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Mathematics

TOP 100

Questions & Solutions

Class 12

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Class - 12

Relations and Functions

1. The functions f: **R** → **R** and g: **R** → **R** are defined by $f(x) = x + a$, $g(x) = 2x + a^2$, where a is constant. If $gof = fog$, then what is the value of a?

2. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b = \min\{a, b\}$. Write the operation table of the operation *.

3. Let * be a binary operation, on the set of all non-zero real numbers, given by $a*b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$. **(Delhi 2014)**

4. Let f : **N** → **R** be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that f : **N** → S, where S is the range of f, is invertible. Also find the inverse of f. **(Abroad 2015)**

5. Consider the binary operations *: R × R → R and o: R × R → R defined as and $a * b = |a - b|$ and $a \ o \ b = a$ for all $a, b \in$ **R**. Show that '*' is commutative but not associative, 'o' is associative but not commutative. **(All India 2012)**

6. Let A = {1, 2, 3,......, 9} and R be the relation in A × A defined by (a, b) R (c, d) if $a + d = b + c$ for (a, b), (c, d) in A × A. Prove that R is an equivalence relation. Also, obtain the equivalence class [(2, 5)]. **(Delhi 2014)**

7. Let A = **R** − {3} and B = **R** − {1}. Consider the function f : A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that is one-one and onto and hence find f^{-1}. **(Delhi 2013)**

8. If f, g : **R** → **R** be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, $\forall x \in \mathbf{R}$. Then find fog and gof. Hence find $fog(-3)$, $fog(5)$ and $gof(-2)$. **(Abroad 2016)**

Inverse Trigonometric Functions

9. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$. **(CBSE 2011)**

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10. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

11. Find the possible values of x if $\tan^{-1}\frac{x^2-1}{x+3}-\tan^{-1}\frac{x^2}{x-3}=\frac{\pi}{2}$.

12. Prove that $\tan^{-1}\frac{3}{5}+\cos^{-1}\frac{15}{17}=\sin^{-1}\frac{5}{\sqrt{34}}$.

13. Prove the following:

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$$

14. Prove that $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$. **(Delhi 2014)**

15. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then find x. **(Delhi 2015)**

16. Prove that :

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

(All India 2015)

Matrices

17. If $(2\ 1\ 3)\begin{pmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}=A$, then write the order of matrix A. **(Abroad 2016)**

18. Write the element a_{23} of a 3×3 matrix $A=(a_{ij})$ whose elements a_{ij} are given by $a_{ij}=\frac{|i-j|}{2}$. **(Delhi 2015)**

19. Find the values of p, q, r, and s if

$$5\begin{bmatrix} 1 & p \\ s & 3 \end{bmatrix} = 2\begin{bmatrix} q & r+1 \\ -7 & r \end{bmatrix} + 3\begin{bmatrix} q & -6 \\ -2 & r \end{bmatrix}$$

20. If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$, then show that $A^2 + 2A + 7I = 0$.

21. Three schools A, B and C organized a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of Rs 25, Rs 100 and Rs 50 each. The number of articles sold are given below:

School			
Article	**A**	**B**	**C**
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by selling the above articles. Also find the total funds collected for the purpose.
Write one value generated by the above situation. **(Delhi 2015)**

22. Obtain the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 6 & -3 & 3 \\ 4 & 3 & 1 \end{bmatrix}$ using elementary row operations.

23. Express the matrix $\begin{bmatrix} 5 & 2 & 7 \\ -4 & 1 & 9 \\ 1 & -3 & -6 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

24. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, then find the values of a and b.

Determinants

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25. Let A be a square matrix of order 3 × 3. Write the value of |2A|, where |A| = 4.

26. Using properties of determinants, show that ΔABC is isosceles if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

27. If a, b and c are all non-zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0,$ then prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1=0$.

28. Using the properties of determinants, prove the following:

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix} = 6x^2\left(1-x^2\right)$$

(All India 2015)

29. Using properties of determinants, show that $\begin{pmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{pmatrix} = 4abc$ **(All India 2012)**

30. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

(All India 2014)

31. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of Rs 6,000. Three times the award money for Hard work added to that given for honesty amounts to Rs 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards. **(Delhi 2013)**

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32. Using elementary transformations, find the inverse of the matrix $A = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ and use it to solve the following system of linear equations :
$8x + 4y + 3z = 19$
$2x + y + z = 5$
$x + 2y + 2z = 7$ **(Delhi 2016)**

Continuity and Differentiability

33. Find the value of k so that the function defined by $f(x) = \begin{cases} \dfrac{1-\cos kx}{\tan^2 x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ is continuous at $x = 0$.

34. Differentiate $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$ w.r.t. x.

35. If $\sin(x+y) + \cos(x+y) = \log(x+y)$, then find the value of $\dfrac{dy}{dx}$.

36. If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\dfrac{d^2y}{dx^2}$.

37. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\dfrac{2x}{1+x^2}$, if $x \in (-1, 1)$

38. If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$.

39. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x. **(All India 2016)**

40. Find the values of a and b, if the function f defined by $f(x) = \begin{cases} x^2 + 3x + a, & x \leqslant 1 \\ bx + 2, & x > 1 \end{cases}$ is differentiable at $x = 1$. **(Abroad 2016)**

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Application of Derivatives

41. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

42. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm ? **(Delhi 2015)**

43. Using differentials, find the approximated value of $\sqrt{49.5}$. **(Delhi 2012)**

44. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

45. Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

46. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is
(a) strictly increasing
(b) strictly decreasing **(Delhi 2014)**

47. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. **(Delhi 2016)**

48. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$.

49. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. **(All India 2012)**

50. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes. **(Abroad 2016)**

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51. The sum of lengths of the hypotenuse and a side of a right triangle is 24 cm. Prove that the maximum area of the triangle is $32\sqrt{3}\ \text{cm}^2$ and the angle between the side and the hypotenuse is $\frac{\pi}{3}$.

52. Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$. **(Abroad 2015)**

Integrals

53. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. **(Delhi 2014)**

54. Evaluate : $\int \sin x \sin 2x \sin 3x\, dx$ **(Delhi 2012)**

55. Evaluate:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

(All India 2013)

56. Evaluate :

$$\int (x-3)\sqrt{x^2+3x-18}\,dx$$

(Delhi 2014)

57. Evaluate :

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$$

(Abroad 2015)

58. Evaluate:

$$\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$$

(All India 2012)

59. Find :

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

(Abroad 2016)

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60. Evaluate:

$$\int \frac{1}{\cos^4 x + \sin^4 x} dx$$

(All India 2014)

61. Evaluate the integral $\int_0^{\frac{\pi}{2}} x\left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$.

62. Evaluate:

$$\int_2^5 \left[|x-2| + |x-3| + |x-5|\right] dx$$

(Delhi 2013)

63. Evaluate :

$$\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$$

(All India 2014)

64. Evaluate :

$$\int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx$$

(All India 2015)

65. Evaluate

$$\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4} + x\right) dx$$

(Delhi 2016)

66. Evaluate :

$$\int_0^{\frac{3}{2}} |x \cos \pi x| dx$$

(All India 2016)

67. Evaluate $\int_1^3 \left(e^{2-3x} + x^2 + 1\right) dx$ as a limit of a sum.

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Application of Integrals

68. Find the area of the region bounded by curve $x^2 + y^2 = 16$ and lines $x = 2$ and $x = 2\sqrt{3}$.

69. Using integration find the area of the triangle formed by positive x-axis and tangent and normal of the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

70. Using integration, find the area of the region bounded by the triangle whose vertices are $(-1, 2)$, $(1, 5)$ and $(3, 4)$. **(All India 2014)**

71. Find the area of the region $\{(x, y): y^2 \leq 6ax$ and $x^2 + y^2 \leq 16a^2\}$ using method of integration.

72. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$. **(Delhi 2013)**

73. Using integration find the area of the region bounded by the curves $y = \sqrt{4 - x^2}$, $x^2 + y^2 - 4x = 0$ and the x-axis. **(Abroad 2016)**

Differential Equations

74. Form a differential equation representing the family of curves $y = ax^2 + bx^3$, where a and b are constants.

75. Solve the differential equation :
$(\tan^{-1}y - x)dy = (1 + y^2)dx$.

76. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ given that $y = 1$, when $x = 0$.

77. Solve the differential equation, $(x^3 + 4xy^2)dy + 5x^2y\,dx = 0$

78. Find the solution of the differential equation,
$x\frac{dy}{dx} + xy\tan x - (x^2 + x)\sec x = 0,\ x \neq 0,\ \frac{\pi}{2}$

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79. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$. **(Delhi 2014)**

80. Find the particular solution of the differential equation $(1 - y^2)(1 + \log x)\,dx + 2xy\,dy = 0$, given that $y = 0$ when $x = 1$. **(Delhi 2016)**

Vector Algebra

81. If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{a}| = 5$, then write the value of $|\vec{b}|$.

82. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. **(Delhi 2015)**

83. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - j + 4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p} \cdot \vec{c} = 18$. **(All India 2012)**

84. Prove that, for any three vectors $\vec{a}, \vec{b}, \vec{c}$
$[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$ **(Delhi 2014)**

85. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, −1, 4) and C(4, 5, −1). **(Delhi 2013)**

Three Dimensional Geometry

86. A plane makes intercepts along positive x-axis, positive y-axis and positive z-axis in the ratio 1: 3: 4. If the distance of the plane from the origin is 6 units, then find the equation of the plane and also the direction cosines of the normal to the plane.

87. A line passing through points (2, −3, −5) and (3, −2, 4) intersects the xy-plane and yz-plane at A and B respectively. Find the equation of plane passing through points A, B, and (1, 2, 3).

88. Show that the lines $\frac{x+1}{6} = \frac{y+3}{-4} = \frac{z-12}{-2}$ and $\frac{x-1}{1} = \frac{y-5}{-3} = \frac{z-2}{2}$ intersect each other. Also, find the coordinates of the point of intersection.

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89. Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}\cdot\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$. **(All India 2013)**

90. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda\left(2\hat{i}+3\hat{j}-\hat{k}\right)$. Also find the image of P in this line. **(All India 2012)**

91. Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0$ and $\vec{r}\cdot\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}\cdot\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$. **(Delhi 2013)**

Linear Programming

92. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below :

$2x + 4y \le 8$

$3x + y \le 6$

$x + y \le 4$

$x \ge 0, y \ge 0$ **(Delhi 2015)**

93. A cement manufacturer has two depots **P** and **Q** with stocks of 60,000 and 40,000 bags of cement. He receives order from three builders **X**, **Y** and **Z** for 30,000, 40,000 and 30,000 bags of cement respectively. The costs of transporting a bag of cement to the builders from the depots are given as:

Transportation cost of a bag of cement (in Rs)			
From/To	**X**	**Y**	**Z**
P	8	4	6
Q	4	12	8

How should the manufacturer supply the order to minimise the transportation cost? Find the minimum transportation cost.

94. A firm plans to purchase at least 200 quintals of scrap comprising high quality metal **X**and low quality metal **Y**. The firm decides that the scrap must contain at least 100 quintals of metal **X** and at most 35 quintals of metal **Y**. The firm purchases the scrap from two suppliers **A** and **B**. The percentage of metal **X** and metal **Y**in terms of weight in the scrap supplied by **A** and **B** is given as follows:

Metal	Supplier	
	A	B
X	25%	75%
Y	10%	20%

The cost per quintal of scraps supplied by suppliers **A** and **B**are Rs 200 and Rs 400 respectively. Find the quantities that the firm should buy from the two suppliers to minimize the purchase cost.

95. A man wants to start a fruit shop where he could sell apples and grapes. He has only Rs 2200 to invest and has a space for at most 60 kg of items. The cost of apples and grapes is Rs 40/kg and Rs 32/kg respectively. He wants to earn a profit of Rs 10 on apples and Rs 8 on grapes. Assuming that he can sell all the apples and grapes, how should he invest his money in order to maximise the profit? Translate this problem as an LPP and solve it.

Probability

96. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B, If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black. **(Delhi 2015)**

97. A and B throw a die alternatively till one of them gets a number greater than four and wins the game. If A starts the game, what is the probability of B winning? **(All India 2015)**

98. Let, X denote the number of colleges where you will apply after your results and $P(X = x)$ denotes your probability of getting admission in x number of colleges. It is given that

$$P(X = x) = \begin{cases} kx & , \text{ if } x = 0 \text{ or } 1 \\ 2\,kx & , \text{ if } x = 2 \\ k(5-x) & , \text{ if } x = 3 \text{ or } 4 \\ 0 & , \text{ if } x > 4 \end{cases}$$

where k is a positive constant. Find the value of k. Also find the probability that you will get admission in (i) exactly one college (ii) at most 2 colleges (iii) at least 2 colleges.

99. Twenty persons are successively selected with replacement from a group of persons of which 30% are vegetarians. Find the probability that there are at least 3 vegetarians.

100. A bottle manufacturing company produces bottles in three plants, A, B, and C, with daily production of 1500, 2000, and 2500 bottles respectively. The fraction of defective bottles produced by plant A, B, and C are 0.006, 0.008, and 0.01 respectively. On a particular day, when a bottle is selected at random, it is found to be defective. What is the probability that it was manufactured by plant C?

Solutions

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Relations and Functions

1.

The functions f: $\mathbf{R} \to \mathbf{R}$ and g: $\mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x + a$, $g(x) = 2x + a^2$, where a is constant.

If $gof = fog$, then

$$gof(x) = fog(x) \ \forall x \in \mathbf{R}$$
$$\Rightarrow g(f(x)) = f(g(x))$$
$$\Rightarrow g(x+a) = f(2x+a^2)$$
$$\Rightarrow 2(x+a)+a^2 = (2x+a^2)+a$$
$$\Rightarrow 2x+2a+a^2 = 2x+a^2+a$$
$$\Rightarrow a = 0$$

Thus, the value of a is 0.

2.

The binary operation * on the set {1, 2, 3, 4, 5} is defined by $a * b = \min\{a, b\}$

$1 * 1 = \min\{1, 1\} = 1$

$1 * 2 = \min\{1, 2\} = 1$

$2 * 3 = \min\{2, 3\} = 2$

$4 * 5 = \min\{4, 5\} = 4$

The operation table for the given operation * on the given set can be written as:

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

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3.

It is given that $a*b=\dfrac{ab}{5}$, where $a,\, b \in R-\{0\}$ and $2*(x*5)=10$.

Using $a*b=\dfrac{ab}{5}$ in the equation $2*(x*5)=10$, we get

$$2*\left(\frac{5x}{5}\right)=10 \qquad (\because x*5=5x)$$

$$\Rightarrow 2*x=10$$

$$\Rightarrow \frac{2x}{5}=10$$

$$\Rightarrow x=25$$

Therefore, the value of x is 25.

4.

The given function is $f(x) = 4x^2 + 12x + 15$.

Let us show that f : **N** → S is a bijection.

f is one-one:

For any $x_1, x_2 \in \mathbf{N}$,

Let $f(x_1)=f(x_2)$

$$\Rightarrow 4x_1^2+12x_1+15=4x_2^2+12x_2+15$$

$$\Rightarrow 4\left(x_1^2-x_2^2\right)+12\left(x_1-x_2\right)=0$$

$$\Rightarrow 4\left(x_1-x_2\right)\left(x_1+x_2+3\right)=0$$

$$\Rightarrow x_1-x_2=0 \qquad \left[\because x_1+x_2+3\neq 0 \text{ for any } x_1, x_2 \in \mathbf{N}\right]$$

$$\Rightarrow x_1=x_2$$

So, f : **N** → S is one-one.

Therefore, f : **N** → S, where S is the range of f, is onto.

Hence, f : **N** → S is invertible.

Let f^{-1} denote the inverse of f.

Then

$fof^{-1}(x) = x \quad \forall\ x \in S$

$\Rightarrow f\left[f^{-1}(x)\right] = x \quad \forall\ x \in S$

$\Rightarrow 4\left[f^{-1}(x)\right]^2 + 12f^{-1}(x) + 15 = x$

$\Rightarrow 4\left[f^{-1}(x)\right]^2 + 12f^{-1}(x) + 15 - x = 0$

$$\Rightarrow f^{-1}(x) = \frac{-12 \pm \sqrt{144 - 4 \times 4 \times (15 - x)}}{8}$$

$$\Rightarrow f^{-1}(x) = \frac{-12 \pm \sqrt{16x - 96}}{8}$$

$$\Rightarrow f^{-1}(x) = \frac{-12 \pm 4\sqrt{x-6}}{8} = \frac{-3 \pm \sqrt{x-6}}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{-3 + \sqrt{x-6}}{2} \qquad \left[\because f^{-1}(x) \in \mathbf{N}, \text{ so } f^{-1}(x) > 0\right]$$

Thus, the inverse of f is $\dfrac{\sqrt{x-6} - 3}{2}$.

5.

It is given that *: **R** × **R** → and o: **R** × **R** → **R** defined as

$a * b = |a - b|$ and $a \ o \ b, = a$, for all $a, b \in \mathbf{R}$.

For $a, b \in \mathbf{R}$ we have:

$a * b = |a - b|$

$b * a = |b - a| = |-(a - b)| = |a - b|$

$\therefore a * b = b * a$

∴The operation * is commutative.

It can be observed that,

$(1*2)*3 = (|1-2|)*3 = 1*3 = |1-3| = 2$

$1*(2*3) = 1*(|2-3|) = 1*1 = |1-1| = 0$

$\therefore (1*2)*3 \neq 1*(2*3) \ \ (\text{where } 1, 2, 3 \in \mathbf{R})$

∴The operation * is not associative.

Now, consider the operation o:

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

$\therefore$ 1 o 2 $\neq$ 2 o 1 (where 1, 2 $\in$ **R**)

$\therefore$ The operation o is not commutative.

Let $a, b, c \in \mathbf{R}$. Then, we have

$(a \text{ o } b) \text{ o } c = a \text{ o } c = a$

$a \text{ o } (b \text{ o } c) = a \text{ o } b = a$

$\Rightarrow (a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c)$

$\therefore$ The operation o is associative.

6.

A = {1, 2, 3, ..., 9} $\subset$ **N**, the set of natural numbers

Let R be the relation in A × A defined by (a, b) R (c, d) if $a + d = b + c$ for (a, b), (c, d) in A × A.

We have to show that R is an equivalence relation.

Reflexivity:

Let (a, b) be an arbitrary element of A × A. Then, we have:

$(a, b) \in A \times A$

$\Rightarrow a, b \in A$

$\Rightarrow a + b = b + a$ (by commutativity of addition on $A \subset \mathbb{N}$)

$\Rightarrow (a, b)$ R (a, b)

Thus, (a, b) R (a, b) for all $(a, b) \in A \times A$.

