Three Dimensional Geometry

- **Direction cosines (d.c.'s) of a line:**
  - d.c.'s of a line are the cosines of angles made by the line with the positive direction of the coordinate axes.
  - If l, m, and n are the d.c.'s of a line, then $l^2 + m^2 + n^2 = 1$
  - d.c.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$, where $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

- **Direction ratios (d.r.'s) of a line:**
  - d.r.'s of a line are the numbers which are proportional to the d.c.'s of the line.
  - d.r.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are given by $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$ or $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

- If a, b, and c are the d.r.'s of a line and l, m, and n are its d.c.'s, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \text{ and } n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of a line through a given point and parallel to a given vector:**
  - **Vector form:** Equation of a line that passes through the given point whose position vector is $\vec{a}$ and which is parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.
  - **Cartesian form:**
    - Equation of a line that passes through a point (x_1, y_1, z_1) having d.r.'s as a, b, c is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
    - Equation of a line that passes through a point (x_1, y_1, z_1) having d.c.'s as l, m, n is given by $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

- **Equation of a line passing through two given points:**
  - **Vector form:** Equation of a line passing through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\lambda \in$ **R**
  - **Cartesian form:** Equation of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by, $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Co-planarity of two lines**
  - **Vector form:** Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are co-planar, if $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0$

- **Cartesian form:** Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are co-planar, if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

- **Angle between two Non-skew lines:**
  - **Cartesian form:**
    - If l_1, m_1, n_1, and l_2, m_2, n_2 are the d.c.'s of two lines and θ is the acute angle between them, then $\cos\theta=|l_1l_2+m_1m_2+n_1n_2|$
    - If a_1, b_1, c_1 and a_2, b_2, c_2 are the d.r.'s of two lines and θ is the acute angle between them, then $\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|$
  - **Vector form:** If θ is the acute angle between the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_1}$, then $\cos\theta=\left|\frac{b_1.b_2}{|b_1||b_2|}\right|$

- Two lines with d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are
  - perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$
  - parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

- Two lines in space are said to be skew lines, if they are neither parallel nor intersecting. They lie in different planes.
- Angle between two skew lines is the angle between two intersecting lines drawn from any point (preferably from the origin) parallel to each of the skew lines.
- **Shortest Distance between two skew lines:** The shortest distance is the line segment perpendicular to both the lines.
  - **Vector form:** Distance between two skew lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ is given by, $d=\left|\frac{\left(\vec{b_1}\times\vec{b_2}\right).\left(\vec{a_2}-\vec{a_1}\right)}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$
  - **Cartesian form:** The shortest distance between two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by,

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}}\right|$$

- The shortest distance between two parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is given by, $d = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|$

- **Equation of a plane in normal form:**
  - **Vector form:** Equation of a plane which is at a distance of *d* from the origin, and the unit vector $\hat{n}$ normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$, where $\vec{r}$ is the position vector of a point in the plane

  - **Cartesian form:** Equation of a plane which is at a distance *d* from the origin and the d.c.'s of the normal to the plane as *l*, *m*, *n* is $lx + my + nz = d$

- **Equation of a plane perpendicular to a given vector and passing through a given point:**
  - **Vector form:** Equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to a given line whose d.r.'s are A, B, C is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non-collinear points:**
  - **Cartesian form:** Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
  - **Vector form:** Equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$, where $\vec{r}$ is the position vector of a point in the plane

- **Planes passing through the intersection of two planes:**
  - **Vector form:** Equation of the plane passing through intersection of two planes $\vec{r} \cdot \vec{n_1} = \vec{d_1}$ and $\vec{r} \cdot \vec{n_2} = \vec{d_2}$ is given by, $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$, where λ is a non-zero constant

  - **Cartesian form:** Equation of a plane passing through the intersection of two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, is given by, $(A_1x + B_1y + C_1z + D_1) + \lambda (A_2x + B_2y + C_2z + D_2) = 0$, where λ is a non-zero constant

- **Angle between two planes:** The angle between two planes is defined as the angle between their normals.
  - **Vector form:** If θ is the angle between the two planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then $\cos\theta=\left|\dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$

    Note that if two planes are perpendicular to each other, then $\vec{n_1}\cdot\vec{n_2}=0$; and if they are parallel to each other, then $\vec{n_1}$ is parallel to $\vec{n_2}$.
  - **Cartesian form:** If θ is the angle between the two planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$, then $\cos\theta=\left|\dfrac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$

    Note that if two planes are perpendicular to each other, then $A_1A_2+B_1B_2+C_1C_2=0$; and if they are parallel to each other, then $\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$

- **Distance of a point from a plane:**
  - **Vector form:** The distance of a point, whose position vector is $\vec{a}$, from the plane $\vec{r}\cdot\hat{n}=d$ is $|d-\vec{a}\cdot\hat{n}|$.

    Note:

    - If the equation of the plane is in the form of $\vec{r}\cdot\vec{N}=d$, where $\vec{N}$ is the normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a}\cdot\vec{N}-d|}{|\vec{N}|}$.
    - Length of the perpendicular from origin to the plane $\vec{r}\cdot\vec{N}=d$ is $\dfrac{|d|}{|\vec{N}|}$.
  - **Cartesian form:** The distance from a point (x_1, y_1, z_1) to the plane $Ax+By+Cz+D=0$ is $\left|\dfrac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$.

- **Angle between a line and a plane:** The angle Φ between a line $\vec{r}=\vec{a}+\lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n}=d$ is the complement of the angle between the line and the normal to the plane and is given by $\Phi=\sin^{-1}\left|\dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$.