.

Power of infinite horizon

Once we detrend the economy; remove the fluctuations from shocks; the economy is essentially on a flat line. In other words, today and tomorrow are the same. Time lost its meaning.

Age doesn't matter. Generation doesn't matter.
All humanity is described by one infinitely lived person, who makes repeated decisions every day.

Hence, "lifetime" value today is exactly the same as "lifetime" value tomorrow, because of the infinite life horizon.

Think: compare $1 + \infty$ and $10000000 + \infty$

Sequential format of optimal growth model in infinite horizon

We can extend directly from a two-period model:

$$\max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} &\text{s.t.} \\ c_t + K_{t+1} =& F(K_t, 1) + (1-d)K_t, \ \text{t} = 0,\, 1,\, ... \\ K_{t+1} \geq& 0, \ \text{t} = 0,\, 1,\, ... \\ c_t \geq& 0, \ \text{t} = 0,\, 1,\, ... \\ &K_0 \text{ given} \end{aligned} \tag{2}$$

At any given time, we are maximizing the (infinite) lifetime utility.
Every period, we need to make the same decision again and again, as shown by the budget constraint.
This format of describing the infinite horizon problem is called - sequential format.

Recursive format of Optimal growth model

Sequential format helps us intuitively see what is happening in an infinite horizon. However, it is impossible to directly solve a model in sequential format with infinite horizon.

We represent it in a *recursive* format:

$$
\begin{aligned}
V(K) &= \max_{c,K'}\{u(c) + \beta V(K')\} \\
&\text{s.t.} \\
c + K' &= F(K,1) + (1-d)K \\
K' &\geq 0 \\
c &\geq 0 \\
K &\text{ given}
\end{aligned} \tag{3}
$$

Value function at any given point in time is **exactly the same** V, just with different capital K and K'. Only today and tomorrow matters.

Method of successive approximations

Algorithm for solving the model in stationary equilibrium

1. Same as the two period model algorithm, our purpose is to find the choice of K' that makes the life time value the highest, and we store the value of K' in decision rule G.
2. Difference is that instead of computing directly the future (second period) value, we assume it a starting value 0, and taken it as given to find K' to maximize $Tv = u(c) + beta*v$ with the assumed v.
3. At the end of each iteration, we update our assumption of V (future value), since we know in steady state (stationary equilibrium), $Tv = V$.
4. We successfully found our solution when $Tv = V$. This is done through updating the "distance" our stopping criteria.

Discrete state-space dynamic programming

setup

Setup before proceeding:

- utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$
- production function: $F(K) = AK^{\alpha}$
- parameters: $\alpha = 0.36$, $d = 0.069$, $\beta = 0.96$, $\sigma = 2$ and $A = 1$.
- Grid space for state variable K:
  $Kgrid = linspace(0.01, 6, 250)$;

Discrete state-space dynamic programming

Pseudo-code

1. Initiate all necessary values for solving the model: current period value: Tv, future period value v, decision rule G.
2. Define stopping rule and iteration counter for while-loop:

$$precision = 1e - 5;$$

$$distance = 2 * precision;$$

$$iteration = 0;$$

3. Initialize all possible candidates of lifetime value, name it $Tv0 = zeros(Knum, Knum)$; and the associated possible current consumption, $c0 = Tv0$;.

Discrete state-space dynamic programming

Pseudo-code

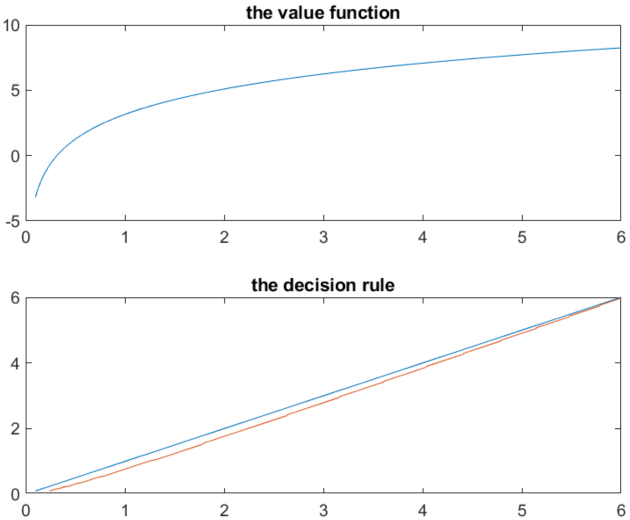
4. Use the initialized future period value v as our starting guess of future value
5. for each current K_i and each possible future K_j, fill-in the corresponding possible consumption, and store it at $c0(i,j)$.
6. for each possible consumption, find possible lifetime value according to value function $U(c) + \beta v$, and store it at the corresponding location $Tv0(i,j)$.

