Econ720 - TA Session 7

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1. Wealth in the Utility Function

Consider the following modification of the standard growth model where household derives utility from holding wealth.

- Demographics: There is a representative household of unit mass who lives forever.
- Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$ where c_t is consumption and k_{t-1} is last period's capital (wealth). The utility function is strictly concave and increasing in both arguments.
- Endowments: At $t = 0$ the household is endowed with capital k_0. In each period the household works 1 unit of time.
- Technologies: $K_{t+1} = AF(K_t, L_t) + (1-\delta)K_t - c_t$. The production function has constant returns to scale.
- Markets: Production takes place in a representative firm which rents capital and labor from households. There are competitive markets for goods (price 1), capital rental (r_t), and labor rental (w_t).

Question:

(1). State the household's dynamic program.

Answer key:
The H.H. maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

subject to the BC

$$c_t + k_{t+1} = w_t + (r_t + 1 - \delta) k_t$$

with initial endowments k_0 given.

What are the state variables? $\rightarrow$ Which variables are predetermined?

1. Problem Set 3 - 1

Define an auxiliary state variable: $z_t = k_{t-1}$.
The law of motion for z is $z_{t+1} = k_t$
State variables: k, z
Control variables: c, k'

- Bellman equation:
$$V(k,z) = max\ u(w + (r+1-\delta)k - k', z) + \beta V(k', k)$$
- FOC:
$$[k']: \quad u_c(c,z) = \beta V_k(k', z')$$
- Envelope condition:
$$[k]: \quad V_k(k,z) = u_c(c,z)(r+1-\delta) + \beta V_z(k', z')$$
$$[z]: \quad V_z(k,z) = u_z(c,z)$$
- EE:
$$u_c(c,z) = \beta u_c(c', z')(r'+1-\delta) + \beta^2 u_z(c'', z'')$$

1. Problem Set 3 - 1

(2). Derive and explain the conditions that characterize a solution to the household problem (in sequence language).

Answer key:

$$max \ \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

$$s.t. \ \ c_t + k_{t+1} = w_t + (r_t + 1 - \delta) k_t$$

$$\mathscr{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1}) + \sum_{t=0}^{\infty} \lambda_t (w_t + (r_t + 1 - \delta) k_t - c_t - k_{t+1})$$

FOC:

$$[c_t]: \quad \beta^t u_c(c_t, k_{t-1}) = \lambda_t$$

$$[k_{t+1}]: \quad \beta^{t+2} u_k(c_{t+2}, k_{t+1}) = \lambda_t - \lambda_{t+1}(r_{t+1} + 1 - \delta)$$

1. Problem Set 3 - 1

EE:

$$u_c(c_t, k_{t-1}) = \beta u_c(c_{t+1}, k_t)(r_{t+1} + 1 - \delta) + \beta^2 u_k(c_{t+2}, k_{t+1})$$

Interpretation:

- If I give up 1 unit consumption at period t, the utility cost is $u_c(c_t, k_{t-1})$.
- I save this 1 unit consumption good as capital investment, hence k_{t+1} increases by 1 unit.

  $\rightarrow$ The additional benefit by increasing k_{t+1} by 1 unit is r_{t+1}. And the undepreciated capital at period $t+1$ is $1-\delta$. So my capital gain period $t+1$ is $(r_{t+1} + 1 - \delta)$. Evaluate this gain in terms of utility and discount it to current value.

  $\rightarrow$ k_{t+1} is increased by 1 unit. Since k_{t+1} is also in my utility function at period $t+2$, my utility at period $t+2$ increases by $u_k(c_{t+2}, k_{t+1})$. Discounting this utility gain to current value.

1. Problem Set 3 - 1

H.H. problem solution:

Sequences $\{c_t, k_{t+1}\}$ that satisfy

- E.E.
- BC
- Boundary conditions:
  Initial endowment k_0 is given;
  TVC: $\lim_{t \to 0} \beta^t u'(c_t) k_{t+1} = 0$

1. Problem Set 3 - 1

(3). Define a competitive equilibrium.

Answer key:

Competitive equilibrium:

Allocations $\{c_t, k_{t+1}, K_t, L_t\}$ and prices $\{r_t, w_t\}$ that satisfy

- H.H.: EE (1), BC (1);
- Firm: FOC (2)
  $r_t = AF_K(K_t, L_t)$, $w_t = AF_L(K_t, L_t)$
- Market clearing:
  Goods market: $AF(K_t, L_t) = c_t + K_{t+1} - (1-\delta)K_t$
  Capital market: $K_t = k_t$
  Labor market: $L_t = 1$

1. Problem Set 3 - 1

(4). Derive a single equation that determines the steady state capital stock.

