Econ720 - TA Session 4

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1. Problem Set 2 - 1

1. Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period, $N_t = (1+n)^t$ persons are born. Each lives for 2 periods. Half of the agents are of type 1, the other half of type 2.

→Heterogeneous agents

Endowments: The initial old hold M_0 units of money, evenly distributed across agents. Each person is endowed with (e_i^y, e_i^o) units of consumption when young and old, respectively.

Preferences: $ln(c_t^y) + \beta ln(c_{t+1}^o)$

Technology: Goods can only be eaten the day they drop from the sky.

→No savings!

1. Problem Set 2 - 1

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_t = M_{t-1} + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu) M_t$.

$\rightarrow$The government transfers $x_t p_t$ units of money to each old

Markets: In each period, agents buy/sell goods and money in spot markets.

Question

1. Define a competitive equilibrium.

Answer key:

- How many sectors? $\rightarrow$ H.H. and Gov.
- What are the choice variables for each sector?
- How many markets

1. Problem Set 2 - 1

$$
\begin{aligned}
&max\ \ ln(c^y_{i,t}) + \beta ln(c^o_{i,t+1}) \\
&s.t.\ \ \ p_t c^y_{i,t} + p_t m^d_{i,t} = p_t e^y_i \\
&\qquad\ \ p_{t+1} c^o_{i,t+1} = p_{t+1} e^o_i + p_t m^d_{i,t} + p_{t+1} x_{t+1} \\
&\mathscr{L} = ln(c^y_{i,t}) + \beta ln(c^o_{i,t+1}) \\
&\qquad + \lambda \{ e^y_i + \frac{p_{t+1}}{p_t}(e^o_i + x_{t+1}) - c^y_{i,t} - \frac{p_{t+1}}{p_t} c^o_{i,t+1} \}
\end{aligned}
$$

Notice that x_{t+1} is an **exogenous** variable for household!

1. Problem Set 2 - 1

$$[c^y_{i,t}]: \ \frac{1}{c^y_{i,t}} = \lambda$$

$$[c^o_{i,t+1}]: \ \beta \frac{1}{c^o_{i,t+1}} = \lambda \frac{p_{t+1}}{p_t}$$

$$\Rightarrow \frac{1}{c^y_{i,t}} = \beta \frac{1}{c^o_{i,t+1}} \frac{p_t}{p_{t+1}}$$

1. Problem Set 2 - 1

Competitive equilibrium:

Allocations $\{c^y_{i,t}, c^o_{i,t}, m^d_{i,t}, x_t, M_t\}$ and prices $\{p_t\}$ that satisfy

- Household: Euler Equation, Budget Constraint
- Government: $M_t = M_{t-1} + N_{t-1} x_t p_t$, $M_{t+1} = (1+\mu) M_t$
- Market Clearing Conditions:

  1.Goods market: $\frac{N_t}{2} c^y_{1,t} + \frac{N_t}{2} c^y_{2,t} + \frac{N_{t-1}}{2} c^o_{1,t} + \frac{N_{t-1}}{2} c^o_{2,t} = \frac{N_t}{2} e^y_1 + \frac{N_t}{2} e^y_2 + \frac{N_{t-1}}{2} e^o_1 + \frac{N_{t-1}}{2} e^o_2$

  2.Money market: $\frac{N_t}{2} m^d_{1,t} p_t + \frac{N_t}{2} m^d_{2,t} p_t = M_t$

2. Derive the household consumption function.
   Tip: log-utility → consumption is a constant fraction of wealth.

Answer key:

From the lifetime budget constraint:

$$c^y_{i,t} + \frac{p_{t+1}}{p_t} c^o_{i,t+1} = e^y_i + \frac{p_{t+1}}{p_t}(e^o_i + x_{t+1})$$

Substitute E.E.

$$c^y_{i,t} + \frac{p_{t+1}}{p_t} \beta c^y_{i,t} \frac{p_t}{p_{t+1}} = e^y_i + \frac{p_{t+1}}{p_t}(e^o_i + x_{t+1})$$

Hence

$$c^y_{i,t} = \frac{1}{1+\beta}\left(e^y_i + \frac{e^o_i}{R_{t+1}} + \frac{x_{t+1}}{R_{t+1}}\right), \text{ where } R_{t+1} = \frac{p_t}{p_{t+1}}$$

$$c^o_{i,t+1} = \beta c^y_{i,t} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e^y_i + e^o_i + x_{t+1})$$

1. Problem Set 2 - 1

3. Derive a difference equation for the equilibrium interest rate when $\mu = 0$.

Logic: $\mu = 0 \rightarrow$ gov. doesn't add money to the economy $\rightarrow x_t = 0$

Answer key:

