Econ720 - TA Session 5

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1. Money and Storage

Consider a two-period OLG model with fiat money and a storage technology.

Demographics: In each period, $N_t = (1+n)^t$ persons are born. Each lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M_0. Each young person is endowed with e units of consumption.

Preferences: $u(c_t^y) + \beta u(c_{t+1}^o)$

Technology: Storing k_t units of the good in t yields $f(k_t)$ units in $t+1$. f obeys Inada conditions. The resource constraint is $N_t k_{t+1} = N_t e + N_{t-1} f(k_t) - C_t$ where $C_t = N_t c_t^y + N_{t-1} c_t^o$

1. Problem Set 2 - 1

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_{t+1} = M_t + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu) M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

1. Problem Set 2 - 1

Timing in period t:

- The old enter period t holding aggregate capital $K_t = N_{t-1}k_t$ and nominal money balances of $M_t = m_t N_{t-1}$.
- Gov pays a lump-sum transfer of $x_t p_t$ units of money to each old person.
- Each old person produces $f(k_t)$.
- The young buy money $\frac{m_{t+1}}{p_t}$ from the old, consume c_t^y and save k_{t+1}.
- The old consume c_t^o.

1. Problem Set 2 - 1

Question

1. State the HH's BC when young and old.

Answer key:

- Young: $p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e$
- Old: $p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1}$

2. Derive the HH's optimality conditions. Define a solution to the HH problem.

Answer key:

$$max \ u(c_t^y) + \beta u(c_{t+1}^o)$$

$$\begin{aligned} s.t. \quad & p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e \\ & p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1} \end{aligned}$$

$$\begin{aligned} \mathscr{L} = & \ u(c_t^y) + \beta u(c_{t+1}^o) \\ & + \lambda \{ e + \frac{p_{t+1}}{p_t} (f(k_{t+1}) + x_{t+1}) - c_t^y - k_{t+1} - \frac{p_{t+1}}{p_t} c_{t+1}^o \} \end{aligned}$$

1. Problem Set 2 - 1

$$[c_t^y]: \ u'(c_t^y) = \lambda$$
$$[c_{t+1}^o]: \ \beta u'(c_{t+1}^o) = \lambda \frac{p_{t+1}}{p_t}$$
$$[k_{t+1}]: \ \frac{p_t}{p_{t+1}} = f'(k_{t+1})$$

Solution to HH problem is given price p_t, a vector $(c_t^y, c_{t+1}^o, m_{t+1}, k_{t+1})$ that satisfies

- 2 BCs
- EE: $u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}}$
- FOC: $\frac{p_t}{p_{t+1}} = f'(k_{t+1})$

3. Define a competitive equilibrium

Answer key:

Competitive equilibrium:

Allocations $\{c_t^y, c_t^o, m_{t+1}, k_{t+1}, x_t, M_t\}$ and prices $\{p_t\}$ that solve

- Household: 1 Euler Equation, 2 Budget Constraint
- Government: $M_{t+1} = M_t + N_{t-1}x_t p_t$, $M_{t+1} = (1+\mu)M_t$
- Market Clearing Conditions:

  1.Money market: $N_{t-1}m_t = M_t$

  2.Goods market: $N_t c_t^y + N_{t-1}c_t^o + N_t k_{t+1} = N_t e + N_{t-1}f(k_t)$
- Accounting identity: $\frac{p_t}{p_{t+1}} = f'(k_{t+1})$

1. Problem Set 2 - 1

4. Does an equilibrium with positive inflation exist? Intuition

Answer key:

No. That implies rate of return dominance, and nobody would hold money.

