Homework 1

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3. Suppose $(\lambda_1, \cdots, \lambda_n)$ and $(v_1, \cdots, v_n)$ are eigenpairs

(a) $\Rightarrow$ If $k \succeq 0$, for any $i = 1, \cdots, n$

$$v_i^T k v_i = v_i^T (\lambda_i v_i) = \lambda_i v_i^T v_i \geq 0$$

$$\Rightarrow \lambda_i \geq 0$$

$\Leftarrow$ If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ $\quad \Sigma = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$k = Q^T \Sigma Q$$

$$\forall x \in \mathbb{R}^n, \quad x^T k x = x^T Q^T \Sigma Q x = (Qx)^T \Sigma Q x$$

Let $Qx = (y_1, \cdots, y_n)^T$, then

$$x^T k x = \sum_{i=1}^{n} \lambda_i y_i^2 \geq 0$$

c) $D = ke^T + ek^T - 2K$, $k = diag(K) \in \mathbb{R}^n$

Suppose $K = X^T X$

$B_\alpha = -\frac{1}{2} H_\alpha D H_\alpha^T = -\frac{1}{2} H_\alpha (ke^T + ek^T - 2K) H_\alpha^T$

$H_\alpha k e^T H_\alpha^T = (I - e\alpha^T) k e^T (I - \alpha e^T)$

$= (I - e\alpha^T) k (e^T - e^T \alpha e^T)$

$= (I - e\alpha^T) k \cdot 0 = 0$

$H_\alpha e k^T H_\alpha^T = 0$

Then $B_\alpha = H_\alpha K H_\alpha^T$

$\forall x \in \mathbb{R}^n$, $x^T B_\alpha x = x^T H_\alpha K H_\alpha^T x$

$= (H_\alpha^T x)^T K (H_\alpha^T x)$

Since $K \succeq$ PSD $\Rightarrow x^T B_\alpha x \geq 0$

$\Rightarrow B_\alpha \succeq$ PSD

(d) ① $\forall x \in \mathbb{R}^n$,

$$x^T(A+B)x = x^TAx + x^TBx \geq 0+0 = 0$$

$$\Rightarrow A+B \succeq 0$$

② $B \succeq 0 \Rightarrow \exists V$ s.t. $B = VV^T$, $V = (v_1, \dots, v_n)$

$\forall x \in \mathbb{R}^n$,

$$x^T(A \circ B)x = x^T(A \circ (VV^T))x$$

$$= \sum_{i,j=1}^{n} x_i a_{ij} \left(\sum_{k=1}^{n} v_{ik} v_{jk}\right) x_j$$

$$= \sum_{k=1}^{n} (x \circ v_k)^T A (x \circ v_k) \geq \sum_{k=1}^{n} 0 = 0$$

$$\Rightarrow A \circ B \succeq 0$$

4. (a) Not.

Pick three points $x_1 = -2$, $x_2 = 0$, $x_3 = 2$

define $d(a,b) = |a-b|$, it's obvious that d is a distance function

Then $d^2(a,b) = |a-b|^2$

$d(x_1, x_2) = 2$, $d^2(x_1, x_2) = 2^2 = 4$

$d(x_2, x_3) = 2$, $d^2(x_2, x_3) = 2^2 = 4$

$d(x_1, x_3) = 4$, $d^2(x_1, x_3) = 4^2 = 16$

$\Rightarrow$ $d^2(x_1, x_3) < d^2(x_1, x_2) + d^2(x_2, x_3)$

$\Rightarrow$ d^2 is not a distance function

(b) Suppose $g(\lambda) = \dfrac{\frac{1}{2}}{\Gamma(\frac{1}{2})} \lambda^{-\frac{1}{2}}$

then $\displaystyle\int_0^\infty \frac{1-\exp(-\lambda d)}{\lambda} g(\lambda) d\lambda$

$$= \int_0^\infty \frac{1 - \exp(-\lambda d)}{\lambda} \frac{\frac{1}{2}}{\Gamma(\frac{1}{2})} \lambda^{-\frac{1}{2}} \, d\lambda$$

$$= d^{\frac{1}{2}}$$

$\Rightarrow$ $\phi(d) = d^{\frac{1}{2}}$ is a schoeberg transform

$\Rightarrow$ $d^{\frac{1}{2}}$ is a distance function