MATH5473/CSIC5011 - Topological and Geometric Data Reduction and Visualization (Homework #6)

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- 1. Order the faces: The following dataset contains 33 faces of the same person $\left(Y \in \mathbb{R}^{112\times 92\times 33}\right)$ in different angles,

  You may create a data matrix $X \in \mathbb{R}^{n\times p}$ where $n = 33, p = 112 \times 92 = 10304$ (e.g. $X =$ reshape $(Y, [10304, 33])'$; in matlab).

  (a) Explore the MDS-embedding of the 33 faces on top two eigenvectors: order the faces according to the top 1st eigenvector and visualize your results with figures.

  (b) Explore the ISOMAP-embedding of the 33 faces on the $k = 5$ nearest neighbor graph and compare it against the MDS results. Note: you may try Tenenbaum's Matlab code

  (c) Explore the LLE-embedding of the 33 faces on the $k = 5$ nearest neighbor graph and compare it against ISOMAP. Note: you may try the following Matlab code

  Solution:

  The visualization of the three projection methods are summarized in Fig. 0.1-Fig .0.3.

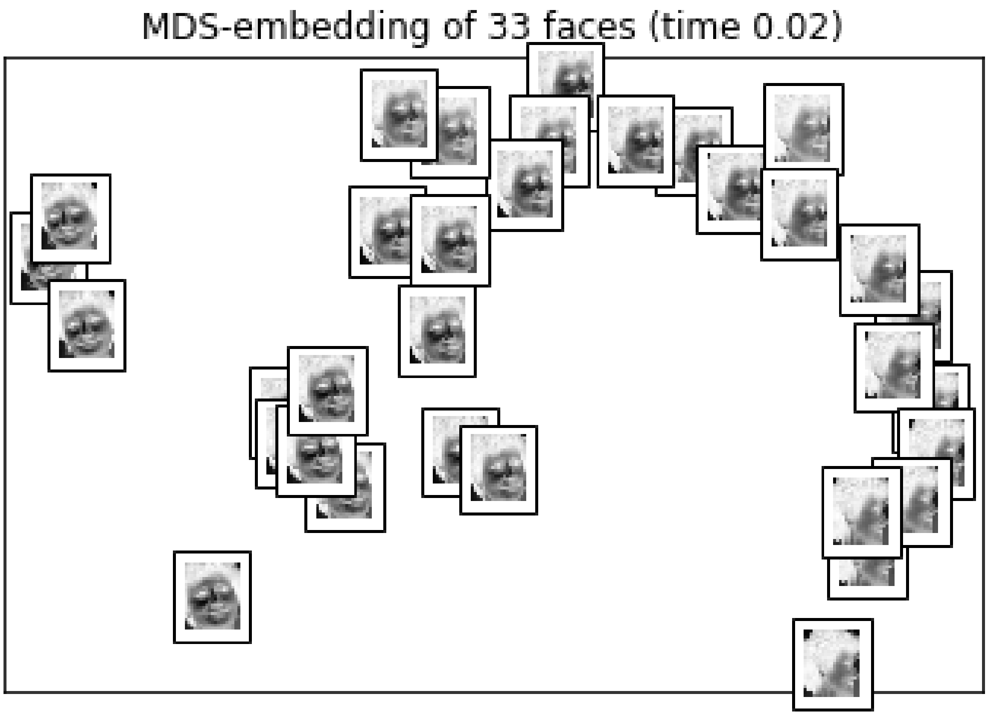

Figure 0.1: The visualization of the MDS-embedding of the 33 faces on top two eigenvectors: order the faces according to the top 1st eigenvector.

