Econ720 - TA Session 8

Yanran Guo

UNC-Chapel Hill

2021. 10. 15

1. Continuous Time vs. Discrete Time

Example:

- Objective: maximize lifetime utility
- BC:

  Income: labor, capital
  Expenditure: consumption, new capital

To solve the model:

- Discrete time:
  Sequential language $\rightarrow$ Lagrangean
  Dynamic programming $\rightarrow$ Bellman equation
- Continuous time:
  Optimal control $\rightarrow$ Hamiltonian (state variable, control variable)

1. Continuous Time vs. Discrete Time

- Discrete time

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.\ c_t + k_{t+1} = w_t + (1 - \delta + r_t)k_t$$

- Continuous time

$$\max \int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt$$

$$s.t.\ \dot{k}_t = w_t + (r_t - \delta)k_t - c_t$$

1. Continuous Time vs. Discrete Time

To discount:

- Discrete time: discount factor β, ($\beta = \frac{1}{1+\rho}$)
- Continuous time: discount rate ρ

In discrete time, $u(t) = \beta u(t+1) = \frac{1}{1+\rho} u(t+1)$. Hence, $\rho = \frac{u(t+1)-u(t)}{u(t)}$

In continuous time, the above equation becomes $\rho = \frac{\frac{d}{dt}u(t)}{u(t)} = \frac{d}{dt} ln u(t)$

Integrating both sides

$$\int_t^{t+\Delta} \rho ds = \int_t^{t+\Delta} \frac{d}{dt} ln u(s) ds$$

$$\rho \Delta = ln u(t+\Delta) - ln u(t) = ln \frac{u(t+\Delta)}{u(t)}$$

$$e^{\rho \Delta} = \frac{u(t+\Delta)}{u(t)} \Rightarrow u(t) = e^{-\rho \Delta} u(t+\Delta)$$

2. Optimal Control: Hamiltonian

1. Read the question carefully, figure out each sector
2. State variable, Control variable
   Only for sectors that have dynamic problem!
3. Write down the objective:
   integral, discount, objective you are trying to maximize
4. For each state var., write down a law of motion for that state var.
5. Hamiltonian or Current Value Hamiltonian
6. Differentiate H w.r.t control var. and set it to be 0
   Differentiate H w.r.t state var. and set it to be $-\dot{\mu}_t$ or $-\dot{\mu}_t + \rho\mu_t$
   Combine them to get rid of co-state variable, then get EE
7. Define the solution to this sector problem

Hamiltonian: $H = e^{-\rho t} u(c_t) + \hat{\mu}_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $e^{-\rho t} u'(c_t) - \hat{\mu}_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\hat{\mu}}_t$
Combine these two equations to substitute out $\hat{\mu}_t$

Current Value Hamiltonian: $H = u(c_t) + \mu_t (w_t + (r_t - \delta) k_t - c_t)$
Differentiate H w.r.t control and set it to 0 $u'(c_t) - \mu_t = 0$
Differentiate H w.r.t state and set it to $-\dot{\mu}_t + \rho \mu_t$
Combine these two equations to substitute out μ_t

2. Optimal Control: Hamiltonian

7. Define the solution:

   The solution to our example problem is a set of $\{c_t, k_t, \mu_t\}$ that satisfy

   - FOC
   - Law of motion for the state (/other constraints)
   - Boundary conditions

     k_0 is given

     TVC

More about TVC

- Finite time
  - With scrap value: $\mu(T) = \phi'(k_T)$
  - Without scrap value: $\mu(T) = 0$
- Infinite time
  - Hamiltonian: $\lim_{t\to\infty} \mu_t k_t = 0$
  - Current value Hamiltonian: $\lim_{t\to\infty} e^{-\rho t}\mu_t k_t = 0$

Ramsey Model

Social Planner Problem

$$max \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$
$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t$$

3. Phase Diagram

Current Value Hamiltonian:

$$H = \frac{c_t^{1-\sigma}}{1-\sigma} + \mu_t(f(k_t) - (n+\delta)k_t - c_t)$$

FOCs

$$\begin{aligned} [c_t]: &\quad c_t^{-\sigma} = \mu_t \\ [k_t]: &\quad \mu_t[f'(k_t) - (n+\delta)] = -\dot{\mu}_t + (\rho - n)\mu_t \end{aligned}$$

3. Phase Diagram

$$c_t^{-\sigma} = \mu_t \quad \rightarrow \quad -\sigma c_t^{-\sigma-1} \dot{c}_t = \dot{\mu}_t$$

Plug into

$$\mu_t[f'(k_t) - (n+\delta)] = -\dot{\mu}_t + (\rho - n)\mu_t$$

Then we have

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - (\delta + \rho)}{\sigma}$$

Define the solution

How to draw phase diagram?

1. Find the dynamics for control and state
   Differential equation for control and for state
   Tips:
   - Usually differential equation for control can be derived by substituting out co-state.
   - Differential equation for state variable is usually given.
2. Think about the steady state $\rightarrow$ "dot" equals zero
3. Plot the two steady state equations separately
4. Decide the movement of control and state respectively.