When $x_t = 0$

$$c_{i,t}^{y} = \frac{1}{1+\beta}(e_i^y + \frac{e_i^o}{R_{t+1}})$$
$$c_{i,t+1}^{o} = \beta c_{i,t}^{y} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e_i^y + e_i^o)$$

1. Problem Set 2 - 1

From goods market clearing condition:

$$\frac{N_t}{2}(c^y_{1,t}+c^y_{2,t})+\frac{N_{t-1}}{2}(c^o_{1,t}+c^o_{2,t})=\frac{N_t}{2}(e^y_1+e^y_2)+\frac{N_{t-1}}{2}(e^o_1+e^o_2)$$

Rearrange this equation by using $N_t=(1+n)^t,\ N_{t-1}=(1+n)^{t-1}$

$$\begin{aligned} e^y_1+e^y_2+\frac{1}{n+1}(e^o_1+e^o_2) &= c^y_{1,t}+c^y_{2,t}+\frac{1}{n+1}(c^o_{1,t}+c^o_{2,t}) \\ &= \frac{1}{1+\beta}(e^y_1+\frac{e^o_1}{R_{t+1}}+e^y_2+\frac{e^o_2}{R_{t+1}}) \\ &+ \frac{1}{n+1}\frac{\beta}{1+\beta}(R_te^y_1+e^o_1+R_te^y_2+e^o_2) \end{aligned}$$

Difference equation of R: $\beta(1+n-R_t)(e^y_1+e^y_2)=\frac{1+n-R_{t+1}}{R_{t+1}}(e^o_1+e^o_2)$

1. Problem Set 2 - 1

4. Is the monetary steady state dynamically efficient?

Review:

- Samuelson case vs. Classical case
- An equilibrium with fiat money exists if and only if the economy without money is dynamically inefficient
- The definition of steady state

Answer key:

In steady state, $m_{t+1} = m_t = \bar{m}$, where $m_t = \frac{M_t}{p_t N_t}$
Hence,

$$\frac{M_{t+1}}{p_{t+1} N_{t+1}} = \frac{M_t}{p_t N_t} \Rightarrow \frac{M_{t+1}}{M_t} = \frac{p_{t+1}}{p_t} \frac{N_{t+1}}{N_t} \Rightarrow 1 + \mu = \frac{1+n}{R_{t+1}}$$

1. Problem Set 2 - 1

In steady state,

$$R = \frac{1+n}{1+\mu}$$

- If $\mu > 0$, this monetary steady state is not dynamically efficient.
- If $\mu = 0$, this monetary steady state is dynamically efficient.

1. Money in the Utility Function in an OLG Model

Question:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household's first-order conditions imply rate of return dominance, i.e. the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium)

2. Problem Set 2 - 2

Answer key:

$$\max\ u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$

$$s.t.\ \ p_t c_t^y + m_t^d + p_t s_{t+1} = W_t$$

$$p_{t+1} c_{t+1}^o = p_{t+1}(1-\delta)s_{t+1} + q_{t+1}s_{t+1} + m_t^d$$

$$\Rightarrow c_t^y + \frac{m_t^d}{p_t} + s_{t+1} = w_t \qquad (1)$$

$$c_{t+1}^o = (1-\delta+r_{t+1})s_{t+1} + \frac{m_t^d}{p_t}\frac{p_t}{p_{t+1}} \qquad (2)$$

where $w_t = \frac{W_t}{p_t}$, $r_{t+1} = \frac{q_{t+1}}{p_{t+1}}$

How to write lifetime budget constraint?
Substitute out s_{t+1} or m_t^d?

2. Problem Set 2 - 2

Since m_t^d is in the objective function (i.e. $v(\frac{m_t^d}{p_t})$), if you substitute out m_t^d when writing LBC, you also need to substitute out m_t^d in the utility function!

But this route will make your math harder. It is easier to build an LBC by substituting out s_{t+1}.

$$\mathscr{L} = u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$
$$+ \lambda\{w_t - c_t^y - \frac{m_t^d}{p_t} - \frac{c_{t+1}^o}{1-\delta+r_{t+1}} + \frac{m_t^d}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+r_{t+1}}\}$$

2. Problem Set 2 - 2

$$[c_t^y]: \ u'(c_t^y) = \lambda$$

$$[c_{t+1}^o]: \ \beta u'(c_{t+1}^o) = \lambda \frac{1}{1-\delta+r_{t+1}}$$

$$[m_t^d]: \ v'(\frac{m_t^d}{p_t})\frac{1}{p_t} = \lambda(\frac{1}{p_t} - \frac{1}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+r_{t+1}})$$

$\Rightarrow$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+r_{t+1}) \quad (3)$$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}} + v'(\frac{m_t^d}{p_t}) \quad (4)$$

Household behavior is characterized by equ. (1), (2), (3), (4)

2. Problem Set 2 - 2

- $u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+r_{t+1})$:
  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of saving in period t. This yields $(1-\delta+r_{t+1})$ income gain in the next period, valued by $u'(c_{t+1}^o)$ and discounted by β.

