Econ720 - TA Session 3

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0. Summary

- OLG model setup and basic assumptions (8/26)
- Solve OLG and define GE, SS (8/26, 9/2)
  - Can't rule out dynamic inefficiency in OLG (9/2)
  - What does efficiency look like? $\rightarrow$ social planner problem (9/7)
- After comparing the two, we find that we need transfer btw young and old to have efficiency. How?
  - An infinitely lived asset: gov bond (9/2)
  - Social security: fully-funded, pay-as-you-go (9/7)
  - Bequest (9/9)
  - Money (9/14)

1. Some takeaways from this week's lecture

- How to find Pareto efficient allocation?
  $\rightarrow$ Solve the social planner's problem.

- Social planner's problem:
  Maximizing some average of individual utilities, subject to **resource constraint**

2. Social security

- Fully-funded: $\tau^o_{t+1} = -(1 + r_{t+1})\tau^y_t$

  The young make contributions to the social security system, and their contributions are paid back to them in their old age.

  $\Rightarrow$ Essentially, the government just relabels some private savings as public. The total amount of capital stock in this economy doesn't change. So fully-funded social security will not improve dynamic inefficiency.

2. Social security

- Pay-as-you-go: $\tau_t^o = -(1+n)\tau_t^y$

  The government collects tax from the young at time t and distributes it directly to the current old. It's a pure transfer system. It may help (or may not) improve dynamic inefficiency.

3. Problem Set 1 - 1

1. OLG with Arrow-Debreu

1. Explain why the following is the correct budget constraint:
$$w_t + q_{t+1}s_{t+1} + (1-\delta)p_{t+1}s_{t+1} = p_t c_t^y + p_{t+1}c_{t+1}^o + p_t s_{t+1}$$

Answer key:

In date t market, rent out labor and earn w_t, buy capital s_{t+1} and consumption c_t^y at price p_t. In date $t+1$ market, earn rental income $q_{t+1}s_{t+1}$, sell the undepreciated capital, and buy c_{t+1}^o at price p_{t+1}.

3. Problem Set 1 - 1

2. Derive the household's FOCs.

Answer key:

$$\begin{aligned}\mathscr{L} &= u(c_t^y) + \beta u(c_{t+1}^o) \\ &\quad + \lambda_t (w_t + q_{t+1} s_{t+1} + (1-\delta) p_{t+1} s_{t+1} - p_t c_t^y - p_t s_{t+1} - p_{t+1} c_{t+1}^o)\end{aligned}$$

$$\begin{aligned}[c_t^y]: &\quad u'(c_t^y) = \lambda_t p_t \\ [c_{t+1}^o]: &\quad \beta u'(c_{t+1}^o) = \lambda_t p_{t+1} \\ [s_{t+1}]: &\quad p_t = q_{t+1} + (1-\delta) p_{t+1}\end{aligned}$$

3. Problem Set 1 - 1

3. Define a solution to the household problem.

Answer key:

$(c_t^y, c_{t+1}^o, s_{t+1})$ that satisfy

- Budget Constraint:

  $p_t c_t^y + p_t s_{t+1} + p_{t+1} c_{t+1}^o = w_t + q_{t+1} s_{t+1} + (1-\delta) p_{t+1} s_{t+1}$
- Euler Equation:

  $u'(c_t^y) = \frac{p_t}{p_{t+1}} \beta u'(c_{t+1}^o)$
- No-arbitrage Condition: $p_t = q_{t+1} + (1-\delta) p_{t+1}$

3. Problem Set 1 - 1

4. What is the real interest rate in this economy?

The relative price between consumption at t and $t+1$ is the **real interest rate**.

Answer key:

The real interest rate is $\frac{p_t}{p_{t+1}}$

- Households can move consumption between dates at the exchange rate $\frac{p_t}{p_{t+1}}$.
- It it derived from E.E.
  The relative price of two goods = The marginal rate of substitution between these two goods

3. Problem Set 1 - 1

5. Interpret the condition $p_t = q_{t+1} + (1-\delta)p_{t+1}$

Answer key:

It is the No-arbitrage condition.

It comes from the equation $\frac{p_t}{p_{t+1}} = \frac{q_{t+1}}{p_{t+1}} + (1-\delta)$.