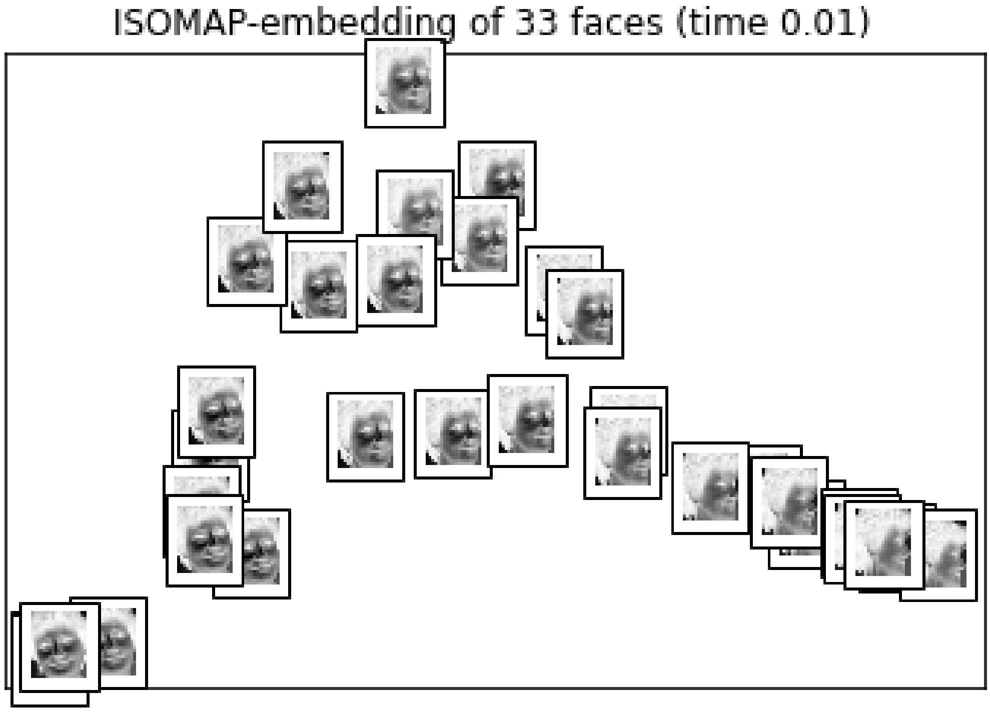

Figure 0.2: The visualization of the ISOMAP-embedding of the 33 faces on the k = 5 nearest neighbor graph.

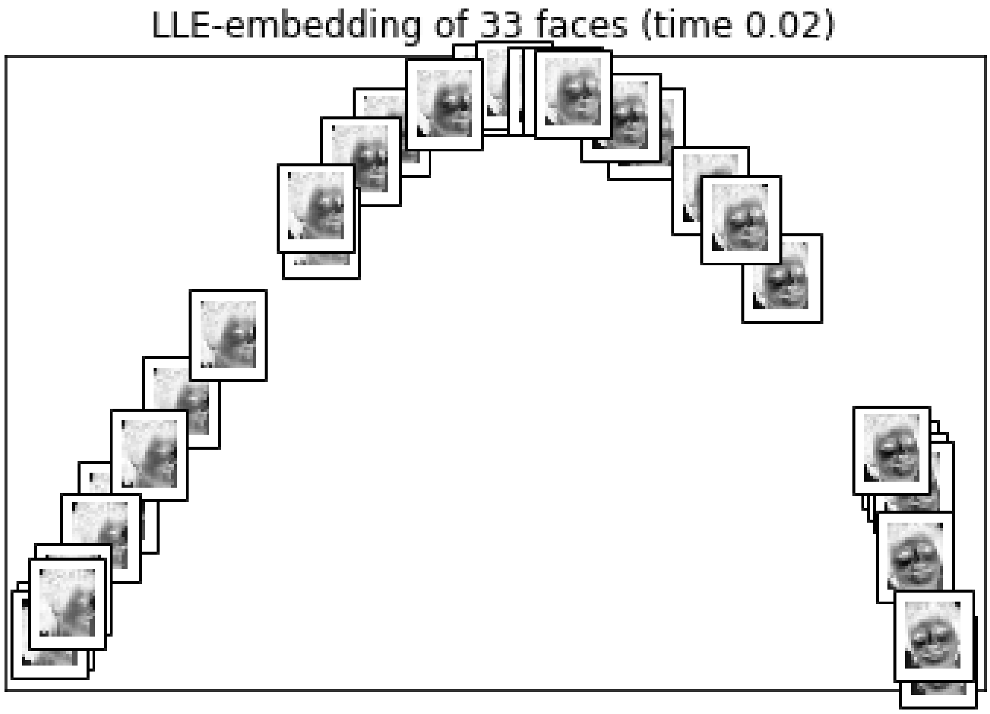

Figure 0.3: The visualization of the LLE-embedding of the 33 faces on the k = 5 nearest neighbor graph.

