Robust Estimation
and
Generative Adversarial Networks

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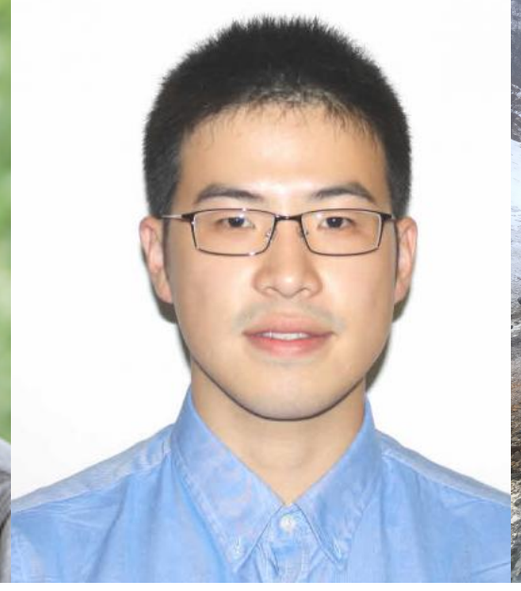

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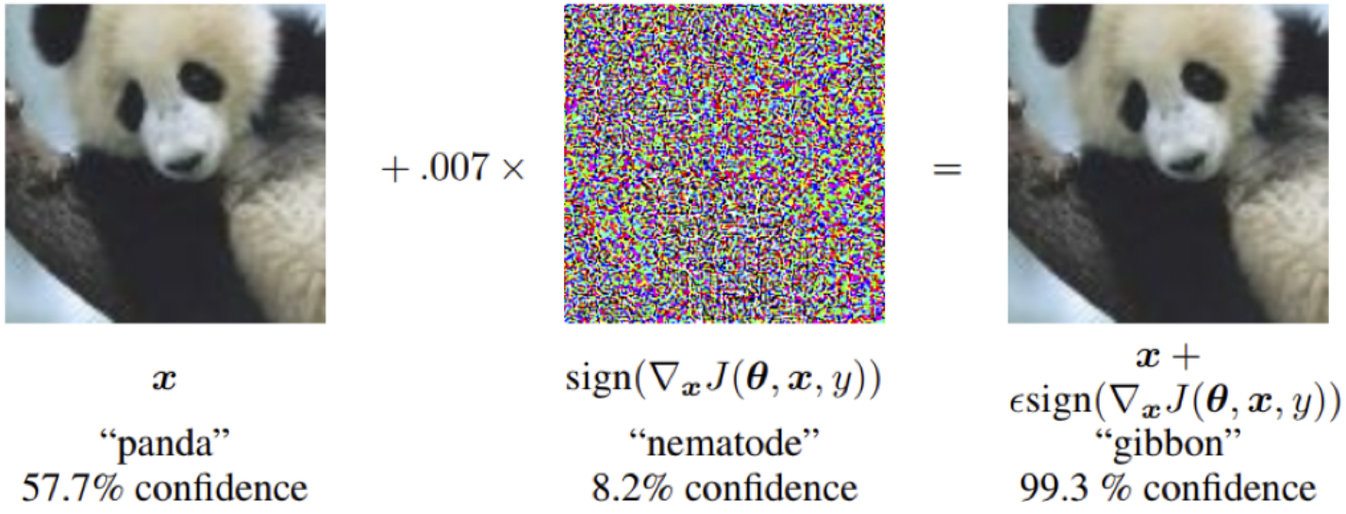
Deep Learning is Notoriously Not Robust!

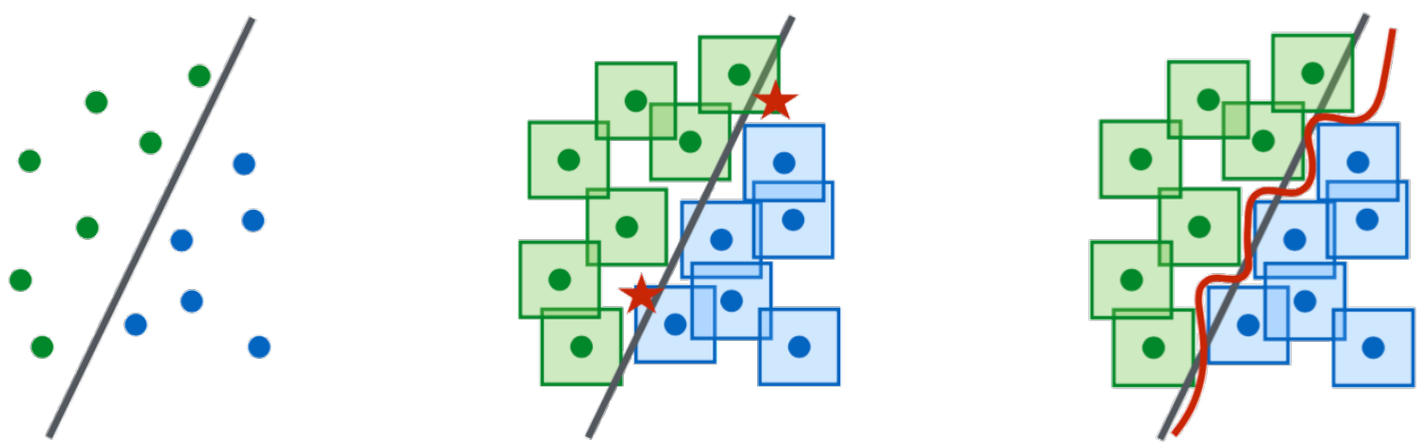


[Goodfellow et al., 2014]

- Imperceivable adversarial examples are ubiquitous to fail neural networks

- How can one achieve robustness?

Robust Optimization



- Traditional training:

$$\min_{\theta} J_n(\theta, \mathbf{z} = (x_i, y_i)_{i=1}^n)$$

  - e.g. square or cross-entropy loss as negative log-likelihood of logit models

- Robust optimization:

$$\min_{\theta} \max_{\|\epsilon_i\| \leq \delta} J_n(\theta, \mathbf{z} = (x_i + \epsilon_i, y_i)_{i=1}^n)$$

  - robust to any distributions, yet perhaps too conservative

Distributional Robust Optimization

- Distributional Robust Optimization:

$$\min_{\theta} \max_{\epsilon} \mathbb{E}_{\mathbf{z} \sim P_\epsilon \in \mathcal{D}}[J_n(\theta, \mathbf{z})]$$

  - $\mathcal{D}$ is a set of ambiguous distributions, e.g. Wasserstein ambiguity set
  - intermediate approach with statistically contaminated distributions
  - *sometimes, contamination might be unstructured...*

Huber's Model

$$X_1, ..., X_n \sim (1 - \epsilon)P_\theta + \epsilon Q$$

[Huber 1964]

Huber's Model

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parameter of interest

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$$X_1, ..., X_n \sim (1-\epsilon)P_\theta + \epsilon Q$$

contamination proportion

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Huber's Model

$$X_1, ..., X_n \sim (1-\epsilon)P_\theta + \epsilon Q$$

contamination proportion

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arbitrary contamination

[Huber 1964]

An Example

$$X_1, ..., X_n \sim (1-\epsilon)N(\theta, I_p) + \epsilon Q.$$

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how to estimate ?

Robust Maxmum-Likelihood Does not work!

