Example: Government Bond

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Government Bonds in an OLG Model

Model

Consider a standard two-period overlapping generations model with the following characteristics:

Demographics

- Each period a cohort of size $N_t = (1+n)^t$ are born. Each cohort lives for two periods.
  $\rightarrow$ 2 periods: c_t^y, c_{t+1}^o; What's population growth rate?
- All cohorts are identical and behave competitively.

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Endowments and Preferences

- Each young cohort is endowed with 1 unit of labor
- At $t = 0$, the old cohort is endowed with s_0 units of capital.
- Each cohort born in generic period t maximizes the following utility function:

$$U = (1-\beta)ln(c_t^y) + \beta ln(c_{t+1}^o)$$

where c_t^y and c_{t+1}^o represent consumption when young and old respectively.
$\rightarrow$ Utility only comes from consumption. So household supplies all their labor endowment, so that they can have more income, thus to support more consumption.
$\rightarrow$ Can we say anything about labor mkt clearing condition already?
$\rightarrow$ Anything else we can get from the utility function?

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Technology

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

$\rightarrow$ Do we have firm sector? Do we have depreciation on capital?

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Government

- The government only rolls over debt from one period to the next

$$B_{t+1} = R_t B_t$$

All the government does: issue new bonds to pay off the old ones.
$\rightarrow$ Can we say anything about bond mkt clearing condition already?

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- How many sectors do we have?
- What's the optimization problem for each sector?
- How many mkts do we have?

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- How many sectors do we have?
- What's the optimization problem for each sector?
- How many mkts do we have? (Save this question to equilibrium)

Markets: goods, bonds, labor, capital rental

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Questions

1. Solve the HH problem for saving function.
2. Write down and solve the firm's problem
3. Define a competitive equilibrium
4. Derive the law of motion for capital stock

$$(b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^{\alpha}$$

   where $b = B/L$, $k = K/L$
5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ.
6. Derive the steady state capital stock for $b > 0$
7. Can you show that the capital stock is lower in the steady state with positive debt?

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1. Solve the HH problem for saving function.

$$\begin{aligned} max \quad & U(\cdot) \\ s.t. \quad & BC \end{aligned}$$

How to write the BC?

$(c_t^y,\ c_{t+1}^o,\ k_{t+1},\ b_{t+1})$

Key idea: Income = Consumption + Saving

Young:

$$w_t = c_t^y + k_{t+1} + b_{t+1}$$

Old:

$$q_{t+1}k_{t+1} + (1-\delta)k_{t+1} + R_{t+1}b_{t+1} = c_{t+1}^o$$

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1. Solve the HH problem for saving function.

$$
\begin{aligned}
max \quad & (1-\beta)ln(c_t^y) + \beta ln(c_{t+1}^o) \\
s.t. \quad & w_t = c_t^y + k_{t+1} + b_{t+1} \\
& q_{t+1}k_{t+1} + (1-\delta)k_{t+1} + R_{t+1}b_{t+1} = c_{t+1}^o
\end{aligned}
$$

Substitute out b_{t+1}, get life-time BC

$$w_t = c_t^y + k_{t+1} + \frac{1}{R_{t+1}}[c_{t+1}^o - (1-\delta+q_{t+1})k_{t+1}]$$

$$
\begin{aligned}
\mathscr{L} = (1-\beta)ln(c_t^y) + \beta ln(c_{t+1}^o) + \lambda_t\{w_t - c_t^y - k_{t+1} \\
- \frac{1}{R_{t+1}}[c_{t+1}^o - (1-\delta+q_{t+1})k_{t+1}]\}
\end{aligned}
$$

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1. Solve the HH problem for saving function.

$$[c_t^y] : (1-\beta)\frac{1}{c_t^y} = \lambda_t$$

$$[c_{t+1}^o] : \beta\frac{1}{c_{t+1}^o} = \frac{\lambda_t}{R_{t+1}}$$

$$[k_{t+1}] : R_{t+1} = 1-\delta+q_{t+1}$$

$\Rightarrow$

$$\text{E.E. for bonds:}(1-\beta)\frac{1}{c_t^y} = \beta\frac{1}{c_{t+1}^o}R_{t+1}$$

$$\text{E.E. for capital:}(1-\beta)\frac{1}{c_t^y} = \beta\frac{1}{c_{t+1}^o}(1-\delta+q_{t+1})$$

No-arbitrage condition/ Accounting identity!

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1. Solve the HH problem for saving function.

$$\text{From the equ.}(1-\beta)\frac{1}{c_t^y} = \beta\frac{1}{c_{t+1}^o}R_{t+1}$$

$$c_{t+1}^o = \frac{\beta}{1-\beta}R_{t+1}c_t^y$$

Plug back to the life-time budget constraint:

$$w_t = c_t^y + k_{t+1} + \frac{1}{R_{t+1}}[c_{t+1}^o - (1-\delta+q_{t+1})k_{t+1}]$$

$$w_t = c_t^y + k_{t+1} + \frac{1}{R_{t+1}}[\frac{\beta}{1-\beta}R_{t+1}c_t^y - R_{t+1}k_{t+1}]$$

$$c_t^y = (1-\beta)w_t$$

Where does saving show up?

