1. Lucas Tree with Crashes

- Cum dividend:

  The asset you buy today gives you dividend today.

- BC with ex-dividend:

$$c_t + p_t k_{t+1} = p_t k_t + d_t k_t$$

- BC with cum-dividend:

$$c_t + p_t k_{t+1} = p_t k_t + d_t k_{t+1}$$

(1). State the HH's dynamic problem.

Models with uncertainty, e.g. Stochastic growth model → RCE.

State variables:
- individual state k_t
- aggregate state d_t, d_{t-1}

HHs are identical, only aggregate uncertainty

Define $x_t = d_{t-1}$

$$V(k; d, x) = \max u(c) + \beta \mathbb{E} V(k'; d', d)$$

$$\text{s.t. } c + pk' = dk' + pk$$

$$\Rightarrow V(k; d, x) = \max u(dk' + pk - pk') + \beta \mathbb{E} V(k'; d', d)$$

FOC: [k'] $\quad u'(c) \cdot (p - d) = \beta \mathbb{E} V_k(k'; d', d)$

EC: [k] $\quad V_k(k; d, x) = u'(c) \cdot p$

$\therefore$ E.E. $\quad u'(c) \cdot (p - d) = \beta \mathbb{E}\{u'(c') \cdot p'\}$

(2). Euler Equation:

$$p_t = \beta \mathbb{E}\left\{\frac{u'(c_{t+1})}{u'(c_t)} \cdot p_{t+1}\right\} + d_t$$

(3). RCE:

Objects: $\underbrace{c(k;d,x),\ k'(k;d,x)}_{\text{policy function}}$, $\underbrace{V(k;d,x)}_{\text{value function}}$

$p(d,x)$

that satisfy

- Solution to HH problem
- Good mkt clearing condition: $c(k;d,x) = d$
- Asset mkt clearing condition: $k'(k;d,x) = 1$
- Law of motion of the aggregate state variable

(4). $$P_t = \beta \mathbb{E}\left\{\frac{u'(C_{t+1})}{u'(C_t)} \cdot P_{t+1}\right\} + d_t$$

utility function

$$= \beta \mathbb{E}\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} \cdot P_{t+1}\right\} + d_t$$

goods mkt clearing

$$= \beta \mathbb{E}\left\{\left(\frac{d_{t+1}}{d_t}\right)^{-\sigma} \cdot P_{t+1}\right\} + d_t$$

① If $d_t = d_{t-1}$, $$P_t = \beta P_{t+1} + d_t$$
$$= \beta[\beta P_{t+2} + d_t] + d_t$$
$$\cdots$$
$$= \cancelto{0}{\beta^{\infty} P_{t+\infty}} + \frac{1}{1-\beta} d_t$$

$\therefore P_t = \frac{1}{1-\beta} d_t \quad \forall t$, when $d_t = d_{t-1}$

② If $d_t \neq d_{t-1}$,
$$P_t = \beta\left\{\pi\left(\frac{\gamma d_t}{d_t}\right)^{-\sigma} \cdot P_{t+1} + (1-\pi)\left(\frac{d_t}{d_t}\right)^{-\sigma} \cdot P_{t+1}\right\} + d_t$$

$$= \beta\left\{\pi \gamma^{-\sigma} P_{t+1} + (1-\pi)\frac{1}{1-\beta} d_t\right\} + d_t$$

$$\therefore \frac{P_t}{d_t} = \beta \left\{ \pi \gamma^{-\sigma} \frac{P_{t+1}}{d_{t+1}} \frac{d_{t+1}}{d_t} + (1-\pi)\frac{1}{1-\beta} \right\} + 1$$

$$= \beta \left\{ \pi \gamma^{-\sigma} \frac{P_{t+1}}{d_{t+1}} \gamma + (1-\pi)\frac{1}{1-\beta} \right\} + 1$$

