An Investment Problem.

$$\max \int_0^\infty e^{-rt} \pi_t \, dt$$

$$\text{s.t.} \quad \dot{k}_t = I_t - \delta k_t$$

$$\Rightarrow \quad \max \int_0^\infty e^{-rt} f(k_t) - I_t - \phi(I_t)$$

$$\text{s.t.} \quad \dot{k}_t = I_t - \delta k_t$$

Control : I_t

State : k_t

- Current Value Hamiltonian :

co-state

H = period objective + μ_t (Law of motion for state)

$$H = f(k_t) - I_t - \phi(I_t) + \mu_t \left(I_t - \delta k_t \right)$$

- Necessary Conditions :

$$\frac{\partial H}{\partial \text{ control}} = 0 \qquad\qquad -1 - \phi'(I_t) + \mu_t = 0$$

$$\frac{\partial H}{\partial \text{ state}} = -\dot{\mu}_t + r\mu_t \qquad\qquad f'(k_t) - \mu_t \delta = -\dot{\mu}_t + r\mu_t$$

- TVC: $\lim_{t\to\infty} e^{-rt}\mu_t k_t = 0$

- Phase Diagram.

Step 1: Get differential equation for control and state.

$$\mu_t = 1+\phi'(I_t) \longrightarrow \dot{\mu}_t = \phi''(I_t)\,\dot{I}_t$$

$$f'(k_t) - \mu_t\delta = -\dot{\mu}_t + r\mu_t$$

$$\therefore f'(k_t) - [1+\phi'(I_t)]\,\delta = -\phi''(I_t)\,\dot{I}_t + r[1+\phi'(I_t)]$$

$$\therefore \dot{I}_t = \frac{(r+\delta)[1+\phi'(I_t)] - f'(k_t)}{\phi''(I_t)}$$

$$\dot{k}_t = I_t - \delta k_t$$

Step 2: Steady state

$$\dot{I}_t = 0 \Rightarrow (r+\delta)[1+\phi'(I)] - f'(k) = 0$$

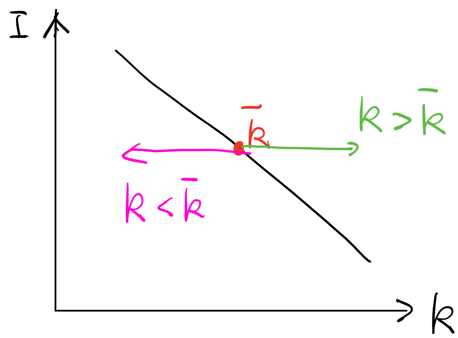
$$\dot{k}_t = 0 \Rightarrow I - \delta k = 0$$

Step 3: Plot the two SS equations separately.

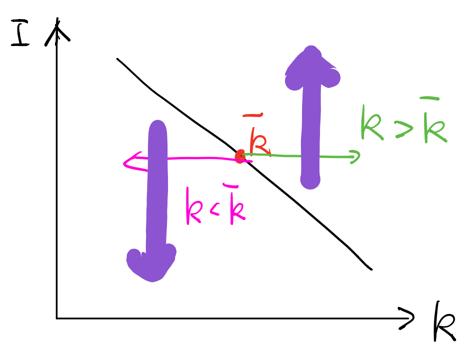
[control: y-axis ; state: x-axis]

Step 4: Decide the movement

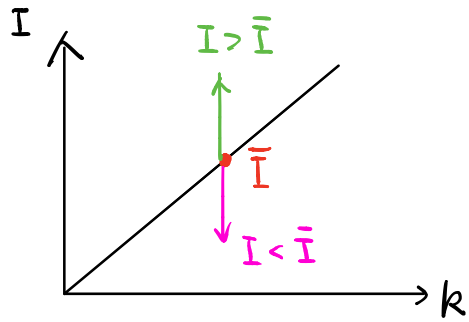
$$\dot{I}_t = 0 \; : \; (r+\delta)[1+\phi'(I)] - f'(k) = 0$$



$$\dot{I}_t = \begin{cases} (r+\delta)[1+\phi'(I)] - f'(\bar{k}) = 0 \\ (r+\delta)[1+\phi'(I)] - f'(k) > 0 \\ (r+\delta)[1+\phi'(I)] - f'(k) < 0 \end{cases}$$



$\dot{k}_t = 0 \quad : \qquad I - \delta k = 0$



$$\dot{k}_t = \begin{cases} \bar{I} - \delta k = 0 \\ I - \delta k > 0 \\ I - \delta k < 0 \end{cases}$$