Parsimonious Deep Learning with Structural Sparsity via Differential Inclusions

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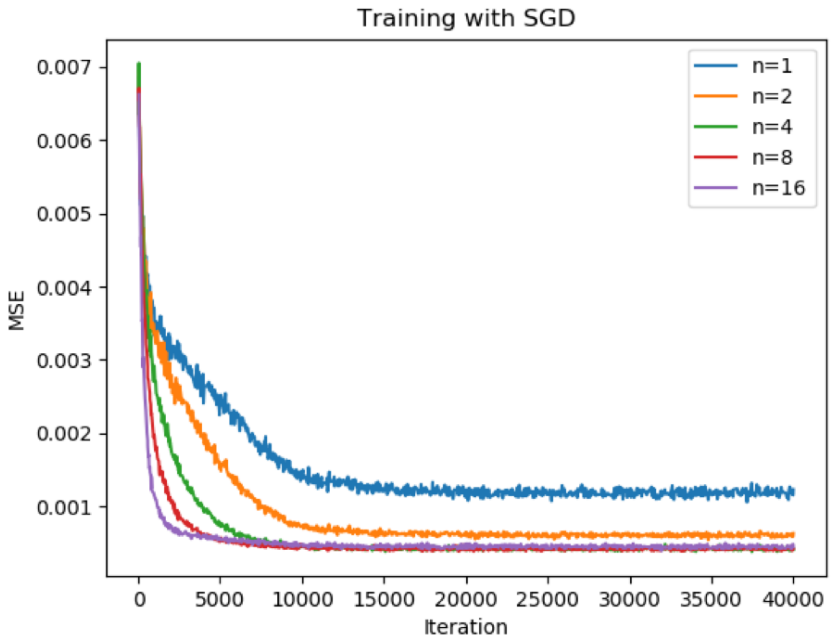
Deep training needs Overparameterization!

Example

Teacher network: 2 layer random ReLU network (size: $150 - 60 - 10$);

Student network: 2 layer ReLU net (size: $150 - 60 * n - 10$)

Training: MSE loss, SGD with i.i.d. batchsize 256 (lr=1e-2, 1/4 for 5000 iter)



The magic of overparameterization

- Overparameterization may help simplify optimization landscape to find global optima;
- Overparameterization may even help generalization without overfitting...

The Lottery Ticket Hypothesis

Conjecture (Frankle-Carbin, ICLR 2019)

Dense, randomly-initialized, feed-forward networks contain subnetworks (winning tickets) that – when trained in isolation – reach test accuracy comparable to the original network in a similar number of iterations.

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- Train a dense, randomly initialized network for certain iterations;
- Prune small weights of the trained network for a sparse subnet;
- Retrain the subnet with the same initialization for a similar number of iterations.

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- Prune small weights of the trained network for a sparse subnet;
- Retrain the subnet with the same initialization for a similar number of iterations.

Yet, is it necessary to fully train a dense, over-parameterized model before finding important sparse subnets?

— *Liu et al., Rethinking the value of network pruning, ICLR 2019*

Training Overparameterized Networks with Sparse Filters

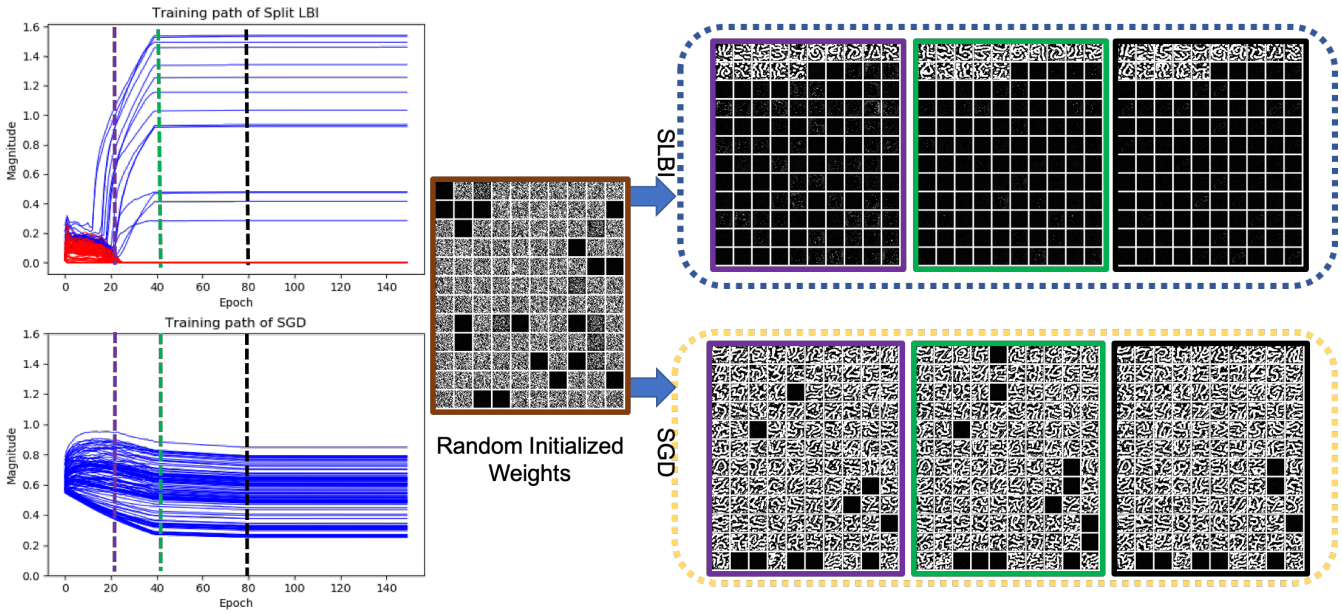


Figure: Visualization of solution path and filter patterns in the third convolutional layer (i.e., conv.c5) of LetNet-5, trained on MNIST. It shows that our algorithm (SplitLBI) enjoys a sparse selection of filters without sacrificing accuracy