Discrete state-space dynamic programming

Pseudo-code

7. for each current k_i, find the value of the $Tv0(i,:)$ that returns the max value, store the value at the corresponding location $Tv(i)$, and save the location *loc*.
8. use *loc* to find the corresponding K' from *Kgrid* that returns the above optimized Tv, and save it at $G(i)$.
9. Now we have just completed one iteration of the "approximation". Let's update the stopping rule $distance = max(max((abs(Tv - v))))$;.
10. We then update our initial guess of future v by our current calculation $v = Tv$;.
11. print the process, so we know how it is doing:
    ```
    s = sprintf ( ' iteration %4d ||Tv-v|| = %8.6f ',
    iteration, distance);
    disp(s)
    ```
12. update the iteration counter $iteration = iteration + 1$;
13. repeat from Step 4 until converges ($Tv = v$)

Result



the value function
10
5
0
-5
0
1
2
3
4
5
6
the decision rule
6
4
2
0
0
1
2
3
4
5
6

Economic fluctuation / shock / uncertainty

- The economy fluctuates between good and bad states, often times not measured by capital and labor. We call such residual term TFP (or Solow residual).
- The fluctuation of the economy, however, is not *iid*. The next state is always somewhat dependent on the current state. We call this a serial correlation.
- Let's discrete the aggregate states for easier explanation, as $z_t \in \{z_1, z_2, z_3, ..., z_N\}$ (arranging it from low to high).

Law of conditional expectation

- If we are currently in the lowest state z_1, it is more likely for tomorrow to be low too (possibly $z_1, z_2, z_3, ...$), than to be high as z_N.
- Overtime, we find probability of the economy transitioning from one state to the next, conditional on where we are now currently. We describe it as: $Pr(z_{t+1}|z_t)$ Or we name it $\pi(z_{t+1}|z_t)$.
- Notice that history carries over. Each realization of state today, comes from a *chain* of realization throughout history. The unconditional probability of where we are today Z^t comes from a series of historically realized z, as $z_{t-1}, z_{t-2}, z_{t-3}, z_{t-1}, ..., z_2, z_1, z_0$.
- Law of conditional expectation says: $\pi(z^t) = \pi(z_{t+1}|z_t)\pi(z^t)$.

Markov Chain

In general, our introduction of uncertainty into dynamic macroeconomic models will rely on exogenous random variables that take on a finite number of values.

Turning to a recursive formulation, we allow for serial correlation in the stochastic process, but of a restricted form where the probability distribution of the random variable next period depends only on its current value. Such stochastic processes are known as **Markov chains**.

Importantly, Markov processes assume that, regardless of the existing history of realizations, next period's probability distribution depends only on the current realization.

Markov Chain

Let $\mathbb{Z} \in \{z_1, z_2, z_3, ..., z_N\}$, the support for Markov Chain.
Let $\pi_{i,j} = Pr(z_{t+1} = z_j | z_t = z_i), i, j = 1, ..., N$ be the transition probability of the economy from state i to state j

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ . & & & \\ . & & & \\ . & & & \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{bmatrix}$$

is the probability distribution over z_{t+1} conditional on $z_t = z_i$, hence $\sum_{j=1}^{N} \pi_{ij} = 1$

Optimal growth model with uncertainty

$$\max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} &\text{s.t.} \\ c_t + K_{t+1} =& z_t K_t^{\alpha} + (1-d)K_t, \ \text{t} = 0, 1, ... \\ K_{t+1} \geq& 0, \ \text{t} = 0, 1, ... \\ c_t \geq& 0, \ \text{t} = 0, 1, ... \\ z_0, K_0 & \text{ given} \end{aligned} \tag{4}$$