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1. Solve the HH problem for saving function.

$$s_{t+1} = w_t - c_t^y = w_t - (1-\beta)w_t = \beta w_t$$

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2. Write down and solve the firm's problem

$$
\begin{aligned}
&max \quad K_t^{\alpha} L_t^{1-\alpha} - q_t K_t - w_t L_t \\
&[K_t] : \alpha K_t^{\alpha-1} L_t^{1-\alpha} = q_t \\
&[L_t] : (1-\alpha) K_t^{\alpha} L_t^{-\alpha} = w_t
\end{aligned}
$$

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3. Define a competitive equilibrium

Allocations $\{c_t^y, c_t^o, s_t, k_t^h, b_t^h, K_t, L_t, B_t\}$ and prices $\{R_t, q_t, w_t\}$ that satisfy

- H.H. Problem: B.C.(2), EE, $s_{t+1} = k_{t+1}^h + b_{t+1}^h$
- Firm Problem: FOC (2)
- Government: $B_{t+1} = R_t B_t$
- Market Clearing Conditions:
  Goods market: $F(K_t,\ L_t) = N_t c_t^y + N_{t-1} c_t^o + K_{t+1} - (1-\delta)K_t$
  Labor rental market: $L_t = N_t$
  Capital rental market: $K_{t+1} = N_t k_{t+1}^h$
  Bonds market: $B_{t+1} = N_t b_{t+1}^h$
- Accounting Identity: $R_{t+1} = 1 - \delta + q_{t+1}$

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4. Derive the law of motion for capital stock

$$(b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^{\alpha}$$

where $b = B/L$, $k = K/L$

- Does $(\frac{K_t}{L_t})^{\alpha}$ look familiar?
- Which equation contains $(\frac{K_t}{L_t})^{\alpha}$?
- Which equation contains b, k and w? Where to look at?

Government Bonds in an OLG Model

4. Derive the law of motion for capital stock

$$(b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^\alpha$$

where $b = B/L$, $k = K/L$

$$\begin{aligned}
s_{t+1} &= k_{t+1}^h + b_{t+1}^h \\
N_t s_{t+1} &= N_t k_{t+1}^h + N_t b_{t+1}^h \\
L_t s_{t+1} &= K_{t+1} + B_{t+1} \\
s_{t+1} &= (k_{t+1} + b_{t+1})(1+n) \\
\beta w_t &= (k_{t+1} + b_{t+1})(1+n) \\
\beta(1-\alpha)k_t^\alpha &= (k_{t+1} + b_{t+1})(1+n)
\end{aligned}$$

k_{t+1}^h and b_{t+1}^h are HH choice for capital and bond. But the k and b in question is capital per labor and bond per labor.

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5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ.

- What is steady state?
- Which equation do we use to get steady state value?

Government Bonds in an OLG Model

5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ.

In steady state, $k_{t+1} = k_t = k^*$

From the law of motion for capital stock

$$(b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^{\alpha}$$

And we assume $b = 0$

$$k^*(1+n) = \beta(1-\alpha)k^*$$

$$k^* = \left(\frac{\beta(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$$

Steady state k^* does not depend on δ. Why?

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5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ.

- How can we link k with δ?
- Who choose k?

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5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ.

→ log utility

→ saving is a constant fraction of income ($s_t = \beta w_t$)

→ saving is independent from rate of return

→ capital is part of saving, δ is part of rate of return. Hence these two things are also independent from each other.

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6. Derive the steady state capital stock for $b > 0$

- To get steady state value, always start from law of motion
- Can we use $(b+k)(1+n) = \beta(1-\alpha)k^{\alpha}$?
- What's wrong with it?
- **Law of Motion!!**

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6. Derive the steady state capital stock for $b > 0$

$$B_{t+1} = R_t B_t$$

$B = RB$ correct? Why?

$$b_{t+1}(1+n) = R_t b_t$$

In steady state, $b_{t+1} = b_t = b^*$

$$b^*(1+n) = Rb^*$$

$$R = 1+n$$

$$1-\delta+q = 1+n$$

$$1-\delta+\alpha k^{\alpha-1} = 1+n$$

$$k^* = \left(\frac{n+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

Golden rule level of capital stock can be obtained with positive debt.

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7. Can you show that the capital stock is lower in the steady state with positive debt?

7. Positive debt: $k_P^* = \left(\frac{\alpha}{n+\delta}\right)^{\frac{1}{1-\alpha}}$

No debt: $k_N^* = \left(\frac{\beta(1-\alpha)}{1+n}\right)^{\frac{1}{1-\alpha}}$

$k_P^* + b^* = \frac{\beta(1-\alpha)}{1+n} k_P^{*\,\alpha}$ ← law of motion of k.

$b^* = \frac{\beta(1-\alpha)}{1+n} k_P^{*\,\alpha} - k_P^* > 0 \quad \Rightarrow \quad \frac{\beta(1-\alpha)}{1+n} k_P^{*\,\alpha-1} - 1 > 0$

$\cancel{\frac{\beta(1-\alpha)}{1+n}\left(\frac{\alpha}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}} \quad \Rightarrow \quad \frac{\beta(1-\alpha)}{1+n}\left(\frac{\alpha}{n+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} > 1$

$\Rightarrow \quad \frac{\beta(1-\alpha)}{1+n} \frac{n+\delta}{\alpha} > 1$

$\Rightarrow \quad \frac{\beta(1-\alpha)}{1+n} > \frac{\alpha}{n+\delta}$

$\therefore \quad k_N^* > k_P^*$

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Take-away from this example

- Introducing an infinitely lived asset fixes dynamic inefficiency.
- Gov is an infinitely lived agent who keeps trading the bonds. Hence the bond fixes dynamic inefficiency even though it lives only for one period.