Econ720 - TA Session 6

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2018. 9. 28

1. Problem Set 3 - 1

1. Bonds of Different Maturities

Consider a standard growth model in discrete time where the government issues two types of bonds:

- b_{t+1} one-period bonds are issued at date t; each has a price of 1 and pay R_{t+1} units of consumption at $t+1$
- $B_{t+1} - B_t$ infinitely lived bonds are issued at date t; each costs p_t and pays one unit of consumption at dates $s \geq t+1$

The government also imposes a lump-sum tax τ_t and spends g_t units of the good on a useless purpose.

Firms are standard with FOCs: $r = f'(k)$ and $w = f(k) - f'(k)k$

The household maximizes: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

What is the gov. budget constraint?
What is the H.H. budget constraint?

1. Problem Set 3 - 1

Question:

(1). Solve the H.H. problem using DP.

(2). Define a CE and show that Ricardian Equivalence holds in this economy.

Answer key:

(1). The H.H. maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the BC

$$c_t + k_{t+1} + b_{t+1} + p_t(B_{t+1} - B_t) + \tau_t = w_t + (r_t + 1 - \delta)k_t + R_t b_t + B_t$$

with initial endowments (k_0, b_0, B_0) given.

What are the state variables?

1. Problem Set 3 - 1

State variables: k, b, B
Control variables: c, k', b', B'
$\Rightarrow$

- Bellman equation:

$$V(k,b,B) = max \ u(w+(r+1-\delta)k+Rb+B-k'-b'- p(B'-B)-\tau)+\beta V(k',b',B')$$

- FOC:

$$\begin{aligned}
[k']: &\quad u'(c) = \beta V_k(k',b',B') \\
[b']: &\quad u'(c) = \beta V_b(k',b',B') \\
[B']: &\quad pu'(c) = \beta V_B(k',b',B')
\end{aligned}$$

1. Problem Set 3 - 1

- Envelope condition:

$$
\begin{aligned}
[k]: &\quad V_k(k,b,B) = u'(c)(r+1-\delta) \\
[b]: &\quad V_b(k,b,B) = u'(c)R \\
[B]: &\quad V_B(k,b,B) = u'(c)(1+p)
\end{aligned}
$$

- EE:

$$
\begin{aligned}
u'(c) &= \beta u'(c')(r'+1-\delta) \\
u'(c) &= \beta u'(c')R' \\
u'(c) &= \beta u'(c')\frac{1+p'}{p}
\end{aligned}
$$

1. Problem Set 3 - 1

H.H. problem solution:

Policy functions: $c = \Gamma(k, b, B)$, $k' = \Lambda(k, b, B)$, $b' = \Psi(k, b, B)$, $B' = \Phi(k, b, B)$
Value function $V(k, b, B)$, that satisfy

- Given policy functions, value function $V(k, b, B)$ is a functional fixed point.
- Given functional form of $V(k, b, B)$, policy functions solve EE and BC.

1. Problem Set 3 - 1

(2). Competitive equilibrium:

Allocations $\{c_t, k_{t+1}, b_{t+1}^{HH}, B_{t+1}^{HH}, K_t, L_t, \tau_t, b_{t+1}^{g}, B_{t+1}^{g}\}$ and prices $\{p_t, r_t, w_t, R_t\}$ that satisfy

- H.H.: EE (3), BC (1);
- Firm: FOC (2);
- Gov.: BC (1);
- Market clearing:

  Goods market: $F(K_t, L_t) = c_t + K_{t+1} - (1-\delta)K_t + g_t$

  Capital market: $K_t = k_t$

  Labor market: $L_t = 1$

  Bonds market: $b_t^g = b_t^{HH}$, $B_t^g = B_t^{HH}$

  No-arbitrage condition: $r_{t+1} + 1 - \delta = R_{t+1} = \frac{1+p_{t+1}}{p_t}$

1. Problem Set 3 - 1

Ricardian Equivalence:

Suppose that markets are perfect and taxes are non-distortionary. Then, equilibrium allocations and prices are independent of either the initial level of public debt, or the mixture of deficits and taxes that the government uses to finance government spending.

Here, we want to show that a change in the timing of τ_t does not affect the equilibrium c_t and k_t for a given sequence g_t.

1. Problem Set 3 - 1

Substitute the gov. present value BC into the H.H. present value BC:

$$\sum_{t=0}^{\infty} \frac{c_t}{D_t} = k_0 + \sum_{t=0}^{\infty} \frac{w_t}{D_t} - \sum_{t=0}^{\infty} \frac{g_t}{D_t}$$

It is then immediate that the household's wealth is independent of either the outstanding level of public debt or the financing of government spending: the RHS does not contain b_t, B_t and τ_t. All that matters is the present value of government spending $(\sum_{t=0}^{\infty} \frac{g_t}{D_t})$, not how this is financed.

