MATH 5473 Homework 1 LUO Yuanhui

3. *Positive Semi-definiteness:* Recall that a n-by-n real symmetric matrix K is called positive semi-definite (*p.s.d.* or $K \succeq 0$) iff for every $x \in \mathbb{R}^n$, $x^T K x \geq 0$.

   (a) Show that $K \succeq 0$ if and only if its eigenvalues are all nonnegative.

   (b) Show that $d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$ is a squared distance function, *i.e.* there exists vectors $u_i, v_j \in \mathbb{R}^n$ $(1 \leq i, j \leq n)$ such that $d_{ij} = \|u_i - u_j\|^2$.

   (c) Let $\alpha \in \mathbb{R}^n$ be a signed measure s.t. $\sum_i \alpha_i = 1$ (or $e^T \alpha = 1$) and $H_\alpha = I - e\alpha^T$ be the Householder centering matrix. Show that $B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T \succeq 0$ for matrix $D = [d_{ij}]$.

   (d) If $A \succeq 0$ and $B \succeq 0$ $(A, B \in \mathbb{R}^{n\times n})$, show that $A + B = [A_{ij} + B_{ij}]_{ij} \succeq 0$ (elementwise sum), and $A \circ B = [A_{ij}B_{ij}]_{ij} \succeq 0$ (Hadamard product or elementwise product).

(a) ($\Rightarrow$) If $K \succeq 0$, for $\forall$ its eigenvalue λ, $x^T K x = \lambda x^T x \geq 0$

$\Rightarrow \lambda \geq 0$

($\Leftarrow$) If all its eigenvalues $\lambda_i \geq 0$, $i=1,2\ldots n$, then

$K = C^T \Lambda C$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots \lambda_n)$, and for $\forall x$,

$x^T K x = (Cx)^T \Lambda (Cx) = \sum_{i=1}^{n} \lambda_i p_i^2 \geq 0$, where $Cx = (p_1, \ldots p_n)^T$

(b) Since $K \succeq 0$, then $\exists\, U = (u_1, \ldots u_n)$, $u_i \in \mathbb{R}^n$, s.t.

$K = U^T U$, then $d_{ij} = K_{ii} + K_{jj} - 2K_{ij} = u_i^T u_i + u_j^T u_j - 2u_i^T u_j =$

$\|u_i - u_j\|^2$

(c) Let $\hat{K} = \mathrm{diag}\, K \in \mathbb{R}^n$, then $D = \hat{K} e^T + e\hat{K}^T - 2\hat{K}$, $\mathrm{Tr}\,\hat{K} \geq 0$,

We have $H_\alpha (\hat{K} e^T) H_\alpha^T = (I - e\alpha^T) \hat{K} e^T (I - \alpha e^T) = (I - e\alpha^T)\hat{K}$

$(e^T - (e^T \alpha) e^T) = 0$, similarly $H_\alpha (e\hat{K}^T) H_\alpha^T = 0$

Then $B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T = -\frac{1}{2} H_\alpha (\hat{K} e^T + e\hat{K}^T - 2\hat{K}) H_\alpha^T =$

$H_\alpha \tilde{K} H_\alpha^T$

For $\forall x$, $x^T B_\alpha x = Tr(x^T B_\alpha x) = Tr(H_\alpha \tilde{K} H_\alpha^T) \geq 0$. then

$B_\alpha \succeq 0$

(d) ①. If $A \succeq 0$, $B \succeq 0$, for $\forall x \in \mathbb{R}^n$, $x^T(A+B)x = x^T A x + x^T B x$

≥ 0. then $A+B \succeq 0$

②. If $A \succeq 0$, $B \succeq 0$, $\exists R, S$ s.t. $A = R^T R$, $B = S^T S$. then for

$\forall x \in \mathbb{R}^n$, $x^T (A \circ B) x = x^T (R^T R) \circ (S^T S) x = (S^T R x)^T (S^T R x) \geq 0$

then $A \circ B \succeq 0$

4. *Distance:* Suppose that $d : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is a distance function.

   (a) Is d^2 a distance function? Prove or give a counter example.

   (b) Is $\sqrt{d}$ a distance function? Prove or give a counter example.

(a) d^2 is not a distance function. for example:

Let $d \triangleq |x-y|$. it can be shown that d is a distance

Consider $x=0$, $y=2$, $z=4$, then $d^2(x,z) = 16 > d^2(x,y) + d^2(y,z)$

$=8$. Therefore, d^2 is not a distance function

(b) $\sqrt{d}$ is a distance function, proof:

Let $g(\lambda) = \frac{\frac{1}{2}}{T(\frac{1}{2})} \lambda^{-\frac{1}{2}}$, then $\int_0^\infty \frac{1-\exp(-\lambda d)}{\lambda} g(\lambda)\, d\lambda$

$$= \frac{1}{2} \int_0^{\infty} \frac{1}{\Gamma(\frac{1}{2})} \lambda^{-\frac{3}{2}} (1 - \exp(-\lambda d)) \, d\lambda = d^{\frac{1}{2}}$$

Therefore, $\Phi(d) = d^{\frac{1}{2}}$ is a Schoenberg Transform, according to the theorem in lecture notes. $d^{\frac{1}{2}}$ is a distance function.