(a) Find λ given $SNR > \sqrt{\lambda}$. By XIA Wencan

suppose $t = \alpha u$, $\alpha \sim N(0, \lambda_0)$, where $\|u\| = 1$

Let $\varepsilon \sim N(0, \sigma^2 I_p)$, $x = t + \varepsilon \sim N(0, \Sigma)$, where $\Sigma = \sigma^2 I_p + \lambda_0 u u^T$

$x_i \sim N(0, \Sigma) \in \mathbb{R}^p$, $X = [x_1 | x_2 | \cdots | x_n] \in \mathbb{R}^{p \times n}$

$SNR = \lambda_0 / \sigma^2$, $S_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$

Let $y_i = \Sigma^{-\frac{1}{2}} x_i$, then $Y = [y_1 | y_2 | \cdots | y_n] = \Sigma^{-\frac{1}{2}} X \sim N(0, I_p)$

$T_n = \frac{1}{n} \sum_{i=1}^{n} y_i y_i^T = \frac{1}{n} Y Y^T$ is a Wishart Matrix.

So the limit dist of T_n's eigenvalues follow a MP dist.

$\because$ $T_n = \frac{1}{n} Y Y^T = \frac{1}{n} (\Sigma^{-\frac{1}{2}} X) \cdot (\Sigma^{-\frac{1}{2}} X)^T$

$= \Sigma^{-\frac{1}{2}} S_n \Sigma^{-\frac{1}{2}}$, thus $S_n = \Sigma^{\frac{1}{2}} T_n \Sigma^{\frac{1}{2}}$

and $T_n \Sigma (\Sigma^{-\frac{1}{2}} v) = \Sigma^{-\frac{1}{2}} \lambda v = \lambda (\Sigma^{-\frac{1}{2}} v)$. λ, v is eigenvalue of S_n

let $v^* = c(\Sigma^{-\frac{1}{2}} v)$ be the normalized eigenvalue of ΣT_n

By $T_n (\sigma^2 I_p + \lambda_0 u u^T) v^* = \lambda v^*$, we have

$u^T v^* = u^T (\lambda I_p - T_n \sigma^2 I_p)^{-1} \lambda_0 T_n u u^T v^*$.

if $u^T v^* \neq 0$, $1 = u^T (\lambda - T_n \sigma^2 I_p)^{-1} \lambda_0 T_n u \quad \cdots \quad (1)$

suppose $T_n = W \Lambda W^T$, $W W^T = I_p$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p)$, $\gamma = \lim_{p,n \to \infty} p/n$

$1 = \lambda_0 \sum_{i=1}^{p} u_i^2 \frac{\lambda_i}{\lambda - \sigma^2 \lambda_i}$, we get

$1 = \frac{\lambda_0}{4\gamma} [2\lambda - (a+b) - 2\sqrt{(\lambda - a)(b - \lambda)}]$ for $\lambda > (1 + \sqrt{\gamma})^2$ and $SNR > \sqrt{\lambda}$.

for $\sigma_\varepsilon^2 = 1$, $\lambda = \lambda_0 + \frac{\gamma}{\lambda_0} + 1 + \gamma = (1 + \lambda_0)(1 + \gamma/\lambda_0)$

So given $SNR > \sqrt{\lambda}$, $\lambda = (1 + \lambda_0)(1 + \gamma/\lambda_0)$

(b) we can estimate $SNR = \sigma_x^2 / \sigma_\varepsilon^2$. W.O.L.G., $\sigma_\varepsilon^2 = 1$.

let $S_n = \frac{1}{n} X X^T$ and $b = (1 + \sqrt{\gamma})^2$.

If $\lambda_{max}(S_n) = b$, then $SNR \in \sqrt{\gamma}$

if $\lambda_{max}(S_n) = (1 + \sigma_x^2)(1 + \gamma / \sigma_x^2)$, then $SNR > \sqrt{\gamma}$.

(c) By (1), $(u^T v^*)^T (u^T v^*) = \lambda_0^2 (u^T v^*)^T u^T T_n (\lambda I_p - T_n \sigma^2 I_p)^{-2} T_n u (u^T v^*)$

So $|u^T v^*|^{-2} = \frac{\lambda_0^2}{4\gamma} \left(-4\gamma + (a+b) + 2\sqrt{(\lambda - a)(\lambda - b)} + \frac{\lambda(2\lambda - (a+b))}{\sqrt{(\lambda - a)(\lambda - b)}} \right)$

since $R = SNR = \sigma_x^2/\sigma_\varepsilon^2 = \lambda_0/\sigma^2 > b = (1+\sqrt{\gamma})^2$

and $\lambda_{max} \to (1+R)(1+\gamma/R)$

Thus $|u^T v^*|^2 = \dfrac{1 - \gamma/R}{1+\sigma + 2\gamma/R}$.

(d) see "PhaseTrans. ipnb."

T2. see "HPA_SP500. ipnb"