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3. *Positive Semi-definiteness:* Recall that a n-by-n real symmetric matrix K is called positive semi-definite (*p.s.d.* or $K \succeq 0$) iff for every $x \in \mathbb{R}^n$, $x^T K x \geq 0$.

(a) Show that $K \succeq 0$ if and only if its eigenvalues are all nonnegative.

$K \succeq 0 \iff \forall x \in \mathbb{R}^n,\ x^T K x \geq 0$

$\iff \forall x \in \mathbb{R}^n,\ x^T P \Lambda P^T x \geq 0$, where $P \Lambda P^T = K$, P is orthogonal matrix.

P full rank, $y = P^T x$

$\iff \forall y \in \mathbb{R}^n,\ y^T \Lambda y \geq 0$

$\iff \forall y \in \mathbb{R}^n,\ y_1^2 \lambda_1 + y_2^2 \lambda_2 + \cdots y_n^2 \lambda_n \geq 0$

$\iff \forall i \in [n],\ \lambda_i \geq 0$

(b) Show that $d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$ is a squared distance function, *i.e.* there exists vectors $u_i, v_j \in \mathbb{R}^n$ $(1 \leq i, j \leq n)$ such that $d_{ij} = \|u_i - u_j\|^2$.

$K \succeq 0 \Rightarrow K = P \Lambda P^T = P \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} P^T = (\Lambda^{\frac{1}{2}} P^T)^T \Lambda^{\frac{1}{2}} P^T$

Let $[u]_{ij} = U = \Lambda^{\frac{1}{2}} P^T$, $K = U^T U$

Thus $d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$

$= u_i^T u_i + u_j^T u_j - 2 u_i^T u_j$

$= \| u_i - u_j \|^2$

(c) Let $\alpha \in \mathbb{R}^n$ be a signed measure s.t. $\sum_i \alpha_i = 1$ (or $e^T \alpha = 1$) and $H_\alpha = I - e\alpha^T$ be the Householder centering matrix. Show that $B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T \succeq 0$ for matrix $D = [d_{ij}]$.

$D = k \cdot e^T + e \cdot k^T - 2K$, $k := \mathrm{diag}(K) \in \mathbb{R}^n$

$B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T = -\frac{1}{2} H_\alpha (k \cdot e^T + e \cdot k^T - 2K) H_\alpha^T$

Since $k \cdot e^T (I - \alpha \cdot e^T) = k \cdot e^T - k \cdot e^T \cdot \alpha \cdot e^T = k \cdot e^T - k e^T = 0$

Thus $B_\alpha = H_\alpha K H_\alpha^T$

$= H_\alpha U^T U H_\alpha^T$

$= (U H_\alpha^T)^T U H_\alpha^T \succeq 0$

(d) If $A \succeq 0$ and $B \succeq 0$ ($A, B \in \mathbb{R}^{n\times n}$), show that $A + B = [A_{ij} + B_{ij}]_{ij} \succeq 0$ (elementwise sum), and $A \circ B = [A_{ij}B_{ij}]_{ij} \succeq 0$ (Hadamard product or elementwise product).

① $A+B$

$A \succeq 0, B \succeq 0 \Rightarrow \forall x \in \mathbb{R}^n, \quad x^T A x \geq 0, \quad x^T B x \geq 0$

$\Rightarrow \forall x \in \mathbb{R}^n, \quad x^T (A+B) x \geq 0$

$\Leftrightarrow (A+B) \succeq 0$

② $A \circ B$

$\forall x \in \mathbb{R}^n, \quad x^T A \circ B x = tr(D_x A D_x B^T)$, $D_x = diag((x_1, x_2, \dots x_n))$

$= tr(D_x L_A L_A^T D_x^T L_B L_B^T)$, $A = L_A L_A^T$, $B = L_B L_B^T$

$= tr(L_B^T D_x L_A L_A^T D_x^T L_B)$

$= tr((L_A^T D_x^T L_B)^T (L_A^T D_x^T L_B))$

≥ 0 , since $(L_A^T D_x^T L_B)^T (L_A^T D_x^T L_B) \succeq 0$, eigenvalues ≥ 0.

Thus $A \circ B \succeq 0$

4. *Distance:* Suppose that $d : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is a distance function.

(a) Is d^2 a distance function? Prove or give a counter example.

(b) Is $\sqrt{d}$ a distance function? Prove or give a counter example.

(a)

No. Let $\|\cdot\|$ be a norm induced by inner product $\langle \cdot, \cdot \rangle$, $d(x, y) := \|x - y\|$

$d^2(x,y) = \|x-y\|^2 = \|x - z + z - y\|^2 = \|x-z\|^2 + \|z-y\|^2 + 2\langle x-z, z-y \rangle$

$= d^2(x,z) + d^2(z,y) + 2\langle x-z, z-y \rangle$

Let $p = 1$, $d(x,y) := |x-y|$

$x = 0$, $y = 1$, $z = \frac{1}{2}$

$d^2(x,y) = |0-1|^2 = 1$

$d^2(x,z) + d^2(z,y) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2} < d^2(x,y)$, which is not a distance.

(b)

Yes.

① $\sqrt{d} \geq 0$,

$\sqrt{d(x,y)} = 0 \iff d(x,y) = 0 \iff x = y \in \mathbb{R}^d$.

② $\sqrt{d(x,y)} = \sqrt{d(y,x)}$

③ $\sqrt{d(x,y)} \leq \sqrt{d(x,z) + d(z,y)} = \sqrt{(\sqrt{d(x,z)})^2 + (\sqrt{d(z,y)})^2} \leq \sqrt{d(x,z)} + \sqrt{d(z,y)}$