1. (a). Given $SNR = \frac{\lambda_0}{\sigma^2} > \sqrt{\gamma}$. Find λ.

By the property of Primary Eigenvalue.

$$1 = \sigma^2 \cdot \frac{1}{p} \sum \frac{\lambda_i}{\hat{\lambda} - \sigma^2 \cdot t} \sim \lambda_0 \int_a^b \frac{t}{\hat{\lambda} - \sigma^2 t} d\mu^{MP}(t)$$

By using Stieltjes transform.

$$1 = \lambda_0 \int_a^b \frac{t}{\hat{\lambda} - \sigma^2 t} \frac{\sqrt{b-t}\sqrt{a-t}}{2\pi \gamma t} dt.$$

$$= \frac{\lambda_0}{4\gamma\sigma^2}\left[2\hat{\lambda} - (a+b) - 2\sqrt{|(\hat{\lambda}-a)(b-\hat{\lambda})|}\right.$$

For $\hat{\lambda} \geq b$, $SNR = \frac{\lambda_0}{\sigma^2} \geq \sqrt{\gamma}$.

$$\hat{\lambda} = \left(1 + \frac{\lambda_0}{\sigma^2}\right)\left(1 + \frac{\gamma}{\lambda_0}\sigma^2\right)\sigma^2$$

(b)

$$\lambda = (1 + SNR)\left(1 + \frac{\gamma}{SNR}\right)\sigma^2.$$

To solve SNR, we need to get.

$\lambda, \gamma, \sigma^2$. And this is a quadratic equation for SNR.

(c)

Since we have Monte-Carlo Integration,

$$|u^T v|^{-2} = \lambda_0^2 \int_a^b \frac{t^2}{(\lambda - \sigma^2 t)^2} d\mu^{MP}(t)$$

For $\lambda \geqslant b$

$$|u^T v|^2 = \frac{1 - \frac{\gamma}{SNR^2}}{1+\gamma + \frac{2\gamma}{SNR}} = \frac{SNR^2 - \gamma}{SNR[(1+\gamma)SNR + 2\gamma]}$$

2. See the code.

3.

Consider the eigenvalue λ with corresponding eigenvector v.

$$(W + \lambda_0 u u^T) v = \lambda v$$

$$(\lambda I_p - W) v = \lambda_0 u u^T v.$$

If $u^T v \neq 0$.

$$u^T v = \underbrace{\lambda_0 u^T (\lambda I_p - W)^{-1} u}(u^T v).$$

$$\lambda_0 u^T (\lambda I_p - W)^{-1} u = 1.$$

Assume $W = U \Lambda U^T$.

where $U = [u_1, u_2, \cdots, u_p]$ is an orthogonal matrix. $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p)$.

$$1 = \lambda_0 u^T U (\lambda I_p - \Lambda)^{-1} U^T u$$

$$= \lambda_0 \sum_{i=1}^{p} \frac{1}{\lambda - \lambda_i} \cdot (u_i^T u).$$

since $\|u\|_2^2 = 1$. $\sum u_i^T u = 1$

$$1 \sim \lambda_0 \int_{-1}^{1} \frac{1}{\lambda - t} \, d\mu.$$

$$= \lambda_0 \int_{-1}^{1} \frac{2\sqrt{1-t^2}}{\pi(\lambda - t)} \, dt.$$

$$= \lambda_0 \left(2\lambda - 2\sqrt{\lambda^2 - 1} \right)$$

$$\lambda = \lambda_0 + \frac{1}{4\lambda_0}$$

To obtain $u^T v$,

$$|u^T v|^{-2} = \lambda_0^2 u^T (\lambda I_p - \Lambda)^{-2} u$$

$$= \lambda_0^2 \int_{-1}^{1} \frac{2(\sqrt{1-t^2})}{\pi(\lambda - t)^2} \, dt.$$

$$= \frac{1}{1-\frac{1}{4\lambda_0^2}}$$

$$|u^T v| = \sqrt{1-\frac{1}{4\lambda_0^2}}$$