MATH5473 Homework 3

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4. When $p=1$, the expected value of $\frac{1}{\|Y\|^2}$ does not exist(the inverse chi square distribution doesn't have a mean for $p=1$). When $p=2$, SURE shows that $R\left(\widehat{\mu}^{\mathrm{JS}}, \mu\right)=R\left(\widehat{\mu}^{\mathrm{MLE}}, \mu\right)$.

   To prove the upper bound of the risk of James-Stein estimator, notice that for $Y \sim N(\mu, I), \|Y\|^2=\sum_{i=1}^p Y_i^2 \sim \chi^2\left(\|\mu\|^2, p\right)$ as noncentral χ^2 -distribution with non-centrality parameter $\|\mu\|^2$ and p degree of freedom, which can be viewed as Poisson-weighted mixture of central χ^2 -distributions. In fact, suppose that a random variable J has a Poisson distribution with mean $\|\mu\|^2/2$, and the conditional distribution of Z given $J=i$ is χ^2 with $p+2i$ degrees of freedom. Then the unconditional distribution of Z is non-central χ^2 with p degrees of freedom, and non-centrality parameter $\|\mu\|^2$, i.e.

$$\chi^2\left(\|\mu\|^2, p\right) \stackrel{d}{=} \chi^2(0, p+2J), \quad J \sim \text{Poisson }\left(\frac{\|\mu\|^2}{2}\right),$$

   we have

$$\begin{aligned}
\mathbb{E}_\mu\left(\frac{1}{\|Y\|^2}\right) &= \mathbb{E}_\mu\left[\frac{1}{\|Y\|^2} \mid J\right]=\mathbb{E} \frac{1}{p+2J-2} \\
&\geq \frac{1}{p+2\mathbb{E}J-2}=\frac{1}{p+\|\mu\|^2-2}
\end{aligned}$$

   where we use Jensen's Inequality. Hence

$$R\left(\widehat{\mu}^{\mathrm{JS}}, \mu\right) \leq p-\frac{(p-2)^2}{p-2+\|\mu\|^2}=2+\frac{(p-2)\|\mu\|^2}{p-2+\|\mu\|^2}$$