Econ720 - TA Session 7

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Key words:

- Optimal control
  $\rightarrow$ Hamilton, current value Hamilton
- BGP
- Phase diagram
- Saddle path

Why continuous time?

- Some pathological results of discrete-time models disappear when using continuous time. (See ‘Introduction to Modern Economic Growth’, Acemoglu, Exercise 2.21)
- Continuous-time models have more flexibility in the analysis of dynamics and allow explicit-form solutions in a wider set of circumstances.

1. Continuous Time vs. Discrete Time

To discount:

- Discrete time: discount factor β, ($\beta = \frac{1}{1+\rho}$)
- Continuous time: discount rate ρ

In discrete time, $u(t) = \beta u(t+1) = \frac{1}{1+\rho} u(t+1)$. Hence, $\rho = \frac{u(t+1)-u(t)}{u(t)}$

In continuous time, the above equation becomes $\rho = \frac{\frac{d}{dt}u(t)}{u(t)} = \frac{d}{dt} ln u(t)$

Integrating both sides

$$\int_t^{t+\Delta} \rho ds = \int_t^{t+\Delta} \frac{d}{dt} ln u(s) ds$$

$$\rho \Delta = ln u(t+\Delta) - ln u(t) = ln \frac{u(t+\Delta)}{u(t)}$$

$$e^{\rho\Delta} = \frac{u(t+\Delta)}{u(t)} \Rightarrow u(t) = e^{-\rho\Delta} u(t+\Delta)$$

1. Continuous Time vs. Discrete Time

To solve the model:

- Discrete time:
  Sequential language $\rightarrow$ Lagrangean
  Dynamic programming
- Continuous time:
  Optimal control $\rightarrow$ Hamiltonian (state variable, control variable)

2. Solow Model vs. Ramsey Model

Instead of solving the model and deriving saving rate endogenously, Solow model assumes a fixed saving rate. Hence in Solow model, households are assumed to save a constant exogenous fraction $s \in (0, 1)$ of their disposable income, regardless of what else is happening in the economy.

$\Rightarrow$ In the Solow model, agents in the economy (and the planner) follow a simplistic linear rule for consumption and investment. In the Ramsey model, agents (and the planner) choose consumption and investment optimally so as to maximize their utility (welfare).

3. BGP and Stationary Transformation

When the economy is experiencing balanced growth, the production function must have a representation of the form $Y_t = F(K_t, A_t L_t)$, with purely labor-augmenting technological progress.

In models with technological progress, we should not look for a steady state where very real variable per capita is constant, but for a BGP, where every variable grows at a constant rate, while stationary transformed variables remain constant.

If a path is balanced growth, we have 2 kinds of stationary transformation to steady state.

- Define $\tilde{x}_t = \frac{x_t}{e^{g_x t}}$, where g_x is the balanced growth rate of x,
- Define $\tilde{x}_t = \frac{x_{1t}}{x_{2t}}$, where $g_{x_1} = g_{x_2}$,

then $\tilde{x}_t$ will be constant in this path.

4. Phase Diagram and Saddle-Path Stability

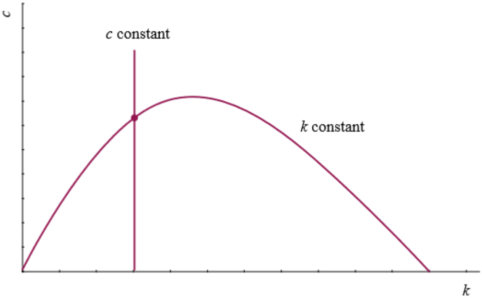
EE: $\frac{\dot{c}_t}{c_t} = \frac{f'(k_t)-\delta-\rho}{\sigma}$

Resource Constraint: $\dot{k}_t = f'(k_t) - (\delta+\rho)k_t - c_t$

$\Rightarrow$

$\dot{c} = 0$: $f'(k) = \delta + \rho$

$\dot{k} = 0$: $c = f'(k) - (\delta+\rho)k$



5. Exercise: An Investment Problem

Macroeconomics Qualifying Examination, January 2012

Consider the problem of an infinitely lived firm that invests in capital K_t subjects to an adjustment cost.

Time is continuous. The profit stream is given by

$$\pi_t = f(k_t) - I_t - \phi(I_t)$$

where f obeys Inada conditions and the adjustment cost is convex: $\phi' > 0$ and $\phi'' > 0$. $\phi(0) = 0$.

The firm maximizes the discounted present value of profits

$$\max_{I_t, K_t; t \geq 0} \int_0^\infty e^{-rt} \pi_t dt$$

subject to the law of motion

$$\dot{k}_t = I_t - \delta k_t$$

5. Exercise: An Investment Problem

Questions:

1. Derive the necessary conditions for the firm's optimal investment plan, including the TVC.
2. From the necessary conditions, derive the differential equation for I_t.
3. Draw a phase diagram in (k_t, I_t) space. For simplicity, assume that the $\dot{I} = 0$ locus is downward sloping.
4. Discuss the stability properties of the steady state.

Please also do Macroeconomics Qualifying Examination, August 2016, 4.Ben-Porath Model