So, R is reflexive.

Symmetry:

Let $(a, b), (c, d) \in A \times A$ such that (a, b) R (c, d).

$a + d = b + c$

$\Rightarrow b + c = a + d$

$\Rightarrow c + b = d + a$ (by commutativity of addition on $A \subset \mathbb{N}$)

$\Rightarrow (c, d)$ R (a, b)

Thus, (a, b) R $(c, d) \Rightarrow (c, d)$ R (a, b) for all $(a, b), (c, d) \in A \times A$.

So, R is symmetric.

Transitivity:

Let $(a, b), (c, d), (e, f) \in A \times A$ such that (a, b) R (c, d) and (c, d) R (e, f). Then, we have

(a, b) R (c, d)

$\Rightarrow a + d = b + c$ ...(1)

(c, d) R (e, f)

$\Rightarrow c + f = d + e$ ...(2)

Adding (1) and (2), we get

$(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b)$ R (e, f)

Thus, (a, b) R (c, d) and (c, d) R $(e, f) \Rightarrow (a, b)$ R (e, f) for all $(a, b), (c, d), (e, f) \in A \times A$.

So, R is transitive on A × A.

Thus, R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

To write the equivalence class of [(2, 5)], we need to search all the elements of the type (a, b) such that $2 + b = 5 + a$.

$\therefore$ Equivalence class of [(2, 5)] = {(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)}

7.

Given, A = **R** − {3} and **B** = R − {1}.

f: A → B is a function defined by $f(x) = \dfrac{x-2}{x-3}$.

Let $x_1, x_2 \in A$.

Now,

$$f(x_1) = f(x_2)$$
$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$
$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$
$$\Rightarrow x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$$
$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

Again, let y be any arbitrary element of B.

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$$f(x) = y$$
$$\Rightarrow \frac{x-2}{x-3} = y$$
$$\Rightarrow (x-2) = y(x-3)$$
$$\Rightarrow x-2 = xy-3y$$
$$\Rightarrow x-xy = 2-3y$$
$$\Rightarrow x(1-y) = 2-3y$$

$\Rightarrow x = \frac{2-3y}{1-y}$, which is a real number for all $y \neq 1$.

Also, $\frac{2-3y}{1-y} \neq 3$ for any y, because if we take $\frac{2-3y}{1-y} = 3$

$$\Rightarrow 2-3y = 3-3y$$

$\Rightarrow 2 = 3$, which is not possible.

$\therefore x \in \mathbf{R} - \{3\}$ for all $y \in \mathbf{R} - \{1\}$.

Thus, for all $y \in \mathbf{R} - \{1\}$ there exists $x = \frac{2-3y}{1-y} \in \mathbf{R} - \{3\}$ such that

$$f(x) = f\left(\frac{2-3y}{1-y}\right) = \left(\frac{\frac{2-3y}{1-y} - 2}{\frac{2-3y}{1-y} - 3}\right) = \left(\frac{2-3y-2+2y}{2-3y-3+3y}\right) = y$$

$\therefore$ Every element y in B has a pre-image x in A which is given by $x = \frac{2-3y}{1-y}$.

So, f is onto.

Hence, f is one-one and onto.

To find f^{-1}:

Let $f(x) = y$ where $x \in \mathbf{R} - \{3\}$ and $y \in \mathbf{R} - \{1\}$.

$\therefore y = \dfrac{x-2}{x-3}$

$\Rightarrow x = \dfrac{2-3y}{1-y}$

$\Rightarrow f^{-1}(y) = \dfrac{2-3y}{1-y}$

$\therefore f^{-1} : \mathbf{R} - \{1\} \to \mathbf{R} - \{3\}$ is defined by $f^{-1}(x) = \dfrac{2-3x}{1-x}$ for all $x \in \mathbf{R} - \{1\}$.

8.

Given: $f(x) = |x| + x$

and $g(x) = |x| - x$, $\forall x \in \mathbf{R}$

$$fog = f(g(x)) = |g(x)| + g(x)$$
$$= ||x| - x| + (|x| - x)$$

Therefore,

$$f(g(x)) = \begin{cases} 0 & x \geq 0 \\ 4x & x < 0 \end{cases}$$

$$fog = \begin{cases} 4x & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$gof = g(f(x)) = |f(x)| - f(x)$$
$$= ||x| + x| - (|x| + x)$$

$$g(f(x)) = \begin{cases} 0 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Therefore, $g(f(x)) = gof = 0$

Now, $fog(-3) = (4)(-3) = -12$ (since, $fog = 4x$ for $x < 0$)

$fog(5) = 0$ (since, $fog = 0$ for $x \geq 0$)

$gof(-2) = 0$ (since, $gof = 0$ for $x < 0$)

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Inverse Trigonometric Functions

9.

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$$

$$= \sin\frac{\pi}{2} = 1$$

10.

$$\tan^{-1}\left(2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right) = \tan^{-1}\left(2\sin\left(2\times\frac{\pi}{6}\right)\right) = \tan^{-1}\left(2\sin\left(\frac{\pi}{3}\right)\right) = \tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

11.

$$\tan^{-1}\frac{x^2-1}{x+3} - \tan^{-1}\frac{x^2}{x-3}$$

$$= \tan^{-1}\left[\frac{\frac{x^2-1}{x+3} - \frac{x^2}{x-3}}{1+\left(\frac{x^2-1}{x+3}\right)\left(\frac{x^2}{x-3}\right)}\right] \qquad \left[\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\left[\frac{\frac{(x^2-1)(x-3) - x^2(x+3)}{x^2-9}}{\frac{x^2-9+x^2(x^2-1)}{x^2-9}}\right]$$

$$= \tan^{-1}\left(\frac{x^3-x-3x^2+3-x^3-3x^2}{x^2-9+x^4-x^2}\right)$$

$$= \tan^{-1}\left(\frac{-6x^2-x+3}{x^4-9}\right)$$

$$\tan^{-1}\left(\frac{-6x^2 - x + 3}{x^4 - 9}\right) = \frac{\pi}{2}$$

$$\frac{-6x^2 - x + 3\pi}{x^4 - 9} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$$\Rightarrow x^4 - 9 = 0$$

$$\Rightarrow x^4 = 9$$

$$\Rightarrow x = \pm\sqrt[4]{9} = \pm\sqrt{3}$$

12.

Let $\cos^{-1}\frac{15}{17} = x$

$$\therefore \cos x = \frac{15}{17}$$

$$\Rightarrow \tan x = \frac{8}{15}$$

$$\Rightarrow x = \tan^{-1}\frac{8}{15} \qquad \text{... (i)}$$

$$\therefore \text{LHS} = \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{8}{15} \qquad [\text{Using (i)}]$$

$$= \tan^{-1}\left[\frac{\frac{3}{5} + \frac{8}{15}}{1 - \left(\frac{3}{5}\right)\left(\frac{8}{15}\right)}\right] \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left[\frac{\frac{9+8}{15}}{1 - \frac{8}{25}}\right]$$

$$= \tan^{-1}\left[\frac{17}{15} \times \frac{25}{17}\right] = \tan^{-1}\frac{25}{15} = \tan^{-1}\frac{5}{3} \qquad \text{...(ii)}$$

Let $\tan^{-1}\frac{5}{3} = y$

$\Rightarrow \tan y = \frac{5}{3}$

$\Rightarrow \sin y = \frac{5}{\sqrt{34}}$

$\Rightarrow y = \sin^{-1}\frac{5}{\sqrt{34}}$

$\therefore \tan^{-1}\frac{5}{3} = \sin^{-1}\frac{5}{\sqrt{34}}$

Substituting this in (ii), we obtain

LHS = $\tan^{-1}\frac{5}{3} = \sin^{-1}\frac{5}{\sqrt{34}} = \text{RHS}$

13.

$$1+\sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$$

$$1-\sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$$

$$\therefore \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

14.

$$2\tan^{-1}\left(\frac{1}{5}\right)+sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1}\right)+2\tan^{-1}\left(\frac{1}{8}\right) \quad \left[\text{Using } sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\right]$$

$$= 2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left(\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right)+\tan^{-1}\left(\frac{1}{7}\right) \quad \left[\text{Using } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= 2\tan^{-1}\left(\frac{13}{39}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right) \qquad \left[\text{Using } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \text{ if } |x| < 1\right]$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$= \text{RHS}$$

Hence proved.

15.

$$(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow (\tan^{-1}x + \cos^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x = \frac{5\pi^2}{8}$$

$$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1}x\left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{5\pi^2}{8}$$

$$\Rightarrow \frac{\pi^2}{4} - \pi\tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x + \frac{\pi^2}{4} - \frac{5\pi^2}{8} = 0$$

$$\Rightarrow 2\left(\tan^{-1}x\right)^2 - \pi\tan^{-1}x - \frac{5\pi^2 + 2\pi^2}{8} = 0$$

$$\Rightarrow 2\left(\tan^{-1}x\right)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$$

Solving the quadratic equation, we get

$$\Rightarrow \tan^{-1}x = \frac{\pi \pm \sqrt{\pi^2 + 4\times 2\times \frac{3\pi^2}{8}}}{2\times 2}$$

$$\Rightarrow \tan^{-1}x = \frac{\pi \pm 2\pi}{4}$$

$$\Rightarrow \tan^{-1}x = \frac{3\pi}{4} \text{ or } \tan^{-1}x = -\frac{\pi}{4}$$

$$\Rightarrow x = \tan\frac{3\pi}{4} \text{ or } x = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow x = -1$$

16.

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

$$= \cos^{-1}\left\{\frac{1-\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^2}{1+\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^2}\right\} \qquad \left[\because 2\tan^{-1}(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \cos^{-1}\left\{\frac{1-\frac{a-b}{a+b}\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}\right\}$$

$$= \cos^{-1}\left\{\frac{a+b-(a-b)\tan^2\frac{x}{2}}{a+b+(a-b)\tan^2\frac{x}{2}}\right\}$$

$$= \cos^{-1}\left\{\frac{a+b-a\tan^2\frac{x}{2}+b\tan^2\frac{x}{2}}{a+b+a\tan^2\frac{x}{2}-b\tan^2\frac{x}{2}}\right\}$$

$$= \cos^{-1}\left\{\frac{a\left(1-\tan^2\frac{x}{2}\right)+b\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}\right\}$$

$$= \cos^{-1}\left\{\frac{a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+b\left(\frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}{a\left(\frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\right\}$$ [Dividing the numerator and denominator by $1+\tan^2\frac{x}{2}$]

$$= \cos^{-1}\left\{\frac{a\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\right\}$$

$$= \cos^{-1}\left\{\frac{a\cos x+b}{a+b\cos x}\right\}$$

Matrices

17.

Consider, $$(2\ 1\ 3)\begin{pmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} = A$$

Order of matrix $(2\ 1\ 3)$ is 1×3.

Order of matrix $\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is 3×3

Order of matrix $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is 3×1

Therefore, order of $(2\ 1\ 3)\begin{pmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is 1×1.

18.

Given:

$$a_{ij} = \frac{|i-j|}{2}$$

$$\therefore a_{23} = \frac{|2-3|}{2} = \frac{|-1|}{2} = \frac{1}{2}$$

19.

The given equation can be simplified as

$$5\begin{bmatrix} 1 & p \\ s & 3 \end{bmatrix} = 2\begin{bmatrix} q & r+1 \\ -7 & r \end{bmatrix} + 3\begin{bmatrix} q & -6 \\ -2 & r \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5p \\ 5s & 15 \end{bmatrix} = \begin{bmatrix} 2q & 2r+2 \\ -14 & 2r \end{bmatrix} + \begin{bmatrix} 3q & -18 \\ -6 & 3r \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5p \\ 5s & 15 \end{bmatrix} = \begin{bmatrix} 5q & 2r-16 \\ -20 & 5r \end{bmatrix}$$

It is known that if two matrices are equal, then their corresponding elements are also equal.

$\therefore 5q = 5 \Rightarrow q = 1$

$5s = -20 \Rightarrow s = -4$

$15 = 5r \Rightarrow r = 3$

$5p = 2r - 16$

$\Rightarrow 5p = 2(3) - 16 = 6 - 16 = -10$

$\Rightarrow p = -2$

Thus, the values of p, q, r, and s are -2, 1, 3, and -4 respectively.

20.

$$A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 4-15 & -6+12 \\ 10-20 & -15+16 \end{bmatrix} = \begin{bmatrix} -11 & 6 \\ -10 & 1 \end{bmatrix}$$

Then, $A^2 + 2A - 7I$

$$= \begin{bmatrix} -11 & 6 \\ -10 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 \\ -10 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ 10 & -8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

21.

The number of articles sold by each school can be written in the matrix form as follows:

$$X = \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

The cost of each article can be written in the matrix form as follows:

$$Y = \begin{bmatrix} 25 & 100 & 50 \end{bmatrix}$$

The fund collected by each school is given by

$$YX = \begin{bmatrix} 25 & 100 & 50 \end{bmatrix} \begin{bmatrix} 40 & 25 & 35 \\ 50 & 40 & 50 \\ 20 & 30 & 40 \end{bmatrix}$$

$$YX = \begin{bmatrix} 7000 & 6125 & 7875 \end{bmatrix}$$

Thus, the funds collected by schools A, B and C are Rs 7,000, Rs 6,125 and Rs 7,875, respectively.

The total fund collected = Rs (7000 + 6125 + 7875)

= Rs 21,000

The situation highlights the helpful nature of the students.

22.

Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & -3 & 3 \\ 4 & 3 & 1 \end{bmatrix}$

A = IA

i.e., $\begin{bmatrix} 3 & 2 & 1 \\ 6 & -3 & 3 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_1 \leftrightarrow R_3$, we obtain

$$\begin{bmatrix} 4 & 3 & 1 \\ 6 & -3 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 6 & -3 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 6R_1$, and $R_3 \to R_3 - 3R_1$ we obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & -9 & 3\\0 & -1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\6 & 1 & -6\\4 & 0 & -3\end{bmatrix}A$$

Applying $R_2 \to R_2 - 10R_3$, we obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & -7\\0 & -1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\-34 & 1 & 24\\4 & 0 & -3\end{bmatrix}A$$

Applying $R_3 \to R_3 + R_2$, we obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & -7\\0 & 0 & -6\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\-34 & 1 & 24\\-30 & 1 & 21\end{bmatrix}A$$

Applying $R_3 \to -\frac{1}{6}R_3$, we obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & -7\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\-34 & 1 & 24\\5 & -\frac{1}{6} & -\frac{7}{2}\end{bmatrix}A$$

Applying $R_2 \to R_2 + 7R_3$, we obtain

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 1\\1 & -\frac{1}{6} & -\frac{1}{2}\\5 & -\frac{1}{6} & -\frac{7}{2}\end{bmatrix}A$$

Applying $R_1 \to R_1 - R_2$, we obtain

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Applying $R_1 \to R_1 - R_2$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{6} & \frac{3}{2} \\ 1 & -\frac{1}{6} & -\frac{1}{2} \\ 5 & -\frac{1}{6} & -\frac{7}{2} \end{bmatrix} A$$

Therefore, $A^{-1} = \begin{bmatrix} -2 & \frac{1}{6} & \frac{3}{2} \\ 1 & -\frac{1}{6} & -\frac{1}{2} \\ 5 & -\frac{1}{6} & -\frac{7}{2} \end{bmatrix}$

23.

Let $A = \begin{bmatrix} 5 & 2 & 7 \\ -4 & 1 & 9 \\ 1 & -3 & -6 \end{bmatrix}$

$\therefore A' = \begin{bmatrix} 5 & -4 & 1 \\ 2 & 1 & -3 \\ 7 & 9 & -6 \end{bmatrix}$

Then, $A + A' = \begin{bmatrix} 5 & 2 & 7 \\ -4 & 1 & 9 \\ 1 & -3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & -4 & 1 \\ 2 & 1 & -3 \\ 7 & 9 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 8 \\ -2 & 2 & 6 \\ 8 & 6 & -12 \end{bmatrix}$

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Let $P=\frac{1}{2}(A+A')$

$$=\frac{1}{2}\begin{bmatrix}10 & -2 & 8\\ -2 & 2 & 6\\ 8 & 6 & -12\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & 4\\ -1 & 1 & 3\\ 4 & 3 & -6\end{bmatrix}$$

Then, $P'=\begin{bmatrix}5 & -1 & 4\\ -1 & 1 & 3\\ 4 & 3 & -6\end{bmatrix}=P$

Thus, $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

$$A-A'=\begin{bmatrix}5 & 2 & 7\\ -4 & 1 & 9\\ 1 & -3 & -6\end{bmatrix}-\begin{bmatrix}5 & -4 & 1\\ 2 & 1 & -3\\ 7 & 9 & -6\end{bmatrix}=\begin{bmatrix}0 & 6 & 6\\ -6 & 0 & 12\\ -6 & -12 & 0\end{bmatrix}$$

Let $Q=\frac{1}{2}(A-A')$

$$=\frac{1}{2}\begin{bmatrix}0 & 6 & 6\\ -6 & 0 & 12\\ -6 & -12 & 0\end{bmatrix}=\begin{bmatrix}0 & 3 & 3\\ -3 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}$$

Then, $Q'=\begin{bmatrix}0 & -3 & -3\\ 3 & 0 & -6\\ 3 & 6 & 0\end{bmatrix}=-Q$

Thus, $Q=\frac{1}{2}(A-A')$ is a skew symmetric matrix.

$$\text{Then, } P+Q=\begin{bmatrix}5 & -1 & 4\\ -1 & 1 & 3\\ 4 & 3 & -6\end{bmatrix}+\begin{bmatrix}0 & 3 & 3\\ -3 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 2 & 7\\ -4 & 1 & 9\\ 1 & -3 & -6\end{bmatrix}=A$$

Thus, A is represented as the sum of a symmetric and a skew symmetric matrix.

24.

Given: $(A+B)^2=A^2+B^2$

$\Rightarrow (A+B)(A+B)=A^2+B^2$

$\Rightarrow A(A+B)+B(A+B)=A^2+B^2$

$\Rightarrow A^2+AB+BA+B^2=A^2+B^2$

$\Rightarrow AB+BA=O$

So,

$$\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}+\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a-b & 2\\ 2a-b & 3\end{bmatrix}+\begin{bmatrix}a+2 & -a-1\\ b-2 & -b+1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2a-b+2 & -a+1\\ 2a-2 & -b+4\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$\Rightarrow 2a-b+2=0,\ -a+1=0,\ 2a-2=0,\ -b+4=0$

$\Rightarrow a=1,\ b=4$

Hence, the respective values of a and b are 1 and 4.