- The household can move consumption between date t market and date $t+1$ market at exchange rate $\frac{p_t}{p_{t+1}}$, which is defined as the real interest rate.
- The household can also move consumption by buying capital and renting it out. $\frac{q_{t+1}}{p_{t+1}} + (1-\delta)$ is the return from investing in capital.

Both approaches must yield the same rate of return.

6. State the firm's FOCs. Watch your units!

Answer key:

$$\max \quad p_t F(K_t, L_t) - w_t L_t - q_t K_t$$
$$q_t = p_t F_1(K_t, L_t)$$
$$w_t = p_t F_2(K_t, L_t)$$

3. Problem Set 1 - 1

7. Define a competitive equilibrium.

Answer key:

Allocations $\{c_t^y, c_t^o, s_{t+1}, K_t, L_t\}$ and prices $\{p_t, w_t, q_t\}$ that satisfy

- Household: Euler Equation, No-arbitrage Condition, Budget Constraint
- Firm: F.O.C (2)
- Market Clearing Conditions:
  1.Goods market: $F(K_t, L_t) = N_t c_t^y + N_{t-1} c_t^o + K_{t+1} - (1-\delta)K_t$
  2.Capital market: $N_t s_{t+1} = K_{t+1}$
  3.Labor market: $L_t = N_t$

3. Problem Set 1 - 1

8. Why don't we lose an equation due to Walras' Law?

Answer key:

We only lose an equation for one t.

9. Where is numeraire?

Answer key:

We can make the price of consumption goods at any date t equal 1.

10. Define a steady state.

Answer key:

At steady state, all per-capita, real variables do not change along with time.

Steady state: $c^y, c^o, k, w/p, q/p, \pi$ that solve the equilibrium conditions without time subscripts. $\pi_{t+1} = p_{t+1}/p_t$

3. Problem Set 1 - 1

11. Under what condition do the Welfare theorems hold/fail? Recall that the Welfare theorems require $\lim_{t\to\infty} p_t = 0$.

Answer key:

- Welfare theorems require $\lim_{t\to\infty} p_t = 0$, indicating that price can not explode.
- From no-arbitrage condition $(\frac{p_t}{p_{t+1}} = \frac{q_{t+1}}{p_{t+1}} + 1 - \delta)$, no price explosion means $\frac{p_t}{p_{t+1}} > 1$, hence $\frac{q_{t+1}}{p_{t+1}} + 1 - \delta > 1$

$$\Rightarrow \frac{q_{t+1}}{p_{t+1}} > \delta \Rightarrow \lim_{t\to\infty} f'(k_{t+1}) > \delta$$

Idea:

- Find a equation that tells me the relationship btw p_t and p_{t+1}
- Equation that is related to k

3. Problem Set 1 - 2

2. OLG with Assets

Demographics: There are two types of households, indexed by h. In each period, a mass of 0.5 households is born of each type. Each person lives for 2 periods.

$\rightarrow$Heterogeneous agents even in same cohort

Endowments: Households receive endowments (e^y, e^o) when young and old, respectively.

Preferences: $ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1})$

$\rightarrow$Log utility, hence consumption is a constant fraction of total income.

Technology: None.

$\rightarrow$It's an endowment economy. Hence no capital accumulation.

3. Problem Set 1 - 2

Markets: Households trade goods and one period bonds that are issued and purchased by households

→Sequential trading setup, not Arrow-Debreu. Why?

What does the budget constraint look like?

- Sequential trading setup
- No capital accumulation
- Endowment economy with bonds

3. Problem Set 1 - 2

$$
\begin{aligned}
c^y_{h,t} + b_{h,t+1} &= e^y \\
c^o_{h,t+1} &= (1 + r_{t+1}) b_{h,t+1} + e^o
\end{aligned}
$$

Lifetime Budget Constraint:

$$
c^y_{h,t} + \frac{c^o_{h,t+1}}{1 + r_{t+1}} = e^y + \frac{e^o}{1 + r_{t+1}}
$$

