Econ720 - TA Session 2

Yanran Guo

UNC-Chapel Hill

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1. We lack one equation. Why?

Overlapping generations model slides $P27$.

Objects: $c_t^y,\ c_{t+1}^o,\ s_{t+1},\ b_{t+1}$
Equations: 2 BC, 1 EE
$\rightarrow$We lack one equation. Why?

Savings and bonds have the **same** rate of return: r_{t+1}. Hence we put them into one portfolio. As a household, when making decisions, we only need to consider how many goods to consume and how many goods to put in the portfolio.
i.e. Define $a_{t+1} = s_{t+1} + b_{t+1}$, we only decide $c_t^y,\ c_{t+1}^o$ and a_{t+1}

2. OLG Model: Dynamic Inefficiency

Overlapping generations model slides $P12$;
Overlapping generations model: equ. and ss slides $P26 - P28$

OLG model: dynamic inefficiency

Intuition:

Intergenerational borrowing and lending (Young $\Leftrightarrow$ Old) is impossible. The old doesn't have other sources of income. They must finance their consumption out of their own saving.

The Fundamental Theorems of Welfare do not necessarily apply to OLG economies. Because in OLG economies we have an infinite number of goods and the competitive equilibrium prices may not be finite in the limit (See PS1-Question 1).

3. Example: OLG with Capital, Land and Bonds

Model

Consider a standard two-period overlapping generations model with the following characteristics:

Demographics

- Each period a cohort of size $N_t = 1$ are born. Each cohort lives for two periods.
  $\rightarrow$ 2 periods: c_t^y, c_{t+1}^o; No population growth
- All cohorts are identical and behave competitively.
  $\rightarrow$ WLOG, we may consider representative households.

3. Example: OLG with Capital, Land and Bonds

Endowments and Preferences

- Each young cohort is endowed with 1 unit of labor which they supply inelastically
  $\rightarrow L_t = N_t = 1$
- At $t = 0$, the old cohort is endowed with k_0 units of capital and x_0 units of land.
- Each cohort born in generic period t maximizes the following utility function:

  $$U = u(c_t^y) + \beta u(c_{t+1}^o)$$

  where c_t^y and c_{t+1}^o represent consumption when young and old respectively and the utility function $u(\cdot)$ satisfies the usual conditions.
  $\rightarrow$ Utility only comes from consumption. So household supplies all their labor endowment, so that they can have more income, thus to support more consumption.

3. Example: OLG with Capital, Land and Bonds

Technology

- Capital k_t, land x_t, and bonds b_t can be traded among households in spot markets. Bonds can be stored intertemporally costlessly.
  No depreciation on bond.
- Capital and consumption goods can be freely transformed one to another (one-to-one)
  $\rightarrow$ k_t and c_t have the same price. Hence if the price of c_t is normalized to 1, the price of capital is also 1.
- Land is available in fixed apply. (Additional land above x_0 cannot be accumulated)
  $\rightarrow$ The total amount of land is always $x_0 N_0 = x_0$.
- Firms are identical and perfectly competitive.
  $\rightarrow$ Factor rental price is determined by FOC.

3. Example: OLG with Capital, Land and Bonds

- Firms **rent capital and land from old households and labor (L_t) from young households** to produce a final good with the following production function:

$$y_t = f(K_t,\ X_t,\ L_t)$$

  where $f(\cdot)$ satisfies the usual Inada conditions and y_t is in units of consumption.
- Capital depreciates after use at rate $0 \leq \delta \leq 1$. Land does not depreciate (Land is a durable good.)

3. Example: OLG with Capital, Land and Bonds

Markets

- Bonds are issued by households with interest rate R_{t+1} (in units of account) and have a one-period maturity.
  $\rightarrow$ A nominal monetary unit of measure
  $\rightarrow$ Buy bonds in current period, get returns in the next period.
- Capital may be traded at price P_t^k and rented to firms at rate R_t^k (in units of account)
- Land may be traded at price P_t^x and rented to firms at rate R_t^x (in units of account)
- Consumption goods may be traded at price P_t^c
- Goods market must hold for consumption and capital
  $\rightarrow$ Goods produced in each period is used for consumption and capital accumulation.

  $\rightarrow$ Goods market clearing condition:

$$y_t = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$$

3. Example: OLG with Capital, Land and Bonds

Questions

1. What are the representative household's budget constraint in each period?
2. Have we defined a numeraire yet? If not, let's do so.
3. What is the representative household's lifetime budget constraint?
4. Write down and solve the representative household's problem
5. Write down and solve the firm's problem
6. Define a competitive equilibrium

3. Example: OLG with Capital, Land and Bonds

1. What are the representative household's budget constraint in each period?

$(c_t^y,\ c_{t+1}^o,\ k_t,\ x_t,\ b_t)$

Key idea: Income = Consumption + Saving

Young:

$$W_t = P_t^c c_t^y + P_t^k k_t + P_t^x x_t + P_t^c b_t$$

Old:

$$R_{t+1}^k k_t + P_{t+1}^k (1-\delta) k_t + R_{t+1}^x x_t + P_{t+1}^x x_t + R_{t+1} b_t + P_{t+1}^c b_t = P_{t+1}^c c_{t+1}^o$$

3. Example: OLG with Capital, Land and Bonds

2. Have we defined a numeraire yet? If not, let's do so.

The budget constraints are in nominal units. But for most cases, it is more convenient to deal with real units.

$\Rightarrow$ We numerate the price of consumption!

$\Rightarrow$ Recall that we can define a numeraire in each period's budget constraint.

