Econ720. HW#7. 12/5/2021.

1. Lucas Tree with Crashes

Cum dividend:

The asset you buy today gives you dividend today.

- BC with ex-dividend:

$$c_t + P_t k_{t+1} = P_t k_t + d_t k_t$$

- BC with cum-dividend:

$$c_t + P_t k_{t+1} = P_t k_t + d_t k_{t+1}$$

(1). State the HH's dynamic problem.

Discussion:

- Since Q(3) asks us to define RCE, we need to state the HH problem here that is consistent with RCE setup, instead of just the standard DP.
- The special part about RCE is that we have two types of state variables: individual state + aggregate state.

e.g. k is individual state, K is aggregate state [Sep-23 slides]

- This model is a "representative HH + aggregate uncertainty" model. Hence, the aggregate state variables ~~is~~ are

the agg. version of the individual state + agg. uncertaity. [Nov-23 slides]

- Following this idea, in this model,

  individual state: k (# of trees, or share of tree)

  aggregate state: $\underbrace{K}_{\text{agg. version of } k}$, $\underbrace{d, d_{-1}}$

  in order to form expectation about d_{t+1}, I need to know d_t and d_{t-1}.

Hence, in this model, agg. state vector is $S = (K, d, d_{-1})$.

However, given by the model setup, there is only 1 tree. $K=1$.

The agg. K is fixed at 1 forever. If the state variable doesn't change, we can drop it. That's why in the AK, there are only exogenous agg. states.

- In summary, in this model, state variables are

  individual state: k

  agg. state vector: $S = (d, d_{-1})$

HH Problem

$$V(k, S) = \max u(c) + \beta \mathbb{E}[V(k', S')]$$

$$\text{s.t.} \quad c + pk' = dk' + pk$$

Bellman Equation:

$$V(k, S) = \max u(dk' + pk - pk') + \beta \mathbb{E}[V(k', S')]$$

FOC: $[k']$: $(-d+p)\, u'(c) = \beta \mathbb{E}\, V_k(k', S')$

EC: $[k]$: $V_k(k, S) = p\, u'(c)$

(2). EE: by combining FOC and EC.

$$(p-d)\, u'(c) = \beta \mathbb{E}\{p' u'(c')\}$$

$$\therefore P_t = \beta \mathbb{E}\left\{\frac{u'(C_{t+1})}{u'(C_t)} P_{t+1}\right\} + d_t$$

(3). RCE

Policy functions : $c(k, S)$, $k' = K(k, S)$

Value function : $V(k, S)$

Price function : $p(S)$

that satisfy

- Household optimality
- Goods mkt clearing: $c(k, S) = d$ → goods produced by the tree will be consumed by HH.
- Asset mkt clearing: $K(k, S) = 1$ → There is only one tree.
- Law of motion of the agg. state variable (d, d_{-1})

(4). $P_t = \beta \mathbb{E}\left\{ \frac{u'(C_{t+1})}{u'(C_t)} P_{t+1} \right\} + d_t$

$= \beta \mathbb{E}\left\{ \left(\frac{C_{t+1}}{C_t}\right)^{-\sigma} P_{t+1} \right\} + d_t$ utility function

$= \beta \mathbb{E}\left\{ \left(\frac{d_{t+1}}{d_t}\right)^{-\sigma} P_{t+1} \right\} + d_t$ goods mkt clearing condition.

① If $d_t = d_{t-1}$, $d_{t+1} = d_t$ forever, hence there is no uncertainty

$\therefore P_t = \beta \cdot \left(\frac{d_t}{d_t}\right)^{-\sigma} P_{t+1} + d_t$

$= \beta P_{t+1} + d_t$

$= \beta \cdot [\beta P_{t+2} + d_t] + d_t$

$\cdots$

$= \cancel{\beta^{\infty} P_{t+\infty}}^{\to 0} + d_t \underbrace{\sum_{t=0}^{\infty} \beta^t}_{} = \frac{1}{1-\beta}$

$\therefore P_t = \frac{1}{1-\beta} d_t, \ \forall t$ when $d_t = d_{t-1}$

② If $d_t \neq d_{t-1}$

$P_t = \beta \left\{ \underbrace{\pi \left(\frac{\gamma d_t}{d_t}\right)^{-\sigma}}_{} \cdot P_{t+1} + \underbrace{(1-\pi)\left(\frac{d_t}{d_t}\right)^{-\sigma}}_{} \cdot P_{t+1} \right\} + d_t$

with prob. π, $d_{t+1} = \gamma d_t$

with prob. $(1-\pi)$ $d_{t+1} = d_t$

$= \beta \left\{ \pi \gamma^{-\sigma} P_{t+1} + (1-\pi) \underbrace{\frac{1}{1-\beta} d_t}_{} \right\} + d_t$

price when d stops growing, obtained in ①

$\therefore \quad \frac{P_t}{d_t} = \beta \left\{ \pi \gamma^{-\sigma} \frac{P_{t+1}}{d_{t+1}} \frac{d_{t+1}}{d_t} + (1-\pi) \frac{1}{1-\beta} \right\} + 1$