Since the representative household's budget constraint is independent of b_t, B_t and τ_t, its optimal consumption, saving, and labor supply decisions are also independent of this. Furthermore, the representative firm's decisions are also independent. But if neither the households' nor the firms' decisions are affected, then the equilibrium prices are also unaffected.

1. Problem Set 3 - 1

Particularly, given a given sequence of g_t,

$$\begin{cases} u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta) \\ u'(c_t) = \beta u'(c_{t+1})\dfrac{1 + p_{t+1}}{p_t} \\ r_t = f'(k_t) \\ w_t = f(k_t) - f'(k_t)k_t \\ f(k_t) = c_t + k_{t+1} - (1 - \delta)k_t + g_t \end{cases}$$

these equations together determine equilibrium $\{c_t, k_t, r_t, w_t, p_t\}$. Hence the change of τ_t does not affect equilibrium allocation.

2. Wealth in the Utility Function

Consider the following modification of the standard growth model where household derives utility from holding wealth.

- Demographics: There is a representative household of unit mass who lives forever.
- Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$ where c_t is consumption and k_{t-1} is last period's capital (wealth). The utility function is strictly concave and increasing in both arguments.
- Endowments: At $t = 0$ the household is endowed with capital k_0. In each period the household works 1 unit of time.
- Technologies: $K_{t+1} = AF(K_t, L_t) + (1 - \delta)K_t - c_t$. The production function has constant returns to scale.
- Markets: Production takes place in a representative firm which rents capital and labor from households. There are competitive markets for goods (price 1), capital rental (r_t), and labor rental (w_t).

2. Problem Set 3 - 2

Question:

(1). State the household's dynamic program.

Answer key:
The H.H. maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

subject to the BC

$$c_t + k_{t+1} = w_t + (r_t + 1 - \delta)k_t$$

with initial endowments k_0 given.

What are the state variables? $\rightarrow$ Which variables are predetermined?

2. Problem Set 3 - 2

Define an auxiliary state variable: $z_t = k_{t-1}$.
The law of motion for z_t is $z_{t+1} = k_t$
State variables: k, z
Control variables: c, k'

- Bellman equation:
$$V(k,z) = max\ u(w + (r+1-\delta)k - k', z) + \beta V(k', k)$$
- FOC:
$$[k']: \quad u_c(c,z) = \beta V_k(k', z')$$
- Envelope condition:
$$[k]: \quad V_k(k,z) = u_c(c,z)(r+1-\delta) + \beta V_z(k', z')$$
$$[z]: \quad V_z(k,z) = u_z(c,z)$$
- EE:
$$u_c(c,z) = \beta u_c(c', z')(r'+1-\delta) + \beta^2 u_z(c'', z'')$$

2. Problem Set 3 - 2

(2). Derive and explain the conditions that characterize a solution to the household problem (in sequence language).

Answer key:

$$max \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

$$s.t. \quad c_t + k_{t+1} = w_t + (r_t + 1 - \delta)k_t$$

$$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1}) + \sum_{t=0}^{\infty} \lambda_t (w_t + (r_t + 1 - \delta)k_t - c_t - k_{t+1})$$

FOC:

$$[c_t]: \quad \beta^t u_c(c_t, k_{t-1}) = \lambda_t$$

$$[k_{t+1}]: \quad \beta^{t+2} u_k(c_{t+2}, k_{t+1}) = \lambda_t - \lambda_{t+1}(r_{t+1} + 1 - \delta)$$

2. Problem Set 3 - 2

EE:

$$u_c(c_t, k_{t-1}) = \beta u_c(c_{t+1}, k_t)(r_{t+1} + 1 - \delta) + \beta^2 u_k(c_{t+2}, k_{t+1})$$

Interpretation:

- If I give up 1 unit consumption at period t, the utility cost is $u_c(c_t, k_{t-1})$.
- I save this 1 unit consumption good as capital investment, hence k_{t+1} increases by 1 unit.

  $\rightarrow$ The additional benefit by increasing k_{t+1} by 1 unit is r_{t+1}. And the undepreciated capital at period $t+1$ is $1-\delta$. So my capital gain period $t+1$ is $(r_{t+1} + 1 - \delta)$. Evaluate this gain in terms of utility and discount it to current value.

