If $Y_t = C_t + I_t$, then on balanced growth path $g_C = g_I = g_Y$

Proof:

$$C_t + I_t = Y_t \Rightarrow \dot{C}_t + \dot{I}_t = \dot{Y}_t \Rightarrow \frac{\dot{C}_t}{Y_t} + \frac{\dot{I}_t}{Y_t} = \frac{\dot{Y}_t}{Y_t}$$

$$\therefore \frac{\dot{C}_t}{C_t}\frac{C_t}{Y_t} + \frac{\dot{I}_t}{I_t}\frac{I_t}{Y_t} = \frac{\dot{Y}_t}{Y_t} \quad (*)$$

$$\frac{\dot{C}_t}{C_t}\frac{C_t}{Y_t} + \frac{\dot{I}_t}{I_t}\frac{Y_t - C_t}{Y_t} = \frac{\dot{Y}_t}{Y_t}$$

$$\left(\frac{\dot{C}_t}{C_t} - \frac{\dot{I}_t}{I_t}\right)\frac{C_t}{Y_t} = \frac{\dot{Y}_t}{Y_t} - \frac{\dot{I}_t}{I_t}$$

On BGP, $\frac{\dot{C}_t}{C_t} = g_c$, $\frac{\dot{I}_t}{I_t} = g_I$, $\frac{\dot{Y}_t}{Y_t} = g_Y$

$\therefore (g_c - g_I)\frac{C_t}{Y_t} = g_Y - g_I$ on BGP

Case 1. If $g_c \neq g_I$

$g_c - g_I \neq 0$ $\quad \therefore \frac{C_t}{Y_t} = \frac{g_Y - g_I}{g_c - g_I}$ is some constant

$\therefore$ C and Y grow at the same rate on BGP. $g_c = g_Y$

Then equation $(*)$ becomes $g_c \frac{C_t}{Y_t} + g_I \frac{I_t}{Y_t} = g_Y$

$\because g_c, g_I, g_Y, \frac{C_t}{Y_t}$ are constant on BGP

$\therefore \frac{I_t}{Y_t}$ must be constant

$\therefore g_I = g_Y$ on BGP $\Rightarrow g_I = g_Y = g_c$ Contradiction!

Case 2. If $g_c = g_I$, then $g_c - g_I = 0$

$\therefore (g_c - g_I)\frac{C_t}{Y_t} = g_Y - g_I \Rightarrow 0 = g_Y - g_I \Rightarrow g_Y = g_I$

$\therefore g_c = g_I = g_Y$