3. Positive semi-definition

(a) when all eigenvaleres are nonnegative,

we have $K = Q^T \Lambda Q$ where Λ is diagonal matrix with nonnegative terms.

so $x^T K x = x^T Q^T \Lambda Q x = (Qx)^T \Lambda (Qx)$

$= \sum_{i=1}^{n} \Lambda_{ii} \cdot (Qx)_i^2 \geq 0.$

On the other hand, when we know $K \succeq 0$,

we first concider eigenvector x and eigenvalue λ,

we have $Kx = \lambda x$, $x^T K x = x^T \lambda x = \lambda x^T x$.

Since we know for all vectors, we have

$x^T K x \geq 0$, we just let x be eigenvectors of K, we can see all the eigenvalues are non-negative,

(b) for K, we have $K = Q^T \Lambda Q$, let

$A = (\Lambda)^{\frac{1}{2}} Q$, we have $K = A^T A$.

Then, we can verify that u_i is the i-th row of A.

$(u_i - u_j)^2 = (u_i)^2 + (u_j)^2 - 2u_i u_j = k_{ii} + k_{jj} - 2k_{ij}$

(c) Since $d_{ij} = k_{ii} + k_{jj} - 2k_{ij}$

let $k = \mathrm{diag}(K)$

then $D = k\mathbb{1}^T + \mathbb{1}k^T - 2K$

So $x^T D x = 0 + 0 - 2x^T K x$

Since $x^T \mathbb{1} = 0$

So D is c.n.d.

by lemma 4.1, we have

$B_2 \succeq 0$

(d) $x^T (A+B) x = x^T A x + x^T B x \geq 0$

for all x.

then $A+B \succeq 0$

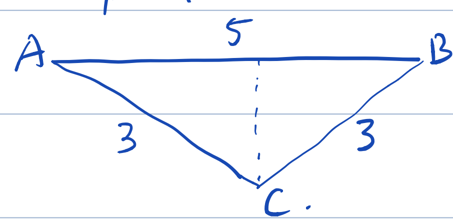
$x^T (A \circ B) x = x^T A x \circ x^T B x \geq 0$

for all x.

then $A \circ B \succeq 0$

4. Distance

(a) counter example:



$d^2(AB) = 25$. $d^2(AC) = 9$ $d^2(BC) = 9$

then $d^2(AB) > d^2(AC) + d^2(BC)$,

the triangular inequality fails, so d^2

is not a distance.

(b) it is a distance, let's check definition for distance.

(1) $\sqrt{d(x,y)} \geqslant 0$ ✓

(2) $\sqrt{d(x,x)} = 0$ ✓

(3) $\sqrt{d(x,y)} = \sqrt{d(y,x)}$ ✓

(4) $\sqrt{d(x,y)} \leq \sqrt{d(x,z) + d(z,y)} \leq \sqrt{d(x,z)} + \sqrt{d(z,y)}$ ✓