Econ720 - TA Session 5

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1. Problem Set 2 - 1

1. Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period, $N_t = (1+n)^t$ persons are born. Each lives for 2 periods. Half of the agents are of type 1, the other half of type 2.

Endowments: The initial old hold M_0 units of money, evenly distributed across agents. Each person is endowed with (e_i^y, e_i^o) units of consumption when young and old, respectively.

Preferences: $ln(c_t^y) + \beta ln(c_{t+1}^o)$

Technology: Goods can only be eaten the day they drop from the sky.
→No savings in the form of goods
→Save by holding money

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Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_t = M_{t-1} + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu)M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

Question

1. Define a competitive equilibrium.

Answer key:

- How many sectors? $\rightarrow$ H.H. and Gov.

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$$
\begin{aligned}
&max \ ln(c^y_{i,t}) + \beta ln(c^o_{i,t+1}) \\
&s.t. \ \ p_t c^y_{i,t} + p_t m^d_{i,t} = p_t e^y_i \\
&\qquad p_{t+1} c^o_{i,t+1} = p_{t+1} e^o_i + p_t m^d_{i,t} + p_{t+1} x_{t+1} \\
&\mathscr{L} = ln(c^y_{i,t}) + \beta ln(c^o_{i,t+1}) \\
&\qquad + \lambda_t \{ e^y_i + \frac{p_{t+1}}{p_t}(e^o_i + x_{t+1}) - c^y_{i,t} - \frac{p_{t+1}}{p_t} c^o_{i,t+1} \}
\end{aligned}
$$

Notice that x_{t+1} is an **exogenous** variable for household!

1. Problem Set 2 - 1

$$[c^y_{i,t}]: \ \frac{1}{c^y_{i,t}} = \lambda_t$$

$$[c^o_{i,t+1}]: \ \beta \frac{1}{c^o_{i,t+1}} = \lambda_t \frac{p_{t+1}}{p_t}$$

$$\Rightarrow \frac{1}{c^y_{i,t}} = \beta \frac{1}{c^o_{i,t+1}} \frac{p_t}{p_{t+1}}$$

1. Problem Set 2 - 1

Competitive equilibrium:

Allocations $\{c^y_{i,t}, c^o_{i,t}, m^d_{i,t}, x_t, M_t\}$ and prices $\{p_t\}$ that satisfy

- Household: 1 Euler Equation, 2 Budget Constraint
- Government: $M_t = M_{t-1} + N_{t-1} x_t p_t,\ M_{t+1} = (1+\mu) M_t$
- Market Clearing Conditions:

  1.Goods market: $\frac{N_t}{2} c^y_{1,t} + \frac{N_t}{2} c^y_{2,t} + \frac{N_{t-1}}{2} c^o_{1,t} + \frac{N_{t-1}}{2} c^o_{2,t} = \frac{N_t}{2} e^y_1 + \frac{N_t}{2} e^y_2 + \frac{N_{t-1}}{2} e^o_1 + \frac{N_{t-1}}{2} e^o_2$

  2.Money market: $\frac{N_t}{2} m^d_{1,t} p_t + \frac{N_t}{2} m^d_{2,t} p_t = M_t$

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2. Derive the household consumption function.
   Tip: log-utility → consumption is a constant fraction of wealth.

Answer key:

From the lifetime budget constraint:

$$c_{i,t}^{y} + \frac{p_{t+1}}{p_t} c_{i,t+1}^{o} = e_i^y + \frac{p_{t+1}}{p_t}(e_i^o + x_{t+1})$$

Substitute E.E.

$$c_{i,t}^{y} + \frac{p_{t+1}}{p_t} \beta c_{i,t}^{y} \frac{p_t}{p_{t+1}} = e_i^y + \frac{p_{t+1}}{p_t}(e_i^o + x_{t+1})$$

Hence

$$c_{i,t}^{y} = \frac{1}{1+\beta}\left(e_i^y + \frac{e_i^o}{R_{t+1}} + \frac{x_{t+1}}{R_{t+1}}\right), \text{ where } R_{t+1} = \frac{p_t}{p_{t+1}}$$

$$c_{i,t+1}^{o} = \beta c_{i,t}^{y} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e_i^y + e_i^o + x_{t+1})$$

1. Problem Set 2 - 1

3. Derive a difference equation for the equilibrium interest rate when $\mu = 0$.

Logic: $\mu = 0 \rightarrow$ gov. doesn't add money to the economy $\rightarrow x_t = 0$

Answer key:

When $x_t = 0$

$$c_{i,t}^{y} = \frac{1}{1+\beta}(e_i^y + \frac{e_i^o}{R_{t+1}})$$

$$c_{i,t+1}^{o} = \beta c_{i,t}^{y} R_{t+1} = \frac{\beta}{1+\beta}(R_{t+1} e_i^y + e_i^o)$$