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Robust Maxmum-Likelihood Does not work!

$$X_1, ..., X_n \sim (1-\epsilon)N(\theta, I_p) + \epsilon Q.$$

$$\ell(\theta, Q) = \text{negative log-likelihood} = \sum_{i=1}(\theta - X_i)^2$$

$$\sim (1-\epsilon)\mathbb{E}_{\mathcal{N}(\theta)}(\theta - X)^2 + \epsilon\mathbb{E}_Q(\theta - X)^2$$

the sample mean

$$\hat{\theta}_{mean} = \frac{1}{n}\sum_{i=1}^{n} X_i = \arg\min_{\theta} \ell(\theta, Q)$$

$$\min_{\theta}\max_{Q} \ell(\theta, Q) \geq \max_{Q}\min_{\theta} \ell(\theta, Q) = \max_{Q} \ell(\hat{\theta}_{mean}, Q) = \infty$$

Medians

1. **Coordinatewise median**

$$\hat{\theta} = (\hat{\theta}_j), \text{ where } \hat{\theta}_j = \text{Median}(\{X_{ij}\}_{i=1}^{n});$$

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2. **Tukey's median**

$$\hat{\theta} = \arg\max_{\eta \in \mathbb{R}^p} \min_{||u||=1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{u^T X_i > u^T \eta\}.$$

Comparisons

	Coordinatewise Median	Tukey's Median
breakdown point	$1/2$	$1/3$
statistical precision (no contamination)	$\frac{p}{n}$	$\frac{p}{n}$
statistical precision (with contamination)	$\frac{p}{n} + p\epsilon^2$	$\frac{p}{n} + \epsilon^2$: minimax [Chen-Gao-Ren'15]
computational complexity	Polynomial	NP-hard [Amenta et al. '00]

Note: R-package for Tukey median can not deal with more than 10 dimensions!
[https://github.com/ChenMengjie/DepthDescent]

Multivariate Location Depth

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{u^T X_i > u^T \eta\} \wedge \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{u^T X_i \leq u^T \eta\} \right\}$$

Multivariate Location Depth

$$\min_{\|u\|=1} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{u^T X_i > u^T \eta\} \wedge \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{u^T X_i \leq u^T \eta\} \right\}$$

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[Tukey, 1975]

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model $y|X \sim N(X^T\beta, \sigma^2)$

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$$\left\{ \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i(y_i - X_i^T\eta) > 0\} \wedge \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i(y_i - X_i^T\eta) \leq 0\} \right\}$$

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$$\hat{\beta} = \underset{\eta\in\mathbb{R}^p}{\operatorname{argmax}} \min_{u\in\mathbb{R}^p} \left\{ \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i(y_i - X_i^T\eta) > 0\} \wedge \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{u^T X_i(y_i - X_i^T\eta) \leq 0\} \right\}$$

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[Rousseeuw & Hubert, 1999]

Tukey's depth is not a special case of regression depth.

Multi-task Regression Depth

$$(X, Y) \in \mathbb{R}^p \times \mathbb{R}^m \sim \mathbb{P}$$

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population version:

$$\mathcal{D}_{\mathcal{U}}(B, \mathbb{P}) = \inf_{U \in \mathcal{U}} \mathbb{P}\left\{\left\langle U^T X, Y - B^T X \right\rangle \geq 0\right\}$$

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[Mizera, 2002]

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$p = 1, X = 1 \in \mathbb{R}$,

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$m = 1$,

$$\mathcal{D}_{\mathcal{U}}(\beta, \mathbb{P}) = \inf_{U \in \mathcal{U}} \mathbb{P}\left\{u^T X (y - \beta^T X) \geq 0\right\}$$

Multi-task Regression Depth

Proposition. For any $\delta > 0$,

$$\sup_{B \in \mathbb{R}^{p \times m}} |\mathcal{D}(B, \mathbb{P}_n) - \mathcal{D}(B, \mathbb{P})| \leq C\sqrt{\frac{pm}{n}} + \sqrt{\frac{\log(1/\delta)}{2n}},$$

with probability at least $1 - 2\delta$.

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Proposition.

$$\sup_{B,Q} |\mathcal{D}(B, (1 - \epsilon P_{B^*}) + \epsilon Q) - \mathcal{D}(B, P_{B^*})| \leq \epsilon$$

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Theorem [G17]. For some $C > 0$,

$$\mathsf{Tr}((\widehat{B}-B)^T\Sigma(\widehat{B}-B)) \le C\sigma^2\left(\frac{pm}{n} \vee \epsilon^2\right),$$

$$\|\widehat{B}-B\|_{\mathrm{F}}^2 \le C\frac{\sigma^2}{\kappa^2}\left(\frac{pm}{n} \vee \epsilon^2\right),$$

with high probability uniformly over B, Q.

Covariance Matrix

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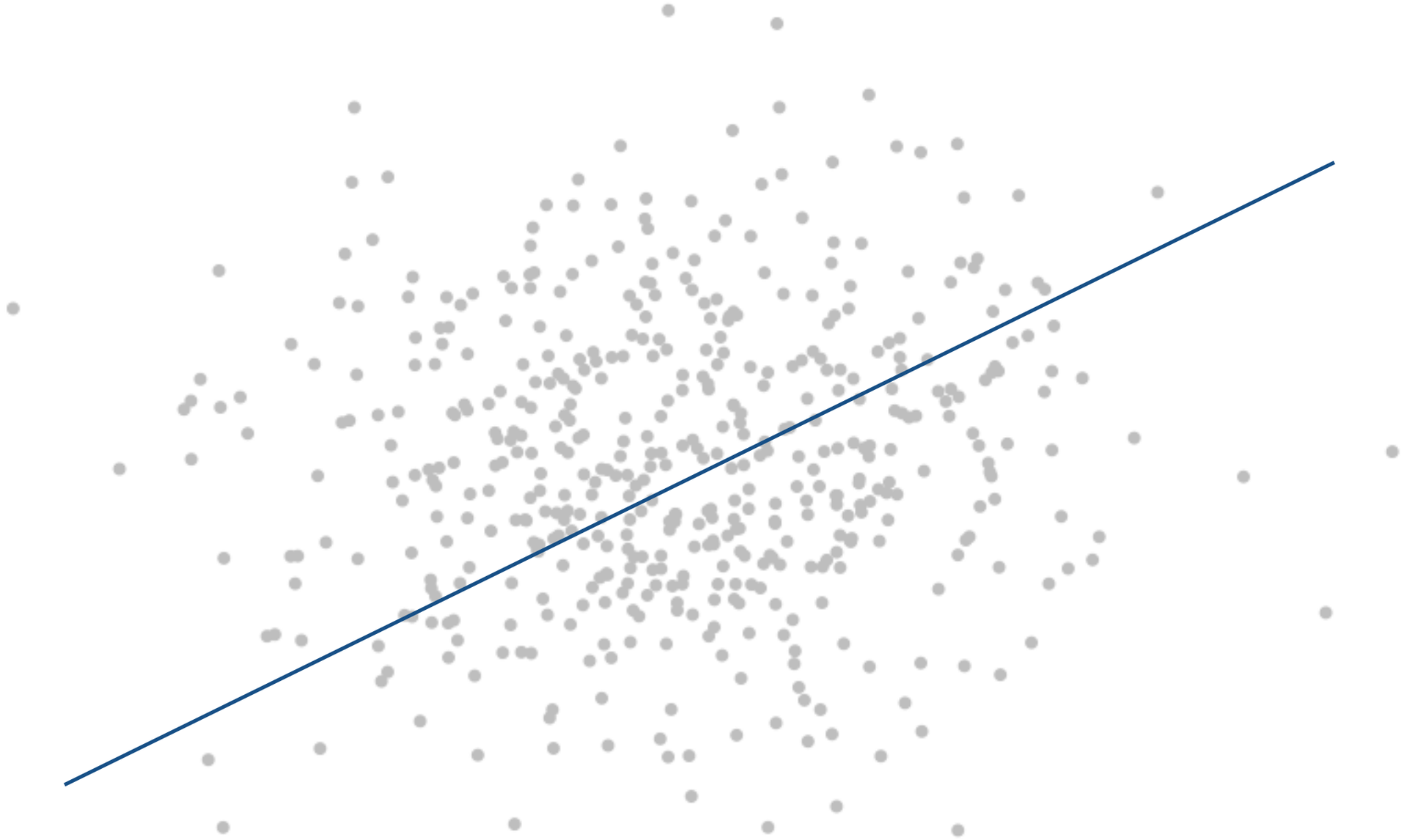

