Econ720 - TA Session 8

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1. Continuous Time vs. Discrete Time

Why continuous time?

- Some pathological results of discrete-time models disappear when using continuous time. (See 'Introduction to Modern Economic Growth', Acemoglu, Exercise 2.21)
- Continuous-time models have more flexibility in the analysis of dynamics and allow explicit-form solutions in a wider set of circumstances.

1. Continuous Time vs. Discrete Time

Example:

- Objective: maximize lifetime utility
- BC:

  Income: labor, capital
  Expenditure: consumption, new capital

To solve the model:

- Discrete time:
  Sequential language $\rightarrow$ Lagrangean
  Dynamic programming $\rightarrow$ Bellman equation
- Continuous time:
  Optimal control $\rightarrow$ Hamiltonian (state variable, control variable)
  Dynamic programming $\rightarrow$ Hamilton-Jacobi-Bellman equation

1. Continuous Time vs. Discrete Time

- Discrete time

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.\ c_t + k_{t+1} = w_t + (1 - \delta + r_t) k_t$$

- Continuous time

$$\max \int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt$$

$$s.t.\ \dot{k}_t = w_t + (r_t - \delta) k_t - c_t$$

1. Continuous Time vs. Discrete Time

To discount:

- Discrete time: discount factor β, ($\beta = \frac{1}{1+\rho}$)
- Continuous time: discount rate ρ

In discrete time, $u(t) = \beta u(t+1) = \frac{1}{1+\rho} u(t+1)$. Hence, $\rho = \frac{u(t+1) - u(t)}{u(t)}$

In continuous time, the above equation becomes $\rho = \frac{\frac{d}{dt} u(t)}{u(t)} = \frac{d}{dt} ln u(t)$

Integrating both sides

$$\int_t^{t+\Delta} \rho \, ds = \int_t^{t+\Delta} \frac{d}{dt} ln u(s) ds$$

$$\rho \Delta = ln u(t+\Delta) - ln u(t) = ln \frac{u(t+\Delta)}{u(t)}$$

$$e^{\rho \Delta} = \frac{u(t+\Delta)}{u(t)} \Rightarrow u(t) = e^{-\rho \Delta} u(t+\Delta)$$

2. Optimal Control: Hamiltonian

1. Read the question carefully, figure out each sector
2. State variable, Control variable
   Only for sectors that have dynamic problem!
3. Write down the objective:
   integral, discount, objective you are trying to maximize
4. For each state var., write down a law of motion for that state var.
5. Hamiltonian or Current Value Hamiltonian
6. Differentiate H w.r.t control var. and set it to be 0
   Differentiate H w.r.t state var. and set it to be $-\dot{\mu}_t$ or $-\dot{\mu}_t + \rho \mu_t$
   Combine them to get rid of co-state variable, then get EE
7. Define the solution to this sector problem

Hamiltonian: $H = e^{-\rho t} u(c_t) + \hat{\mu}_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $e^{-\rho t} u'(c_t) - \hat{\mu}_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\hat{\mu}}_t$
Combine these two equations to substitute out $\hat{\mu}_t$

Current Value Hamiltonian: $H = u(c_t) + \mu_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $u'(c_t) - \mu_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\mu}_t + \rho \mu_t$
Combine these two equations to substitute out μ_t

2. Optimal Control: Hamiltonian

7. Define the solution:

   The solution to our example problem is a set of $\{c_t, k_t, \mu_t\}$ that satisfy

   - FOC
   - Law of motion for the state (/other constraints)
   - Boundary conditions

     k_0 is given

     TVC

More about TVC

- Finite time
  - With scrap value: $\mu(T) = \phi'(k_T)$
  - Without scrap value: $\mu(T) = 0$
- Infinite time
  - Hamiltonian: $\lim_{t\to\infty} \mu_t k_t = 0$
  - Current value Hamiltonian: $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = 0$