- 2. Manifold Learning: The following codes by Todd Wittman contain major manifold learning algorithms talked on class.

  Precisely, eight algorithms are implemented in the codes: MDS, PCA, ISOMAP, LLE, Hessian Eigenmap, Laplacian Eigenmap, Diffusion Map, and LTSA. The following nine examples are given to compare these methods,

(a) Swiss roll;

(b) Swiss hole;

(c) Corner Planes;

(d) Punctured Sphere;

(e) Twin Peaks;

(f) 3D Clusters;

(g) Toroidal Helix;

(h) Gaussian;

(i) Occluded Disks.

Run the codes for each of the nine examples, and analyze the phenomena you observed.

Solutions:

The numerical results of the nine examples given to compare eight algorithms are summarized in Fig.0.4 - Fig.0.12.

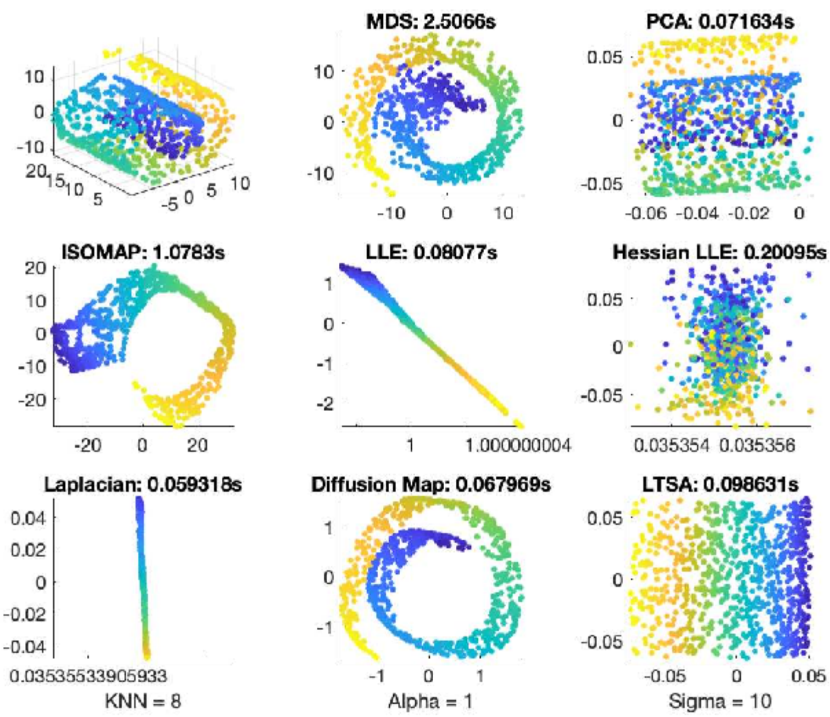

Figure 0.4: Example of Swiss Roll.

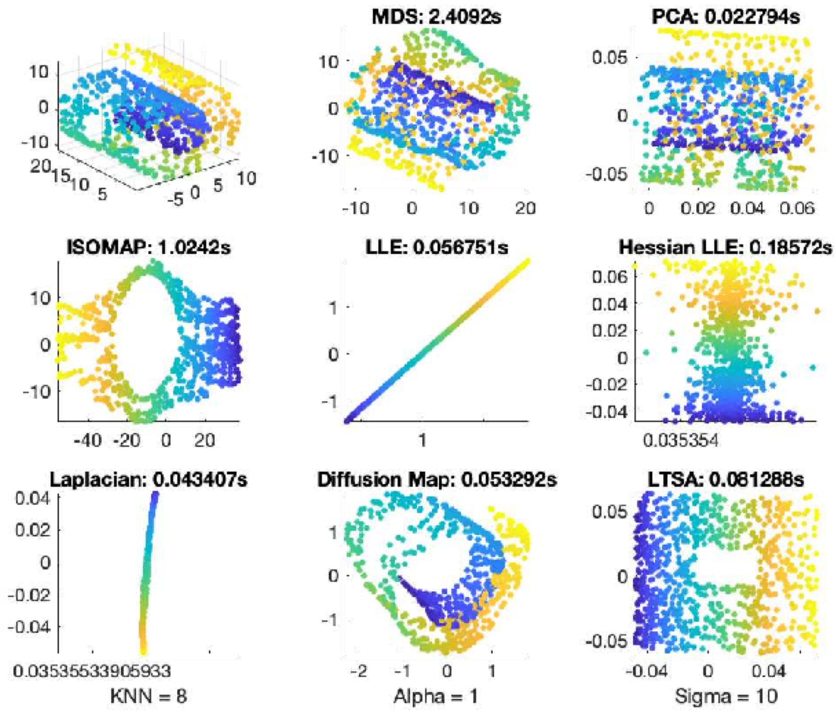

Figure 0.5: Example of Swiss Hole.