Young: $(P_t^y = 1)$

$$w_t = c_t^y + k_t + p_t^x x_t + b_t$$

Old: $(P_{t+1}^o = 1)$

$$r_{t+1}^k k_t + (1-\delta)k_t + r_{t+1}^x x_t + p_{t+1}^x x_t + r_{t+1} b_t + b_t = c_{t+1}^o$$

3. What is the representative household's lifetime budget constraint?

Substitute out b_t

$$\begin{aligned} w_t = c_t^y + p_t^x x_t + k_t \\ + \frac{1}{1+r_{t+1}}[c_{t+1}^o - (p_{t+1}^x + r_{t+1}^x)x_t - (1-\delta + r_{t+1}^k)k_t] \end{aligned}$$

3. Example: OLG with Capital, Land and Bonds

4. Write down and solve the representative household's problem

$$
\begin{aligned}
max \quad & u(c_t^y) + \beta u(c_{t+1}^o) \\
s.t. \quad & w_t = c_t^y + p_t^x x_t + k_t \\
& \quad + \frac{1}{1+r_{t+1}}[c_{t+1}^o - (p_{t+1}^x + r_{t+1}^x)x_t - (1-\delta + r_{t+1}^k)k_t]
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L} = & u(c_t^y) + \beta u(c_{t+1}^o) + \lambda \{w_t - c_t^y - p_t^x x_t - k_t \\
& - \frac{1}{1+r_{t+1}}[c_{t+1}^o - (p_{t+1}^x + r_{t+1}^x)x_t - (1-\delta + r_{t+1}^k)k_t]\}
\end{aligned}
$$

3. Example: OLG with Capital, Land and Bonds

$$[c_t^y] : u'(c_t^y) = \lambda$$

$$[c_{t+1}^o] : \beta u'(c_{t+1}^o) = \frac{\lambda}{1 + r_{t+1}}$$

$$[x_t] : \frac{1}{1 + r_{t+1}}(p_{t+1}^x + r_{t+1}^x) = p_t^x$$

$$[k_t] : \frac{1}{1 + r_{t+1}}(1 - \delta + r_{t+1}^k) = 1$$

$\Rightarrow$

E.E. for bonds: $u'(c_t^y) = \beta u'(c_{t+1}^o)(1 + r_{t+1})$

E.E. for land: $u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_{t+1}^x + r_{t+1}^x}{p_t^x}$

E.E. for capital: $u'(c_t^y) = \beta u'(c_{t+1}^o)(1 - \delta + r_{t+1}^k)$

INTERPRETATION!!

3. Example: OLG with Capital, Land and Bonds

5. Write down and solve the firm's problem

$$
\begin{aligned}
& max \quad P_t^c y_t - R_t^k K_t - W_t L_t - R_t^x X_t \\
\rightarrow & max \quad f(K_t,\ L_t,\ X_t) - r_t^k K_t - w_t L_t - r_t^x X_t \\
& [K_t] : f_K = r_t^k \\
& [L_t] : f_L = w_t \\
& [X_t] : f_X = r_t^x
\end{aligned}
$$

3. Example: OLG with Capital, Land and Bonds

6. Define a competitive equilibrium

Allocations $\{c_t^y, c_{t+1}^o, k_t, x_t, b_t, K_t, L_t, X_t\}$ and prices $\{p_t^x, r_t, r_t^k, r_t^x, w_t\}$ that satisfy

- H.H. Problem: B.C.(1), FOC(3);
- Firm Problem: FOC (3);
- Market Clearing Conditions:
  Goods market:
  $f(K_t,\ L_t,\ X_t) = N_t c_t^y + N_{t-1} c_t^o + K_{t+1} - (1-\delta)K_t$
  Capital rental market: $K_t = N_{t-1}k_{t-1} = k_{t-1}$
  Land rental market: $X_t = N_{t-1}x_{t-1} = x_0$
  Labor rental market: $L_t = N_t = 1$
  Bonds market: $b_t = 0$
- Accounting Identity: $1 + r_{t+1} = 1 - \delta + r_{t+1}^k = \frac{p_{t+1}^x + r_{t+1}^x}{p_t^x}$

Appendix: Interest Rate vs. Capital Rental Price

The households own the capital stock of the economy and rent it to firms. The rental price of capital at time t is denoted by q_t. If firms are perfectly competitive, q_t is equal to the marginal productivity of capital $q_t = f_K$.

Assume capital depreciates, meaning that machines that are used in production lose some of their value because of wear an tear. Particularly, let us assume that capital depreciates (exponentially) at rate δ, so that out of 1 unit of capital this period, only $1 - \delta$ is left for next period.

The loss of part of the capital stock affects the interest rate (rate of return on capital stock) faced by households.

Appendix: Interest Rate vs. Capital Rental Price

Given the assumption of exponential depreciation at the rate δ and the normalization of the price of final good to 1, the interest rate faced by the households is $r_t = q_t - \delta$, where recall that q_t is the capital rental price at time t ($q_t = f_K$).

A unit of final good can be consumed now or used as capital and rented to firms. In the latter case, a household receives q_t units of good in the next period as the rental price for its savings, but loses δ units of its capital holdings, since δ fraction of capital depreciates over time. Thus the household has given up one unity of commodity dated $t - 1$ and receives $1 + r_t = q_t + 1 - \delta$ units of commodity dated t, so that $r_t = q_t - \delta$.

– *Introduction to Modern Economic Growth, Daron Acemoglu, P31∼P32*