Homework 2

2. (a)

$$SNR = \frac{\lambda_0}{\sigma^2} > \sqrt{\gamma}$$

The Marchenko-Pastur law states that the empirical spectral distribution of S_n converges to the probability measure with density

$$\rho(\lambda) = \frac{\gamma}{2\pi\sigma^2} \cdot \sqrt{\frac{(\lambda_{max}-\lambda)(\lambda-\lambda_{min})}{\lambda}}$$

on the interval $[\lambda_{min}, \lambda_{max}]$, where $\lambda_{min} = \sigma^2(1-\sqrt{\gamma})^2$ and $\lambda_{max} = \sigma^2(1+\sqrt{\gamma})^2$.

Given $SNR > \sqrt{\gamma}$, we have $\frac{\lambda_0}{\sigma^2} > \sqrt{\gamma} \Rightarrow \lambda_0 > \sigma^2\sqrt{\gamma}$.

Using the formula for the sample covariance matrix,

$$S_n = \frac{1}{n} X X^T,$$

where X is the matrix whose rows are X_i.

The largest eigenvalue λ of S_n is given by

$$\lambda = \lambda_{max}(S_n).$$

$$\lambda_{max} = \lambda_{max}(S_n) \approx \lambda_{max}(\rho) = \lambda_{max}\left(\frac{\gamma}{2\pi\sigma^2}\right) \cdot \sqrt{\frac{(\lambda_{max}-\lambda_{min})(\lambda_{max}-\lambda)}{\lambda_{max}}}$$

$$\lambda_{max} = \sigma^2(1+\sqrt{\gamma})^2.$$

(b) We can use the largest eigenvalue λ of the sample covariance matrix S_n to estimate SNR.

Under the Marchenko-Pastur law, the eigenvalues of S_n are distributed with density

$$\rho(\lambda) = \frac{\gamma}{2\pi\sigma^2} \sqrt{\frac{(\lambda_{max}-\lambda)(\lambda-\lambda_{min})}{\lambda}}$$

We can estimate the SNR as

$$\frac{\lambda_0}{\sigma^2} = \frac{\Lambda}{\lambda_{max}}.$$

Substituting the expression for λ_{max}, we obtain

$$\frac{\lambda_0}{\sigma^2} = \frac{\lambda}{\sigma^2[(1+\sqrt{r})^2}.$$

We can estimate the SNR by computing

$$\frac{\lambda_0}{\sigma^2} = \frac{\lambda}{\sigma^2[(1+\sqrt{r})^2}.$$

(c) The Rayleigh-Ritz thm states that for any matrix A and vector x, we have

$$\min_x(x^T A x) \leq x^T A x \leq \max_x(x^T A x).$$

We apply the Rayleigh-Ritz thm to the sample covariance matrix S_n, we obtain

$$\min_x(x^T S_n x) \leq x^T S_n x \leq \max_x(x^T S_n x).$$

Since S_n is a positive semi-definite matrix, the maximum value of $x^T S_n x$ is attained when x is a linear combination.
we have

$$\min_x(x^T S_n x) = V^T S_n V = V^T(\sigma^2 I + \lambda_0 u u^T)v$$
$$= \sigma^2 + \lambda_0(u^T v)^2.$$

Therefore, we have

$$\sigma^2 + \lambda_0(u^T v)^2 \leq \lambda,$$
$$|u^T v|^2 \leq \frac{\lambda - \sigma^2}{\lambda_0}$$

Substituting the expression for λ, we obtain

$$|u^T v|^2 \leq \frac{\sigma^2[(1+\sqrt{r})^2 - \sigma^2}{\lambda_0} = \frac{\sigma^2(2\sqrt{r}+r)}{\lambda_0}.$$

(b) Suppose we have a Wigner matrix W, we perturb it by adding a rank-1 matrix uv^T. Then the perturbed matrix is given by $M = W + uv^T$.

The largest eigenvalue of a symmetric matrix A is given by

$$\lambda_{max}(A) = \max_x \frac{x^T A x}{\|x\|^2}.$$

Let x be the eigenvector corresponding to the largest eigenvalue of M. Then we have

$$Mx = \lambda_{max}(M)\, x$$

$$x^T M x = \lambda_{max}(M)\, x^T x.$$

Since M is symmetric, we have

$$x^T M x = x^T W x + x^T (uv^T) x = x^T W x + (v^T x)(u^T x).$$

Since W is a Wigner matrix, its eigenvalues are typically of order $\sqrt{n}$, and its largest eigenvalue is of order $\sqrt{n}$.

We can approximate the largest eigenvalue of M as

$$\lambda_{max}(M) = \lambda_{max}(W) + \|u\|\,\|v\|.$$