Determinants

25.

In a square matrix of order 3 × 3, $|kA| = k^3\,|A|$

$\therefore |2A|=2^3|A|=8\times 4=32$

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26.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix}=0$$

Performing $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & \cos B-\cos A & \cos C-\cos A \\ \cos^2\mathrm{A}+\cos A & (\cos B-\cos A)(\cos A+\cos B+1) & (\cos C-\cos A)(\cos C+\cos A+1) \end{vmatrix}=0$$

$$(\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\cos A & 1 & 1 \\ \cos^2\mathrm{A}+\cos A & (\cos A+\cos B+1) & (\cos C+\cos A+1) \end{vmatrix}=0$$

$(\cos B-\cos A)(\cos C-\cos A)(\cos C-\cos B)=0$

$\therefore\ \cos B=\cos A$

$\Rightarrow B=A$

or $\cos C=\cos A$

$\Rightarrow C=A$

or $\cos C=\cos B$

$\Rightarrow C=B$

$\therefore\ \triangle$ABC is an isosceles triangle.

27.

It is given that, $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}=0,$

Apply elementary row operations $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

Therefore,

$$\begin{vmatrix} a & -b & 0 \\ 0 & b & -c \\ 1 & 1 & 1+c \end{vmatrix} = 0$$

$\Rightarrow a(b+bc+c)+b(0+c)=0$

$\Rightarrow ab+bc+ac+abc=0$

Dividing both sides by *abc*, we get

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1=0$

Hence proved.

28.

$$\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(1-x) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$$

$$= x^2(1-x)\begin{vmatrix} 1 & x & x+1 \\ 2 & (x-1) & (x+1) \\ 3 & -(x-2) & -(x+1) \end{vmatrix}$$

[Taking out x common from R_2 and $x(1-x)$ common form R_3]

$$= x^2(1-x)(1+x)\begin{vmatrix} 1 & x & 1 \\ 2 & (x-1) & 1 \\ 3 & -(x-2) & -1 \end{vmatrix}$$ [Taking out $(1+x)$ common form C_3]

$$= x^2(1-x^2)\begin{vmatrix} 1 & x & 1 \\ 1 & -1 & 0 \\ 4 & 2 & 0 \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + R_1$]

$= x^2(1-x^2)[1\times(2+4)-0+0]$ [Expanding along C_3]

$= 6x^2(1-x^2)$

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29.

$$\text{Let } \Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Applying $R_1 \to R_1 - R_2 - R_3$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & 2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\left(ab + b^2 - bc\right) - 2b\left(bc - c^2 - ac\right)$$

$$= 2abc + 2cb^2 - 2bc^2 - 2b^2c + 2bc^2 + 2abc$$

$$= 4abc$$

30.

Taking LHS, we get:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$$

$$= x^3\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + x^2y\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

$$= x^3\begin{vmatrix} 0 & 0 & 1 \\ 3 & 2 & 2 \\ 7 & 5 & 3 \end{vmatrix} + 0$$

(Using $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, in the first determinant)

(2nd determinant is equal to zero as C_1 and C_2 are equal)

$$= x^3(15 - 14) = x^3$$

$= RHS$

Hence proved.

31.

Let the award money given for Honesty, Regularity and Hard work be x, y and z respectively.

Since total cash award is Rs 6,000.

$\therefore x + y + z =$ Rs 6,000 ...(1)

Three times the award money for Hard work and Honesty is Rs 11,000.

$\therefore x + 3z =$ Rs 11,000

$\Rightarrow x + 0.y + 3z =$ Rs 11,000 ...(2)

Award money for Honesty and Hard work is double the one given for regularity.

$\therefore x + z = 2y$

$\Rightarrow x - 2y + z = 0$ ...(3)

The above system can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

Or $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$|A| = 6 \neq 0$

Thus, A is non-singular. Hence, it is invertible.

$$\text{Adj } A = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

Hence, $x = 500$, $y = 2000$, and $z = 3500$.

Thus, award money given for Honesty, Regularity and Hard work are Rs 500, Rs 2000 and Rs 3500 respectively.

School can include sincerity for awards.

32.

A = IA

i.e.

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 8R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 12R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying $R_3 \to -R_3$ and $R_2 \to R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Thus, we have

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equations is

$8x+4y+3z=19$

$2x+y+z=5$

$x+2y+2z=7$

This system of equations can be written as AX = B, where

$$A=\begin{bmatrix}8&4&3\\2&1&1\\1&2&2\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix}\text{ and }B=\begin{bmatrix}19\\5\\7\end{bmatrix}.$$

$$\therefore\ X=A^{-1}B$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&\frac{2}{3}&-\frac{1}{3}\\1&-\frac{13}{3}&\frac{2}{3}\\-1&4&0\end{bmatrix}\begin{bmatrix}19\\5\\7\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0+\frac{10}{3}-\frac{7}{3}\\19-\frac{65}{3}+\frac{14}{3}\\-19+20+0\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$\therefore x=1,\ y=2$ and $z=1$

Continuity and Differentiability

33.

It is given that function f is continuous at $x = 0$.

$$\therefore\lim_{x\to0}f(x)=f(0)$$

$$\Rightarrow\lim_{x\to0}\frac{1-\cos kx}{\tan^2 x}=2\qquad\ldots(1)$$

$$\text{Now, }\lim_{x\to0}\frac{1-\cos kx}{\tan^2 x}=\lim_{x\to0}\frac{2\sin^2\left(\frac{kx}{2}\right)}{\tan^2 x}=2\lim_{x\to0}\frac{\frac{\sin^2\left(\frac{kx}{2}\right)}{x^2}}{\frac{\tan^2 x}{x^2}}$$

$$= 2\left(\frac{k}{2}\right)^2 \frac{\left(\lim\limits_{\frac{kx}{2}\to 0} \frac{\sin\frac{kx}{2}}{\frac{kx}{2}}\right)^2}{\left(\lim\limits_{x\to 0} \frac{\tan x}{x}\right)^2}$$

$$= \frac{2\left(\frac{k}{2}\right)^2 \times 1^2}{(1)^2} \qquad \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 2\times\frac{k^2}{4} = \frac{k^2}{2}$$

Therefore, equation (1) reduces to $\frac{k^2}{2} = 2$.

$\Rightarrow k^2 = 4$

$\Rightarrow k = \pm 2$

Thus, the value of k is 2 or -2.

34.

$x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Let $y = x^{x\cos x}$ and $z = \frac{x^2+1}{x^2-1}$

$y = x^{x\cos x}$

Taking log of both sides:

$\log y = \log\left(x^{x\cos x}\right)$

$\Rightarrow \log y = x\cos x \log x$

Differentiating with respect to x, we obtain:

$$\frac{1}{y}.\frac{dy}{dx} = (x\cos x)\frac{1}{x} + \log x\frac{d}{dx}(x\cos x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x + \log x(\cos x - x\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\{\cos x + \log x(\cos x - x\sin x)\}$$

$$\Rightarrow \frac{dy}{dx} = x^{x\cos x}\{\cos x + \log x(\cos x - x\sin x)\} \qquad \ldots \qquad (1)$$

$$z = \frac{x^2+1}{x^2-1}$$

Differentiating with respect to x, we obtain:

$$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{1}{x^2-1}\frac{d}{dx}(x^2+1) + (x^2+1)\frac{d}{dx}\left(\frac{1}{x^2-1}\right)$$

$$= \frac{1}{x^2-1}\cdot(2x) + (x^2+1)\left\{\frac{-2x}{(x^2-1)^2}\right\}$$

$$= \frac{2x}{x^2-1} - (x^2+1)\frac{2x}{(x^2-1)^2}$$

$$= \frac{2x}{(x^2-1)}\left\{1 - \frac{x^2+1}{x^2-1}\right\}$$

$$= \frac{2x}{(x^2-1)}\cdot\left(\frac{-2}{x^2-1}\right)$$

$$= \frac{-4x}{(x^2-1)^2}$$

$$\Rightarrow \frac{dz}{dx} = -\frac{4x}{(x^2-1)^2} \qquad \ldots \qquad (2)$$

On adding (1) and (2):

$$\frac{dy}{dx} + \frac{dz}{dx} = x^{x\cos x}\{\cos x + \log x(\cos x - x\sin x)\} - \frac{4x}{(x^2-1)^2}$$

$$\Rightarrow \frac{d}{dx}\left\{x^{x\cos x} + \frac{x^2+1}{x^2-1}\right\} = x^{x\cos x}\{\cos x + \log x(\cos x - x\sin x)\} - \frac{4x}{(x^2-1)^2}$$

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35.

It is given that $\sin(x+y)+\cos(x+y)=\log(x+y)$.

On differentiating both sides of this relationship with respect to x, we obtain

$$\cos(x+y)\left(1+\frac{dy}{dx}\right)-\sin(x+y)\left(1+\frac{dy}{dx}\right)=\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \cos(x+y)+\cos(x+y)\frac{dy}{dx}-\sin(x+y)-\sin(x+y)\frac{dy}{dx}=\frac{1}{x+y}+\frac{1}{x+y}\frac{dy}{dx}$$

$$\Rightarrow \left[\cos(x+y)-\sin(x+y)-\frac{1}{x+y}\right]\frac{dy}{dx}=\sin(x+y)-\cos(x+y)+\frac{1}{x+y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin(x+y)-\cos(x+y)+\frac{1}{x+y}}{\cos(x+y)-\sin(x+y)-\frac{1}{x+y}}$$

Thus, $\dfrac{dy}{dx}=\dfrac{\sin(x+y)-\cos(x+y)+\frac{1}{x+y}}{\cos(x+y)-\sin(x+y)-\frac{1}{x+y}}$

36.

$$x=a(\theta-\sin\theta),\ y=a(1+\cos\theta)$$

On differentiating x and y with respect to θ, it is obtained:

$$\frac{dx}{d\theta}=a(1-\cos\theta) \quad \ldots \quad (1)$$

$$\text{and } \frac{dy}{d\theta}=a(0-\sin\theta)$$

$$\Rightarrow \frac{dy}{d\theta}=-a\sin\theta \quad \ldots \quad (2)$$

Dividing equation (2) by equation (1), we obtain:

$$\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}$$

$$\Rightarrow \frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{-\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-\left(1-2\sin^2\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\frac{dy}{dx}=-\cot\frac{\theta}{2}$$

On differentiating again with respect to x, we obtain

$$\frac{d^2y}{dx^2}=-\left(-\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{a(1-\cos\theta)}$$

$$=\frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a(1-\cos\theta)}$$

$$=\frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a.2\sin^2\frac{\theta}{2}}$$

$$=\frac{1}{4a}\cdot\operatorname{cosec}^4\frac{\theta}{2}$$

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37.

Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta$

$\therefore\ u = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$\Rightarrow u = \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$\Rightarrow u = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow u = \tan^{-1}\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\Rightarrow u = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

Here, $-1 < x < 1$

$\Rightarrow -1 < \tan\theta < 1$

$\Rightarrow -\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$

$\therefore\ u = \dfrac{\theta}{2}$ $\qquad \left[\because \tan^{-1}(\tan\theta) = \theta, \text{ if } \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right]$

$\Rightarrow u = \dfrac{1}{2}\tan^{-1}x$ $\qquad [\because x = \tan\theta]$

Differentiating both sides with respect to x, we get

$\dfrac{du}{dx} = \dfrac{1}{2(1+x^2)}$

Let $v = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow v = 2\tan^{-1}x$

Differentiating both sides with respect to x, we get

$\frac{dv}{dx} = \frac{2}{1+x^2}$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{4}$$

38.

$x = \sin t$ and $y = \sin pt$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = p\cos pt$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^2 t} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t}{\cos^3 t} + \frac{p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t}$$

$$\Rightarrow \cos^2 t\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow \left(1-\sin^2 t\right)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

39.

$y = x^{\sin x} + (\sin x)^{\cos x}$

$\Rightarrow y = e^{\sin x \log x} + e^{\cos x \log \sin x}$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\sin x \log x} \times \frac{d}{dx}(\sin x \log x) + e^{\cos x \log \sin x} \times \frac{d}{dx}(\cos x \log \sin x)$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\sin x \times \frac{1}{x} + \log x \times \cos x\right) + (\sin x)^{\cos x}\left(\cos x \times \frac{1}{\sin x} \times \cos x + \log \sin x \times (-\sin x)\right)$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) + (\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right)$$

40.

Given that $f(x)$ is differentiable at $x = 1$. Therefore, $f(x)$ is continuous at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1}\left(x^2 + 3x + a\right) = \lim_{x\to 1}(bx + 2) = 1 + 3 + a$$

$$\Rightarrow 1 + 3 + a = b + 2$$

$$\Rightarrow a - b + 2 = 0 \qquad .....(1)$$

Again, $f(x)$ is differentiable at $x = 1$. So,

(LHD at $x = 1$) = (RHD at $x = 1$)

$$\Rightarrow \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$\lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1}=\lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{(bx-2-a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1}=\lim_{x\to 1}\frac{bx-b}{x-1}$$

$$\Rightarrow \lim_{x\to 1}(x+4)=\lim_{x\to 1}\frac{b(x-1)}{x-1}$$

$$\Rightarrow 5=b$$

Putting $b = 5$ in (1), we get
$a = 3$
Hence, $a = 3$ and $b = 5$.

Application of Derivatives

41.

The volume (V) of a cone with radius (r) and height (h) is given by,

$$V=\frac{1}{3}\pi r^2 h$$

It is given that,

$$h=\frac{1}{6}r \Rightarrow r=6h$$

$$\therefore V=\frac{1}{3}\pi(6h)^2 h=12\pi h^3$$

The rate of change of the volume with respect to time (t) is given by,

$$\frac{dV}{dt} = 12\pi \frac{d}{dh}\left(h^3\right) \cdot \frac{dh}{dt} \qquad \text{[By chain rule]}$$

$$= 12\pi\left(3h^2\right)\frac{dh}{dt}$$

$$= 36\pi h^2 \frac{dh}{dt}$$

It is also given that $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$.

Therefore, when $h = 4$ cm, we have:

$$12 = 36\pi\left(4\right)^2 = \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is 4 cm, its height is increasing at the rate of $\frac{1}{48\pi}$ cm/s.

42.

Area of an equilateral triangle, $A = \frac{\sqrt{3}}{4}a^2$ where a = Side of an equilateral triangle

Given:

$$\frac{da}{dt} = 2 \text{ cm/s}$$

Now,

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{\sqrt{3}}{4}a^2\right)$$

$$= \frac{\sqrt{3}}{4} \times 2 \times a \times \frac{da}{dt}$$

$$= \frac{\sqrt{3}a}{2}\frac{da}{dt}$$

$$= \frac{\sqrt{3}a}{2} \times 2$$

$$= \sqrt{3}a \text{ cm}^2/\text{s}$$

$$\therefore \left[\frac{dA}{dt}\right]_{a=20} = 20\sqrt{3}\ \text{cm}^2/\text{s}$$

Hence, the area is increasing at the rate of $20\sqrt{3}\ \text{cm}^2/\text{s}$ when the side of the triangle is 20 cm.

43.

Let $y = f(x) = \sqrt{x}$

For $x = 49$,

$x + \Delta x = 49.5$

$\Rightarrow \Delta x = 49.5 - 49$

$\Rightarrow \Delta x = 0.5$

For $x = 49$, we have $y = \sqrt{x} = \sqrt{49} = 7$.

Let $dx = \Delta x = 0.5$

Now,

$$y = \sqrt{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=49} = \frac{1}{2\sqrt{49}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=49} = \frac{1}{14}$$

We have $\Delta y = \frac{dy}{dx}\Delta x$

$$\Rightarrow \Delta y = \frac{1}{14}(0.5)$$

$$\Rightarrow \Delta y = \frac{1}{28}$$

$$\therefore \sqrt{49.5} = y + \Delta y$$

$$\Rightarrow \sqrt{49.5} = 7 + \frac{1}{28}$$

$$\Rightarrow \sqrt{49.5} = 7.036 \qquad \text{(approximately)}$$

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44.

Let P (x, y) be any point on the given curve $x^2 + y^2 - 2x - 3 = 0$.

Tangent to the curve at the point (x, y) is given by $\frac{dy}{dx}$.

Differentiating the equation of the curve on both the sides with respect to x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{2y} = \frac{1 - x}{y}$$

Let $\mathrm{P}(x_1, y_1)$ be the point on the given curve at which the tangents are parallel to the x-axis.

$$\therefore \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = 0$$
$$\Rightarrow \frac{1 - x_1}{y_1} = 0$$
$$\Rightarrow 1 - x_1 = 0$$
$$\Rightarrow x_1 = 1$$

When $x_1 = 1$,

$$(1)^2 + y_1^2 - 2 \times 1 - 3 = 0$$
$$\Rightarrow y_1^2 - 4 = 0$$
$$\Rightarrow y_1^2 = 4$$
$$\Rightarrow y_1 = \pm 2$$

So, the required points are (1, 2) and (1, −2).

Thus, the points on the given curve at which the tangents are parallel to x-axis are (1, 2) and (1, −2).

45.

The equation of the given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating with respect to x, we get:

$$\frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$$

$$\left(\frac{dy}{dx}\right)_{(\sqrt{2}a,b)} = \frac{\sqrt{2}b}{a}$$

Therefore, the slope of the tangent is $\frac{\sqrt{2}b}{a}$ and of the normal is $\frac{-a}{\sqrt{2}b}$.
Thus, the equation of the tangent is

$$y-b = \frac{\sqrt{2}b}{a}\left(x-\sqrt{2}a\right)$$

$$\Rightarrow \sqrt{2}bx - ay - ab = 0$$

Equation of the normal is

$$y-b = \frac{-a}{\sqrt{2}b}\left(x-\sqrt{2}a\right)$$

$$\Rightarrow ax + \sqrt{2}by - \sqrt{2}\left(a^2+b^2\right) = 0$$

46.

We have:

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Now, $f'(x) = 12x^3 - 12x^2 - 24x$

Now, $f'(x) = 0$

$\Rightarrow 12x^3 - 12x^2 - 24x = 0$

$\Rightarrow 12x\left(x^2 - x - 2\right) = 0$

$\Rightarrow 12x\left(x^2 - 2x + x - 2\right) = 0$

$\Rightarrow 12x[x(x-2) + 1(x-2)] = 0$

$\Rightarrow 12x\ (x+1)(x-2) = 0$

$\Rightarrow x = 0\ ;\ x = -1;\ x = 2$

So, the points $x = -1$, $x = 0$ and $x = 2$ divide the real line into four disjoint intervals, namely $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$.

