CSIC 5011 Homework 1 LIU YUNQIN

3. (a) ($\Rightarrow$) If $K \geq 0$, for $\forall x$ $x^T K x = \lambda x^T x \geq 0$.

$\Rightarrow \lambda \geq 0$.

($\Leftarrow$) If $\forall \lambda_i \geq 0$, $i = 1, 2, \cdots, n$. then

$K = C^T \Lambda C$. $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$.

for $\forall x$ $x^T K x = (Cx)^T \Lambda (Cx) = \sum_{i=1}^{n} \lambda_i p_i \geq 0$. $Cx = (p_1, \cdots, p_n)^T$

(b) Since $K \geq 0$. $\exists U = (u_1, \cdots, u_n)$, $u_i \in R^n$, st

$K = U^T U$

then $d_{ij} = k_{ii} + k_{jj} - 2k_{ij} = u_i^T u_i + u_j^T u_j - 2u_i^T u_j = ||u_i - u_j||^2$

(c) $d_{ij} = k_{ii} + k_{jj} - 2k_{ij}$

$D = diag(K) e^T + e\, diag^T(K) - 2K$ $diag(K) = (K_{11}\ K_{22} \cdots K_{nn})^T$

for $\forall x \neq 0$ $x^T B_a x = -\frac{1}{2} x^T H_a D H_a^T x = -\frac{1}{2} (H_a^T x)^T D H_a^T x$

$= -\frac{1}{2} (H_a^T x)^T [diag(K) e^T + e\, diag^T(K) - 2K] H_a^T x$

Consider $H_a^T x = (I - a e^T) x$.

Thus $(H_a^T x)^T diag(K) e^T H_a^T x = x^T (I - e a^T) diag(K) e^T (I - a e^T) x$

$= x^T (I - e a^T) diag(K) (e^T - e^T) x = 0$

$(H_a^T x)^T e\, diag^T(K) H_a^T x = x^T (I - e a^T) e\, diag^T(K) (I - a e^T) x = 0$.

Thus $x^T B_a x = (H_a^T x)^T K H_a^T x \geq 0$.

(d) for $\forall x \in R^n$ $x^T (A+B) x = x^T A x + x^T B x \geq 0$.

$\Rightarrow A + B \geq 0$ $x^T A \circ B x = x^T diag(A D_x B^T) \geq 0$.

4. (a) d^2 is not a distance function.

Let $d(x,y) = |x-y|$. d is a distance function.

Let $x=0$, $y=2$, $z=4$.

$d^2 = |x-y|^2$.

$d^2(x,z) = 4^2 > d^2(x,y) + d^2(y,z) = 8$.

Thus, d^2 is not a distance function.

(b) $\sqrt{d}$ is a distance function.

Because d is a distance function

$d(x,y) \geq 0 \Rightarrow \sqrt{d(x,y)} \geq 0$

$d(x,y) = 0 \Rightarrow \sqrt{d(x,y)} = 0$

$d(x,y) = d(y,x) \Rightarrow \sqrt{d(x,y)} = \sqrt{d(y,x)}$.

$d(x,y) \leq d(x,z) + d(z,y) + 2\sqrt{d(x,z)\, d(z,y)}$

$\Rightarrow (\sqrt{d(x,y)})^2 \leq (\sqrt{(x,z)} + \sqrt{(z,y)})^2$.

$\sqrt{d(x,y)} \leq \sqrt{(x,z)} + \sqrt{(z,y)}$.