Integrals

- Integration is the inverse process of differentiation. If $\frac{d}{dx} f(x) = g(x)$, then we can write $\int g(x)\, dx = f(x) + C$. This is called the general or the indefinite integral and C is called the constant of integration.
- Some standard indefinite integrals are given as follows:
  - $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
  - $\int dx = x + C$
  - $\int \sin x dx = -\cos x + C$
  - $\int \cos x dx = \sin x + C$
  - $\int \sec^2 x dx = \tan x + C$
  - $\int \text{cosec}^2 x dx = -\cot x + C$
  - $\int \sec x \tan x dx = \sec x + C$
  - $\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ or $-\cos^{-1} x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \tan^{-1} x + C$ or $-\cot^{-1} x + C$
  - $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ or $-\text{cosec}^{-1} x + C$
  - $\int e^x dx = e^x + C$
  - $\int a^x dx = \frac{a^x}{\log a} + C$
  - $\int \frac{1}{x} dx = \log |x| + C$
  - $\int e^{ax} dx = \frac{e^{ax}}{a} + C$
- Properties of indefinite integrals:
  - $\frac{d}{dx} \int f(x) dx = f(x)$ and $\int f'(x) dx = f(x) + C$
  - If the derivative of two indefinite integrals is the same, then they belong to same family of curves and hence they are equivalent.
  - $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
  - $\int k f(x) dx = k \int f(x) dx$, where k is any constant

- There are three important methods of integration, namely, integration by substitution, integration using partial fractions, and integration by parts.
- **Integration by substitution:** A change in the variable of integration often reduces an integral to one of the fundamental integrals, which can be easily found out. The method in which we change the variable to some other variable is called the method of substitution.
- Using substitution method of integration, we obtain the following standard

  integrals:
  - $\int \tan x dx = -\log |\cos x| + C$ or $\log |\sec x| + C$
  - $\int \cot x dx = \log |\sin x| + C$
  - $\int \sec x dx = \log |(\sec x + \tan x)| + C$
  - $\int \text{cosec}\, x dx = \log |\text{cosec}\, x - \cot x)| + C$

- Integration by partial fractions: The following table shows how a function of the form $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$ and degree of $Q(x)$ is greater than the degree of $P(x)$, is broken by the concept of partial fractions. After doing this, we find the integration of the given function by integrating the right hand side $i.e., partial fractional form$.

Function	Form of partial fraction
$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

$\frac{px^2+qx+r}{(x-a)\left(x^2+bx+c\right)}$, where x^2+bx+c cannot be factorised	$\frac{A}{x-a}+\frac{Bx+c}{x^2+bx+c}$

Here, A, B, C are constants that are to be determined.

Example: Integrate the function $\frac{8x^2-17x+11}{(2x-3)\left(2x^2-4x+5\right)}$.

Solution:

Let $\frac{8x^2-17x+11}{(2x-3)\left(2x^2-4x+5\right)}=\frac{A}{2x-3}+\frac{Bx+C}{2x^2-4x+5}$ ...(1)

where A, B and C are constants.

$$\frac{8x^2-17x+11}{(2x-3)\left(2x^2-4x+5\right)}=\frac{A\left(2x^2-4x+5\right)+(Bx+C)(2x-3)}{(2x-3)\left(2x^2-4x+5\right)}$$

$$=\frac{2(A+B)x^2+(-4A-3B+2C)x+(5A-3C)}{(2x-3)\left(2x^2-4x+5\right)}$$

Comparing L.H.S. and R.H.S. of the above equation, we obtain

$2A+B=8 \Rightarrow A+B=4$ … 2

$-4A-3B+2C=-17$ … 3

and, $5A-3C=11$ … 4

Solving equations 2, 3, and 4, we obtain

A = 1, B = 3, and C = –2

Substituting these values in equation 1, we obtain

$$\int \frac{1}{2x-3}+\frac{3x-2}{2x^2-4x+5}\,dx=\int \frac{1}{2x-3}+\frac{3x-3+1}{2x^2-4x+5}\,dx=\int \frac{1}{2x-3}+\frac{3x-3}{2x^2-4x+5}+\int \frac{1}{2x^2-4x+5}\,dx$$