Flow Chart for Finding Sparse Structure

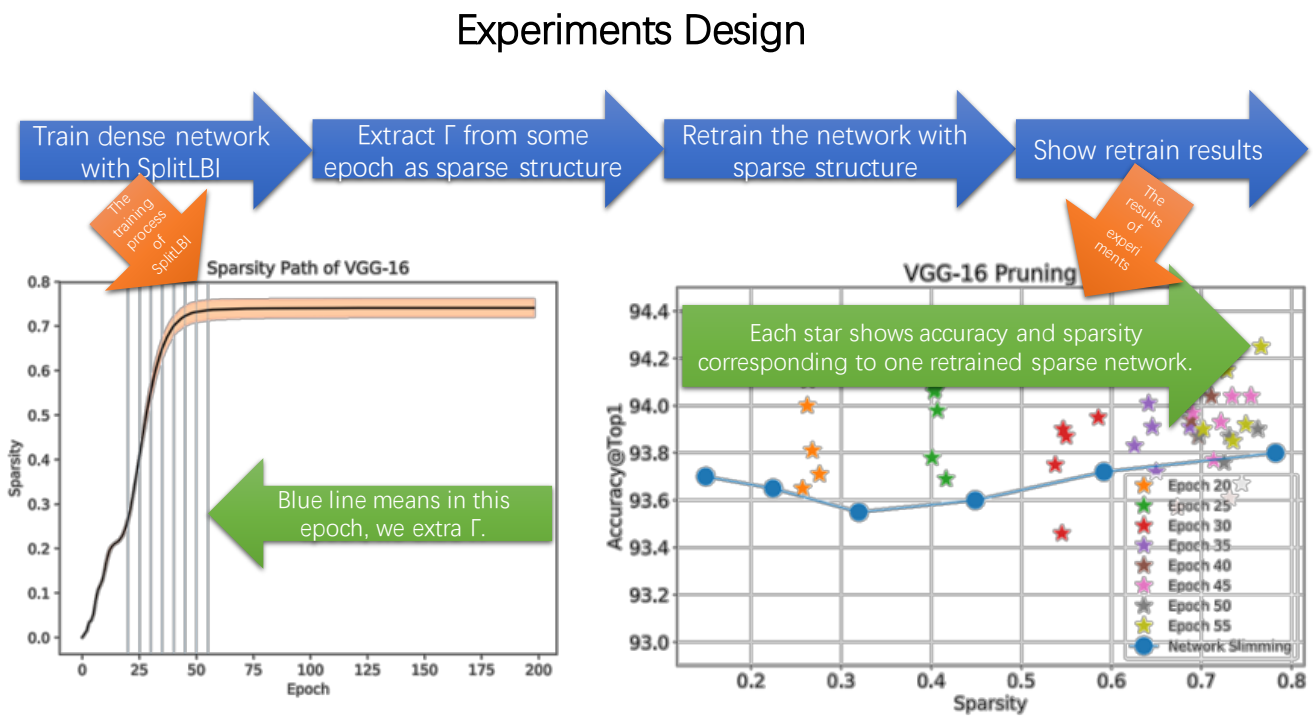


Figure: with Yanwei Fu, Donghao Li, Chen Liu, Xinwei Sun, 2019

Sparse Structure Results

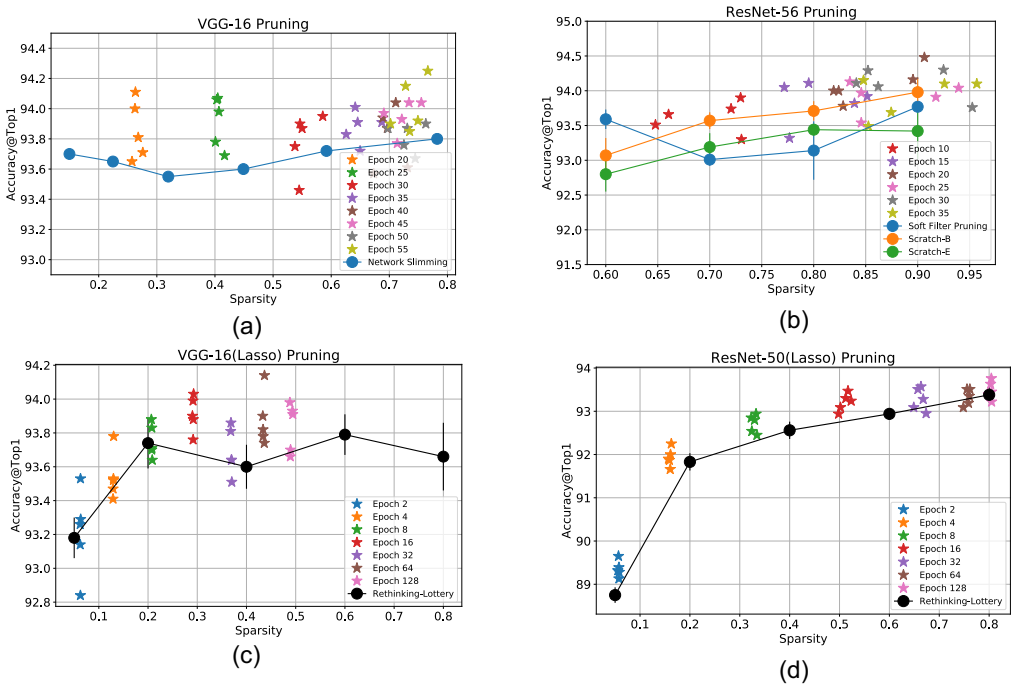


Figure: SplitLBI with early stopping finds sparse subnets whose test accuracies (stars) after retrain are comparable or even better than the baselines (Network Slimming, Soft-Filter Pruning, Scratch-B, Scratch-E, and "Rethinking-Lottery" as reported in Rethink the Value of Pruning. Sparse filters of VGG-16 and ResNet-56 are show in (a) and (b), while sparse weights of VGG-16 and ResNet-50 are shown in (c) and (d).

Algorithm: Split Linearized Bregman Iterations (SplitLBI)

$$W_{k+1} = W_k - \kappa\alpha_k \cdot \nabla_W \bar{\mathcal{L}}\left(W_k, \Gamma_k\right), \tag{1a}$$

$$V_{k+1} = V_k - \alpha_k \cdot \nabla_\Gamma \bar{\mathcal{L}}\left(W_k, \Gamma_k\right), \tag{1b}$$

$$\Gamma_{k+1} = \kappa \cdot \mathrm{Prox}_{\Omega_\lambda}\left(V_{k+1}\right), \tag{1c}$$

where

$$\mathrm{Prox}_{\Omega_\lambda}(V) = \arg\min_{\Gamma} \left\{ \frac{1}{2}\|\Gamma - V\|_2^2 + \Omega_\lambda(\Gamma) \right\}, \tag{2}$$

- For group Lasso penalty $\Omega_1(\Gamma) = \sum_g \|\Gamma^g\|_2$, Eq. (1c) has a closed form solution $\Gamma^g = \kappa \cdot \max\left(0, 1 - 1/\|V^g\|_2\right) V^g$ for the g-th filter.
  - Convolutional filters take group Lasso penalty;
  - Fully connected layers reduce it to Lasso penalty.

Differential Inclusion Method

It is Euler forward discretization of differential inclusion

$$\frac{\dot{W}_t}{\kappa} = -\nabla_W \bar{\mathcal{L}}(W_t, \Gamma_t) \tag{3a}$$

$$\dot{V}_t = -\nabla_\Gamma \bar{\mathcal{L}}(W_t, \Gamma_t) \tag{3b}$$

$$V_t \in \partial\bar{\Omega}(\Gamma_t) \tag{3c}$$

where

- the augmented empirical loss

$$\bar{\mathcal{L}}(W, \Gamma) = \widehat{\mathcal{L}}_n(W) + \frac{1}{2\nu}\|W - \Gamma\|_2^2, \quad \nu > 0, \tag{4}$$

- the weight W_t goes *gradient descent flow* in ℓ_2-proximity of Γ
- the dual weight V goes *mirror descent flow*, as a sub-gradient of $\bar{\Omega}(\Gamma) := \Omega_\lambda(\Gamma) + \frac{1}{2\kappa}\|\Gamma\|^2$ for sparse regularization $\Omega_\lambda(\Gamma) = \lambda\Omega_1(\Gamma)$ ($\lambda \in \mathbb{R}_+$) (e.g. group lasso $\Omega_1(\Gamma)$)

Exploring Sparse Structures of Overparameterized Models

- Gradient descent flow W_t in the proximity of Γ_t explores *overparameterized* models;
- In mirror descent flow, gradient descent goes on the dual space consisting of sub-gradients V_t, driving the flow in primal space Γ_t that is *sparse*:
  - $\mathrm{supp}(\Gamma)$ gives us a sequence of sparsity structure at different levels,
  - Sparsity structure is more important than the weights of the network

  and it can be evaluated by retrain the network with that structure.

How does it work? Convergence theory is not enough!

Theorem (Global Convergence, with Jinshan Zeng et al., 2019)

Assume:

(a) ℓ is any smooth definable loss function, such as the square loss (t^2), exponential loss (e^t), logistic loss $\log(1+e^{-t})$, and cross-entropy loss;

(b) σ_i is any smooth definable activation, such as linear activation (t), sigmoid ($\frac{1}{1+e^{-t}}$), hyperbolic tangent ($\frac{e^t-e^{-t}}{e^t+e^{-t}}$), and softplus ($\frac{1}{c}\log(1+e^{ct})$ for some $c>0$) as a smooth approximation of ReLU;

(c) Ω is the group Lasso.

For any finite initialization and

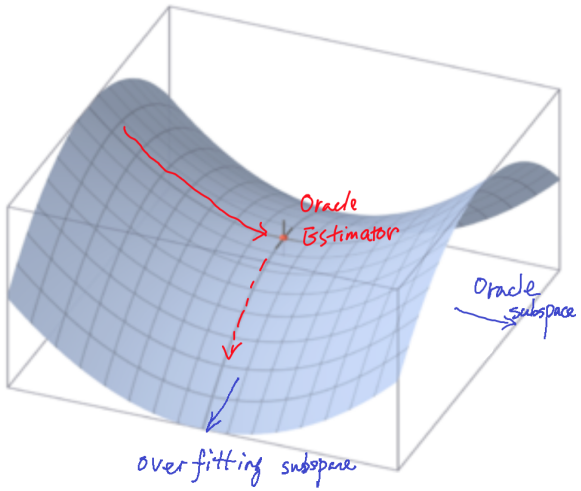
$$0 < \alpha_k = \alpha < \frac{2}{\kappa(Lip_{\nabla \widehat{\mathcal{L}}_n} + \nu^{-1})},$$

(W_k, Γ_k) converges to a critical point of $\bar{\mathcal{L}}$ defined in Eq. (4), and $\{W^k\}$ converges to a critical point of $\widehat{\mathcal{L}}_n(W)$.

How does it work?

A preliminary statistical theory for linear models brings more insights:

- Differential inclusion here is a restricted gradient descent flow (Figure)
- Under nearly the same condition as LASSO, it reaches variable selection consistency, though may incur less bias than LASSO
- Equipped with variable splitting, SplitLBI weakens the conditions of Generalized LASSO in variable selection



Sparse Linear Regression

Assume that $\beta^* \in \mathbb{R}^p$ is sparse and unknown. Consider recovering β^* from n linear measurements

$$y = X\beta^* + \epsilon, \qquad y \in \mathbb{R}^n$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$ is **noise**.