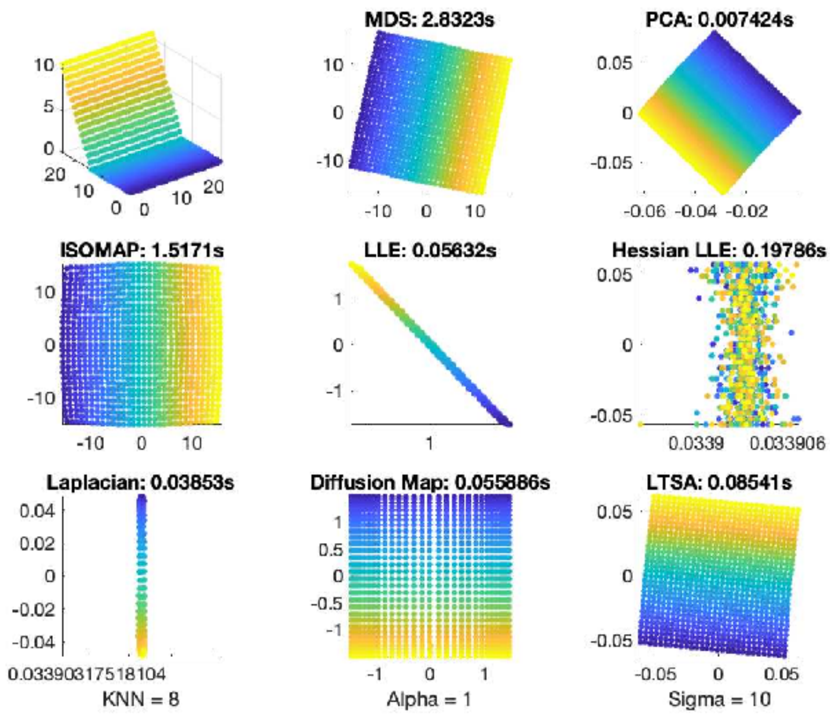

Figure 0.6: Example of Corner Planes.

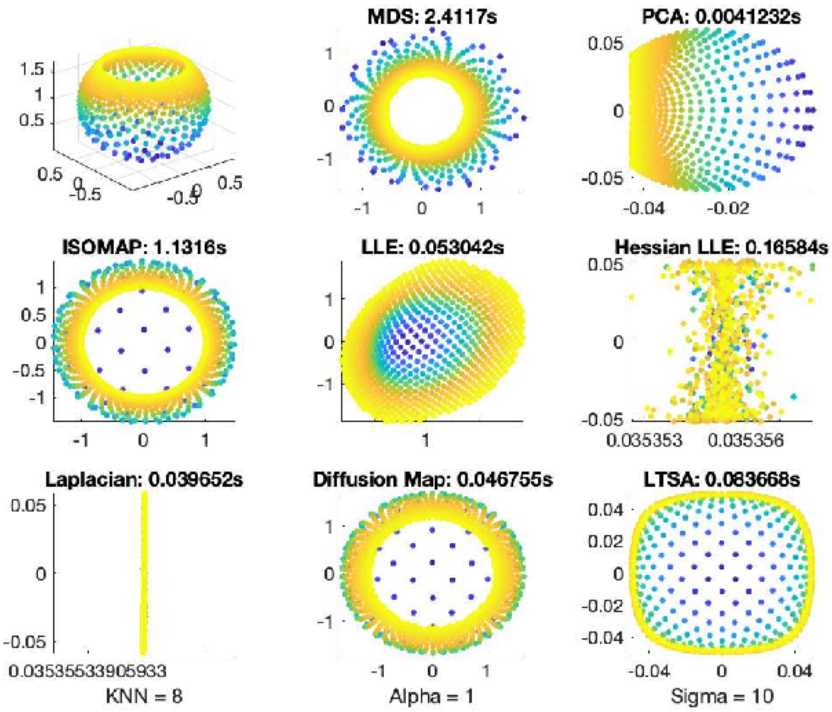

Figure 0.7: Example of Punctured Sphere.

