8) a) $x_i \sim N(0, \sigma^2 I_p + \lambda_0 u u^T)$, $u \in \mathbb{R}^p$

$$S_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$$

$\gamma = p/n$ $SNR = \lambda_0/\sigma^2$

$$\lambda_{max}(S_n) \longrightarrow \begin{cases} (1+\sqrt{\gamma})^2 = b & \text{if } SNR \le \sqrt{\gamma} \\ (1+SNR)\left(1+\frac{\gamma}{SNR}\right)\sigma^2, & \text{if } SNR > \sqrt{\gamma} \end{cases}$$

From the derivatives in the slides, replacing σ_x^2 by SNR & $\hat{\lambda}$ by $\frac{\hat{\lambda}}{\sigma^2}$, it can be shown that for $SNR > \sqrt{\gamma}$

$$\lambda = \left(1+\frac{\lambda_0}{\sigma^2}\right)\left(1+\frac{\gamma\sigma^2}{\lambda_0}\right)\sigma^2$$

b) let $SNR = R$

from the sample covariance matrix S_n, calculate the maximum eigenvalue $\hat{\lambda}_{max}$

then solve $\hat{\lambda}_{max} = (1+R)\left(1+\frac{\gamma}{R}\right)\sigma^2$

for this, the noise covariance σ^2 should be known

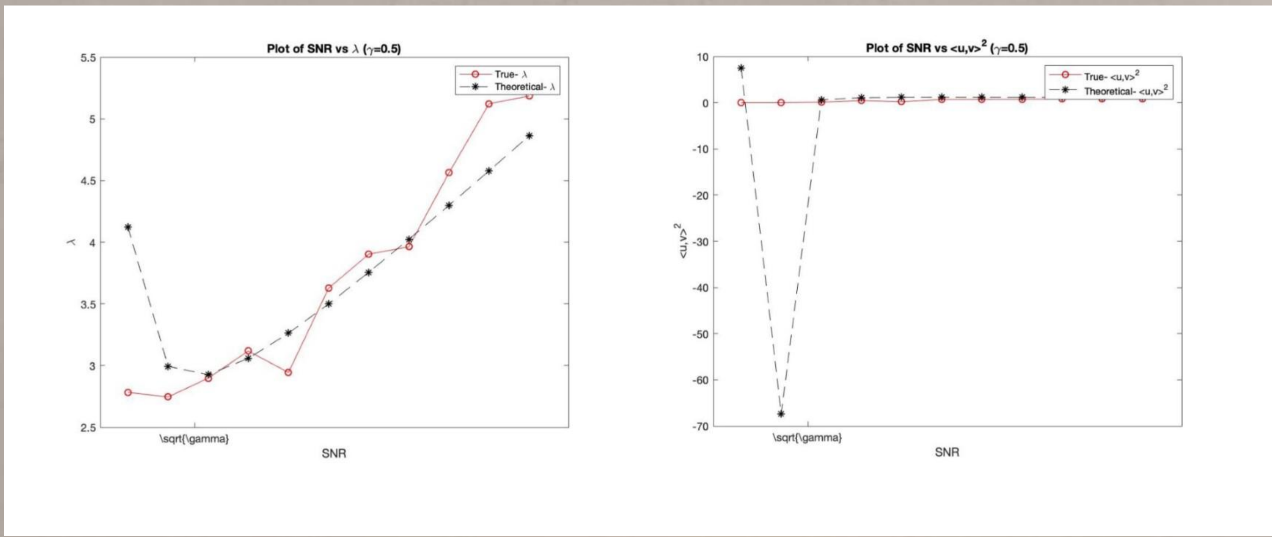
c) find $|\langle u, v\rangle|^2$: ~~from the~~

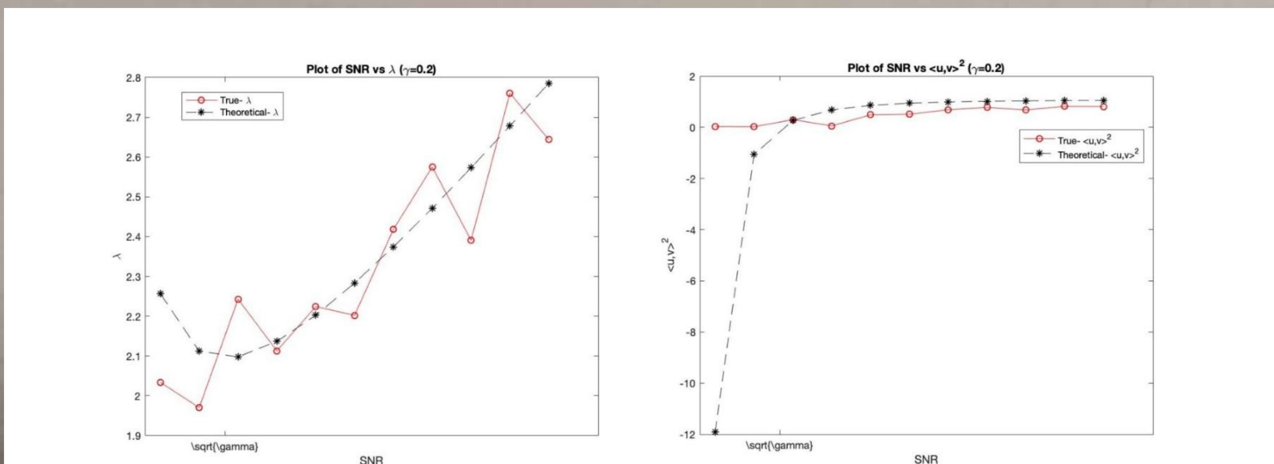
$$(u^T v)^2 = \frac{1-\frac{\gamma}{R^2}}{1-\frac{\gamma}{R}} = \frac{1-\frac{p}{nR^2}}{1-\frac{p}{nR}}$$

d) verification of the theoretical results using Matlab simulations

$u = c = [1, 0, 0 \ldots 0]$, $\sigma = 1$, $\sqrt{\gamma} - \gamma \le \lambda_0 \le \sqrt{\gamma} + 5\gamma$