$\because \frac{P}{d}$ is constant during the phase with growth

$$\therefore \frac{P_t}{d_t} = \frac{P_{t+1}}{d_{t+1}} \equiv \frac{P_g}{d}$$

$$\therefore \frac{P_g}{d} = \beta \left\{ \pi \gamma^{1-\sigma} \frac{P_g}{d} + (1-\pi)\frac{1}{1-\beta} \right\} + 1$$

$$\therefore \frac{P_g}{d} = \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1 - \beta\pi\gamma^{1-\sigma}}$$

$$\therefore P_t = \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1-\beta\pi\gamma^{1-\sigma}} d_t \quad \text{when } d_t \neq d_{t-1}$$

Markov process: a discrete-time process for which the future behavior, given the past and present, only depends on the present and not the past.

$$P_t \begin{cases} = \frac{1}{1-\beta} d_t & \text{if } d_t = d_{t-1} \\ = \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1-\beta\pi\gamma^{1-\sigma}} d_t & \text{if } d_t \neq d_{t-1} \end{cases}$$

$\therefore P_t$ depends on (d_t, d_{t-1}), which is the current agg. state.

$\therefore P_t$ is Markov

(5). The price when growing is $\frac{p_t^g}{d_t} = \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1 - \beta\pi\gamma^{1-\sigma}}$

The price when stopping is $\frac{p_t^s}{d_t} = \frac{1}{1-\beta}$

"Crash" means $\frac{p_t^s}{d_t} < \frac{p_t^g}{d_t}$

$$\therefore \quad \frac{1}{1-\beta} < \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1 - \beta\pi\gamma^{1-\sigma}}$$

2. Two stocks.

(1). HH problem in recursive form

State variables: individual states: b, k_j, $j=1,2$.
aggregate states: θ_j, $j=1,2$.

$$V(b,k_1,k_2;\theta_1,\theta_2) = \max\ u(c) + \beta \mathbb{E}\, V(b,k_1,k_2;\theta_1,\theta_2)$$

$$\text{s.t.}\quad c + \Sigma p_j k_j' + q b' = b + \Sigma p_j k_j + \Sigma k_j \theta_j y$$

$$\Rightarrow$$

$$V(b,k_1,k_2;\theta_1,\theta_2) = \max\ u\left(b + \Sigma p_j k_j + \Sigma k_j \theta_j y - \Sigma p_j k_j' - q b'\right) + \beta \mathbb{E}\, V(b',k_1',k_2';\theta_1',\theta_2')$$

(2). Euler Equation

FOC $[b']$: $q\, u'(c) = \beta \mathbb{E} V_b(\,')$

$[k_1']$: $p_1 u'(c) = \beta \mathbb{E} V_{k_1}(\,')$

$[k_2']$: $p_2 u'(c) = \beta \mathbb{E} V_{k_2}(\,')$

EC. $[b]$: $V_b = u'(c)$

$[k_1]$: $V_{k_1} = (p_1 + \theta_1 y)\cdot u'(c)$

$[k_2]$: $V_{k_2} = (p_2 + \theta_2 y)\cdot u'(c)$

$\therefore$ EE.

$$q\, u'(c) = \beta \mathbb{E}\, u'(c')$$

$$p_1 u'(c) = \beta \mathbb{E}\, (p_1' + \theta_1' y)\cdot u'(c')$$

$$p_2 u'(c) = \beta \mathbb{E}\, (p_2' + \theta_2' y)\cdot u'(c')$$

(3). Good mkt clearing: $C_t = \sum_j \theta_{j,t} \cdot y = y$, $\forall t$

$\therefore$ EE for risk free bond becomes

$$q = \beta \mathbb{E} \left\{ \frac{u'(c')}{u'(c)} \right\} = \beta \mathbb{E} \left\{ \frac{u'(y)}{u'(y)} \right\} = \beta$$

