Homework 7

2. (a) If $Av^* \neq \lambda^* v^*$, then for some i, $[Av^*]_i > \lambda^* v_i^*$. Below we will find an increase of λ^*, which is thus not optimal.
Define $\tilde{v} = v^* + \varepsilon e_i$ with $\varepsilon > 0$ and e_i denotes the vector which is one on the i-th component and zero otherwise.
For those $j \neq i$,
$$(A\tilde{v})_j = (Av^*)_j + \varepsilon (Ae_i)_j = \lambda^* v_j^* + \varepsilon A_{ji} > \lambda^* v_j^* = \lambda^* \tilde{v}_j$$
where the last inequality is due to $A > 0$.
For those $j = i$,
$$(A\tilde{v})_i = (Av^*)_i + \varepsilon (Ae_i)_i > \lambda^* v_i^* + \varepsilon A_{ii}.$$
Since $\lambda^* \tilde{v}_i = \lambda^* v_i^* + \varepsilon \lambda^*$, we have
$$(A\tilde{v})_i - (\lambda^* \tilde{v})_i + \varepsilon (A_{ii} - \lambda^*) = (Av^*)_i - (\lambda^* v_i^*) - \varepsilon (\lambda^* - A_{ii}) > 0,$$
where the last inequality holds for small enough $\varepsilon > 0$.
$\exists \varepsilon > 0$, $(A\tilde{v}) > \lambda^* \tilde{v}$. Thus λ^* is not optimal, which leads to a contradiction.

(b) Assume on the contrary, for some k, $v_k^* = 0$. Then $(Av^*)_k = \lambda^* v_k^* = 0$. But $A > 0$, $v^* \geq 0$ and $v^* \neq 0$, so there $\exists i$, $v_i^* > 0$, which implies that $Av^* > 0$. That contradicts to the previous conclusion. So $v^* > 0$, which followed by $\lambda^* > 0$.

(c) For every $v > 0$, $Av = uv \Rightarrow u = \lambda^*$. Following the same reasoning above, A must have a left Perron vector $w^* > 0$, s.t. $A^T w^* = \lambda^* w^*$. Then
$\lambda^* (w^{*T} v) = w^{*T} A v = u (w^{*T} v)$. Since $w^{*T} v > 0$, there must be $\lambda^* = u$, i.e. λ^* is unique, v^* is unique.

(d) For any other eigenvalue $Az = \lambda z$, $A|z| \geq |Az| = |\lambda||z|$, so $|\lambda| \leq \lambda^*$. Since we have the lemma

$$Az = \lambda z, |\lambda| = \lambda^*, z \neq 0 \Rightarrow A|z| = \lambda^* |z|, \lambda^* = \max_i |\lambda_i(A)|.$$

Assume that we have $Az = \lambda z (z \neq 0)$ with $|\lambda| = \lambda^*$,

$$A|z| = \lambda^* |z| = |\lambda||z| = |Az|$$

$$\Rightarrow |\sum_j \bar{a}_{ij} z_j| = \sum_j \bar{a}_{ij} |z_j|, \bar{A} = \frac{A}{\lambda^*}$$

which implies that z_j has the same sign, i.e. $z_j \geq 0$ or $z_j \leq 0$.

3. (a) The probability of transition from i to j in exactly n steps is the (i,j)-entry of P^n.

$$P^2 = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}^2 = \begin{bmatrix} Q^2 & R+QR \\ 0 & I \end{bmatrix}$$

$$\cdots$$

$$P^n = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}^n = \begin{bmatrix} Q^n & (I+Q+\cdots+Q^{n-1})R \\ 0 & I \end{bmatrix}$$

$$N = I + Q + Q^2 + \cdots$$

$$QN = \quad Q + Q^2 + \cdots$$

$$(I-Q)N = I$$

$$N = (I-Q)^{-1}$$

$\therefore N(i,i) = 1 + \sum_k N(i,k) Q(k,i)$ and

$N(i,j) = \sum_k N(i,k) Q(k,j)$ for $i \neq j$.

(d) By Induction, $P^n = \begin{bmatrix} Q^n & (I-Q^n)NR \\ 0 & I \end{bmatrix}$

The probability of eventually being absorbed in the absorbing state j when starting from transient state i is given by

$$B = NR.$$

7. Step (7) is wrong. $(g(x)+g(y))^2 \leq 2(g^2(x)+g^2(y))$

$$\sum_{x\sim y}(g(x)+g(y))^2 \leq \sum_{x\sim y} 2(g^2(x)+g^2(y))$$

$$\sum_{x\sim y} 2(g^2(x)+g^2(y)) \leq 4\sum_{x\in V} g^2(x) d_x$$

Therefore

$$\frac{\left(\sum_{x\sim y}|g^2(x)-g^2(y)|\right)^2}{\left(\sum_{x\in V} g^2(x) d_x\right)\left(\sum_{x\sim y}(g(x)+g(y))^2\right)} \geq \frac{\left(\sum_{x\sim y}|g^2(x)-g^2(y)|\right)^2}{4\left(\sum_{x\in V} g^2(x) d_x\right)^2} \geq \frac{h_G^2}{4}$$