MATH5473/CSIC5011 - Topological and Geometric Data Reduction and Visualization (Homework #1)

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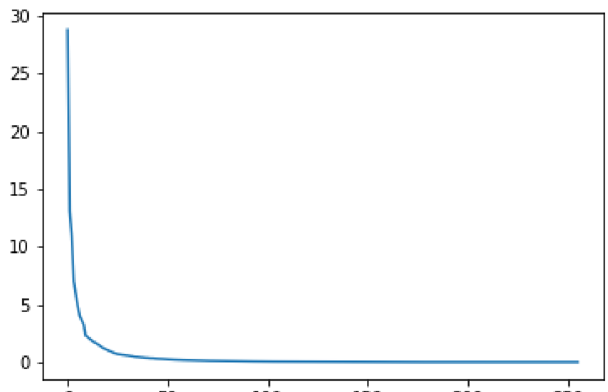
Programming Problem

- 1. PCA experiments:

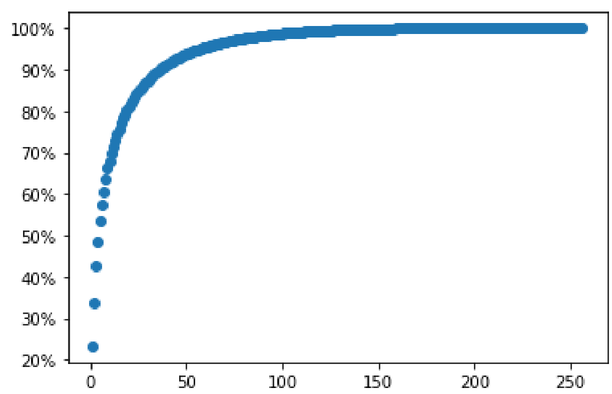
  In (a) and (b), we visualize the eigenvalues and plot the ratio of variance explained.

  In (c), we visualize top-20 right singular vector.

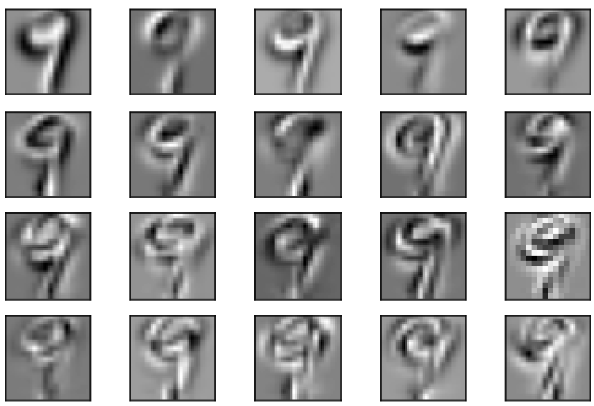
  In (d), samples projected onto 2-D by PCA are plotted.



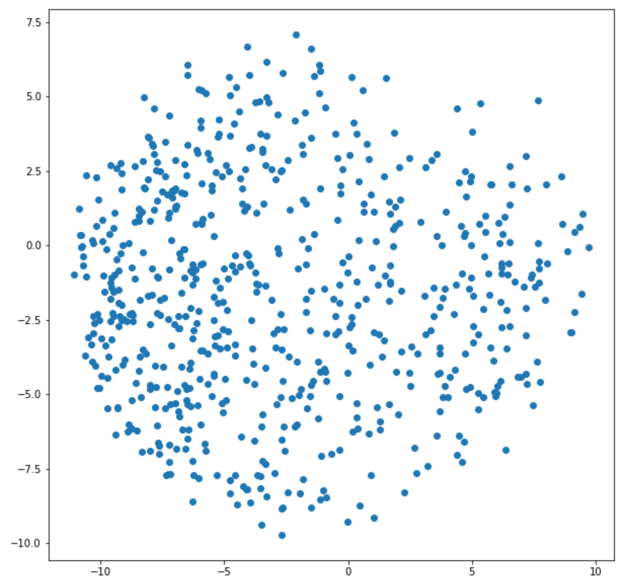
(a) Eigenvalue Curve



(b) Variance Ratio

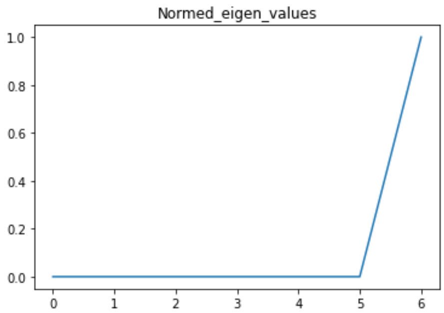


(c) Visualize top-20 right singular vector

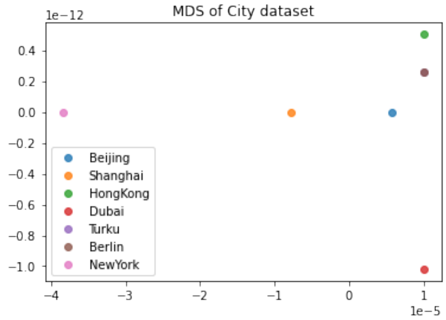


(d) Samples projected onto 2-D by PCA

- 2. MDS of cities:



(e) Normalized Eigenvalues



(f) Top Cities

In (a), the normalized eigenvalues are plotted in a descending order of magnitudes. And, Yes, there are some negative eigenvalues.