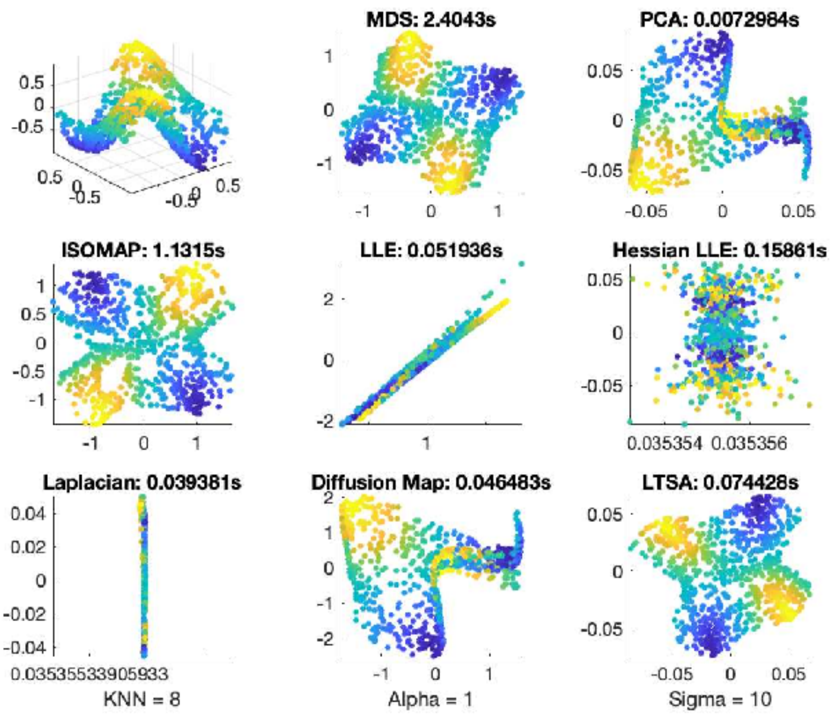

Figure 0.8: Example of Twin Peaks.

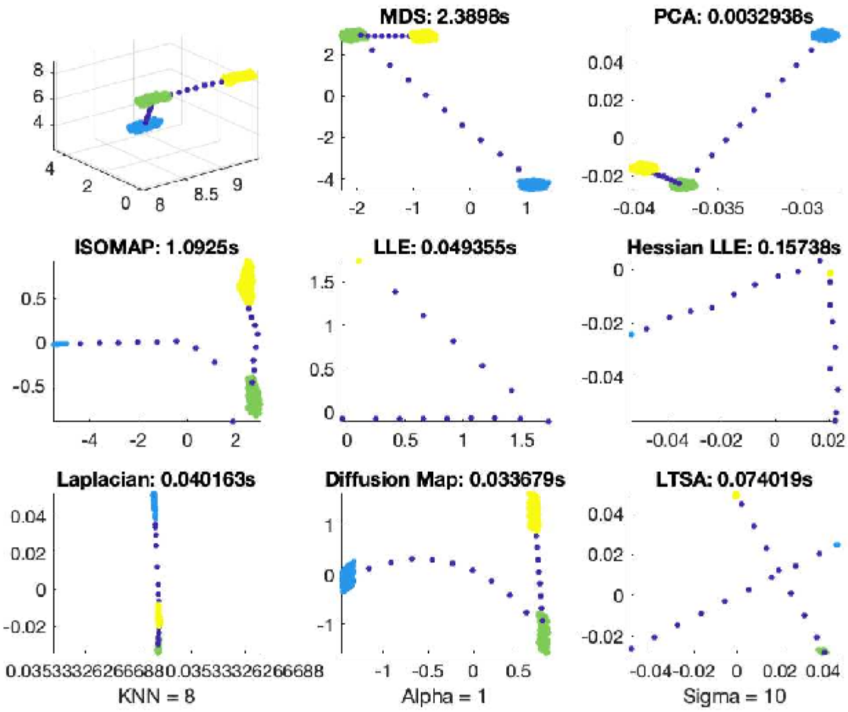

Figure 0.9: Example of 3D Clusters.