- Basic Sparsity: $S := \mathrm{supp}(\beta^*)$ ($s = |S|$) and T be its complement.
  - X_S (X_T) be the columns of X with indices restricted on S (T)
  - X is n-by-p, with $p \gg n \geq s$.
- Or Structural Sparsity: $\gamma^* = D\beta^*$ is sparse, where D is a linear transform (wavelet, gradient, etc.), $S = \mathrm{supp}(\gamma^*)$
- *How to recover β^* (or γ^*) sparsity pattern (sparsistency) and estimate values with variations (consistency)?*

Best Possible in Basic Setting: The Oracle Estimator

Had God revealed S to us, the *oracle estimator* was the subset least square solution (MLE) with $\tilde{\beta}^*_T = 0$ and

$$\tilde{\beta}^*_S = \beta^*_S + \frac{1}{n}\Sigma_n^{-1} X_S^T \epsilon, \quad \text{where } \Sigma_n = \tfrac{1}{n} X_S^T X_S \tag{5}$$

"Oracle properties"

- **Model selection consistency**: $\mathrm{supp}(\tilde{\beta}^*) = S$;
- **Normality**: $\tilde{\beta}^*_S \sim \mathcal{N}(\beta^*, \frac{\sigma^2}{n}\Sigma_n^{-1})$.

So $\tilde{\beta}^*$ is unbiased, i.e. $\mathbb{E}[\tilde{\beta}^*] = \beta^*$.

Recall LASSO

LASSO:

$$\min_{\beta} \|\beta\|_1 + \frac{t}{2n}\|y - X\beta\|_2^2.$$

optimality condition:

$$\frac{\rho_t}{t} = \frac{1}{n}X^T(y - X\beta_t), \tag{6a}$$

$$\rho_t \in \partial\|\beta_t\|_1, \tag{6b}$$

where $\lambda = 1/t$ is often used in literature.

- Chen-Donoho-Saunders'1996 (BPDN)
- Tibshirani'1996 (LASSO)

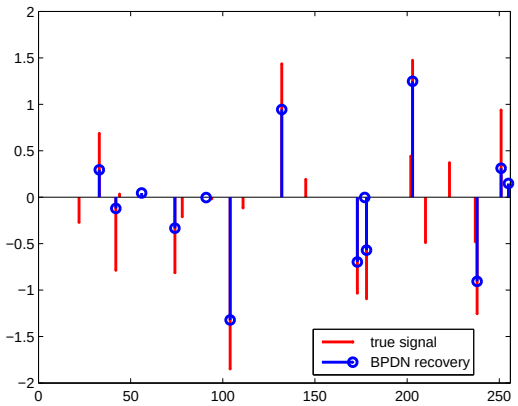
The Bias of LASSO

LASSO is biased, i.e. $\mathbb{E}(\hat{\beta}) \neq \beta^*$

- e.g. $X = Id$, $n = p = 1$, LASSO is soft-thresholding

$$\hat{\beta}_\tau = \begin{cases} 0, & \text{if } \tau < 1/\tilde{\beta}^*; \\ \tilde{\beta}^* - \frac{1}{\tau}, & \text{otherwise,} \end{cases}$$

- e.g. $n = 100$, $p = 256$, $X_{ij} \sim \mathcal{N}(0,1)$, $\epsilon_i \sim \mathcal{N}(0, 0.1)$



True vs LASSO (t hand-tuned)

LASSO Estimator is Biased at Path Consistency

Even when the following **path consistency** (conditions given by Zhao-Yu'06, Zou'06, Yuan-Lin'07, Wainwright'09, etc.) is reached at τ_n:

$$\exists \tau_n \in (0, \infty) \text{ s.t. } \operatorname{supp}(\hat{\beta}_{\tau_n}) = S,$$

LASSO estimate is biased away from the oracle estimator

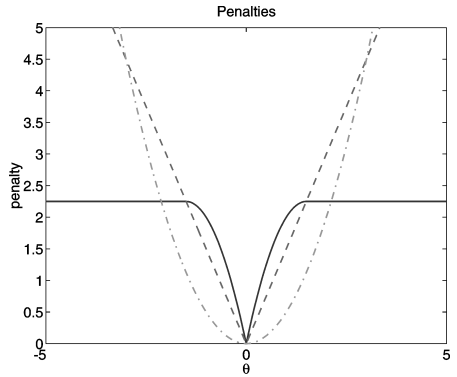
$$(\hat{\beta}_{\tau_n})_S = \tilde{\beta}^*_S - \frac{1}{\tau_n} \Sigma^{-1}_{n,S} \operatorname{sign}(\beta^*_S), \quad \tau_n > 0.$$

How to remove the bias and return the Oracle Estimator?

Nonconvex Regularization?

- To reduce bias, **non-convex** regularization was proposed (Fan-Li's SCAD, Zhang's MPLUS, Zou's Adaptive LASSO, l_q $(q < 1)$, etc.)

$$\min_{\beta} \sum_i p(|\beta_i|) + \frac{t}{2n}\|y - X\beta\|_2^2.$$



- Yet it is generally hard to locate the global optimizer
- *Any other simple scheme?*

New Idea

- LASSO:

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- Taking derivative (assuming differentiability) w.r.t. t

$$\Rightarrow \dot{\rho}_t = \frac{1}{n}X^T(y - X(\dot{\beta}_t t + \beta_t)), \quad \rho_t \in \partial\|\beta_t\|_1$$

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- Assuming sign-consistency in a neighborhood of τ_n,

$$\text{for } i \in S,\ \rho_{\tau_n}(i) = \text{sign}(\beta^*(i)) \in \pm 1 \Rightarrow \dot{\rho}_{\tau_n}(i) = 0,$$

$$\Rightarrow \dot{\beta}_{\tau_n} \tau_n + \beta_{\tau_n} = \tilde{\beta}^*$$

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- Equivalently, the blue part removes bias of LASSO automatically

$$\beta^{lasso}_{\tau_n} = \tilde{\beta}^* - \frac{1}{\tau_n}\Sigma_n^{-1}\text{sign}(\beta^*) \Rightarrow \dot{\beta}^{lasso}_{\tau_n}\tau_n + \beta^{lasso}_{\tau_n} = \tilde{\beta}^*(oracle)!$$

Differential Inclusion: Inverse Scaled Spaces (ISS)

Differential inclusion replacing $\dot{\beta}^{lasso}_{\tau_n}\tau_n + \beta^{lasso}_{\tau_n}$ by β_t

$$\dot{\rho}_t = \frac{1}{n}X^T(y - X\beta_t), \tag{7a}$$

$$\rho_t \in \partial\|\beta_t\|_1. \tag{7b}$$

starting at $t = 0$ and $\rho(0) = \beta(0) = \mathbf{0}$.

- Replace ρ/t in LASSO KKT by $\mathrm{d}\rho/\mathrm{d}t$

$$\frac{\rho_t}{t} = \frac{1}{n}X^T(y - X\beta_t)$$

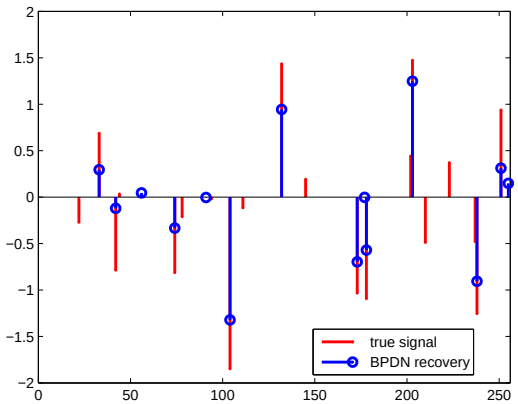
- Burger-Gilboa-Osher-Xu'06 (in image recovery it recovers the objects in an inverse-scale order as t increases (larger objects appear in β_t first))

Examples

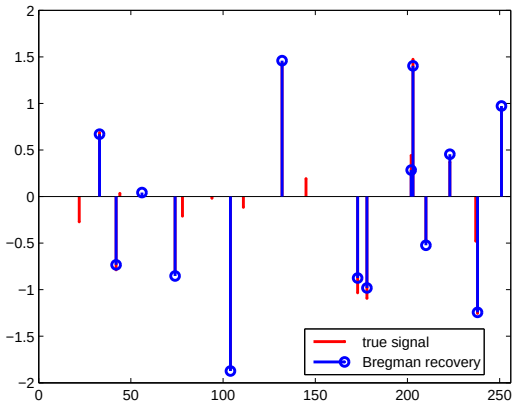
- e.g. $X = Id$, $n = p = 1$, hard-thresholding

$$\beta_\tau = \begin{cases} 0, & \text{if } \tau < 1/(\tilde{\beta}^*); \\ \tilde{\beta}^*, & \text{otherwise,} \end{cases}$$

- the same example shown before



True vs LASSO



True vs ISS

Solution Path: Sequential Restricted Maximum Likelihood Estimate

- ρ_t is piece-wise linear in t,

$$\rho_t = \rho_{t_k} + \frac{t - t_k}{n} X^T (y - X\beta_{t_k}), \quad t \in [t_k, t_{k+1})$$

where $t_{k+1} = \sup\{t > t_k : \rho_{t_k} + \frac{t-t_k}{n} X^T (y - X\beta_{t_k}) \in \partial \|\beta_{t_k}\|_1\}$