Tip: to find steady state, start from EE and resource constraint

Answer key:

From resource constraint and other market clearing conditions,

$$k_{t+1} = Af(k_t) + (1-\delta)k_t - c_t$$

At steady state

$$k = Af(k) + (1-\delta)k - c$$

Hence

$$c = Af(k) - \delta k \qquad (*)$$

1. Problem Set 3 - 1

In steady state, the EE becomes

$$u_c(c,k) = \beta u_c(c,k)(r+1-\delta) + \beta^2 u_k(c,k)$$

Substitute equ. $(*)$ into steady state EE,

$$\begin{aligned} & u_c(Af(k)-\delta k,k) \\ & = \beta u_c(Af(k)-\delta k,k)(Af'(k)+1-\delta) + \beta^2 u_k(Af(k)-\delta k,k) \end{aligned}$$

Notice that $r = Af'(k)$
Hence

$$1 = \beta(Af'(k)+1-\delta) + \beta^2 \tfrac{u_k(Af(k)-\delta k,k)}{u_c(Af(k)-\delta k,k)}$$

(5). Is the steady state unique? Explain the intuition why the steady state is or is not unique.

Answer key:

Steady state is generally not unique. Household may choose low c and high k or vice versa.

2. Problem Set 3 - 2

2. Ben-Porath Model

We study the decision problem of an infinitely lived agent in discrete time.

- At $t = 0$, the agent is endowed with h_0 units of human capital.
- In each period, he can invest l_t units of time, so that human capital evolves according to

$$\begin{aligned} h_{t+1} &= (1-\delta)h_t + F(h_t l_t) \\ F(h_t l_t) &= (h_t l_t)^\alpha \end{aligned}$$

  with $0 < \alpha, \delta < 1$.
- The objective is to maximize the present value of lifetime earnings, given by

$$Y = \sum_{t=0}^{\infty} R^{-t} w h_t (1 - l_t)$$

  where $R > 0$ and $w > 0$ are taken as given.

2. Problem Set 3 - 2

Question:

(1). Write down the agent's Dynamic Program

Answer key:
The agent maximizes

$$\sum_{t=0}^{\infty} R^{-t} w h_t (1 - l_t)$$

subject to

$$h_{t+1} = (1-\delta)h_t + F(h_t l_t)$$
$$F(h_t l_t) = (h_t l_t)^{\alpha}$$

with initial endowments h_0 given.

What are the state variables? $\rightarrow$ Which variables are predetermined?

2. Problem Set 3 - 2

State variables: h

- Bellman equation:

$$V(h) = max \ wh(1-l) + R^{-1}V((1-\delta)h + (hl)^{\alpha})$$

  Notice in this case, there is only one control variable l

- If we write Bellman equation in the following way

$$V(h) = max \ wh\left[1 - \frac{(h'-(1-\delta)h)^{1/\alpha}}{h}\right] + R^{-1}V(h')$$

  The control variable in this case is h'

- The third way to write Bellman equation is

$$V(h) = max \ wh(1-l) + R^{-1}V(h') + \lambda(h' - (1-\delta)h - (hl)^{\alpha})$$

  In this case, there are two control variables l and h'

2. Problem Set 3 - 2

(2). Derive and interpret the first order condition for l.

Answer key:

Notice that from this question, it is better to write Bellman equation as

$$V(h) = max\ wh(1-l) + R^{-1}V((1-\delta)h + (hl)^{\alpha})$$

FOC:

$$[l] : wh = R^{-1}V'(h')\alpha h^{\alpha} l^{\alpha-1}$$

Interpretation

- LHS: If I increase my studying time by 1 unit, I have 1 unit less of working time, then my labor income will decrease by wh. It's the cost of studying.
- RHS: By having 1 unit more of studying time today, my human capital tomorrow will increase by $\alpha h^{\alpha} l^{\alpha-1}$. Measuring this increase in terms of value and discounting it back to present value, the benefit of studying is $R^{-1}V'(h')\alpha h^{\alpha} l^{\alpha-1}$

2. Problem Set 3 - 2

(3). Derive $V'(h) = w + (1-\delta)R^{-1}V'(h')$.