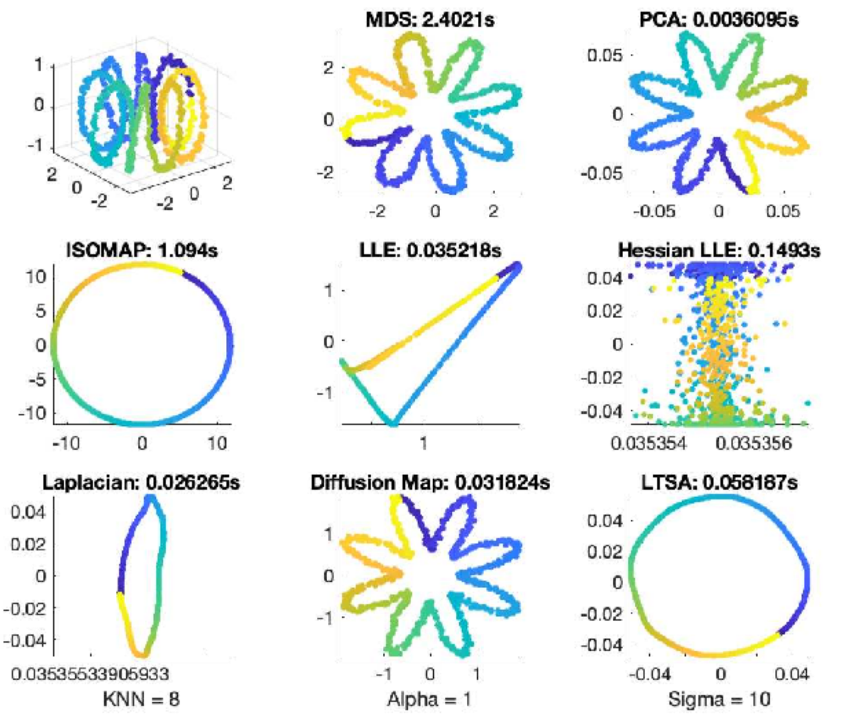

Figure 0.10: Example of Toroidal Helix.

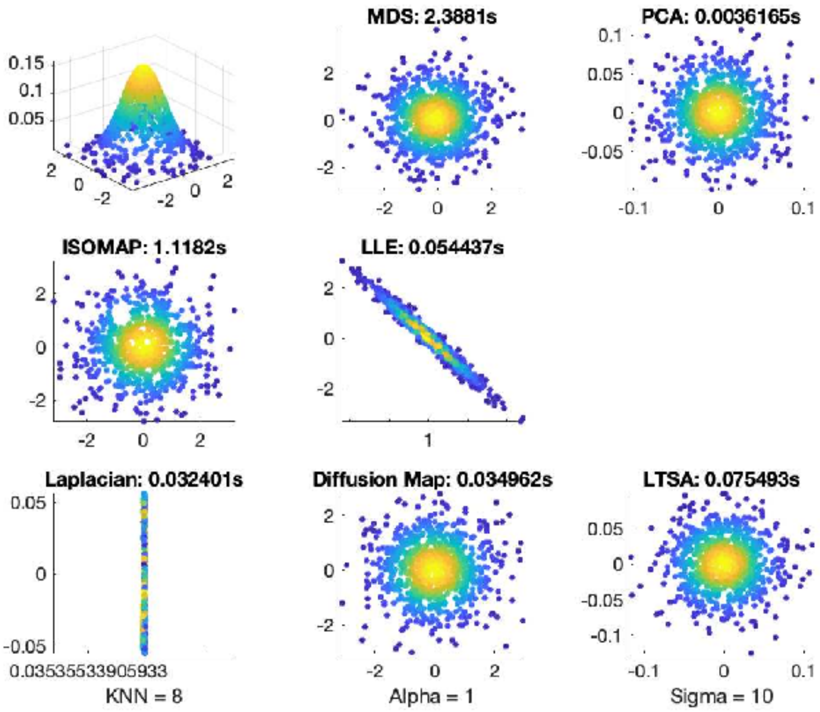

Figure 0.11: Example of Gaussian.