INTERVAL	SIGN OF $f'(x) = 12x(x+1)(x-2)$	NATURE OF FUNCTION
$(-\infty, -1)$	$(-)(-)(-) = -$ or < 0	Strictly decreasing
$(-1, 0)$	$(-)(+)(-) = +$ or > 0	Strictly increasing
$(0, 2)$	$(+)(+)(-) = -$ or < 0	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) = +$ or > 0	Strictly increasing

(a) The given function is strictly increasing in the intervals $(-1, 0) \cup (2, \infty)$.

(b) The given function is strictly decreasing in the intervals $(-\infty, -1) \cup (0, 2)$.

47.

Consider the function $f(x) = \sin 3x - \cos 3x$.

$f'(x) = 3\cos 3x + 3\sin 3x$

$= 3(\sin 3x + \cos 3x)$

$= 3\sqrt{2}\left\{\sin 3x \cos\left(\frac{\pi}{4}\right) + \cos 3x \sin\left(\frac{\pi}{4}\right)\right\}$

$= 3\sqrt{2}\left\{\sin\left(3x + \frac{\pi}{4}\right)\right\}$

For the increasing interval $f'(x) > 0$.

$3\sqrt{2}\left\{\sin\left(3x + \frac{\pi}{4}\right)\right\} > 0$

$\sin\left(3x + \frac{\pi}{4}\right) > 0$

$\Rightarrow 0 < 3x + \frac{\pi}{4} < \pi$

$\Rightarrow 0 < 3x < \frac{3\pi}{4}$

$\Rightarrow 0 < x < \frac{\pi}{4}$

Also,

$$\sin\left(3x+\frac{\pi}{4}\right)>0$$

when, $2\pi < 3x+\frac{\pi}{4} < 3\pi$

$$\Rightarrow \frac{7\pi}{4} < 3x < \frac{11\pi}{4}$$

$$\Rightarrow \frac{7\pi}{12} < x < \frac{11\pi}{12}$$

Therefore, intervals in which function is strictly increasing in $0 < x < \frac{\pi}{4}$ and $\frac{7\pi}{12} < x < \frac{11\pi}{12}$.

Similarly, for the decreasing interval $f'(x) < 0$.

$$3\sqrt{2}\left\{\sin\left(3x+\frac{\pi}{4}\right)\right\}<0$$

$$\sin\left(3x+\frac{\pi}{4}\right)<0$$

$$\Rightarrow \pi < 3x+\frac{\pi}{4} < 2\pi$$

$$\Rightarrow \frac{3\pi}{4} < 3x < \frac{7\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} < x < \frac{7\pi}{12}$$

Also,

$$\sin\left(3x+\frac{\pi}{4}\right)<0$$

When $3\pi < 3x+\frac{\pi}{4} < 4\pi$,

$$\Rightarrow \frac{11\pi}{4} < 3x < \frac{15\pi}{4}$$

$$\Rightarrow \frac{11\pi}{12} < x < \frac{15\pi}{12}$$

The function is strictly decreasing in $\frac{\pi}{4} < x < \frac{7\pi}{12}$ and $\frac{11\pi}{12} < x < \pi$.

48.

We have

$f(x) = \sin x - \cos x \qquad 0 < x < 2\pi$

$f'(x) = \frac{d}{dx}(\sin x - \cos x)$

$= \cos x + \sin x$

For maxima and minima, we have

$f'(x) = 0$

$\Rightarrow \cos x + \sin x = 0$

$\Rightarrow \cos x = -\sin x$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Now, $f''(x) = \frac{d}{dx}(\cos x + \sin x)$

$= -\sin x + \cos x$

At $x = \frac{3\pi}{4}$,

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

$= -\sqrt{2}$

$\Rightarrow f''\left(\frac{3\pi}{4}\right) < 0$

Thus, $x = \frac{3\pi}{4}$ is the point of local maxima.

Local maximum value $= f\left(\frac{3\pi}{4}\right)$

$$= \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

At $x = \frac{7\pi}{4}$,

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\Rightarrow f''\left(\frac{7\pi}{4}\right) > 0$$

Thus, $x = \frac{7\pi}{4}$ is the point of local minima.

Local minimum value of $f(x) = f\left(\frac{7\pi}{4}\right)$

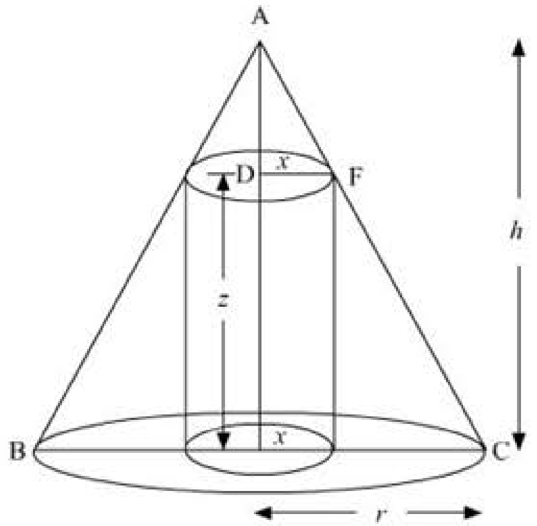
$$= \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\sqrt{2}$$

49.

Let h and r be the height and radius of the base of the cone ABC respectively. Suppose the radius and height of the cylinder inscribed in the cone be x and z respectively.



Now DF = x,

AD = AL − DL = $h - z$

Here, ΔADF and ΔALC are similar.

$$\frac{AD}{AL} = \frac{DF}{LC}$$

$$\text{i.e.,} \frac{h-z}{h} = \frac{x}{r}$$

$$\Rightarrow \left(1 - \frac{x}{r}\right)h = z$$

Let S be the curved surface area of the cylinder.

$$\therefore S = 2\pi xz = 2\pi x\left(1 - \frac{x}{r}\right)h = 2\pi h\left(x - \frac{x^2}{r}\right) \qquad [\text{Using } (1)]$$

$$\frac{dS}{dx} = 2\pi h\left(1 - \frac{2x}{r}\right)$$

$$\text{and} \frac{d^2S}{dx^2} = -\frac{4\pi h}{r}$$

For maximum or minimum, $\frac{dS}{dx} = 0$

i.e., $2\pi h\left(1-\frac{2x}{r}\right)=0 \Rightarrow x=\frac{r}{2}$

At $x=\frac{r}{2}, \frac{d^2S}{dx^2}=\frac{-4\pi h}{r}=-\text{ve}$

$\therefore$ Curved surface area of the cylinder is maximum at $x=\frac{r}{2}$.

Hence, the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

50.

Surface area of cuboid = 2 ($lb + bh + hl$)

$$=\left(2x^2+\frac{2x^2}{3}+\frac{x^2}{3}\right)$$

$$=6x^2$$

Let radius of the sphere be r,

Surface area of sphere = $4\pi r^2$

Therefore, $6x^2+4\pi r^2=k(\text{constant})$ .....(i)

Now, volume of both figures will be

$$V=\frac{2}{3}x^3+\frac{4}{3}\pi r^3$$

Putting the value of r from the equation (i),

$$V=\frac{2}{3}x^3+\frac{4}{3}\pi\left(\frac{k-6x^2}{4\pi}\right)^{\frac{3}{2}}$$

For minimum volume $\frac{dV}{dx}=0$, so

$$\frac{dV}{dx} = 2x^2 + \left(\frac{4}{3}\pi\right)\left(\frac{1}{4\pi}\right)^{\frac{3}{2}} . \frac{3}{2}\left(k - 6x^2\right)^{\frac{1}{2}}(-12x) = 0$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\left(k - 6x^2\right)^{\frac{1}{2}}(6x)$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\left(4\pi r^2\right)^{\frac{1}{2}}(6x) \qquad \left[\text{since, } k - 6x^2 = 4\pi r^2\right]$$

$$\Rightarrow x = 3r$$

Hence proved.

Further, minimum value of sum of their volume is

$$V_{\min} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{x}{3}\right)^3 \qquad \left[r = \frac{x}{3}\right]$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\frac{x^3}{27}$$

$$= \frac{2}{3}x^3\left(1 + \frac{2}{27}\right)$$

$$= \frac{58}{81}x^3$$

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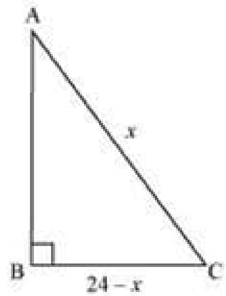
51.

Let ΔABC be right-angled at B. Let AC = $x$cm and BC = $(24-x)$ cm so that BC + AC = 24 cm



$$AB=\sqrt{AC^2-BC^2}=\sqrt{(x)^2-(24-x)^2}=\sqrt{48x-576}\text{ cm}=4\sqrt{3x-36}\text{ cm}$$

Area (A) of ΔABC is given by,

$$A(x)=\frac{1}{2}AB\times BC=\frac{1}{2}\cdot\left(4\sqrt{3x-36}\right)\cdot(24-x)\text{ cm}^2=2\sqrt{3x-36}\cdot(24-x)\text{ cm}^2$$

$$A'(x)=2\cdot\left[\frac{3}{2\sqrt{3x-36}}(24-x)+\sqrt{3x-36}(-1)\right]$$

$$=\frac{3(24-x)-2(3x-36)}{\sqrt{3x-36}}$$

$$=\frac{144-9x}{\sqrt{3x-36}}$$

$$A''(x)=\frac{-9\sqrt{3x-36}-(144-9x)\cdot\frac{1}{2\sqrt{3x-36}}\cdot 3}{(3x-36)}$$

$$=\frac{-18(3x-36)-3(144-9x)}{2(3x-36)^{\frac{3}{2}}}$$

$$=\frac{-27x+216}{2(3x-36)^{\frac{3}{2}}}$$

Putting $A'(x)=0$

$$\frac{144-9x}{\sqrt{3x-36}}=0\Rightarrow x=16$$

At $x = 16$, $A''(x) = \frac{-27(16)+216}{2(3\times16-36)^{\frac{3}{2}}} = \frac{-216}{2(12)^{\frac{3}{2}}} < 0$

Thus, $x = 16$ is the point of maxima. Therefore, the area of the triangle is maximum at $x = 16$ and is given by,

$[A]_{x=16} = 2\sqrt{3\times16-36}\cdot(24-16) = 16\sqrt{48-36} = 16\times2\sqrt{3} = 32\sqrt{3}$

Also, if $\theta$is the angle between BC and AC, then

$$\theta = \cos^{-1}\left(\frac{BC}{AC}\right) = \cos^{-1}\left(\frac{8}{16}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Thus, the maximum area of the triangle is $32\sqrt{3}\ \text{cm}^2$ and the angle between the side and the hypotenuse is $\frac{\pi}{3}$.

52.

Let the coordinates of the required point of the parabola $y = x^2 + 7x + 2$ be (h, k).

$$\therefore\ k = h^2 + 7h + 2 \qquad .....(1)$$

Distance of the point (h, k) from the straight line $y = 3x - 3$ is given by

$$D = \frac{|3h - k - 3|}{\sqrt{(3)^2 + (-1)^2}} = \frac{|3h - k - 3|}{\sqrt{10}} \qquad .....(2)$$

From (1) and (2), we get

$$D = \frac{|3h - h^2 - 7h - 2 - 3|}{\sqrt{10}}$$

$$= \frac{|-h^2 - 4h - 5|}{\sqrt{10}}$$

$$= \frac{h^2 + 4h + 5}{\sqrt{10}}$$

Differentiating both sides w.r.t. h, we get

$$\frac{dD}{dh} = \frac{2h+4}{\sqrt{10}}$$

For maxima and minima,

$$\frac{dD}{dh} = 0$$

$$\Rightarrow \frac{2h+4}{\sqrt{10}} = 0$$

$$\Rightarrow 2h = -4$$

$$\Rightarrow h = -2$$

Now,

$$\frac{d^2D}{d^2h} = \frac{2}{\sqrt{10}} > 0$$

So, $h = -2$ is the point of minima.

When $h = -2$,

$$k = (-2)^2 + 7\times(-2) + 2 = 4 - 14 + 2 = -8 \qquad \text{[Using (1)]}$$

Thus, the coordinates of a point of the given parabola that is closest to the given straight line is $(-2, -8)$.

Integrals

53.

Antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \int\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

Now, we have:

$$\int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx=\int 3x^{\frac{1}{2}}dx+\int x^{-\frac{1}{2}}dx$$

$$=3\times\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c \qquad \left(\text{Using }\int x^n dx=\frac{x^{n+1}}{n+1}+c\right)$$

$$=2x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$$

$$=2\sqrt{x}(x+1)+c$$

Thus, the antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ is $2\sqrt{x}(x+1)+c$, where c is the constant of integration.

54.

$$I=\int(\sin x\sin 2x\sin 3x)dx$$

$$=\frac{1}{2}\int\sin x(2\sin 2x\sin 3x)dx$$

$$=\frac{1}{2}\int\sin x(2\sin 3x\sin 2x)dx$$

$$=\frac{1}{2}\int\sin x\left[\cos(3x-2x)-\cos(3x+2x)\right]dx$$

$$=\frac{1}{2}\int[\sin x\cos x-\sin x\cos 5x]dx$$

$$=\frac{1}{2}\int\frac{2\sin x\cos x}{2}dx-\frac{1}{2}\int\frac{2\sin x\cos 5x}{2}dx$$

$$=\frac{1}{4}\int\sin 2x dx-\frac{1}{4}\int 2\cos 5x\sin x dx$$

$$=\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)-\frac{1}{4}\int\left[\sin(5x+x)-\sin(5x-x)\right]dx$$

$$=\frac{-\cos 2x}{8}-\frac{1}{4}\int(\sin 6x-\sin 4x)dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$$
$$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$
$$= \frac{1}{48}(2\cos 6x - 3\cos 4x - 6\cos 2x) + C$$

55.

$$I = \int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx = \int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx + \int \frac{1}{\sqrt{x^2+2x+3}}\,dx = I_1 + I_2$$
$$I_1 = \int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$$
$$Let, \sqrt{x^2+2x+3} = t \Rightarrow \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = dt$$
$$I_1 = \int \frac{dt}{t} = \log t = \log\left|\sqrt{x^2+2x+3}\right|$$
$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}\,dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right|$$
$$I = I_1 + I_2 = \log\left|\sqrt{x^2+2x+3}\right| + \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + c$$

56.

Let I = $\int (x-3)\sqrt{x^2+3x-18}\,\mathrm{d}x$

Put $\sqrt{x^2+3x-18} = t \;\Rightarrow (x^2+3x-18) = t^2$

On differentiating with respect to x, we get:

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$$2x+3=2t\frac{dt}{dx}$$

$$\Rightarrow x+\frac{3}{2}=t\frac{dt}{dx}$$

$$\Rightarrow x+\frac{3}{2}+3-3=t\frac{dt}{dx}$$

$$\Rightarrow x-3+\frac{9}{2}=t\frac{dt}{dx} \quad \ldots(1)$$

The given integral can be rewritten as follows:

$$\mathrm{I}=\int\left(x-3+\frac{9}{2}-\frac{9}{2}\right)\sqrt{x^2+3x-18}\,dx$$

$$=\int\left(x-3+\frac{9}{2}\right)\sqrt{x^2+3x-18}\,dx-\frac{9}{2}\int\sqrt{x^2+3x-18}\,dx \quad \ldots(2)$$

Suppose that $\mathrm{I}_1=\int\left(x-3+\frac{9}{2}\right)\sqrt{x^2+3x-18}\,dx$

On using equation (1), we get

$$\mathrm{I}_1=\int t^2dt=\frac{t^3}{3}+C_1=\frac{\left(x^2+3x-18\right)^{\frac{3}{2}}}{3}+C_1$$

Suppose that $\mathrm{I}_2=\int\sqrt{\left(x^2+3x-18\right)}\,dx$

$$\int\sqrt{\left(x^2+3x-18\right)}\,dx=\int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\,dx$$

$$=\left(\frac{2x+3}{4}\right)\sqrt{\left(x^2+3x-18\right)}-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{\left(x^2+3x-18\right)}\right|+C_2$$

$\therefore \mathrm{I}=\dfrac{\left(x^2+3x-18\right)^{\frac{3}{2}}}{3}-\dfrac{9}{8}(2x+3)\sqrt{\left(x^2+3x-18\right)}+\dfrac{729}{16}\log\left|\dfrac{2x+3}{2}+\sqrt{\left(x^2+3x-18\right)}\right|+C$, where $C=C_1+C_2$ is a constant.

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57.

Let $I=\int \frac{x^2}{\left(x^2+4\right)\left(x^2+9\right)} \mathrm{d}x$

Putting $x^2=z$, we have

$$\frac{x^2}{\left(x^2+4\right)\left(x^2+9\right)}=\frac{z}{(z+4)(z+9)}$$

Using partial fraction, we have

$$\frac{z}{(z+4)(z+9)}=\frac{A}{(z+4)}+\frac{B}{(z+9)}$$

$$\Rightarrow \frac{z}{(z+4)(z+9)}=\frac{(A+B)z+(9A+4B)}{(z+4)(z+9)}$$

$$\Rightarrow z=(A+B)z+(9A+4B)$$

Comparing the respective coefficients, we get

$A+B=1$ and $9A+4B=0$

$\Rightarrow A=-\frac{4}{5}$ and $B=\frac{9}{5}$

So,

$$I=\int \frac{x^2}{\left(x^2+4\right)\left(x^2+9\right)} \mathrm{d}x$$

$$=\int\left(\frac{-\frac{4}{5}}{x^2+4}+\frac{\frac{9}{5}}{x^2+9}\right)\mathrm{d}x$$

$$=-\frac{4}{5}\int \frac{1}{\left(x^2+4\right)}\mathrm{d}x+\frac{9}{5}\int \frac{1}{\left(x^2+9\right)}\mathrm{d}x$$

$$=-\frac{4}{5}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{9}{5}\times\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$$

$$\therefore I=-\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right)+C$$

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58.

$$I = \int \frac{x^2+1}{(x-1)^2(x+3)}dx$$

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$

Multiplying both sides by $(x-1)^2(x+3)$, we have

$x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \ldots(1)$

Putting $x = 1$ in (1), we get

$2 = 4B$

$\Rightarrow B = \frac{1}{2}$

Putting $x = -3$ in (1), we get

$(-3)^2 + 1 = C(-3-1)^2$

$\Rightarrow 10 = 16C$

$\Rightarrow C = \frac{5}{8}$

Comparing the coefficients of x^2 on both sides of equation (1), we get $A + C = 1$

$\Rightarrow A = 1 - C = 1 - \frac{5}{8}$

$\Rightarrow A = \frac{3}{8}$

From (1), we have

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{\frac{3}{8}}{(x-1)} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{5}{8}}{(x+3)}$$

$$\Rightarrow \int \frac{x^2+1}{(x-1)^2(x+3)}dx = \frac{3}{8}\int \frac{1}{(x-1)}dx + \frac{1}{2}\int \frac{1}{(x-1)^2}dx + \frac{5}{8}\int \frac{1}{(x+3)}dx$$

$$\Rightarrow I = \frac{3}{8}\log|x-1| + \frac{1}{2} \times \frac{(x-1)^{-2+1}}{(-2+1)} + \frac{5}{8}\log|x+3| + C$$

$$\Rightarrow I = \frac{3}{8}\log|x-1| + \frac{5}{8}\log|x+3| - \frac{1}{2(x-1)} + C$$

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59.

