Econ720 - TA Session 8

Yanran Guo

UNC-Chapel Hill

2019. 10. 25

1. Continuous Time vs. Discrete Time

Why continuous time?

- Some pathological results of discrete-time models disappear when using continuous time. (See 'Introduction to Modern Economic Growth', Acemoglu, Exercise 2.21)
- Continuous-time models have more flexibility in the analysis of dynamics and allow explicit-form solutions in a wider set of circumstances.

1. Continuous Time vs. Discrete Time

Example:

- Objective: maximize lifetime utility
- BC:

  Income: labor, capital
  Expenditure: consumption, new capital

To solve the model:

- Discrete time:
  Sequential language $\rightarrow$ Lagrangean
  Dynamic programming $\rightarrow$ Bellman equation
- Continuous time:
  Optimal control $\rightarrow$ Hamiltonian (state variable, control variable)
  Dynamic programming $\rightarrow$ Hamilton-Jacobi-Bellman equation

1. Continuous Time vs. Discrete Time

To discount:

- Discrete time: discount factor β, ($\beta = \frac{1}{1+\rho}$)
- Continuous time: discount rate ρ

In discrete time, $u(t) = \beta u(t+1) = \frac{1}{1+\rho} u(t+1)$. Hence, $\rho = \frac{u(t+1)-u(t)}{u(t)}$

In continuous time, the above equation becomes $\rho = \frac{\frac{d}{dt}u(t)}{u(t)} = \frac{d}{dt} ln u(t)$

Integrating both sides

$$\int_t^{t+\Delta} \rho ds = \int_t^{t+\Delta} \frac{d}{dt} ln u(s) ds$$

$$\rho\Delta = ln u(t+\Delta) - ln u(t) = ln \frac{u(t+\Delta)}{u(t)}$$

$$e^{\rho\Delta} = \frac{u(t+\Delta)}{u(t)} \Rightarrow u(t) = e^{-\rho\Delta} u(t+\Delta)$$

2. Optimal Control: Hamiltonian

1. Read the question carefully, figure out each sector
2. State variable, Control variable
   Only for sectors that have dynamic problem!
3. Write down the objective:
   integral, discount, objective you are trying to maximize
4. For each state var., write down a law of motion for that state var.
5. Hamiltonian or Current Value Hamiltonian
6. Differentiate H w.r.t control var. and set it to be 0
   Differentiate H w.r.t state var. and set it to be $-\dot{\mu}_t$ or $-\dot{\mu}_t + \rho\mu_t$
   Combine them to get rid of co-state variable, then get EE
7. Define the solution to this sector problem

Hamiltonian: $H = e^{-\rho t} u(c_t) + \hat{\mu}_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $e^{-\rho t} u'(c_t) - \hat{\mu}_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\hat{\mu}}_t$
Combine these two equations to substitute out $\hat{\mu}_t$

Current Value Hamiltonian: $H = u(c_t) + \mu_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $u'(c_t) - \mu_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\mu}_t + \rho \mu_t$
Combine these two equations to substitute out μ_t

2. Optimal Control: Hamiltonian

7. Define the solution:

   The solution to our example problem is a set of $\{c_t, k_t, \mu_t\}$ that satisfy

   - FOC
   - Law of motion for the state (/other constraints)
   - Boundary conditions

     k_0 is given

     TVC

More about TVC

- Finite time
  - With scrap value: $\mu(T) = \phi'(k_T)$
  - Without scrap value: $\mu(T) = 0$
- Infinite time
  - Hamiltonian: $\lim_{t\to\infty} \mu_t k_t = 0$
  - Current value Hamiltonian: $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = 0$

3. Example

Consider a version of the standard one sector, neoclassical growth model with no technological progress, inelastic labor supply and zero population growth. We will examine the planner's problem in an economy in which capital is costly to adjust.

The planner maximizes the lifetime utility of the representative household:

$$max \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

There is one final good which is produced using labor and capital using a production technology which can be written in intensive form as $f(k_t)$, where $f'(\cdot) > 0$, $f''(\cdot) < 0$, and the usual Inada conditions apply.

3. Example

Capital is costly to adjust however, so the resource constraint is given by

$$f(k_t) = c_t + i_t(1 + T(\frac{i_t}{k_t}))$$

where i_t is investment, and $T(\cdot)$ represents additional costs incurred whenever capital is adjusted. We assume that $T(0) = 0$ and $T'(\cdot) > 0$. Capital depreciates at rate δ, implying that

$$\dot{k}_t = i_t - \delta k_t$$

The initial capital stock, k_0, is given.
Solve the social planner's problem and define the solution