- β_t is piece-wise constant in t: $\beta_t = \beta_{t_k}$ for $t \in [t_k, t_{k+1})$ and $\beta_{t_{k+1}}$ is the **sequential restricted Maximum Likelihood Estimate** by solving nonnegative least square (Burger et al.'13; Osher et al.'16)

$$\begin{aligned} \beta_{t_{k+1}} = \arg\min_\beta \quad & \|y - X\beta\|_2^2 & & \\ \text{subject to} \quad & (\rho_{t_{k+1}})_i \beta_i \geq 0 & \forall\ i \in S_{k+1}, & \\ & \beta_j = 0 & \forall\ j \in T_{k+1}. & \end{aligned} \tag{8}$$

- Note: Sign consistency $\rho_t = \mathrm{sign}(\beta^*) \Rightarrow \beta_t = \tilde{\beta}^*$ the oracle estimator

Example: Regularization Paths of LASSO vs. ISS

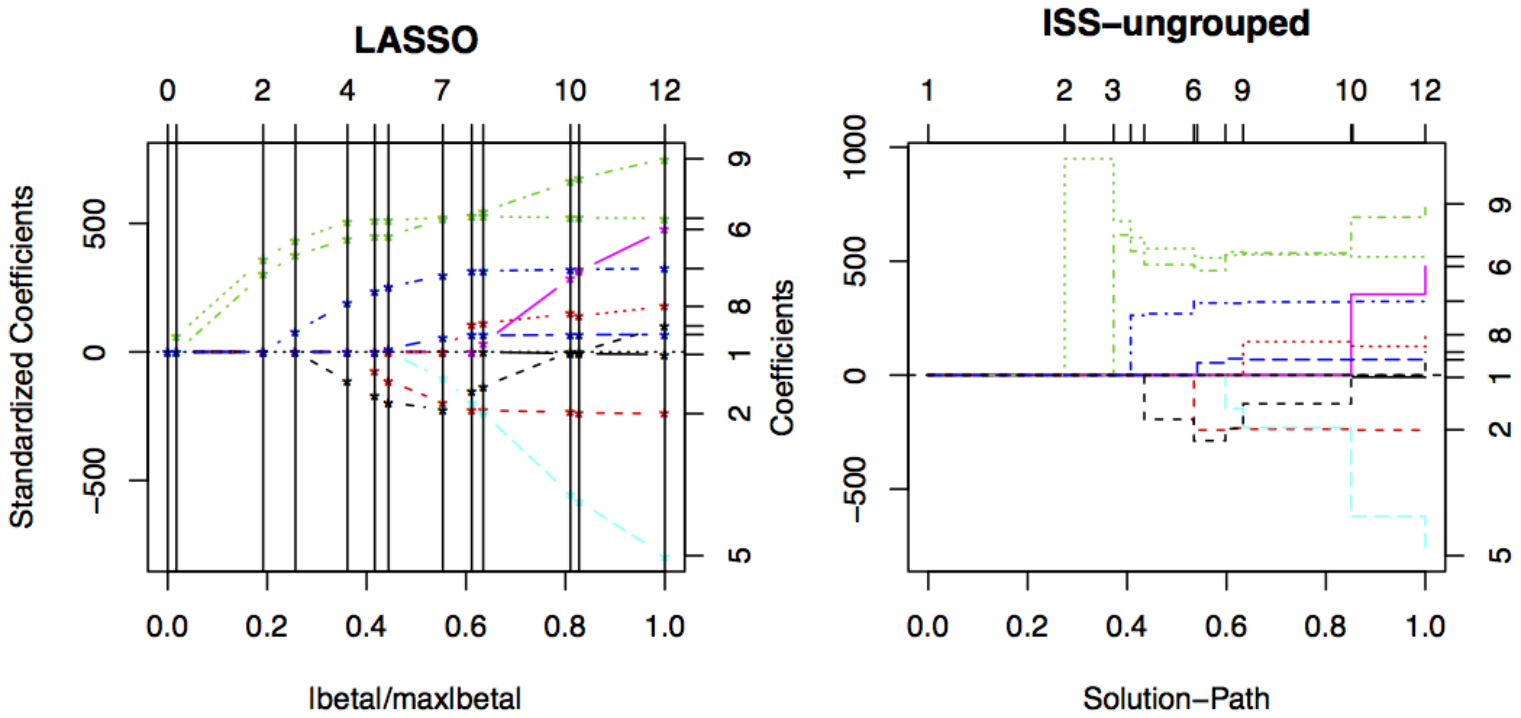


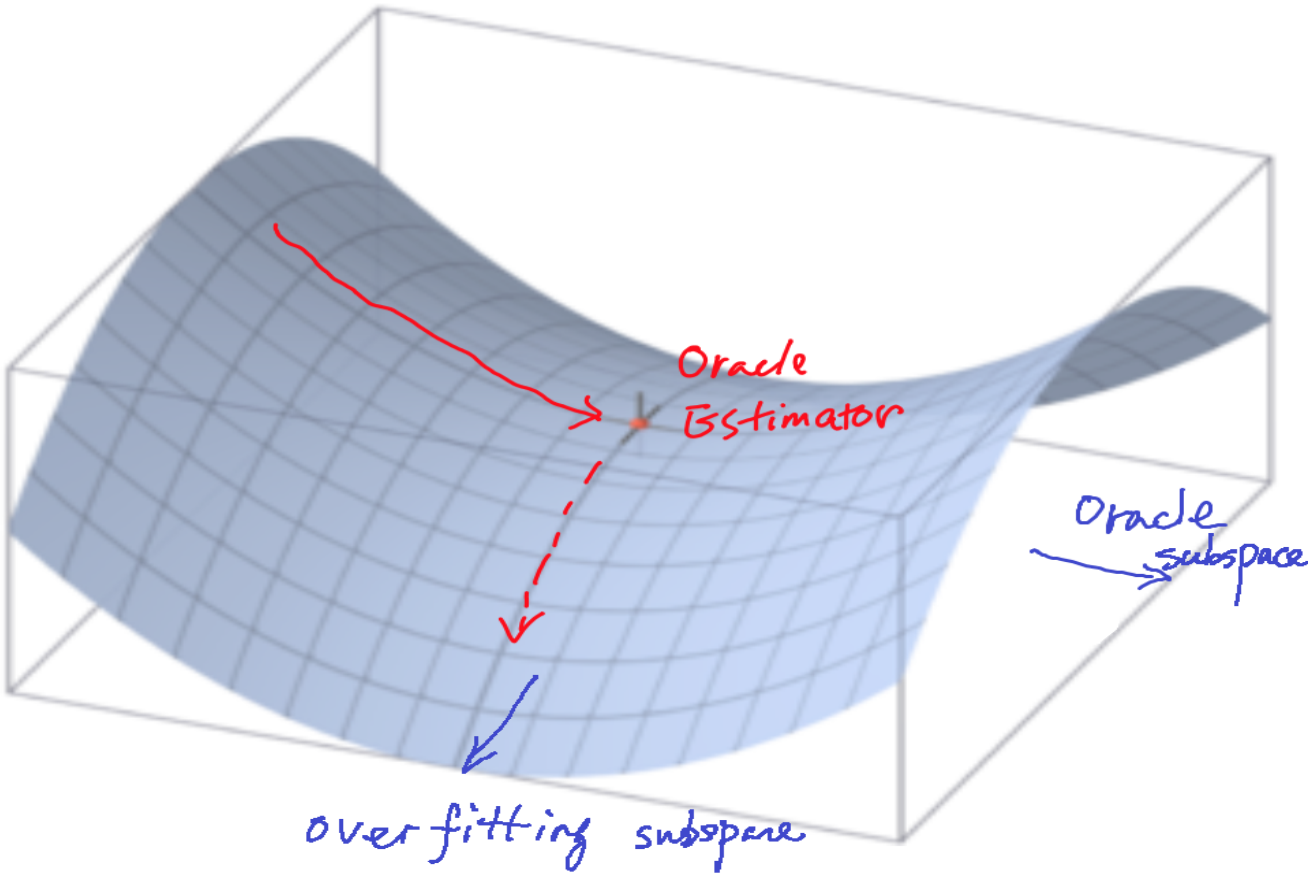
Figure: Diabetes data (Efron et al.'04) and regularization paths are different, yet bearing similarities on the order of parameters being nonzero

How does it work? A Path Consistency Theory

Our aim is to show that under nearly the same conditions for sign-consistency of LASSO, there exists points on their paths $(\beta(t), \rho(t))_{t \geq 0}$, which are

- sparse
- sign-consistent (the same sparsity pattern of nonzeros as true signal)
- the oracle estimator which is unbiased, better than the LASSO estimate.
- Early stopping regularization is necessary to prevent overfitting noise!

