Econ720 - Problem Set 5

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2021. 11. 5

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1. Relative Wealth Preferences

Consider the following version of the growth model in continuous time.

Notation: $\bar{k}$: average capital in the economy.

Demographics: There is one representative household who lives forever.

Preferences:

$$\int_0^\infty e^{-\rho t}[u(c_t) + v(\frac{k_t}{\bar{k}_t})]dt$$

Endowments: The household starts with k_0.

Technology:

$$\dot{k}_t = f(k_t) - c_t$$

Government budget constraint: The government taxes consumption at rate τ_c and lump-sum rebates the revenues R_t to the household.

$$R_t = \tau_c c_t$$

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Markets: Goods (numeraire)

Household budget constraint:

$$\dot{k}_t = f(k_t) - (\tau_c + 1)c_t + R_t$$

Assumptions: u, v, f are strictly increasing and strictly concave.
$f'(0) = \infty$, $f'(\infty) = 0$

Questions:

1. State the household's current value Hamilton and derive the FOCs. Do not yet substitute out the co-state. Define a solution to the household problem.

$\bar{k}_t$ **is exogenously given in HH problem.**

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Answer key:

$$\mathscr{H} = u(c_t) + v(\frac{k_t}{\bar{k}_t}) + \lambda_t[f(k_t) - (\tau_c + 1)c_t + R_t]$$

$$[c_t]: \quad u'(c_t) = \lambda_t(\tau_c + 1)$$

$$[k_t]: \quad v'(\frac{k_t}{\bar{k}_t})\frac{1}{\bar{k}_t} + \lambda_t f'(k_t) = \rho\lambda_t - \dot{\lambda}_t$$

Solution to H.H. problem:

$\{c_t, k_t, \lambda_t\}$ that satisfy:

- FOC (2)
- BC
- Boundary conditions:
  k_0 is given
  TVC: $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$

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2. Define a competitive equilibrium.

Answer key:

Objectives $\{c_t, k_t, \lambda_t, \bar{k}_t, R_t\}$that satisfy

- Household: FOC (2), BC
- Government: $R_t = \tau_c c_t$
- Goods Market Clearing Conditions: $\dot{k}_t = f(k_t) - c_t$
- Identity: $k_t = \bar{k}_t$

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3. Derive an equation that implicitly solves for the steady state capital stock.

Answer key:

From the two FOCs,

$$v'(\frac{k_t}{\bar{k}_t})\frac{1}{\bar{k}_t} + \frac{u'(c_t)}{1+\tau_c}f'(k_t) = \rho\frac{u'(c_t)}{1+\tau_c} - \frac{u''(c_t)}{1+\tau_c}\dot{c}_t$$

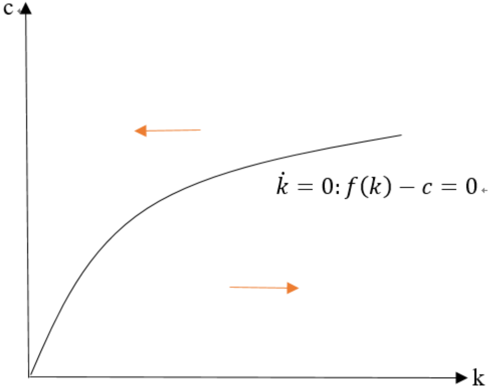
In steady state, $\dot{c}_t = 0$, $\dot{k}_t = 0$, and we can use the equilibrium condition $k_t = \bar{k}_t$

$$v'(1)\frac{1}{k} + \frac{u'(f(k))}{1+\tau_c}f'(k) = \rho\frac{u'(f(k))}{1+\tau_c}$$

4. Derive $\dot{k}_t = 0$ and discuss its shape.

Answer key:

$\dot{k}_t = 0$: $f(k) - c = 0$



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5. Derive $\dot{c}_t = 0$ and discuss its slope/ intercept. For which values of k does $\dot{c}_t = 0$ have a solution?

Answer key:

$\dot{c}_t = 0$: $v'(1)\frac{1}{k} + \frac{u'(c)}{1+\tau_c} f'(k) = \rho \frac{u'(c)}{1+\tau_c}$

- Slope: (Want to know $\frac{\partial c}{\partial k} < 0$ or > 0)

$$\mathscr{F} \equiv v'(1)\frac{1}{k} + \frac{u'(c)}{1+\tau_c}(f'(k) - \rho) = 0$$

Apply Implicit Function Theorem,

$$\frac{\partial \mathscr{F}}{\partial k} = -\frac{v'(1)}{k^2} + \frac{u'(c)}{1+\tau_c} f''(k) < 0$$

$$\frac{\partial \mathscr{F}}{\partial c} = \frac{u''(c)}{1+\tau_c}\underline{(f'(k) - \rho)}_{?}$$

