The file `phddoc.dtx` for use with $\LaTeX\,2_{\varepsilon}$.[*] It contains the code for `phddoc.cls`

Yannis Lazarides

2005/11/02

[*] This file has version number v4.41, dated 2005/11/02.

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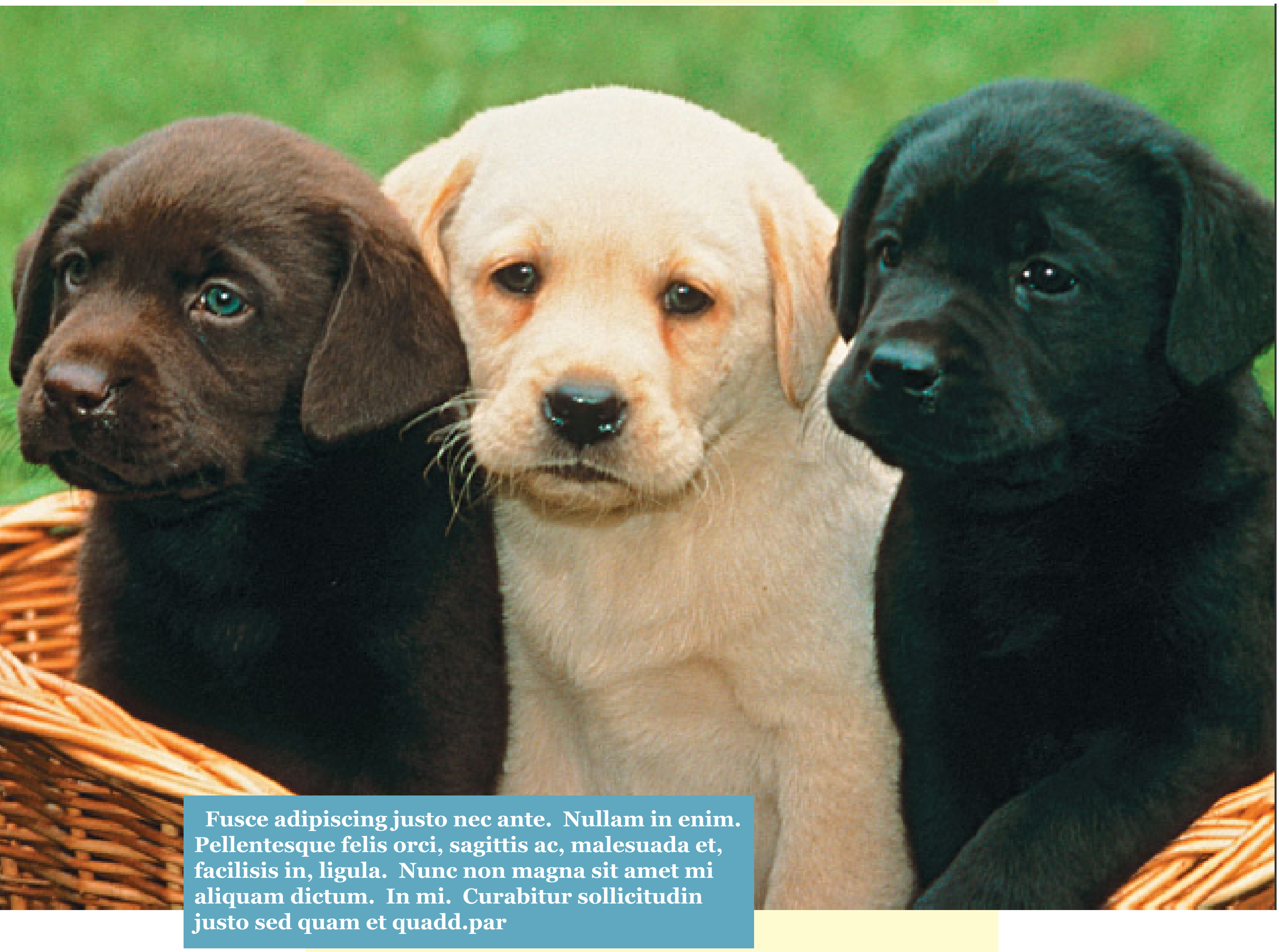
Latex Classes

I Latex Classes



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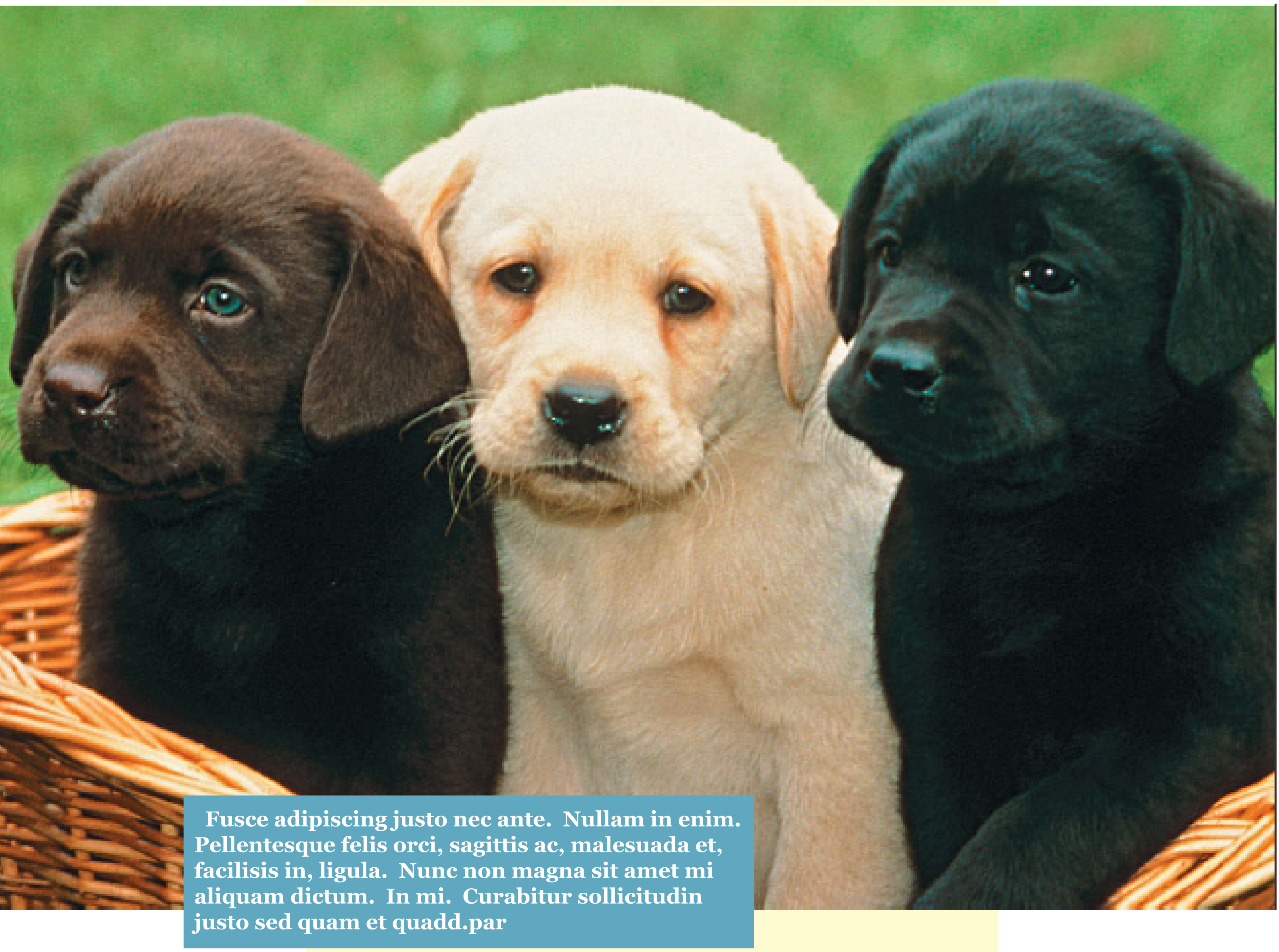
Book Design

II Book Design



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48

Creating Book Designs

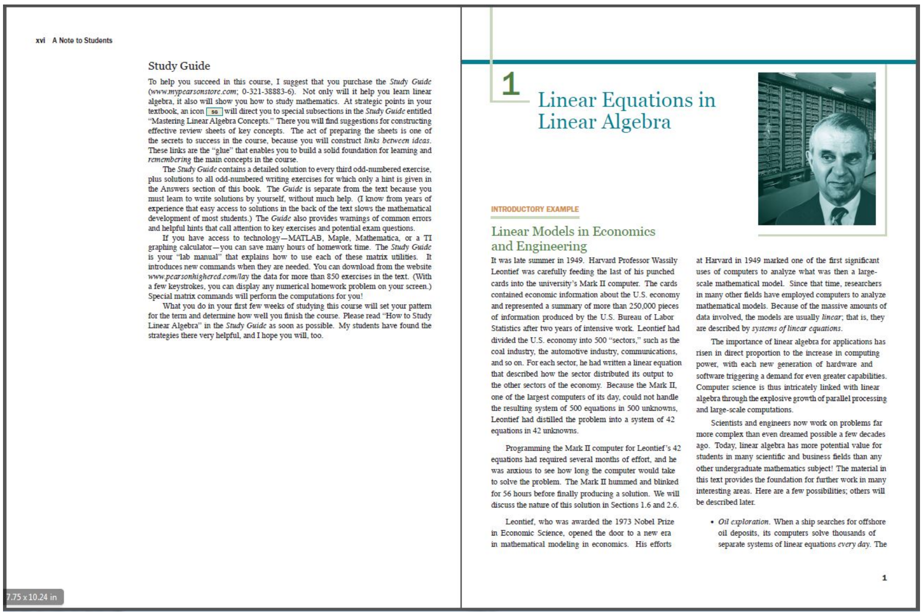
In antiquity men and women saw each other as different; accordingly, they developed complex taxonomies (philosophical explanations) for understanding anatomical, physiological, emotional, and rational differences.