Intuition



History: two traditions of regularizations

- Penalty functions
  - ℓ_2: Ridge regression/Tikhonov regularization: $\frac{1}{n}\sum_{i=1}^n \ell(y_i, x_i^T\beta) + \lambda\|\beta\|_2^2$
  - ℓ_1 (sparse): Basis Pursuit/LASSO (ISTA): $\frac{1}{n}\sum_{i=1}^n \ell(y_i, x_i^T\beta) + \lambda\|\beta\|_1^2$
- Early stopping of dynamic regularization paths
  - ℓ_2-equivalent: Landweber iterations/gradient descent/ℓ_2-Boost

$$\frac{d\beta_t}{dt} = -\frac{1}{n}\sum_{i=1}^n \nabla_\beta \ell(y_i, x_i^T\beta), \quad \beta_t = \nabla\left\{\frac{1}{2}\|\beta_t\|^2\right\}$$

  - ℓ_1 (sparse)-equiv.: Orthogonal Matching Pursuit, Linearized Bregman Iteration (sparse Mirror Descent) (not ISTA! – later)

$$\frac{d\rho_t}{dt} = -\frac{1}{n}\sum_{i=1}^n \nabla_\beta \ell(y_i, x_i^T\beta), \quad \rho_t \in \partial\|\beta_t\|_1$$

Assumptions

(A1) **Restricted Strongly Convex**: $\exists \gamma \in (0, 1]$,

$$\frac{1}{n} X_S^T X_S \geq \gamma I$$

(A2) **Incoherence/Irrepresentable** Condition: $\exists \eta \in (0, 1)$,

$$\left\| \frac{1}{n} X_T^T X_S^\dagger \right\|_\infty = \left\| \frac{1}{n} X_T^T X_S \left(\frac{1}{n} X_S^T X_S \right)^{-1} \right\|_\infty \leq 1 - \eta$$

- "Irrepresentable" means that one can not represent (regress) column vectors in X_T by covariates in X_S.
- The incoherence/irrepresentable condition is used independently in Tropp'04, Yuan-Lin'05, Zhao-Yu'06, and Zou'06, Wainwright'09, etc.

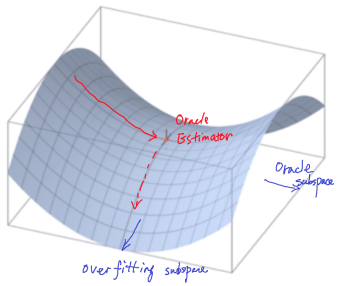
Understanding the Dynamics

ISS as restricted gradient descent:

$$\dot{\rho}_t = -\nabla L(\beta_t) = \frac{1}{n} X^T (y - X\beta_t), \quad \rho_t \in \partial \|\beta_t\|_1$$

such that

- incoherence condition and strong signals ensure it firstly evolves on index set S to reduce the loss
- strongly convex in subspace restricted on index set $S \Rightarrow$ fast decay in loss
- early stopping after all strong signals are detected, before picking up the noise



Path Consistency

Theorem (Osher-Ruan-Xiong-Y.-Yin'2016)

Assume (A1) and (A2). Define an early stopping time

$$\bar{\tau} := \frac{\eta}{2\sigma}\sqrt{\frac{n}{\log p}}\left(\max_{j\in T}\|X_j\|\right)^{-1},$$

and the smallest magnitude $\beta^*_{\min} = \min(|\beta^*_i| : i \in S)$. Then

- No-false-positive: for all $t \le \bar{\tau}$, the path has no-false-positive with high probability, $\mathrm{supp}(\beta(t)) \subseteq S$;
- Consistency: moreover if the signal is strong enough such that

$$\beta^*_{min} \ge \left(\frac{4\sigma}{\gamma^{1/2}} \vee \frac{8\sigma(2+\log s)\left(\max_{j\in T}\|X_j\|\right)}{\gamma\eta}\right)\sqrt{\frac{\log p}{n}},$$

  there is $\tau \le \bar{\tau}$ such that solution path $\beta(t)) = \tilde{\beta}^*$ for every $t \in [\tau, \bar{\tau}]$.

Note: equivalent to LASSO with $\lambda^* = 1/\bar{\tau}$ (Wainwright'09) up to $\log s$.

Large scale algorithm: Linearized Bregman Iteration

Damped Dynamics: continuous solution path

$$\dot{\rho}_t + \frac{1}{\kappa}\dot{\beta}_t = \frac{1}{n}X^T(y - X\beta_t), \quad \rho_t \in \partial\|\beta_t\|_1. \tag{9}$$

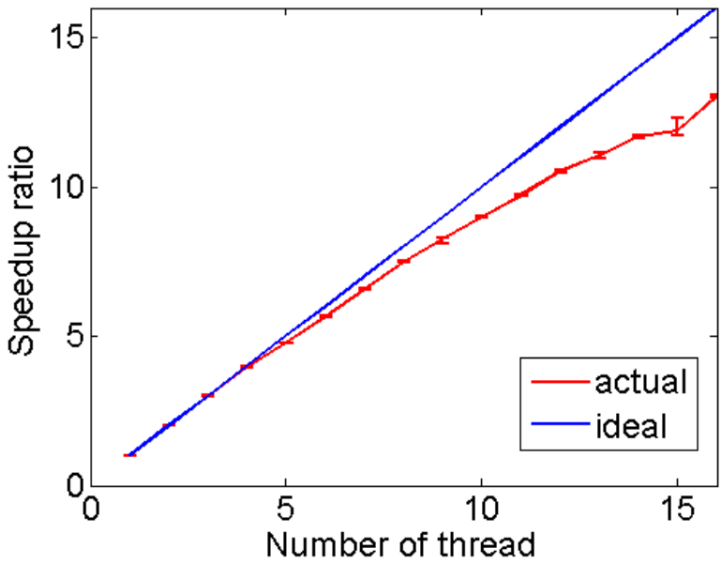
Linearized Bregman Iteration as forward Euler discretization proposed even earlier than ISS dynamics (Osher-Burger-Goldfarb-Xu-Yin'05, Yin-Osher-Goldfarb-Darbon'08): for $\rho_k \in \partial\|\beta_k\|_1$,

$$\rho_{k+1} + \frac{1}{\kappa}\beta_{k+1} = \rho_k + \frac{1}{\kappa}\beta_k + \frac{\alpha_k}{n}X^T(y - X\beta_k), \tag{10}$$

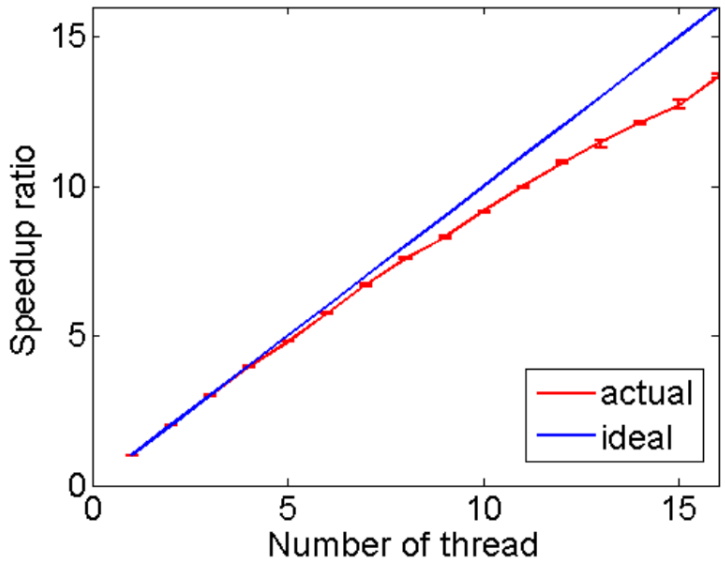
where

- Damping factor: $\kappa > 0$
- Step size: $\alpha_k > 0$ s.t. $\alpha_k\kappa\|\Sigma_n\| \leq 2$
- Moreau Decomposition: $z_k := \rho_k + \frac{1}{\kappa}\beta_k \Leftrightarrow \beta_k = \kappa \cdot \mathit{Shrink}(z_k, 1)$

Easy for Parallel Implementation



(a) n=1000



(b) n=2000

Figure: Linear speed-ups on a 16-core machine with synchronized parallel computation of matrix-vector products.