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

Integrating by parts by taking $\sin^{-1}x$ as 1st function and $\frac{x}{\sqrt{1-x^2}}$ as 2nd function.

$$I = \sin^{-1}x \int \frac{x}{\sqrt{1-x^2}} - \int \left[\frac{d}{dx}(\sin^{-1} x) \int \frac{x}{\sqrt{1-x^2}} dx \right] dx \quad ...(1)$$

Consider the integral, $\int \frac{x}{\sqrt{1-x^2}} dx$

Put $1 - x^2 = z^2$

Differentiating both sides, we have

$-2xdx = 2zdz$

$\Rightarrow xdx = -zdz$

$$\therefore \int \frac{x}{\sqrt{1-x^2}} dx$$
$$= \int \frac{-zdz}{z}$$
$$= -\int dz$$
$$= -z$$
$$= -\sqrt{1-x^2} \quad ...(2)$$

From (1), and (2), we have

$$I = \sin^{-1}x\left(-\sqrt{1-x^2}\right) - \int \left[\frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right) \right] dx$$

$$= -\sin^{-1} x\sqrt{1-x^2} + \int dx + C$$
$$= -\sin^{-1} x\sqrt{1-x^2} + x + C$$

60.

Let $I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$

Dividing the numerator and denominator by $\cos^4 x$, we get:

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$$I = \int \frac{sec^4 x}{1+\tan^4 x} dx$$

$$= \int \frac{sec^2 x \cdot sec^2 x}{1+\tan^4 x} dx$$

$$= \int \frac{sec^2 x \cdot (1+\tan^2 x)}{1+\tan^4 x} dx$$

Putting $\tan x = t$,
$\sec^2 x \, dx = dt$

$$\therefore I = \int \frac{1+t^2}{1+t^4} dt$$

Dividing the numerator and denominator by t^2, we get:

$$I = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dx$$

$$I = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2} dt$$

Let $t - \frac{1}{t} = u$

$$\Rightarrow \left(1+\frac{1}{t^2}\right) dt = du$$

$$\therefore I = \int \frac{du}{u^2+(\sqrt{2})^2}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - 1/t}{\sqrt{2}}\right) + C$$

$\Rightarrow \text{I} = \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{t^2-1}{\sqrt{2}t}\right)+C$

$\Rightarrow \text{I} = \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan^2 x-1}{\sqrt{2}\tan x}\right)+C$

61.

Let $I = \int_0^{\frac{\pi}{2}} x\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ ... (1)

It is known that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

$\therefore I = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\left(\sqrt{\tan\left(\frac{\pi}{2}-x\right)}+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}\right)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$ ... (2)

On adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}}\frac{\pi}{2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

$$\Rightarrow I = \frac{\pi}{4}\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx = \frac{\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sqrt{1+2\sin x\cos x-1}}dx = \frac{\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{(\sin x+\cos x)\,dx}{\sqrt{1-(\sin x-\cos x)^2}}$$

Let $\sin x - \cos x = z$

$\Rightarrow (\cos x + \sin x)\,dx = dz$

When $x = 0$, $z = 0 - 1 = -1$

When $x = \frac{\pi}{2}$, $z = 1 - 0 = 1$

$$I=\frac{\pi}{2\sqrt{2}}\int_{-1}^{1}\frac{dz}{\sqrt{1-z^2}}$$

$$=\frac{\pi}{2\sqrt{2}}(\sin^{-1}z)\Big|_{-1}^{1}$$

$$=\frac{\pi}{2\sqrt{2}}[\sin^{-1}1-\sin^{-1}(-1)]$$

$$=\frac{\pi}{2\sqrt{2}}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]$$

$$=\frac{\pi^2}{2\sqrt{2}}$$

62.

$$\int_2^5\left[|x-2|+|x-3|+|x-5|\right]dx$$

Let, $f(x)=|x-2|+|x-3|+|x-5|$

$$I=\int_2^5 f(x)\,dx=\int_2^3 f(x)\,dx+\int_3^5 f(x)\,dx$$

$$\Rightarrow I=\int_2^3(x-2+3-x+5-x)\,dx+\int_3^5(x-2+x-3+5-x)\,dx$$

$$\Rightarrow I=\int_2^3(6-x)\,dx+\int_3^5 x\,dx=\left[6x-\frac{x^2}{2}\right]_2^3+\left[\frac{x^2}{2}\right]_3^5=\left[18-\frac{9}{2}-12+2\right]+\left[\frac{25}{2}-\frac{9}{2}\right]=\frac{23}{2}$$

63.

$$\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}\,dx \qquad \text{.........(1)}$$

Using $f(x)=f(a-x)$, we get

$$I=\int_0^{\pi}\frac{4(\pi-x)\sin x}{1+\cos^2 x}\,dx \qquad \text{.........(2)}$$

Adding (1) and (2), we get

$$2I=4\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Let $\cos x = t$.

$$\Rightarrow -\sin x dx = dt$$

$$\Rightarrow I = 2\pi \int_1^{-1} \frac{-1}{1+t^2}\, dt$$

$$\Rightarrow I = -2\pi \tan^{-1} t \Big|_1^{-1}$$

$$\Rightarrow I = -2\pi \left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$\Rightarrow I = \pi^2$$

64.

Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx \qquad .....(1)$

Then,

$$I = \int_0^{\frac{\pi}{2}} \frac{2^{\sin\left(\frac{\pi}{2}-x\right)}}{2^{\sin\left(\frac{\pi}{2}-x\right)} + 2^{\cos\left(\frac{\pi}{2}-x\right)}} dx \qquad \left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\frac{\pi}{2}} \frac{2^{\cos x}}{2^{\cos x} + 2^{\sin x}} dx \qquad .....(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx + \int_0^{\frac{\pi}{2}} \frac{2^{\cos x}}{2^{\cos x} + 2^{\sin x}} dx$$

$$2I = \int_0^{\frac{\pi}{2}} dx$$

$$2I = x\Big|_0^{\frac{\pi}{2}}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx = \frac{\pi}{4}$$

65.

Let $I = \int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx$

Integrating by parts, we get

$$I = \frac{1}{2}\left[e^{2x} \sin\left(\frac{\pi}{4} + x\right)\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} e^{2x} \cos\left(\frac{\pi}{4} + x\right) dx$$

Now, integrating the second term by parts, we get

$$\Rightarrow I = \frac{1}{2}\left[e^{2x} \sin\left(\frac{\pi}{4} + x\right)\right]_0^{\pi} - \frac{1}{2}\left\{\left[\frac{1}{2}e^{2x}\cos\left(\frac{\pi}{4} + x\right)\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4} + x\right) dx\right\}$$

$$\Rightarrow I = \frac{1}{2}\left[e^{2x} \sin\left(\frac{\pi}{4} + x\right)\right]_0^{\pi} - \frac{1}{4}\left[e^{2x}\cos\left(\frac{\pi}{4} + x\right)\right]_0^{\pi} - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2}\left[e^{2\pi}\sin\left(\pi + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] - \frac{1}{4}\left[e^{2\pi}\cos\left(\pi + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2}\left[-e^{2\pi} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right] - \frac{1}{4}\left[-e^{2\pi} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \frac{5}{4}I = -\frac{1}{2\sqrt{2}}e^{2\pi} - \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}e^{2\pi} + \frac{1}{4\sqrt{2}}$$

$$\Rightarrow I = -\frac{1}{5\sqrt{2}}\left(e^{2\pi} + 1\right)$$

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66.

$$f(x) = |x \cdot \cos(\pi x)| = \begin{cases} x \cdot \cos(\pi x), & \text{for } 0 < x \le \frac{1}{2} \\ -x \cdot \cos(\pi x), & \text{for } \frac{1}{2} < x \le \frac{3}{2} \end{cases}$$

$$\therefore \int_0^{3/2} |x \cdot \cos(\pi x)| dx = \int_0^{1/2} x \cdot \cos(\pi x) dx + \int_{\frac{1}{2}}^{3/2} -x \cdot \cos(\pi x) dx$$

Integrating both integrals on right hand side, we get

$$\int_0^{3/2} |x \cdot \cos(\pi x)| dx$$

$$= \left[x \frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^{\frac{1}{2}} - \left[x \frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \left[\left(\frac{\frac{1}{2} \times 1}{\pi} + 0\right) - \left(0 + \frac{1}{\pi^2}\right)\right] - \left[\left(\frac{\frac{3}{2} \times (-1)}{\pi} + 0\right) - \left(\frac{\frac{1}{2} \times 1}{\pi} + 0\right)\right]$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi}$$

$$= \frac{5}{2\pi} - \frac{1}{\pi^2}$$

67.

Here, $a = 1$, $b = 3$ and $f(x) = e^{2-3x} + x^2 + 1$.

$$\therefore nh = b - a = 3 - 1 = 2$$

We know

$$\int_a^b f(x) dx = \lim_{h \to 0} h\{f(a) + f(a+h) + f(a+2h) + \ldots + f[a + (n-1)h]\}$$

Now,

$f(a)=f(1)=e^{2-3\times 1}+1^2+1$

$f(a+h)=f(1+h)=e^{2-3(1+h)}+(1+h)^2+1$

$f(a+2h)=f(1+2h)=e^{2-3(1+2h)}+(1+2h)^2+1$

.

.

.

$f[a+(n-1)h]=f[1+(n-1)h]=e^{2-3[1+(n-1)h]}+[1+(n-1)h]^2+1$

Adding these equations, we get

$f(a)+f(a+h)+f(a+2h)+\ldots+f[a+(n-1)h]$

$=\left[e^{2-3\times 1}+1^2+1\right]+\left[e^{2-3\times(1+h)}+(1+h)^2+1\right]+\left[e^{2-3\times(1+2h)}+(1+2h)^2+1\right]+\ldots+$

$\{e^{2-3[1+(n-1)h]}+[1+(n-1)h]^2+1\}$

$$\therefore \int_1^3\left(e^{2-3x}+x^2+1\right)dx$$

$$=\lim_{h\to 0} h\left[e^2.\left(e^{-3}+e^{-3(1+h)}+e^{-3(1+2h)}+\ldots+e^{-3(1+(n-1)h)}\right)\right]$$

$$+\lim_{h\to 0} h\left[1^2+(1+h)^2+(1+2h)^2+\ldots+[1+(n-1)h]^2\right]$$

$$+\lim_{h\to 0} h\left[\underbrace{1+1+1+\ldots+1}_{n\text{ terms}}\right]$$

$$=\lim_{h\to 0} h\left\{e^2\times\frac{e^{-3}\left(1-e^{3nh}\right)}{1-e^{-3h}}\right\}$$

$$+\lim_{h\to 0} h\left\{\underbrace{(1+1+1+\ldots+1)}_{n\text{ terms}}+2h[1+2+3+\ldots+(n-1)]+h^2\left(1^2+2^2+3^2+\ldots+(n-1)^2\right)\right\}$$

$$+\lim_{h\to 0} h\left(\underbrace{1+1+1+\ldots+1}_{n\text{ terms}}\right)$$

$$=\lim_{h\to 0} h\left\{\frac{e^{-1}\left(1-e^{-6}\right)}{1-e^{-3h}}\right\}+\lim_{h\to 0} h\left[n+2h\times\frac{(n-1)n}{2}+h^2\times\frac{(n-1)n(2n-1)}{6}\right]+\lim_{h\to 0} hn$$

$$=\lim_{h\to 0} h\left\{\frac{e^{-1}\left(1-e^{-6}\right)}{1-e^{-3h}}\right\}+\lim_{h\to 0}\left[2nh+(nh-h)nh+\frac{(nh-h)nh(2nh-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{\lim\limits_{h\to 0} e^{3h}}{3\times\lim\limits_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)}+\lim_{h\to 0}\left[2\times 2+(2-h)\times 2+\frac{(2-h)\times 2\times(2\times 2-h)}{6}\right]$$

$$= \frac{1}{e}\left(1-\frac{1}{e^6}\right)\times\frac{1}{3\times 1}+\left(4+4+\frac{8}{3}\right)$$

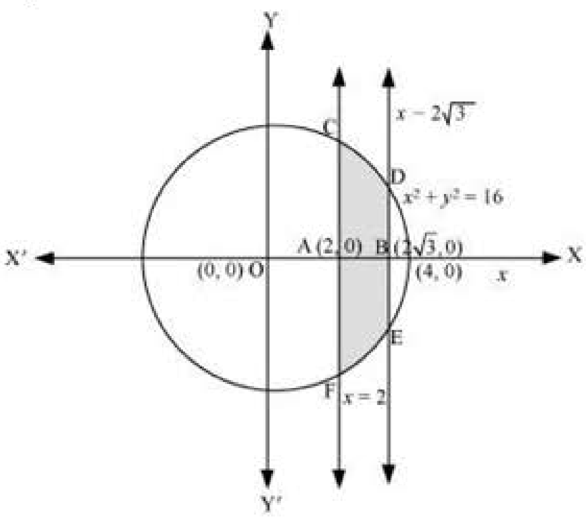
$$= \frac{1}{3e}\left(1-\frac{1}{e^6}\right)+\frac{32}{3}$$

Application of Integrals

68.

The curve $x^2 + y^2 = 16$ represents a circle with centre (0, 0) and radius = 4.

The region bounded by curve $x^2 + y^2 = 16$ and lines $x = 2$ and $x = 2\sqrt{3}$ can be graphically represented as:



Area of the region bounded by the given curve and lines

= Area of the shaded region

= Area (ACDBEFA)

= 2 × area (ACDBA)

$$= 2\int_{2}^{2\sqrt{3}} \sqrt{16-x^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_2^{2\sqrt{3}}$$
$$= 2\left[\left\{\frac{2\sqrt{3}}{2}\sqrt{16-\left(2\sqrt{3}\right)^2}+8\sin^{-1}\left(\frac{2\sqrt{3}}{4}\right)\right\}-\left\{\frac{2}{2}\sqrt{16-(2)^2}+\frac{16}{2}\sin^{-1}\left(\frac{2}{4}\right)\right\}\right]$$
$$= 2\left[\left\{\sqrt{3}\times 2+8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\}-\left\{1\times 2\sqrt{3}+8\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$
$$= 2\left[\left(2\sqrt{3}+8\times\frac{\pi}{3}\right)-\left(2\sqrt{3}+8\times\frac{\pi}{6}\right)\right]$$
$$= 2\left[2\sqrt{3}+\frac{8\pi}{3}-2\sqrt{3}-\frac{4\pi}{3}\right]$$
$$= 2\times\frac{4\pi}{3}$$
$$= \frac{8\pi}{3}\text{ sq units}$$

Thus, the area of the region bounded by curve $x^2 + y^2 = 16$ and lines $x = 2$ and $x = 2\sqrt{3}$ is $\frac{8\pi}{3}$ sq units.

69.

The given equation of the circle is $x^2+y^2=4$.

The equation of the normal to the circle at $(1,\sqrt{3})$ is same as the line joining the points $(1,\sqrt{3})$ and (0, 0), which is given by

$$\frac{y-\sqrt{3}}{x-1}=\frac{\sqrt{3}-0}{1-0}$$
$$\Rightarrow \frac{y-\sqrt{3}}{x-1}=\sqrt{3}$$
$$\Rightarrow y-\sqrt{3}=\sqrt{3}x-\sqrt{3}$$
$$\Rightarrow y=\sqrt{3}x \qquad .....(1)$$

So, the slope of normal is $\sqrt{3}$.

We know that the product of the slopes of the normal and the tangent is -1.

Therefore, the slope of tangent is $\frac{-1}{\sqrt{3}}$.

Now, the equation of the tangent to the circle at $(1, \sqrt{3})$ is given by

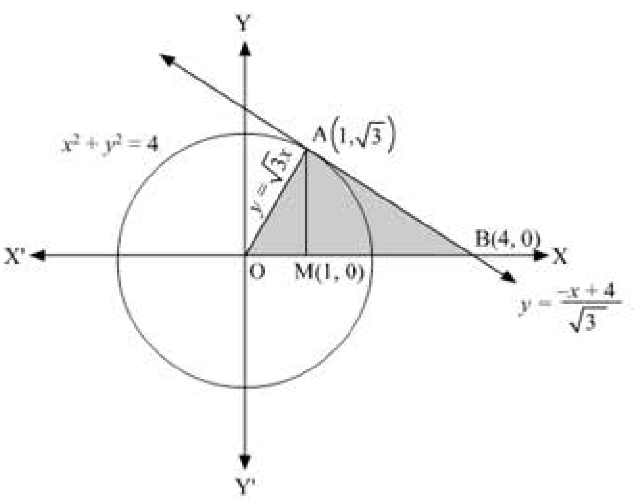
$$\frac{y-\sqrt{3}}{x-1} = \frac{-1}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}y - 3 = -x + 1$$

$$\Rightarrow y = \frac{-x+4}{\sqrt{3}} \qquad .....(2)$$

Putting $y = 0$ in (2), we get $x = 4$.

Thus, ABC is the triangle formed by the positive x-axis and tangent and normal to the given circle at $(1, \sqrt{3})$.



Now,

Area of ΔAOB = Area of ΔAOM + Area of ΔAMB

$$= \int_0^1 y \, dx + \int_1^4 y \, dx$$

$$= \int_0^1 \sqrt{3}x \, dx + \int_1^4 \left(\frac{-x+4}{\sqrt{3}}\right) dx$$

$$= \left[\frac{\sqrt{3}\,x^2}{2}\right]_0^1 + \int_1^4 -\frac{x}{\sqrt{3}} dx + \int_1^4 \frac{4}{\sqrt{3}} dx$$

$$= \left(\frac{\sqrt{3}}{2} - 0\right) - \left[\frac{x^2}{2\sqrt{3}}\right]_1^4 + \left[\frac{4}{\sqrt{3}}x\right]_1^4$$

$$= \frac{\sqrt{3}}{2} - \frac{16}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} + \frac{16}{\sqrt{3}} - \frac{4}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= 2\sqrt{3}$$

Thus, the area of the triangle so formed is $2\sqrt{3}$ square units.