Some of these differences seem profoundly odd to us moderns. Modern discussions about erotic art have often concerned the place of women: to what extent are they objects of social manipulation, to what extent can they be subjects?

48.1 First Steps

In this chapter we will develop a full book template from scratch. Before we delve into it further, I would like to emphasize that the phd system is a bit different from classes. A phd style includes all the information necessary for the typesetting of a document. I have called this a style template. It is slightly different from a class system where generic commands might be included that can develop a totally different look. An identical design with perhaps different colors and fonts and other minor changes, is termed a *theme*.

Unlike book designers who would first focus on fonts, we will first give our attention to the structural elements of the book. I will be using as an example the *Linear Algebra*.



xvi A Note to Students

Study Guide

To help you succeed in this course, I suggest that you purchase the *Study Guide* (*www.mypearsonstore.com*; 0-321-38883-6). Not only will it help you learn linear algebra, it also will show you how to study mathematics. At strategic points in your textbook, an icon SG will direct you to special subsections in the *Study Guide* entitled "Mastering Linear Algebra Concepts." There you will find suggestions for constructing effective review sheets of key concepts. The act of preparing the sheets is one of the secrets to success in the course, because you will construct *links between ideas*. These links are the "glue" that enables you to build a solid foundation for learning and *remembering* the main concepts in the course.

The *Study Guide* contains a detailed solution to every third odd-numbered exercise, plus solutions to all odd-numbered writing exercises for which only a hint is given in the Answers section of this book. The *Guide* is separate from the text because you must learn to write solutions by yourself, without much help. (I know from years of experience that easy access to solutions in the back of the text slows the mathematical development of most students.) The *Guide* also provides warnings of common errors and helpful hints that call attention to key exercises and potential exam questions.

If you have access to technology—MATLAB, Maple, Mathematica, or a TI graphing calculator—you can save many hours of homework time. The *Study Guide* is your "lab manual" that explains how to use each of these matrix utilities. It introduces new commands when they are needed. You can download from the website *www.pearsonhighered.com/lay* the data for more than 850 exercises in the text. (With a few keystrokes, you can display any numerical homework problem on your screen.) Special matrix commands will perform the computations for you!

What you do in your first few weeks of studying this course will set your pattern for the term and determine how well you finish the course. Please read "How to Study Linear Algebra" in the *Study Guide* as soon as possible. My students have found the strategies there very helpful, and I hope you will, too.

7.75 x 10.24 in

1 Linear Equations in Linear Algebra

INTRODUCTORY EXAMPLE

Linear Models in Economics and Engineering

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained economic information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration*. When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*. The

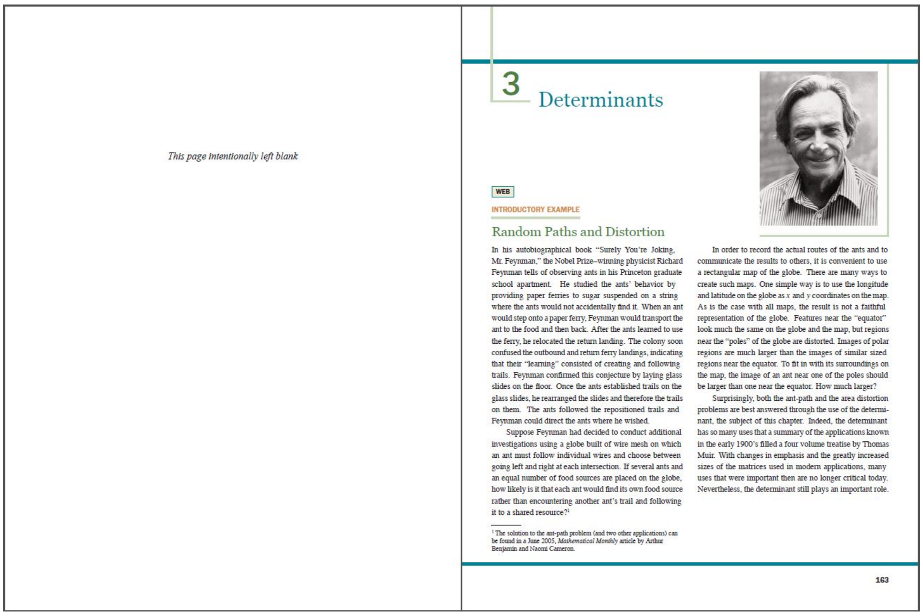
1

Figure 48.1: The opening chapter can leave a blank page.

Figure 48.1 shows the chapter head design. This is an interesting and challenging design that we will not easily make with the phd standard chapter head routines. The chapter starts with a full line and a structural element that is called *Introductory Example*. The heading of this also goes to the Table of Contents. So the chapter opening page starts with a rule and end with a rule. The ending rule in Figure 48.2 can be seen in the next figure.

48.1.1 Chapter Opening

One of the first things you will need to take care of, is to design if the style template should cater for opening at right or if it is to open at any place. Another decision you will need to make, is what to do with blank pages. Personally I dislike them and suggest, if you are going to have them to either introduce epigraphs or full page images.



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3 Determinants

WEB

INTRODUCTORY EXAMPLE

Random Paths and Distortion

In his autobiographical book "Surely You're Joking, Mr. Feynman," the Nobel Prize–winning physicist Richard Feynman tells of observing ants in his Princeton graduate school apartment. He studied the ants' behavior by providing paper ferries to sugar suspended on a string where the ants would not accidentally find it. When an ant would step onto a paper ferry, Feynman would transport the ant to the food and then back. After the ants learned to use the ferry, he relocated the return landing. The colony soon confused the outbound and return ferry landings, indicating that their "learning" consisted of creating and following trails. Feynman confirmed this conjecture by laying glass slides on the floor. Once the ants established trails on the glass slides, he rearranged the slides and therefore the trails on them. The ants followed the repositioned trails and Feynman could direct the ants where he wished.

Suppose Feynman had decided to conduct additional investigations using a globe built of wire mesh on which an ant must follow individual wires and choose between going left and right at each intersection. If several ants and an equal number of food sources are placed on the globe, how likely is it that each ant would find its own food source rather than encountering another ant's trail and following it to a shared resource?[1]

In order to record the actual routes of the ants and to communicate the results to others, it is convenient to use a rectangular map of the globe. There are many ways to create such maps. One simple way is to use the longitude and latitude on the globe as x and y coordinates on the map. As is the case with all maps, the result is not a faithful representation of the globe. Features near the "equator" look much the same on the globe and the map, but regions near the "poles" of the globe are distorted. Images of polar regions are much larger than the images of similar sized regions near the equator. To fit in with its surroundings on the map, the image of an ant near one of the poles should be larger than one near the equator. How much larger?

Surprisingly, both the ant-path and the area distortion problems are best answered through the use of the determinant, the subject of this chapter. Indeed, the determinant has so many uses that a summary of the applications known in the early 1900's filled a four volume treatise by Thomas Muir. With changes in emphasis and the greatly increased sizes of the matrices used in modern applications, many uses that were important then are no longer critical today. Nevertheless, the determinant still plays an important role.

[1] The solution to the ant-path problem (and two other applications) can be found in a June 2005, *Mathematical Monthly* article by Arthur Benjamin and Naomi Cameron.

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Figure 48.2: The opening chapter can leave a blank page.

48.1.2 The User Commands

It is always best to start thinking about the user commands, as we go along in order to provide a user friendly interface, without the introduction of too many keys. We also need to name our template. We will name it *andrea* in honour of the Designer of the book, who was Andrea Nix. Andrea designed many of the Pearson books that were mostly textbooks and has a unique distinctive design style that can make a mathematics book fun to read.

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\cxset{chapter template = andrea,
          chapter opening = right}
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\paperheight 845.04684pt
\textwidth 345.0pt
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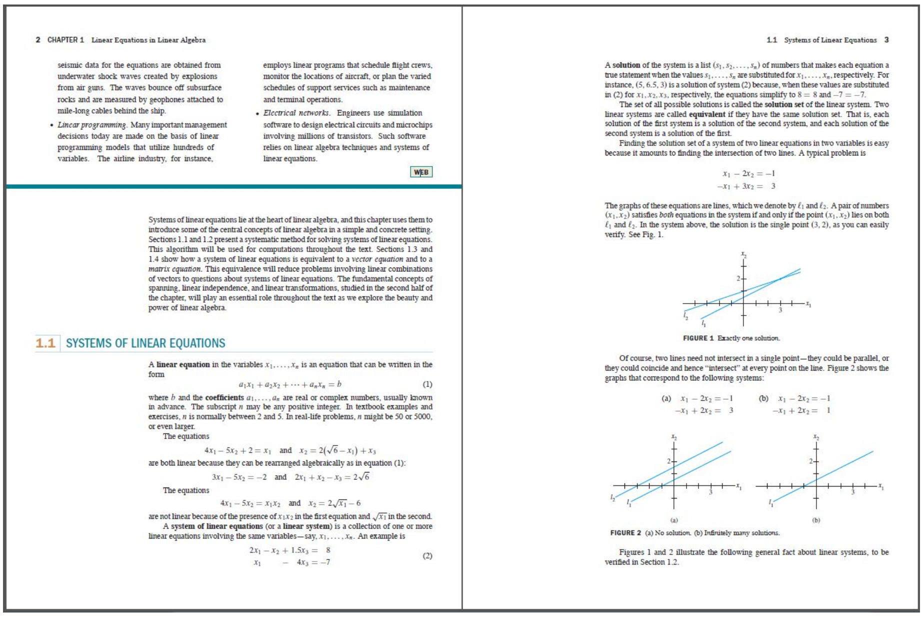
We also need to check on oddside pages as well. Remember the switch `\@mparswitchfalse` will set the margin pars to be on the same size. This layout only has them on the right pages. We need to set it to false.
Another decision we need to make is if we going to draw the layout using TeX commands or one of the graphic units. Using TikZ, can be much easier, but we need to ensure we know where we are on the page. Alternatively we can use the remember picture, overlay hack to accomplish it. We will first give it a try with rules and boxes.
Now we have the dimensions of the left margin and right margin width right we can continue with the layout.
The next item we will draw is the corner frame.