Comparison with ISTA

Linearized Bregman (LB) iteration:

$$z_{t+1} = z_t - \alpha_t X^T(\kappa X \mathit{Shrink}(z_t, 1) - y)$$

which is not **ISTA**:

$$z_{t+1} = \mathit{Shrink}(z_t - \alpha_t X^T(Xz_t - y), \lambda).$$

Comparison:

- **ISTA**:
  - as $t \to \infty$ solves **LASSO**: $\frac{1}{n}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1$
  - parallel run ISTA with $\{\lambda_k\}$ for LASSO regularization paths
- **LB**: a single run generates the whole regularization path at same cost of ISTA-LASSO estimator for a fixed regularization

LB generates regularization paths

$n = 200$, $p = 100$, $S = \{1, \ldots, 30\}$, $x_i \sim N(0, \Sigma_p)$ ($\sigma_{ij} = 1/(3p)$ for $i \neq j$ and 1 otherwise)

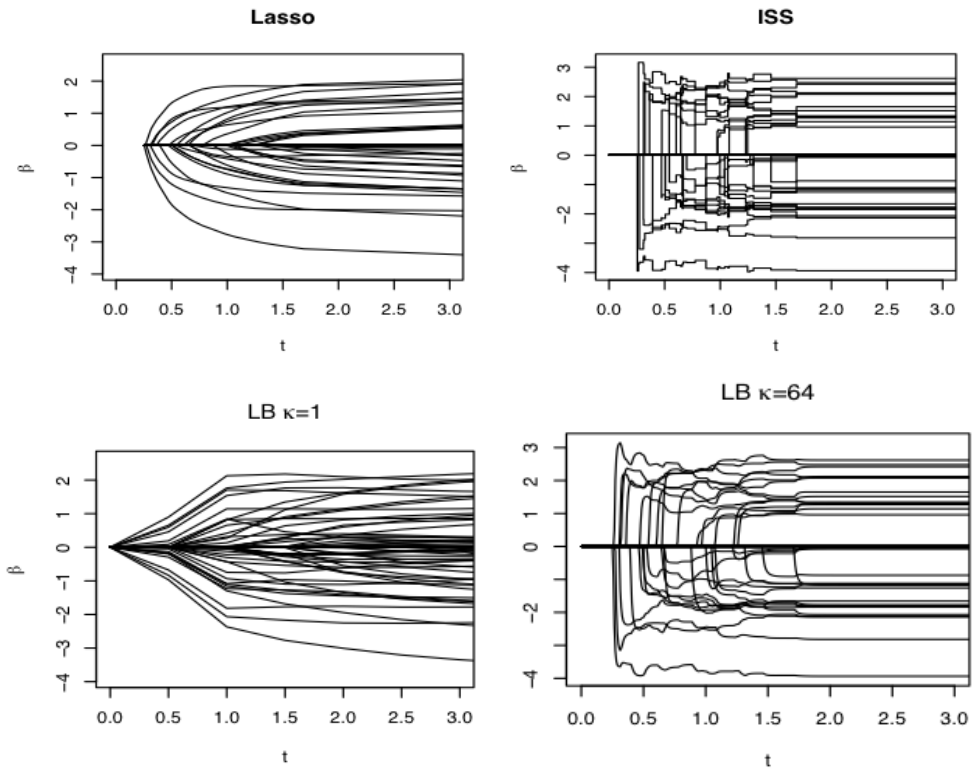
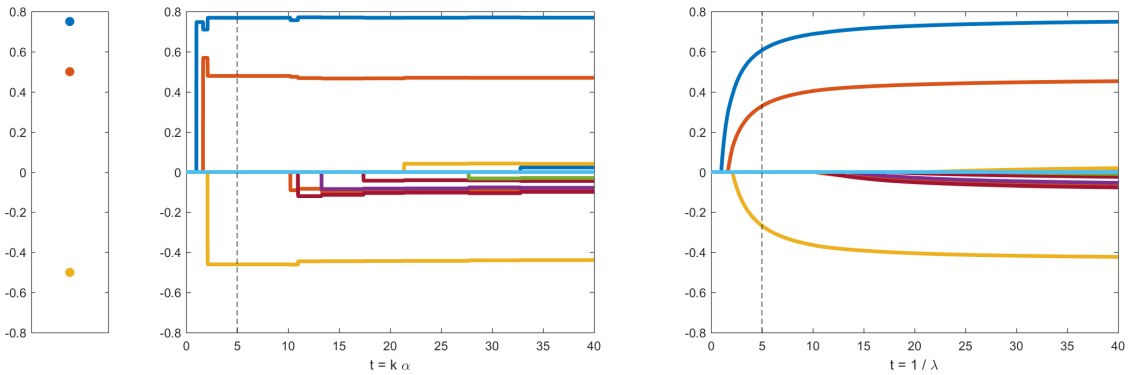


Figure: As $\kappa \to \infty$, LB paths have a limit as piecewise-constant ISS path

Accuracy: LB may be less biased than LASSO



- Left shows (the magnitudes of) nonzero entries of $\beta^\star$.
- Middle shows the regularization path of LB.
- Right shows the regularization path of LASSO vs. $t = 1/\lambda$.

Path Consistency in Discrete Setting

Theorem (Osher-Ruan-Xiong-Y.-Yin'2016)

Assume that κ is large enough and α is small enough, with $\kappa\alpha\|X_S^* X_S\| < 2$,

$$\overline{\tau} := \frac{(1 - B/\kappa\eta)\eta}{2\sigma}\sqrt{\frac{n}{\log p}}\left(\max_{j\in T}\|X_j\|\right)^{-1}$$

$$\beta^*_{max} + 2\sigma\sqrt{\frac{\log p}{\gamma n}} + \frac{\|X\beta^*\|_2 + 2s\sqrt{\log n}}{n\sqrt{\gamma}} \triangleq B \leq \kappa\eta,$$

then all the results for ISS can be extended to the discrete algorithm.

Note: it recovers the previous theorem as $\kappa \to \infty$ and $\alpha \to 0$, so LB can be less biased than LASSO.

General Loss and Regularizer

$$\dot{\eta}_t = -\frac{\kappa_0}{n}\sum_{i=1}^{n}\nabla_\eta \ell(x_i, \theta_t, \eta_t) \tag{11a}$$

$$\dot{\rho}_t + \frac{\dot{\theta}_t}{\kappa_1} = -\frac{1}{n}\sum_{i=1}^{n}\nabla_\theta \ell(x_i, \theta_t, \eta_t) \tag{11b}$$

$$\rho_t \in \partial\|\theta_t\|_* \tag{11c}$$

where

- $\ell(x_i, \theta)$ is a loss function: negative logarithmic likelihood, non-convex loss (neural networks), etc.
- $\|\theta_t\|_*$ is the Minkowski-functional (gauge) of dictionary convex hulls:

$$\|\theta\|_* := \inf\{\lambda \geq 0 : \theta \in \lambda K\}, \quad K \text{ is a symmetric convex hull of } \{a_i\}$$

  - it can be generalized to non-convex regularizers

Linearized Bregman Iteration Algorithms

Differential inclusion (11) admits the following Euler Forward discretization

$$\eta_{t+1} = \eta_t - \frac{\alpha_k \kappa_0}{n} \sum_{i=1}^{n} \nabla_\eta \ell(x_i, \theta_t, \eta_t) \tag{12a}$$

$$z_{t+1} = z_t - \frac{\alpha_k}{n} \sum_{i=1}^{n} \nabla_\theta \ell(x_i, \theta_t, \eta_t) \tag{12b}$$

$$\theta_{t+1} = \kappa_1 \cdot \text{prox}_{\|\cdot\|_*}(z_{t+1}) \tag{12c}$$

where (12c) is given by Moreau Decomposition with

$$\text{prox}_{\|\cdot\|_*}(z_t) = \arg\min_x \frac{1}{2}\|x - z_t\|^2 + \|x\|_*,$$

and

- $\alpha_k > 0$ is step-size while $\alpha_k \kappa_i \|\nabla_\theta^2 \hat{\mathbb{E}} \ell(x, \theta)\| < 2$
- as simple as ISTA, easy to parallel implementation

More reference

- **Logistic Regression**: loss – conditional likelihood, regularizer – l_1 (Shi-Yin-Osher-Saijda'10,Huang-Yao'18)
- **Graphical Models** (Gaussian/Ising/Potts Model): loss – likelihood, composite conditional likelihood, regularizer – l_1 and group l_1 (Huang-Yao'18)
- **Fused LASSO/TV**: split Bregman with composite l_2 loss and l_1 gauge (Osher-Burger-Goldfarb-Xu-Yin'06, Burger-Gilboa-Osher-Xu'06, Yin-Osher-Goldfarb-Darbon'08, Huang-Sun-Xiong-Yao'16)
- **Matrix Completion/Regression**: gauge – the matrix nuclear norm (Cai-Candès-Shen'10)

Split LB vs. Generalized LASSO

Structural Sparse Regression:

$$y = X\beta^\star + \epsilon,\ \gamma^\star = D\beta^\star\ \ (S = \mathrm{supp}\,(\gamma^\star)\,,\ s = |S| \ll p)\,, \tag{13}$$

Loss that splits prediction vs. sparsity control

$$\ell\,(\beta, \gamma) := \frac{1}{2n}\,\|y - X\beta\|_2^2 + \frac{1}{2\nu}\,\|\gamma - D\beta\|_2^2 \quad (\nu > 0). \tag{14}$$

Split LBI:

$$\beta_{k+1} = \beta_k - \kappa\alpha\nabla_\beta \ell(\beta_k, \gamma_k), \tag{15a}$$