3. Phase Diagram

Step-1: Find differential equation for control and for state

Control variable c_t

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - (\delta + \rho)}{\sigma}$$

State variable k_t

$$\dot{k}_t = f(k_t) - (n + \delta)k_t - c_t$$

3. Phase Diagram

Step-2: How do these two equations look like in SS

$$\frac{\dot{c}_t}{c_t} = 0 \quad \Rightarrow \quad f'(k^*) = \delta + \rho$$

Hence k^* is a constant

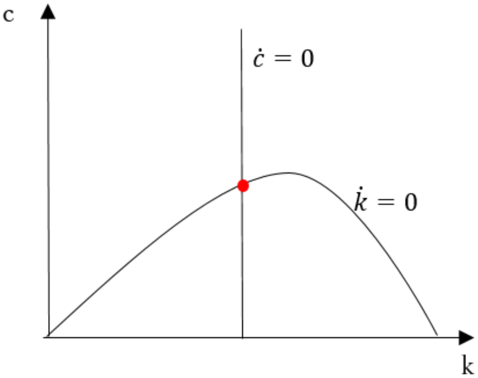
$$\dot{k}_t = 0 \quad \Rightarrow \quad c^* = f(k^*) - (n + \delta)k^*$$

Hence c^* is a function of k^*

3. Phase Diagram

Step-3: Plot the two SS equations

$$\dot{c} = 0: \quad f'(k^*) = \delta + \rho$$
$$\dot{k} = 0: \quad c^* = f(k^*) - (n+\delta)k^*$$



Step-4: Decide the movement of these two variables separately

- c is the vertical axis $\rightarrow$ c moves up and down
- k is the horizontal axis $\rightarrow$ k moves left and right

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - (\delta + \rho)}{\sigma}$$

$$\dot{k}_t = f(k_t) - (n + \delta)k_t - c_t$$

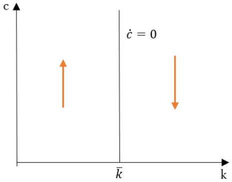
1. Use $\dot{c}$ equation and change the value of k to study the movement of c
2. Use $\dot{k}$ equation and change the value of c to study the movement of k
3. Put them together

3. Phase Diagram

Use $\dot{c}$ equation and change the value of k to study the movement of c

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - (\delta + \rho)}{\sigma}$$

- Take one k on $\dot{c} = 0$ curve, $k = \overline{k}$, $\frac{\dot{c}}{c} = \frac{f'(\overline{k}) - (\delta+\rho)}{\sigma} = 0$
- For each k to the left of $\overline{k}$, $k < \overline{k}$,
  $\frac{\dot{c}}{c} = \frac{f'(k) - (\delta+\rho)}{\sigma} > \frac{f'(\overline{k}) - (\delta+\rho)}{\sigma} = 0$ c increases
- For each k to the right of $\overline{k}$, $k > \overline{k}$,
  $\frac{\dot{c}}{c} = \frac{f'(k) - (\delta+\rho)}{\sigma} < \frac{f'(\overline{k}) - (\delta+\rho)}{\sigma} = 0$ c decreases

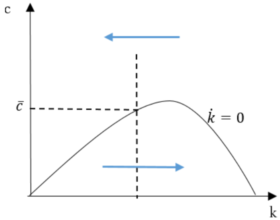


3. Phase Diagram

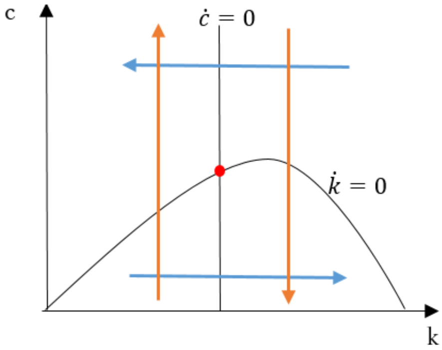
Use $\dot{k}$ equation and change the value of c to study the movement of k

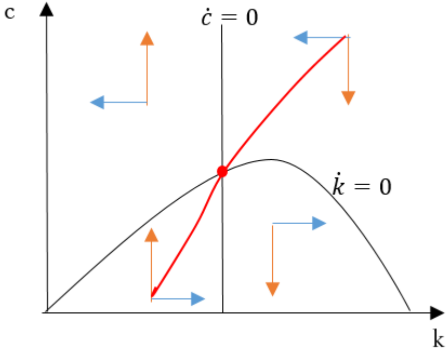
$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t$$

- Take one c on $\dot{k} = 0$ curve, $c = \bar{c}$, $\dot{k} = f(k) - (n+\delta)k - \bar{c} = 0$
- For each c above $\bar{c}$, $c > \bar{c}$,
  $\dot{k} = f(k) - (n+\delta)k - c < f(k) - (n+\delta)k - \bar{c}$ k decreases
- For each c under $\bar{c}$, $c < \bar{c}$,
  $\dot{k} = f(k) - (n+\delta)k - c > f(k) - (n+\delta)k - \bar{c}$ k increases



Put them together





Saddle path stable

4. Detrending

- Balanced growth path: real variables keep growing at constant growth rates
- Want SS
- Detrend

Fall 2017 Final

Question-1. Continuous time growth model

5. Example

An Investment Problem (Macro Quality, Jan 2012)

Consider the problem of an infinitely lived firm that invests in capital K_t subjects to an adjustment cost.

Time is continuous. The profit stream is given by

$$\pi_t = f(k_t) - I_t - \phi(I_t)$$

where f obeys Inada conditions and the adjustment cost is convex: $\phi' > 0$ and $\phi'' > 0$. $\phi(0) = 0$.

The firm maximizes the discounted present value of profits

$$\max_{I_t, K_t; t \geq 0} \int_0^\infty e^{-rt} \pi_t dt$$

subject to the law of motion

$$\dot{k}_t = I_t - \delta k_t$$

5. Example

Questions:

1. Derive the necessary conditions for the firm's optimal investment plan, including the TVC.
2. From the necessary conditions, derive the differential equation for I_t.
3. Draw a phase diagram in (k_t, I_t) space. For simplicity, assume that the $\dot{I} = 0$ locus is downward sloping.
4. Discuss the stability properties of the steady state.