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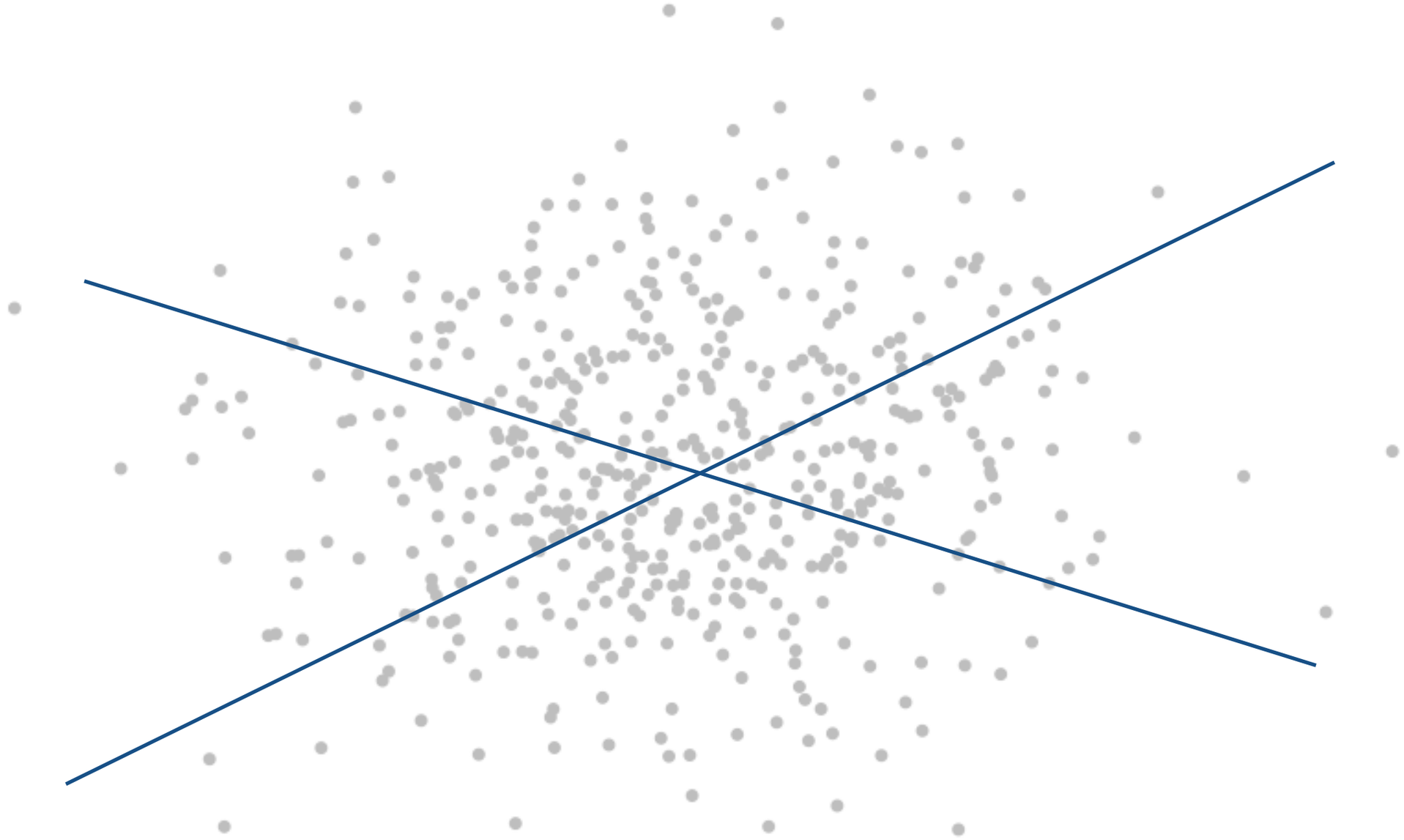

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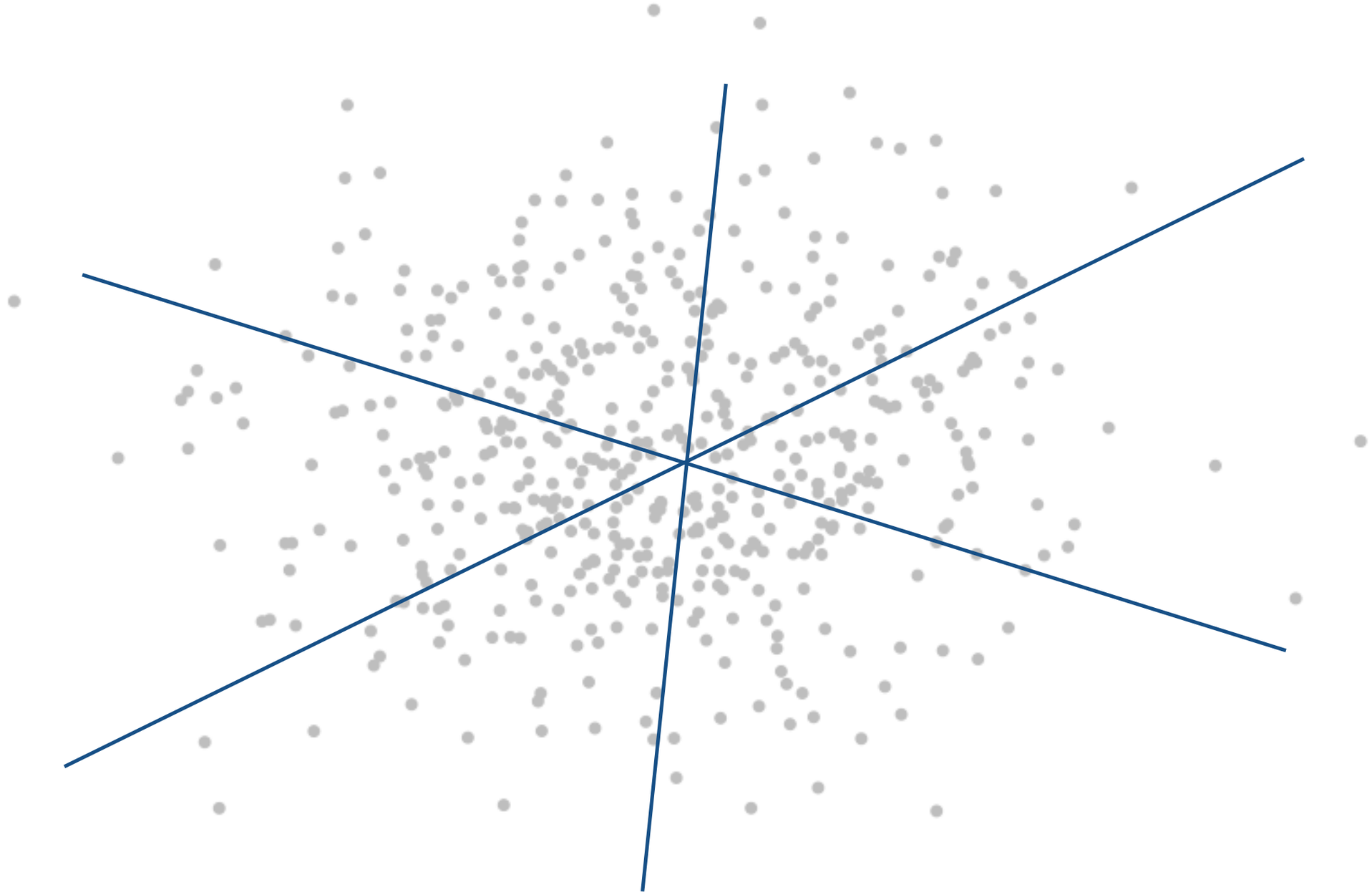

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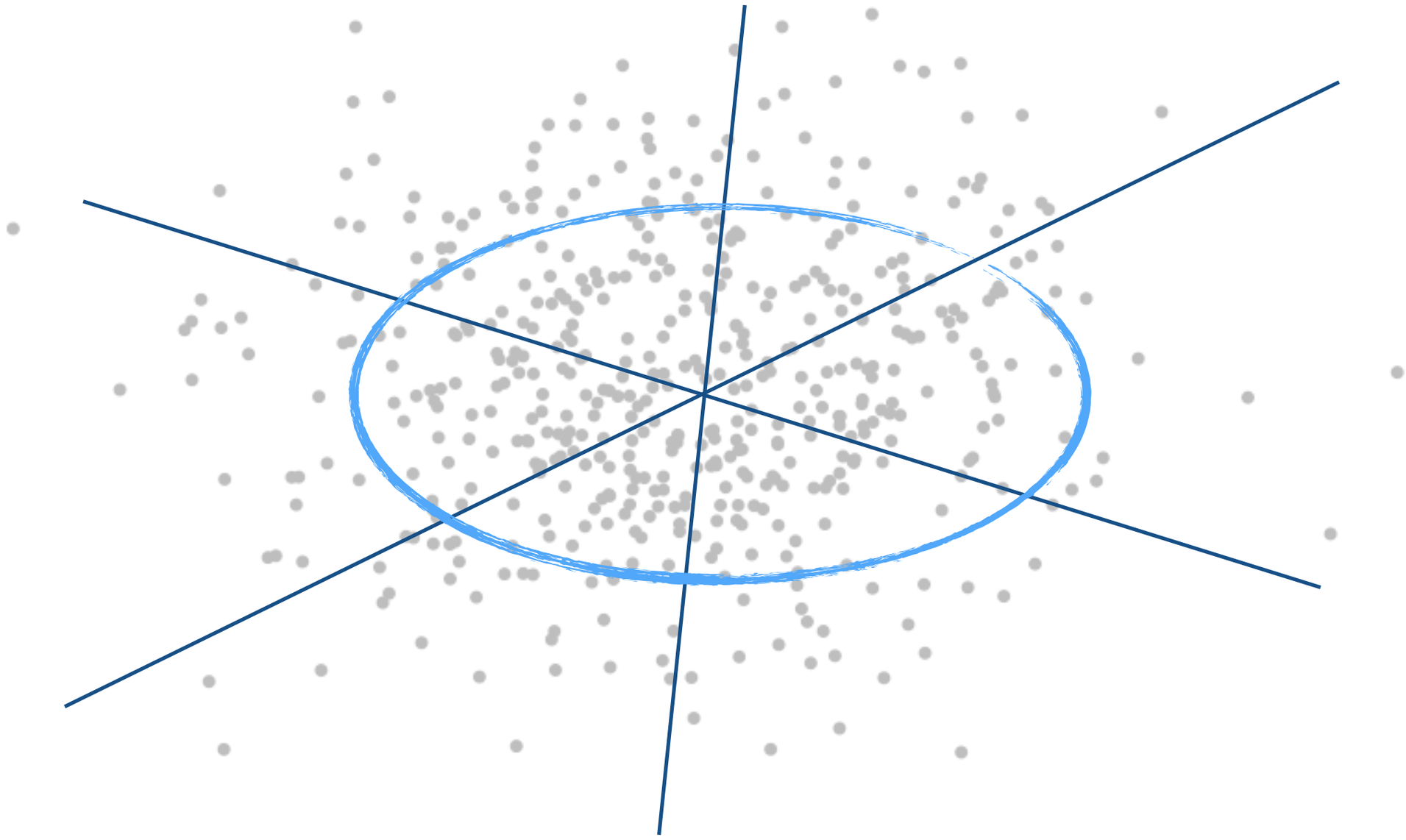

