Econ720 - TA Session 5

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1. Problem Set 2 - 1

1. Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period, $N_t = (1+n)^t$ persons are born. Each lives for 2 periods. Half of the agents are of type 1, the other half of type 2.

Endowments: The initial old hold M_0 units of money, evenly distributed across agents. Each person is endowed with (e_i^y, e_i^o) units of consumption when young and old, respectively.

Preferences: $ln(c_t^y) + \beta ln(c_{t+1}^o)$

Technology: Goods can only be eaten the day they drop from the sky.
$\rightarrow$Save by holding money

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Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_t = M_{t-1} + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu)M_t$.

How many choices do government make?

Markets: In each period, agents buy/sell goods and money in spot markets.

$\rightarrow$What comes into HH BC?

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Question

1. Define a competitive equilibrium.

Answer key:

- How many sectors? $\rightarrow$ H.H. and Gov.

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$$
\begin{aligned}
& max \;\; ln(c^{y}_{i,t}) + \beta\, ln(c^{o}_{i,t+1}) \\
& s.t. \;\; p_t c^{y}_{i,t} + p_t m^{d}_{i,t} = p_t e^{y}_{i} \\
& \qquad p_{t+1} c^{o}_{i,t+1} = p_{t+1} e^{o}_{i} + p_t m^{d}_{i,t} + p_{t+1} x_{t+1} \\
& \mathscr{L} = ln(c^{y}_{i,t}) + \beta\, ln(c^{o}_{i,t+1}) \\
& \qquad + \lambda_{it}\{e^{y}_{i} + \frac{p_{t+1}}{p_t}(e^{o}_{i} + x_{t+1}) - c^{y}_{i,t} - \frac{p_{t+1}}{p_t} c^{o}_{i,t+1}\}
\end{aligned}
$$

Notice that x_{t+1} is an **exogenous** variable for household!

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$$[c^y_{i,t}]: \ \frac{1}{c^y_{i,t}} = \lambda_{it}$$

$$[c^o_{i,t+1}]: \ \beta \frac{1}{c^o_{i,t+1}} = \lambda_{it} \frac{p_{t+1}}{p_t}$$

$$\Rightarrow \frac{1}{c^y_{i,t}} = \beta \frac{1}{c^o_{i,t+1}} \frac{p_t}{p_{t+1}}$$

1. Problem Set 2 - 1

Competitive equilibrium:

Allocations $\{c_{i,t}^y, c_{i,t}^o, m_{i,t+1}^d, x_t, M_t\}$ and prices $\{p_t\}$ that satisfy

- Household: 1 Euler Equation, 2 Budget Constraint
- Government: $M_t = M_{t-1} + N_{t-1}x_t p_t$, $M_{t+1} = (1+\mu)M_t$
- Market Clearing Conditions:

  1.Goods market: $\frac{N_t}{2}c_{1,t}^y + \frac{N_t}{2}c_{2,t}^y + \frac{N_{t-1}}{2}c_{1,t}^o + \frac{N_{t-1}}{2}c_{2,t}^o = \frac{N_t}{2}e_1^y + \frac{N_t}{2}e_2^y + \frac{N_{t-1}}{2}e_1^o + \frac{N_{t-1}}{2}e_2^o$

  2.Money market: $\frac{N_t}{2}m_{1,t}^d p_t + \frac{N_t}{2}m_{2,t}^d p_t = M_t$

2. Derive the household consumption function.
   Tip: log-utility $\rightarrow$ consumption is a constant fraction of income.

Answer key:

From the lifetime budget constraint:

$$c_{i,t}^{y} + \frac{p_{t+1}}{p_t} c_{i,t+1}^{o} = e_i^{y} + \frac{p_{t+1}}{p_t}(e_i^{o} + x_{t+1})$$

Substitute E.E.

$$c_{i,t}^{y} + \frac{p_{t+1}}{p_t} \beta c_{i,t}^{y} \frac{p_t}{p_{t+1}} = e_i^{y} + \frac{p_{t+1}}{p_t}(e_i^{o} + x_{t+1})$$

Hence

$$c_{i,t}^{y} = \frac{1}{1+\beta}\left(e_i^{y} + \frac{e_i^{o}}{R_{t+1}} + \frac{x_{t+1}}{R_{t+1}}\right), \text{ where } R_{t+1} = \frac{p_t}{p_{t+1}}$$

$$c_{i,t+1}^{o} = \beta c_{i,t}^{y} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e_i^{y} + e_i^{o} + x_{t+1})$$