45.1 Sections

48.2 Sections

The sections follow a very similar style to that of the chapter heading with rules and similar colours.



2 CHAPTER 1 Linear Equations in Linear Algebra

seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming*. Many important management decisions today are made on the basis of linear programming models that utilize hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks*. Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

WEB

Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

1.1 SYSTEMS OF LINEAR EQUATIONS

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \dots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - 4x_3 &= -7 \end{aligned} \tag{2}$$

1.1 Systems of Linear Equations 3

A **solution** of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2. A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Fig. 1.

FIGURE 1 Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

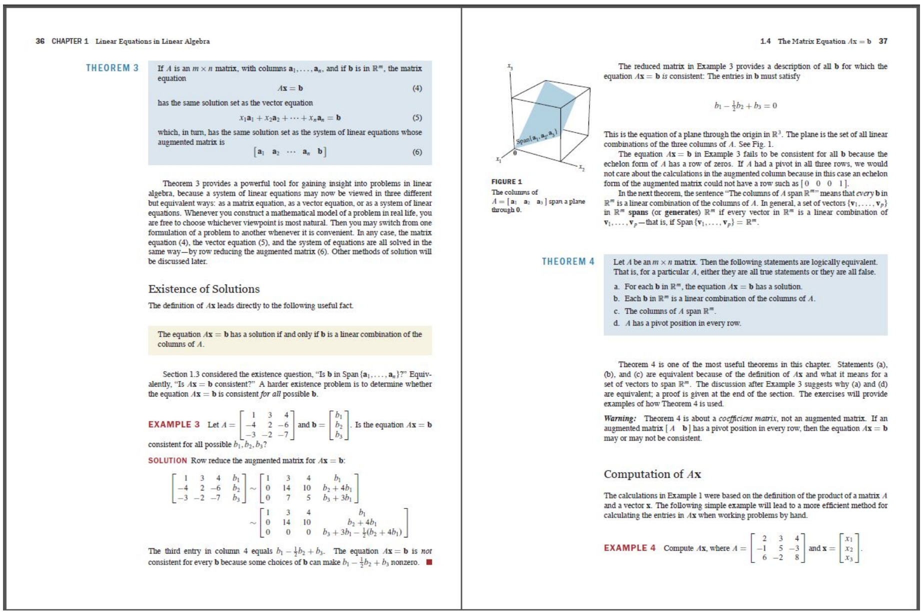
$$\text{(a)} \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

FIGURE 2 (a) No solution. (b) Infinitely many solutions.

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.

Figure 48.3: The opening chapter can leave a blank page.

The book does not use subsection. As a matter of fact most books don't consider that numbering of subsections offers an advantage to the reader.



THEOREM 3 If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of $A\mathbf{x}$ leads directly to the following useful fact.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Section 1.3 considered the existence question, "Is $\mathbf{b}$ in Span $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?" Equivalently, "Is $A\mathbf{x} = \mathbf{b}$ consistent?" A harder existence problem is to determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent *for all* possible $\mathbf{b}$.

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

SOLUTION Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is *not* consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

FIGURE 1 The columns of $A = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]$ span a plane through **0**.

The reduced matrix in Example 3 provides a description of all $\mathbf{b}$ for which the equation $A\mathbf{x} = \mathbf{b}$ *is* consistent: The entries in $\mathbf{b}$ must satisfy

$$b_1 - \tfrac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in $\mathbb{R}^3$. The plane is the set of all linear combinations of the three columns of A. See Fig. 1.

The equation $A\mathbf{x} = \mathbf{b}$ in Example 3 fails to be consistent for all $\mathbf{b}$ because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[\,0 \;\; 0 \;\; 0 \;\; 1\,]$.

In the next theorem, the sentence "The columns of A span $\mathbb{R}^m$" means that *every* $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A. In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$—that is, if Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

THEOREM 4 Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of $A\mathbf{x}$ and what it means for a set of vectors to span $\mathbb{R}^m$. The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[\,A \;\; \mathbf{b}\,]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ may or may not be consistent.

Computation of $A\mathbf{x}$

The calculations in Example 1 were based on the definition of the product of a matrix A and a vector $\mathbf{x}$. The following simple example will lead to a more efficient method for calculating the entries in $A\mathbf{x}$ when working problems by hand.

EXAMPLE 4 Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Figure 48.4: The opening chapter can leave a blank page.

48.3 Examples and Solutions

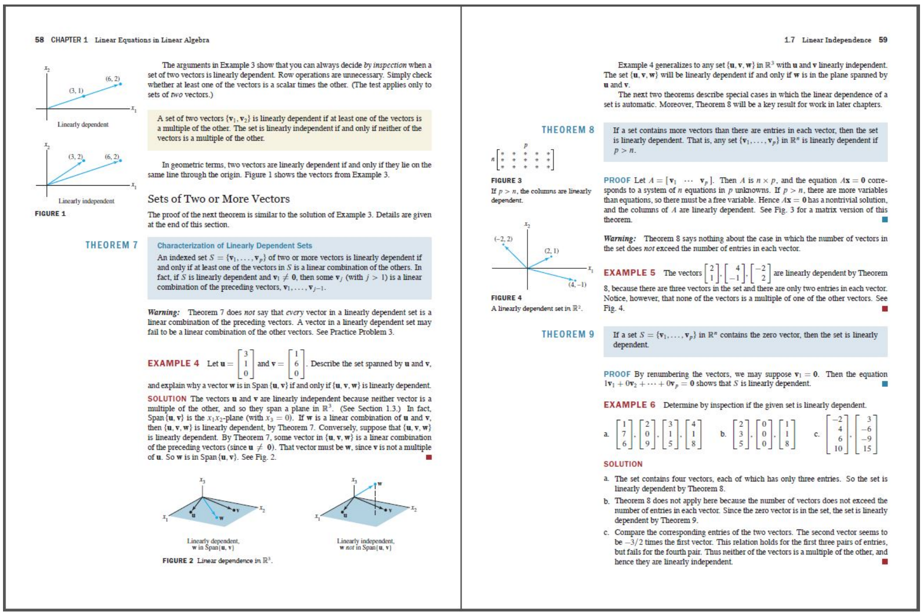
The examples are straight forward typesetting and numbering. The specification should be that they be numbered consequently with the example and solution in capital letters to be distinguished by the type size. The colour is to be identical. The example heading is inlined with about a quadd of space between it and the text that follow. The solution is on its own line and it is followed normally by a list which is numbered alphabetically. In other cases it is in-lined see the page at the left. We can perhaps handle this with a starred command, one for stand alone heading and another for an inlined. I will come back with some suggestions for this before, we delve into codin.

48.4 Exercises

These are modelled after sections and are also numbered. They are numbered in a different counter from that of sections and are reset at every chapter.

48.5 Figures and diagrams

The user commands should also be minimized and would follow normal LaTeX conventions, with the exception we will redefine an environment `\begin`⟨*marginfigure*⟩.... The margin figures are both numbered as well as unnumbered, so we will use normal LaTeX conventions to both define them as well as



58 CHAPTER 1 Linear Equations in Linear Algebra

Linearly dependent

Linearly independent

FIGURE 1

The arguments in Example 3 show that you can always decide *by inspection* when a set of two vectors is linearly dependent. Row operations are unnecessary. Simply check whether at least one of the vectors is a scalar times the other. (The test applies only to sets of *two* vectors.)

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

In geometric terms, two vectors are linearly dependent if and only if they lie on the same line through the origin. Figure 1 shows the vectors from Example 3.

Sets of Two or More Vectors

The proof of the next theorem is similar to the solution of Example 3. Details are given at the end of this section.

THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Warning: Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors. See Practice Problem 3.

EXAMPLE 4 Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by $\mathbf{u}$ and $\mathbf{v}$, and explain why a vector $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

SOLUTION The vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$. (See Section 1.3.) In fact, Span $\{\mathbf{u}, \mathbf{v}\}$ is the x_1x_2-plane (with $x_3 = 0$). If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, by Theorem 7. Conversely, suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. By Theorem 7, some vector in $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$). That vector must be $\mathbf{w}$, since $\mathbf{v}$ is not a multiple of $\mathbf{u}$. So $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$. See Fig. 2. ■

Linearly dependent, $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$

Linearly independent, $\mathbf{w}$ *not* in Span $\{\mathbf{u}, \mathbf{v}\}$

FIGURE 2 Linear dependence in $\mathbb{R}^3$.

1.7 Linear Independence 59

Example 4 generalizes to any set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly dependent if and only if $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

The next two theorems describe special cases in which the linear dependence of a set is automatic. Moreover, Theorem 8 will be a key result for work in later chapters.

THEOREM 8 If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

FIGURE 3 If $p > n$, the columns are linearly dependent.

PROOF Let $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_p]$. Then A is $n \times p$, and the equation $A\mathbf{x} = \mathbf{0}$ corresponds to a system of n equations in p unknowns. If $p > n$, there are more variables than equations, so there must be a free variable. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and the columns of A are linearly dependent. See Fig. 3 for a matrix version of this theorem. ■

Warning: Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

EXAMPLE 5 The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent by Theorem 8, because there are three vectors in the set and there are only two entries in each vector. Notice, however, that none of the vectors is a multiple of one of the other vectors. See Fig. 4. ■

FIGURE 4 A linearly dependent set in $\mathbb{R}^2$.