$I=I_1+I_2+I_3$

$I_1=\int \frac{1}{2x-3}\,dx I_2=\int \frac{3x-3}{2x^2-4x+5}\,dx I_3=\int \frac{1}{2x^2-4x+5}\,dx$

On solving I_1, we get $I_1=\int \frac{1}{2x-3}\,dx = \frac{1}{2}\log|2x-3|+c_1$

On solving I_2, we get

$$I_2=\int \frac{3x-3}{2x^2-4x+5}\,dx let\, 2x^2-4x+5 = t4\left(x-1\right)dx=dt therefore\, I_2=\frac{3}{4}\log|t|+c_2 = \frac{3}{4}log|2x^2-4x+5|+c_2$$

On solving I_3,

$$I_3=\int \frac{1}{2x^2-4x+5}\,dx=\int \frac{1}{x^2-2x+\frac{5}{2}}\,dx = \int \frac{1}{\left(x-1\right)^2+\left(\sqrt{\frac{3}{2}}\right)^2}\,dx=\sqrt{\frac{2}{3}}\tan^{-1}\sqrt{\frac{2}{3}}\left(x-1\right)+c_3$$

$$So,\ \int \frac{8x^2-17+11}{(2x-3)(2x^2-4x+5)}\,dx = \frac{1}{2}log|2x-3| + \frac{3}{4}log|2x^2-4x+5|+\sqrt{\frac{2}{3}}tan^{-1}\sqrt{\frac{2}{3}}\left(x-1\right)+C$$

Integration by parts: For given functions $f(x)$ and $g(x)$, $\int f(x)\cdot g(x)\,dx=f(x)\int g(x)\,dx-\int[f'(x)\cdot \int g(x)\,dx]\,dx$

In other words, the integral of the product of two functions is equal to first function × integral of the second function – integral of {differential of the first function × integral of the second function}.

Here, the functions f and g have to be taken in proper order with respect to the ILATE rule, where I, L, A, T, and E respectively represent inverse, logarithm, arithmetic, trigonometric, and exponential function.

Example: Evaluate $\int x^2\sin^{-1}x\,dx$

Solution:

Integrating by parts, taking $\sin^{-1}x$ as the first function, we get

$$\sin^{-1}x\left(\frac{x^3}{3}\right)-\int \frac{1}{\sqrt{1-x^2}}\,\frac{x^3}{3}\,dx$$

$$=\frac{x^3}{3}\sin^{-1}x-\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$Let\, I ==\frac{x^3}{3}\sin^{-1}x-\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx where\, I_1=\frac{x^3}{3}\,sin^{-1}x,\, I_2=\frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx Therefore,\, I = I_1+I_2$$

$$let\, 1-x^2=t1-t=x^2-dt=2xdx$$

Putting these values in the equation we get,

$$I_2=-\frac{1}{3}\int \frac{-(1-t)}{\sqrt{t}}\,\frac{dt}{2}=\frac{1}{3}\times\frac{1}{2}\int \frac{1}{\sqrt{t}}-\frac{t}{\sqrt{t}}\,dt=\frac{1}{6}\left[2t^{\frac{1}{2}}-\frac{2}{3}t^{\frac{3}{2}}\right]$$

$$I = \frac{x^3}{3}\sin^{-1}x + \frac{1}{3}\sqrt{1-x^2}-\frac{1}{9}\left(1-x^2\right)^{\frac{3}{2}}+C$$

- The definite integral $\int_a^b f(x)\,dx$ can be expressed as the sum of limits as

$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\ldots+f(a+(n-1)h)]$, where $h=\frac{b-a}{n}\to 0$ as $n\to\infty$

- First fundamental theorem of integral calculus: Let *f* be a continuous function on the closed interval [*a*, *b*] and let *A* (*x*) be the area function. Then, $A'(x)=f(x)\ \forall x\in[a,b]$
- Second fundamental theorem of integral calculus: Let *f* be a continuous function on the closed interval [*a*, *b*] and let *F* be an anti-derivative of *f*. Then,