Exogenous stochastic process, $\{z_t\}$, with

$$z_{t+1} = z_t^{\rho} e^{\epsilon_t}$$

, where $\rho \in (0, 1)$, and $\epsilon_t \sim N(0, \sigma^2)$

Optimal growth model with uncertainty

$$
\begin{aligned}
V(K, z) &= \max_{c, K'} \{u(c) + \beta \, \mathbb{E}_{z'|z} \, V(K', z')\} \\
&\text{s.t.} \\
c + K' &= zK^{\alpha} + (1 - d)K \\
K' &\geq 0 \\
c &\geq 0 \\
K &\text{ given}
\end{aligned} \tag{5}
$$

Now, we have **two** state variables:

K as endogenous state

Z **as exogenous state**

Optimal growth model with uncertainty

$$
\begin{aligned}
V(K, z_i) &= \max_{c,K'} \{u(c) + \beta \sum_{j=1}^{N} \pi_{ij} V(K', z_j)\} \\
&\text{s.t.} \\
c + K' &= z_i K^{\alpha} + (1-d)K \\
K' &\geq 0 \\
c &\geq 0 \\
K &\text{ given}
\end{aligned}
\tag{6}
$$

Exogenous stochastic process, $\{z_t\}$, with $z_{t+1} = z_t^{\rho} e^{\epsilon_t}$, where $\rho \in (0, 1)$, and $\epsilon_t \sim N(0, \sigma^2)$
And if we take the log of the shock process, we have:

$$logz_{t+1} = \rho logz_t + \epsilon_t$$

, an **AR(1) process**.

Algorithm for infinite horizon model with uncertainty

- First, we need to discretize the aggregate state/shock process into a finite grid.
- Second, we need to find the Markov transition probability of the shock grid.
- Step 1 and 2 are done through the *Tauchen* Algorithm. I attached it in the Sakai folder, where you can use directly.
- Third, modify step (4) from optimal growth model with certainty to incorporate shock process in the initial guess.
- Everything else follows the same as the algorithm for model with certainty.

Code for using Tauchen algorithm function

% grid on total factor productivity, using Tauchen method

$meanz = 0.0; znum = 5; rho = 0.859; sigma_e = 0.014;$

$[zgrid, piz] = tauchen(rho, sigma_e, znum, meanz, 2.575);$

```
zgrid = exp(zgrid); % zgrid is the grid for TFP shock level; and piz is the transition matrix for all z values.
```

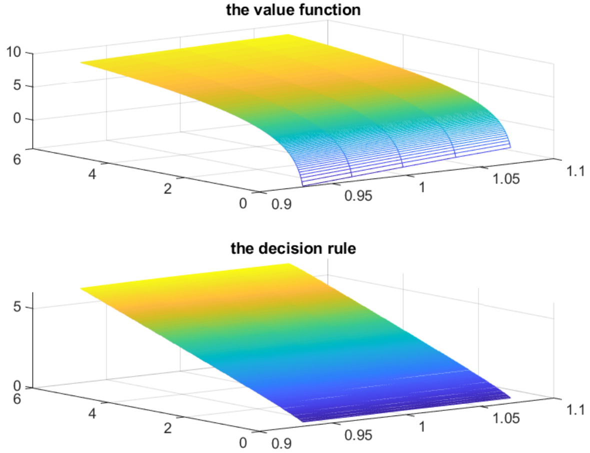
Pseudo-code

Since the rest of the program is exactly the same (except for one more dimension of state variable), I will only outline the part where we find expected future value.

- Just like previous algorithm, we initiated a guess of future value $V(K', z')$.
- Model with certainty directly takes $V(K', z')$ into solving, then update $V = Tv$.
- Model with uncertainty needs to adjust $V(K', z')$ by the conditional probability associated with each z'.

1. Initiate expected future value: $ev = zeros(Knum, znum)$
2. For each current capital K_i, and for each current shock state z_i, the probability of being at each future shock state z_j is $pi(iz, jz)$ (pi was obtained from Tauchen function.
3. The expected future value is to sum up all probability adjusted future value $ev(i, iz) = ev(i, iz) + pi(iz, jz) * v(i, jz$ for each current state (i, iz).

Result



I set shock process parameters with
$\rho = 0.859; \sigma_e = 0.014; znum = 5$; in addition to the parameters from the certainty model.