3. Derive a difference equation for the equilibrium interest rate when $\mu = 0$.

Logic: $\mu = 0 \rightarrow$ gov. doesn't add money to the economy $\rightarrow x_t = 0$

Answer key:

When $x_t = 0$

$$c^y_{i,t} = \frac{1}{1+\beta}(e^y_i + \frac{e^o_i}{R_{t+1}})$$

$$c^o_{i,t+1} = \beta c^y_{i,t} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e^y_i + e^o_i)$$

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From goods market clearing condition:

$$\frac{N_t}{2}(c_{1,t}^y + c_{2,t}^y) + \frac{N_{t-1}}{2}(c_{1,t}^o + c_{2,t}^o) = \frac{N_t}{2}(e_1^y + e_2^y) + \frac{N_{t-1}}{2}(e_1^o + e_2^o)$$

Rearrange this equation by using $N_t = (1+n)^t$, $N_{t-1} = (1+n)^{t-1}$

$$\begin{aligned} e_1^y + e_2^y + \frac{1}{n+1}(e_1^o + e_2^o) = & \quad c_{1,t}^y + c_{2,t}^y + \frac{1}{n+1}(c_{1,t}^o + c_{2,t}^o) \\ = & \quad \frac{1}{1+\beta}(e_1^y + \frac{e_1^o}{R_{t+1}} + e_2^y + \frac{e_2^o}{R_{t+1}}) \\ + & \quad \frac{1}{n+1}\frac{\beta}{1+\beta}(R_t e_1^y + e_1^o + R_t e_2^y + e_2^o) \end{aligned}$$

Difference equation of R: $\beta(1+n-R_t)(e_1^y + e_2^y) = \frac{1+n-R_{t+1}}{R_{t+1}}(e_1^o + e_2^o)$

1. Problem Set 2 - 1

4. Is the monetary steady state dynamically efficient?

Answer key:

In steady state, $m_{t+1} = m_t = \bar{m}$, where $m_t = \frac{M_t}{p_t N_t}$

Hence,

$$\frac{M_{t+1}}{p_{t+1} N_{t+1}} = \frac{M_t}{p_t N_t} \Rightarrow \frac{M_{t+1}}{M_t} = \frac{p_{t+1}}{p_t} \frac{N_{t+1}}{N_t} \Rightarrow 1 + \mu = \frac{1+n}{R_{t+1}}$$

1. Problem Set 2 - 1

In steady state,

$$R = \frac{1+n}{1+\mu}$$

- If $\mu > 0$, this monetary steady state is not dynamically efficient.
- If $\mu = 0$, this monetary steady state is dynamically efficient.

2. Problem Set 2 - 2

1. Money and Heterogeneity

Consider a two-period OLG model with fiat money and a storage technology.

Demographics: In each period, $N_t = (1+n)^t$ persons are born. Each lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M_0. Each young person is endowed with e units of consumption.

Preferences: $u(c_t^y) + \beta u(c_{t+1}^o)$

Technology: Storing k_t units of the good in t yields $f(k_t)$ units in $t+1$. f obeys Inada conditions. The resource constraint is $N_t K_{t+1} = N_t e + N_{t-1} f(k_t) - C_t$ where $C_t = N_t c_t^y + N_{t-1} c_t^o$

$\rightarrow$Save by holding money and storing capital.

2. Problem Set 2 - 2

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_{t+1} = M_t + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu)M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

2. Problem Set 2 - 2

Timing in period t:

- The old enter period t holding aggregate capital $K_t = N_{t-1}k_t$ and nominal money balances of $M_t = m_t N_{t-1}$.
- Gov pays a lump-sum transfer of $x_t p_t$ units of money to each old person.
- Each old person produces $f(k_t)$.
- The young buy money $\frac{m_{t+1}}{p_t}$ from the old, consume c_t^y and save k_{t+1}.
- The old consume c_t^o.

