Econ720 - TA Session 3

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1. Some takeaways from this week's lecture

- How to find Pareto efficient allocation?
  $\rightarrow$ Solve the social planner's problem.
- Social planner's problem:
  Maximizing some average of individual utilities, subject to **resource constraint**

2. Social security

- Fully-funded: $\tau^o_{t+1} = -(1 + r_{t+1})\tau^y_t$

  The young make contributions to the social security system, and their contributions are paid back to them in their old age.

  $\Rightarrow$ Essentially, the government just relabels some private savings as public. The total amount of capital stock in this economy doesn't change. So fully-funded social security will not improve dynamic inefficiency.

2. Social security

- Pay-as-you-go: $\tau_t^o = -(1+n)\tau_t^y$

  The government collects tax from the young at time t and distributes it directly to the current old. It's a pure transfer system.

3. Problem Set 1 - 1

1. OLG with Arrow-Debreu

1. Explain why the following is the correct budget constraint:
$$w_t + q_{t+1}s_{t+1} + (1-\delta)p_{t+1}s_{t+1} = p_t c_t^y + p_{t+1}c_{t+1}^o + p_t s_{t+1}$$

Answer key:

- What does the B.C. look like when it's a sequential trading setup?
- Why the price of saving is p_t?

3. Problem Set 1 - 1

In an Arrow-Debreu world, markets only open once before the actual start of the economy. Once trades are completed, markets close, and agents keep executing their contracts over time.

Interpretation:

The agent lives two periods. At period t, the only endowment for her is 1 unit of labor. She supplies this 1 unit of labor inelastically to get w_t income. She consumes c_t^y units of goods with price p_t. Since she can freely decide whether to consume 1 unit of goods or to use this one unit of goods as saving, the price of consumption goods should be equal to the price of saving. At the second period, the agent will consume all of her income with price p_{t+1} and will not save. Because this is her last period. Her income in the second period comes from two resources: the return of her saving $q_{t+1}s_{t+1}$ and she can also sell her undepreciated capital at price p_{t+1} $((1-\delta)p_{t+1}s_{t+1})$.

3. Problem Set 1 - 1

2. Derive the household's FOCs.

Answer key:

$$\begin{aligned}\mathscr{L} &= u(c_t^y) + \beta u(c_{t+1}^o) \\ &\quad + \lambda(w_t + q_{t+1}s_{t+1} + (1-\delta)p_{t+1}s_{t+1} - p_t c_t^y - p_t s_{t+1} - p_{t+1}c_{t+1}^o)\end{aligned}$$

$$\begin{aligned}[c_t^y]: &\quad u'(c_t^y) = \lambda p_t \\ [c_{t+1}^o]: &\quad \beta u'(c_{t+1}^o) = \lambda p_{t+1} \\ [s_{t+1}]: &\quad p_t = q_{t+1} + (1-\delta)p_{t+1}\end{aligned}$$

3. Problem Set 1 - 1

3. Define a solution to the household problem.

Answer key:

$(c_t^y,\ c_{t+1}^o,\ s_{t+1})$ that satisfy

- Budget Constraint:

  $p_t c_t^y + p_t s_{t+1} + p_{t+1} c_{t+1}^o = w_t + q_{t+1} s_{t+1} + (1-\delta) p_{t+1} s_{t+1}$

- Euler Equation:

  $u'(c_t^y) = \frac{p_t}{p_{t+1}} \beta u'(c_{t+1}^o)$

- No-arbitrage Condition: $p_t = q_{t+1} + (1-\delta) p_{t+1}$

3. Problem Set 1 - 1

4. What is the real interest rate in this economy?

Answer key:

The real interest rate is $\frac{p_t}{p_{t+1}}$

- Households can move consumption between dates at the exchange rate $\frac{p_t}{p_{t+1}}$.
- It it derived from E.E.

3. Problem Set 1 - 1

5. Interpret the condition $p_t = q_{t+1} + (1-\delta)p_{t+1}$

Answer key:

It is the No-arbitrage condition.