Covariance Matrix

$$\mathcal{D}(\Gamma, \{X_i\}_{i=1}^n) = \min_{\|u\|=1} \min \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{|u^T X_i|^2 \geq u^T \Gamma u\}, \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{|u^T X_i|^2 < u^T \Gamma u\} \right\}$$

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$$\hat{\Gamma} = \arg\max_{\Gamma \succeq 0} \mathcal{D}(\Gamma, \{X_i\}_{i=1}^n) \qquad \hat{\Sigma} = \hat{\Gamma}/\beta$$

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Theorem [CGR15]. For some $C > 0$,

$$\|\hat{\Sigma} - \Sigma\|_{\text{op}}^2 \leq C \left(\frac{p}{n} \vee \epsilon^2 \right)$$

with high probability uniformly over Σ, Q .

Summary

mean	$\|\cdot\|^2$	$\frac{p}{n} \vee \epsilon^2$
reduced rank regression	$\|\cdot\|_{\mathrm{F}}^2$	$\frac{\sigma^2}{\kappa^2}\frac{r(p+m)}{n} \vee \frac{\sigma^2}{\kappa^2}\epsilon^2$
Gaussian graphical model	$\|\cdot\|_{\ell_1}^2$	$\frac{s^2\log(ep/s)}{n} \vee s\epsilon^2$
covariance matrix	$\|\cdot\|_{\mathrm{op}}^2$	$\frac{p}{n} \vee \epsilon^2$
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Computation

Computational Challenges

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- Polynomial algorithms are proposed [Diakonikolas et al.'16, Lai et al. 16] of minimax optimal statistical precision
  - needs information on second or higher order of moments
  - some priori knowledge about ϵ

Advantages of Tukey Median

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- **A well-defined objective function**

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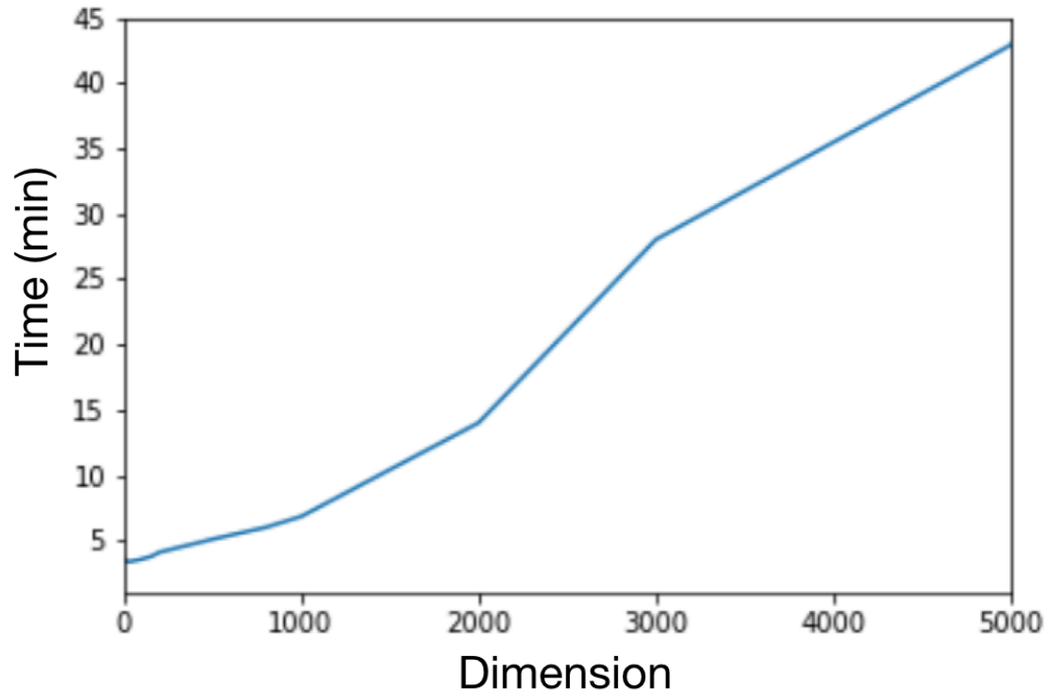
Advantages of Tukey Median

- **A well-defined objective function**
- **Adaptive to ϵ and Σ**
- **Optimal for any elliptical distribution**

A practically good algorithm?

Generative Adversarial Networks [Goodfellow et al. 2014]



Note: R-package for Tukey median can not deal with more than 10 dimensions [https://github.com/ChenMengjie/DepthDescent]

Robust Learning of Cauchy Distributions

Table 4: Comparison of various methods of robust location estimation under Cauchy distributions. Samples are drawn from $(1-\epsilon)\text{Cauchy}(0_p, I_p) + \epsilon Q$ with $\epsilon = 0.2, p = 50$ and various choices of Q. Sample size: 50,000. Discriminator net structure: 50-50-25-1. Generator $g_\omega(\xi)$ structure: 48-48-32-24-12-1 with absolute value activation function in the output layer.

Contamination Q	JS-GAN (G_1)	JS-GAN (G_2)	Dimension Halving	Iterative Filtering
$\text{Cauchy}(1.5 * 1_p, I_p)$	**0.0664 (0.0065)**	0.0743 (0.0103)	0.3529 (0.0543)	0.1244 (0.0114)
$\text{Cauchy}(5.0 * 1_p, I_p)$	**0.0480 (0.0058)**	0.0540 (0.0064)	0.4855 (0.0616)	0.1687 (0.0310)
$\text{Cauchy}(1.5 * 1_p, 5 * I_p)$	0.0754 (0.0135)	**0.0742 (0.0111)**	0.3726 (0.0530)	0.1220 (0.0112)
$\text{Normal}(1.5 * 1_p, 5 * I_p)$	**0.0702 (0.0064)**	0.0713 (0.0088)	0.3915 (0.0232)	0.1048 (0.0288))

- *Dimension Halving:* [Lai et al.'16]
  `https://github.com/kal2000/AgnosticMeanAndCovarianceCode`.
- *Iterative Filtering:* [Diakonikolas et al.'17]
  `https://github.com/hoonose/robust-filter`.

f-GAN

Given a strictly convex function f that satisfies $f(1) = 0$, the f-divergence between two probability distributions P and Q is defined by

$$D_f(P\|Q) = \int f\left(\frac{p}{q}\right) dQ. \tag{8}$$

Let f^* be the convex conjugate of f. A variational lower bound of (8) is

$$D_f(P\|Q) \geq \sup_{T\in\mathcal{T}} \left[\mathbb{E}_P T(X) - \mathbb{E}_Q f^*(T(X))\right]. \tag{9}$$

where equality holds whenever the class $\mathcal{T}$ contains the function $f'(p/q)$.