3. Problem Set 1 - 2

1. Define a solution to the household problem. Solve for the household's bond supply function.

Answer key:

$$\max \; ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1})$$

$$s.t. \;\; c^y_{h,t} + b_{h,t+1} = e^y$$

$$c^o_{h,t+1} = (1 + r_{t+1})b_{h,t+1} + e^o$$

$$\mathscr{L} = ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1}) + \lambda_{h,t}(e^y + \frac{e^o}{1 + r_{t+1}} - c^y_{h,t} - \frac{c^o_{h,t+1}}{1 + r_{t+1}})$$

3. Problem Set 1 - 2

$$[c^y_{h,t}]: \ \frac{1}{c^y_{h,t}} = \lambda_{h,t}$$

$$[c^o_{h,t+1}]: \ \frac{\beta_h}{c^o_{h,t+1}} = \frac{\lambda_{h,t}}{1 + r_{t+1}}$$

$$\Rightarrow c^o_{h,t+1} = \beta_h(1 + r_{t+1})c^y_{h,t}$$

From budget constraints

$$c^y_{h,t} + \frac{c^o_{h,t+1}}{1 + r_{t+1}} = e^y + \frac{e^o}{1 + r_{t+1}}$$

$$c^y_{h,t} + \frac{\beta_h(1 + r_{t+1})c^y_{h,t}}{1 + r_{t+1}} = e^y + \frac{e^o}{1 + r_{t+1}}$$

$$c^y_{h,t} = \frac{1}{1 + \beta}(e^y + \frac{e^o}{1 + r_{t+1}})$$

Hence

$$
\begin{aligned}
b_{h,t+1} &= e^y - c^y_{h,t} \\
&= e^y - \frac{1}{1+\beta}(e^y + \frac{e^o}{1+r_{t+1}}) \\
&= \frac{\beta_h}{1+\beta_h} e^y - \frac{e^o}{(1+\beta_h)(1+r_{t+1})}
\end{aligned}
$$

Solution: given r_{t+1}, $(c^y_{h,t}, c^o_{h,t+1}, b_{h,t+1})$ that satisfy 1 E.E. and 2 BCs.

3. Problem Set 1 - 2

2. Solve for the equilibrium bond interest rate.

Answer key:

We stated the budget constraint as

$$c^y_{h,t} + b_{h,t+1} = e^y$$
$$c^o_{h,t+1} = (1 + r_{t+1})b_{h,t+1} + e^o$$

In equilibrium, it must be the case that $\sum_h b_{h,t+1} = 0$.

3. Problem Set 1 - 2

$$\sum b_{h,t+1} = 0$$

$$\sum\left(\frac{\beta_h}{1+\beta_h}e^y - \frac{e^o}{(1+\beta_h)(1+r_{t+1})}\right) = 0$$

$$e^y \sum \frac{\beta_h}{1+\beta_h} = \frac{e^o}{1+r_{t+1}} \sum \frac{1}{1+\beta_h}$$

$$1 + r_{t+1} = \frac{e^o}{e^y} \frac{\sum \frac{1}{1+\beta_h}}{\sum \frac{\beta_h}{1+\beta_h}}$$

3. Problem Set 1 - 2

3. Provide intuitions for the features of equilibrium bond interest rate.

Answer key:

(1). If old endowments are larger, r is higher.

If old endowments are larger, households have less incentive to buy bonds when they are young. Hence bond supply exceeds demand. As a result, the return of bond (r_{t+1}) increases, in order to attract more people to buy bonds and to clear the bond market.

3. Problem Set 1 - 2

(2). If β_h increases, r decreases.

When β_h increases, households become more patient, which also means they value their old consumption more. Hence, households will reduce their young consumption and buy more bonds in order to consume more when they are old. Bond demand exceeds bond supply, leading to a decrease in bond return. r_{t+1} decreases.

(3). Because there is no intergenerational trade, and the endowment for each cohort and the parameters don't vary overtime, r is time invariant.

4. Now add a durable good to the economy. It is in fixed supply, K. It pays a dividend d per period (in units of consumption goods). Households trade shares of this good in an asset market at price p_t, measured in units of consumption goods. Define a competitive equilibrium for this economy.