It comes from the equation $\frac{p_t}{p_{t+1}} = \frac{q_{t+1}}{p_{t+1}} + (1-\delta)$.

$\frac{p_t}{p_{t+1}}$ is the exchange rate of moving consumption between dates.

$\frac{q_{t+1}}{p_{t+1}} + (1-\delta)$ is the rate of return from investing in capital.

Both approaches must yield the same rate of return.

3. Problem Set 1 - 1

6. State the firm's FOCs. Watch your units!

Answer key:

$$
\begin{aligned}
&max \quad p_t F(K_t, L_t) - w_t L_t - q_t K_t \\
&q_t = p_t F_1(K_t, L_t) \\
&w_t = p_t F_2(K_t, L_t)
\end{aligned}
$$

3. Problem Set 1 - 1

7. Define a competitive equilibrium.

Answer key:

Allocations $\{c_t^y, c_{t+1}^o, s_{t+1}, K_t, L_t\}$ and prices $\{p_t, w_t, q_t\}$ that satisfy

- Household: Euler Equation, No-arbitrage Condition, Budget Constraint
- Firm: F.O.C (2)
- Market Clearing Conditions:
  1.Goods market: $F(K_t, L_t) = N_t c_t^y + N_{t-1} c_t^o + K_{t+1} - (1-\delta)K_t$
  2.Capital market: $N_t s_{t+1} = K_{t+1}$
  3.Labor market: $L_t = N_t$

3. Problem Set 1 - 1

8. Why don't we lose an equation due to Walras' Law?

Answer key:

Neither equation is redundant by Walras' Law. We only lose an equation for one t.

9. Where is numeraire?

Answer key:

We can make the price of consumption goods at any date t equal 1.

10. Define a steady state.

Answer key:

At steady state, all per-capita, real variables do not change along with time.

Steady state: $c^y, c^o, k, w/p, q/p, \pi$ that solve the equilibrium conditions without time subscripts. $\pi_{t+1} = p_{t+1}/p_t$

11. Under what condition do the Welfare theorems hold/fail? Recall that the Welfare theorems require $\lim_{t\to\infty} p_t = 0$.

Answer key:

Welfare theorems require $\lim_{t\to\infty} p_t = 0$, indicating that price can not explode. From no-arbitrage condition ($\frac{p_t}{p_{t+1}} = q_{t+1} + 1 - \delta$), no price explosion means $\frac{p_t}{p_{t+1}} > 1$, hence $q_{t+1} + 1 - \delta > 1 \Rightarrow q_{t+1} > \delta$ $\Rightarrow \lim_{t\to\infty} f'(k_{t+1}) > \delta$

2. OLG with Assets

Demographics: There are two types of households, indexed by h. In each period, a mass of 0.5 households is born of each type. Each person lives for 2 periods.

$\rightarrow$Heterogeneous agents

Endowments: Households receive endowments (e^y, e^o) when young and old, respectively.

Preferences: $ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1})$

$\rightarrow$Log utility, hence consumption is a constant fraction of total income.

Technology: None.

$\rightarrow$It's an endowment economy. Hence no capital accumulation.

3. Problem Set 1 - 2

Markets: Households trade goods and one period bonds that are issued and purchased by households

$\rightarrow$Sequential trading setup, not Arrow-Debreu. Why?

What is the budget constraint?

- Sequential trading setup
- No capital accumulation
- Endowment economy with bonds

3. Problem Set 1 - 2

Two ways to include bonds into budget constraint.

