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3. *Positive Semi-definiteness:* Recall that a n-by-n real symmetric matrix K is called positive semi-definite ($p.s.d.$ or $K \succeq 0$) iff for every $x \in \mathbb{R}^n$, $x^T K x \geq 0$.

   (a) Show that $K \succeq 0$ if and only if its eigenvalues are all nonnegative.

   (b) Show that $d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$ is a squared distance function, $i.e.$ there exists vectors $u_i, v_j \in \mathbb{R}^n$ ($1 \leq i, j \leq n$) such that $d_{ij} = \|u_i - u_j\|^2$.

   (c) Let $\alpha \in \mathbb{R}^n$ be a signed measure s.t. $\sum_i \alpha_i = 1$ (or $e^T\alpha = 1$) and $H_\alpha = I - e\alpha^T$ be the Householder centering matrix. Show that $B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T \succeq 0$ for matrix $D = [d_{ij}]$.

   (d) If $A \succeq 0$ and $B \succeq 0$ ($A, B \in \mathbb{R}^{n\times n}$), show that $A + B = [A_{ij} + B_{ij}]_{ij} \succeq 0$ (elementwise sum), and $A \circ B = [A_{ij}B_{ij}]_{ij} \succeq 0$ (Hadamard product or elementwise product).

Proof:

(1) Since λ is an eigenvalue and v is the corresbounding eigenvector, we have

$$Kv = \lambda v$$

"$\Rightarrow$" $K \succeq 0$

$$v^T K v = v^T \lambda v$$
$$= \lambda v^T v \geq 0$$

Thus $\lambda \geq 0$

"$\Leftarrow$" If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

Denote $T = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$K = Q^T T Q$

$\forall x \in \mathbb{R}^n$. $x^T K x = (Qx)^T T (Qx)$ where $Qx = (p_1 \cdots p_n)^T$

$x^T K x = \sum_{i=1}^n \lambda_i p_i^2 \geq 0$

Thus, $K \succeq 0$

(2) Denote u_i as the i-th row of K

v_j as the n dimension vectors with only j-th element to be 1

Then,

$$\|u_i - v_j\|^2 = (u_i - v_j)(u_i - v_j)$$
$$= u_i^T v_i + u_j^T v_j - u_i^T v_j - u_i v_j^T$$
$$= K_{ii} + K_{jj} - 2K_{ij}$$
$$= d_{ij}$$

(3) Denote $k = \operatorname{diag}(K) \in \mathbb{R}^n$

$D = ke^T + ek^T - 2k$

Suppose $k = X^T X$

$$B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T$$
$$= -\frac{1}{2} H_\alpha (ke^T + ek^T - 2k) H_\alpha^T$$

$$H_\alpha K e^T H_\alpha^T = (I - e\alpha^T) K e^T (I - \alpha e^T)$$
$$= (I - e\alpha^T) K (e^T - (e^T\alpha) e^T)$$
$$= 0$$

Similarly.

$$H_\alpha e K^T H_\alpha^T = (e - e(\alpha^T e)) K H e^T = 0$$

Thus

$$B_\alpha = H_\alpha K H_\alpha^T$$

$\forall x \in \mathbb{R}^n$. $x^T B_\alpha x = x^T H_\alpha K H_\alpha^T x$

$$= (H_\alpha^T x)^T K (H_\alpha^T x)$$
$$\geq 0$$

Since K is p.s.d. B_α is p.s.d.

(4) $A \geq 0$. $B \geq 0$

① $\forall x \in \mathbb{R}^n$. $x^T (A+B) x = x^T A x + x^T B x \geq 0$

Thus $A + B \geq 0$

② $B \geq 0 \Rightarrow \exists T$ s.t. $B = TT^T$

$\forall x \in \mathbb{R}^n$. $x^T (A \circ B) x = x^T (A \circ TT^T) x$

$$= \sum_{i,j=1}^n x_i a_{ij} \left(\sum_{k=1}^n t_{ik} t_{jk} \right) x_j$$
$$= \sum_{k=1}^n (x * t_k)' A (x * t_k)$$
$$\geq \sum_{k=1}^n 0 = 0$$

where t_k is the k-th column of T

Thus $A \otimes B \geq 0$

$\square$

4. *Distance:* Suppose that $d : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is a distance function.

(a) Is d^2 a distance function? Prove or give a counter example.

(b) Is $\sqrt{d}$ a distance function? Prove or give a counter example.

(1) No.

Define $d(a,b) = |b-a|$

Obviously, d is a distance function.

Let $x=0$. $y=2$. $z=4$.

$d^2 = (d(a,b))^2 = |b-a|^2$

$d^2(x,z)^2 = 4^2 > d^2(x,y) + d^2(y,z) = 8$

Thus d^2 is not a distance function.

(2) Yes.

From Schoenberg transform we have $d^{\frac{1}{2}}$ is a distance function.

$\square$