Answer key:

EC:

$$
\begin{aligned}
[h] : V'(h) &= w(1-l) + R^{-1}V'(h')\left[(1-\delta) + \alpha h^{\alpha-1} l^{\alpha}\right] \\
&= w + (1-\delta)R^{-1}V'(h') + \frac{l}{h}\Big(\underbrace{R^{-1}V'(h')\alpha h^{\alpha} l^{\alpha-1} - wh}_{=0 \text{ by FOC}}\Big) \\
&= w + (1-\delta)R^{-1}V'(h')
\end{aligned}
$$

2. Problem Set 3 - 2

Interpretation: $V'(h) = w + (1-\delta)R^{-1}V'(h')$

If my human capital increases by 1 unit today, my value will increase by $V'(h)$. This comes from two parts:

- Today:
  The higher human capital brings me more labor income, $w(1-l)$.
- Tomorrow:
  Since human capital tomorrow depends on today's human capital, 1 more unit of human capital today increases my human capital tomorrow by $(1-\delta)+\alpha h^{\alpha-1}l^{\alpha}$. Measure this increment in human capital in terms of value and discount it back to present value, then we have $R^{-1}V'(h')\left[(1-\delta)+\alpha h^{\alpha-1}l^{\alpha}\right]$.

2. Problem Set 3 - 2

(4). Derive and interpret $V'(h) = w\frac{R}{r+\delta}$ where $R = 1 + r$.

Answer key:

$$\begin{aligned}
V'(h_t) &= w + (1-\delta)R^{-1}V'(h_{t+1}) \\
&= w + (1-\delta)R^{-1}\Big[w + (1-\delta)R^{-1}V'(h_{t+2})\Big] \\
&\dots \\
&= \sum_{j=0}^{T}(1-\delta)^j(R^{-1})^j w + (1-\delta)^{T+1}(R^{-1})^{T+1}V'(h_{T+1})
\end{aligned}$$

When T goes to infinity, $(1-\delta)^{T+1}(R^{-1})^{T+1}V'(h_{T+1}) \to 0$

$$V'(h) = \sum_{j=0}^{\infty}(1-\delta)^j(R^{-1})^j w = w\frac{R}{r+\delta}$$

2. Problem Set 3 - 2

Interpretation:

- Note that, $V'(h) = w\frac{R}{r+\delta}$ indicates $V(h) = w\frac{R}{r+\delta}h$
- Given my current human capital level h, if I work full time from now on without investing in human capital accumulation any more, the present value I get is

$$V^w = \sum_{j=0}^{\infty}(R^{-1})^j wh(1-\delta)^j = w\frac{R}{r+\delta}h$$

- Hence, $V(h) = V^w$, the value of human capital equals the present value earnings when the agent works full time. This is because, at the margin, the values of working and studying are the same.

2. Problem Set 3 - 2

Discussion

The increase in present value of my lifetime earning brought by an additional unit of human capital today is $V'(h) = w\frac{R}{r+\delta}$. Since $0 < \delta < 1$, $V'(h) = w\frac{r+1}{r+\delta} > w$. The marginal benefit of studying is greater than w, because the benefit brought by an extra unit of h comes from two parts:

- It will increase my labor income today.
- It will increase my future labor income by giving me a higher h tomorrow.

2. Problem Set 3 - 2

(5). How do the wage and interest rate affect steady state h and l?

Answer key:

$$V'(h_t) = w\frac{R}{r+\delta}$$
$$wh = R^{-1}V'(h')\alpha h^{\alpha} l^{\alpha-1}$$

Combine these two equations, we get

$$1 = \frac{\alpha}{r+\delta} h^{\alpha-1} l^{\alpha-1} \qquad (1)$$

Since in steady state, $h' = h$, from law of motion for h we get

$$h = (1-\delta)h + h^{\alpha} l^{\alpha}$$
$$\text{Hence, } h^{\alpha-1} l^{\alpha-1} = \frac{\delta}{l} \qquad (2)$$

2. Problem Set 3 - 2

Combine (1) and (2), we get steady state l

$$l = \frac{\alpha\delta}{r+\delta}$$

Plug s.s. equation for l into equation (1), we get steady state h

$$h = \delta^{\frac{1}{\alpha-1}}\left(\frac{\alpha\delta}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Hence the wage doesn't affect s.s. l and h. The interest rate reduces s.s. l and h.

2. Problem Set 3 - 2

Reason:

- Wage doesn't affect s.s. l and h, because wage affects the marginal cost as much as the marginal benefit of studying. These two effects are canceled out.
- Interest rate reduces s.s. l and h, because it lowers the value of future earnings. The future earnings are discounted by R^{-1}. When R is high, it implies that the discounted value of future earnings become lower. Agent will have less incentive to invest in her human capital. The higher future earnings brought by human capital investment will be discounted heavily by a high interest rate. Hence, agent will spend less time in investing human capital. And a result, lower l leads to lower h.