THEOREM 9 If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

PROOF By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. ■

EXAMPLE 6 Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ b. $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$ c. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

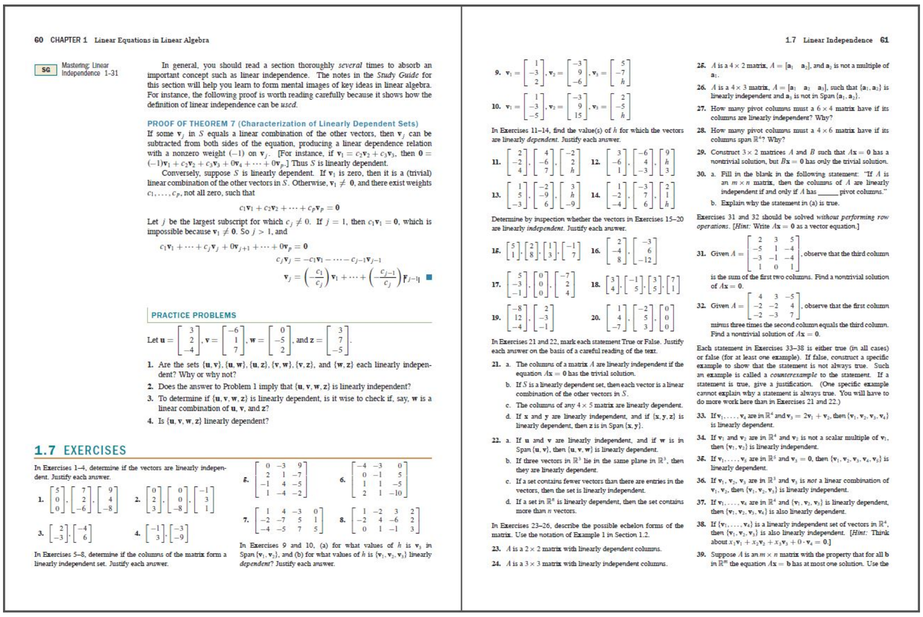
SOLUTION

a. The set contains four vectors, each of which has only three entries. So the set is linearly dependent by Theorem 8.

b. Theorem 8 does not apply here because the number of vectors does not exceed the number of entries in each vector. Since the zero vector is in the set, the set is linearly dependent by Theorem 9.

c. Compare the corresponding entries of the two vectors. The second vector seems to be $-3/2$ times the first vector. This relation holds for the first three pairs of entries, but fails for the fourth pair. Thus neither of the vectors is a multiple of the other, and hence they are linearly independent. ■

Figure 48.5: The book will have a lot of examples and their solutions.



60 CHAPTER 1 Linear Equations in Linear Algebra

SG Mastering: Linear Independence 1-31

In general, you should read a section thoroughly *several* times to absorb an important concept such as linear independence. The notes in the *Study Guide* for this section will help you learn to form mental images of key ideas in linear algebra. For instance, the following proof is worth reading carefully because it shows how the definition of linear independence can be *used*.

PROOF OF THEOREM 7 (Characterization of Linearly Dependent Sets)

If some $\mathbf{v}_j$ in S equals a linear combination of the other vectors, then $\mathbf{v}_j$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on $\mathbf{v}_j$. [For instance, if $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, then $\mathbf{0} = (-1)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\mathbf{v}_4 + \cdots + 0\mathbf{v}_p$.] Thus S is linearly dependent.

Conversely, suppose S is linearly dependent. If $\mathbf{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S. Otherwise, $\mathbf{v}_1 \neq \mathbf{0}$, and there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Let j be the largest subscript for which $c_j \neq 0$. If $j = 1$, then $c_1\mathbf{v}_1 = \mathbf{0}$, which is impossible because $\mathbf{v}_1 \neq \mathbf{0}$. So $j > 1$, and

$$\begin{aligned} c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p &= \mathbf{0} \\ c_j\mathbf{v}_j &= -c_1\mathbf{v}_1 - \cdots - c_{j-1}\mathbf{v}_{j-1} \\ \mathbf{v}_j &= \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1} \end{aligned}$$ ■

PRACTICE PROBLEMS

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$.

1. Are the sets $\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{w}\}, \{\mathbf{u}, \mathbf{z}\}, \{\mathbf{v}, \mathbf{w}\}, \{\mathbf{v}, \mathbf{z}\}$, and $\{\mathbf{w}, \mathbf{z}\}$ each linearly independent? Why or why not?
2. Does the answer to Problem 1 imply that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly independent?
3. To determine if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ is linearly dependent, is it wise to check if, say, $\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$?
4. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ linearly dependent?

1.7 EXERCISES

In Exercises 1–4, determine if the vectors are linearly independent. Justify each answer.

1. $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$
3. $\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix}$
4. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

In Exercises 5–8, determine if the columns of the matrix form a linearly independent set. Justify each answer.

5. $\begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix}$
6. $\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$
8. $\begin{bmatrix} 1 & -2 & 3 & 2 \\ -2 & 4 & -6 & 2 \\ 0 & 1 & -1 & 3 \end{bmatrix}$

In Exercises 9 and 10, (a) for what values of h is $\mathbf{v}_3$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, and (b) for what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*? Justify each answer.

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$
10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ 15 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$

In Exercises 11–14, find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$
12. $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$
13. $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$
14. $\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$
16. $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$
17. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$
18. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$
19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$
20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

21. a. The columns of a matrix A are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.
    b. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
    c. The columns of any 4×5 matrix are linearly dependent.
    d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.
22. a. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and if $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.
    b. If three vectors in $\mathbb{R}^3$ lie in the same plane in $\mathbb{R}^3$, then they are linearly dependent.
    c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
    d. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. A is a 2×2 matrix with linearly dependent columns.
24. A is a 3×3 matrix with linearly independent columns.

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25. A is a 4×2 matrix, $A = [\mathbf{a}_1 \ \ \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.
26. A is a 4×3 matrix, $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.
27. How many pivot columns must a 6×4 matrix have if its columns are linearly independent? Why?
28. How many pivot columns must a 4×6 matrix have if its columns span $\mathbb{R}^4$? Why?
29. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, but $B\mathbf{x} = \mathbf{0}$ has only the trivial solution.
30. a. Fill in the blank in the following statement: "If A is an $m \times n$ matrix, then the columns of A are linearly independent if and only if A has _____ pivot columns."
    b. Explain why the statement in (a) is true.

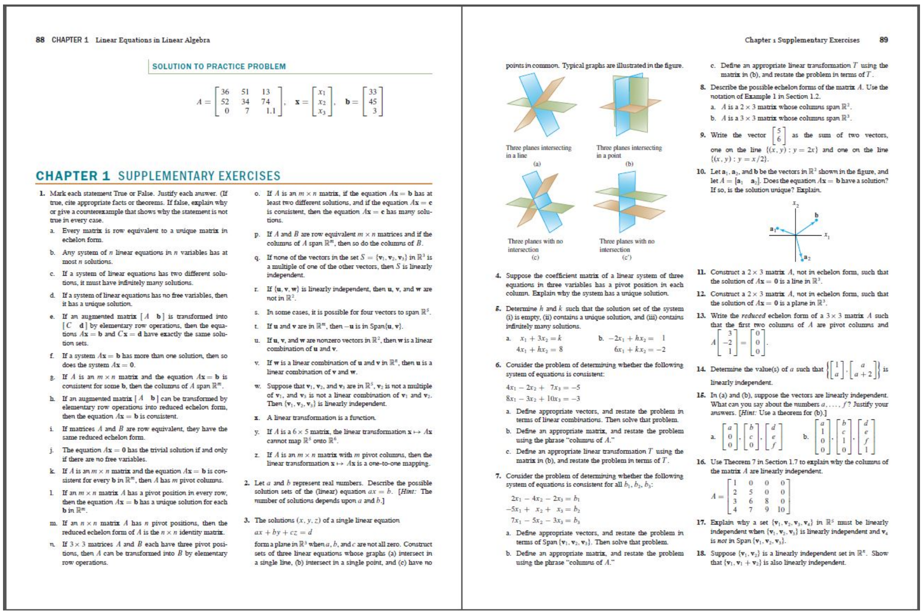
Exercises 31 and 32 should be solved *without performing row operations*. [*Hint:* Write $A\mathbf{x} = \mathbf{0}$ as a vector equation.]

31. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.
32. Given $A = \begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}$, observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.
34. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.
35. If $\mathbf{v}_1, \dots, \mathbf{v}_5$ are in $\mathbb{R}^5$ and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is linearly dependent.
36. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in $\mathbb{R}^3$ and $\mathbf{v}_3$ is *not* a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
37. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.
38. If $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. [*Hint:* Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.]
39. Suppose A is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the

Figure 48.6: The book will have a lot of examples and their solutions.



88 CHAPTER 1 Linear Equations in Linear Algebra

SOLUTION TO PRACTICE PROBLEM

$$A = \begin{bmatrix} 36 & 51 & 13 \\ 52 & 34 & 74 \\ 0 & 7 & 1.1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 33 \\ 45 \\ 3 \end{bmatrix}$$

CHAPTER 1 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

   a. Every matrix is row equivalent to a unique matrix in echelon form.

   b. Any system of n linear equations in n variables has at most n solutions.

   c. If a system of linear equations has two different solutions, it must have infinitely many solutions.

   d. If a system of linear equations has no free variables, then it has a unique solution.

   e. If an augmented matrix $[A \quad \mathbf{b}]$ is transformed into $[C \quad \mathbf{d}]$ by elementary row operations, then the equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ have exactly the same solution sets.

   f. If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x} = \mathbf{0}$.

   g. If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$.

   h. If an augmented matrix $[A \quad \mathbf{b}]$ can be transformed by elementary row operations into reduced echelon form, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

   i. If matrices A and B are row equivalent, they have the same reduced echelon form.

   j. The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if there are no free variables.

   k. If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, then A has m pivot columns.

   l. If an $m \times n$ matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

   m. If an $n \times n$ matrix A has n pivot positions, then the reduced echelon form of A is the $n \times n$ identity matrix.

   n. If 3×3 matrices A and B each have three pivot positions, then A can be transformed into B by elementary row operations.

   o. If A is an $m \times n$ matrix, if the equation $A\mathbf{x} = \mathbf{b}$ has at least two different solutions, and if the equation $A\mathbf{x} = \mathbf{c}$ is consistent, then the equation $A\mathbf{x} = \mathbf{c}$ has many solutions.

   p. If A and B are row equivalent $m \times n$ matrices and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

   q. If none of the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.

   r. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not in $\mathbb{R}^2$.

   s. In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

   t. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^m$, then $-\mathbf{u}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

   u. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

   v. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, then $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

   w. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are in $\mathbb{R}^5$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   x. A linear transformation is a function.

   y. If A is a 6×5 matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^5$ onto $\mathbb{R}^6$.

   z. If A is an $m \times n$ matrix with m pivot columns, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one mapping.