[Nowozin-Cseke-Tomioka'16] f-GAN minimizes the variational lower bound (9)

$$\widehat{P} = \arg\min_{Q\in\mathcal{Q}} \sup_{T\in\mathcal{T}} \left[\frac{1}{n}\sum_{i=1}^{n} T(X_i) - \mathbb{E}_Q f^*(T(X))\right]. \tag{10}$$

with i.i.d. observations $X_1, ..., X_n \sim P$.

From f-GAN to Tukey's Median: f-learning

Consider the special case

$$\mathcal{T} = \left\{ f'\left(\frac{\widetilde{q}}{q}\right) : \widetilde{q} \in \widetilde{\mathcal{Q}} \right\}. \tag{11}$$

which is tight if $P \in \widetilde{\mathcal{Q}}$. The sample version leads to the following f-learning

$$\widehat{P} = \underset{Q \in \mathcal{Q}}{\arg\min} \sup_{\widetilde{Q} \in \widetilde{\mathcal{Q}}} \left[\frac{1}{n} \sum_{i=1}^{n} f'\left(\frac{\widetilde{q}(X_i)}{q(X_i)}\right) - \mathbb{E}_Q f^*\left(f'\left(\frac{\widetilde{q}(X)}{q(X)}\right)\right) \right]. \tag{12}$$

- If $f(x) = x \log x$, $\mathcal{Q} = \widetilde{\mathcal{Q}}$, (12) $\Rightarrow$ Maximum Likelihood Estimate
- If $f(x) = (x-1)+$, then $D_f(P\|Q) = \frac{1}{2}\int |p-q|$ is the TV-distance, $f^*(t) = t\mathbb{I}\{0 \le t \le 1\}$, f-GAN $\Rightarrow$ TV-GAN
  - $\mathcal{Q} = \{N(\eta, I_p) : \eta \in \mathbb{R}^p\}$ and $\widetilde{\mathcal{Q}} = \{N(\widetilde{\eta}, I_p) : \|\widetilde{\eta} - \eta\| \le r\}$, (12) $\overset{r \to 0}{\Rightarrow}$ Tukey's Median

f-Learning

f-Learning

Jensen-Shannon	$f(x) = x \log x - (x+1)\log(x+1)$	**GAN**

[Goodfellow et al.

f-Learning

Jensen-Shannon	$f(x) = x \log x - (x+1) \log(x+1)$	**GAN**
Kullback-Leibler	$f(x) = x \log x$	**MLE**

[Goodfellow et al.

f-Learning

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Hellinger Squared	$f(x) = 2 - 2\sqrt{x}$	**rho**

[Goodfellow et al., Baraud and Birge]

f-Learning

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Kullback-Leibler	$f(x) = x \log x$	**MLE**
Hellinger Squared	$f(x) = 2 - 2\sqrt{x}$	**rho**
Total Variation	$f(x) = (x-1)_+$	**depth**

[Goodfellow et al., Baraud and Birge]

TV-Learning

$$\min_{Q \in \mathcal{Q}} \max_{\tilde{Q} \in \tilde{\mathcal{Q}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left\{ \frac{\tilde{q}(X_i)}{q(X_i)} \geq 1 \right\} - Q\left(\frac{\tilde{q}}{q} \geq 1 \right) \right\}$$

TV-Learning

$$\min_{Q\in\mathcal{Q}} \max_{\tilde{Q}\in\tilde{\mathcal{Q}}} \left\{ \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\left\{ \frac{\tilde{q}(X_i)}{q(X_i)} \geq 1 \right\} - Q\left(\frac{\tilde{q}}{q} \geq 1 \right) \right\}$$

$$\mathcal{Q} = \left\{ N(\theta, I_p) : \theta \in \mathbb{R}^p \right\} \qquad \tilde{\mathcal{Q}} = \left\{ N(\tilde{\theta}, I_p) : \tilde{\theta} \in \mathcal{N}_r(\theta) \right\}$$

TV-Learning

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$r \to 0$

TV-Learning

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$$r \to 0$$

Tukey depth $$\max_{\theta\in\mathbb{R}} \min_{\|u\|=1} \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\left\{ u^T X_i \geq u^T \theta \right\}$$

TV-Learning

$$\min_{Q \in \mathcal{Q}} \max_{\tilde{Q} \in \tilde{\mathcal{Q}}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left\{ \frac{\tilde{q}(X_i)}{q(X_i)} \geq 1 \right\} - Q\left(\frac{\tilde{q}}{q} \geq 1 \right) \right\}$$

TV-Learning

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$$\mathcal{Q} = \left\{ N(0,\Sigma) : \Sigma \in \mathbb{R}^{p\times p} \right\} \qquad \tilde{\mathcal{Q}} = \left\{ N(0,\tilde{\Sigma}) : \tilde{\Sigma} = \Sigma + ruu^T, \|u\| = 1 \right\}$$

TV-Learning

$$\min_{Q\in\mathcal{Q}} \max_{\tilde{Q}\in\tilde{\mathcal{Q}}} \left\{ \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\left\{ \frac{\tilde{q}(X_i)}{q(X_i)} \geq 1 \right\} - Q\left(\frac{\tilde{q}}{q} \geq 1 \right) \right\}$$

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TV-Learning

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$r \to 0$

(related to) matrix depth

$$\max_{\Sigma} \min_{\|u\|=1} \left[\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{|u^T X_i|^2 \leq u^T\Sigma u\} - \mathbb{P}(\chi_1^2 \leq 1) \right) \wedge \left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{|u^T X_i|^2 > u^T\Sigma u\} - \mathbb{P}(\chi_1^2 > 1) \right) \right]$$

robust statistics community

deep learning community

robust statistics community

f-Learning

f-GAN

deep learning community

robust statistics community

f-Learning

f-GAN

deep learning community

practically good algorithms

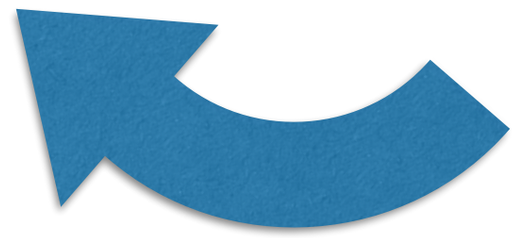
theoretical foundation

robust statistics community

f-Learning

f-GAN

deep learning community



practically good algorithms

TV-GAN

$$\widehat{\theta} = \underset{\eta}{\operatorname{argmin}} \sup_{w,b} \left[\frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + e^{-w^T X_i - b}} - E_\eta \frac{1}{1 + e^{-w^T X - b}} \right]$$

TV-GAN

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$$N(\eta, I_p)$$

TV-GAN

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$N(\eta, I_p)$

logistic regression classifier

TV-GAN

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logistic regression classifier

$N(\eta, I_p)$

Theorem [GLYZ18]. For some $C > 0$,

$$\|\widehat{\theta} - \theta\|^2 \leq C \left(\frac{p}{n} \vee \epsilon^2 \right)$$

with high probability uniformly over $\theta \in \mathbb{R}^p, Q$.

TV-GAN

rugged landscape!