In (b), a scatter plot of those cities using top 2 or 3 eigenvectors is also made.If you revert the graph above 90 degree to the right. Then the graph will basically reflect the position relationship like Dubai is in the West of east Asian countries like Beijing and Hong Kong. However, there is not much cities near New york, so New york looks liek an outlier and lacks information to help it being embedded better.

MATH 5473 HW1. (proof part)

T3. (a) $\Rightarrow$ If $K \geq 0$, $v^T K v = v^T \lambda v = \lambda \cdot v^T v \geq 0 \Rightarrow \lambda \geq 0$

$\Leftarrow$ If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, $T = \mathrm{diag}(\lambda_i)$, $1 \leq i \leq n$,

then $K = Q^T T Q$, here $Q = (\lambda_1, \cdots \lambda_n)$

$\forall x \in \mathbb{R}^n$, $x^T K x = (Qx)^T T (Qx)$, let $Qx = (P_1, \cdots P_n)^T$.

$\Rightarrow x^T K x = \sum_{i=1}^{n} \lambda_i P_i^2 \geq 0$

$\Rightarrow K \geq 0$

(b) $\| u_i - v_j \|^2 = (u_i - v_j)^T (u_i - v_j) = u_i^T v_i + u_j^T v_i - u_i^T v_j - u_i v_j^T$

Let u_i: i-th row of K

v_j: $(0, 0, \cdots 1, 0, \cdots 0)$ only j-th non-zero.

Then $\forall i, j$. $\| u_i - v_j \|^2 = d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$.

(c) Let $D = ke^T + ek^T - 2K$, suppose $K = X^T X$.

$B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T = -\frac{1}{2} H_\alpha (ke^T + ek^T - 2K) H_\alpha^T$,

here $H_\alpha k e^T H_\alpha^T = (I - e\alpha^T) k e^T (I - \alpha e^T) = 0$,

$H_\alpha e k^T H_\alpha^T = [e - e(\alpha^T e)] k H_\alpha^T = 0$.

Then, $B_\alpha = H_\alpha K H_\alpha^T$, for $\forall x \in \mathbb{R}^n$,

$x^T B_\alpha x = x^T H_\alpha K H_\alpha^T x = (H_\alpha^T x)^T K (H_\alpha^T x) \geq 0$,

here K is p.s.d. So, B_α is also p.s.d.

(d). ① $\forall x \in \mathbb{R}^n$,

$$x^T(A+B)x = x^TAx + x^TBx \geq 0 \Rightarrow A+B \succeq 0$$

② If $B \succeq 0$, then $\exists T$ s.t. $B = TT^T$. $T = (t_{ij})$ $A = (a_{ij})$

$\forall x \in \mathbb{R}^n$,

$$x^T(A \circ B)x = x^T(A \circ (TT^T))x$$

$$= \sum_{i,j} x_i a_{ij} \left(\sum_k t_{ik} t_{jk}\right) x_j$$

$$= \sum_k (x * t_k)' A (x * t_k)$$

$$\geq \sum_k 0 = 0$$

here t_k is the k-column of T.

Thus, $A \circ B \succeq 0$.

T4. (a) d^2 is not a distance fuction.

Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $d(a,b) \triangleq |a-b|$.

d is a distance function.

Let $x=0$, $y=2$, $z=4$, then $d^2(a,b) = (a-b)^2$

$$d^2(x,z) = 16 > 8 = d^2(x+y) + d^2(y+z)$$

So, d^2 is not a distance function.

(b) $\sqrt{d}$ is a distance function.

If d is a distance fuction, then

1) $d(a,b) \geq 0$ and $d(a,b) = 0$ iff $a = b$.

$\Rightarrow \sqrt{d(a,b)} \geq 0$ and $\sqrt{d(a,b)} = 0$ iff $a = b$.

2) $d(a,b) = d(b,a) \Rightarrow \sqrt{d(a,b)} = \sqrt{d(b,a)}$

3) $d(a,c) + d(b,c) \geq d(a,b)$

$\Rightarrow \sqrt{d(a,c) + d(b,c)} \geq \sqrt{d(a,b)}$

$\sqrt{d(a,c)} + \sqrt{d(b,c)} \geq \sqrt{d(a,c) + d(b,c)} \geq \sqrt{d(a,b)}$ $\square$.