EE for tree j becomes

$$P_j = \beta \mathbb{E} \left\{ \frac{u'(c')}{u'(c)} \left(P_j' + \theta_j' y \right) \right\}$$

$$= \beta \mathbb{E} \left\{ \frac{u'(y)}{u'(y)} \left(P_j' + \theta_j' y \right) \right\}$$

$$= \beta P_j' + \beta y \mathbb{E} (\theta_j')$$

$\because$ θ_j is iid

$\therefore$ $\mathbb{E}(\theta_{jt})$ is constant $\forall t$

$$\therefore P_{jt} = \beta P_{jt+1} + \beta y \mathbb{E}(\theta_j)$$

$$= \beta \left(\beta P_{jt+2} + \beta y \mathbb{E}(\theta_j) \right) + \beta y \mathbb{E}(\theta_j)$$

$$\cdots$$

$$= \cancel{\beta^{\infty} P_{jt+\infty}}^{\to 0} + \frac{\beta}{1-\beta} y \mathbb{E}(\theta_j)$$

$\therefore$ Stock price is constant, $P_{jt} = \frac{\beta}{1-\beta} y \mathbb{E}(\theta_j)$ $\forall t$, $j=1,2$.

(4). Equity Premium

Stock price $P_j = \frac{\beta}{1-\beta} y \, \mathbb{E}(\theta_j)$

and $\Sigma \theta_j = 1$

$\therefore$ (stock 1 & stock 2) have constant price and constant dividend.

$\therefore$ (Stock 1 & Stock 2) is a risk free asset

Then in equilibrium, $R^s \equiv R^f = \frac{1}{q} = \frac{1}{\beta}$.

(5) From EE.

$$P_{j,t} = \mathbb{E}\left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \left(P_{j\,t+1} + \theta_{j\,t+1} y_{t+1} \right) \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{c_t}{c_{t+1}} \left(P_{j\,t+1} + \theta_{j\,t+1} y_{t+1} \right) \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{y_t}{y_{t+1}} P_{j\,t+1} + \beta y_t \theta_{j\,t+1} \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{y_t}{y_{t+1}} \left(\beta \frac{y_{t+1}}{y_{t+2}} + \beta y_{t+1} \theta_{j\,t+2} \right) + \beta y_t \theta_{j\,t+1} \right\}$$

$$= y_t \left\{ \underbrace{\beta^{\infty} \mathbb{E}\left(\frac{1}{y_{t+\infty}} \right)}_{=0} + \sum_{s=1}^{\infty} \beta^s \theta_{j\,t+s} \right\}$$

$=0 \because y$ is from a finite Markov chain

$$= y_t + \mathbb{E}\left(\sum_{s=1}^{\infty} \beta^s \theta_{j,t+s}\right)$$

$\because \theta_j$ is iid

$$= y_t + \mathbb{E}(\theta_j) \cdot \sum_{s=1}^{\infty} \beta^s$$

$$= y_t + \frac{\beta}{1-\beta} \mathbb{E}(\theta_j)$$

The portfolio of two stocks : (Stock 1 & stock 2).

- Price : $P_t = P_{1t} + P_{2t}$

$$= y_t \frac{\beta}{1-\beta} \mathbb{E}(\theta_1) + y_t \frac{\beta}{1-\beta} \mathbb{E}(\theta_2)$$

$$= y_t \frac{\beta}{1-\beta} \left(\mathbb{E}(\theta_1) + \mathbb{E}(\theta_2) \right)$$

$$= y_t \frac{\beta}{1-\beta} \mathbb{E}\left(\underbrace{\theta_1 + \theta_2}_{=1}\right) = y_t \frac{\beta}{1-\beta}$$

- Dividend : y_t

$$\therefore \mathbb{E}(R_t^S) = \mathbb{E}\left(\frac{P_{t+1} + y_{t+1}}{P_t}\right) = \mathbb{E}\left(\frac{y_{t+1}\frac{\beta}{1-\beta} + y_{t+1}}{y_t \frac{\beta}{1-\beta}}\right)$$

$$\therefore \mathbb{E}(R_t^S) = \frac{1}{\beta} \mathbb{E}\left(\frac{y_{t+1}}{y_t}\right)$$