MATH5473 Homework 6

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3. (d) Since A is invertible we have

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^T & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & C - B^T A^{-1} B \end{bmatrix}$$

Taking determinants on both sides, we obtain

$$\det(K) = \det(A) \cdot \det(C - B^T A^{-1} B) = \det(A) \cdot \det(K/A)$$

(e) Since A is invertible, We have

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

Therefore

$$\begin{aligned} \operatorname{rank}(K) &= \operatorname{rank}\left(\begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix} \right) \\ &= \operatorname{rank}(A) \cdot \operatorname{rank}(C - B^T A^{-1} B) = \operatorname{rank}(A) \cdot \operatorname{rank}(K/A) \end{aligned}$$

(f) Define the generalized inverse A^- to be any solution of

$$AA^-A = A$$

Lemma 1. *For matrices over an arbitrary field,*

$$r(A, B) = r(A) + r\left(\left[I - AA^-\right] B\right) = r\left(\left[I - BB^-\right] A\right) + r(B) \quad (1)$$

for every A^-, B^-, and

$$r\begin{pmatrix} A \\ B^T \end{pmatrix} = r(A) + r\left(B^T\left[I - A^-A\right]\right) = r\left(A\left[I - (B^T)^- B^T\right]\right) + r(B^T) \quad (2)$$

for every $A^-, (B^T)^-$.

Proof. We may write

$$
\begin{aligned}
r(A,B) &= r\left[(A,B)\begin{pmatrix} I & -A^{-}B \\ 0 & I \end{pmatrix}\right] \\
&= r\left(A, \left[I - A^{-}\right]B\right) \\
&= r(A) + r\left(\left[I - AA^{-}\right]B\right),
\end{aligned}
$$

since the column spaces of A and $(I - AA^{-})\,B$ are virtually disjoint: if $a = Ab = (I - AA^{-})\,B$, then $(I - AA^{-})\,a = 0 = (I - AA^{-})\,Bc = a$, as $I - AA^{-}$is idempotent. This proves the first equation in (1). The second equation in (1) and both equations in (2) may be proved similarly. □

Theorem 1. *Let the matrix*

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

have elements over an arbitrary field, and suppose that both K and A are square. If either

$$r(A,B) = r(A) \tag{3}$$

or

$$r\begin{pmatrix} A \\ B^T \end{pmatrix} = r(A) \tag{4}$$

then

$$|K| = |A| \cdot \left|C - B^T A^{-} B\right| \tag{5}$$

for every g-inverse A^{-}.

Proof. It follows from Lemma 1 that (3) implies

$$AA^{-}B = B$$

for every g-inverse A^{-}. In this event, writing

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ B^T & I \end{pmatrix}\begin{pmatrix} I & A^{-}B \\ 0 & C - B^T A^{-} B \end{pmatrix}$$

and taking determinants yields (5). A similar proof works when (4) holds. □

Lemma 2. *For matrices over an arbitrary field,*

$$r\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = r(A) + r\begin{pmatrix} 0 & \left(I - AA^{-}\right)B \\ B^T\left(I - A^{-}A\right) & C - B^T A^{-} B \end{pmatrix}$$

Three different choices of A^{-}may be made.

Proof. We note that

$$\begin{pmatrix} I & 0 \\ -B^T A & I \end{pmatrix} \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} I & -A^{\sim} B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & X \\ Y & J \end{pmatrix},$$

where $A^{\sim}$ is a g-inverse of A, possibly different to A^{-},

$$X = (I - AA^{\sim})\,B, \quad Y = B^T\left(I - A^{-}A\right)$$

and

$$J = C - B^T A^{-} B - YA^{\sim}B$$

Then

$$r\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = r\begin{pmatrix} A & X \\ Y & J \end{pmatrix} = r\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + r\begin{pmatrix} 0 & X \\ Y & J \end{pmatrix}$$

since the columns (rows) of A are linearly independent of the columns of X (rows of Y). Since

$$\begin{pmatrix} 0 & X \\ Y & J \end{pmatrix} = \begin{pmatrix} 0 & X \\ Y & C - B^T A - B \end{pmatrix} \begin{pmatrix} I & -A^{\sim} B \\ 0 & I \end{pmatrix},$$