2. Let a and b represent real numbers. Describe the possible solution sets of the (linear) equation $ax = b$. [*Hint:* The number of solutions depends upon a and b.]

3. The solutions (x, y, z) of a single linear equation

   $ax + by + cz = d$

   form a plane in $\mathbb{R}^3$ when a, b, and c are not all zero. Construct sets of three linear equations whose graphs (a) intersect in a single line, (b) intersect in a single point, and (c) have no points in common. Typical graphs are illustrated in the figure.

Three planes intersecting in a line
(a)
Three planes intersecting in a point
(b)
Three planes with no intersection
(c)
Three planes with no intersection
(c')

Chapter 1 Supplementary Exercises 89

4. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot position in each column. Explain why the system has a unique solution.

5. Determine h and k such that the solution set of the system (i) is empty, (ii) contains a unique solution, and (iii) contains infinitely many solutions.

   a. $\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$ b. $\begin{aligned} -2x_1 + hx_2 &= 1 \\ 6x_1 + kx_2 &= -2 \end{aligned}$

6. Consider the problem of determining whether the following system of equations is consistent:

   $$\begin{aligned} 4x_1 - 2x_2 + 7x_3 &= -5 \\ 8x_1 - 3x_2 + 10x_3 &= -3 \end{aligned}$$

   a. Define appropriate vectors, and restate the problem in terms of linear combinations. Then solve that problem.

   b. Define an appropriate matrix, and restate the problem using the phrase "columns of A."

   c. Define an appropriate linear transformation T using the matrix in (b), and restate the problem in terms of T.

7. Consider the problem of determining whether the following system of equations is consistent for all b_1, b_2, b_3:

   $$\begin{aligned} 2x_1 - 4x_2 - 2x_3 &= b_1 \\ -5x_1 + x_2 + x_3 &= b_2 \\ 7x_1 - 5x_2 - 3x_3 &= b_3 \end{aligned}$$

   a. Define appropriate vectors, and restate the problem in terms of Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Then solve that problem.

   b. Define an appropriate matrix, and restate the problem using the phrase "columns of A."

   c. Define an appropriate linear transformation T using the matrix in (b), and restate the problem in terms of T.

8. Describe the possible echelon forms of the matrix A. Use the notation of Example 1 in Section 1.2.

   a. A is a 2×3 matrix whose columns span $\mathbb{R}^2$.

   b. A is a 3×3 matrix whose columns span $\mathbb{R}^3$.

9. Write the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as the sum of two vectors, one on the line $\{(x, y) : y = 2x\}$ and one on the line $\{(x, y) : y = x/2\}$.

10. Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ be the vectors in $\mathbb{R}^2$ shown in the figure, and let $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$. Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? If so, is the solution unique? Explain.

11. Construct a 2×3 matrix A, not in echelon form, such that the solution of $A\mathbf{x} = \mathbf{0}$ is a line in $\mathbb{R}^3$.

12. Construct a 2×3 matrix A, not in echelon form, such that the solution of $A\mathbf{x} = \mathbf{0}$ is a plane in $\mathbb{R}^3$.

13. Write the *reduced* echelon form of a 3×3 matrix A such that the first two columns of A are pivot columns and $A\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

14. Determine the value(s) of a such that $\left\{\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix}\right\}$ is linearly independent.

15. In (a) and (b), suppose the vectors are linearly independent. What can you say about the numbers $a, \ldots, f$? Justify your answers. [*Hint:* Use a theorem for (b).]

    a. $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ b. $\begin{bmatrix} a \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix}$

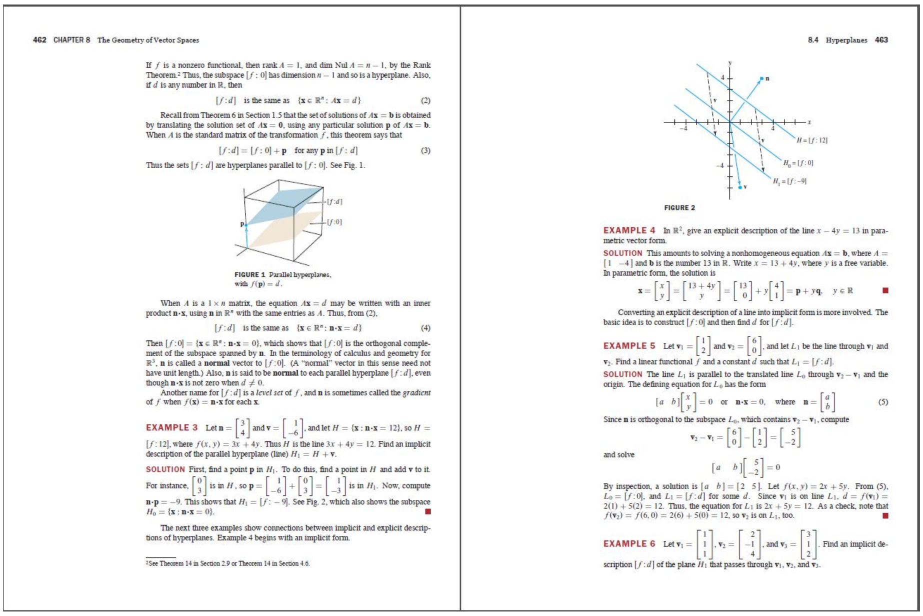
16. Use Theorem 7 in Section 1.7 to explain why the columns of the matrix A are linearly independent.

    $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 8 & 0 \\ 4 & 7 & 9 & 10 \end{bmatrix}$$

17. Explain why a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbb{R}^5$ must be linearly independent when $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and $\mathbf{v}_4$ is *not* in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

18. Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is also linearly independent.

Figure 48.7: The book will have a lot of examples and their solutions.



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If f is a nonzero functional, then rank $A = 1$, and dim Nul $A = n - 1$, by the Rank Theorem.[2] Thus, the subspace $[f : 0]$ has dimension $n - 1$ and so is a hyperplane. Also, if d is any number in $\mathbb{R}$, then

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = d\} \tag{2}$$

Recall from Theorem 6 in Section 1.5 that the set of solutions of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$. When A is the standard matrix of the transformation f, this theorem says that

$$[f : d] = [f : 0] + \mathbf{p} \quad \text{for any } \mathbf{p} \text{ in } [f : d] \tag{3}$$

Thus the sets $[f : d]$ are hyperplanes parallel to $[f : 0]$. See Fig. 1.

FIGURE 1 Parallel hyperplanes, with $f(\mathbf{p}) = d$.

When A is a $1 \times n$ matrix, the equation $A\mathbf{x} = d$ may be written with an inner product $\mathbf{n} \cdot \mathbf{x}$, using $\mathbf{n}$ in $\mathbb{R}^n$ with the same entries as A. Thus, from (2),

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = d\} \tag{4}$$

Then $[f : 0] = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = 0\}$, which shows that $[f : 0]$ is the orthogonal complement of the subspace spanned by $\mathbf{n}$. In the terminology of calculus and geometry for $\mathbb{R}^3$, $\mathbf{n}$ is called a **normal** vector to $[f : 0]$. (A "normal" vector in this sense need not have unit length.) Also, $\mathbf{n}$ is said to be **normal** to each parallel hyperplane $[f : d]$, even though $\mathbf{n} \cdot \mathbf{x}$ is not zero when $d \neq 0$.

Another name for $[f : d]$ is a *level set* of f, and $\mathbf{n}$ is sometimes called the *gradient* of f when $f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x}$ for each $\mathbf{x}$.

EXAMPLE 3 Let $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$, and let $H = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 12\}$, so $H = [f : 12]$, where $f(x, y) = 3x + 4y$. Thus H is the line $3x + 4y = 12$. Find an implicit description of the parallel hyperplane (line) $H_1 = H + \mathbf{v}$.

SOLUTION First, find a point $\mathbf{p}$ in H_1. To do this, find a point in H and add $\mathbf{v}$ to it. For instance, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ is in H, so $\mathbf{p} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is in H_1. Now, compute $\mathbf{n} \cdot \mathbf{p} = -9$. This shows that $H_1 = [f : -9]$. See Fig. 2, which also shows the subspace $H_0 = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} = 0\}$. ■

The next three examples show connections between implicit and explicit descriptions of hyperplanes. Example 4 begins with an implicit form.

[2] See Theorem 14 in Section 2.9 or Theorem 14 in Section 4.6.

8.4 Hyperplanes 463

FIGURE 2

EXAMPLE 4 In $\mathbb{R}^2$, give an explicit description of the line $x - 4y = 13$ in parametric vector form.

SOLUTION This amounts to solving a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$, where $A = [1 \quad -4]$ and $\mathbf{b}$ is the number 13 in $\mathbb{R}$. Write $x = 13 + 4y$, where y is a free variable. In parametric form, the solution is

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 + 4y \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \end{bmatrix} + y\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{p} + y\mathbf{q}, \quad y \in \mathbb{R} \qquad ■$$

Converting an explicit description of a line into implicit form is more involved. The basic idea is to construct $[f : 0]$ and then find d for $[f : d]$.

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and let L_1 be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Find a linear functional f and a constant d such that $L_1 = [f : d]$.

