Example 1. Capital Adjustment Costs.

$$\max \int_0^\infty e^{-\rho t} u(c_t)\, dt$$

$$\text{s.t.} \quad f(k_t) = c_t + i_t \left(1 + T\left(\frac{i_t}{k_t}\right)\right)$$

$$\dot{k}_t = i_t - \delta k_t$$

Current Value Hamiltonian:

$$H = u(c_t) + \mu_t (i_t - \delta k_t) + \lambda_t \left(f(k_t) - c_t - i_t - i_t T\left(\frac{i_t}{k_t}\right)\right)$$

$[c_t]$: $\quad u'(c_t) = \lambda_t$

$[i_t]$: $\quad \mu_t = \lambda_t + \lambda_t T\left(\frac{i_t}{k_t}\right) + \lambda_t \frac{i_t}{k_t} T'\left(\frac{i_t}{k_t}\right)$ Notice that i_t is a control variable

$[k_t]$: $\quad -\delta \mu_t + \lambda_t \left(f'(k_t) + \left(\frac{i_t}{k_t}\right)^2 T'\left(\frac{i_t}{k_t}\right)\right) = -\dot{\mu}_t + \rho \mu_t$

The solution to this planner problem is $\{c_t, i_t, k_t, \mu_t, \lambda_t\}$ that satisfy

- FOCs
- Resource Constraint: $f(k_t) = c_t + i_t \left(1 + T\left(\frac{i_t}{k_t}\right)\right)$
- Law of motion for k
- Boundary Condition $\begin{cases} k_0 \text{ is given} \\ \text{TVC: } \lim_{t \to \infty} e^{-\rho t} \mu_t k_t = 0 \end{cases}$

Example 2. An Investment Problem.

$$\max \int_0^\infty e^{-rt} \left(f(k_t) - I_t - \phi(I_t) \right) dt$$

$$\text{s.t.} \quad \dot{k}_t = I_t - \delta k_t$$

Current Value Hamiltonian

$$H = f(k_t) - I_t - \phi(I_t) + \mu_t (I_t - \delta k_t)$$

$$[I_t]: \quad \mu_t = 1 + \phi'(I_t) \quad \Rightarrow \dot{\mu}_t = \phi''(I_t)\, \dot{I}_t \quad \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = \frac{\phi''(I_t)\, \dot{I}_t}{1+\phi'(I_t)}$$

$$[k_t]: \quad f'(k_t) - \delta \mu_t = -\dot{\mu}_t + r\mu_t \quad \Rightarrow \frac{\dot{\mu}_t}{\mu_t} = \delta + r - \frac{f'(k_t)}{\mu_t} = \delta + r - \frac{f'(k_t)}{1+\phi'(I_t)}$$

$$\therefore \quad \frac{\phi''(I_t)\, \dot{I}_t}{1+\phi'(I_t)} = \delta + r - \frac{f'(k_t)}{1+\phi'(I_t)} \quad \Rightarrow \dot{I}_t = \frac{(\delta + r)(1+\phi'(I_t)) - f'(k_t)}{\phi''(I_t)}$$

1. Necessary conditions for the firm's optimal investment plan.

$$\mu_t = 1 + \phi'(I_t)$$

$$f'(k_t) - \delta \mu_t = -\dot{\mu}_t + r\mu_t$$

$$\lim_{t \to \infty} e^{-rt} \mu_t k_t = 0$$

2. Differential equation for I_t: $\quad \dot{I}_t = \dfrac{(\delta + r)(1+\phi'(I_t)) - f'(k_t)}{\phi''(I_t)}$

3. Phase diagram in (k_t, I_t) space.

① Two differential equations

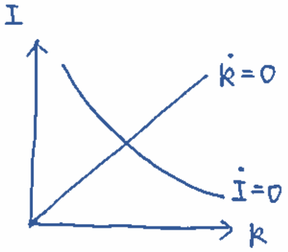
$$\dot{k}_t = I_t - \delta k_t$$

$$\dot{I}_t = \frac{(\delta + r)(1+\phi'(I_t)) - f'(k_t)}{\phi''(I_t)}$$

② Think about steady state and plot $\dot{k}=0$, $\dot{I}=0$ curve separately.

$\dot{k}=0 \Rightarrow I^* = \delta k^* \Rightarrow I^*$ is a function of k^*

$\dot{I}=0 \Rightarrow (\delta+r)(1+\phi'(I^*)) = f'(k^*) \Rightarrow$ Recall that $f''(\cdot)<0$ by Inada condition, $\phi''(\cdot)>0$ by assumption.



Larger $k^* \rightarrow$ smaller $f'(k^*) \rightarrow$ smaller $\phi'(I^*)$ $\rightarrow$ smallar I^*.

Hence I^* is decreasing in k^*, implying that $\dot{I}=0$ is a downward sloping curve.