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Due to $\dot{c}_t = 0$: $v'(1)\frac{1}{k} + \frac{u'(c)}{1+\tau_c} f'(k) = \rho \frac{u'(c)}{1+\tau_c}$

$$u'(c) = \frac{v'(1)(1+\tau_c)}{k(\rho - f'(k))}$$

Since u is strictly increasing and strictly concave, it must be that case that $u'(c) > 0$.
This is only defined for sufficiently high k, such that $\rho - f'(k) > 0$

Hence

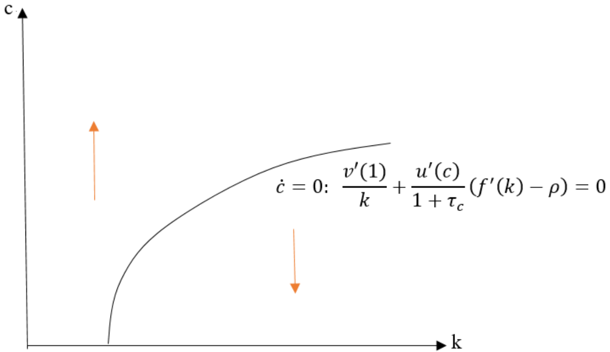
$$\frac{\partial \mathscr{F}}{\partial c} = \frac{u''(c)}{1+\tau_c}(f'(k) - \rho) > 0$$

$$\Rightarrow$$

$$\frac{\partial c}{\partial k} = -\frac{\partial \mathscr{F}/\partial k}{\partial \mathscr{F}/\partial c} > 0$$

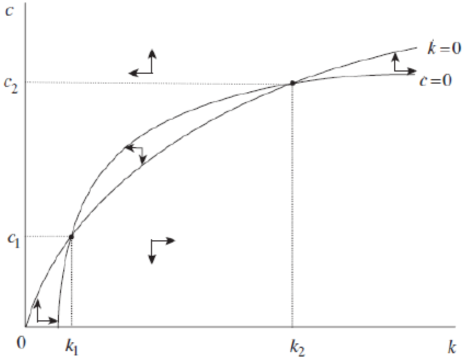
- Intercept:

  From the above analysis about $\dot{c}_t = 0$, we know that k must be sufficiently high, such that $\rho > f'(k)$. It implies that on $\dot{c}_t = 0$ curve, no matter what value c takes, the corresponding k must be some positive number, such that $\rho > f'(k)$.



6. Assume that $\dot{c}_t = 0$ is concave,
$$\frac{\partial^2 c}{\partial^2 k}\Big|_{\dot{c}=0} < 0$$
and that it intersects $\dot{k}_t = 0$ twice. Discuss the stability properties of the two steady states.

Answer key:



2. Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

Preferences: $\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt$
where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Technology: $f(k_t) - \delta k_t = c_t + \dot{k}_t$

Money: nominal money grows at exogenous rate $g(M)$. New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$

Market: money (numeraire), goods, capital rental (price r), labor (w)

Questions:

1. The household's budget constraint is given by
$$\dot{k}_t + \dot{m}_t = w_t + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t)$$
where $\phi(\dot{m}_t)$ is the cost of adjusting the money stock. $\phi'(0) = 0$ and $\phi''(\dot{m}_t) > 0$. State the Hamiltonian.

Answer Key:

Define a new variable z_t as $z_t = \dot{m}_t$

The budget constraint becomes:

$\dot{k}_t + z_t = w_t + r_t k_t + x_t - c_t - \pi_t m_t - \phi(z_t)$

Notice that we have two 'dot' equations, one is for k, one is for m

$\mathscr{H} = e^{-\rho t} u(c_t, m_t) + \lambda_t [w_t + r_t k_t + x_t - c_t - \pi_t m_t - \phi(z_t) - z_t] + \mu_t z_t$

Questions:

2. State the first-order conditions

Answer Key:

Two choice variables: c_t, z_t
Two state variables: k_t, m_t

$$[c_t]\ e^{-\rho t} u_c(c_t, m_t) = \lambda_t \tag{1}$$

$$[z_t]\ \lambda_t(\phi'(z_t) + 1) = \mu_t \tag{2}$$

$$[k_t]\ \lambda_t r_t = -\dot{\lambda}_t \tag{3}$$

$$[m_t]\ e^{-\rho t} u_m(c_t, m_t) - \lambda_t \pi_t = -\dot{\mu}_t \tag{4}$$

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Derive the EEs

From (1): take derivative w.r.t time t

$$-\rho e^{-\rho t} u_c + e^{-\rho t} u_{cc} \dot{c}_t + e^{-\rho t} u_{cm} \dot{m}_t = \dot{\lambda}_t \tag{5}$$

