Q.2

$\ker \Delta = \ker A^T + \ker B$

Let $x \in \ker A^T \cap \ker B$, then $A^T x = Bx = 0 \Rightarrow \Delta x = 0$

$x \in \ker \Delta \Rightarrow \ker \Delta \supseteq \ker A^T \cap \ker B$

Suppose $x \in \ker \Delta$, then $\Delta x = 0$

Hence $\langle \Delta x, x \rangle = \langle AA^T x + B^T Bx, x \rangle$

$= \langle AA^T x, x \rangle + \langle B^T Bx, x \rangle$

$= \langle A^T x, A^T x \rangle + \langle Bx, Bx \rangle$

$= \|A^T x\|_2 + \|Bx\|_2$

$A^T x = Bx = 0 \quad \therefore \quad x \in \ker A^T \cap \ker B$

$\Rightarrow \ker \Delta \subseteq \ker A^T \cap \ker B$

$\mathrm{Im}\, A \oplus \ker \Delta \oplus \mathrm{Im}\, B \subseteq Y \quad -(1)$

Let $d \in \ker(\Delta)$, $A^T x = Bx = 0$

Assume that $y \in \mathrm{Im}\, A$, we write $y = Ax$ for $x \in X$

$y' \in \mathrm{Im}\, B$, we write $y' = B^T z$ for some $z \in Z$

$\ker(\Delta) \perp \mathrm{Im}\, A$,

$\langle d, y \rangle = \langle d, Ax \rangle = \langle A^T d, x \rangle = \langle 0, x \rangle = 0$

Similarly, we can prove $\ker \Delta \perp \mathrm{Im}\, B^T$ since

$\langle d, y' \rangle = \langle d, B^T z \rangle = \langle Bd, z \rangle = \langle 0, z \rangle = 0$

$\langle y, y' \rangle = \langle Ax, B^T z \rangle = \langle BAx, z \rangle$

$= \langle 0, z \rangle = 0$

$\operatorname{Im} A \perp \operatorname{Im} B^T$

$\operatorname{Im} A \oplus \ker \Delta \oplus \operatorname{Im} B \subseteq Y$

$\operatorname{im} A \oplus \ker \Delta \oplus \operatorname{im} B \supseteq Y$ — (2)

$\Delta : Y \to Y$ is self-adjoint operator.

Spectral theorem. Y can be decomposed into orthogonal direct sum of eigenspaces of Δ i.e.

$$Y = \bigoplus E_\Delta(\lambda) = \ker \Delta \oplus E_\Delta(\lambda)$$

$E_\Delta(\lambda) = \ker(\Delta - \lambda I)$ for each $x \in Y$ we can write

$x = x_0 + \sum_{\lambda \neq 0} x_\lambda$, $x_0 \in \ker \Delta$ and $x_\lambda \in E_\Delta(\lambda)$ $\lambda \neq 0$

$$\Delta x = \sum_\lambda \lambda x_\lambda \in \bigoplus E_\Delta(\lambda)$$

In other words $\operatorname{Im} \Delta \subseteq \bigoplus_{\lambda \neq 0} E_\Delta \lambda$

For each $x = x_\lambda \in E_\Delta(\lambda)$, $\Delta x = \lambda x$ which implies that

$x = \Delta(x/\lambda) \in \operatorname{Im} \Delta$ we find that $E_\Delta(\lambda) \subseteq \operatorname{Im} \Delta$ for $\lambda \neq 0$

Since $\operatorname{Im}\Delta$ is vector subspace of Y, $\bigoplus E_\Delta(\lambda) \subseteq \operatorname{Im}\Delta$

We can prove $\operatorname{Im}\Delta = \bigoplus_{\lambda \neq 0} E_\Delta(\lambda)$

$Y = \ker\Delta \oplus \operatorname{Im}\Delta$

For each $x \in Y$, $x = x_1 + x_2$, $x_1 \in \ker\Delta$ & $x_2 \in \operatorname{Im}\Delta$

By definition, choose $y \in Y$ so that

$$x_2 = \Delta y = AA^T y + B^T B y \in \operatorname{Im} A^T \oplus \operatorname{Im} B$$

$x \in \ker\Delta \oplus \operatorname{Im} A^T \oplus \operatorname{Im} B$ for $x \in Y$

We can find $Y \subseteq \ker\Delta \oplus \operatorname{Im} A^T \oplus \operatorname{Im} B$

From equation (1) and (2)

$$\boxed{Y = \operatorname{im} A + \ker\Delta + \operatorname{im} B}$$

3 <u>Hodge Decomposition</u> of <u>Prisoners Dilemma</u>

(C, C) → 2 → (C, D)

(C, C) → 2 → (D, C)

(C, D) → 1 → (D, D)

(D, C) → 1 → (D, D)

Quadrangular curl free ⇒ Potential game

(~~Attack~~, Expand) → 12 → (~~Attack~~, Stop)

(Quit, expand) → 3 → (Quit, stop)

(Attack, quit) → 21 → (Quit, stop)

(Attack, expand) → 12 → (Quit, stop)

Potential game