③ Decide the movement of I and k

- Movement of I: use $\dot{I}$ equation and change the value of k.

$$\dot{I} = \frac{(\delta+r)(1+\phi'(I)) - f'(k)}{\phi''(I)}$$

For each k on $\dot{I}=0$ curve, $\dot{I} = \dfrac{(\delta+r)(1+\phi'(I)) - f'(\bar{k})}{\phi''(I)} = 0$

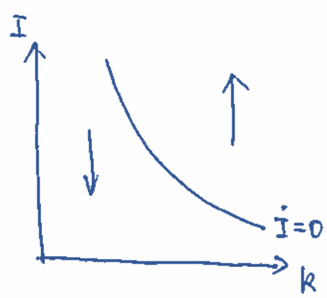
given each value of I, $\bar{k}$ is the value of k which makes $\dot{I}=0$

For each k on the right hand side of $\dot{I}=0$ curve, $k>\bar{k} \Rightarrow (\delta+r)(1+\phi'(I)) - f'(k) > (\delta+r)(1+\phi'(I)) - f'(\bar{k})$

$\Rightarrow \dot{I}>0$. hence I increases

For each k on the left hand side of $\dot{I}=0$ curve, $k<\bar{k} \Rightarrow (\delta+r)(1+\phi'(I)) - f'(k) < (\delta+r)(1+\phi'(I)) - f'(\bar{k})$

$\Rightarrow \dot{I}<0$, hence I decreases



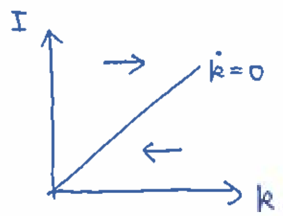
- Movement of k: use $\dot{k}=0$ equation and change the value of I

$$\dot{k} = I - \delta k$$

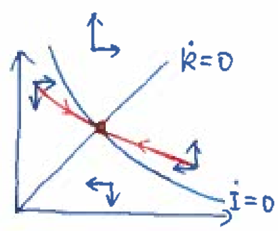
For each I on $\dot{k}=0$ curve, $\dot{k} = \bar{I} - \delta k = 0$ Given each value of k, $\bar{I}$ is the value of I which make $\dot{k}=0$

For each I above $\dot{k}=0$ curve, $I > \bar{I} \Rightarrow \dot{k} = I - \delta k > \bar{I} - \delta k \Rightarrow \dot{k} > 0$. hence k increases

For each I below $\dot{k}=0$ curve, $I < \bar{I} \Rightarrow \dot{k} = I - \delta k < \bar{I} - \delta k \Rightarrow \dot{k} < 0$. hence k decreases



④. Combine them together.



4. Discuss the stability properties of the steady state.

This question wants us to talk about saddle path. The answer key talked about saddle path in an analytical way, as Professor Hendricks' lecture notes. Please check the answer key, and make sure that you understand it. Here, I provide another way to check saddle path (solve it numerically).

$$\dot{k}_t = I_t - \delta k_t$$

$$\dot{I}_t = \frac{(r+\delta)(1+\phi'(I_t)) - f'(k_t)}{\phi''(I_t)}$$

First-order Taylor expansion around steady state (k^*, I^*)

$$\dot{k} = -\delta(k-k^*) + (I-I^*)$$

$$\dot{I} = -\frac{f''(k^*)}{\phi''(I^*)}(k-k^*) + \frac{(r+\delta)\phi''(I^*)\cancel{\phi''(I^*)} - \overbrace{[(r+\delta)(1+\phi'(I^*)) - f'(k^*)]}^{=0 \ \because (k^*, I^*) \text{ is steady state, } \dot{I}=0}\phi'''(I^*)}{\phi''(I^*)\cancel{\phi''(I^*)}}(I-I^*)$$

$$= -\frac{f''(k^*)}{\phi''(I^*)}(k-k^*) + (r+\delta)(I-I^*)$$

Write these two equations in matrix form

$$\dot{k} = -\delta(k-k^*) + (I-I^*)$$

$$\dot{I} = -\frac{f''(k^*)}{\phi''(I^*)}(k-k^*) + (r+\delta)(I-I^*)$$

$$\Rightarrow \begin{pmatrix} \dot{k} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} -\delta & 1 \\ -\frac{f''(k^*)}{\phi''(I^*)} & r+\delta \end{pmatrix} \begin{pmatrix} k-k^* \\ I-I^* \end{pmatrix}$$

Solve for eigenvalue

$$\det \begin{bmatrix} -\delta-\lambda & 1 \\ -\frac{f''(k^*)}{\phi''(I^*)} & r+\delta-\lambda \end{bmatrix} = 0$$

$$(-\delta-\lambda)(r+\delta-\lambda) - \left(-\frac{f''(k^*)}{\phi''(I^*)}\right)\cdot 1 = 0$$

$$\lambda^2 - r\lambda + \frac{f''(k^*)}{\phi''(I^*)} - \delta r - \delta^2 = 0$$

$$\therefore \lambda = \frac{r \pm \sqrt{r^2 - 4\left(\frac{f''(k^*)}{\phi''(I^*)} - \delta r - \delta^2\right)}}{2} = \frac{r \pm \sqrt{r^2 - 4\frac{f''(k^*)}{\phi''(I^*)} + 4(r\delta+\delta^2)}}{2}$$

$$\because f''(\cdot) < 0, \quad \phi''(\cdot) > 0 \qquad \therefore -4\frac{f''(k^*)}{\phi''(I^*)} > 0$$

$$\therefore r^2 \underbrace{- 4\frac{f''(k^*)}{\phi''(I^*)}}_{>0} + \underbrace{4(r\delta+\delta^2)}_{>0} > r^2$$

$$\therefore \sqrt{r^2 - 4\frac{f''(k^*)}{\phi''(I^*)} + 4(r\delta+\delta^2)} > r$$

$$\therefore \lambda_1 = \frac{r + \sqrt{r^2 - 4\frac{f''(k^*)}{\phi''(I^*)} + 4(r\delta+\delta^2)}}{2} > 0$$

$$\lambda_2 = \frac{r - \sqrt{r^2 - 4\frac{f''(k^*)}{\phi''(I^*)} + 4(r\delta+\delta^2)}}{2} < 0$$

We have two eigenvalues.
One is positive, one is negative

$\Downarrow$

Saddle Path !