Econ720 - TA Session 12

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Stochastic Patent Duration

1. State the household problem and its solution

Answer key:

Standard household problem with EE

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$$

Notice!

~~Reduced form BC:~~ $\dot{a}_t = r_t a_t + w_t - c_t$

Problem Set 6

2. Solve the problem of the final goods sector

Answer key:

- Final goods sector: perfect competition
- Goods producers are price takers

Normalize final goods price to be 1

(Need to say this before stating the problem)

$$\max \; AL_t^{1-\alpha} \sum_{j=1}^{N_t} x_{jt}^{\alpha} - w_t L_t - \sum_{j=1}^{N_t} p_{jt} x_{jt}$$

$$[L_t]: \quad (1-\alpha) Y_t / L_t = w_t$$
$$[x_{jt}]: \quad \alpha A L_t^{1-\alpha} x_{jt}^{\alpha-1} = p_{jt}$$

Hence the demand function for intermediate goods is $x_{jt} = (\frac{\alpha A}{p_{jt}})^{\frac{1}{1-\alpha}} L_t$

Problem Set 6

3. Solve the problem of the intermediate input producer

Answer key:

There are two types of intermediate producers!!!

- Monopoly

$$\max \; p_{jt}^m x_{jt}^m - x_{jt}^m \quad \text{with } x_{jt} = \left(\frac{\alpha A}{p_{jt}}\right)^{\frac{1}{1-\alpha}} L_t$$

$$[p_{jt}^m]: \quad p_{jt}^m = 1/\alpha$$

$$x_{jt}^m = \left(\frac{\alpha A}{p_{jt}^m}\right)^{\frac{1}{1-\alpha}} L_t = (\alpha^2 A)^{\frac{1}{1-\alpha}} L_t$$

$$\pi_{jt}^m = \frac{1-\alpha}{\alpha}(\alpha^2 A)^{\frac{1}{1-\alpha}} L_t$$

- Perfect Competition

  PRICE TAKER! PRICE TAKER! PRICE TAKER!

  price=MC $\rightarrow$ plug price into demand function, $x_{jt}^c = (\alpha A)^{\frac{1}{1-\alpha}} L_t$

Problem Set 6

4. State the free entry condition for innovation

Answer key:

Cost=Benefit

- η units of final goods create a new type of intermediate good
- Benefit is V

$$V = \int_t^{\infty} e^{-\int_t^{v} r(s)ds} e^{-p(v-t)} \pi_v^m dv$$

Hence

$$\eta = \int_t^{\infty} e^{-\int_t^{v} r(s)ds} e^{-p(v-t)} \pi_v^m dv$$

Problem Set 6

5. Define an equilibrium

Answer key:

Allocations $\{c_t, L_t, x^m_{jt}, x^c_{jt}, Y_t, N_t, N^c_t\}$, and
Prices $\{w_t, r_t, p^c_{jt}, p^m_{jt}\}$, that satisfy

- HH: solution to HH problem
- Final goods: 2FOC + 1 production function
- Intermediates:
  - Monopoly: profit maximization $p^m_{jt} \to x^m_{jt}$
  - Competition: profit maximization x^m_{jt}
- R&D: free entry $\eta = \int_t^\infty e^{-r(\nu-t)} e^{-p(\nu-t)} \left(\frac{1-\alpha}{\alpha} (\alpha^2 A)^{\frac{1}{1-\alpha}} L \right) d\nu$
- Goods market: $Y_t = c_t + \sum_{j=1}^{N^c_t} x^c_{jt} + \sum_{j=N^c_t+1}^{N_t} x^m_{jt} + \eta \dot{N}_t$
- Labor market: $L_t = L$
- Asset market: omitted, since I did not write out the BC
- Intermediates market: implicit in notation

6. Derive the quantity of x_j produced when the jth producer is a monopolist

Answer key:

$$x_{jt}^{m} = (\alpha^2 A)^{\frac{1}{1-\alpha}} L_t$$

Problem Set 6

7. Derive the quantity of x_j produced when the jth intermediate good is produced competitively

Answer key:

$$x_{jt}^{c} = (\alpha A)^{\frac{1}{1-\alpha}} L_t$$

Problem Set 6

8. Using free entry and the definition of profits, show that

$$r = (L/\eta)A^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)} - p$$

Answer key:

From the free entry condition

$$\begin{aligned}
\eta &= \int_t^\infty e^{-r(v-t)}e^{-p(v-t)}\pi_v^m dv \\
&= \int_t^\infty e^{-r(v-t)}e^{-p(v-t)}\left(\frac{1-\alpha}{\alpha}(\alpha^2 A)^{\frac{1}{1-\alpha}} L\right) dv \\
&= \left(\frac{1-\alpha}{\alpha}(\alpha^2 A)^{\frac{1}{1-\alpha}} L\right)\int_t^\infty e^{-r(v-t)}e^{-p(v-t)} dv \\
&= \left(\frac{1-\alpha}{\alpha}(\alpha^2 A)^{\frac{1}{1-\alpha}} L\right)\frac{1}{p+r}
\end{aligned}$$

Notice that this is in fact BGP interest rate

Problem Set 6

9. Solve for a balanced growth values of $\dot{c}/c$, N^c/N, and Y/N

Answer key:

- $\dot{c}/c$: due to EE and BGP r

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\theta} = \frac{(L/\eta)A^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)} - p - \rho}{\theta}$$

Problem Set 6

- N^c/N:

Can't use $\dot{c}/c = \dot{Y}/Y = \dot{N}/N = \dot{N}^c/N^c$ directly without proof