  $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b)-F(a)$

  Example 2: Find: $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1-x^2}}\,dx$

  Solution:
  $\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}x=F(x)$
  By second fundamental theorem, we have
  $\therefore \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1-x^2}}\,dx=\left[\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$

  $=\frac{\pi}{3}-\frac{\pi}{6}$

  $=\frac{\pi}{6}$

- Definite integral: A definite integral is denoted by $\int_a^b f(x)\,dx$, where *a* is the lower limit and *b* is the upper limit of the integral. If $\int f(x)\,dx = F(x)+C$, then

  $\int_a^b f(x)\,dx = F(b)-F(a)$

- The definite integral $\int_a^b f(x)\,dx$ represents the area function A(*x*) since $\int_a^b f(x)\,dx$ is the area bounded by the curve $y = f(x)$, $x \in [a, b]$, the *x*-axis, and the ordinates $x = a$ and $x = b$

- The steps for evaluating $\int_a^b f(x)\,dx$ by substitution method can be listed as:
- **Step I:** Considering the integral without limits, substitute $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form and the limits of integral are accordingly changed.
  Step 2: Integrate the new integrand with respect to the new variable, and then find the difference of the values at the obtained upper and lower limits.

  Example:

  Evaluate: $\int_1^2\frac{3x^2}{1+x^3}dx$

  Solution:

  Put $1 + x^3 = t$

  Then, $3x^2dx = dt$

  When $x = 1$, $t = 2$

  $x= 2$, $t = 9$

  $\therefore \int_1^2\frac{3x^2}{4+x^3}dx=\int_2^9\frac{dt}{t}=\left[\log t\right]_2^9=\log 9-\log 2=\log\frac{9}{2}$

  - Some useful properties of definite integrals are as follows:
    - $\int_a^b f(x)dx=\int_a^b f(t)dt$
    - $\int_a^b f(x)dx=-\int_a^b f(x)dx \cdot \text{In particular, } \int_a^a f(x)dx=0$

- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$
- $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, \text{if } & f(2a-x) = -f(x) \end{cases}$
- $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f \text{ is an even function i.e., if } f(-x) = f(x) \\ 0, \text{if } & f \text{ is an odd function i.e., if } f(-x) = -f(x) \end{cases}$

Example 3: Evaluate: $\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx$

Solution:

Let *I* = $\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx$ … 1

Using the property of definite integrals, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{\sin^6 x}{\sin^4 x - \cos^4 x} dx I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{\cos^6 x}{\cos^4 x - \sin^4 x} dx \text{Adding the above 2 equations we get}, 2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{sin^6 x - cos^6 x}{sin^4 x - cos^4 x} dx = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{(sin^2 x)^3 - (cos^2 x)^3}{(sin^2 x - cos^2 x)(sin^2 x + cos^2 x)} dx$

$2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{(sin^2 x - cos^2 x)(sin^4 x + cos^4 x + \sin^2 x\, cos^2 x)}{(sin^2 x - cos^2 x)(sin^2 x + cos^2 x)} dx 2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left(1 - sin^2 x\, cos^2 x\right) dx 2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left(1 - \frac{sin^2 2x}{4}\right) dx 2I = \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} 1 - \frac{\left(\frac{1 - cos^4 x}{2}\right)^2}{4}$

$2I = \frac{1}{16} \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \left[\frac{29}{2} + 2\cos 4x - \frac{\cos 8x}{2}\right] dx I = \frac{1}{32} \left[\frac{29x}{2} + \frac{\sin 4x}{2} - \frac{\sin 8x}{16}\right]_{\frac{\pi}{2}}^{\frac{5\pi}{2}} = \frac{1}{32} \left[29\pi + \frac{\sin 10\pi}{2} - \frac{\sin 20\pi}{16} - \frac{\sin 2\pi}{2} + \frac{\sin 4\pi}{16}\right] = \frac{29\pi}{32}$