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Q2.

(a) From lecture notes, $\hat{\Sigma}_n = \frac{1}{n} YY^T = \Sigma^{\frac{1}{2}} S_n \Sigma^{-\frac{1}{2}}$ where $(\hat{\lambda}, \hat{v})$ is eigenvalue-eigenvector pair of $\hat{\Sigma}_n$.

$$v = c\Sigma^{-\frac{1}{2}}\hat{v}.$$

Assume that $\hat{\lambda}^T - p - \sigma_\varepsilon^2 S_n$ is inevitable and $u^T v \neq 0$.

$$I = \sigma_x^2 \cdot u^T(\hat{\lambda} I_p - \sigma_\varepsilon^2 S_n)^{-1} S_n u$$

$$S_n = W\hat{\Lambda}W^T \quad \text{for } \Lambda = \mathrm{diag}(\lambda_i ; i = 1, \cdots, p)$$

$$a_i = w_i^T u$$

Then for large p, $\lambda_i \sim \mu^{MP}(\lambda_i)$

$$I = \sigma_x^2 \cdot \frac{1}{p}\sum_{i=1}^{p} \frac{\lambda_i}{\hat{\lambda} - \sigma_\varepsilon^2 \lambda_i} \sim \sigma_x^2 \int_a^b \frac{t}{\hat{\lambda} - \sigma_\varepsilon^2 t} d\mu^{MP}(t)$$

For $\frac{\lambda}{\sigma^2} > \sqrt{r}$, $1 \sim \frac{\lambda}{\sigma^2}\int_a^b \frac{\sigma^2 t}{\sigma^2(\frac{\hat{\lambda}}{\sigma^2} \sim 1)} d\mu^{MP}(t)$

Using the stieltjes transform,

$$1 = \frac{\lambda}{\sigma^2}\int_a^b \frac{t}{\frac{\hat{\lambda}}{\sigma} - t} \frac{\sqrt{(b-t)(t-a)}}{2\pi r t}$$

$$= \frac{\lambda}{4r\sigma^2}\left[2\hat{\lambda} - (a+b) - 2\sqrt{1\left(\frac{\hat{\lambda}}{\sigma} - a\right)\left(b - \frac{\hat{\lambda}}{\sigma}\right)}\right], \quad \frac{\lambda}{\sigma^2} = \left(1 + \frac{\lambda_0}{\sigma^2}\right)\left(1 + \frac{r\sigma^2}{\lambda_0}\right)$$

$$\therefore \lambda = (\sigma^2 + \lambda_0)\left(1 + \frac{r\sigma^2}{\lambda_0}\right)$$

(b) $SNR = \frac{\lambda_0}{\sigma^2}$, from (a), we can get:

$$\lambda = \left(\sigma^2 + \frac{r\sigma^2}{\lambda_0} + \lambda_0 + r\sigma^2\right) = \sigma^2 + r\left(\frac{1}{SNR}\right) + \lambda_0 + r\sigma^2$$

$$SNR^2 + \left(1 + r - \frac{\lambda}{\sigma^2}\right) SNR + r = 0,$$

$$\therefore SNR = \frac{\left(\frac{\lambda}{\sigma^2} - 1 - r\right) \pm \sqrt{\left(1 + r - \frac{\lambda}{\sigma}\right)^2 - 4r}}{2}$$

(c) From the lecture notes, $\|v\|_2 = 1$, and

$$|u^T v|^{-2} = \sigma_x^4 [u^T S_n (\lambda I_p - \sigma_\omega^2 S_n)^{-2} S_n u] \sim \sigma_x^4 \int_a^b \frac{t^2}{(\lambda - \sigma_\omega^2 t)^2} d\mu^{mp}(t)$$

Here, $$|u^T \Sigma^{-\frac{1}{2}} v|^{-2} \sim \frac{x_0^2}{\sigma^4} \int_a^b \frac{\sigma^4 t^2}{|\sigma^2(\frac{\lambda}{\sigma^2} - t)|^2} d\mu^{mp}(t)$$

$$|u^T \Sigma^{-\frac{1}{2}} v|^2 = \frac{1 - \frac{r}{SNR^2}}{1 + r + \frac{2r}{SNR}}$$

$$\therefore \; |u^T v|^2 = \frac{1 - \frac{r}{SNR^2}}{1 + \frac{r}{SNR}}$$

Q3. (b) λ = eigenvalue

v = eigenvector for $W \to \lambda_0 uu^T$.

Then, $(W + \lambda_0 uu^T) = \lambda v$

$$v(\lambda I_p - W) = \lambda_0 u (u^T v)$$

$$v = (\lambda I_p - W)^{-1} \lambda_0 u (u^T v)$$

$$I = u^T (\lambda I_p - W)^{-1} \lambda_0 u$$

As $W = D \Sigma D^T$, $a_i = w_i^T \cdot u \cdot a$

$$\therefore I = u^T D (\lambda I_p - \Sigma)^{-1} \lambda_0 D^T u$$

$$= \lambda_0 \sum_{i=1}^{p} \frac{1}{\lambda - \lambda_i} a_i^2$$

$$\therefore I \sim \lambda_0 \int_a^b \frac{1}{\lambda - t} d\mu(t) = \lambda_0 \int_a^b \frac{2\sqrt{1-t^2}}{\pi(\lambda - t)} dt$$

for $a = -1, b = 1$, $I \sim \lambda_0 (2\lambda - 2\sqrt{\lambda^2 - 1})$

$$\Rightarrow \lambda = \lambda_0 + \frac{1}{4\lambda_0}$$

$$\therefore |u^T v|^{-2} = \lambda_0^2 u^T (\lambda I_p - \Sigma^{-2}) u$$

$$= \frac{4\lambda_0^2}{4\lambda_0^2 - 1}$$

$$\therefore |u^T v|^2 = 1 - \frac{1}{4\lambda_0^2 - 1}$$