(1)

$$c^y_{h,t} + b_{h,t+1} = e^y$$
$$c^o_{h,t+1} = (1 + r_{t+1})b_{h,t+1} + e^o$$

(2)

$$c^y_{h,t} + q_t b_{h,t+1} = e^y$$
$$c^o_{h,t+1} = b_{h,t+1} + e^o$$

They are different in interpretation. But essentially they are equivalent: $q_t = \frac{1}{1+r_{t+1}}$

3. Problem Set 1 - 2

1. Define a solution to the household problem. Solve for the household's bond supply function.

Answer key:

$$\begin{aligned} & max\ ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1}) \\ & s.t.\ \ c^y_{h,t} + b_{h,t+1} = e^y \\ & \qquad c^o_{h,t+1} = (1 + r_{t+1})b_{h,t+1} + e^o \end{aligned}$$

$$\mathscr{L} = ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1}) + \lambda(e^y + \frac{e^o}{1+r_{t+1}} - c^y_{h,t} - \frac{c^o_{h,t+1}}{1+r_{t+1}})$$

3. Problem Set 1 - 2

$$[c^y_{h,t}]: \ \frac{1}{c^y_{h,t}} = \lambda$$

$$[c^o_{h,t+1}]: \ \frac{\beta_h}{c^o_{h,t+1}} = \frac{\lambda}{1 + r_{t+1}}$$

$$\Rightarrow c^o_{h,t+1} = \beta_h(1 + r_{t+1})c^y_{h,t}$$

From budget constraints

$$c^y_{h,t} + \frac{c^o_{h,t+1}}{1 + r_{t+1}} = e^y + \frac{e^o}{1 + r_{t+1}}$$

$$c^y_{h,t} + \frac{\beta_h(1 + r_{t+1})c^y_{h,t}}{1 + r_{t+1}} = e^y + \frac{e^o}{1 + r_{t+1}}$$

$$c^y_{h,t} = \frac{1}{1 + \beta}(e^y + \frac{e^o}{1 + r_{t+1}})$$

3. Problem Set 1 - 2

Hence

$$
\begin{aligned}
b_{h,t+1} &= e^y - c^y_{h,t} \\
&= e^y - \frac{1}{1+\beta}(e^y + \frac{e^o}{1+r_{t+1}}) \\
&= \frac{\beta_h}{1+\beta_h} e^y - \frac{e^o}{(1+\beta_h)(1+r_{t+1})}
\end{aligned}
$$

Solution: given r_{t+1}, $(c^y_{h,t}, c^o_{h,t+1}, b_{h,t+1})$ that satisfy 1 E.E. and 2 BCs.

3. Problem Set 1 - 2

2. Solve for the equilibrium bond interest rate.

Answer key:

Since we stated the budget constraint as

$$c^y_{h,t} + b_{h,t+1} = e^y$$
$$c^o_{h,t+1} = (1 + r_{t+1})b_{h,t+1} + e^o$$

for both types of households, when $b_{h,t+1}$ is negative, it means this household is selling bonds; when $b_{h,t+1}$ is positive, it means this household is buying bonds. In equilibrium, it must be the case that $\sum b_{h,t+1} = 0$.

3. Problem Set 1 - 2

$$\sum b_{h,t+1} = 0$$

$$\sum\left(\frac{\beta_h}{1+\beta_h} e^y - \frac{e^o}{(1+\beta_h)(1+r_{t+1})}\right) = 0$$

$$e^y \sum \frac{\beta_h}{1+\beta_h} = \frac{e^o}{1+r_{t+1}} \sum \frac{1}{1+\beta_h}$$

$$1 + r_{t+1} = \frac{e^o}{e^y} \frac{\sum \frac{1}{1+\beta_h}}{\sum \frac{\beta_h}{1+\beta_h}}$$

3. Problem Set 1 - 2

3. Provide intuitions for the features of equilibrium bond interest rate.

Answer key:

(1). If old endowments are larger, r is higher.

If old endowments are larger, households have less incentive to buy bonds when they are young. Hence bond supply exceeds demand. As a result, the price of bond decreases ($q_t = \frac{1}{1+r_{t+1}}$), indicating an increasing in r_{t+1}.

3. Problem Set 1 - 2

(2). If β_h increases, r decreases.

When β_h increases, households become more patient, which also means they value their old consumption more. Hence, households will reduce their young consumption and buy more bonds in order to consume more when they are old. Bond demand exceeds bond supply, leading to an increase in bond price ($q_t = \frac{1}{1+r_{t+1}}$). r_{t+1} decreases.