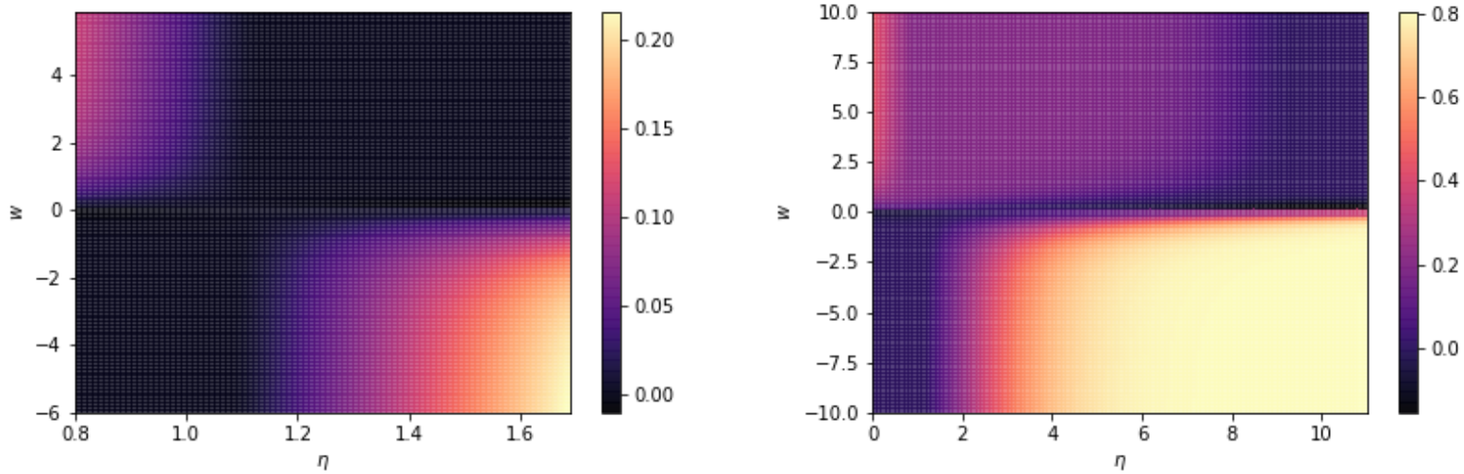

Figure: Heatmaps of the landscape of $F(\eta, w) = \sup_b [E_P \text{sigmoid}(wX + b) - E_{N(\eta,1)} \text{sigmoid}(wX + b)]$, where b is maximized out for visualization. Left: samples are drawn from $P = (1 - \epsilon)N(1, 1) + \epsilon N(1.5, 1)$ with $\epsilon = 0.2$. Right: samples are drawn from $P = (1 - \epsilon)N(1, 1) + \epsilon N(10, 1)$ with $\epsilon = 0.2$. Left: the landscape is good in the sense that no matter whether we start from the left-top area or the right-bottom area of the heatmap, gradient ascent on η does not consistently increase or decrease the value of η. This is because the signal becomes weak when it is close to the saddle point around $\eta = 1$. Right: it is clear that $\tilde{F}(w) = F(\eta, w)$ has two local maxima for a given η, achieved at $w = +\infty$ and $w = -\infty$. In fact, the global maximum for $\tilde{F}(w)$ has a phase transition from $w = +\infty$ to $w = -\infty$ as η grows. For example, the maximum is achieved at $w = +\infty$ when $\eta = 1$ (blue solid) and is achieved at $w = -\infty$ when $\eta = 5$ (red solid). Unfortunately, even if we initialize with $\eta_0 = 1$ and $w_0 > 0$, gradient ascents on η will only increase the value of η (green dash), and thus as long as the discriminator cannot reach the global maximizer, w will be stuck in the positive half space $\{w : w > 0\}$ and further increase the value of η.

The Original JS-GAN

[Goodfellow et al. 2014] For $f(x) = x \log x - (x+1) \log \frac{x+1}{2}$,

$$\widehat{\theta} = \arg\min_{\eta \in \mathbb{R}^p} \max_{D \in \mathcal{D}} \left[\frac{1}{n} \sum_{i=1}^{n} \log D(X_i) + \mathbb{E}_{\mathcal{N}(\eta, I_p)} \log(1 - D(X)) \right] + \log 4. \quad (15)$$

What are $\mathcal{D}$, the class of discriminators?

- Single layer (no hidden layer):

$$\mathcal{D} = \left\{ D(x) = \text{sigmoid}(w^T x + b) : w \in \mathbb{R}^p, b \in \mathbb{R} \right\}$$

- One-hidden or Multiple layer:

$$\mathcal{D} = \left\{ D(x) = \text{sigmoid}(w^T g(X)) \right\}$$

JS-GAN

$$\widehat{\theta} = \underset{\eta \in \mathbb{R}^p}{\operatorname{argmin}} \max_{T \in \mathcal{T}} \left[\frac{1}{n} \sum_{i=1}^{n} \log T(X_i) + E_\eta \log(1 - T(X)) \right] + \log 4$$

JS-GAN

$$\widehat{\theta} = \underset{\eta \in \mathbb{R}^p}{\operatorname{argmin}} \max_{T \in \mathcal{T}} \left[\frac{1}{n} \sum_{i=1}^{n} \log T(X_i) + E_\eta \log(1 - T(X)) \right] + \log 4$$

numerical experiment

$$X_1, ..., X_n \sim (1-\epsilon)N(\theta, I_p) + \epsilon N(\widetilde{\theta}, I_p)$$

JS-GAN

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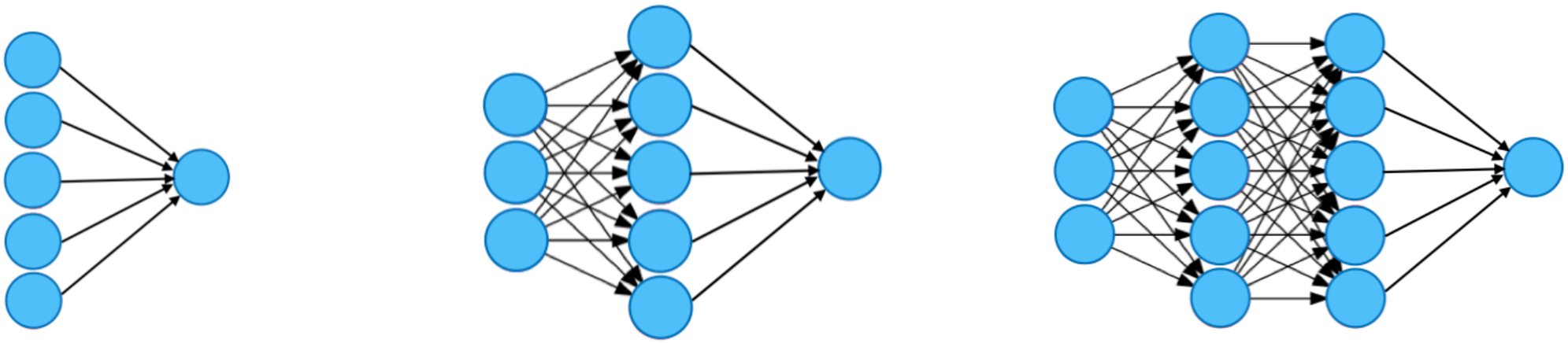

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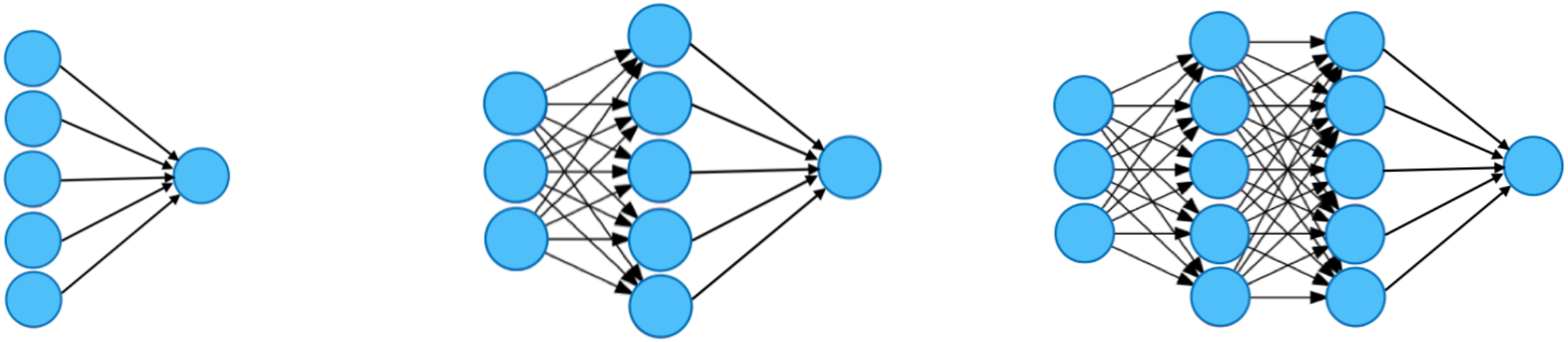

$$\widehat{\theta} \approx (1-\epsilon)\theta + \epsilon\widetilde{\theta}$$

JS-GAN

$$\widehat{\theta} = \underset{\eta \in \mathbb{R}^p}{\operatorname{argmin}} \max_{T \in \mathcal{T}} \left[\frac{1}{n} \sum_{i=1}^{n} \log T(X_i) + E_\eta \log(1 - T(X)) \right] + \log 4$$

