MATH5473 Homework 7

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April 5, 2023

2. (a) If $A\nu^* \neq \lambda^*\nu^*$, then for some i, $[A\nu^*]_i > \lambda^*\nu_i^*$. Below we will find an increase of λ^*, which is thus not optimal. Define $\tilde{\nu} = \nu^* + \epsilon e_i$ with $\epsilon > 0$ and e_i denotes the vector which is one on the i^{th} component and zero otherwise. For those $j \neq i$

$$(A\tilde{\nu})_j = (A\nu^*)_j + \epsilon (Ae_i)_j = \lambda^*\nu_j^* + \epsilon A_{ji} > \lambda^*\nu_j^* = \lambda^*\tilde{\nu}_j$$

where the last inequality is due to $A > 0$. For those $j = i$

$$(A\tilde{\nu})_i = (A\nu^*)_i + \epsilon (Ae_i)_i > \lambda^*\nu_i^* + \epsilon A_{ii}.$$

Since $\lambda^*\tilde{\nu}_i = \lambda^*\nu_i^* + \epsilon\lambda^*$, we have

$$(A\tilde{\nu})_i - (\lambda^*\tilde{\nu})_i + \epsilon (A_{ii} - \lambda^*) = (A\nu^*)_i - (\lambda^*\nu_i^*) - \epsilon (\lambda^* - A_{ii}) > 0,$$

where the last inequality holds for small enough $\epsilon > 0$. That means, for some small $\epsilon > 0, (A\tilde{\nu}) > \lambda^*\tilde{\nu}$. Thus λ^* is not optimal, which leads to a contradiction.

(b) Assume on the contrary, for some $k, \nu_k^* = 0$, then $(A\nu^*)_k = \lambda^*\nu_k^* = 0$. But $A > 0, \nu^* \geq 0$ and $\nu^* \neq 0$, so there $\exists i, \nu_i^* > 0$, which implies that $A\nu^* > 0$. That contradicts to the previous conclusion. So $\nu^* > 0$, which followed by $\lambda^* > 0$ (otherwise $A\nu^* > 0 = \lambda^*\nu^* = A\nu^*$).

(3) We are going to show that for every $\nu \geq 0, A\nu = \mu\nu \Rightarrow \mu = \lambda^*$. Following the same reasoning above, A must have a left Perron vector $\omega^* > 0$, s.t. $A^T\omega^* = \lambda^*\omega^*$. Then $\lambda^* \left(\omega^{*T}\nu\right) = \omega^{*T}A\nu = \mu \left(\omega^{*T}\nu\right)$. Since $\omega^{*T}\nu > 0 \, (\omega^* > 0, \quad \nu \geq 0)$, there must be $\lambda^* = \mu$, i.e. λ^* is unique, and ν^* is unique.

(4) For any other eigenvalue $Az = \lambda z, A|z| \geq |Az| = |\lambda||z|$, so $|\lambda| \leq \lambda^*$. Then we prove that $|\lambda| < \lambda^*$. Before proceeding, we need the following lemma.

Lemma 1. $Az = \lambda z, |\lambda| = \lambda^*, z \neq 0 \quad \Rightarrow \quad A|z| = \lambda^*|z|. \quad \lambda^* = \max_i |\lambda_i(A)|$

Proof. Since $|\lambda| = \lambda^*$

$$A|z| = |A||z| \geq |Az| = |\lambda||z| = \lambda^*|z|$$

Assume that $\exists k, \quad \frac{1}{\lambda^*} A\,|z_k| > |z_k|$. Denote $Y = \frac{1}{\lambda^*} A|z| - |z| \geq 0$, then $Y_k > 0$. Using that $A > 0, x \geq 0, x \neq 0, \Rightarrow Ax > 0$, we can get

$$\Rightarrow \frac{1}{\lambda^*} AY > 0, \quad \frac{1}{\lambda^*} A|z| > 0$$

$$\Rightarrow \exists \epsilon > 0, \quad \frac{A}{\lambda^*} Y > \epsilon \frac{A}{\lambda^*} |z|$$

$$\Rightarrow \bar{A} Y > \epsilon \bar{A} |z|, \quad \bar{A} = \frac{A}{\lambda^*}$$

$$\Rightarrow \bar{A}^2 |z| - \bar{A}|z| > \epsilon \bar{A}|z|$$

$$\Rightarrow \frac{\bar{A}^2}{1+\epsilon} |z| > \bar{A}|z|$$

$$\Rightarrow B = \frac{\bar{A}}{1+\epsilon}, \quad 0 = \lim_{m \to \infty} B^m \bar{A}|z| \geq \bar{A}|z|$$

$$\Rightarrow \bar{A}|z| = 0 \quad |z| = 0 \quad \Rightarrow \quad Y = 0 \quad \Rightarrow \quad \bar{A}|z| = \lambda^* |z|$$

$\square$

Equipped with this lemma, assume that we have $Az = \lambda z (z \neq 0)$ with $|\lambda| = \lambda^*$, then

$$A|z| = \lambda^* |z| = |\lambda||z| = |Az| \Rightarrow \left| \sum_j \bar{a}_{ij} z_j \right| = \sum_j \bar{a}_{ij} \, |z_j| \, , \quad \bar{A} = \frac{A}{\lambda^*}$$

which implies that z_j has the same sign, i.e. $z_j \geq 0$ or $z_j \leq 0 (\forall j)$. In both cases $|z|$ $(z \neq 0)$ is a nonnegative eigenvector $A|z| = \lambda|z|$ which implies $\lambda = \lambda^*$ by (c).

3. (a) Intuitively, if the chain begins in the ith non-absorbing state, then it must (obviously) occupy the ith state for the one initial period. Further, for each of the $N(i,k)$ periods that the chain is expected to occupy the kth non-absorbing state, the chain transitions back to the ith non-absorbing state with probability $Q(k,i)$. Summing over all non-absorbing states, we obtain the total number of periods (in addition to the first) that the process is expected to occupy the ith non-absorbing state.

   (b) This equation is similar to (a), but omits the initial period since the chain did not begin in the jth non-absorbing state.

   (c) Rewriting (a) and (b) in matrix notation, we obtain $N = I + QN$, and hence $N = (I - Q)^{-1}$.

   (d) Conditioning on the first step we have that,

$$B(i) = P(i, n+1) + \sum_{k=1}^{n} P(i,k) B(k)$$

   Thus $B = R + QB$, that is, $(I - Q)B = R$. Therefore $B = (I - Q)^{-1} R = NR$.