Homework 2

WU Jiamin 20659352

1. (a). From the lecture notes:

$$\hat{\Sigma}_n = \frac{1}{n} Y Y^T = \Sigma^{\frac{1}{2}} S_n \Sigma^{\frac{1}{2}}$$

$(\hat{\lambda}, \hat{v})$ is eigenvalue-eigenvector pair of $\hat{\Sigma}_n$

$$v = c\,\Sigma^{-\frac{1}{2}} \hat{v}$$

Assume $\hat{\lambda} I_p - \sigma_\varepsilon^2 S_n$ is invertible and $u^T v \neq 0$.

$$1 = \sigma_x^2 \cdot u^T (\hat{\lambda} I_p - \sigma_\varepsilon^2 S_n)^{-1} S_n u$$

$$S_n = W \hat{\Lambda} W^T \quad \text{for } \Lambda = diag(\lambda_i : i = 1, \dots p)$$

$$\alpha_i = w_i^T u$$

Then for large p $\quad \lambda_i \sim \mu^{mp}(\lambda_i)$

$$1 = \sigma_x^2 \cdot \frac{1}{p} \sum_{i=1}^{p} \frac{\lambda_i}{\hat{\lambda} - \sigma_\varepsilon^2 \lambda_i} \sim \sigma_x^2 \cdot \int_a^b \frac{t}{\hat{\lambda} - \sigma_\varepsilon^2 t} d\mu^{mp}(t)$$

For $\frac{\lambda}{\sigma^2} > \sqrt{\gamma}$

$$1 \sim \frac{\lambda}{\sigma^2} \int_a^b \frac{\sigma^2 t}{\sigma^2 (\frac{\hat{\lambda}}{\sigma^2} - 1)} d\mu^{mp}(t)$$

Using the Stieltjes transform

$$1 = \frac{\lambda}{\sigma^2} \int_a^b \frac{t}{\frac{\hat{\lambda}}{\sigma^2} - t} \frac{\sqrt{(b-t)(t-a)}}{2\pi \gamma t}$$

$$= \frac{\lambda}{4\gamma \sigma^2} \left[2\hat{\lambda} - (a+b) - 2\sqrt{\left|\left(\frac{\hat{\lambda}}{\sigma^2} - a\right)\left(b - \frac{\hat{\lambda}}{\sigma^2}\right)\right|} \right]$$

$$\Rightarrow \frac{\lambda}{\sigma^2} = \left(1 + \frac{\lambda_0}{\sigma^2}\right)\left(1 + \frac{\gamma \sigma^2}{\lambda_0}\right)$$

$$\Rightarrow \lambda = (\sigma^2 + \lambda_0)\left(1 + \frac{\gamma \sigma^2}{\lambda_0}\right)$$

$$\sigma_x^2 = \frac{\lambda_0}{\sigma^2}$$

(b) $SNR = \frac{\lambda_0}{\sigma^2}$

from (a) we get

$$\lambda = \sigma^2 + \lambda_0 + \frac{\gamma \sigma^4}{\lambda_0} + \gamma \sigma^2$$

$$\frac{\lambda}{\sigma^2} = 1 + SNR + \frac{r}{SNR} + r$$

$$SNR^2 + \left(1 + r - \frac{\lambda}{\sigma^2}\right) SNR + r = 0$$

$$SNR = \frac{\left(\frac{\lambda}{\sigma^2} - 1 - r\right) + \sqrt{\left(1 + r - \frac{\lambda}{\sigma^2}\right)^2 - 4r}}{2}$$

(c) From the lecture notes $\|v\|_2 = 1$ and

$$|u^T v|^{-2} = \sigma_x^4 \left[u^T S_n (\lambda I_p - \sigma_\varepsilon^2 S_n)^{-2} S_n u\right]$$

$$\sim \sigma_x^4 \int_a^b \frac{t^2}{(\lambda - \sigma_\varepsilon^2 t)^2} \, d\mu^{mp}(t)$$

In this question

$$|u^T \Sigma^{-\frac{1}{2}} v|^{-2} \sim \frac{\lambda_0^2}{\sigma^4} \int_a^b \frac{\sigma^4 t^2}{\left(\sigma^2\left(\frac{\lambda}{\sigma^2} - t\right)\right)^2} \, d\mu^{mp}(t)$$

$$|u^T \Sigma^{-\frac{1}{2}} v|^2 = \frac{1 - \frac{r}{SNR^2}}{1 + r + \frac{2r}{SNR}}$$

$$\Rightarrow |u^T v|^2 = \frac{1 - \frac{r}{SNR^2}}{1 + \frac{r}{SNR}}$$

3. (b) λ is the eigenvalue and v is the eigenvector for $W + \lambda_0 u u^T$,

then $(W + \lambda_0 u u^T) v = \lambda v$

$$\Rightarrow v(\lambda I_p - W) = \lambda_0 u (u^T v)$$

$$\Rightarrow v = (\lambda I_p - W)^{-1} \lambda_0 u (u^T v)$$

$$u^T v = u^T (\lambda I_p - W)^{-1} \lambda_0 u (u^T v)$$

$$I = u^T (\lambda I_p - W)^{-1} \lambda_0 u$$

Set $W = D \Sigma D^T$, $\alpha_i = w_i^T u$, $\alpha = (\alpha_i)$

$$I = u^T D (\lambda I_p - \Sigma)^{-1} \lambda_0 D^T u$$

$$= \lambda_0 \sum_{i=1}^{p} \frac{1}{\lambda - \lambda_i} \alpha_i^2$$

$$\Rightarrow \quad I \sim \lambda_0 \int_a^b \frac{1}{\lambda - t} \, d\mu(t)$$

$$= \lambda_0 \int_a^b \frac{2\sqrt{1-t^2}}{\pi(\lambda - t)} \, dt$$

for $a = -1 \quad b = 1$

$$I \sim \lambda_0 (2\lambda - 2\sqrt{\lambda^2 - 1})$$

$$\Rightarrow \lambda = \lambda_0 + \frac{1}{4\lambda_0}$$

$$|u^T v|^{-2} = \lambda_0^2 \, u^T (\lambda I_p - \Sigma^{-2}) u$$

$$= \frac{4\lambda_0^2}{4\lambda_0^2 - 1}$$

$$|u^T v|^2 = 1 - \frac{1}{4\lambda_0^2}$$