(4.28) follows, except that the choice of A^{-}in Y is the same as that in $C - B^T A^{-} B$. To relax this condition we note that with $A^{\#}$ as a g-inverse of A (possibly different to A^{-}), we have that

$$\begin{pmatrix} 0 & X \\ B^T\left(I - A^{-}A\right) & S \end{pmatrix} = \begin{pmatrix} 0 & X \\ B^T\left(I - A^{\#}A\right) & S \end{pmatrix} \begin{pmatrix} I - A^{-}\mathbf{E} & 0 \\ 0 & I \end{pmatrix},$$

$$\begin{pmatrix} 0 & X \\ B^T\left(I - A^{\#}A\right) & S \end{pmatrix} = \begin{pmatrix} 0 & X \\ B^T\left(I - A^{-}A\right) & S \end{pmatrix} \begin{pmatrix} I - A^{\#}\mathbf{E} & 0 \\ 0 & I \end{pmatrix},$$

where $S = C - B^T A^{-} B$, and hence

$$r\begin{pmatrix} 0 & X \\ B^T\left(I - A^{-}A\right) & S \end{pmatrix} = r\begin{pmatrix} 0 & X \\ B^T\left(I - A^{\#}A\right) & S \end{pmatrix}$$

is invariant under choice of A^{-}. This completes the proof. □

Corollary 1. *For matrices over an arbitrary field,*

$$r\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = r(C) + r\begin{pmatrix} A - BC^{-}B^T & B\left(I - C^{-}C\right) \\ \left(I - CC^{-}\right)B^T & 0 \end{pmatrix}$$

Three different choices of C^{-}may be made.

We may expand the rank in Lemma 2 using Corollary 1 to obtain

$$r\begin{pmatrix} 0 & \left(I - AA^{-}\right)B \\ B^T\left(I - A^{-}A\right) & S \end{pmatrix} = r(S) + r\begin{pmatrix} U & V \\ W & 0 \end{pmatrix} \quad (6)$$

where

$$\begin{aligned} U &= -\left(I - AA^{-}\right) BS^{-}B^{T}\left(I - A^{-}A\right), \\ V &= \left(I - AA^{-}\right) B\left(I - S^{-}S\right), \\ W &= \left(I - SS^{-}\right) B^{T}\left(I - A^{-}A\right). \end{aligned} \tag{7}$$

Lemma 3. *For matrices over an arbitrary field,*

$$r\begin{pmatrix} 0 & X \\ Y & S \end{pmatrix} = r(X) + r(Y) + r\left[\left(I - Y^{-}\right) S\left(I - X^{-}X\right)\right]$$

Any choices of X^{-} and Y^{-} may be made.

Proof. Using Lemma 1 yields

$$\begin{aligned} r\begin{pmatrix} 0 & X \\ Y & S \end{pmatrix} &= r(X) + r(Y, S)\begin{pmatrix} I & 0 \\ 0 & I - X^{-}X \end{pmatrix} \\ &= r(X) + r\left(Y, S\left(I - X^{-}X\right)\right) \end{aligned}$$

Applying (1) completes the proof. □

Combining Lemma 2, (6) and Lemma 3, we obtain

Lemma 4. *For matrices over an arbitrary field,*

$$r\begin{pmatrix} A & B \\ B^{T} & C \end{pmatrix} = r(A) + r(S) + r(V) + r(W) + r(Z)$$

where

$$Z = (I - VV^{-})U\left(I - W^{-}W\right)$$

while U, V and W are as in (7). The g-inverses may be any choices.

Theorem 2. *For matrices over an arbitrary field, rank is additive on the Schur complement:*

$$r\begin{pmatrix} A & B \\ B^{T} & C \end{pmatrix} = r(A) + r(C - B^{T}A^{\sim}B)$$

where $A^{\sim}$ is a particular g-inverse of A, if and only if

$$\begin{aligned} \left(I - AA^{-}\right) B\left(I - S^{-}S\right) &= 0 \\ \left(I - SS^{-}\right) B^{T}\left(I - A^{-}A\right) &= 0 \\ \left(I - AA^{-}\right) BS^{-}B^{T}\left(I - A^{-}A\right) &= 0, \end{aligned}$$

where $S = C - B^{T}A^{\sim}B$, while A^{-} and S^{-} are any choices of g-inverses.

Proof. Immediate from Lemma 4. □