1. (a). Let λ be the largest eigenvalue of the sample covariance matrix $S_n = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T$ and let v be the corresponding unit eigenvector.

Then, $S_n v = \lambda v$.

Let $\Sigma = \sigma^2 I_{p\times p} + \lambda_0 u u^T$, $\quad z_i = \Sigma^{-\frac{1}{2}} x_i$.

then $z_i \sim N(0, I_p)$,

and $P_n = \frac{1}{n}\sum_{i=1}^{n} z_i z_i^T = \frac{1}{n} Z Z^T$ is a Wishart random matrix whose eigenvalues follow the Marčenko-Pastur distribution.

$$S_n = \frac{1}{n} X X^T = \frac{1}{n}\Sigma^{\frac{1}{2}} Z Z^T \Sigma^{\frac{1}{2}} = \Sigma^{\frac{1}{2}} P_n \Sigma^{\frac{1}{2}}$$

$$\Rightarrow \Sigma^{\frac{1}{2}} P_n \Sigma^{\frac{1}{2}} v = \lambda v \Rightarrow P_n \Sigma (\Sigma^{-\frac{1}{2}} v) = \lambda (\Sigma^{-\frac{1}{2}} v)$$

So λ and $\Sigma^{-\frac{1}{2}} v$ are the eigenvalue and eigenvector of $P_n \Sigma$.

Suppose $c\Sigma^{-\frac{1}{2}} v = w$, where c makes w a unit eigenvector,

so $$c^2 = c^2 v^T v = w^T \Sigma w = w^T (\sigma^2 I_p + \lambda_0 u u^T) w$$
$$= \lambda_0 (u^T w)^2 + \sigma^2$$

and $P_n \Sigma w = \lambda w. \Rightarrow P_n(\lambda_0 u u^T + \sigma^2 I_p) w = \lambda w$

$$\Rightarrow (\lambda I_p - \sigma^2 P_n) w = \lambda_0 P_n u (u^T w)$$

If $\lambda I_p - \sigma^2 P_n$ is invertible,

then $$w = \lambda_0 (\lambda I_p - \sigma^2 P_n)^{-1} P_n u (u^T w) \quad (1)$$

$$u^T w = \lambda_0 u^T (\lambda I_p - \sigma^2 P_n)^{-1} P_n u (u^T w),$$

if $u^T w \neq 0$, $\quad 1 = \lambda_0 u^T (\lambda I_p - \sigma^2 P_n)^{-1} P_n u \quad (*)$

the eigenvalue decomposition of P_n is: $P_n = L \Lambda L^T$, $\Lambda = \mathrm{diag}(\lambda_i)$, λ_i in descending order,
$L^T L = L L^T = I_p$, $w = [l_1, \cdots, l_p] \in \mathbb{R}^{p\times p}$

Let $\alpha_i = l_i^T u$, $\alpha = [\alpha_1 \cdots \alpha_p]^T \in \mathbb{R}^{p\times 1}$,

then $u = \sum_{i=1}^{p} \alpha_i l_i = L^T \alpha$.

$$(*) \Rightarrow \quad 1 = \lambda_0 u^T [L(\lambda I_p - \sigma^2 \Lambda)^{-1} L^T](L \Lambda L^T) u$$
$$= \lambda_0 \alpha^T (\lambda I_p - \sigma^2 \Lambda)^{-1} \Lambda \alpha$$
$$= \lambda_0 \sum_{i=1}^{p} \frac{\lambda_i}{\lambda - \sigma^2 \lambda_i} \alpha_i^2, \quad \text{where } \sum_{i=1}^{p} \alpha_i^2 = 1 \quad (**)$$

Since L consists of a random orthonormal basis on a sphere, α_i will concentrate on its mean $\alpha_i = \frac{1}{\sqrt{p}}$.

For large p, $\lambda_i \sim \mu^{MP}(\lambda_i)$ can be thought sampled from μ^{MP}.

