Dynamic Programming

Step 1: Write down the problem

Step 2: Write down the basic structure of Bellman Equation.

Step 3: What are the state variables ?

Step 4: What are the constraints?

Step 5: Put state variables and BCs into BE.

Step 6: What are the control variables in your BE ?

Step 7: FOC: take derivative of BE w.r.t. control

Step 8: EC: take derivative of BE w.r.t. state

Step 9: Combine FOC + EC $\Rightarrow$ EE.

Example: HW #3 Q1.

Preference: $u(c_t, k_{t-1})$

BC: $c_t + k_{t+1} = w_t + (1-\delta+r_t)k_t$

Step 1:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

$$\text{s.t. } c_t + k_{t+1} = w_t + (1-\delta+r_t)k_t$$

Step 2: $V(\quad) = \max u(\quad) + \beta V(\quad)$

Step 3: $k_t, \ k_{t-1} \qquad z_t = k_{t-1}$

Step 4: $V(k, z) = \max u(c, z) + \beta V(k', z')$

$\text{s.t. } c + k' = w + (1-\delta+r)k$ ✗

Step 5: $V(k, z) = \max u(w + (1-\delta+r)k - k', z) + \beta V(k', k)$

Step 6: k' is control variable in this Bellman Equation

Three Period OLG Model.

- HH Problem

$$\max \; u(c_t^y) + \beta u(c_{t+1}^m) + \beta^2 u(c_{t+2}^o)$$

$$\text{s.t.} \quad c_t^y = -b_{t+1}^m - s_t$$

$$c_{t+1}^m = w_{t+1} h_{t+1}^m + R_{t+1} b_{t+1}^m - b_{t+2}^o$$

$$c_{t+2}^o = w_{t+2} h_{t+1}^m + R_{t+2} b_{t+2}^o$$

$$h_{t+1}^m = h^y + f(s_t)$$

Bellman equation: $\underbrace{\text{this period objective}}_{\text{variable itself}}$ + $\underbrace{\text{continuation value}}_{\text{variable with prime}}$.

Young: $V^y(\quad) = \max u(\quad) + \beta V^m(\quad)$

Middle: $V^m(\quad) = \max u(\quad) + \beta V^o(\quad)$

Old: $V^o(\quad) = \max u(\quad) + \beta V(\quad)$

① Young

Step 1.

$$u(c_t^y)$$

$$c_t^y = -b_{t+1}^m - S_t$$

$$h_{t+1}^m = h^y + f(S_t)$$

Step 2. $V^y(\quad) = \max u(\quad) + \beta V^m(\quad)$

Step 3. h^y is state variable

$$\Rightarrow V^y(h) = \max u(-b' - s) + \beta V^m(\ ?\ )$$

② Middle-aged

Step 1.

$$u(c_{t+1}^m)$$

$$c_{t+1}^m = w_{t+1} h_{t+1}^m + R_{t+1} b_{t+1}^m - b_{t+2}^o$$

Note: for middle-aged, $t+1$ is "this period"
$t+2$ is "next period"

Step 2. $V^m(\quad) = \max u(\quad) + \beta V^o(\quad)$

Step 3. h_{t+1}^m, b_{t+1}^m are state variables

$$\Rightarrow V^m(h, b) = \max u(wh + Rb - b') + \beta V^o(\ ?\ )$$

Since we figured out the state variables for V^m, we can complete V^y

$$V^y(h) = \max u(-b' - s) + \beta V^m(h', b')$$

$$= \max u(-b' - s) + \beta V^m(h + f(s), b')$$

③ Old

Step 1. $u(c^o_{t+2})$

$$c^o_{t+2} = R_{t+2} b^o_{t+2} + W_{t+2} h_{t+2} = h^m_{t+1}$$

Note: for old, t+2 is "this period"

Step 2. $V^o(\quad) = \max u(\quad) + \cancel{\beta V}$ no next period

Step 3. b^o_{t+2}, h_{t+2} are state variables

$$\Rightarrow V^o(h, b) = u(Rb + wh)$$

Since we figured out the state variables for V^o,

We can complete V^m

$$V^m(h, b) = \max u(wh + Rb - b') + \beta V^o(h', b')$$
$$= \max u(wh + Rb - b') + \beta V^o(h, b')$$

Hence the Bellman Equations are

$$V^y(h) = \max u(-b' - s) + \beta V^m(h + f(s), b')$$
$$V^m(h, b) = \max u(wh + Rb - b') + \beta V^o(h, b')$$
$$V^o(h, b) = u(Rb + wh)$$

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Young: $V^h(h) = \max u(-b' - s) + \beta V^m(h + f(s), b')$

State controls

FOC $[b']$: $u'(c) = \beta V^m_b(h', b')$ (1)

$[s]$: $u'(c) = \beta V^m_h(h', b') \cdot f'(s)$ (2)

EC $[h]$: $V^y_h(h) = \beta V^m_h(h', b')$ (3)

Middle-age: $V^m(h, b) = \max u(wh + Rb - b') + \beta V^o(h, b')$

FOC $[b']$: $u'(c) = \beta V^o_b(h', b')$ (4)

EC $[h]$: $V^m_h(h, b) = u'(c) \cdot w + \beta V^o_h(h', b')$ (5)

$[b]$: $V^m_b(h, b) = u'(c) \cdot R$ (6)

Old: $V^o(h, b) = u(Rb + wh)$

EC $[h]$: $V^o_h(h, b) = w \cdot u'(c)$ (7)

$[b]$: $V^o_b(h, b) = R \cdot u'(c)$ (8)

(1) + (6): $u'(c) = \beta R' u'(c')$

(4) + (8): $u'(c) = \beta R' u'(c')$

Standard EE.

(2) + (5) + (7):

$$u'(c) = \beta \left\{ u'(c') \cdot w' + \beta w'' \cdot u'(c'') \right\} \cdot f'(s)$$

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- Firm Problem

$$\max \quad h_t^m L_t^m + h_{t-1}^m L_t^o - W_t \left(h_t^m L_t^m + h_{t-1}^m L_t^o \right)$$

- A-D setup

$$c_t^y = -b_{t+1}^m - S_t$$

$$c_{t+1}^m = W_{t+1} h_{t+1}^m + R_{t+1} b_{t+1}^m - b_{t+2}^o$$

$$c_{t+2}^o = W_{t+2} h_{t+1}^m + R_{t+2} b_{t+2}^o$$

$$h_{t+1}^m = h^y + f(S_t)$$

No bond !

Have price !