SOLUTION The line L_1 is parallel to the translated line L_0 through $\mathbf{v}_2 - \mathbf{v}_1$ and the origin. The defining equation for L_0 has the form

$$[a \quad b]\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = 0, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{5}$$

Since $\mathbf{n}$ is orthogonal to the subspace L_0, which contains $\mathbf{v}_2 - \mathbf{v}_1$, compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and solve

$$[a \quad b]\begin{bmatrix} 5 \\ -2 \end{bmatrix} = 0$$

By inspection, a solution is $[a \quad b] = [2 \quad 5]$. Let $f(x, y) = 2x + 5y$. From (5), $L_0 = [f : 0]$, and $L_1 = [f : d]$ for some d. Since $\mathbf{v}_1$ is on line L_1, $d = f(\mathbf{v}_1) = 2(1) + 5(2) = 12$. Thus, the equation for L_1 is $2x + 5y = 12$. As a check, note that $f(\mathbf{v}_2) = f(6, 0) = 2(6) + 5(0) = 12$, so $\mathbf{v}_2$ is on L_1, too. ■

EXAMPLE 6 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Find an implicit description $[f : d]$ of the plane H_1 that passes through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Figure 48.8: The book will have a lot of examples and their solutions.

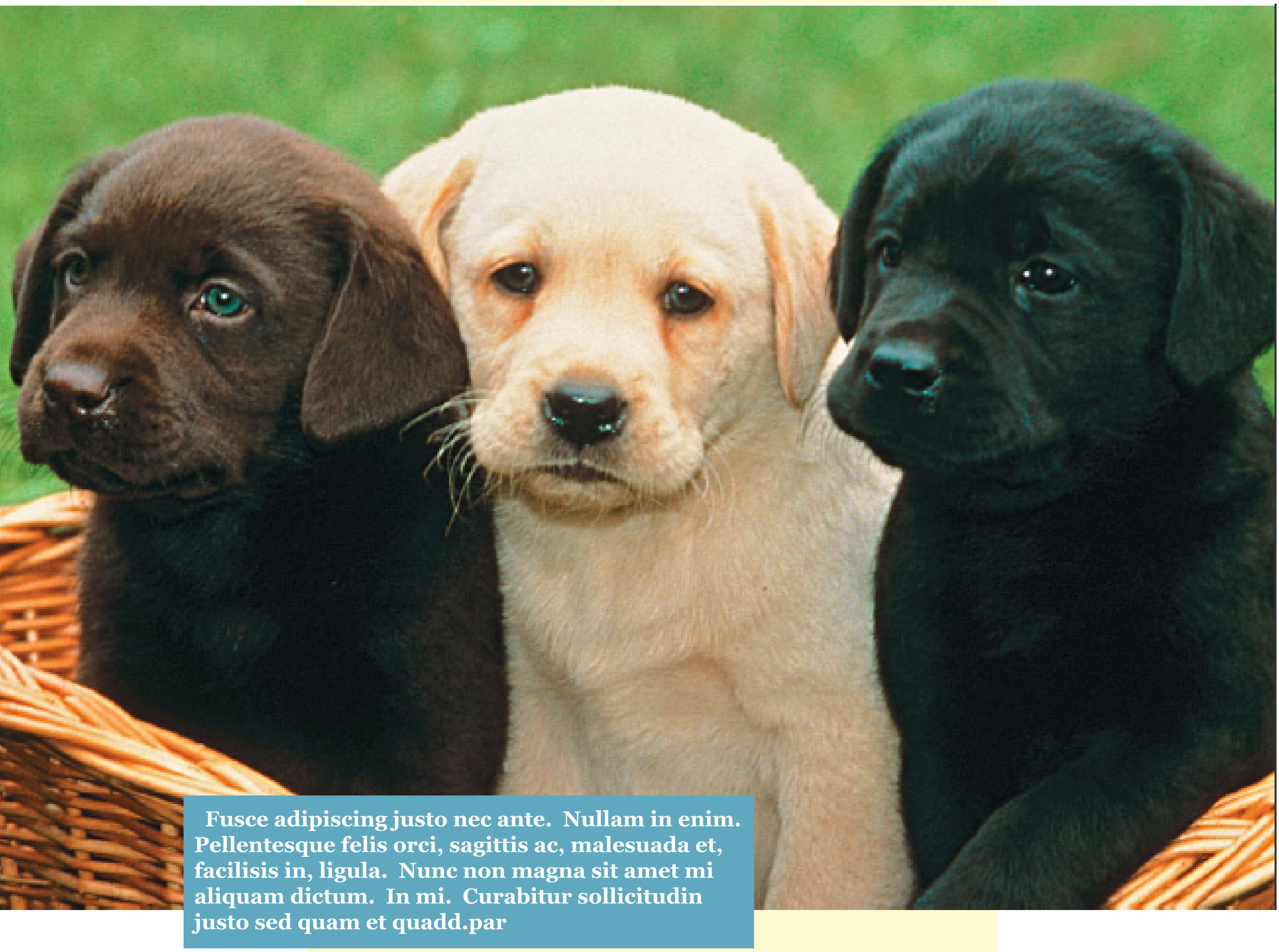
for author commands.

48.6 Geometry

Although we spend a good part of the Chapter on page design, reviewing historical typographical paper sizes, modern book production of text books is not bound with tradition but economics. High speed printing technology uses rolls and pages can be printed up to 64 pages at a time. We will follow the books dimensions which are 7.75x10.25in. The text area occupies approximately 0.67 of the textwidth and is particularly well balanced. Many mathematical text books come out too dense and are difficult to be used by students.

The Ecosystem

III The Ecosystem

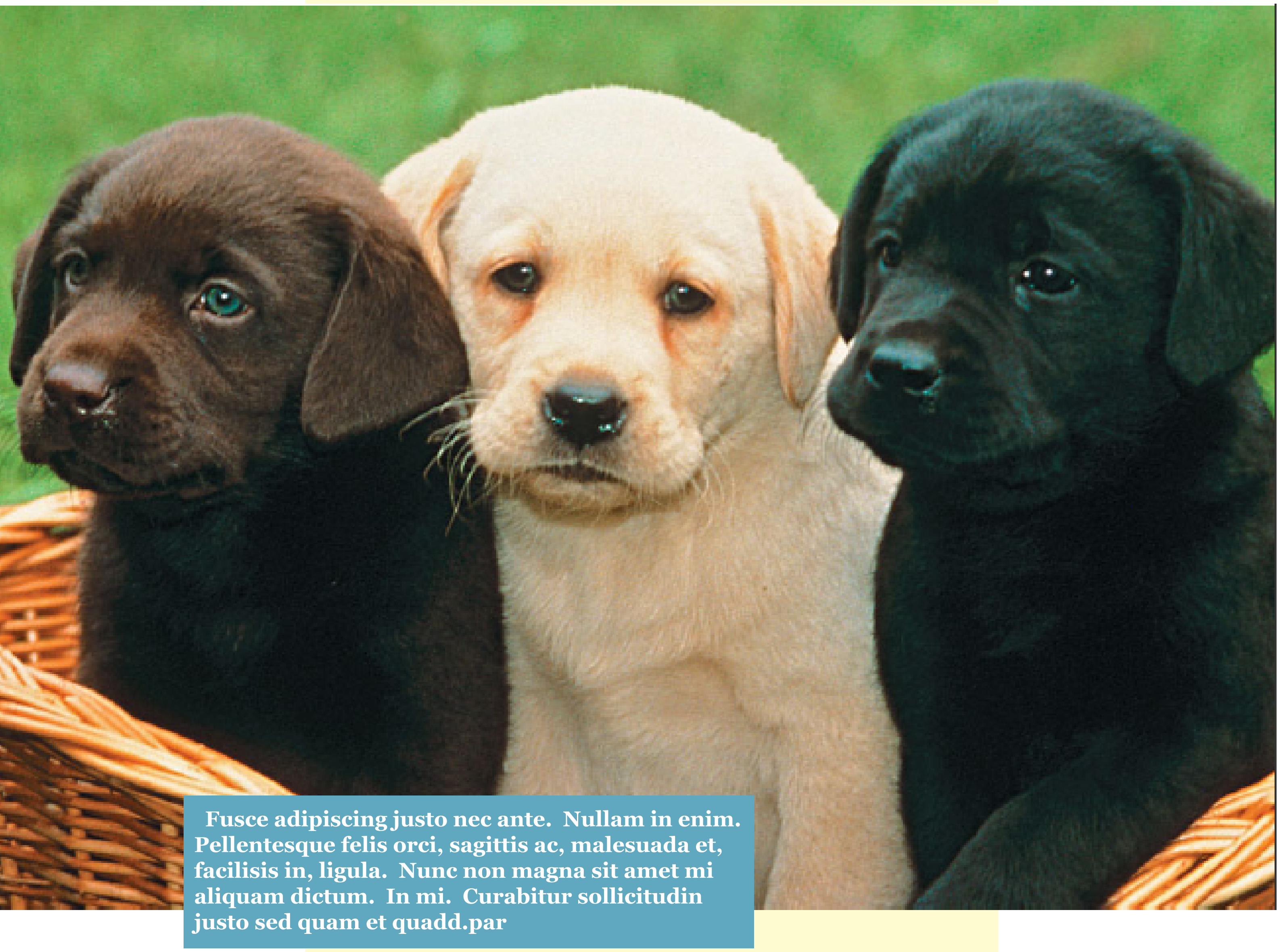


Fusce adipiscing justo nec ante. Nullam in enim. Pellentesque felis orci, sagittis ac, malesuada et, facilisis in, ligula. Nunc non magna sit amet mi aliquam dictum. In mi. Curabitur sollicitudin justo sed quam et quadd.par

Fusce adipiscing justo nec ante. Nullam in enim. Pellentesque felis orci, sagittis ac, malesuada et, facilisis in, ligula. Nunc non magna sit amet mi aliquam dictum. In mi. Curabitur sollicitudin justo sed quam et quadd.par

expl3

IV expl3



Fusce adipiscing justo nec ante. Nullam in enim. Pellentesque felis orci, sagittis ac, malesuada et, facilisis in, ligula. Nunc non magna sit amet mi aliquam dictum. In mi. Curabitur sollicitudin justo sed quam et quadd.par

Fusce adipiscing justo nec ante. Nullam in enim. Pellentesque felis orci, sagittis ac, malesuada et, facilisis in, ligula. Nunc non magna sit amet mi aliquam dictum. In mi. Curabitur sollicitudin justo sed quam et quadd.par

A

User Manual

In antiquity men and women saw each other as different; accordingly, they developed complex taxonomies (philosophical explanations) for understanding anatomical, physiological, emotional, and rational differences.