70.

Let ABC be the required triangle with vertices A (−1, 2), B (1, 5) and C (3, 4).
Now, the equation of the side AB is

$$y - 2 = \frac{5-2}{1-(-1)}(x-(-1))$$

$$\Rightarrow 3x - 2y + 7 = 0 \qquad ...(1)$$

The equation of BC is

$$y - 5 = \frac{4-5}{3-1}(x-1)$$

$$\Rightarrow x + 2y - 11 = 0 \qquad ...(2)$$

The equation of AC is

$$y - 2 = \frac{4-2}{3-(-1)}(x-(-1))$$

$$\Rightarrow x - 2y + 5 = 0 \qquad ...(3)$$

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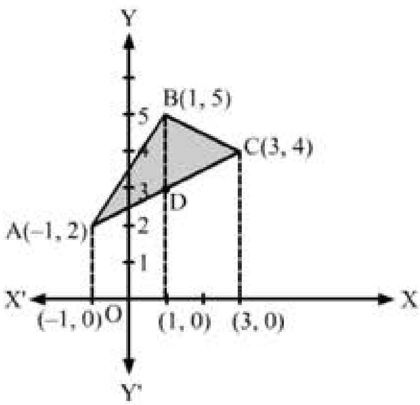


Now, area of ΔABC = $\int_{-1}^{1} y_{AB}dx + \int_{1}^{3} y_{BC}dx - \int_{-1}^{3} y_{AC}dx$

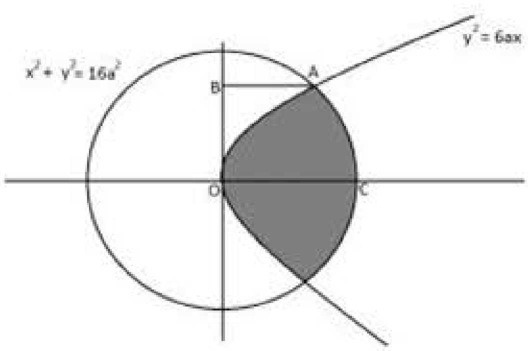
$= \int_{-1}^{1} \frac{3x+7}{2}dx + \int_{1}^{3} \frac{11-x}{2}dx - \int_{-1}^{3} \frac{x+5}{2}dx$

$= \frac{1}{2}\left[\left(\frac{3x^2}{2}+7x\right)_{-1}^{1} + \left(11x-\frac{x^2}{2}\right)_{1}^{3} - \left(\frac{x^2}{2}+5x\right)_{-1}^{3}\right]$

$= \frac{1}{2}\left[\left(\frac{3}{2}+7-\frac{3}{2}+7\right)+\left(33-\frac{9}{2}-11+\frac{1}{2}\right)-\left(\frac{9}{2}+15-\frac{1}{2}+5\right)\right]$

$= \frac{1}{2} \times 8 = 4$ sq. units

71.



Coordinates of point C is $(4a, 0)$. Let the coordinates of point A be (x_1, y_1).

$\therefore x_1^2 + 6ax_1 = 16a^2 \Rightarrow (x_1 - 2a)(x_1 + 8a) = 0 \Rightarrow x_1 = 2a, x_1 \neq -8a$ *as* x_1 *lies in the first quadrant.*

$\therefore y_1^2 = 6ax_1 = 12a^2 \Rightarrow y_1 = 2\sqrt{3}a$

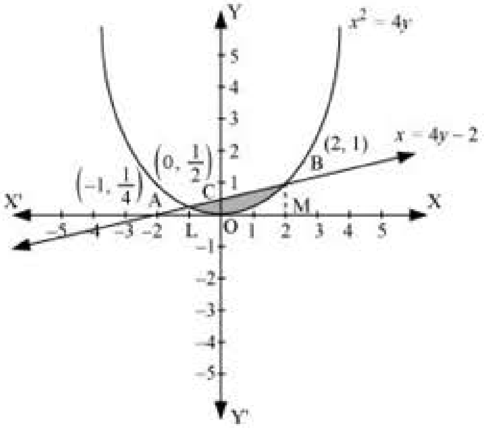
Area of the shaded region

$$= 2\left(\left|\int_0^{2\sqrt{3}a}\left(\sqrt{16a^2-y^2}\right)dy\right| - \left|\int_0^{2\sqrt{3}a}\left(\frac{y^2}{6a}\right)dy\right|\right)$$

$$= 2\left(\left(2\sqrt{3}a^2 + \frac{8\pi a^2}{3}\right) - \left(\frac{4\sqrt{3}}{3}a^2\right)\right) = \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right)$$ square units

72.

The area bounded by the curve, $x^2 = 4y$, and line, $x = 4y - 2$, is represented by the shaded area OBAO.



Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are $\left(-1, \frac{1}{4}\right)$.

Coordinates of point B are (2, 1).

We draw AL and BM perpendicular to x-axis.

It can be observed that,

Area OBAO = Area OBCO + Area OACO … (1)

Then, Area OBCO = Area OMBC − Area OMBO

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{4}[2+4] - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

Similarly, Area OACO = Area OLAC − Area OLAO

$$= \int_{-1}^{0} \frac{x+2}{4} dx - \int_{-1}^{0} \frac{x^2}{4} dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{0} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= -\frac{1}{4}\left[\frac{(-1)^2}{2} + 2(-1)\right] - \left[-\frac{1}{4}\left(\frac{(-1)^3}{3}\right)\right] = \frac{7}{24}$$

Therefore, required area = $\left(\frac{5}{6} + \frac{7}{24}\right) = \frac{9}{8}$ units

73.

The given curves are $y = \sqrt{4-x^2}$ and $x^2 + y^2 - 4x = 0$.

$y = \sqrt{4-x^2} \Rightarrow x^2 + y^2 = 4$ .....(1)

This represents a circle with centre O(0, 0) and radius = 2 units.

Also,

$x^2 + y^2 - 4x = 0 \Rightarrow (x-2)^2 + y^2 = 4$ .....(2)

This represents a circle with centre B(2, 0) and radius = 2 units.

Solving (1) and (2), we get

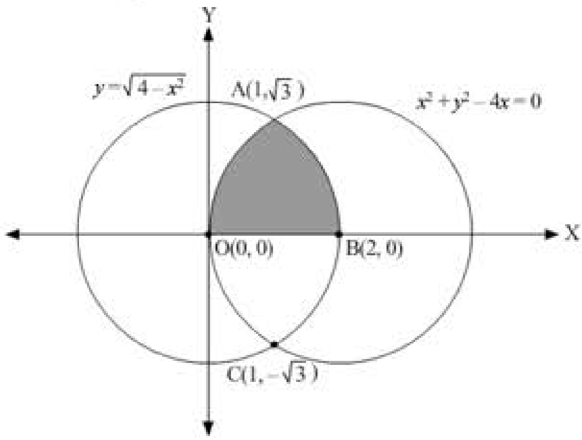
$(x-2)^2 = x^2$

$\Rightarrow x^2 - 4x + 4 = x^2$

$\Rightarrow x = 1$

$\therefore y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

Thus, the given circles intersect at $A(1, \sqrt{3})$ and $C(1, -\sqrt{3})$.



∴ Required area

= Area of the shaded region OABO

$= \int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx$

$= \left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$

$+ \left[\frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$

$= \left[-\frac{\sqrt{3}}{2} + 2\sin^{-1}\left(-\frac{1}{2}\right)\right] - \left[0 + 2\sin^{-1}(-1)\right]$

$+ \left(0 - \frac{1}{2}\sqrt{3}\right) + 2\left[\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right]$

$= -\frac{\sqrt{3}}{2} - 2\times\frac{\pi}{6} + 2\times\frac{\pi}{2} - \frac{\sqrt{3}}{2} + 2\times\frac{\pi}{2} - 2\times\frac{\pi}{6}$

$= -\sqrt{3} + 2\pi - \frac{2\pi}{3}$

$= \left(\frac{4\pi}{3} - \sqrt{3}\right)$ square units

Differential Equations

74.

The given family of curves is $y = ax^2 + bx^3$ … (1)

On differentiating both sides of (1) with respect to x, we obtain

$$\frac{dy}{dx} = 2ax + 3bx^2 \quad \text{...(2)}$$

Again differentiating both sides with respect to x:

$$\frac{d^2y}{dx^2} = 2a + 6bx \quad \text{...(3)}$$

Equations (2) and (3) can be solved as:

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}=\left(2ax+6bx^2\right)-\left(2ax+3bx^2\right)=3bx^2$$
$$\Rightarrow b=\frac{1}{3x^2}\left(x\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)$$

On substituting the value of b in equation (2), we obtain

$$\frac{dy}{dx}=2ax+3\times\frac{1}{3x^2}\left(x\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)x^2$$
$$\Rightarrow\frac{dy}{dx}=2ax+x\frac{d^2y}{dx^2}-\frac{dy}{dx}$$
$$\Rightarrow a=\frac{1}{2x}\left(2\frac{dy}{dx}-x\frac{d^2y}{dx^2}\right)$$

On substituting the value of a and b in equation (1), we obtain

$$y=\frac{1}{2x}\left(2\frac{dy}{dx}-x\frac{d^2y}{dx^2}\right)x^2+\frac{1}{3x^2}\left(x\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)x^3$$
$$\Rightarrow y=x\frac{dy}{dx}-\frac{1}{2}x^2\frac{d^2y}{dx^2}+\frac{1}{3}x^2\frac{d^2y}{dx^2}-\frac{1}{3}x\frac{dy}{dx}$$
$$\Rightarrow y=\frac{2}{3}x\frac{dy}{dx}-\frac{1}{6}x^2\frac{d^2y}{dx^2}$$
$$\Rightarrow 6y=4x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}$$
$$\Rightarrow x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0$$

Thus, the differential equation representing the given family of curves is

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=0$$

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75.

$(\tan^{-1}y - x)dy = (1 + y^2)dx$

Dividing both sides by $(1 + y^2)$, we get

$$\frac{\tan^{-1}y\, dy}{1+y^2} - \frac{xdy}{1+y^2} = dx \qquad .....(1)$$

Let $t = \tan^{-1}y$

Differentiating both sides with respect to y, we have

$$\frac{dt}{dy} = \frac{1}{1+y^2}$$

$$\Rightarrow dt = \frac{1}{1+y^2}dy$$

From (1), we have

$$tdt - xdt = dx$$

$$\Rightarrow (t-x)dt = dx$$

$$\Rightarrow \frac{dx}{dt} = t - x$$

$$\therefore \frac{dx}{dt} + x = t \qquad .....(2)$$

Here, I.F. $= e^{\int (1)dt} = e^t$

Hence, the solution of the differential equation (2) is given by

$$x(\text{I.F.}) = \int (\text{I.F.})tdt$$

$$\Rightarrow xe^t = \int e^t.tdt + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow xe^t = te^t - \int e^t dt + C$$

$$\Rightarrow xe^t = te^t - e^t + C$$

$$\Rightarrow x = t - 1 + Ce^{-t}$$

Substituting $t = \tan^{-1}y$, we have

$$x = \tan^{-1}y - 1 + Ce^{-\tan^{-1}y}$$

This is the general solution of the given differential equation.

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76.

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \qquad .....(1)$$

This is a homogeneous differential equation.

Substitute $y = vx$ .....(2)

$$\Rightarrow \frac{dy}{dx} = v + \frac{x\,dv}{dx} \qquad .....(3)$$

From (1), (2) and (3), we have

$$\frac{x\,dv}{dx} + v = \frac{x\,(vx)}{x^2+(vx)^2} = \frac{vx^2}{x^2\left(1+v^2\right)}$$

$$\Rightarrow \frac{x\,dv}{dx} + v = \frac{v}{1+v^2}$$

$$\Rightarrow \frac{x\,dv}{dx} = \frac{v}{1+v^2} - v = \frac{v-v-v^3}{1+v^2}$$

$$\Rightarrow \frac{x\,dv}{dx} = -\frac{v^3}{1+v^2}$$

$$\Rightarrow \frac{\left(1+v^2\right)}{v^3}dv = -\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{v^3}+\frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both sides, we have

$$\frac{v^{-3+1}}{-3+1} + \ln v = -\ln x + C$$

$$\Rightarrow -\frac{1}{2v^2} + \ln v = -\ln x + C$$

$$\Rightarrow -\frac{1}{2v^2} + \ln vx = C$$

$$\Rightarrow -\frac{x^2}{2y^2} + \ln y = C$$

Given: $y = 1$ when $x = 0$

$\Rightarrow C = 0$

Thus, the particular solution of the given differential equation is given by

$$\ln y = \frac{x^2}{2y^2}$$

or

$$x^2 = 2y^2 \ln y$$

77.

The given differential equation can be written as

$$\left(x^3 + 4xy^2\right)dy = -5x^2 y\, dx$$

$$\frac{dy}{dx} = \frac{-5x^2 y}{x^3 + 4xy^2} = \mathrm{F}\left(x, y\right) \ \ (\text{say})$$

Dividing the numerator and denominator by x^3, we obtain

$$\frac{dy}{dx} = \frac{-5\frac{y}{x}}{1 + 4\frac{y^2}{x^2}} = g\left(\frac{y}{x}\right) \qquad \text{...(i)}$$

Therefore, $\mathrm{F}(x, y)$ is a homogeneous function of degree 2.

Thus, the given differential equation is a homogeneous differential equation.

Substituting $y = vx$ in equation (i), we obtain

$$v + x\frac{dv}{dx} = \frac{-5v}{1 + 4v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-5v}{1 + 4v^2} - v = \frac{-5v - v - 4v^3}{1 + 4v^2} = \frac{-6v - 4v^3}{1 + 4v^2}$$

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$$\Rightarrow \frac{1+4v^2}{3v+2v^3}dv = -\frac{2dx}{x}$$

$$\therefore \int \frac{1+4v^2}{3v+2v^3}dv = -2\int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1+4v^2}{3v+2v^3}dv = -2\log x - \log C$$

$$\Rightarrow \int \frac{1+2v^2}{3v+2v^3}dv + 2\int \frac{v^2}{3v+2v^3}dv = -2\log x - \log C$$

Substitute $3v+2v^3 = t \Rightarrow \left(3+6v^2\right)dv = dt$ or $3\left(1+2v^2\right)dv = dt$

$$\therefore \int \frac{dt}{3t} + 2\int \frac{v}{3+2v^2}dv = -2\log x - \log C$$

$$\Rightarrow \frac{1}{3}\log t + \frac{1}{2}\int \frac{4v}{3+2v^2}dv = -2\log x - \log C$$

$$\Rightarrow \frac{1}{3}\log\left(3v+2v^3\right) + \frac{1}{2}\int \frac{du}{u} = -2\log x - \log C \qquad \left(\text{By taking } 3+2v^2 = u \Rightarrow 4v\,dv = du\right)$$

$$\therefore \frac{1}{3}\log\left(3v+2v^3\right) + \frac{1}{2}\log\left(3+2v^2\right) = -2\log x - \log C$$

$$\Rightarrow \log\left[\frac{\left(3v+2v^3\right)^{\frac{1}{3}}}{\left(3+2v^2\right)^{\frac{1}{2}}}\right] = -2\log x - \log C$$

$$\Rightarrow \log\left[\frac{\left(3\frac{y}{x}+2\frac{y^3}{x^3}\right)^{\frac{1}{3}}}{\left(3+2\frac{y^2}{x^2}\right)^{\frac{1}{2}}}\right] + 2\log x + \log C = 0$$

$$\Rightarrow \log\left[\frac{\left(3yx^2+2y^3\right)^{\frac{1}{3}}}{\left(3x^2+2y^2\right)^{\frac{1}{2}}}\right] + \log\left(x^2C\right) = 0$$

$$\Rightarrow \log\left[\frac{x^2C\left(3x^2y+2y^3\right)^{\frac{1}{3}}}{\left(3x^2+2y^2\right)^{\frac{1}{2}}}\right] = 0$$

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78.

The given differential equation can be written as

$$\frac{dy}{dx}+y\tan x=(x+1)\sec x$$

It is of the type $\frac{dy}{dx}+\text{P}y=\text{Q}$, where P = tan x and $\text{Q}=(x+1)\sec x$

$$\therefore \text{I.F}=e^{\int \tan x\,dx}=e^{\log \sec x}=\sec x$$

The solution of the differential equation is given by,

$$\begin{aligned} y\cdot\sec x &= \int (x+1)\sec x\cdot\sec x\ dx+C \\ &= \int x\sec^2 x\ dx+\int \sec^2 x+C \\ &= I_1+\tan x+C \qquad \text{(say)} \qquad ...(\text{i}) \end{aligned}$$

$$\begin{aligned} \text{Now, I}_1 &= \int x\sec^2 x\ dx \\ &= x\int \sec^2 x dx-\int\left\{\left(\frac{d}{dx}x\right)\int \sec^2 x dx\right\}dx \\ &= x\tan x-\int \tan x\ dx \\ &= x\tan x-\log(\sec x) \qquad ...(\text{ii}) \end{aligned}$$

From (i), and (ii), we obtain

$$y\sec x=x\tan x-\log\sec x+\tan x+C$$
$$y\sec x=(x+1)\tan x-\log\sec x+C$$

This is the required solution of the given differential equation.

79.

The given differential equation is

$$\frac{dy}{dx}=\frac{x(2\log x+1)}{\sin y+y\cos y} \qquad ...(1)$$

Separating the variables in (1), we get

$$(\sin y+y\cos y)dy=x(2\log x+1)dx \qquad ...(2)$$

Integrating both sides of (2), we have

$\int(\sin y + y\cos y)dy = \int x(2\log x + 1)dx$ ...(3)

Now, $\int \sin y\, dy = -\cos y + C$ and $\int y\cos y\, dy = y\sin y + \cos y + C$ (Using by parts)

$\therefore \int(\sin y + y\cos y)dy = -\cos y + y\sin y + \cos y + C_1 = y\sin y + C_1$ ...(4)

Let $I = \int(2x\log x + x)dx$

$= \int 2x\log x\, dx + \int x\, dx$

$= 2\left[\log x\left(\int x\, dx\right) - \int\left(\frac{d}{dx}(\log x).\int x\, dx\right)dx\right] + \frac{x^2}{2} + C_2$

$= 2\left[\log x \times \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} dx\right] + \frac{x^2}{2} + C_2$

$= 2\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + \frac{x^2}{2} + C_2$

$= x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C_2$

$= x^2\log x + C_2$ ...(5) (Using by parts)

Putting the values in equation (3), we get

$y\sin y = x^2\log x + \text{C}$, where $\text{C} = \text{C}_2 - \text{C}_1$ ...(6)

On putting $y = \frac{\pi}{2}$ and $x = 1$ in equation (6), we get

$\text{C} = \frac{\pi}{2}$

$\therefore$ The particular solution of the given differential equation is $y\sin y = x^2\log x + \frac{\pi}{2}$.