$= \beta \left\{ \pi \gamma^{-\sigma} \frac{P_{t+1}}{d_{t+1}} \gamma + (1-\pi) \frac{1}{1-\beta} \right\} + 1$

$\because$ We assume p/d is constant during the phase with growth.

$\therefore \quad \frac{P_t}{d_t} = \frac{P_{t+1}}{d_{t+1}} \equiv \frac{P_g}{d}$

$\therefore \quad \frac{P_g}{d} = \beta \left\{ \pi \gamma^{1-\sigma} \frac{P_g}{d} + (1-\pi) \frac{1}{1-\beta} \right\} + 1$

$\therefore \quad (1 - \beta \pi \gamma^{1-\sigma}) \frac{P_g}{d} = (1-\pi) \frac{\beta}{1-\beta} + 1$

$\therefore \quad P_t = \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1 - \beta\pi\gamma^{1-\sigma}} d_t$, when $d_t \neq d_{t-1}$

Markov process: a discrete - time process for which the future behavior, given the past and present, only depends on the present and not the past.

$$P_t = \begin{cases} \frac{1}{1-\beta} d_t, & \text{if } d_t = d_{t-1} \\ \frac{(1-\pi)\frac{\beta}{1-\beta} + 1}{1-\beta\pi\gamma^{1-\sigma}}, & \text{if } d_t \neq d_{t-1} \end{cases}$$

$\therefore$ P_t depends on (d_t, d_{t-1}), which is the current agg. state vector $\underset{\sim}{S}$

$\therefore$ P_t is Markov.

(5). The price when growing is $\frac{p_t^g}{d_t} = \frac{(1-\pi)\frac{\beta}{1-\beta}+1}{1-\beta\pi\gamma^{1-\sigma}}$

The price when stopping is $\frac{p_t^s}{d_t} = \frac{1}{1-\beta}$

"Crash" means $\frac{p_t^s}{d_t} < \frac{p_t^g}{d_t}$

$$\therefore \frac{1}{1-\beta} < \frac{(1-\pi)\frac{\beta}{1-\beta}+1}{1-\beta\pi\gamma^{1-\sigma}}$$

"in recursive form" means to use DP,
you don't have to use RCE.
So we just go with the standard DP

2. Two stocks.

(1). HH problem in recursive form

State variables: b, k_1, k_2

θ_1, θ_2 (uncertainty, exogenous states)

$$V(b, k_1, k_2; \theta_1, \theta_2) = \max u(c) + \beta \mathbb{E} V(b, k_1, k_2; \theta_1, \theta_2)$$

$$\text{s.t. } c + \sum p_j k_j' + q b' = b + \sum p_j k_j + \sum k_j \theta_j y$$

$$\Rightarrow V(b, k_1, k_2; \theta_1, \theta_2) = \max u\left(b + \sum p_j k_j + \sum k_j \theta_j y - \sum p_j k_j' - q b'\right) + \beta \mathbb{E} V(b', k_1', k_2'; \theta_1', \theta_2')$$

(2). Euler Equation

FOC $[b']$: $q u'(c) = \beta \mathbb{E} V_b(')$

$[k_1']$: $p_1 u'(c) = \beta \mathbb{E} V_{k_1}(')$

$[k_2']$: $p_2 u'(c) = \beta \mathbb{E} V_{k_2}(')$

EC. $[b]$: $V_b = u'(c)$

$[k_1]$: $V_{k_1} = (p_1 + \theta_1 y) \cdot u'(c)$

$[k_2]$: $V_{k_2} = (p_2 + \theta_2 y) \cdot u'(c)$

$\therefore$ EE. $q u'(c) = \beta \mathbb{E} u'(c')$

$p_1 u'(c) = \beta \mathbb{E}\{(p_1' + \theta_1' y) \cdot u'(c')\}$

$p_2 u'(c) = \beta \mathbb{E}\{(p_2' + \theta_2' y) \cdot u'(c')\}$

(3). Good mkt clearing: $C_t = \sum_j \theta_{j,t} \cdot y = y$, $\forall t$

$\therefore$ EE for risk free bond becomes

$$q = \beta \mathbb{E}\left\{\frac{u'(c')}{u'(c)}\right\} = \beta \mathbb{E}\left\{\frac{u'(y)}{u'(y)}\right\} = \beta$$