Answer key:

$$\begin{aligned} & max \;\; ln(c^{y}_{h,t}) + \beta_h ln(c^{o}_{h,t+1}) \\ & s.t. \;\; c^{y}_{h,t} + b_{h,t+1} + p_t k_{h,t+1} = e^{y} \\ & \qquad c^{o}_{h,t+1} = (1 + r_{t+1}) b_{h,t+1} + d k_{h,t+1} + p_{t+1} k_{h,t+1} + e^{o} \end{aligned}$$

3. Problem Set 1 - 2

$$\mathscr{L} = ln(c^y_{h,t}) + \beta_h ln(c^o_{h,t+1})$$

$$+\lambda_{h,t}(e^y + \frac{e^o}{1+r_{t+1}} - c^y_{h,t} - \frac{c^o_{h,t+1}}{1+r_{t+1}} - (p_t - \frac{p_{t+1}+d}{1+r_{t+1}})k_{h,t+1})$$

E.E. $\frac{c^o_{h,t+1}}{c^y_{h,t}} = \beta_h(1+r_{t+1})$

No-arbitrage condition: $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$

3. Problem Set 1 - 2

Competitive equilibrium:

Allocations $\{c^y_{h,t}, c^o_{h,t}, b_{h,t+1}, k_{h,t+1}\}$ and prices $\{r_t, p_t\}$ that satisfy

- Household problem: E.E. (2), BC (4);
- Market clearing condition:
  Goods market: $e^y + e^o + dK = 0.5\sum c^y_{h,t} + 0.5\sum c^o_{h,t}$
  Bonds market: $\sum b_{h,t+1} = 0$
  Durable Good Market: $0.5\sum k_{h,t+1} = K$
- No-arbitrage Condition: $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$

10 objects, 10 equations

5. Why do you find that the number of equations equals the number of objects to be determined? Usually, we find that we have one additional equation, which is redundant by Walras' law.

Answer key:

Because due to No-arbitrage condition, the real return of bonds is equal to the real return of durable goods, hence households do not need to specify the particular amount of $b_{h,t+1}$ and $k_{h,t+1}$, and can put them into one portfolio.

3. Problem Set 1 - 2

6. Derive an equation that determines the equilibrium price sequence p_t.

Logic

- Equation for price sequence
  $\rightarrow$ Equation that shows the relationship between today's p and tomorrow's p: $F(p_t, p_{t+1})$
- Which equilibrium condition contains both p_t and p_{t+1}
  $\rightarrow$ No-arbitrage condition
- Which condition can be further developed using no-arbitrage condition?
  $\rightarrow$ Lifetime budget constraint
- Two endogenous variables $c^y_{h,t}$ and $c^o_{h,t+1}$, how to get rid of one?
  $\rightarrow$ E.E.
- Equation for $c^y_{h,t}$
  $\rightarrow$ Which condition contains $c^y_{h,t}$
- How to take $b_{h,t+1}$ away from young HH's BC?
- How to take $k_{h,t+1}$ away from young HH's BC?

3. Problem Set 1 - 2

Answer key:

In equilibrium, $1 + r_{t+1} = \frac{p_{t+1}+d}{p_t}$. Hence the life-time budget constraint of household

$$e^y + \frac{e^o}{1 + r_{t+1}} = c^y_{h,t} + \frac{c^o_{h,t+1}}{1 + r_{t+1}} + (p_t - \frac{p_{t+1} + d}{1 + r_{t+1}})k_{h,t+1}$$

becomes

$$e^y + \frac{e^o}{1 + r_{t+1}} = c^y_{h,t} + \frac{c^o_{h,t+1}}{1 + r_{t+1}}$$

3. Problem Set 1 - 2

Substituting Euler Equation $\frac{c^o_{h,t+1}}{c^y_{h,t}} = \beta_h(1 + r_{t+1})$ into life-time budget constraint,

$$c^y_{h,t} = \frac{e^y + \frac{e^o}{1+r_{t+1}}}{1+\beta_h} = \frac{e^y + \frac{e^o p_t}{p_{t+1}+d}}{1+\beta_h}$$

From the durable goods market clearing condition

$$\begin{aligned}
2K &= \sum k_{h,t+1} \\
2p_t K &= \sum p_t k_{h,t+1} \\
&= \sum p_t k_{h,t+1} + \sum b_{h,t+1} \\
&= 2e^y - \sum c^y_{h,t} \\
&= 2e^y - \sum \frac{e^y + \frac{e^o p_t}{p_{t+1}+d}}{1+\beta_h}
\end{aligned}$$