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80.

Given:

$$\left(1-y^2\right)(1+\log x)\mathrm{d}x+2xy\mathrm{d}y=0$$

$$\Rightarrow\left(1-y^2\right)(1+\log x)\mathrm{d}x=-2xy\mathrm{d}y$$

$$\Rightarrow\left(\frac{1+\log x}{2x}\right)\mathrm{d}x=-\left(\frac{y}{1-y^2}\right)\mathrm{d}y \qquad \ldots\ldots(1)$$

Let:

$1+\log x=t$

and

$\left(1-y^2\right)=p$

$$\Rightarrow\frac{1}{x}\mathrm{d}x=\mathrm{d}t \text{ and } -2y\mathrm{d}y=\mathrm{d}p$$

Therefore, (1) becomes

$$\int\frac{t}{2}\mathrm{d}t=\int\frac{1}{2p}\mathrm{d}p$$

$$\Rightarrow\frac{t^2}{4}=\frac{\log p}{2}+C \qquad \ldots\ldots(2)$$

Substituting the values of t and p in (2), we get

$$\frac{(1+\log x)^2}{4}=\frac{\log\left(1-y^2\right)}{2}+C \qquad \ldots\ldots(3)$$

At $x=1$ and $y=0$, (3) becomes

$$C=\frac{1}{4}$$

Substituting the value of C in (3), we get

$$\frac{(1+\log x)^2}{4}=\frac{\log\left(1-y^2\right)}{2}+\frac{1}{4}$$

$$\Rightarrow(1+\log x)^2=2\log\left(1-y^2\right)+1$$

Or

$$(\log x)^2+\log x^2=\log\left(1-y^2\right)^2$$

It is the required particular solution.

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Vector Algebra

81.

$|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$

$\Rightarrow \{|\vec{a}||\vec{b}|\sin\theta\}^2+\{|\vec{a}||\vec{b}|\cos\theta\}^2=400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta=400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2=400$

$\Rightarrow 25\times|\vec{b}|^2=400$

$\Rightarrow |\vec{b}|^2=16$

$\Rightarrow |\vec{b}|=4$

82.

Given:

$\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$

$\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$

The projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})=\frac{(7\times2)+(1\times6)+(-4\times3)}{\sqrt{(2)^2+(6)^2+(3)^2}}=\frac{8}{7}$$

83.

Let $\vec{p}=p_1\hat{i}+p_2\hat{j}+p_3\hat{k}$

Since $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, $\vec{p}\cdot\vec{a}=0$ and $\vec{p}\cdot\vec{b}=0$.

$\vec{p}\cdot\vec{a}=0$

$\Rightarrow\left(p_1\hat{i}+p_2\hat{j}+p_3\hat{k}\right)\cdot\left(\hat{i}+4\hat{j}+2\hat{k}\right)=0$

$\Rightarrow p_1\cdot1+p_2.4+p_3\cdot2=0$

$\Rightarrow p_1+4p_2+2p_3=0 \qquad ...(1)$

$\vec{p}\cdot\vec{b}=0$

$\Rightarrow 3p_1-2p_2+7p_3=0 \qquad ...(2)$

Also, it is given that $\vec{p}\cdot\vec{c}=18$.

$\Rightarrow 2p_1-p_2+4p_3=18 \qquad ...(3)$

Solving (1) and (2), we have

$\frac{p_1}{28+4}=\frac{p_2}{6-7}=\frac{p_3}{-2-12}=k$

$\Rightarrow p_1=32k, p_2=-k, p_3=-14$

Substituting the values of p_1, p_2 and p_3 in (3), we get $2\times(32k)-(-k)+4\times(-14k)=18$

$\Rightarrow 64k+k=56k=18$

$\Rightarrow 9k=18$

$\Rightarrow k=2$

$\therefore p_1=64, p_2=-2$ and $p_3=-28$

Hence, the required vector $\vec{p}$ is $64\hat{i}-2\hat{j}-28\hat{k}$.

84.

We have:

$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$=(\vec{a}+\vec{b}).[\vec{b}\times\vec{c}\ +\ \vec{b}\times\vec{a}\ +\ \vec{c}\times\vec{c}\ +\ \vec{c}\times\vec{a}]$ (By distributive law)

$=(\vec{a}+\vec{b}).[\vec{b}\times\vec{c}\ +\ \vec{b}\times\vec{a}\ +\ \vec{c}\times\vec{a}]$ $(\because \vec{c}\times\vec{c}=0)$

$=\vec{a}.(\vec{b}\times\vec{c})+\vec{b}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{b}\times\vec{a})+\vec{b}.(\vec{b}\times\vec{a})+\vec{a}.(\vec{c}\times\vec{a})+\vec{b}.(\vec{c}\times\vec{a})$

$=[\vec{a},\vec{b},\vec{c}]+[\vec{b},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{a}]+[\vec{b},\vec{b},\vec{a}]+[\vec{a},\vec{c},\vec{a}]+[\vec{b},\vec{c},\vec{a}]$

$=[\vec{a},\vec{b},\vec{c}]+[\vec{b},\vec{c},\vec{a}]$ ($\because$ scalar triple product with two equal vectors is 0)

$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]$ $(\because [\vec{b},\vec{c},\vec{a}]=[\vec{a},\vec{b},\vec{c}])$

$=2[\vec{a},\vec{b},\vec{c}]$

Hence, $[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]=2[\vec{a},\vec{b},\vec{c}]$

85.

The vertices of ΔABC are A (1, 2, 3), B (2, −1, 4) and C (4, 5, −1).

$\therefore \overrightarrow{AB}$ = Position vector of B – Position vector of A

$$=\left(2\hat{i}-\hat{j}+4\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=\hat{i}-3\hat{j}+\hat{k}$$

$\overrightarrow{AC}$ = Position vector of C – Position vector of A

$$=\left(4\hat{i}+5\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$
$$=3\hat{i}+3\hat{j}-4\hat{k}$$

Now,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\left(\hat{i}-3\hat{j}+\hat{k}\right)\times\left(3\hat{i}+3\hat{j}-4\hat{k}\right)$$
$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=9\hat{i}+7\hat{j}+12\hat{k}$$
$$\Rightarrow\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(9)^2+(7)^2+(12)^2}=\sqrt{274}$$

$\therefore$ Area of the $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{274}$

Thus, the area of ΔABC is $\frac{1}{2}\sqrt{274}$ square units.

Three Dimensional Geometry

86.

Let the plane make intercepts a, $3a$ and $4a$ along positive x, y and z-axes respectively, where $a > 0$.

$\therefore$ Equation of the plane is $\frac{x}{a}+\frac{y}{3a}+\frac{z}{4a}=1$.

It is known that the distance (D) of a plane $a_1x + b_1y + c_1z = d$ from point (x_1, y_1, z_1) is given by

$$D=\left|\frac{a_1x_1+b_1y_1+c_1z_1-d}{\sqrt{a_1^2+b_1^2+c_1^2}}\right|.$$

It is given that the distance of plane $\frac{x}{a}+\frac{y}{3a}+\frac{z}{4a}=1$ from the origin is 6 units.

$$\therefore \left|\frac{\frac{0}{a}+\frac{0}{3a}+\frac{0}{4a}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{3a}\right)^2+\left(\frac{1}{4a}\right)^2}}\right|=6$$

$$\Rightarrow \left|\frac{-1}{\sqrt{\frac{169}{144a^2}}}\right|=6$$

$$\Rightarrow \left|\frac{-12a}{13}\right|=6$$

$$\Rightarrow |-12a|=78$$

$$\Rightarrow -12a=78 \text{ or } -12a=-78$$

$$\Rightarrow a=-\frac{13}{2} \text{ or } \frac{13}{2}$$

$\therefore a=\frac{13}{2}$ ($a>0$)

Therefore, the equation of the plane is $\frac{2x}{13}+\frac{2y}{39}+\frac{2z}{52}=1$ i.e., $12x+4y+3z-78=0$

The direction ratios of normal to this plane are 12, 4, 3.

Therefore, the direction cosines of the normal to the plane are given by

$$\frac{12}{\sqrt{12^2+4^2+3^2}},\frac{4}{\sqrt{12^2+4^2+3^2}},\frac{3}{\sqrt{12^2+4^2+3^2}} \text{ i.e., } \frac{12}{13},\frac{4}{13},\frac{3}{13}.$$

87.

The equation of line passing through points (2, −3, 5) and (3, −2, 4) is given by

$$\frac{x-2}{3-2}=\frac{y+3}{-2+3}=\frac{z-5}{4-5}$$

$$\Rightarrow \frac{x-2}{1}=\frac{y+3}{1}=\frac{z-5}{-1}=k \text{ (say)} \quad \ldots (1)$$

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Any point on the line joining points (2, −3, 5) and (3, −2, 4) is of the form $(k+2, k-3, 5-k)$.
The z-coordinate of any point on the xy-plane is 0.
$\therefore 5 - k = 0$
$\Rightarrow k = 5$
Therefore, line (1) intersects the xy-plane at A $(5 + 2, 5 - 3, 0)$ i.e., at A (7, 2, 0).
The x-coordinate of any point on the yz-plane is 0.
$\therefore k + 2 = 0$
$\Rightarrow k = -2$
Therefore, line (1) intersects the yz-plane at B $(0, -2-3, 5+2)$ i.e., at B (0, −5, 7).
The equation of any plane passing through points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Therefore, the equation of the plane through points A (7, 2, 0), B (0, −5, 7) and (1, 2, 3) is given by

$$\begin{vmatrix} x-7 & y-2 & z \\ 0-7 & -5-2 & 7-0 \\ 1-7 & 2-2 & 3-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-7 & y-2 & z \\ -7 & -7 & 7 \\ -6 & 0 & 3 \end{vmatrix} = 0$$

$$\Rightarrow -7\times 3\begin{vmatrix} x-7 & y-2 & z \\ 1 & 1 & -1 \\ -2 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (-21)\left[(x-7)\cdot(1-0)-(y-2)(1-2)+z(0+2)\right]=0$$

$$\Rightarrow (x-7)+(y-2)+2z=0$$

$$\Rightarrow x+y+2z-9=0$$

Thus, the equation of the required plane is $x+y+2z-9=0$.

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88.

The given lines are $\frac{x+1}{6}=\frac{y+3}{-4}=\frac{z-12}{-2}$ and $\frac{x-1}{1}=\frac{y-5}{-3}=\frac{z-2}{2}$.

Two lines, $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, intersect each other if the shortest distance between them is zero.

$$\text{i.e., } d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}=0$$

Comparing the equation of the given lines with

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, we obtain

$x_1=-1,\ y_1=-3,\ z_1=12,\ a_1=6,\ b_1=-4,\ c_1=-2$
$x_2=1,\ y_2=5,\ z_2=2,\ a_2=1,\ b_2=-3,\ c_2=2$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 2 & 8 & -10 \\ 6 & -4 & -2 \\ 1 & -3 & 2 \end{vmatrix}$$

$=2(-8-6)-8(12+2)-10(-18+4)$
$=-28-112+140$
$=0$

$\therefore d=0$

This proves that the given lines intersect each other. The coordinates of any point on the line $\frac{x+1}{6}=\frac{y+3}{-4}=\frac{z-12}{-2}$ are $(6k-1,-4k-3,-2k+12)$.

If the point $(6k-1,-4k-3,-2k+12)$ is the point of intersection, then it satisfies the other line.

$$\therefore \frac{6k-2}{1}=\frac{-4k-8}{-3}=\frac{-2k+10}{2}$$
$$\Rightarrow \frac{6k-2}{1}=\frac{4k+8}{3}$$
$$\Rightarrow k=1$$

Thus, the coordinates of the point of intersection of the given lines are

$$\left(6\times1-1,-4\times1-3,-2\times1+12\right)=(5,-7,10)$$

89.

The vector equation of the line passing through $\hat{i}+2\hat{j}+3\hat{k}$ and parallel to the vector $a\hat{i}+b\hat{j}+c\hat{k}$ is $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+k\left(a\hat{i}+b\hat{j}+c\hat{k}\right)$ ...(1)

The equations of the given planes are

$$\vec{r}\cdot\left(\hat{i}-\hat{j}+2\hat{k}\right)=5 \quad ...(2)$$
$$\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6 \quad ...(3)$$

Since line (1) is parallel to the planes (2) and (3), so

$$a\times1+b\times(-1)+c\times2=0\Rightarrow a-b+2c=0 \quad ...(4)$$
$$a\times3+b\times1+c\times1=0\Rightarrow 3a+b+c=0 \quad ...(5)$$

Solving (4) and (5), we have

$$\frac{a}{-1-2}=\frac{b}{6-1}=\frac{c}{1+3}$$
$$\Rightarrow \frac{a}{-3}=\frac{b}{5}=\frac{c}{4}$$
$$\Rightarrow a=-3\mu, b=5\mu, \text{ and } c=4\mu$$

Substituting the values of a, b and c in (1), we have

$$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+k\left(-3\mu\hat{i}+5\mu\hat{j}+4\mu\hat{k}\right)$$
$$\Rightarrow \vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\mu k\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$$
$$\Rightarrow \vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right), \text{ where } \lambda=\mu k$$

Thus, the vector equation of the line is $\vec{r}=\hat{i}+2\hat{j}+3k+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$

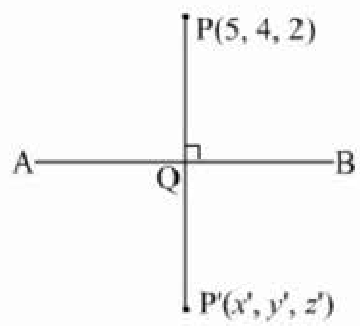
90.

The given point is (5, 4, 2) and the given line is $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda\left(2\hat{i}+3\hat{j}-\hat{k}\right)$.

This line passes through the point (−1, 3, 1) and is parallel to the vector $2\hat{i}+3\hat{j}-\hat{k}$.
Cartesian equation of line is

$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\lambda(\text{say})$$
$$\therefore x=2\lambda-1, y=3\lambda+3, z=-\lambda+1$$

These are the coordinates of any general point on the line.



Let Q be the foot of the perpendicular on the line. Then, for some value of λ, the coordinates of Q are (2λ − 1, 3λ + 3, − λ + 1).
Direction ratios of PQ are 2λ − 1 − 5, 3λ + 3 − 4, − λ + 1 −2 i.e. 2λ − 6, 3λ − 1, − λ − 1.
PQ is perpendicular to given line.

$\therefore 2(2\lambda-6)+3(3\lambda-1)-1(-\lambda-1)=0$
$\Rightarrow 4\lambda-12+9\lambda-3+\lambda+1=0$
$\Rightarrow 14\lambda-14=0$
$\Rightarrow \lambda=1$

∴ Coordinates of Q = (2 × 1 − 1, 3 × 1 + 3, −1 + 1) = (1, 6, 0)
Hence, the coordinates of the foot of perpendicular are (1, 6, 0).
Using distance formula, we have

$$\begin{aligned} PQ = \left|\overrightarrow{PQ}\right| &= \sqrt{(1-5)^2+(6-4)^2+(0-2)^2} \\ &= \sqrt{16+4+4} \\ &= \sqrt{24} \\ &= 2\sqrt{6} \text{ units} \end{aligned}$$

Let $P'(x', y', z')$ be the image of point P (5, 4, 2) in the given line.
Then, Q is midpoint of PP'.

$$\therefore \frac{x'+5}{2} = 1 \Rightarrow x'+5 = 2 \Rightarrow x' = -3$$

$$\frac{y'+4}{2} = 6 \Rightarrow y'+4 = 12 \Rightarrow y' = 8$$

$$\frac{z'+2}{2} = 0 \Rightarrow z'+2 = 0 \Rightarrow z' = -2$$

Hence, the image of P in the given line is (−3, 8, −2).

91.

The equations of the given planes are

$$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0 \qquad ...(1)$$

$$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0 \qquad ...(2)$$

The equation of the plane passing through the intersection of the planes (1) and (2) is

$$\left[\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]=0$$

$$\Rightarrow \vec{r}.\left[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right]=4-5\lambda \qquad ...(3)$$

Given, the plane (3) is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$.

$$(1+2\lambda)\times 5+(2+\lambda)\times 3+(3-\lambda)\times(-6)=0 \Rightarrow 19\lambda-7=0 \Rightarrow \lambda=\frac{7}{19}$$

Putting $\lambda = \frac{7}{19}$ in (3), we get

$$\vec{r}.\left[\left(1+\frac{14}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]=4-\frac{35}{19}\Rightarrow \vec{r}.\left(\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right)=\frac{41}{19}$$
$$\Rightarrow \vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41$$

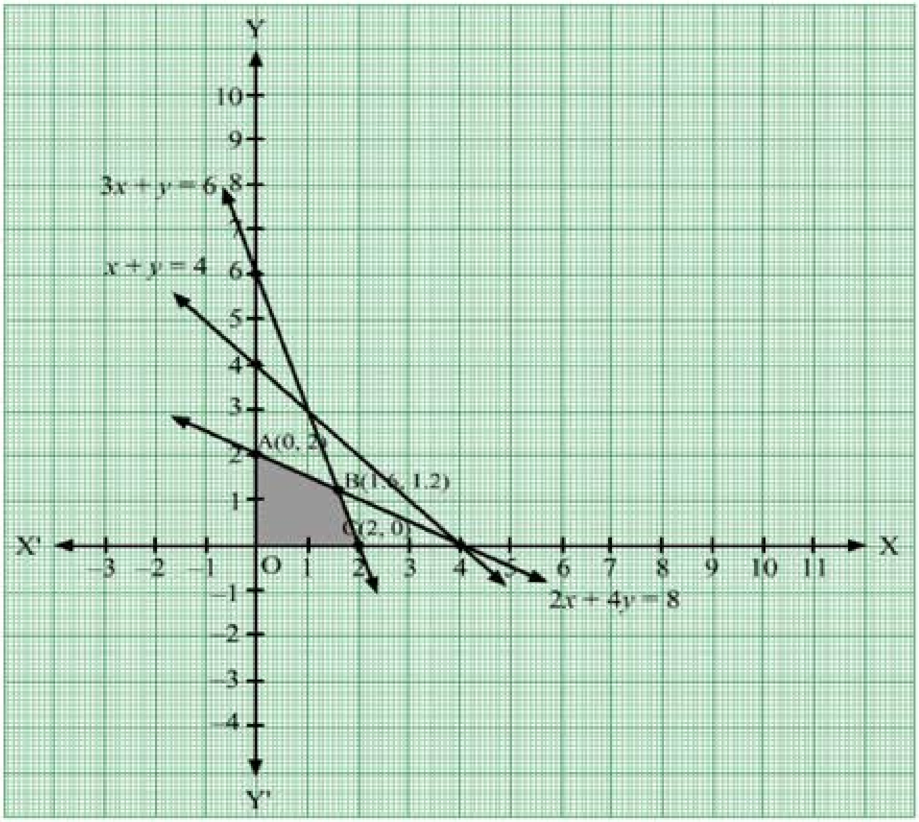
Thus, the equation of the required plane is $\vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41$.

