Econ.720 Recitation#7

Discount Factor

- Discrete time: discount factor β, ($\beta = \frac{1}{1+\rho}$)
- Continuous time: discount rate ρ

In discrete time, $u(t) = \beta u(t+1) = \frac{1}{1+\rho} u(t+1)$. Hence, $\rho = \frac{u(t+1)-u(t)}{u(t)}$

In continuous time, the above equation becomes $\rho = \frac{\frac{d}{dt}u(t)}{u(t)} = \frac{d}{dt} ln u(t)$

Integrating both sides

$$\int_t^{t+\Delta} \rho ds = \int_t^{t+\Delta} \frac{d}{dt} ln u(s) ds$$

$$\rho\Delta = ln u(t+\Delta) - ln u(t) = ln \frac{u(t+\Delta)}{u(t)}$$

$$e^{\rho\Delta} = \frac{u(t+\Delta)}{u(t)} \Rightarrow u(t) = e^{-\rho\Delta} u(t+\Delta)$$

Optimal Control

1. Objective function

$$max \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

2. State variables: cannot jump (sluggish variable)

   Control variables: can jump (jumpable variable)

   For each state variable, there should be a law of motion for that state variable.

$$\dot{k}_t = w_t + (r_t - \delta) k_t - c_t$$

3. Solve it

Hamiltonian: $H = e^{-\rho t} u(c_t) + \hat{\mu}_t (w_t + (r_t - \delta) k_t - c_t)$	**Current Value Hamiltonian**: $H = u(c_t) + \mu_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0. FOC: $e^{-\rho t} u'(c_t) - \hat{\mu}_t = 0$	Differentiate H w.r.t control and set it to 0. FOC: $u'(c_t) - \mu_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\hat{\mu}}_t$	Differentiate H w.r.t state and set it to $-\dot{\mu}_t + \rho \mu_t$
Combine these two equations to substitute out $\hat{\mu}_t$	Combine these two equations to substitute out μ_t

4. Define the solution to H.H. problem

   The solution to H.H. problem is a set of $\{c_t, k_t, \mu_t\}$ that satisfy

   - FOC
   - Law of motion for the state (/other constraints)
   - Boundary conditions

     k_0 is given

     TVC

Example

Consider a version of the standard one sector, neoclassical growth model with no technological progress, inelastic labor supply and zero population growth. We will examine the planner's problem in an economy in which capital is costly to adjust.

The planner maximizes the lifetime utility of the representative household:

$$max \int_0^\infty e^{-\rho t} u(c_t) dt$$

There is one final good which is produced using labor and capital using a production technology which can be written in intensive form as $f(k_t)$, where $f'(\cdot) > 0$, $f''(\cdot) < 0$, and the usual Inada conditions apply.

Capital is costly to adjust however, so the resource constraint is given by

$$f(k_t) = c_t + i_t(1 + T(\frac{i_t}{k_t}))$$

where i_t is investment, and $T(\cdot)$ represents additional costs incurred whenever capital is adjusted. We assume that $T(0) = 0$ and $T'(\cdot) > 0$. Capital depreciates at rate δ, implying that

$$\dot{k}_t = i_t - \delta k_t$$

The initial capital stock, k_0, is given.

Solve the social planner's problem