5. Define a steady sate as a system of 6 equations with 6 unknowns

1. Problem Set 2 - 1

Answer key:

CE: $\{c_t^y, c_t^o, m_{t+1}, k_{t+1}, x_t, M_t\}$, $\{p_t\}$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}} \tag{1}$$

$$p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e \tag{2}$$

$$p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1} \tag{3}$$

$$M_{t+1} = M_t + N_{t-1} x_t p_t \tag{4}$$

$$M_{t+1} = (1+\mu) M_t \tag{5}$$

$$N_{t-1} m_t = M_t \tag{6}$$

$$N_t c_t^y + N_{t-1} c_t^o + N_t k_{t+1} = N_t e + N_{t-1} f(k_t) \tag{7}$$

$$\frac{p_t}{p_{t+1}} = f'(k_{t+1}) \tag{8}$$

1. Problem Set 2 - 1

A steady state consists of constants $(c^y, c^o, m/p, k, x, \pi)$ that satisfy

- $\frac{1}{1+\pi} = \frac{1+n}{1+\mu} \rightarrow \pi$
- $f'(k) = \frac{1}{1+\pi} \rightarrow k$
- $u'(c^y) = \beta u'(c^o)\frac{1}{1+\pi}$ and
  $c^y + \frac{1}{1+n}c^o + k = e + \frac{1}{1+n}f(k) \rightarrow c^y$ and c^o
- $c^y + k + m/p(1+\pi) = e \rightarrow m/p$
- $x = \mu(m/p) \rightarrow x$

1. Problem Set 2 - 1

6. Find the money growth rate (μ) that maximizes steady state consumption per young person $\frac{N_t c^y + N_{t-1} c^o}{N_t}$

Answer key:

$$max\ c^y + \frac{1}{1+n} c^o \Leftrightarrow max\ e + \frac{1}{1+n} f(k) - k$$

Hence,

$$\frac{1}{1+n} f'(k^*) - 1 = 0$$

$$f'(k^*) = 1+n \quad \rightarrow \quad \frac{1}{1+\pi^*} = 1+n \quad \rightarrow \quad \frac{1+n}{1+\mu^*} = 1+n$$

$\mu^* = 0$

2. Problem Set 2 - 2

2. Money in the Utility Function in an OLG Model

Demographics: In each period a cohort of constant size N is born. Each lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M. No new money is ever issued. The young are endowed with one unit of work time.

Preferences: $u(c_t^y) + \beta u(c_{t+1}^o) + v(m_t^d / p_t)$. Assume $v' > 0$. Agents derive utility from real money balances as defined above.

Technology: Output is produced with a constant returns to scale production function $F(K_t, L_t)$. The resource constraint is standard. Capital depreciates at rate δ.

Markets: There are spot markets for goods (price p_t), money, labor (wage w_t), and capital rental (price q_t).

2. Problem Set 2 - 2

Timing in period t:

- The old enter period t holding capital K_t and money M.
- Production takes place.
- The old sell money to the young. m_t^d is the nominal per capita money holding of a young person.
- Consumption takes place.

2. Problem Set 2 - 2

Question:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household's first-order conditions imply rate of return dominance, i.e. the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium)

2. Problem Set 2 - 2

Answer key:

$$max \;\; u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$

$$s.t. \;\; p_t c_t^y + m_t^d + p_t s_{t+1} = p_t w_t$$

$$p_{t+1} c_{t+1}^o = p_{t+1}(1-\delta)s_{t+1} + p_{t+1} q_{t+1} s_{t+1} + m_t^d$$

$$\Rightarrow c_t^y + \frac{m_t^d}{p_t} + s_{t+1} = w_t \qquad (1)$$

$$c_{t+1}^o = (1-\delta+q_{t+1})s_{t+1} + \frac{m_t^d}{p_t}\frac{p_t}{p_{t+1}} \qquad (2)$$

Write lifetime budget constraint by substituting out s_{t+1}

2. Problem Set 2 - 2

$$\mathscr{L} = u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$
$$+ \lambda \{ w_t - c_t^y - \frac{m_t^d}{p_t} - \frac{c_{t+1}^o}{1-\delta+q_{t+1}} + \frac{m_t^d}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+q_{t+1}} \}$$

2. Problem Set 2 - 2

$$[c_t^y]: \; u'(c_t^y) = \lambda$$

$$[c_{t+1}^o]: \; \beta u'(c_{t+1}^o) = \lambda \frac{1}{1-\delta+q_{t+1}}$$

$$[m_t^d]: \; v'(\frac{m_t^d}{p_t})\frac{1}{p_t} = \lambda(\frac{1}{p_t} - \frac{1}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+q_{t+1}})$$