$\gamma < 1$





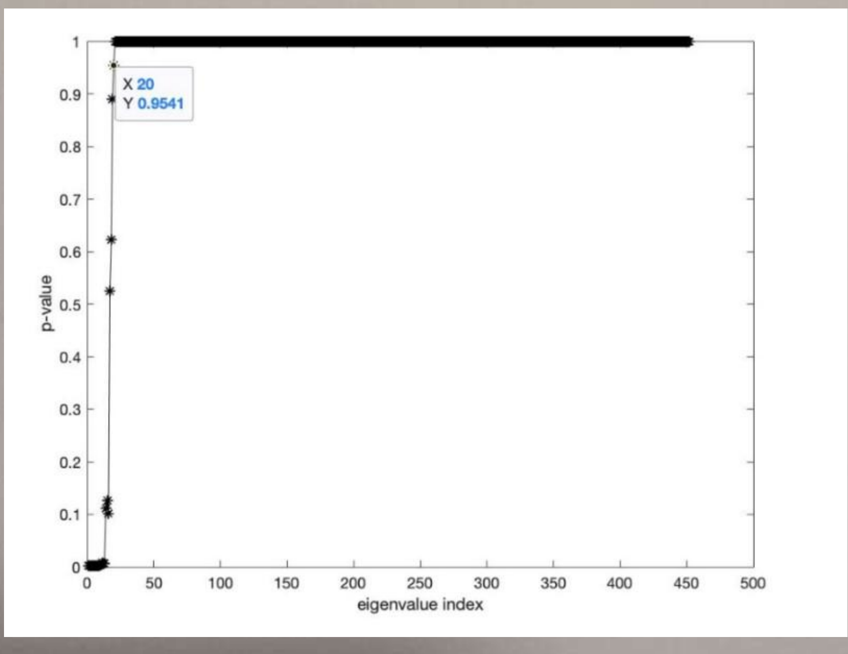
It is observed that when $SNR(\lambda_0) > \sqrt{\gamma}$, then the theoretical results max eigenvalue & $\langle u, v\rangle^2$ match to the simulated ones.

See the Matlab code attached for parts (a) - (e)

8)) 1)
e) Result of Horns Parallel Analysis

Low p value indicates that these eigen values come from true signal data, rather than from random noise in data.

The plot below shows that the top 20 eigen value convey useful regarding the stock market, the rest of the eigenvalue come from in stock market. Top 20 eigen values tell about the correlation the stocks.

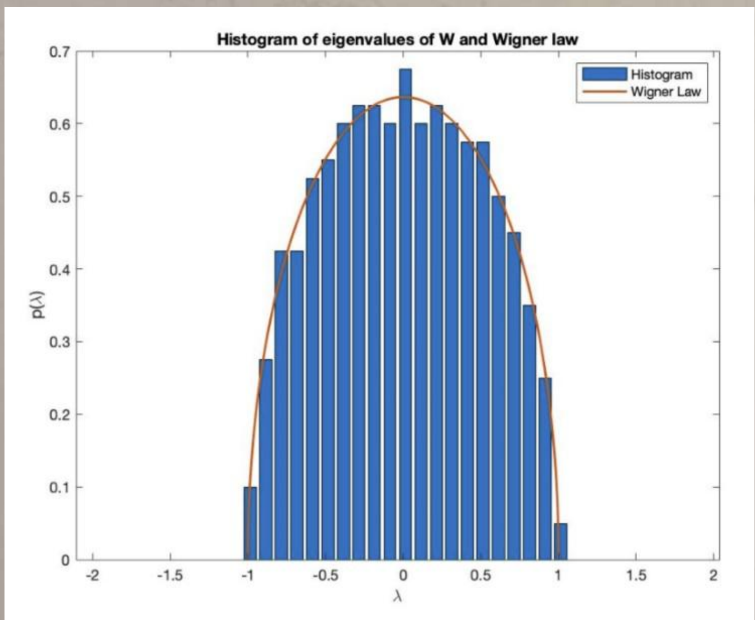


8) $W \to n \times n$ symmetric matrix , $w_{ij} = w_{ji}$

$w_{ij} \sim N(0, 1/4n)$

limiting eigenvalue distribution $\Rightarrow p(t) = \frac{2}{\pi}\sqrt{1-t^2}$

a) from Matlab simultions, we see that the theoretical eigenvalue distribution from wigner's semi circle law, perfectly aligns with simulated histogram the eigenvalues of W.



b) Using the results from Theorem 2.1 & Theorem 2.2, 2.3, It can be shown that for

$$W1 = W + \lambda_0 u u^T$$

$$\lambda(W1) \xrightarrow{a.s} \begin{cases} \lambda_0 + \dfrac{\sigma^2}{\lambda_0} & , \text{if } \lambda_0 > \sigma, \text{ where } w_{ij} \sim N \\ 2\sigma & , o.w \end{cases}$$

$$\langle u, v\rangle^2 \xrightarrow{a.s} \begin{cases} 1 - \dfrac{\sigma^2}{\lambda_0^2} & , \text{if } \lambda_0 \geq \sigma \\ 0 & \text{if } \lambda_0 < \sigma \end{cases}$$

The above results are also given in section 3.1 of [1]

It can be verified using Matlab simulations as shown below: The theoretical & simulated results agree when $\lambda_0 \geq \sigma$.