Some of these differences seem profoundly odd to us moderns. Modern discussions about erotic art have often concerned the place of women: to what extent are they objects of social manipulation, to what extent can they be subjects?

A.1 Documentation of the LaTeX sources

This is a class for documenting the **phd** bundle, a collection of packages and classes that enables the typesetting of documents using a flexible user interface, You may however find it generally useful as a class for typesetting the documentation of files produced in 'doc' format.

The class is written as a "self-contained" docstrip file: executing `latex phddoc.dtx` generates the **`phddoc.cls`** file and typesets this documentation; execute `tex l3doc.dtx` to only generate **`phddoc.cls`**.

Each documented file in the standard distribution comes with extension `dtx`. The appropriate class package or initex file will be extracted from the source by the docstrip system. Each `dtx` file may be directly processed with LaTeX 2_ε, for example

```
% latex2e docclass.dtx
%
```

would produce the documentation of the Class and package interface.

Each file that is used in producing the LaTeX 2_ε format (ie not including the standard class and packages) will be printed together in one document if you LaTeX the file `sources2e.tex`. This has the advantage that one can produce a full index of macro usage across all the source files.

If you need to customise the typesetting of any of these files, there are two options:

- You can use DOCSTRIP with the module 'driver' to extract a small LaTeX file that you may edit to use whatever class or package options you require, before inputting the source file.

- You can create a file `phddoc.cfg`. This configuration file will be read whenever the `phddoc` class is used, and so can be used to customise the typesetting of all the source files, without having to edit lots of small driver files.

The second option is usually more convenient. Various possibilities are discussed in the next section.

A.2 Specification

The class builds on the **ltxdoc**[1] class and the **doc**[2] package, but since they were written many authors have come up with different ideas, as to how these documents should be produced.

The LaTeX3 Team has also more recently developed the `l3doc` class and `l3docstrip` package for documenting the l3 sources. Other Teams such as the developers of **pgf** prefer not to use `docstrip` and document the code and user manuals in a more traditional way, as normal documents in conjuction with external scripts written in python.

My objectives in writing this package, was to integrate the ability of the other packages in this series to document code in a flexible way. For longer books, such as a thesis, where the author might use their own developed macros, it also enables one to use such a method.

[1] Carlisle, David (Mar. 2018). *The file ltxdoc.dtx for use with LaTeX2e.*

[2] Mittelbach, Frank (May 2018). *The doc and shortvrb package.*

This class can be considered as a framework, as it can be used to produce almost any type of document.
The objectives are as follows:

Flexibilty Provide flexibility to use one of the standard LaTeX 2e classes article, book report or the KOMA classes scrartcl, scrbook, scrreprt as the main class. In addition to classes normally used for documents, the class also can be used for documenting LaTeX or LaTeX 3 packages and classes.

Style Enable the use of a fully featured key value interface for documenting the code.

Tools Provide a series of tools to create new documents, formatting and scaffolding. Currently LaTeX distributions have a plethora of tools, mostly using perl and lately l3build using Lua. Perl tools have served the community well for many years. One such tool ctanify does not work using normal dstributions as the Perl bundled in the distributions has some missiong modules. Go is a cross-compliation systems language enabling scripts to be bundled for different operating systems easily, hence the choice here.[3] Some of these problems with Perl on Windows can be overcome using Strawberry Perl[4] For any conflicts follow the guidelines in penwatch.[5]

```
phd ctanify  myclass.dtx  myclass.ins  README
```

A.3 Customisation

The simplest form of customisation is to pass more options to the article class which is loaded by **phddoc**. For instance if you wish all the documentation to be formated for A4 paper, add the following line to phddoc.cfg:

```
% \PassOptionsToClass{a4paper}{article}
%
```

All the source files are in two parts, separated by \StopEventually. The first part (should) contain 'user' documentation. The second part is a full documented listing of the source code. The doc package provides the command \OnlyDescription which suppresses the code listings. This may also be used in the configuration file, but as the doc package is read later, you must delay the execution of \OnlyDescription until after the doc package has been read. The simplest way is to use \AtBeginDocument. Thus you could put the following in your phddoc.cfg.

```
% \AtBeginDocument{\OnlyDescription}
```

If the full source listing sources2e.tex is processed, then an index and change history are produced by default, however indices are not normally produced for individual files.

[3] See for example https://tex.stackexchange.com/questions/256096/which-perl-to-install-for-xindy-with-miktex-on-windows

[4] Donload at http://strawberryperl.com/. This will also enable xindy to work on a MikTeX distribution.

[5] https://www.penwatch.net/cms/pip_conflict/

As an example, consider `ltclass.dtx`, which contains the sources for the new class and package interface commands. With no `cfg` file, a 19 page document is produced. With the above configuration a slightly more readable document (4 pages) is produced.
Conversely, if you really want to read the source listings in detail, you will want to have an index. Again the index commands provided by the doc package may be used, but their execution must be delayed.

```
% \AtBeginDocument{\CodelineIndex\EnableCrossrefs}
% \AtEndDocument{\PrintIndex}
```

The `doc` package writes index files to be sorted using MakeIndex with the `gind` style, so one would then use a command such as

```
% makeindex -s gind.ist ltclass.idx
%
```

and re-run LaTeX.
Similarly to print a Change history, you would add

```
% \AtBeginDocument{\RecordChanges}
% \AtEndDocument{\PrintChanges}
%
```

to `phddoc.cfg`, and use MakeIndex with a comand such as

```
% makeindex -s gglo.ist -o ltclass.gls ltclass.glo
%
```

Finally if you do not want to list all the sections of `source2e.tex`, you can use `\includeonly` in the `cfg` file:

```
% \includeonly{ltvers,ltboxes}
%
```

B

Implementation Code

In antiquity men and women saw each other as different; accordingly, they developed complex taxonomies (philosophical explanations) for understanding anatomical, physiological, emotional, and rational differences.

Some of these differences seem profoundly odd to us moderns. Modern discussions about erotic art have often concerned the place of women: to what extent are they objects of social manipulation, to what extent can they be subjects?

B.1 Options

```
*class

%\RequirePackage{underscore}

\ExplSyntaxOn
\cs_gset:Npn \l_phd_version{1.00}
\ExplSyntaxOff
```

1.00

rocode_colorize_bool

Boolean to switch off the colorizing of code appearing within a macrocode environment.
Define the prefix of the module

```
@@=phdcl
```

```
\ExplSyntaxOn
\bool_new:N \g__phdcl_macrocode_colorize_bool
\bool_new:N \g__phdcl_book_bool
\bool_new:N \g__phdcl_article_bool
\bool_new:N \g__phdcl_report_bool
\bool_new:N \g__phdcl_scrbook_bool
\bool_new:N \g__phdcl_scrartcl_bool
\bool_new:N \g__phdcl_scrreprt_bool
\bool_new:N \g__phdcl_masterthesis_bool
\bool_new:N \g__phdcl_ldoc_bool
\bool_new:N \g__phdcl_ldociii_bool
\ExplSyntaxOff
```

```
\ExplSyntaxOn
\bool_if:NTF \g__phdcl_macrocode_colorize_bool
  {\bool_gset_true:N \phdd_code_colorize_bool}
  {\bool_gset_false:N \phdd_code_colorize_bool}
```

(*End definition for* `\g_@@_macrocode_colorize_bool`.)

B.1.1 Geometry parameters

```
\DeclareOption{a5paper}{\@latexerr{Option not supported}%
   {}}

\DeclareOption { full }
  {
    \bool_gset_true:N \g_phdd_typeset_documentation_bool
    \bool_gset_true:N \g_phdd_typeset_implementation_bool
  }
\DeclareOption { onlydoc }
  {
    \bool_gset_true:N \g_phdd_typeset_documentation_bool
    \bool_gset_false:N \g_phdd_typeset_implementation_bool
  }
\DeclareOption{colorize}
  {\bool_gset_true:N \phdd_code_colorize_bool}
```

```
\DeclareOption { check }
```

```
  { \bool_gset_true:N \g__phdcl_checkfunc_bool }
\DeclareOption { nocheck }
  { \bool_gset_false:N \g__phdcl_checkfunc_bool }
\DeclareOption { checktest }
  { \bool_gset_true:N \g__phdcl_checktest_bool }
\DeclareOption { nochecktest }
  { \bool_gset_false:N \g__phdcl_checktest_bool }
\DeclareOption { kernel }
  { \bool_gset_true:N \g__phdcl_kernel_bool }
\DeclareOption { stdmodule }
  { \bool_gset_false:N \g__phdcl_kernel_bool }
\DeclareOption { cm-default }
  { \bool_gset_false:N \g__phdcl_lmodern_bool }
\DeclareOption { lm-default }
  { \bool_gset_true:N \g__phdcl_lmodern_bool }
\DeclareOption { cs-break-off }
  { \bool_gset_false:N \g__phdcl_cs_break_bool }
\DeclareOption { cs-break-nohyphen }
  { \PassOptionsToPackage{nohyphen}{underscore} }
\tl_new:N \g__phdcl_doctype_tl
\DeclareOption {book}
  {\tl_gput_right:Nn \g__phdcl_doctype_tl{book} }
\DeclareOption {scrbook}
  {\tl_gput_right:Nn \g__phdcl_doctype_tl{book} }
\DeclareOption* { \PassOptionsToClass { \CurrentOption } { book } }
\ExecuteOptions { full, kernel, nocheck, nochecktest, lm-default }
\PassOptionsToClass { a4paper } { book }
```