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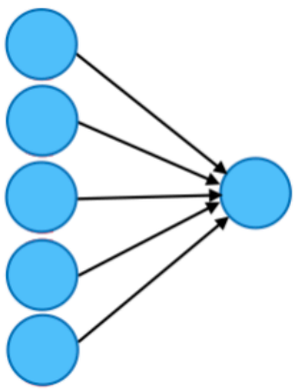
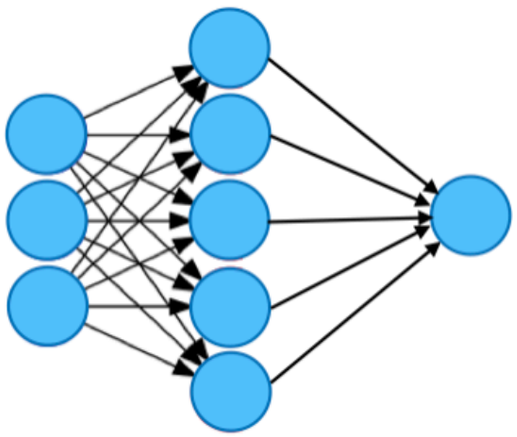
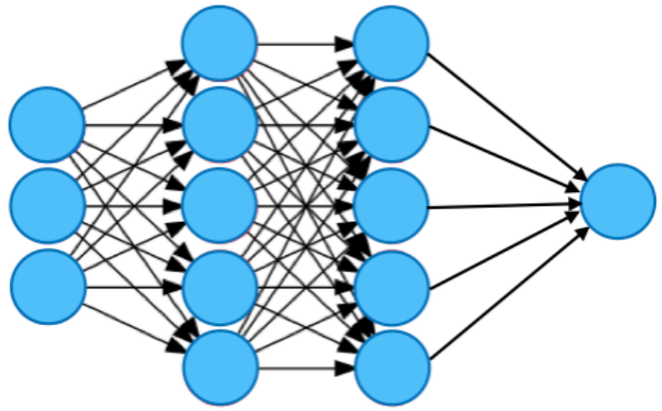


$\widehat{\theta} \approx (1-\epsilon)\theta + \epsilon\widetilde{\theta}$



$\widehat{\theta} \approx \theta$



$\widehat{\theta} \approx \theta$

JS-GAN

A classifier with hidden layers leads to robustness. Why?

JS-GAN

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$$\mathsf{JS}_g(\mathbb{P},\mathbb{Q}) = \max_{w\in\mathbb{R}^d}\left[\mathbb{P}\log\frac{1}{1+e^{-w^T g(X)}} + \mathbb{Q}\log\frac{1}{1+e^{w^T g(X)}}\right] + \log 4.$$

JS-GAN

A classifier with hidden layers leads to robustness. Why?

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Proposition.

$$\mathsf{JS}_g(\mathbb{P},\mathbb{Q}) = 0 \iff \mathbb{P}g(X) = \mathbb{Q}g(X)$$

JS-GAN

$$\widehat{\theta} = \underset{\eta \in \mathbb{R}^p}{\operatorname{argmin}} \max_{T \in \mathcal{T}} \left[\frac{1}{n} \sum_{i=1}^{n} \log T(X_i) + E_\eta \log(1 - T(X)) \right] + \log 4$$

Theorem [GLYZ18]. For a neural network class $\mathcal{T}$ with at least one hidden layer and appropriate regularization, we have

$$\|\widehat{\theta} - \theta\|^2 \lesssim \begin{cases} \dfrac{p}{n} + \epsilon^2 & \text{(indicator/sigmoid/ramp)} \\ \dfrac{p \log p}{n} + \epsilon^2 & \text{(ReLUs+sigmoid features)} \end{cases}$$

with high probability uniformly over $\theta \in \mathbb{R}^p, Q$.

JS-GAN

unknown covariance?

$$X_1, ..., X_n \sim (1-\epsilon)N(\theta, \Sigma) + \epsilon Q$$

JS-GAN

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JS-GAN

unknown covariance?

$$X_1, ..., X_n \sim (1-\epsilon)N(\theta, \Sigma) + \epsilon Q$$

$$(\widehat{\theta}, \widehat{\Sigma}) = \operatorname*{argmin}_{\eta,\Gamma} \max_{T \in \mathcal{T}} \left[\frac{1}{n} \sum_{i=1}^{n} \log T(X_i) + \mathbb{E}_{X \sim N(\eta,\Gamma)} \log(1 - T(X)) \right]$$

no need to change the discriminator class

Robust Learning of Gaussian Distributions

Q	n	p	ϵ	TV-GAN	JS-GAN	Dimension Halving	Iterative Filtering
$N(0.5 * 1_p, I_p)$	50,000	100	.2	**0.0953 (0.0064)**	0.1144 (0.0154)	0.3247 (0.0058)	0.1472 (0.0071)
$N(0.5 * 1_p, I_p)$	5,000	100	.2	**0.1941 (0.0173)**	0.2182 (0.0527)	0.3568 (0.0197)	0.2285 (0.0103)
$N(0.5 * 1_p, I_p)$	50,000	200	.2	**0.1108 (0.0093)**	0.1573 (0.0815)	0.3251 (0.0078)	0.1525 (0.0045)
$N(0.5 * 1_p, I_p)$	50,000	100	.05	0.0913 (0.0527)	0.1390 (0.0050)	0.0814 (0.0056)	**0.0530 (0.0052)**
$N(5 * 1_p, I_p)$	50,000	100	.2	2.7721 (0.1285)	**0.0534 (0.0041)**	0.3229 (0.0087)	0.1471 (0.0059)
$N(0.5 * 1_p, \Sigma)$	50,000	100	.2	0.1189 (0.0195)	**0.1148 (0.0234)**	0.3241 (0.0088)	0.1426 (0.0113)
Cauchy$(0.5 * 1_p)$	50,000	100	.2	0.0738 (0.0053)	**0.0525 (0.0029)**	0.1045 (0.0071)	0.0633 (0.0042)

Table: Comparison of various robust mean estimation methods. The smallest error of each case is highlighted in bold.

- *Dimension Halving:* [Lai et al.'16]
  `https://github.com/kal2000/AgnosticMeanAndCovarianceCode`.
- *Iterative Filtering:* [Diakonikolas et al.'17]
  `https://github.com/hoonose/robust-filter`.

