Probability

- If E and F are two events associated with the sample space of a random experiment, then the conditional probability of event E, given that F has already occurred, is denoted by P(E/F) and is given by the formula:

  $$P(E/F) = \frac{P(E \cap F)}{P(F)}, \text{ where } P(F) \neq 0$$

  Example:
  A die is rolled twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 3 has appeared at-least once?

  Solution:
  Let E: Event of getting the sum as 7 and F: Event of appearing 3 at-least once

  Then $, E = \{(1,6),(2,5)(3,4),(4,3),(5,2),(6,1)\}$ and

  $F = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3),(3,1),(3,2),(3,4),(3,5),(3,6)\}$

  $\therefore E \cap F = \{(3,4),(4,3)\}$

  $n(E) = 6, n(F) = 11$ and $n(E \cap F) = 2$

  $$P\left(\frac{F}{E}\right) = \frac{P(E \cap F)}{P(E)} = \frac{n(E \cap F)}{n(E)} = \frac{2}{6} = \frac{1}{3}$$

- If E and F are two events of a sample space S of an experiment, then the following are the properties of conditional probability:
  - $0 \leq P(E/F) \leq 1$
  - $P(F/F) = 1$
  - $P(S/F) = 1$
  - $P(E'/F) = 1 - P(E/F)$
  - If A and B are two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then
    - $P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$
    - $P((A \cup B)/F) = P(A/F) + P(B/F)$, if the events A and B are disjoint.

- **Multiplication theorem of probability:** If E, F, and G are events of a sample space S of an experiment, then
  - $P(E \cap F) = P(E). P(F/E)$, if $P(E) \neq 0$
  - $P(E \cap F) = P(F). P(E/F)$, if $P(F) \neq 0$
  - $P(E \cap F \cap G) = P(E). P(F/E). P(G/(E \cap F)) = P(E). P(F/E). P(G/EF)$

- Two events E and F are said to be independent events, if the probability of occurrence of one of them is not affected by the occurrence of the other.

- If E and F are two independent events, then
  - $P(F/E) = P(F)$, provided $P(E) \neq 0$
  - $P(E/F) = P(E)$, provided $P(F) \neq 0$
- If three events A, B, and C are independent events, then
  $P(A \cap B \cap C) = P(A).P(B).P(C)$

- If the events E and F are independent events, then
  - E' and F are independent
  - E' and F' are independent
- A set of events $E_1, E_2, \ldots E_n$ is said to represent a partition of the sample space S, if
  - $E_i \cap E_j = \varnothing$, $i \neq j$, $i, j = 1, 2, 3, \ldots n$
  - $E_1 \cup E_2 \cup \ldots \cup E_n = S$
  - $P(E_i) > 0, \forall\, i = 1, 2, 3, \ldots n$

- **Bayes' Theorem:** If $E_1, E_2, \ldots E_n$ are n non-empty events, which constitute a partition of sample space S, then

$$P(E_i / A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)}, i = 1, 2, 3, ..n$$

Example:
There are three urns. First urn contains 3 white and 2 red balls, second urn contains 2 white and 3 red balls, and third urn contains 4 white and 1 red balls. A white ball is drawn at random. Find the probability that the white ball is drawn from the third urn?

Solution:
Let E_1, E_2 and E_3 be the events of choosing the first second and third urn respectively.

Then, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Let A be the event that a white ball is drawn.

Then, $P\left(\frac{A}{E_1}\right) = \frac{3}{5}, P\left(\frac{A}{E_2}\right) = \frac{2}{5}$ and $P\left(\frac{A}{E_3}\right) = \frac{4}{5}$

By the theorem of total probability,

$$P(A) = P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right) + P(E_3)\times P\left(\frac{A}{E_3}\right)$$

$$= \frac{1}{3}\times\frac{3}{5} + \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{4}{5}$$

$$= \frac{3}{5}$$

By Bayes' theorem,
probability of getting the ball from third urn given that it is white

$$= P\left(\frac{E_3}{A}\right) = \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(A)} = \frac{\frac{1}{3}\times\frac{4}{5}}{\frac{3}{5}} = \frac{4}{9}$$

- A random variable is a real-valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers:

X:	x_1	x_2	…	x_n
P(X):	p_1	p_2	…	p_n

Where, $P_i > 0 = \sum_{i=1}^{n} p_i = 1, i = 1, 2, ..n$

Here, the real numbers $x_1, x_2, \ldots, x_n$ are the possible values of the random variable X and p_i ($i = 1, 2, \ldots, n$) is the probability of the random variable X taking the value of x_i i.e., $P(X = x_i) = p_i$

- **Mean/expectation of a random variable:** Let X be a random variable whose possible values $x_1, x_2, x_3 \ldots x_n$ occur with probabilities $p_1, p_2, p_3 \ldots p_n$ respectively. The mean of X (denoted by m) or the expectation of X (denoted by E(X)) is the number $\sum_{i=1}^{n} x_i p_i$.

  That is, $\mathrm{E}(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + \ldots x_n p_n$

- **Variance of a random variable:** Let X be a random variable whose possible values $x_1, x_2 \ldots x_n$ occur with probabilities $p(x_1), p(x_2) \ldots p(x_n)$ respectively. Let $m = \mathrm{E}(X)$ be the mean of X. The variance of X denoted by Var (X) or ${\sigma_x}^2$ is calculated by any of the following formulae:
  - ${\sigma_x}^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$
  - ${\sigma_x}^2 = \mathrm{E}(X - \mu)^2$
  - ${\sigma_x}^2 = \sum_{i=1}^{n} x_i^2 p(x_i) - \left[\sum_{i=1}^{n} x_i p(x_i)\right]^2$
  - ${\sigma_x}^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ where $[\mathrm{E}(X)]^2 \sum_{i=1}^{n} x_i^2 p(x_i)$

  It is advisable to students to use the fourth formula.

- **Binomial distribution:** For binomial distribution B(n, p), the probability of x successes is denoted by P($X = x$) or P(X) and is given by
  $P(X = x) = {}^{n}C_x q^{n-x} p^x, x = 0, 1, 2, ..n, q = 1 - p$

  Here, P(X) is called the probability function of the binomial distribution.

Example:

An unbiased coin is tossed 5 times. Find the probability of getting atleast 4 heads.

Solution:

Let the random variable X denotes the number of heads.

Here, $n = 5$ and P (getting a head) $= \frac{1}{2}$

$\therefore p = \frac{1}{2}$ and $q = 1 - \frac{1}{2} = \frac{1}{2}$

$P(X = r) = {}^{n}C_r p^r q^{n-r} = {}^{5}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^{5}C_r \left(\frac{1}{2}\right)^5$

P (getting at-least 4 heads)

$= \mathrm{P}(X \geq 4)$

$= \mathrm{P}(X = 4) + \mathrm{P}(X = 5)$

$= {}^{5}\mathrm{C}_{4}\left(\frac{1}{2}\right)^{5} + {}^{5}\mathrm{C}_{4}\left(\frac{1}{2}\right)^{5}$

$= (5 + 1)\left(\frac{1}{2}\right)^{5}$

$= 6\times\frac{1}{32}$

$= \frac{3}{16}$