Linear Programming

92.

The given constraints are
$2x + 4y \leq 8$ .....(1)
$3x + y \leq 6$ .....(2)
$x + y \leq 4$ .....(3)
$x \geq 0, y \geq 0$
We need to maximise the objective function $z = 2x + 5y$.
Converting the inequations into equations, we obtain the lines $2x + 4y = 8$, $3x + y = 6$, $x + y = 4$, $x = 0$ and $y = 0$.
These lines are drawn and the feasible region of the LPP is shaded.

**Meritnation Top 100 Questions
(Solutions)
Class - 12**



The coordinates of the corner points of the feasible region are O(0, 0), A(0, 2), B(1.6, 1.2) and C(2, 0).
The value of the objective function at these points are given in the following table.

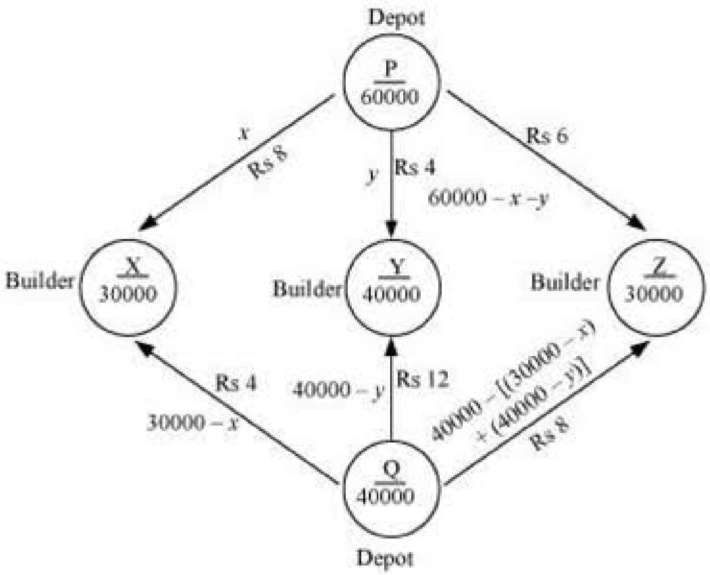
Points	Value of the Objective Function, $z = 2x + 5y$
O(0, 0)	$2 \times 0 + 5 \times 0 = 0$
A(0, 2)	$2 \times 0 + 5 \times 2 = 10$
B(1.6, 1.2)	$2 \times 1.6 + 5 \times 1.2 = 9.2$
C(2, 0)	$2 \times 2 + 5 \times 0 = 4$

Out of these values of z, the maximum value of z is 10, which is attained at the point (0, 2).
Hence, the maximum value of z is 10.

93.

Let x bags and y bags of cement be transported from depot **P** to builders **X** and **Y** respectively.
Accordingly, $(60000 - x - y)$ bags of cements are transported from depot **P** to builder **Z**.
The given problem can be represented in the form of a diagram as:

Meritnation Top 100 Questions
(Solutions)
Class - 12



From the figure, it is clear that

$x, y \geq 0$

$60000 - x - y \geq 0$

$\Rightarrow x + y \leq 60000$

$30000 - x \geq 0$

$\Rightarrow x \leq 30000$

$40000 - y \geq 0$

$\Rightarrow y \leq 40000$

$40000 - [(30000 - x) + (40000 - y)] \geq 0$

$\Rightarrow x + y - 30000 \geq 0$

$\Rightarrow x + y \geq 30000$

Total cost of transportation to **Z**is given by

$$\begin{aligned} Z &= 8x + 4y + 6(60000 - x - y) + 4(30000 - x) + 12(40000 - y) + 8\left[40000 - \{(30000 - x) + (40000 - y)\}\right] \\ &= 6x - 6y + 720000 \\ &= 6(x - y + 120000) \end{aligned}$$

Therefore, the given linear programming problem is

Minimize Z = 6 ($x - y$+ 120000) …(1)

Subject to the constraints

$x \leq 30000$ …(2)

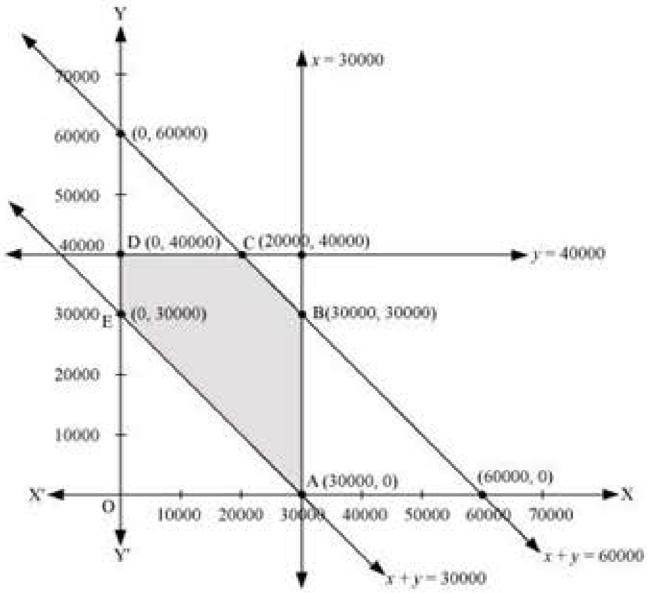
$y \le 40000 \quad \ldots(3)$

$x + y \ge 30000 \quad \ldots(4)$

$x + y \le 60000 \quad \ldots(5)$

$x, y \ge 0 \quad \ldots(6)$

The feasible region bounded by constraints (2) to (6) is represented using a figure as:



The corner points of the feasible region are A (30000, 0), B (30000, 30000), C (20000, 40000), D (0, 40000) and E (0, 30000).

The values of Z at these corner points are calculated as:

Corner point	**Z = 6 ($x - y$ + 120000)**	
A (30000, 0)	900000	
B (30000, 30000)	720000	
C (20000, 40000)	600000	

D (0, 40000)	480000	← Minimum
E (0, 30000)	540000	

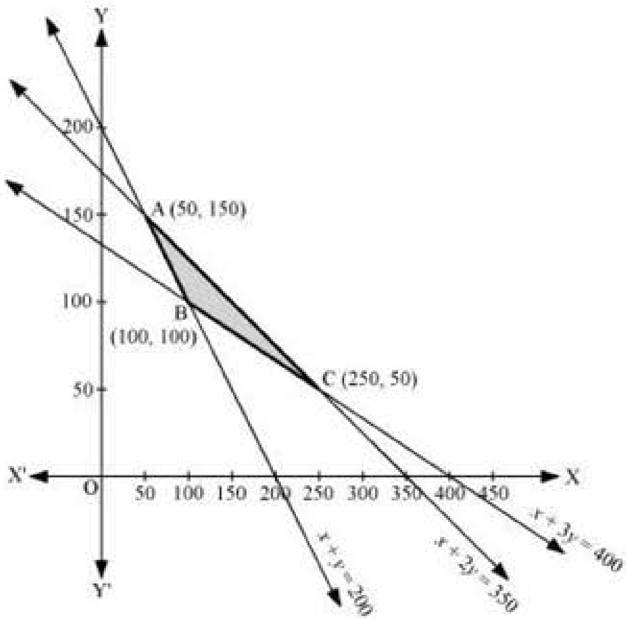
The minimum value of Z is 4,80,000, which is attained at $x = 0$ and $y = 40000$.
Therefore, 40000 and 20000 bags of cements are to be transported from depot **P** to dealers **Y** and **Z** respectively and 30000 and 10000 bags of cements are to be transported from depot **Q** to dealers **X** and **Z** respectively to minimize the cost of transportation.
Thus, the minimum cost of transportation is Rs 4,80,000.

94.

Let the quantity of scrap purchased by the firm from suppliers **A** and **B** be x quintals and y quintals respectively.
It is given that the firm plans to purchase at least 200 quintals of scrap.
$\therefore x + y \geq 200$
It is also given that the scrap must contain at least 100 quintals of metal **X** and at most 35 quintals of metal **Y**.

$$\therefore \frac{1}{4}x + \frac{3}{4}y \geq 100 \text{ and } \frac{x}{10} + \frac{y}{5} \leq 35$$
$$\Rightarrow x + 3y \geq 400 \text{ and } x + 2y \leq 350$$

The cost of scrap from suppliers **A** and **B** are Rs 200 and Rs 400 per quintal respectively.
Therefore, the objective function Z is given as $Z = 200x + 400y$.
Therefore, the given linear programming problem can be mathematically formulated as:
Minimize $Z = 200x + 400y$ subject to the constraints
$x + y \geq 200$
$x + 3y \geq 400$
$x + 2y \leq 350$
$x \geq 0, y \geq 0$
The feasible region determined by the constraints is graphically represented as:



The feasible region is ABCA.

The values of Z at the corner points of this bounded feasible region are given in tabular form as:

Corner point	$Z = 200x + 400y$	
A (50, 150)	70,000	
B (100, 100)	**60,000**	$\rightarrow$ **Minimum**
C (250, 50)	70,000	

The minimum value of Z is 60000, which is attained at $x = 100$ and $y = 100$.

Thus, to minimise the total purchase cost, the quantities of scrap that the firm should purchase from suppliers **A** and **B** are 100 quintals and 100 quintals respectively.

95.

Let the apples and grapes required to maximise the profit be x kg and y kg respectively.

The given information can be represented by an LPP as:

Max $Z = \text{Rs}\ (10x + 8y)$ …(1)

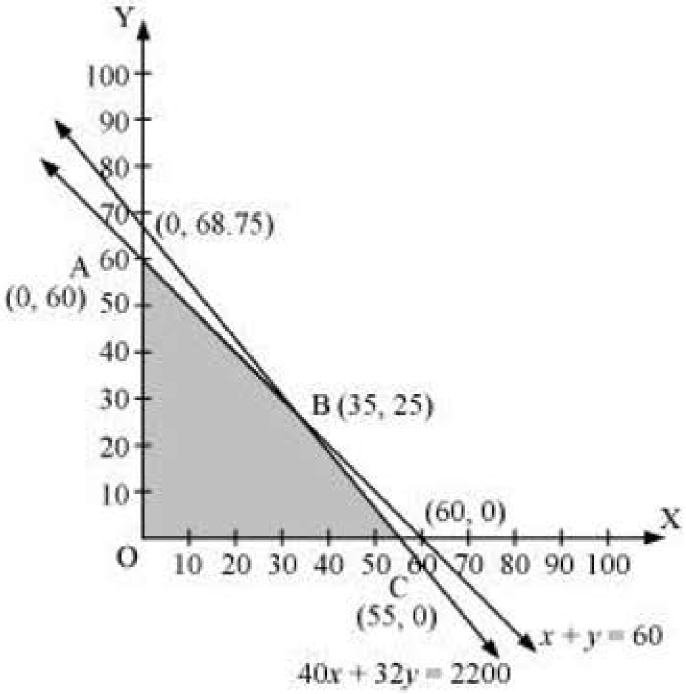
Subject to

$x + y \leq 60$ ...(2)

$40x + 32y \leq 2200$ ...(3)

$x, y \geq 0$ ...(4)

The feasible region determined by the system of constraints is as follows:



The corner points are O (0, 0), A (0, 60), B (35, 25) and C (55, 0) respectively.

The values of Z at these corner points are as follows:

Corner point	$Z = 10x + 8y$	
O (0, 0)	0	
A (0, 60)	480	
B (35, 25)	550	→ Maximum

C (55, 0)	550	→ Maximum

Therefore, Z is maximum at (35, 25) and (55, 0).
Thus, in order to maximise his profit to Rs 550, the man should invest his money either in 35 kg of apples and 25 kg of grapes or in 55 kg of apples.

Probability

96.
Let the event be defined as follows:
E_1 = The die shows 1 or 2

E_2 = The die shows 3, 4, 5 or 6

E = One of the ball drawn is red and another is black

$P(E_1) = \frac{2}{6} = \frac{1}{3}$ and $P(E_2) = \frac{4}{6} = \frac{2}{3}$

The probability of drawing a red and a black ball from bag A is given by

$P(E|E_1) = \frac{6}{10} \times \frac{4}{9} + \frac{4}{10} \times \frac{6}{9} = \frac{8}{15}$

The probability of drawing a red and a black ball from bag B is given by

$P(E|E_2) = \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9} = \frac{7}{15}$

Using the theorem of total probability, we have

$$P(E) = P(E_1)P(E|E_1) + P(E_2)P(E|E_2)$$
$$= \frac{1}{3} \times \frac{8}{15} + \frac{2}{3} \times \frac{7}{15}$$
$$= \frac{22}{45}$$

97.
Let S denote the success, i.e. getting a number greater than four and F denote the failure, i.e. getting a number less than four.

$\therefore$ P(S) $= \frac{2}{6} = \frac{1}{3}$, P(F) $= 1 - \frac{1}{3} = \frac{2}{3}$

Now, B gets the second throw, if A fails in the first throw.

$\therefore$ P(B wins in the second throw) = P(FS) = P(F)P(S) = $\frac{2}{3} \times \frac{1}{3}$

Similarly, P(B wins in the fourth throw) = P(FFFS) = P(F)P(F)P(F)P(S) = $\left(\frac{2}{3}\right)^3 \times \frac{1}{3}$

P(B wins in the sixth throw) = P(FFFFFS) = P(F)P(F)P(F)P(F)P(F)P(S) = $\left(\frac{2}{3}\right)^5 \times \frac{1}{3}$ and so on.

Hence,

$$\text{P(B wins)} = \frac{2}{3} \times \frac{1}{3} + \left(\frac{2}{3}\right)^3 \times \frac{1}{3} + \left(\frac{2}{3}\right)^5 \times \frac{1}{3} + \ldots$$

$$= \frac{2}{3} \times \frac{1}{3} \times \left[1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \ldots\right]$$

$$= \frac{2}{3} \times \frac{1}{3} \times \left(\frac{1}{1 - \frac{4}{9}}\right) \qquad \left[\because a + ar + ar^2 + \ldots = \frac{a}{1-r}\right]$$

$$= \frac{2}{5}$$

Thus, the probability that B wins is $\frac{2}{5}$.

98.

The probability distribution of X is

X	0	1	2	3	4
P(X)	0	k	$4k$	$2k$	k

The given distribution is a probability distribution.

$\therefore \sum p_i = 1$

$\Rightarrow 0 + k + 4k + 2k + k = 1$

$\Rightarrow 8k = 1$

$\Rightarrow k = 0.125$

(i) P(getting admission in exactly one college) = P($X = 1$) = k = 0.125

(ii) P(getting admission in at most 2 colleges) = P($X \leq 2$) = 0 + k + $4k$ = $5k$ = 0.625

(iii) P(getting admission in at least 2 colleges) = P($X \geq 2$) = $4k + 2k + k = 7k$ = 0.875

99.

Let X denote the number of persons who are vegetarians.

Since the selection of persons is done with replacement, the trails are Bernoulli's trials.

∴ X has the binomial distribution with $n = 20$ and $p = \frac{30}{100} = \frac{3}{10}$.

$$\therefore q = 1 - p = 1 - \frac{3}{10} = \frac{7}{10}$$

∴ P (at least 3 persons are vegetarians)

= P ($X \geq 3$)

= 1 − [P ($X = 0$) + P ($X = 1$) + P ($X = 2$)]

$$= 1 - \left[{}^{20}C_0 \left(\frac{7}{10}\right)^{20} + {}^{20}C_1 \cdot \left(\frac{7}{10}\right)^{19} \cdot \left(\frac{3}{10}\right) + {}^{20}C_2 \left(\frac{7}{10}\right)^{18} \cdot \left(\frac{3}{10}\right)^2 \right]$$

$$= 1 - \left[\left(\frac{7}{10}\right)^{20} + 20 \cdot \frac{7^{19} \cdot 3}{10^{20}} + 190 \cdot \frac{7^{18}}{10^{18}} \cdot \frac{9}{10^2} \right]$$

$$= 1 - (49 + 420 + 1710) \cdot \left[\frac{7^{18}}{10^{20}} \right]$$

$$= 1 - \frac{2179 \times 7^{18}}{10^{20}}$$

100.

Let the events, E_1, E_2, E_3 and A be defined as

E_1: The bottle is manufactured in plant A

E_2: The bottle is manufactured in plant B

E_3: The bottle is manufactured in plant C

A: The bottle is defective

Total number of bottles manufactured in a day = 1500 + 2000 + 2500 = 6000

$$\therefore P(E_1) = \frac{1500}{6000} = \frac{1}{4},\ P(E_2) = \frac{2000}{6000} = \frac{1}{3},\ P(E_3) = \frac{2500}{6000} = \frac{5}{12}$$

$P(A|E_1)$ = Probability that the defective bottle was manufactured in plant A = 0.006

$P(A|E_2)$ = Probability that the defective bottle was manufactured in plant B = 0.008

$P(A|E_3)$ = Probability that the defective bottle was manufactured in plant C = 0.01

The probability that a randomly chosen bottle is from plant C, given that it is defective, is $P(E_3|A)$.

By using Bayes' theorem, we obtain

$$P(E_3|A) = \frac{P(E_3)\cdot P(A|E_3)}{P(E_1)\cdot P(A|E_1)+P(E_2)\cdot P(A|E_2)+P(E_3)\cdot P(A|E_3)}$$

$$= \frac{\frac{5}{12}\times 0.01}{\frac{1}{4}\times 0.006+\frac{1}{3}\times 0.008+\frac{5}{12}\times 0.01}$$

$$= 0.5$$

Thus, the probability that the defective bottle was manufactured in plant C is 0.5.