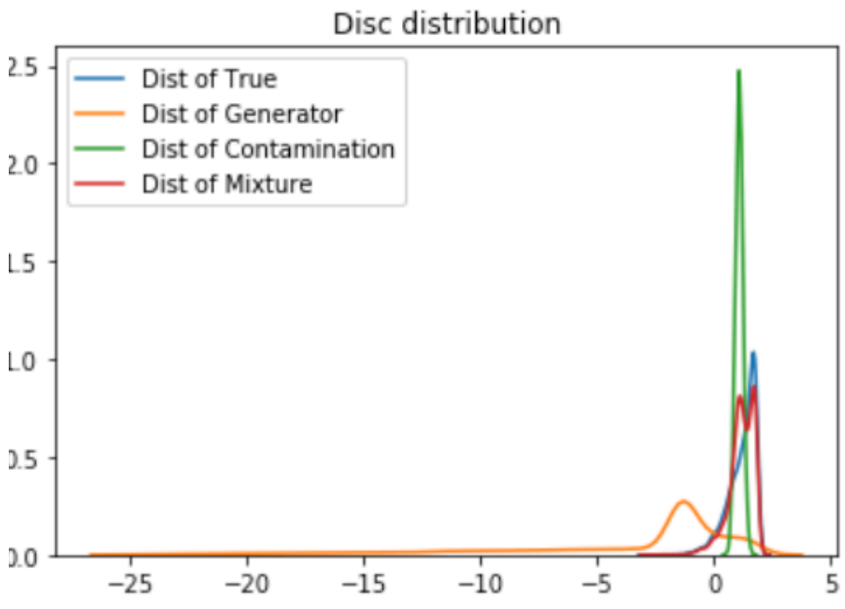
Robust Learning of Cauchy Distributions

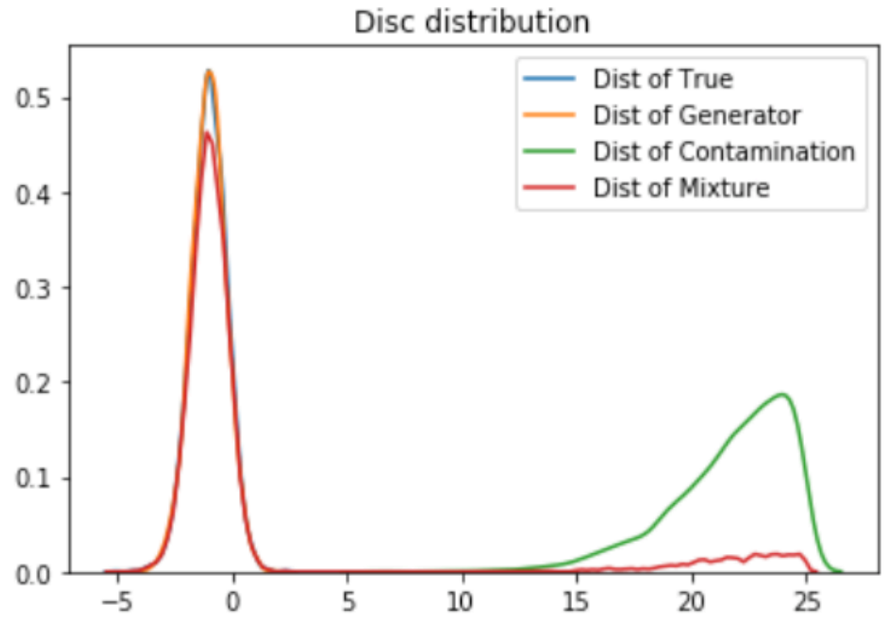
Table 4: Comparison of various methods of robust location estimation under Cauchy distributions. Samples are drawn from $(1-\epsilon)\text{Cauchy}(0_p, I_p) + \epsilon Q$ with $\epsilon = 0.2, p = 50$ and various choices of Q. Sample size: 50,000. Discriminator net structure: 50-50-25-1. Generator $g_\omega(\xi)$ structure: 48-48-32-24-12-1 with absolute value activation function in the output layer.

Contamination Q	JS-GAN (G_1)	JS-GAN (G_2)	Dimension Halving	Iterative Filtering
$\text{Cauchy}(1.5 * 1_p, I_p)$	**0.0664 (0.0065)**	0.0743 (0.0103)	0.3529 (0.0543)	0.1244 (0.0114)
$\text{Cauchy}(5.0 * 1_p, I_p)$	**0.0480 (0.0058)**	0.0540 (0.0064)	0.4855 (0.0616)	0.1687 (0.0310)
$\text{Cauchy}(1.5 * 1_p, 5 * I_p)$	0.0754 (0.0135)	**0.0742 (0.0111)**	0.3726 (0.0530)	0.1220 (0.0112)
$\text{Normal}(1.5 * 1_p, 5 * I_p)$	**0.0702 (0.0064)**	0.0713 (0.0088)	0.3915 (0.0232)	0.1048 (0.0288))

- *Dimension Halving:* [Lai et al.'16]
  `https://github.com/kal2000/AgnosticMeanAndCovarianceCode`.
- *Iterative Filtering:* [Diakonikolas et al.'17]
  `https://github.com/hoonose/robust-filter`.

Discriminator identifies outliers



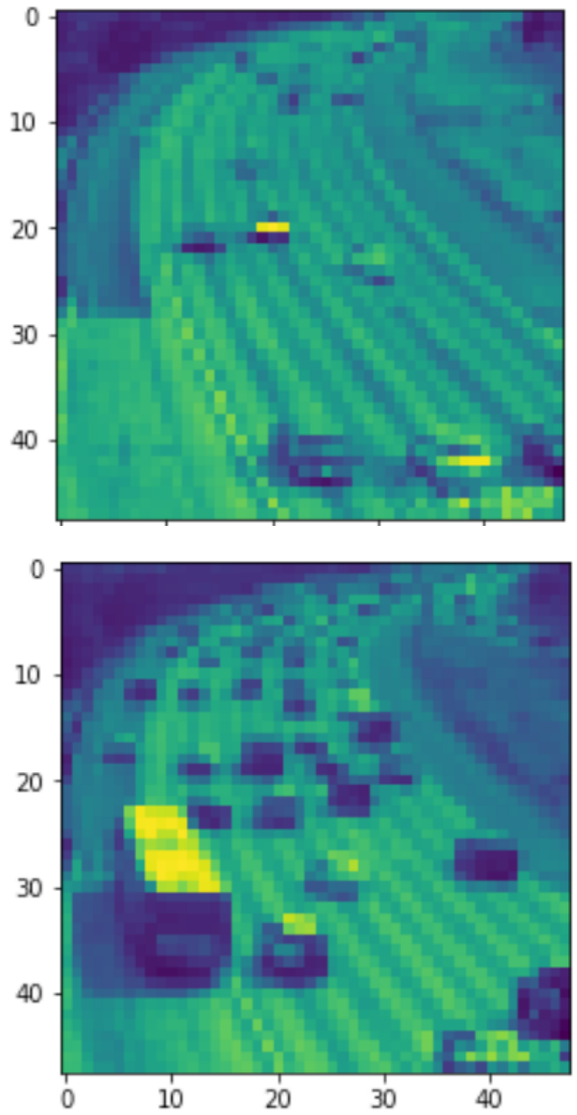


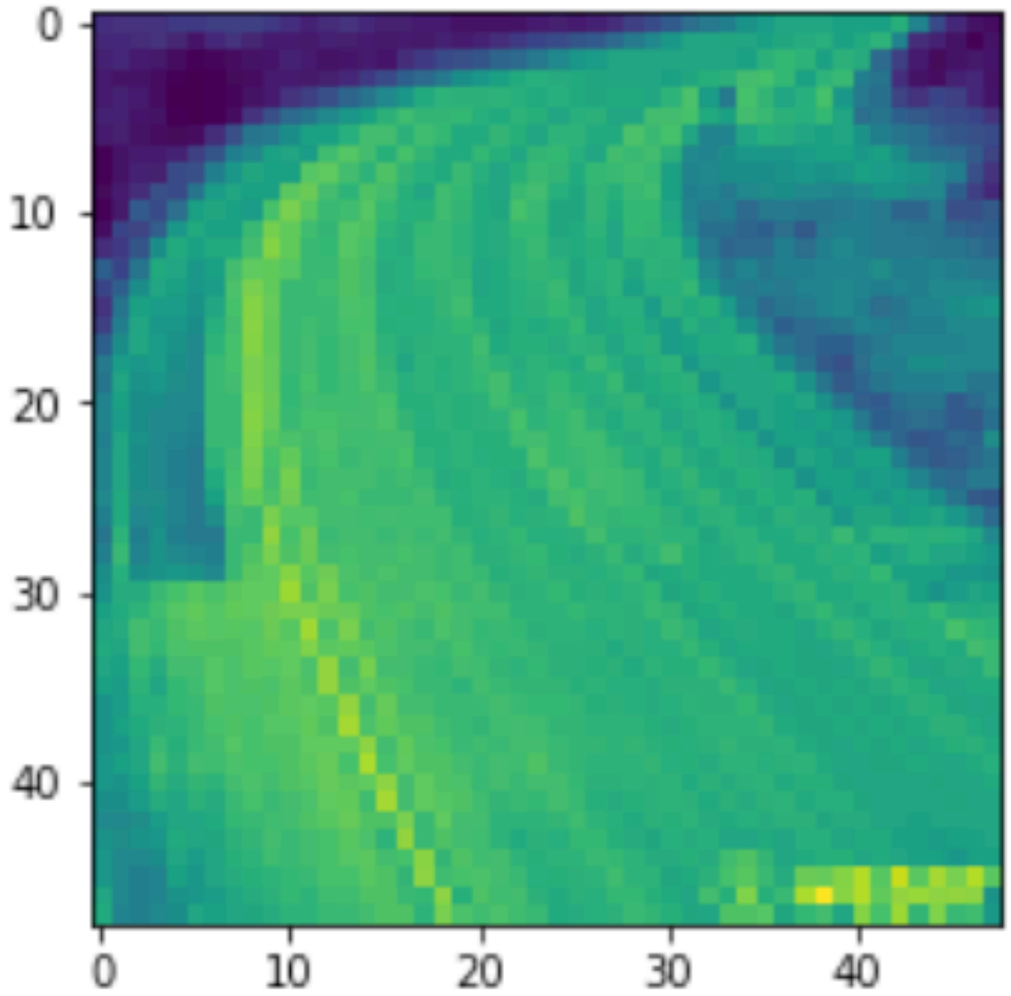
$$(1-\epsilon)N(0_p, I_p) + \epsilon Q$$

$$N(5 * 1_p, I_p)$$

- Discriminator helps identify outliers or contaminated samples
- Generator fits uncontaminated portion of true samples

App: Robust Mean





by Weizhi ZHU

Reference

- Gao, Liu, Yao, Zhu, Robust Estimation and Generative Adversarial Networks, ICLR 2019, https://arxiv.org/abs/1810.02030

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Thank You