(3). Because of a lack of intergenerational trade, r is time invariant.

3. Problem Set 1 - 2

4. Now add a durable good to the economy. It is in fixed supply, K. It pays a dividend d per period (in units of consumption goods). Households trade shares of this good in an asset market at price p_t, measured in units of consumption goods. Define a competitive equilibrium for this economy.

Answer key:

$$
\begin{aligned}
& max \ ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1}) \\
& s.t. \quad c^y_{h,t} + b_{h,t+1} + p_t k_{h,t+1} = e^y \\
& \qquad\quad c^o_{h,t+1} = (1 + r_{t+1}) b_{h,t+1} + d k_{h,t+1} + p_{t+1} k_{h,t+1} + e^o
\end{aligned}
$$

3. Problem Set 1 - 2

$$\mathscr{L} = ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1})$$
$$+ \lambda(e^y + \frac{e^o}{1+r_{t+1}} - c^y_{h,t} - \frac{c^o_{h,t+1}}{1+r_{t+1}} - (p_t - \frac{p_{t+1}+d}{1+r_{t+1}})k_{h,t+1})$$

E.E. $\frac{c^o_{h,t+1}}{c^y_{h,t}} = \beta_h(1+r_{t+1})$

No-arbitrage condition: $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$

3. Problem Set 1 - 2

Competitive equilibrium:

Allocations $\{c^y_{h,t}, c^o_{h,t+1}, b_{h,t+1}, k_{h,t+1}\}$ and prices $\{r_{t+1}, p_t\}$ that satisfy

- Household problem: E.E. (2), BC (4);
- Market clearing condition:
  Goods market: $e^y + e^o + dK = 0.5\sum c^y_{h,t} + 0.5\sum c^o_{h,t}$
  Bonds market: $\sum b_{h,t+1} = 0$
  Durable Good Market: $0.5\sum k_{h,t+1} = K$
- No-arbitrage Condition: $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$

10 objects, 10 equations

3. Problem Set 1 - 2

5. Why do you find that the number of equations equals the number of objects to be determined? Usually, we find that we have one additional equation, which is redundant by Walras' law.

Answer key:

Because due to No-arbitrage condition, the real return of bonds is equal to the real return of durable goods, hence households do not need to specify the particular amount of $b_{h,t+1}$ and $k_{h,t+1}$, and can put them into one portfolio.

6. Derive an equation that determines the equilibrium price sequence p_t.

Answer key:

In equilibrium, $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$. Hence the life-time budget constraint of household

$$e^y + \frac{e^o}{1+r_{t+1}} = c^y_{h,t} + \frac{c^o_{h,t+1}}{1+r_{t+1}} + (p_t - \frac{p_{t+1}+d}{1+r_{t+1}})k_{h,t+1}$$

becomes

$$e^y + \frac{e^o}{1+r_{t+1}} = c^y_{h,t} + \frac{c^o_{h,t+1}}{1+r_{t+1}}$$

3. Problem Set 1 - 2

Substituting Euler Equation $\frac{c^o_{h,t+1}}{c^y_{h,t}} = \beta_h(1+r_{t+1})$ into life-time budget constraint,

$$c^y_{h,t} = \frac{e^y + \frac{e^o}{1+r_{t+1}}}{1+\beta_h} = \frac{e^y + \frac{e^o p_t}{p_{t+1} d}}{1+\beta_h}$$

From the durable goods market clearing condition

$$\begin{aligned}
2K &= \sum k_{h,t+1} \\
2p_t K &= \sum p_t k_{h,t+1} \\
&= \sum p_t k_{h,t+1} + \sum b_{h,t+1} \\
&= 2e^y - \sum c^y_{h,t} \\
&= 2e^y - \sum \frac{e^y + \frac{e^o p_t}{p_{t+1} d}}{1+\beta_h}
\end{aligned}$$