EE for tree j becomes

$$P_j = \beta \mathbb{E}\left\{\frac{u'(c')}{u'(c)} \left(P_j' + \theta_j' y\right)\right\}$$

$$= \beta \mathbb{E}\left\{\frac{u'(y)}{u'(y)} \left(P_j' + \theta_j' y\right)\right\}$$

$$= \beta P_j' + \beta y \mathbb{E}\left(\theta_j'\right)$$

$\because$ θ_j is iid

$\therefore$ $\mathbb{E}(\theta_{jt})$ is constant $\forall t$

$$\therefore P_{jt} = \beta P_{jt+1} + \beta y \mathbb{E}(\theta_j)$$

$$= \beta\left(\beta P_{jt+2} + \beta y \mathbb{E}(\theta_j)\right) + \beta y \mathbb{E}(\theta_j)$$

$$\cdots$$

$$= \cancel{\beta^{\infty} P_{jt+\infty}}^{\to 0} + \frac{\beta}{1-\beta} y \mathbb{E}(\theta_j)$$

$\therefore$ Stock price is constant, $P_{jt} = \frac{\beta}{1-\beta} y \mathbb{E}(\theta_j)$ $\forall t$, $j = 1, 2$.

(4). Equity Premium

Stock price $P_j = \frac{\beta}{1-\beta} y \mathbb{E}(\theta_j)$

and $\Sigma \theta_j = 1$

$\therefore$ (Stock 1 & Stock 2) have constant price and constant dividend.

$\therefore$ (Stock 1 & Stock 2) is a risk free asset

Then in equilibrium, $R^s \equiv R^f = \frac{1}{q} = \frac{1}{\beta}$.

(5) From EE.

$$P_{j,t} = \mathbb{E}\left\{ \beta \frac{u'(C_{t+1})}{u'(C_t)} \left(P_{j\,t+1} + \theta_{j\,t+1}\, y_{t+1}\right) \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{C_t}{C_{t+1}} \left(P_{j\,t+1} + \theta_{j\,t+1}\, y_{t+1}\right) \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{y_t}{y_{t+1}} P_{j\,t+1} + \beta y_t \theta_{j\,t+1} \right\}$$

$$= \mathbb{E}\left\{ \beta \frac{y_t}{y_{t+1}} \left(\beta \frac{y_{t+1}}{y_{t+2}} + \beta y_{t+1} \theta_{j\,t+2} \right) + \beta y_t \theta_{j\,t+1} \right\}$$

$$= y_t \left\{ \underbrace{\beta^\infty \mathbb{E}\left(\frac{1}{y_{t+\infty}}\right)}_{} + \sum_{s=1}^{\infty} \beta^s \theta_{j\,t+s} \right\}$$

$=0$ $\because$ y is from a finite Markov chain

$$= y_t \cdot \mathbb{E}\left(\sum_{s=1}^{\infty} \beta^s \theta_{j,t+s}\right)$$

$\because \theta_j$ is iid

$$= y_t \cdot \mathbb{E}(\theta_j) \cdot \sum_{s=1}^{\infty} \beta^s$$

$$= y_t \cdot \frac{\beta}{1-\beta} \mathbb{E}(\theta_j)$$

The portfolio of two stocks : (Stock 1 & Stock 2).

- Price: $P_t = P_{1t} + P_{2t}$

$$= y_t \frac{\beta}{1-\beta} \mathbb{E}(\theta_1) + y_t \frac{\beta}{1-\beta} \mathbb{E}(\theta_2)$$

$$= y_t \frac{\beta}{1-\beta} \left(\mathbb{E}(\theta_1) + \mathbb{E}(\theta_2)\right)$$

$$= y_t \frac{\beta}{1-\beta} \mathbb{E}\underbrace{(\theta_1 + \theta_2)}_{=1} = y_t \frac{\beta}{1-\beta}$$

- Dividend : y_t

$$\therefore \mathbb{E}(R_t^s) = \mathbb{E}\left(\frac{P_{t+1} + y_{t+1}}{P_t}\right) = \mathbb{E}\left(\frac{y_{t+1}\frac{\beta}{1-\beta} + y_{t+1}}{y_t \frac{\beta}{1-\beta}}\right)$$

$$\therefore \mathbb{E}(R_t^s) = \frac{1}{\beta} \mathbb{E}\left(\frac{y_{t+1}}{y_t}\right)$$

See AK for explanation and intuition.