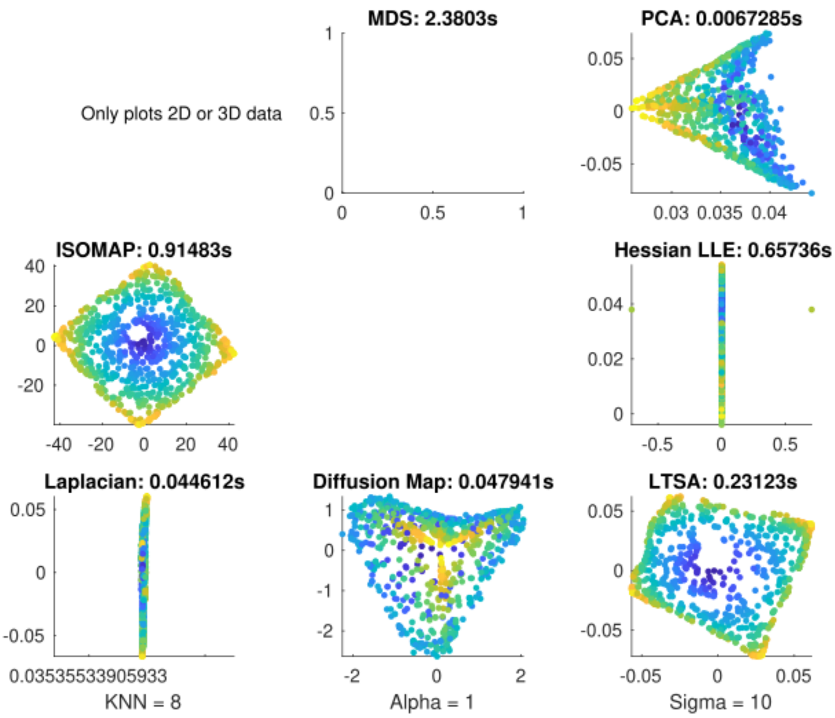

Figure 0.12: Example of Occluded Disks.

MATH 5473 HW6

3. (a) Since $A = U \Lambda U^T$, $\Lambda = diag(\lambda_i)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \cdots = 0$

Then $A = U_k \Lambda_k U_k^T$, $U_k = [u_1, u_2, \cdots u_k]$ top k eigenvectors.

$\Lambda_k = diag(\lambda_1, \lambda_2, \cdots \lambda_k)$

By $K = XX^T$, $X = [X_1 ; X_2] \Rightarrow K = \begin{pmatrix} X_1 X_1^T & X_1 X_2^T \\ X_2 X_1^T & X_2 X_2^T \end{pmatrix}$

By $X_1 X_1^T = A \Rightarrow X_1 X_1^T = U_k \Lambda_k^{\frac{1}{2}} \Lambda_k^{\frac{1}{2}} U_k^T = (U_k \Lambda_k^{\frac{1}{2}})(U_k \Lambda_k^{\frac{1}{2}})^T$

Thus $X_1 = U_k \Lambda_k^{\frac{1}{2}}$.

By $X_1 X_2^T = B \Rightarrow X_2^T = X_1^{-1} B = (\Lambda_k^{\frac{1}{2}})^{-1} U_k^{-1} B = \Lambda_k^{-\frac{1}{2}} U_k^T B$

Thus $X_2 = B^T U_k \Lambda_k^{-\frac{1}{2}}$.

(b) Given A, B from (a), we have

$$\hat{K} = \begin{bmatrix} A & B \\ B^T & \hat{C} \end{bmatrix} = \begin{bmatrix} X_1 X_1^T & X_1 X_2^T \\ X_2 X_1^T & X_2 X_2^T \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & B^T U_k \Lambda_k U_k^{-1} B \end{bmatrix}$$

Assume $A^* = U_k \Lambda_k^{-1} U_k^T$, and A^* is unique.

Thus $\|K - \hat{K}\|_F = \sqrt{\|A - A\|_F^2 + \|B - B\|_F^2 + \|B^T - B^T\|_F^2 + \|C - B^T A^* B\|_F^2}$

$$= \sqrt{\|C - B^T A^* B\|_F^2}$$