  $\rightarrow$ k_{t+1} is increased by 1 unit. Since k_{t+1} is also in my utility function at period $t+2$, my utility at period $t+2$ increases by $u_k(c_{t+2}, k_{t+1})$. Discounting this utility gain to current value.

2. Problem Set 3 - 2

H.H. problem solution:

Allocations c_t, k_{t+1} that satisfy

- E.E.
- BC
- Boundary conditions:
  Initial endowment k_0 is given;
  TVC: $\lim_{t \to 0} \beta^t u'(c_t) k_{t+1} = 0$

(3). Define a competitive equilibrium.

Answer key:

Competitive equilibrium:

Allocations $\{c_t, k_{t+1}, K_t, L_t\}$ and prices $\{r_t, w_t\}$ that satisfy

- H.H.: EE (1), BC (1);
- Firm: FOC (2)
  $r_t = AF_K(K_t, L_t)$, $w_t = AF_L(K_t, L_t)$
- Market clearing:
  Goods market: $AF(K_t, L_t) = c_t + K_{t+1} - (1 - \delta)K_t$
  Capital market: $K_t = k_t$
  Labor market: $L_t = 1$

2. Problem Set 3 - 2

(4). Derive a single equation that determines the steady state capital stock.

Tip: to find steady state, start from EE and resource constraint

Answer key:

From resource constraint and other market clearing conditions,

$$k_{t+1} = Af(k_t) + (1-\delta)k_t - c_t$$

At steady state

$$k = Af(k) + (1-\delta)k - c$$

Hence

$$c = Af(k) - \delta k \qquad (*)$$

2. Problem Set 3 - 2

In steady state, the EE becomes

$$u_c(c,k) = \beta u_c(c,k)(r+1-\delta) + \beta^2 u_k(c,k)$$

Substitute equ. $(*)$ into steady state EE,

$$\begin{aligned} & u_c(Af(k)-\delta k,k) \\ & = \beta u_c(Af(k)-\delta k,k)(Af'(k)+1-\delta) + \beta^2 u_k(Af(k)-\delta k,k) \end{aligned}$$

Notice that $r = Af'(k)$

Hence

$$1 = \beta(Af'(k)+1-\delta) + \beta^2 \frac{u_k(Af(k)-\delta k,k)}{u_c(Af(k)-\delta k,k)}$$

2. Problem Set 3 - 2

(5). Is the steady state unique? Explain the intuition why the steady state is or is not unique.

Answer key:

Steady state is generally not unique. Household may choose low c and high k or vice versa.

Shopping time

- Demographics: There is a single representative household who lives forever.
- Preferences: The household values consumption (c) and leisure (l) according to $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$
- Endowments: In each period the household is endowed with 1 unit of time that can be used for leisure (l), work (n), and shopping (s).

$$1 = l_t + n_t + s_t$$

At $t = 0$ the household is endowed with k_0 units of capital and M_0 units of money.

- Technology: The transactions technology is such that s_t units of time are required to purchase c_t given money balances $m_t = \frac{M_t}{p_t}$: $s_t = g(c_t, m_t)$, where p_t is the price of the good. Obviously, $g_c > 0$ and $g_m < 0$.
  Goods are produced from capital and labor with the production function $f(k_t, n_t)$, which has nice properties. The resource constraint is $f(k, n) + (1 - \delta)k = c + k'$
- Markets: The usual markets for goods, money, capital and labor rental operate. There is no government and the money supply is constant.

3. Problem Set 4

Question:

(1). Define a solution to the H.H. problem using DP.

Answer key:

(1). The H.H. maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to the BC

$$1 = l_t + n_t + s_t$$
$$s_t = g(c_t, m_t)$$
$$p_t c_t + p_t k_{t+1} + M_{t+1} = W_t n_t + Q_t k_t + (1-\delta) p_t k_t + M_t$$

$\Rightarrow$

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t - g(c_t, m_t))$$

subject to

$$c_t + k_{t+1} + m_{t+1} \frac{p_{t+1}}{p_t} = w_t n_t + q_t k_t + (1-\delta) k_t + m_t$$

where $m_t = \frac{M_t}{p_t}$, $w_t = \frac{W_t}{p_t}$, $q_t = \frac{Q_t}{p_t}$

What are the state variables?

3. Problem Set 4

Bellman equ.

$$V(k,m) = max\ u(c, 1-n-g(c,m)) + \beta V(k',m')$$
$$+\lambda(w_t n_t + q_t k_t + (1-\delta)k_t + m_t - c - k' - m'\frac{p'}{p})$$

What are the control variables?