1. Problem Set 2 - 1

From goods market clearing condition:

$$\frac{N_t}{2}(c^y_{1,t}+c^y_{2,t})+\frac{N_{t-1}}{2}(c^o_{1,t}+c^o_{2,t})=\frac{N_t}{2}(e^y_1+e^y_2)+\frac{N_{t-1}}{2}(e^o_1+e^o_2)$$

Rearrange this equation by using $N_t=(1+n)^t,\ N_{t-1}=(1+n)^{t-1}$

$$\begin{aligned}
e^y_1+e^y_2+\frac{1}{n+1}(e^o_1+e^o_2) = & \quad c^y_{1,t}+c^y_{2,t}+\frac{1}{n+1}(c^o_{1,t}+c^o_{2,t}) \\
= & \quad \frac{1}{1+\beta}(e^y_1+\frac{e^o_1}{R_{t+1}}+e^y_2+\frac{e^o_2}{R_{t+1}}) \\
+ & \quad \frac{1}{n+1}\frac{\beta}{1+\beta}(R_te^y_1+e^o_1+R_te^y_2+e^o_2)
\end{aligned}$$

Difference equation of R: $\beta(1+n-R_t)(e^y_1+e^y_2)=\frac{1+n-R_{t+1}}{R_{t+1}}(e^o_1+e^o_2)$

1. Problem Set 2 - 1

4. Is the monetary steady state dynamically efficient?

Answer key:
In steady state, $m_{t+1} = m_t = \bar{m}$, where $m_t = \frac{M_t}{p_t N_t}$
Hence,

$$\frac{M_{t+1}}{p_{t+1} N_{t+1}} = \frac{M_t}{p_t N_t} \Rightarrow \frac{M_{t+1}}{M_t} = \frac{p_{t+1}}{p_t}\frac{N_{t+1}}{N_t} \Rightarrow 1 + \mu = \frac{1+n}{R_{t+1}}$$

1. Problem Set 2 - 1

In steady state,

$$R = \frac{1+n}{1+\mu}$$

- If $\mu > 0$, this monetary steady state is not dynamically efficient.
- If $\mu = 0$, this monetary steady state is dynamically efficient.

What is the intuition??

1. Money in the Utility Function in an OLG Model

Question:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household's first-order conditions imply rate of return dominance, i.e. the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium)

2. Problem Set 2 - 2

Answer key:

$$max\ u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$

$$s.t.\ \ p_t c_t^y + m_t^d + p_t s_{t+1} = p_t w_t$$

$$p_{t+1} c_{t+1}^o = p_{t+1}(1-\delta)s_{t+1} + p_{t+1} q_{t+1} s_{t+1} + m_t^d$$

$$\Rightarrow c_t^y + \frac{m_t^d}{p_t} + s_{t+1} = w_t \qquad (1)$$

$$c_{t+1}^o = (1 - \delta + q_{t+1}) s_{t+1} + \frac{m_t^d}{p_t} \frac{p_t}{p_{t+1}} \qquad (2)$$

Write lifetime budget constraint by substituting out s_{t+1}

2. Problem Set 2 - 2

$$\mathscr{L} = u(c_t^y) + \beta u(c_{t+1}^o) + v(\frac{m_t^d}{p_t})$$
$$+ \lambda_t \{ w_t - c_t^y - \frac{m_t^d}{p_t} - \frac{c_{t+1}^o}{1-\delta+q_{t+1}} + \frac{m_t^d}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+q_{t+1}} \}$$

2. Problem Set 2 - 2

$$[c_t^y]: \; u'(c_t^y) = \lambda_t$$

$$[c_{t+1}^o]: \; \beta u'(c_{t+1}^o) = \lambda_t \frac{1}{1-\delta+q_{t+1}}$$

$$[m_t^d]: \; v'(\frac{m_t^d}{p_t})\frac{1}{p_t} = \lambda_t(\frac{1}{p_t} - \frac{1}{p_t}\frac{p_t}{p_{t+1}}\frac{1}{1-\delta+q_{t+1}})$$

$\Rightarrow$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+q_{t+1}) \quad (3)$$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}} + v'(\frac{m_t^d}{p_t}) \quad (4)$$

Household behavior is characterized by equ. (1), (2), (3), (4)

2. Problem Set 2 - 2

- $u'(c_t^y) = \beta u'(c_{t+1}^o)(1-\delta+q_{t+1})$:
  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of saving in period t. This yields $(1-\delta+r_{t+1})$ income gain in the next period, valued by $u'(c_{t+1}^o)$ and discounted by β.