$\Rightarrow$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+q_{t+1}) \quad (3)$$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}} + v'(\frac{m_t^d}{p_t}) \quad (4)$$

Household behavior is characterized by equ. (1), (2), (3), (4)

2. Problem Set 2 - 2

- $u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+q_{t+1})$:

  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of saving in period t. This yields $(1-\delta+r_{t+1})$ income gain in the next period, valued by $u'(c_{t+1}^o)$ and discounted by β.

- $u'(c_t^y) - v'(\frac{m_t^d}{p_t}) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}}$

  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of real money in period t. But this cost is mitigated by utility gain from holding an extra unit of real money $v'(\frac{m_t^d}{p_t})$. So the net change in utility today is $u'(c_t^y) - v'(\frac{m_t^d}{p_t})$. Hold one unit of real money today yields $\frac{p_t}{p_{t+1}}$ gain tomorrow, valued by $u'(c_{t+1}^o)$ and discounted by β.

From equ. (3) and (4), we obtain

$$v'(\frac{m_t^d}{p_t}) = \beta u'(c_{t+1}^o)(1 - \delta + q_{t+1} - \frac{p_t}{p_{t+1}})$$

Since $u'(\cdot) > 0$, $v'(\cdot) > 0$, in order to have this equation to hold, it must be the case that

$$1 - \delta + q_{t+1} > \frac{p_t}{p_{t+1}}$$

$\Rightarrow$ Rate of return dominance

2. Problem Set 2 - 2

2. Solve the firm's problem.

Answer key:

$$\begin{aligned}
&max \quad F(K_t, L_t) - w_t L_t - q_t K_t \\
&q_t = F_1(K_t, L_t) \\
&w_t = F_2(K_t, L_t)
\end{aligned}$$

3. Define a competitive equilibrium.

Answer key:

Allocations $\{c_t^y, c_t^o, s_{t+1}, m_t^d, K_t, L_t\}$ and prices $\{p_t, w_t, q_t\}$ that satisfy

- Household: Euler Equation (2), Budget Constraint (2)
- Firm: F.O.C (2)
- Market Clearing Conditions:
  1.Goods market: $F(K_t, L_t) = Nc_t^y + Nc_t^o + K_{t+1} - (1-\delta)K_t$
  2.Capital market: $Ns_{t+1} = K_{t+1}$
  3.Labor market: $L_t = N$
  4.Money market: $M = Nm_t^d$

2. Problem Set 2 - 2

4. Assume that the utility functions u and v are logarithmic. Solve in closed form for the household's money demand function, $\frac{m_t^d}{p_t} = \varphi(w_t, q_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, q_{t+1}, \pi_{t+1})$, where $\pi_{t+1} = \frac{p_{t+1}}{p_t}$

Answer key:
Combing equ. (3), (4) and two BCs, we can obtain

$$\frac{m_t^d}{p_t} = \frac{\pi_{t+1}(1-\delta+q_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}$$

$$s_{t+1} = \{\frac{\beta}{\beta+2} - \frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}\}w_t$$

2. Problem Set 2 - 2

From equ. (3), we can obtain

$$\frac{1}{c_t^y} = (1-\delta+q_{t+1})\beta\frac{1}{c_{t+1}^o}$$
$$\rightarrow \ c_{t+1}^o = (1-\delta+q_{t+1})\beta c_t^y$$

From equ. (4), we can obtain

$$\frac{1}{c_t^y} = \frac{1}{\pi_{t+1}}\beta\frac{1}{c_{t+1}^o} + \frac{1}{m_t}$$
$$\rightarrow \ c_{t+1}^o = \beta c_t^y m_t \frac{1}{\pi_{t+1}(m_t - c_t^y)}$$

where $m_t \equiv \frac{m_t^d}{p_t}$

2. Problem Set 2 - 2

$$c^o_{t+1} = (1-\delta+q_{t+1})\beta c^y_t$$
$$c^o_{t+1} = \beta c^y_t m_t \frac{1}{\pi_{t+1}(m_t - c^y_t)}$$