Question

1. State the HH's BC when young and old.

Answer key:

- Young:
- Old:

2. Problem Set 2 - 2

Question

1. State the HH's BC when young and old.

Answer key:

- Young: $p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e$
- Old: $p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1}$

2. Problem Set 2 - 2

2. Derive the HH's optimality conditions. Define a solution to the HH problem.

Answer key:

$$max\ \ u(c_t^y) + \beta u(c_{t+1}^o)$$

$$\begin{aligned} s.t.\ \ & p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e \\ & p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1} \end{aligned}$$

$$\begin{aligned} \mathscr{L} = {} & u(c_t^y) + \beta u(c_{t+1}^o) \\ & + \lambda_t \{ e + \frac{p_{t+1}}{p_t} (f(k_{t+1}) + x_{t+1}) - c_t^y - k_{t+1} - \frac{p_{t+1}}{p_t} c_{t+1}^o \} \end{aligned}$$

2. Problem Set 2 - 2

$$[c_t^y]: \; u'(c_t^y) = \lambda_t$$

$$[c_{t+1}^o]: \; \beta u'(c_{t+1}^o) = \lambda_t \frac{p_{t+1}}{p_t}$$

$$[k_{t+1}]: \; \frac{p_t}{p_{t+1}} = f'(k_{t+1})$$

Solution to HH problem is given price p_t, a vector $(c_t^y, c_{t+1}^o, m_{t+1}, k_{t+1})$ that satisfies

- 2 BCs
- EE: $u'(c_t^y) = \beta u'(c_{t+1}^o) \frac{p_t}{p_{t+1}}$
- FOC: $\frac{p_t}{p_{t+1}} = f'(k_{t+1})$

3. Define a competitive equilibrium

Answer key:

Competitive equilibrium:

Allocations $\{c_t^y, c_t^o, m_{t+1}, k_{t+1}, x_t, M_t\}$ and prices $\{p_t\}$ that solve

- Household: 1 Euler Equation, 2 Budget Constraint
- Government: $M_{t+1} = M_t + N_{t-1}x_t p_t$, $M_{t+1} = (1+\mu)M_t$
- Market Clearing Conditions:

  1.Money market: $N_{t-1}m_t = M_t$

  2.Goods market: $N_t c_t^y + N_{t-1}c_t^o + N_t k_{t+1} = N_t e + N_{t-1}f(k_t)$
- Accounting identity: $\frac{p_t}{p_{t+1}} = f'(k_{t+1})$

2. Problem Set 2 - 2

4. Does an equilibrium with positive inflation exist? Intuition

Answer key:

No. That implies rate of return dominance, and nobody would hold money.

2. Problem Set 2 - 2

5. Define a steady sate as a system of 6 equations with 6 unknowns

2. Problem Set 2 - 2

Answer key:

CE: $\{c_t^y, c_t^o, m_{t+1}, k_{t+1}, x_t, M_t\}$, $\{p_t\}$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}}$$
$$p_t c_t^y + p_t k_{t+1} + m_{t+1} = p_t e$$
$$p_{t+1} c_{t+1}^o = p_{t+1} f(k_{t+1}) + m_{t+1} + p_{t+1} x_{t+1}$$
$$M_{t+1} = M_t + N_{t-1} x_t p_t$$
$$M_{t+1} = (1+\mu) M_t$$
$$N_{t-1} m_t = M_t$$
$$N_t c_t^y + N_{t-1} c_t^o + N_t k_{t+1} = N_t e + N_{t-1} f(k_t)$$
$$\frac{p_t}{p_{t+1}} = f'(k_{t+1})$$

2. Problem Set 2 - 2

A steady state consists of constants $(c^y, c^o, m/p, k, x, \pi)$ that satisfy

- $\frac{1}{1+\pi} = \frac{1+n}{1+\mu} \rightarrow \pi$
- $f'(k) = \frac{1}{1+\pi} \rightarrow k$
- $u'(c^y) = \beta u'(c^o)\frac{1}{1+\pi}$ and
  $c^y + \frac{1}{1+n}c^o + k = e + \frac{1}{1+n}f(k) \rightarrow c^y$ and c^o
- $c^y + k + m/p(1+\pi) = e \rightarrow m/p$
- $x = \mu(m/p) \rightarrow x$

6. Find the money growth rate (μ) that maximizes steady state consumption per young person $\frac{N_t c^y + N_{t-1} c^o}{N_t}$

Answer key:

$$max\ c^y + \frac{1}{1+n} c^o \Leftrightarrow max\ e + \frac{1}{1+n} f(k) - k$$

Hence,

$$\frac{1}{1+n} f'(k^*) - 1 = 0$$

$$f'(k^*) = 1+n \rightarrow \frac{1}{1+\pi^*} = 1+n \rightarrow \frac{1+n}{1+\mu^*} = 1+n$$

$\mu^* = 0$