- $u'(c_t^y) - v'(\frac{m_t^d}{p_t}) = \beta u'(c_{t+1}^o)\frac{p_t}{p_{t+1}}$
  $u'(c_t^y)$ is the marginal utility cost of increasing 1 unit of real money in period t. But this cost is mitigated by utility gain from holding an extra unit of real money $v'(\frac{m_t^d}{p_t})$. So the net change in utility today is $u'(c_t^y) - v'(\frac{m_t^d}{p_t})$. Hold one unit of real money today yields $\frac{p_t}{p_{t+1}}$ gain tomorrow, valued by $u'(c_{t+1}^o)$ and discounted by β.

2. Problem Set 2 - 2

From equ. (3) and (4), we obtain

$$v'(\frac{m_t^d}{p_t}) = \beta u'(c_{t+1}^o)(1 - \delta + q_{t+1} - \frac{p_t}{p_{t+1}})$$

Since $u'(\cdot) > 0$, $v'(\cdot) > 0$, in order to have this equation to hold, it must be the case that

$$1 - \delta + q_{t+1} > \frac{p_t}{p_{t+1}}$$

$\Rightarrow$ Rate of return dominance

2. Problem Set 2 - 2

2. Solve the firm's problem.

Answer key:

$$
\begin{aligned}
&max \quad F(K_t, L_t) - w_t L_t - q_t K_t \\
&q_t = F_1(K_t, L_t) \\
&w_t = F_2(K_t, L_t)
\end{aligned}
$$

2. Problem Set 2 - 2

3. Define a competitive equilibrium.

Answer key:

Allocations $\{c_t^y, c_t^o, s_{t+1}, m_t^d, K_t, L_t\}$ and prices $\{p_t, w_t, q_t\}$ that satisfy

- Household: Euler Equation (2), Budget Constraint (2)
- Firm: F.O.C (2)
- Market Clearing Conditions:
  1.Goods market: $F(K_t, L_t) = Nc_t^y + Nc_t^o + K_{t+1} - (1-\delta)K_t$
  2.Capital market: $Ns_{t+1} = K_{t+1}$
  3.Labor market: $L_t = N$
  4.Money market: $M = Nm_t^d$

2. Problem Set 2 - 2

4. Assume that the utility functions u and v are logarithmic. Solve in closed form for the household's money demand function, $\frac{m_t^d}{p_t} = \varphi(w_t, q_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, q_{t+1}, \pi_{t+1})$, where $\pi_{t+1} = \frac{p_{t+1}}{p_t}$

Answer key:
Combing equ. (3), (4) and two BCs, we can obtain

$$\frac{m_t^d}{p_t} = \frac{\pi_{t+1}(1-\delta+q_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}$$

$$s_{t+1} = \{\frac{\beta}{\beta+2} - \frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}\}w_t$$

2. Problem Set 2 - 2

From equ. (3), we can obtain

$$\frac{1}{c_t^y} = (1-\delta+q_{t+1})\beta\frac{1}{c_{t+1}^o}$$
$$\rightarrow \; c_{t+1}^o = (1-\delta+q_{t+1})\beta c_t^y$$

From equ. (4), we can obtain

$$\frac{1}{c_t^y} = \frac{1}{\pi_{t+1}}\beta\frac{1}{c_{t+1}^o} + \frac{1}{m_t}$$
$$\rightarrow \; c_{t+1}^o = \beta c_t^y m_t \frac{1}{\pi_{t+1}(m_t - c_t^y)}$$

where $m_t \equiv \frac{m_t^d}{p_t}$

2. Problem Set 2 - 2

$$c^o_{t+1} = (1 - \delta + q_{t+1})\beta c^y_t$$

$$c^o_{t+1} = \beta c^y_t m_t \frac{1}{\pi_{t+1}(m_t - c^y_t)}$$

$\Rightarrow$

$$c^y_t = \frac{[(1 - \delta + q_{t+1})\pi_{t+1} - 1]m_t}{(1 - \delta + q_{t+1})\pi_{t+1}}$$

Substitute this equation into B.C.

$$c^y_t + s_{t+1} + m_t = w_t$$

$$(1 - \delta + q_{t+1})\beta c^y_t = \frac{m_t}{\pi_{t+1}} + (1 - \delta + q_{t+1})s_{t+1}$$

2. Problem Set 2 - 2

$$\frac{[(1-\delta+q_{t+1})\pi_{t+1}-1]m_t}{(1-\delta+q_{t+1})\pi_{t+1}}+s_{t+1}+m_t=w_t$$

$$\beta\frac{[(1-\delta+q_{t+1})\pi_{t+1}-1]m_t}{\pi_{t+1}}=\frac{m_t}{\pi_{t+1}}+(1-\delta+q_{t+1})s_{t+1}$$

Two equations, two unknowns $(m_t,\, s_{t+1})$

$$\frac{m_t^d}{p_t}=\frac{\pi_{t+1}(1-\delta+q_{t+1})w_t}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}$$

$$s_{t+1}=\{\frac{\beta}{\beta+2}-\frac{1}{(\beta+2)[\pi_{t+1}(1-\delta+q_{t+1})-1]}\}w_t$$