From (1), (3), (5)

$$-u_c(r_t - \rho) = u_{cc} \dot{c}_t + u_{cm} \dot{m}_t \tag{6}$$

From (2): take derivative w.r.t time t

$$\dot{\mu}_t = \dot{\lambda}_t(\phi'(z_t) + 1) + \lambda_t \phi''(z_t) \dot{z}_t \tag{7}$$

Plug (7), (1), (3) into (4)

$$\frac{u_m}{u_c} - \pi_t = r_t(g'(z_t) + 1) - \phi''(z_t) \dot{z}_t \tag{8}$$

Questions:

3. Define a competitive equilibrium
   The question says there are capital rental market and labor market. Hence there should be a firm that demands capital (K) and labor (L): $\max F(K,L) - (r+\delta)K - wL$

Answer key:

Objectives $\{c_t, z_t, k_t, m_t, \lambda_t, \mu_t, K_t, L_t, M_t, x_t, r_t, w_t, \pi_t\}$that satisfy

- Household: FOC (4), BC (2)
- Firm: standard conditions (2)
- Goods Market Clearing Condition: $\dot{k}_t = f(k_t) - \delta k_t - c_t$
- Money market: $M_t = m_t p_t$
- Capital Market Clearing Condition: $K_t = k_t$
- Labor Market Clearing: $L_t = 1$
- Money Growth: $g = \frac{\dot{M}_t}{M_t} = \frac{\dot{m}_t}{m_t} + \pi_t$
- $\dot{M}_t = p_t x_t$

Questions:

4. Characterize the steady state to the extent possible. What is the effect of a permanent change in $g(M)$?

Answer key:

- Alway start with $\dot{k}_t = 0$, $\dot{c}_t = 0$, $\dot{m}_t = 0$, $\dot{z}_t = 0$ etc.
  WARNING!!! It's dangerous to start with $\dot{\lambda}_t = 0$, $\dot{\mu}_t = 0$
- Equations with "dot"
  1. $f(k_t) - \delta k_t = c_t + \dot{k}_t$
  2. $\dot{m}_t = z_t$
  3. $\lambda_t r_t = -\dot{\lambda}_t$
  4. $e^{-\rho t} u_m(c_t, m_t) - \lambda_t \pi_t = -\dot{\mu}_t$
  5. $-u_c(r_t - \rho) = u_{cc}\dot{c}_t + u_{cm}\dot{m}_t$
  6. $\frac{u_m}{u_c} - \pi_t = r_t(\phi'(z_t) + 1) - \phi''(z_t)\dot{z}_t$

  Notice that I didn't include HH BC

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$\dot{k}_t = 0,\ \dot{c}_t = 0,\ \dot{m}_t = 0,\ \dot{z}_t = 0$

1. $f(k_t) - \delta k_t = c_t + \dot{k}_t \qquad f(k) - \delta k = c$
2. $\dot{m}_t = z_t \qquad z = 0$
3. $-u_c(r_t - \rho) = u_{cc}\dot{c}_t + u_{cm}\dot{m}_t \qquad -u_c(r - \rho) = 0 \to r = \rho$
4. $\frac{u_m}{u_c} - \pi_t = r_t(\phi'(z_t) + 1) - \phi''(z_t)\dot{z}_t \qquad \frac{u_m}{u_c} - \pi = r$

Get r (we already found $r = \rho$), get k (using firm FOC), get c (from the equation $f(k) - \delta k = c$). Hence $g(M)$ doesn't affect steady state capital and consumption. Money is super-neutral. It only affects steady state money holding by the following two equations

$$g(M) = \overbrace{\dot{m}_t / m_t}^{=0 \text{ in steady state}} + \pi_t$$

$$\frac{u_m}{u_c} - \pi = r$$

Notice that in this case $\dot{\lambda}_t \neq 0$

- FOC $\lambda_t r_t = -\dot{\lambda}_t$ implies that in steady state $\dot{\lambda}/\lambda = -r = -\rho$
- λ changes at a constant growth rate $-\rho$ in steady state
- But under setup of Current Value Hamiltonian, we can derive $\dot{\lambda}_t = 0$
- Where does the divergence come from?
  Hamiltonian: $H = e^{-\rho t} u(.) + \mu_t(...)$
  The costate variable μ_t represents the value of the state variable at time t in units of time zero levels of utility

Rule of thumb:
When characterizing steady state, always start with $\dot{k}_t = 0$, $\dot{c}_t = 0$, $\dot{m}_t = 0$, etc.

Questions:

5. What is the optimal rate of inflation? Explain

Logic:

Inflation comes from holding money.

→ What's the optimal money holding?

→ What's the money holding that can maximize utility?

→ $u_m = 0$

→ $u_m = (r + \pi)u_c = 0$, hence $\pi = -r$

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Questions:

5. What is the optimal rate of inflation? Explain

Answer key:

- $\pi = -r$ Friedman Rule
- Nominal interest rate is $i = r + \pi = 0$
- Money holding is costless