- $u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}} + v'(\frac{m_t^d}{p_t})$
  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of real money in period t. The benefit comes from two parts: direct utility and discounted marginal value of more future consumption.

2. Problem Set 2 - 2

From equ. (3) and (4), we obtain

$$v'(\frac{m_t^d}{p_t}) = \beta u'(c_{t+1}^o)(1 - \delta + r_{t+1} - \frac{p_t}{p_{t+1}})$$

Since $u'(\cdot) > 0$, $v'(\cdot) > 0$, in order to have this equation to hold, it must be the case that

$$1 - \delta + r_{t+1} > \frac{p_t}{p_{t+1}}$$

$\Rightarrow$ Rate of return dominance

2. Problem Set 2 - 2

2. Solve the firm's problem.

Answer key:

$$max \quad p_t F(K_t, L_t) - W_t L_t - q_t K_t$$
$$r_t = \frac{q_t}{p_t} = F_1(K_t, L_t)$$
$$w_t = \frac{W_t}{p_t} = F_2(K_t, L_t)$$

2. Problem Set 2 - 2

3. Define a competitive equilibrium.

Answer key:

Allocations $\{c_t^y, c_t^o, s_{t+1}, m_t^d, K_t, L_t\}$ and prices $\{p_t, w_t, r_t\}$ that satisfy

- Household: Euler Equation (2), Budget Constraint (2)
- Firm: F.O.C (2)
- Market Clearing Conditions:
  1.Goods market: $F(K_t, L_t) = Nc_t^y + Nc_t^o + K_{t+1} - (1-\delta)K_t$
  2.Capital market: $Ns_{t+1} = K_{t+1}$
  3.Labor market: $L_t = N$
  4.Money market: $M = Nm_t^d$

4. Assume that the utility functions u and v are logarithmic. Solve in closed form for the household's money demand function, $\frac{m_t^d}{p_t} = \varphi(w_t, r_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, r_{t+1}, \pi_{t+1})$, where $\pi_{t+1} = \frac{p_{t+1}}{p_t}$

Answer key:
Combing equ. (3), (4) and two BCs, we can obtain

$$\frac{m_t^d}{p_t} = \frac{\pi_{t+1}(1-\delta+r_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+r_{t+1})-1]}$$

$$s_{t+1} = \{\frac{\beta}{\beta+2} - \frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+r_{t+1})-1]}\}w_t$$

2. Problem Set 2 - 2

From equ. (3), we can obtain

$$\frac{1}{c_t^y} = (1-\delta+r_{t+1})\beta\frac{1}{c_{t+1}^o}$$
$$\rightarrow \; c_{t+1}^o = (1-\delta+r_{t+1})\beta c_t^y$$

From equ. (4), we can obtain

$$\frac{1}{c_t^y} = \frac{1}{\pi_{t+1}}\beta\frac{1}{c_{t+1}^o} + \frac{1}{m_t}$$
$$\rightarrow \; c_{t+1}^o = \beta c_t^y m_t \frac{1}{\pi_{t+1}(m_t - c_t^y)}$$

where $m_t \equiv \frac{m_t^d}{p_t}$

2. Problem Set 2 - 2

$$c^o_{t+1} = (1-\delta+r_{t+1})\beta c^y_t$$
$$c^o_{t+1} = \beta c^y_t m_t \frac{1}{\pi_{t+1}(m_t - c^y_t)}$$

$\Rightarrow$

$$c^y_t = \frac{[(1-\delta+r_{t+1})\pi_{t+1} - 1]m_t}{(1-\delta+r_{t+1})\pi_{t+1}}$$

Substitute this equation into B.C.

$$c^y_t + s_{t+1} + m_t = w_t$$
$$(1-\delta+r_{t+1})\beta c^y_t = \frac{m_t}{\pi_{t+1}} + (1-\delta+r_{t+1})s_{t+1}$$

$$\frac{[(1-\delta+r_{t+1})\pi_{t+1}-1]m_t}{(1-\delta+r_{t+1})\pi_{t+1}}+s_{t+1}+m_t=w_t$$
$$\beta\frac{[(1-\delta+r_{t+1})\pi_{t+1}-1]m_t}{\pi_{t+1}}=\frac{m_t}{\pi_{t+1}}+(1-\delta+r_{t+1})s_{t+1}$$

Two equations, two unknowns (m_t, s_{t+1})

$$\frac{m_t^d}{p_t}=\frac{\pi_{t+1}(1-\delta+r_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+r_{t+1})-1]}$$
$$s_{t+1}=\{\frac{\beta}{\beta+2}-\frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+r_{t+1})-1]}\}w_t$$