From final goods production function

$$
\begin{aligned}
Y_t &= AL^{1-\alpha}\sum_{j=1}^{N_t} x_{jt}^{\alpha} \\
&= AL^{1-\alpha}[\sum_{j=1}^{N_t^c}(x_{jt}^c)^{\alpha} + \sum_{j=N_t^c+1}^{N_t}(x_{jt}^m)^{\alpha}] \\
&= AL^{1-\alpha}\left\{N_t^c[(\alpha A)^{\frac{1}{1-\alpha}}L]^{\alpha} + (N_t - N_t^c)[(\alpha^2 A)^{\frac{1}{1-\alpha}}L]^{\alpha}\right\} \\
&= (\alpha^{\alpha}A)^{\frac{1}{1-\alpha}}LN_t^c + (\alpha^{2\alpha}A)^{\frac{1}{1-\alpha}}L(N_t - N_t^c) \\
&= \underbrace{(\alpha^{2\alpha}A)^{\frac{1}{1-\alpha}}L}_{\Omega_1}N_t + \underbrace{[(\alpha^{\alpha}A)^{\frac{1}{1-\alpha}}L - (\alpha^{2\alpha}A)^{\frac{1}{1-\alpha}}L]}_{\Omega_2}N_t^c
\end{aligned}
$$

Problem Set 6

$$Y_t = \Omega_1 N_t + \Omega_2 N_t^c$$

$$\dot{Y}_t = \Omega_1 \dot{N}_t + \Omega_2 \dot{N}_t^c$$

$$\frac{\dot{Y}_t}{Y_t} = \Omega_1 \frac{\dot{N}_t}{Y_t} + \Omega_2 \frac{\dot{N}_t^c}{Y_t}$$

$$\frac{\dot{Y}_t}{Y_t} = \Omega_1 \frac{\dot{N}_t}{N_t} \frac{N_t}{Y_t} + \Omega_2 \frac{\dot{N}_t^c}{N_t^c} \frac{N_t^c}{Y_t}$$

$$g(Y) = \Omega_1 g(N) \frac{N_t}{Y_t} + \Omega_2 g(N^c) \frac{N_t^c}{Y_t}$$

$$g(Y) = \Omega_1 g(N) \frac{N_t}{Y_t} + \Omega_2 g(N^c) \frac{(Y_t - \Omega_1 N_t)/\Omega_2}{Y_t}$$

$$g(Y) = \Omega_1 \frac{N_t}{Y_t} \Big(g(N) - g(N^c) \Big) + g(N^c)$$

Problem Set 6

$$\frac{N_t}{Y_t} = \frac{g(Y) - g(N^c)}{\Omega_1 \Big(g(N) - g(N^c) \Big)}$$

$\rightarrow$ N_t and Y_t grow at the same rate on BGP

$\rightarrow$ N_t^c, N_t and Y_t grow at the same rate on BGP

$$\dot{Y}/Y = \dot{N}/N = \dot{N}^c/N^c$$

Following the same logic, you can show $\dot{Y}/Y = \dot{N}/N = \dot{N}^c/N^c = \dot{c}/c$ by using goods market clearing condition

Exercise

Final Exam 2016. Asset Pricing with Habit Formation

Demographics: A unit mass of infinitely lived, identical households

Preferences:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t, c_{t-1})$$

Endowments: At $t = 0$, the household owns one tree.

Technology: The tree produces a random dividend that follows $d_t = G_t d_{t-1}$ with $E(G) = \overline{G} > 0$ and $G \sim iid$.

Markets: There are competitive markets for goods (numeraire), trees (p_t), and one period discount bonds (price 1; return R_t)

Questions:

1. State the household's dynamic program.
2. Derive firs-order and envelope conditions.
3. Derive Lucas asset pricing equations.

Answer Key

1. State the household's dynamic program.

The HH problem is

$$\max \ \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$
$$s.t. \ c_t + p_t k_{t+1} + b_{t+1} = (p_t + d_t) k_t + R_t b_t$$

Define $z_t = c_{t-1}$
The states variables are: k_t, b_t, z_t, d_t
Hence the HH problem in DP language is

$$V(k, b, z, d) = \max \ u(c, z) + \beta \mathbb{E} V(k', b', z', d')$$
$$s.t. \ c + pk' + b' = (p + d)k + Rb$$
$$z' = c$$

Exercise

2. Derive firs-order and envelope conditions.

Bellman Equation

$$V(k,b,z,d) = \max \; u(c,z) + \beta \mathbb{E} V(k',b',c,d') \\ + \lambda\Big((p+d)k + Rb - c - pk' - b'\Big)$$

First-order conditions:

$$\begin{aligned} [c] :& \quad u_c(c,z) + \beta \mathbb{E} V_z(k',b',c,d') = \lambda \\ [k'] :& \quad \beta \mathbb{E} V_k(k',b',c,d') = \lambda p \\ [b'] :& \quad \beta \mathbb{E} V_b(k',b',c,d') = \lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} [k] :& \quad V_k(k,b,z,d) = \lambda(p+d) \\ [b] :& \quad V_b(k,b,z,d) = \lambda R \\ [z] :& \quad V_z(k,b,z,d) = u_z(c,z) \end{aligned}$$

Exercise

3. Derive Lucas asset pricing equations.

Combine the FOCs and Envelope conditions, we have

$$u_c(c,z) + \beta \mathbb{E} u_z(c',z') = \lambda$$
$$\beta \mathbb{E} \lambda' (p' + d') = \lambda p$$
$$\beta \mathbb{E} \lambda' R' = \lambda$$

The last two equations are the Lucas asset pricing equations