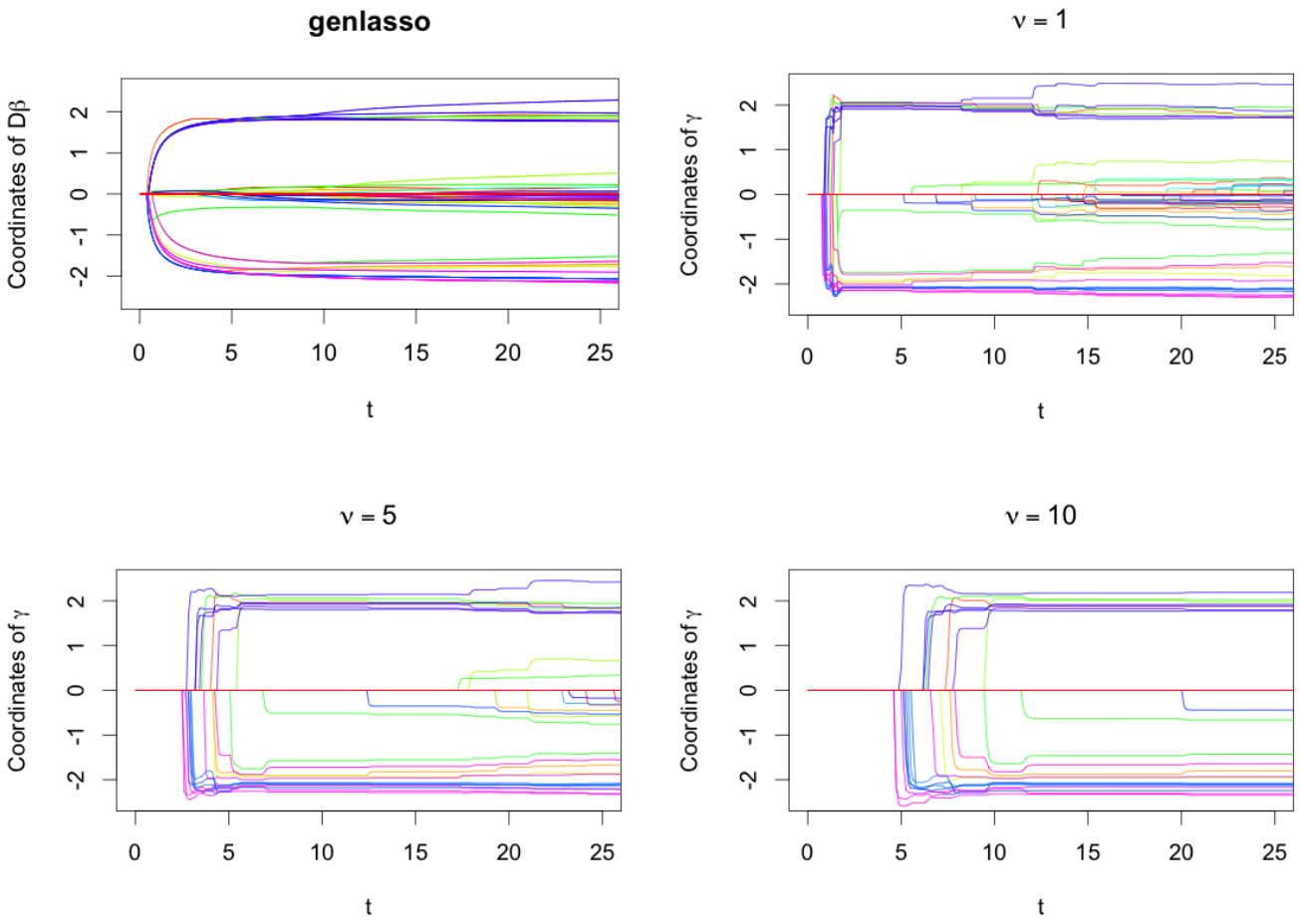
$$z_{k+1} = z_k - \alpha\nabla_\gamma \ell(\beta_k, \gamma_k), \tag{15b}$$

$$\gamma_{k+1} = \kappa \cdot \mathrm{prox}_{\|\cdot\|_1}(z_{k+1}), \tag{15c}$$

Generalized LASSO (`genlasso`):

$$\arg\min_\beta \left(\frac{1}{2n}\,\|y - X\beta\|_2^2 + \lambda\,\|D\beta\|_1\right). \tag{16}$$

Split LBI vs. Generalized LASSO paths



Split LB may beat Generalized LASSO in Model Selection

genlasso	Split LBI			genlasso	Split LBI		
	$\nu = 1$	$\nu = 5$	$\nu = 10$		$\nu = 1$	$\nu = 5$	$\nu = 10$
.9426	.9845	.9969	**.9982**	.9705	.9955	.9996	**.9998**
(.0390)	(.0185)	(.0065)	(**.0043**)	(.0212)	(.0056)	(.0014)	(**.0009**)

- Example: $n = p = 50$, $X \in \mathbb{R}^{n \times p}$ with $X_j \sim N(0, I_p)$, $\epsilon \sim N(0, I_n)$
- (Left) $D = I$ (LASSO vs. Split LB)
- (Right) 1-D fused (generalized) LASSO vs. Split LB (next page).
- In terms of Area Under the ROC Curve (AUC), LB has less false discoveries than `genlasso`
- *Why*? Split LB may need weaker irrepresentable conditions than generalized LASSO...

Structural Sparsity Assumptions

- Define $\Sigma(\nu) := (I - D(\nu X^* X + D^T D)^\dagger D^T)/\nu$.
- **Assumption 1**: Restricted Strong Convexity (RSC).

$$\Sigma_{S,S}(\nu) \succeq \lambda \cdot I. \tag{17}$$

- **Assumption 2**: Irrepresentable Condition (IRR).

$$\mathrm{IRR}(\nu) := \|\Sigma_{S^c,S}(\nu) \cdot \Sigma_{S,S}^{-1}(\nu)\|_\infty \leq 1 - \eta. \tag{18}$$

- $\nu \to 0$: RSC and IRR above reduce to the RSC and IRR neccessary and sufficient for consistency of `genlasso` (Vaiter'13,LeeSunTay'13).
- $\nu \neq 0$: by allowing variable splitting in proximity, IRR above can be weaker than literature, bringing better variable selection consistency than `genlasso` (observed before)!

Identifiable Condition (IC) and Irrepresentable Condition (IRR)

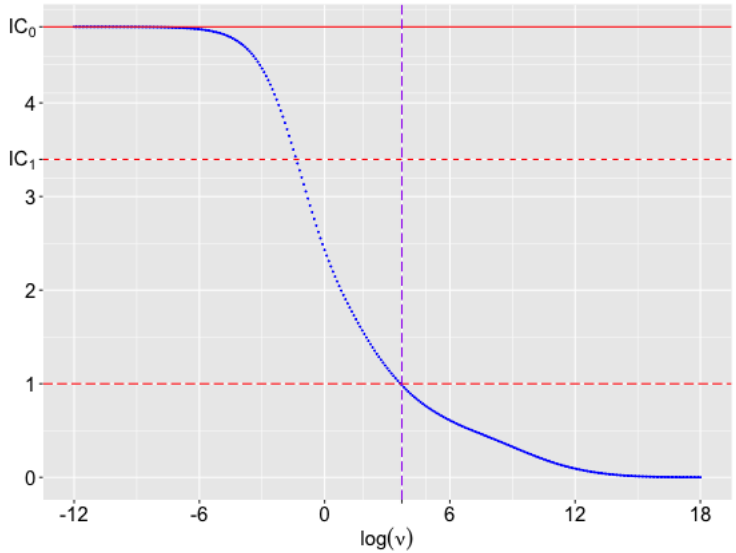
- Let the columns of W form an orthogonal basis of $\ker(D_{S^c})$.

$$\Omega^S := \left(D_{S^c}^{\dagger}\right)^T \left(X^*XW\left(W^TX^*XW\right)^{\dagger}W^T - I\right)D_S^T, \tag{19}$$

$$\mathrm{IC}_0 := \left\|\Omega^S\right\|_\infty, \ \mathrm{IC}_1 := \min_{u\in\ker(D_{S^c})}\left\|\Omega^S\mathrm{sign}\left(D_S\beta^\star\right) - u\right\|_\infty. \tag{20}$$

- The sign consistency of `genlasso` has been proved, under $\mathrm{IC}_1 < 1$ (Vaiter et al. 2013).
- We will show the sign consistency of Split LBI, under $\mathrm{IRR}(\nu) < 1$.
- If $\mathrm{IRR}(\nu) < \mathrm{IC}_1$, then our IRR is easier to be met?

Split LB improves Irrepresentable Condition (Huang-Sun-Xiong-Y.'16)



Theorem (Huang-Sun-Xiong-Y.'2016)

- $\mathrm{IC}_0 \geq \mathrm{IC}_1$.
- $\mathrm{IRR}(\nu) \to \mathrm{IC}_0$ $(\nu \to 0)$.
- $\mathrm{IRR}(\nu) \to C$ $(\nu \to \infty)$. $C = 0 \Longleftrightarrow \ker(X) \subseteq \ker(D_S)$.

Consistency

Theorem (Huang-Sun-Xiong-Y.'2016)

Under RSC and IRR, with large κ and small δ, there exists K such that with high probability, the following properties hold.

