HW1

Q3_ (a) $\Rightarrow$ $K \succeq 0$ $x^T K x \geq 0$

suppose $U K U^T = D$ ($D = diag(\lambda_1, \cdots \lambda_n)$)

we can pick $x_i \cdots x_k$ that

$x_i^T U K U^T x_i = \lambda_i \cdot c_i^2$

$K \succeq 0 \Rightarrow \lambda_i \geq 0 \quad \forall i$

$\Leftarrow$ $x^T K x = x^T U D U^T x$

Here view $U^T x$ as y

so $x^T K x$ can be rewrite as losing information

$\sum_{i=1}^{n} c_i^2 \cdot \lambda_i$ it is clearly that

$\sum c_i^2 \lambda_i \geq 0$

so $K \succeq 0$

(b) Let $x = e_i - e_j$ $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \rightarrow i$

$x^T K x = d_{ij}$

$x^T K x = (x^T U) D (U^T x)$

so it can be viewed as

the squared distance function

(c) similar to MDS derivation

let $K = x^T x$

$D = K \cdot e^T + e \cdot K^T - 2K$

$$(I - e\alpha^T)(k \cdot e^T + ek^T - 2k)(I - e\alpha^T)^T$$

$$(I - e\alpha^T)\, ek = 0$$

so it can converted to $H_\alpha k H_\alpha^T = (H_\alpha x^T)(x H_\alpha^T)$

so it is SPD

(d) for $A+B$, $\forall x \in \mathbb{R}^n$

$$x^T(A+B)x = \underbrace{x^TAx}_{\geq 0} + \underbrace{x^TBx}_{\geq 0} \geq 0$$

So $A+B$ is SPD

for $A \odot B$

$$A = \sum a_i u_i u_i^T \qquad B = \sum b_i v_i v_i^T$$

$$A \odot B = \sum_{ij} a_i b_j (u_i u_i^T) \odot (v_j v_j^T)$$

$$= \sum_{ij} a_i b_j (u_i \odot v_j)(u_i \odot v_j)^T$$

for $(u_i \odot v_j)(u_i \odot v_j)^T$ is SPD

$A \odot B$ is SPD

Q4. (a) ① $d^2(x,y) = d^2(y,x)$

② $d^2 > 0$

from property of $d(\cdot,\cdot)$

let $d(x,y) = |x-y|$

pick $x=0$ $y=1$ $z=\frac{1}{2}$

$d(x,y)=1$ $d(x,z)+d(y,z) = 1$

$d^2(x,y) = 1$ $d^2(x,z) + d^2(y,z) = \frac{1}{2}$

so it has counter example

(b) ① $\sqrt{d}(x,y) = \sqrt{d}(y,x)$

② $\sqrt{d}(x,y) > 0$

if $\forall x, y, z$

$\sqrt{d}(x,y) \leq \sqrt{d}(x,z) + \sqrt{d}(y,z)$

$\Leftrightarrow$ $d(x,y) \leq d(x,z) + d(y,z) + C$ $(C \geqslant 0)$

we have $d(x,z) + d(y,z) \geq d(x,y)$

so $\sqrt{d}$ is a distance function