Homework 9

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2. $B \circ A = 0 \Rightarrow A^T w = Bw = 0 \Rightarrow w \in \ker(AA^T + B^T B)$

$\Rightarrow w \in \ker(\Delta)$

$\langle Ax, y \rangle = \langle x, A^T y \rangle$

$\langle y, B^T z \rangle = \langle By, z \rangle$

For $\Delta = AA^T + B^T B$

$x \in \ker(A) \cap \ker(B^T) \Rightarrow Ax = B^T x = 0$

$\Rightarrow \Delta x = 0 \Rightarrow x \in \ker(\Delta)$

$\Rightarrow \ker(\Delta) \supseteq \ker(A^T) \cap \ker(B)$

Suppose $x \in \ker(\Delta) \Rightarrow \Delta x = 0$

$\langle \Delta x, x \rangle = \langle A \circ B^T x + B^T \circ A x, x \rangle$

$= \langle B^T x, B^T x \rangle + \langle Ax, Ax \rangle$

$= \|B^T x\|^2 + \|Ax\|^2$

$= 0$

$\Rightarrow \|B^T x\|^2 = \|Ax\|^2 = 0 \Rightarrow B^T x = Ax = 0$

$\Rightarrow x \in \ker(A^T) \cap \ker(B)$

$\Rightarrow \ker \Delta \subseteq \ker(A^T) \cap \ker(B)$

So $\ker \Delta = \ker(A^T) \cap \ker(B)$

Let $x \in \ker(\Delta)$ $y \in \mathrm{Im}(A)$

$\Rightarrow Ax = B^T x = 0$, $y = Az$

$\langle x, y \rangle = \langle x, Az \rangle = \langle B^T x, z \rangle = \langle 0, z \rangle = 0$

Assume $y' \in \mathrm{Im}(B)$, $y' = B^T z'$

$\langle x, y' \rangle = \langle x, B^T z' \rangle = \langle Ax, z' \rangle = \langle 0, z' \rangle = 0$

$\langle y, y' \rangle = \langle Az, B^T z' \rangle = \langle A \circ B z, z' \rangle = \langle 0, z' \rangle = 0$

$\Rightarrow$ $im(A) + ker(\Delta) + im(B^T) \subseteq Y$ ①

$Y = ker(\Phi) + Im(\Delta)$

for $x = x_1 + x_2$, $x_1 \in ker(\Delta)$, $x_2 \in im(\Delta)$

for $y \in Y$

$x_2 = \Delta y = A(B^T y) + (B^T(Ay)) \in Im(A) + Im(B)$

$\Rightarrow x \in ker(\Delta) + Im(A) + Im(B^T)$

$\Rightarrow Y \subseteq im(A) + ker(\Delta) + im(B)$ ②

①, ② $\Rightarrow$ $Y = im(A) + ker(\Delta) + im(B)$.