(a) $K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \mathbb{R}^{N\times N}$

Assume that $K = XX^T$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}^{N\times k}$, $X_1 \in \mathbb{R}^{n\times k}$.

$A = U\Lambda U^T$, $\Lambda = \mathrm{diag}(\lambda_i)$ $(\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k > \lambda_{k+1} = \cdots = 0)$,

$U = [u_1 \cdots u_n]$, $UU^T = I$

Now let $U_k = [u_1 \cdots u_k] \in \mathbb{R}^{n\times k}$, $\Lambda_k = \mathrm{diag}(\lambda_1 \cdots \lambda_k) \in \mathbb{R}^{k\times k}$.

then we know $U_k U_k^T = I$. Let $U_{n-k} = [u_{k+1} \cdots u_n]$.

What's more, $A = [U_k \ \ U_{n-k}] \begin{bmatrix} \Lambda_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_k^T \\ U_{n-k}^T \end{bmatrix} = U_k \Lambda_k U_k^T$

$= (U_k \Lambda_k^{\frac{1}{2}})(U_k \Lambda_k^{\frac{1}{2}})^T$.

On the other hand, since $K = XX^T = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [X_1^T \ \ X_2^T] = \begin{bmatrix} X_1X_1^T & X_1X_2^T \\ X_2X_1^T & X_2X_2^T \end{bmatrix}$

$= \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$,

we have $X_1X_1^T = A = (U_k\Lambda_k^{\frac{1}{2}})(U_k\Lambda_k^{\frac{1}{2}})^T$,

$X_2X_1^T = B^T$

So $X_1 = U_k \Lambda_k^{\frac{1}{2}}$. $\Rightarrow X_1^T = \Lambda_k^{\frac{1}{2}} U_k^T$, $(X_1^T)^{-1} = U_k \Lambda_k^{-\frac{1}{2}}$.

$\Rightarrow X_2 = B^T (X_1^T)^{-1} = B^T U_k \Lambda_k^{-\frac{1}{2}}$.

(b). $A = U\Lambda U^T = U_k \Lambda_k U_k^T$, so $A^\dagger = U_k \Lambda_k^{-1} U_k^T$.

From (a), $C = X_2X_2^T = B^T U_k \Lambda_k^{-\frac{1}{2}} \Lambda_k^{-\frac{1}{2}} U_k^T B = B^T U_k \Lambda_k^{-} U_k^T B$

$= B^T A^\dagger B$

So obviously $\| K - \hat{K} \|_F^2 = \| A - A \|_F^2 + \| B - B \|_F^2 + \| B^T - B^T \|_F^2 + \| C - B^T A^\dagger B \|_F^2$

$= \| C - B^T A^\dagger B \|_F^2$

$\Rightarrow \| K - \hat{K} \|_F = \| C - B^T A^\dagger B \|_F$.

(d). Since $\begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$

$= \begin{bmatrix} A & B \\ 0 & C - B^T A^{-1} B \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$

and $A^{-1} = A^{+}$ because A is invertible,

$$K/A = C - B^T A^{-1} B.$$

And $\det\left(\begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix}\right) = 1$, $\det\left(\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}\right) = 1$

so $\det\left(\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\right) = \det(A) \cdot \det(C - B^T A^{-1} B)$

i.e. $\det(K) = \det(A) \cdot \det(K/A)$.

(e) Since $\begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix}$, $\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}$ are invertible,

we have $\operatorname{rank}\left(\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} A & 0 \\ 0 & K/A \end{bmatrix}\right) = \operatorname{rank} A + \operatorname{rank}(K/A)$

i.e. $\operatorname{rank}(K) = \operatorname{rank}(A) + \operatorname{rank}(K/A)$