(**) can be considered as MC integration w.r.t. MP distribution.

$$1 = \lambda_0 \frac{1}{p}\sum_{i=1}^{p} \frac{\lambda_i}{\lambda - \sigma^2 \lambda_i} \sim \lambda_0 \int_a^b \frac{t}{\lambda - \sigma^2 t} \, d\mu^{MP}(t),$$

where $\mu^{MP}(t) = (1-\frac{1}{\gamma})\delta(t) I(\gamma>1) + \begin{cases} 0 & t \notin [a,b] \\ \frac{\sqrt{(b-t)(t-a)}}{2\pi\gamma t} dt & t\in[a,b] \end{cases}$.

WLOG, assume $\sigma^2 = 1$.

$$1 = \lambda_0 \int_a^b \frac{t}{\lambda - t} d\mu^{MP}(t) \ \underset{\text{transform}}{\overset{\text{stieltjes}}{=\!=}} \ \frac{\lambda_0}{4\gamma}\left[2\lambda - (a+b) - 2\sqrt{|(\lambda-a)(b-\lambda)|}\right]$$

If $SNR = \lambda_0 > \sqrt{\gamma}$,

1) if $\lambda \geq b$, then $1 = \frac{\lambda_0}{4\gamma}\left[2\lambda - (a+b) - 2\sqrt{(\lambda-a)(\lambda-b)}\right]$

and $a = (1-\sqrt{\gamma})^2$, $b = (1+\sqrt{\gamma})^2$

then $\lambda = \lambda_0 + 1 + \gamma + \frac{\gamma}{\lambda_0} = (1+\lambda_0)(1+\frac{\gamma}{\lambda_0})$

2) if $a \leq \lambda \leq b$, then S_n has its primary eigenvalue λ within supp (μ^{MP}), so it's undistinguishable from the noise P_n.

In case 1), for general $\sigma^2 \neq 1$, $\frac{\lambda}{\sigma^2} = (1+\frac{\lambda_0}{\sigma^2})(1+\frac{\gamma}{\lambda_0/\sigma^2})$

$\Rightarrow \lambda = (1+R)(1+\frac{\gamma}{R})\sigma^2$,

where $R = \lambda_0/\sigma^2$ is the SNR.

(b) Generate n i.i.d samples from the distribution, then find the largest eigenvalue λ of the sample covariance matrix, and then solve Eq: $\lambda = (1+R)(1+\frac{\gamma}{R})$,

R is the estimation of SNR

(c). From (1), $1 = w^T w = \lambda_0^2 (w^T u) u^T P_n (\lambda I_p - \sigma^2 P_n)^{-2} P_n u (u^T w)$

$= \lambda_0^2 |u^T w|^2 u^T P_n (\lambda I_p - \sigma^2 P_n)^{-2} P_n u$

$\Rightarrow |u^T w|^{-2} = \lambda_0^2 u^T P_n (\lambda I_p - \sigma^2 P_n)^{-2} P_n u$

(MC-integration) $\sim \lambda_0^2 \int_a^b \frac{t^2}{(\lambda - \sigma^2 t)^2} d\mu^{MP}(t)$ (WLOG, $\sigma^2 = 1$)

$$\underset{\text{stieltjes transform}}{=\!=\!=} \frac{\lambda_0^2}{4\gamma}\left(-4\lambda + (a+b) + 2\sqrt{(\lambda-a)(\lambda-b)} + \frac{\lambda(2\lambda-(a+b))}{\sqrt{(\lambda-a)(\lambda-b)}}\right)$$

Using $\lambda = (1+R)(1+\frac{\gamma}{R})$, then $|u^T w|^2 = \frac{1-\frac{\gamma}{R^2}}{1+\gamma+\frac{2\gamma}{R}}$.

$|u^T v|^2 = (\frac{1}{c} u^T \Sigma^{\frac{1}{2}} w)^2 = \frac{1}{c^2}\left(((R u u^T + I_p)^{\frac{1}{2}} u)^T w\right)^2$

$$(\Sigma^{\frac{1}{2}} u = \sqrt{1+R}\, u) \qquad = \frac{1}{C^2}\left(\left(\sqrt{(1+R)}\, u\right)^T w\right)^2$$

$$= \frac{(1+R)\,(u^T w)^2}{R(u^T w)^2+1} = \frac{1-\frac{\gamma}{R^2}}{1+\frac{\gamma}{R}} = |\langle u, v\rangle|^2$$