- No-false-positive property: γ_k $(k \le K)$ has no false-positive, i.e. $\mathrm{supp}(\gamma_k) \subseteq S = \mathrm{supp}(\gamma^\star)$.
- Sign consistency of γ_k: If $\gamma^\star_{\min} := \min(|\gamma^\star_j| : j \in S)$ (the minimal signal) is not weak, then $\mathrm{supp}(\gamma_K) = \mathrm{supp}(\gamma^\star)$.
- ℓ_2 consistency of γ_k: $\|\gamma_K - \gamma^\star\|_2 \le C_1\sqrt{s \log m/n}$.
- ℓ_2 "consistency" of β_k: $\|\beta_K - \beta^\star\|_2 \le C_2\sqrt{s \log m/n} + C_3\nu$.

- Issues due to variable splitting (despite benefit on IRR):
  - $D\beta_K$ does not follow the sparsity pattern of $\gamma^\star = D\beta^\star$.
  - β_K incurs an additional loss $C_3\nu$ ($\nu \sim \sqrt{s \log m/n}$ minimax optimal).

Consistency

Theorem (Huang-Sun-Xiong-Y.'2016)

Define

$$\tilde{\beta}_k := \mathrm{Proj}_{\ker(D_{S_k^c})}(\beta_k) \ \ (S_k = \mathrm{supp}(\gamma_k)) \tag{21}$$

Under RSC and IRR, with large κ and small δ, there exists K such that with high probability, the following properties hold, if $\gamma^\star_{\min}$ is not weak.

- Sign consistency of $D\tilde{\beta}_K$: $\mathrm{supp}(D\tilde{\beta}_K) = \mathrm{supp}(D\beta^\star)$.
- ℓ_2 consistency of $\tilde{\beta}_K$: $\left\|\tilde{\beta}_K - \beta^\star\right\|_2 \leq C_4\sqrt{s \log m/n}$.

Application: Partial Order of Basketball Teams

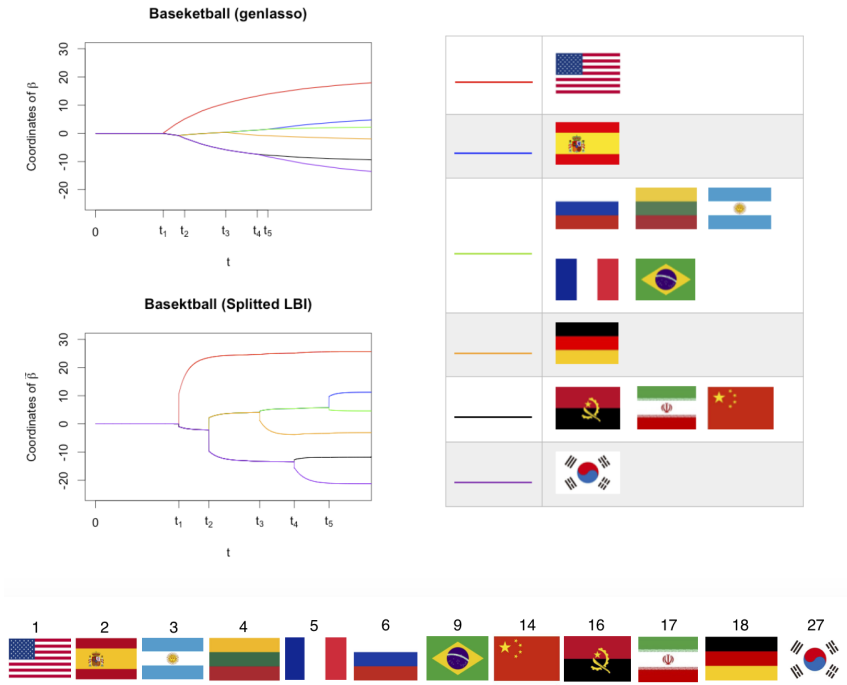


Figure: Partial order ranking for basketball teams. Top left shows $\{\beta_\lambda\}$ $(t = 1/\lambda)$ by `genlasso` and $\tilde{\beta}_k$ $(t = k\alpha)$ by Split LBI. Top right shows the same grouping result just passing t_5. Bottom is the FIBA ranking of all teams.

Application: Parsimonious Deep Learning (Image Classification)

Dataset		MNIST	Cifar-10	ImageNet-2012	
Models	Variants	LeNet	ResNet-20	AlexNet	ResNet-18
SGD	Naive	98.87	86.46	–/–	60.76/79.18
	l_1	98.52	67.60	–/–	–/–
	Mom	99.16	89.44	55.14/78.09	66.98/86.97
	Mom-W$d^\star$	99.23	90.31	56.55/79.09	69.76/89.18
	Nesterov	99.23	90.18	-/-	70.19/89.30
Adam	Naive	99.19	89.14	–/–	59.66/83.28
	Adabound	–/–	87.89	–/–	–/–
	Adagrad	–/–	88.17	–/–	–/–
	Amsgrad	–/–	88.68	–/–	–/–
	Radam	–/–	88.44	–/–	–/–
SplitLBI	Naive	–/–	–/–	55.06/77.69	65.26/86.57
	Mom	99.19	89.72	56.23/78.48	68.55/87.85
	Mom-Wd	99.20	89.95	**57.09/79.86**	**70.55/89.56**

Table: Top-1/Top-5 accuracy(%) on ImageNet-2012 and test accuracy on MNIST/Cifar-10. *: results from the official pytorch website. We use the official pytorch codes to run the competitors. All models are trained by 100 epochs.

Non-semantic Features Learned on ImageNet

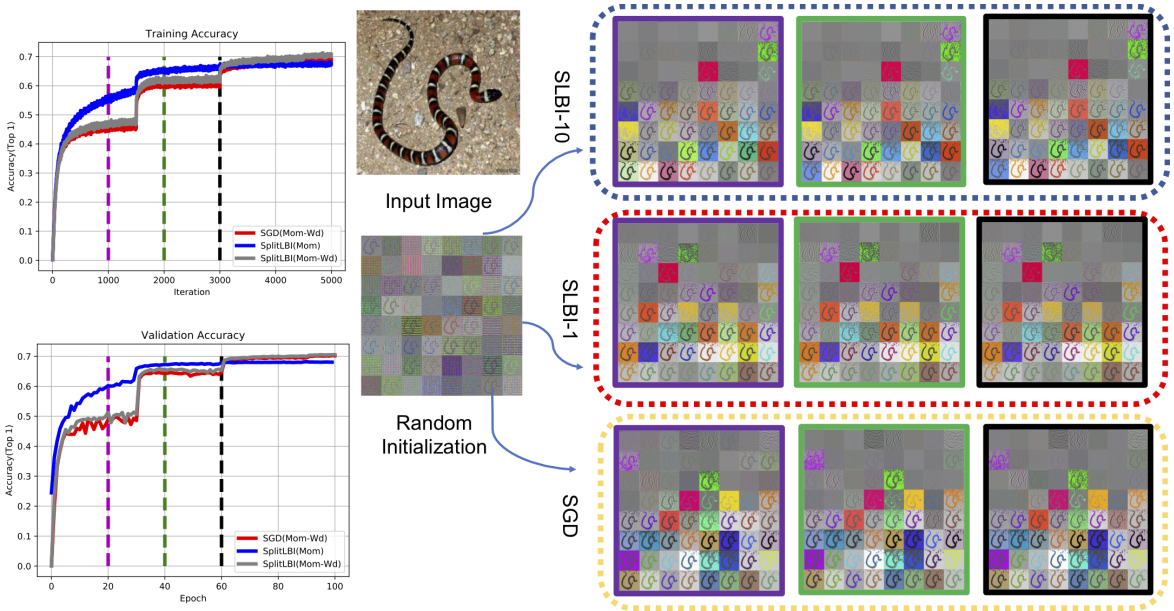


Figure: Visualization of the first convolutional layer filters of ResNet-18 trained on ImageNet-2012. Given the input image and initial weights visualized in the middle, filter response gradients at 20 (purple), 40 (green), and 60 (black) epochs are visualized. SGD with Momentum (Mom) and Weight Decay (WD), is compared with SLBI.

Important Property of SPLIT LBI

- Split LBI can achieve similar results on classification tasks compared with SGD sometimes even better.

- Split LBI can be used to find sparse structure when training the network, the sparse structure and final accuracy is robust to hyperparameters.

- Split LBI can find winning ticket as early stage instead of fully training a neural network which is costly in terms of training time.
  - Sparse subnets found by early stopping of SplitLBI achieve remarkably good accuracy after retrain from scratch, with comparable or even better accuracy than SOTA algorithms.

Summary

We have seen:

- The limit of Linearized Bregman iterations follows a restricted gradient flow: differential inclusions dynamics
- It passes the unbiased *Oracle Estimator* under sign-consistency
- Sign consistency under nearly the same condition as LASSO
  - Restricted Strongly Convex + Irrepresentable Condition
- Split extension: sign consistency under a weaker condition than generalized LASSO
  - under a provably weaker Irrepresentable Condition
- Early stopping regularization is exploited against overfitting under noise

A Renaissance of Boosting as restricted gradient descent ...

Some Reference

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- R package:
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- Pytorch package for deep learning:
  - `https://github.com/yao-lab/FSplitLBI`

The END


thank you
danke
謝謝
рахмат
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
спасибо
Баярлалаа
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