- FOC:

$$\begin{aligned}
[c]: &\quad u_c(c,l) - u_l(c,l)g_c(c,m) = \lambda \\
[n]: &\quad u_l = \lambda w \\
[k']: &\quad \beta V_k(k',m') = \lambda \\
[m']: &\quad \beta V_m(') = \lambda \frac{p'}{p}
\end{aligned}$$

- Envelope condition:

$$
\begin{aligned}
[k]: \quad & V_k = \lambda(q+1-\delta) \\
[m]: \quad & V_m = -u_l g_m + \lambda
\end{aligned}
$$

- EE:

$$
\begin{aligned}
u_c - u_l g_c &= \beta(q'+1-\delta)[u_c(') - u_l(')g_c(')] \\
u_c - u_l g_c &= \beta \frac{p}{p'}[u_c(') - u_l(')g_c(')] - \beta \frac{p}{p'} u_l(')g_m(')
\end{aligned}
$$

Define $U(c,n,m) = u(c, 1-n-g(c,m))$, hence

$$
\begin{aligned}
U_c(c,n,m) &= u_c - u_l g_c \\
U_n(c,n,m) &= -u_l \\
U_m(c,n,m) &= -u_l g_m
\end{aligned}
$$

3. Problem Set 4

Then the first EE can be rewritten as

$$U_c(c,n,m) = \beta(q' + 1 - \delta)U_c(c',n',m')$$

By combining the two EEs,

$$\begin{aligned}
\beta(q'+1-\delta)U_c(') &= \beta\frac{p}{p'}U_c(') + \beta\frac{p}{p'}U_m(') \\
(q'+1-\delta)\frac{p'}{p} &= \frac{U_c(') + U_m(')}{U_c(')} \\
(q'+1-\delta)\frac{p'}{p} - 1 &= \frac{U_m(')}{U_c(')}
\end{aligned}$$

This equation governs the allocation of assets.

3. Problem Set 4

H.H. problem solution:

Policy functions: $c = \Gamma_1(k,m)$, $l = \Gamma_2(k,m)$, $n = \Gamma_3(k,m)$, $s = \Gamma_4(k,m)$, $k' = \Gamma_5(k,m)$, $m' = \Gamma_6(k,m)$

Value function $V(k,m)$, that satisfy

- Given policy functions, value function $V(k,m)$ is a functional fixed point.
- Given functional form of $V(k,m)$, policy functions solve EE and BC.

(2). Define a competitive equilibrium.

Answer key:
Competitive equilibrium:
Allocations $\{c_t, l_t, n_t, s_t, k_{t+1}, m_{t+1}, k_t^f, n_t^f\}$ and prices $\{p_t, q_t, w_t\}$ that satisfy

- H.H.: EE, BC
- Firm: FOC
  $q_t = f_k(k_t, n_t)$, $w_t = f_n(k_t, n_t)$
- Market clearing:
  Goods market: $f(k_t, n_t) + (1-\delta)k_t = c_t + k_{t+1}$
  Capital market: $k_t^f = k_t$
  Labor market: $n_t^f = n_t$
  Money market: $p_t m_t = \bar{M}$

(3). Is money neutral in this economy? Prove your answer using the system of equations that define a competitive equilibrium.

Recall:

Money is called neutral if changing the level of M does not affect the real allocation.

It is called super neutral if changing the growth rate of M does not affect the real allocation.

Answer key:

Money is neutral in this economy. A change in the level of the money supply (in all periods) causes a proportional increase in all nominal prices but leaves the equilibrium values of real variables unaffected. We can see this by inspecting the equilibrium conditions and observing that nominal variables always appear in ratios.

3. Problem Set 4

Define the inflation rate as $1+\pi_t = \frac{p_{t+1}}{p_t}$

Define the growth rate of money supply as $1+g_t = \frac{M_{t+1}}{M_t}$

In steady state, $m_{t+1} = m_t = m_{ss} \rightarrow \frac{M_{t+1}}{p_{t+1}} = \frac{M_{t+1}}{p_t} \rightarrow g = \pi$

- Due to EE $U_c = \beta(q' + 1 - \delta)U_c(')$, in steady state we have $f'(k_{ss}) = \frac{1}{\beta} + \delta - 1$. k_{ss} does not depend on M.
- Due to goods market clearing condition, in steady state we have $c_{ss} = f(k_{ss}) - \delta k_{ss}$. c_{ss} does not depend on M.

(4). Would money still be neutral if the transactions technology used nominal money balances i.e. $s_t = g(c_t, M_t)$? Explain the intuition. You need not derive your answer.

Answer key:

Money is not neutral. Think about what happens when M and p double in every period. This could not be an equilibrium because the household now needs less time for shopping. Increasing M makes shopping time more 'productive'.