B.2 Configuration

Input a local configuration file, if it exists, with a message to the console that this has happened. Since we distribute a `.cfg` file with the class, this should usually always be true. Therefore, check for `\ExplMakeTitle` (defined in “our” `.cfg` file) and only output the informational message if it’s not found.

```
\ExplSyntaxOn
\msg_new:nnn { phdcl } { input-cfg }
  { Local~config~file~phdcl.cfg~loaded. }
\file_if_exist:nTF {phdcl.cfg}
  {
    \file_input:n {phdcl.cfg}
  }
  { \msg_info:nn { phdcl } { input-cfg }  }
  {}
\ExplSyntaxOff
```

```
\ExplSyntaxOn
\str_case_e:nnTF { \g__phdcl_doctype_tl }
   {
     { book     }    {
                       %\LoadClass{book}
                       \bool_gset_true:N \g__phdcl_book_bool

                     }
     { article  }    {
                       %\LoadClass{article}
                       \bool_gset_true:N \g__phdcl_article_bool
                     }
     { report         } {
      %\LoadClass{report}
      \bool_gset_true:N \g__phdcl_report_bool
       }
     { scrbook        } {
                         %\LoadClass{scrbook}
                         \KOMAoptions{twoside = false}
                         \bool_gset_true:N \g__phdcl_scrbook_bool
                       }
     { scrartcl       } {
                         %\LoadClass{scrartcl}
                         \bool_gset_true:N \g__phdcl_scrartcl_bool
                       }
     { l3doc          } { %\LoadClass{l3doc}
                       }
     { masterthesis   } {                         }
     { tufte          } {                         }
     { ltxdoc         } {                         }
     { l3doc          } {                         }
     { memoir         } { %\LoadClass{memoir}
                                                }
   }
   {                                             }
   { }

\ExplSyntaxOff
```

B.3 Option Processing

```
\ExplSyntaxOn
\ProcessOptions
%\ProcessKeysPackageOptions
\LoadClass{\g__phdcl_doctype_tl}
\ExplSyntaxOff

```

B.4 Loading book and doc

The original `ltxdoc` uses the article class. For longer documentation it is preferable to use the book, so for the **phddoc** class I have opted to default it to book.

```
% hypdoc is loaded with the phd-packagemanager so that
% the right order for packages and patches can be provided
```

```
\RequirePackage{doc}
\RequirePackage{phd}
\RequirePackage{phd-pkgmanager}
\sethyperref
\RequirePackage{phd-documentation} %modifies doc as necessary
\RequirePackage{phd-colorpalette}
\RequirePackage{phd-runningheads}
\RequirePackage{phd-toc}

```

Make | be a ‘short verb’ character, but not in the document preamble, where an active character may interfere with packages that are loaded.

```
\AtBeginDocument{\MakeShortVerb{\|}}
```

As ‘doc’ documents tend to have a lot of monospaced material, Set up some tt substitutions to occur silently.

```
%\DeclareFontShape{OT1}{cmtt}{bx}{n}{<-> ssub * cmtt/m/n}{}
%\DeclareFontFamily{OMS}{cmtt}{\skewchar\font 48}  % '60
%\DeclareFontShape{OMS}{cmtt}{m}{n}{<-> ssub * cmsy/m/n}{}
%\DeclareFontShape{OMS}{cmtt}{bx}{n}{<-> ssub * cmsy/b/n}{}
```

This substitution is in the standard fd file, but not silent.

```
\DeclareFontShape{OT1}{cmss}{m}{it}{<->ssub*cmss/m/sl}{}
```

```
\CodelineIndex
\CodelineNumbered
\EnableCrossrefs
```

Increase the text width slightly so that width the standard fonts 72 columns of code may appear in a macrocode environment.

```
%\setlength{\textwidth}{375pt}
```

Increase the marginpar width slightly, for long command names. And increase the left margin by a similar amount

```
%\addtolength\marginparwidth{40pt}
%\addtolength\oddsidemargin{40pt}
%\addtolength\evensidemargin{40pt}
```

```
\setcounter{StandardModuleDepth}{1}
```

B.5 Useful abbreviations

The **phd-documentation** provides numerous commands for typesetting LaTeX code. It is imported automatically by the phddoc class and hence the following macros are described here for convenience.

\cmd{\foo} Prints \foo verbatim. It may be used inside moving arguments. It can *not* be use to record commands that are defined as “\outer” nor is it possible to use it on conditionals such as \iftrue or defined by \newif. \cs{foo} also prints \foo, for those who prefer that syntax. (This second form can be used to record all type of commends so the above restrictions do not apply.

\cmd
\cs

```
\def\cmd#1{\cs{\expandafter\cmd@to@cs\string#1}}
\def\cmd@to@cs#1#2{\char\number`#2\relax}

```

```
\DeclareRobustCommand\cls{\textcolor{thered}}
%\newcommand\cs[1]{\color{blue}{\texttt{\char`\ #1}}}

```

phddoc

(*End definition for* `\cmd` *and* `\cs`.)

`\marg` `\marg{text}` prints `{text}`, ‘mandatory argument’.

```
%\providecommand\marg[1]{%
%   {\ttfamily\char`\{}\meta{#1}{\ttfamily\char`\}}}
```

(*End definition for* `\marg`.)

`\oarg` `\oarg{text}` prints `[`⟨*text*⟩`]`, ‘optional argument’.

```
%\providecommand\oarg[1]{%
%   {\ttfamily[}\meta{#1}{\ttfamily]}}
```

(*End definition for* `\oarg`.)

`\parg` `\parg{te,xt}` prints `(`⟨*te,xt*⟩`)`, ‘picture mode argument’.

```
\providecommand\parg[1]{%
  {\ttfamily(}\meta{#1}{\ttfamily)}}
```

(*End definition for* `\parg`.)

B.6 DocInclude

```
%\@addtoreset{CodelineNo}{part}
\@addtoreset{CodelineNo}{chapter}
```

`\DocInclude` More or less exactly the same as `\include`, but uses `\DocInput` on a `dtx` file, not `\input` on a `tex` file.

```
\def\partname{File}
```

(*End definition for* `\DocInclude`.)

```
\def\task#1#2{}
\endinput

\newcommand*{\DocInclude}[1]{%
  \relax
  \clearpage
  \docincludeaux
  \IfFileExists{#1.fdd}{%
    \def\currentfile{#1.fdd}}{\def\currentfile{#1.dtx}
   }%
  \ifnum\@auxout=\@partaux
    \@latexerr{\string\include\space cannot be nested}\@eha
  \else \@docinclude#1 \fi}
%
\def\@docinclude#1 {\clearpage
  \if@filesw \immediate\write\@mainaux{\string\@input{#1.aux}}\fi
\@tempswatrue\if@partsw \@tempswafalse\edef\@tempb{#1}\@for
\@tempa:=\@partlist\do{\ifx\@tempa\@tempb\@tempswatrue\fi}\fi
\if@tempswa \let\@auxout\@partaux \if@filesw
\immediate\openout\@partaux #1.aux
\immediate\write\@partaux{\relax}\fi
```

We need to save (and later restore) various index-related commands which might be changed by the included file.

```
\let\@phddoc@PrintIndex\PrintIndex
\let\PrintIndex\relax
\let\@phddoc@PrintChanges\PrintChanges
\let\PrintChanges\relax
\let\@phddoc@theglossary\theglossary
\let\@phddoc@endtheglossary\endtheglossary
\part{\currentfile}%
  {\let\ttfamily\relax
  \xdef\filekey{\filekey, \thepart={\ttfamily\currentfile}}}%
\DocInput{\currentfile}%
\let\PrintIndex\@phddoc@PrintIndex
\let\PrintChanges\@phddoc@PrintChanges
\let\theglossary\@phddoc@theglossary
\let\endtheglossary\@phddoc@endtheglossary
\clearpage
\@writeckpt{#1}\if@filesw \immediate\closeout\@partaux \fi
\else\@nameuse{cp@#1}\fi\let\@auxout\@mainaux}
```

`\codeline@wrindex`

```
\gdef\codeline@wrindex#1{\if@filesw
        \immediate\write\@indexfile
            {\string\indexentry{#1}%
            {\filesep\number\c@CodelineNo}}\fi}%
```

(*End definition for* `\codeline@wrindex`.)

```
\let\filesep\@empty
```

`\aalph` Special form of `\alph` as currently `source2e.tex` includes more than 26 files .

```
\def\aalph#1{\@aalph{\csname c@#1\endcsname}}
\def\@aalph#1{%
  \ifcase#1\or a\or b\or c\or d\or e\or f\or g\or h\or i\or
         j\or k\or l\or m\or n\or o\or p\or q\or r\or s\or
         t\or u\or v\or w\or x\or y\or z\or A\or B\or C\or
         D\or E\or F\or G\or H\or I\or J\or K\or L\or M\or
         N\or O\or P\or Q\or R\or S\or T\or U\or V\or W\or
         X\or Y\or Z\else\@ctrerr\fi}
```

(*End definition for* `\aalph`.)

`\docincludeaux`

```
\def\docincludeaux{%
  \def\thepart{\aalph{part}}\def\filesep{\thepart-}%
  \let\filekey\@gobble
% add to index prologue
  \g@addto@macro\index@prologue{%
    \gdef\@oddfoot{\parbox{\textwidth}{\strut\footnotesize
       \raggedright{\bfseries File Key:} \filekey}}%
    \let\@evenfoot\@oddfoot}%
  \global\let\docincludeaux\relax
%
 \gdef\@oddfoot{%
   \expandafter\ifx\csname ver@\currentfile\endcsname\relax
```

```
    File \thepart: {\ttfamily\currentfile} %
   \else
    \GetFileInfo{\currentfile}%
    File \thepart: {\ttfamily\filena
    me} %
    Date: \filedate\ %
    Version \fileversion
    \fi
    \hfill\thepage}%
% one sided paper
 \let\@evenfoot\@oddfoot}%
```

(*End definition for* `\docincludeaux`.)

```
\def\task#1#2{}

/class
```

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