$\Rightarrow$

$$c^y_t = \frac{[(1-\delta+q_{t+1})\pi_{t+1}-1]m_t}{(1-\delta+q_{t+1})\pi_{t+1}}$$

Substitute this equation into B.C.

$$c^y_t + s_{t+1} + m_t = w_t$$
$$(1-\delta+q_{t+1})\beta c^y_t = \frac{m_t}{\pi_{t+1}} + (1-\delta+q_{t+1})s_{t+1}$$

2. Problem Set 2 - 2

$$\frac{[(1-\delta+q_{t+1})\pi_{t+1}-1]m_t}{(1-\delta+q_{t+1})\pi_{t+1}}+s_{t+1}+m_t=w_t$$

$$\beta\frac{[(1-\delta+q_{t+1})\pi_{t+1}-1]m_t}{\pi_{t+1}}=\frac{m_t}{\pi_{t+1}}+(1-\delta+q_{t+1})s_{t+1}$$

Two equations, two unknowns $(m_t,\, s_{t+1})$

$$\frac{m_t^d}{p_t}=\frac{\pi_{t+1}(1-\delta+q_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}$$

$$s_{t+1}=\{\frac{\beta}{\beta+2}-\frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}\}w_t$$

3. Dynamic Programming

Two ways to solve the model:

- Sequence language /sequential solution - set up Lagrangean to find sequences of real variables.
  - In infinite horizon problem, when defining the solution to planner or household problem, always remember to write boundary conditions: initial condition + TVC (9/16, P13)
- Recursive formulation - use Dynamic Programming (DP) and set up Bellman equation to find a sequence of value functions and policy functions

3. Dynamic Programming

- Define **state variables**
  Variables carried over into the current period from the last period
  Variables that are predetermind in the current period
- Define control variables
- Value function: V(state variables)
- Utility + continuation value

$$\Rightarrow \begin{cases} \text{Bellman equation} \\ \text{FOC} \\ \text{Envelope condition} \\ \text{Euler equation} \\ \text{Law of motion} \end{cases}$$

3. Dynamic Programming - Example

$$max \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.t. \quad k_{t+1} = f(k_t) - c_t$$

What are the state variables?

What are the control variables?

How to write the Bellman equation?

3. Dynamic Programming - Example

Bellman equ.

$$V(k) = max\ u(c) + \beta V(k') + \lambda(f(k) - c - k') \quad \text{(state } k\text{, control } c, k'\text{)}$$

or

$$V(k) = max\ u(f(k) - k') + \beta V(k') \quad \text{(state } k\text{, control } k'\text{)}$$

or

$$V(k) = max\ u(c) + \beta V(f(k) - c) \quad \text{(state } k\text{, control } c\text{)}$$

Notice:

- It is conventional to drop the time subscript and use a prime to denote next period variables.
- Remember to write 「max」! The value function tells us the maximum utility obtainable from tomorrow onwards for any value of the state variables.
- Remember to write β in front of next period's V

3. Dynamic Programming - Solution

Solve the DP problem

- FOC w.r.t control variables, Envelope condition w.r.t state variables $\Rightarrow$ EE

The solution of a DP problem is

- Policy function: each control variable has a policy function, which is a function of state variables.

  e.g. $c = \phi(k)$, notice that it should be a function of k, not k'!
- Value function: $V(\cdot)$

3. Dynamic Programming - Some Tips

1. Finite horizon

   Time consistency and stationarity $\rightarrow$ one Bellman equation. However, stationary doesn't hold with finite horizon. Value function changes over time.
   Write Bellman equation for each period (DP slides P13-16), or add t as a state variable (DP slides P33)
   Exercise: 2020 Midterm Q1

2. Lagged variable in utility or BC, e.g. $u(c_t, c_{t-1})$

   Define a new state variable $s_t = c_{t-1}$ and add it to the value function. Remember to define the law of motion for s! (DP slides P36-44)