Derivability

Exercise - 1 (Objective Questions)

Part : (A) Only one correct option

1. If $f(x) = \begin{cases} e^{-(1/x^2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$ then the value of $f'(0)$ is:

(A) e (B) 1 (C) 0 (D) undefined

2. Given $f(x) = \begin{cases} x^2 e^{2(x-1)} & \text{for } 0 \le x \le 1 \\ a\,\text{sgn}(x+1)\cos(2x-2) + bx^2 & \text{for } 1 < x \le 2 \end{cases}$ f(x) is differentiable at x = 1 provided:

(A) a = – 1, b = 2 (B) a = 1, b = – 2 (C) a = – 3, b = 4 (D) a = 3, b = – 4

3. If $f(x) = p\,|\sin x| + q.e^{|x|} + r\,|x|^3$ and f(x) is differentiable at x = 0, then

(A) p = q = r = 0 (B) p = 0, q = 0, $r \in R$
(C) q = 0, r = 0, $p \in R$ (D) p + q = 0, $r \in R$

4. Let f(x) = sin x, g(x) = [x + 1] and g(f(x)) = h(x), where [.] is the greatest integer function. Then $h'\left(\frac{\pi}{2}\right)$ is

(A) nonexistent (B) 1 (C) – 1 (D) none of these

5. A function $f(x) = x[1 + (1/3)\sin(\ln x^2)]$, $x \neq 0$. [x] denotes the greatest integer less than or equal to x and f(0) = 0. Then the function:

(A) is continuous at x = 0 (B) is monotonic
(C) is derivable at x = 0 (D) can not be defined for x < –1

6. The number of points at which the function f(x) = max. {a – x, a + x, b}, $-\infty < x < \infty$, 0 < a < b cannot be differentiable is:

(A) 1 (B) 2 (C) 3 (D) none

7. For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$$f(x) = \begin{cases} x & , \quad x \le 1 \\ ax^2 + bx + c & , \quad \text{otherwise} \end{cases}$$ is differentiable for all real x?

(A) $\{(a, 1–2a, a) \mid a \in R, a \neq 0\}$ (B) $\{(a, 1–2a, c) \mid a, c \in R, a \neq 0\}$
(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$ (D) $\{(a, 1–2a, 0) \mid a \in R, a \neq 0\}$

8. The functions defined by f(x) = max {x^2, $(x – 1)^2$, 2x (1 – x)}, $0 \le x \le 1$

(A) is differentiable for all x
(B) is differentiable for all x except at one point
(C) is differentiable for all x except at two points
(D) is not differentiable at more than two points.

9. Consider $f(x) = \left[\frac{2(\sin x - \sin^3 x) + |\sin x - \sin^3 x|}{2(\sin x - \sin^3 x) - |\sin x - \sin^3 x|}\right]$, $x \neq \frac{\pi}{2}$ for $x \in (0, \pi)$

$f(\pi/2) = 3$ where [ ] denotes the greatest integer function then,
(A) f is continuous & differentiable at $x = \pi/2$
(B) f is continuous but not differentiable at $x = \pi/2$
(C) f is neither continuous nor differentiable at $x = \pi/2$
(D) none of these

10. Given $f(x) = \begin{cases} \log_a\left(a\left|[x]+[-x]\right|\right)^x \left(\dfrac{a^{\left(\frac{2}{\frac{[x]+[-x]}{|x|}}\right)}-5}{3+a^{\frac{1}{|x|}}}\right) & \text{for } |x| \neq 0;\ a > 1 \\ 0 & \text{for } x = 0 \end{cases}$

where [ ] represents the integral part function, then:
(A) f is continuous but not differentiable at x = 0
(B) f is continuous & differentiable at x = 0
(C) the differentiability of 'f' at x = 0 depends on the value of a
(D) f is continuous & differentiable at x = 0 and for a = e only.

11. Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) \text{ for } 0 \le t \le x\} & \text{for } 0 \le x \le 1 \\ 3 - x + x^2 & \text{for } 1 < x \le 2 \end{cases}$ then:

(A) g(x) is continuous & derivable at x = 1
(B) g(x) is continuous but not derivable at x = 1
(C) g(x) is neither continuous nor derivable at x = 1
(D) g(x) is derivable but not continuous at x = 1

12. Let f (x) be defined in [– 2 , 2] by $f(x) = \begin{cases} \max\left(\sqrt{4-x^2},\ \sqrt{1+x^2}\right) & , -2 \le x \le 0 \\ \min\left(\sqrt{4-x^2},\ \sqrt{1+x^2}\right) & , 0 < x \le 2 \end{cases}$

then f (x) :
(A) is continuous at all points
(B) is not continuous at more than one point .
(C) is not differentiable only at one point
(D) is not differentiable at more than one point

13. Suppose that f is a differentiable function with the property that $f(x + y) = f(x) + f(y) + xy$ and

$\lim\limits_{h\to 0} \frac{1}{h} f(h) = 3$ then

(A) f is a linear function
(B) $f(x) = 3x + x^2$
(C) $f(x) = 3x + \frac{x^2}{2}$
(D) none of these

14. Let $f(x) = x - x^2$ and $g(x) \begin{cases} \max f(t), 0 \le t \le x, 0 \le x \le 1 \\ \sin \pi x \quad , x > 1 \end{cases}$

Then in the interval $[0, \infty]$
(A) g(x) is everywhere continuous except at two points
(B) g(x) is everywhere differentiable except at two points
(C) g(x) is everywhere differentiable except at x = 1

(D) none of these

15. If f: R $\to$ R be a differentiable function, such that $f(x + 2y) = f(x) + f(2y) + 4xy$ $\forall$ x, y $\in$ R. then

(A) $f'(1) = f'(0) + 1$ (B) $f'(1) = f'(0) - 1$ (C) $f'(0) = f'(1) + 2$ (D) $f'(0) = f'(1) - 2$

16. Let $f(x + y) = f(x)\, f(y)$ for all x and y. Suppose that $f(3) = 3$ and $f'(0) = 11$ then $f'(3)$ is given by

(A) 22 (B) 44 (C) 28 (D) none of these

17. Let $f''(x)$ be continuous x = 0 and $f''(0) = 4$ the value of $\lim\limits_{x\to 0} \dfrac{2f(x) - 3f(2x) + f(4x)}{x^2}$ is

(A) 11 (B) 2 (C) 12 (D) none of these

18. If f (x) is differentiable everywhere, then:

(A) $|f|$ is differentiable everywhere (B) $|f|^2$ is differentiable everywhere

(C) $f\,|f|$ is not differentiable at some point (D) $f + |f|$ is differentiable everywhere

19. Let f: R $\to$ R be any function and $g(x) = \dfrac{1}{f(x)}$. Then g is

(A) onto if f is onto (B) one-one if f is one-one

(C) continuous if f is continuous (D) differentiable if f is differentiable

20. Let $f(x) = [n + p \sin x]$, $x \in (0, p)$, $n \in Z$, p is a prime number and [x] = then greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is

(A) p (B) p – 1 (C) 2p + 1 (D) 2p – 1

Part : (B) May have more than one options correct

21. If $f(x) = \sum\limits_{k=0}^{n} a_k\, |x|^k$, where a_i's are real constants, then f(x) is

(A) continuous at x = 0 for all a_i (B) differentiable at x = 0 for all $a_i \in R$

(C) differentiable at x = 0 for all $a_{2k+1} = 0$ (D) none of these

Exercise - 2 (Subjective Questions)

1. Let $f(x) = \begin{cases} \dfrac{x\left(e^{1/x} - e^{-1/x}\right)}{e^{1/x} + e^{-1/x}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$. Check differentiability of f at x = 0.

2. Examine the differentiability of $f(x) = \sqrt{1 - e^{-x^2}}$ at x = 0.

3. A function is defined as follows: $f(x) = \begin{cases} x^3 & ; \ x^2 < 1 \\ x & ; \ x^2 \geq 1 \end{cases}$. Draw the graph of the function & discuss continuity & differentiability at x = 1.

4. Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{x-1} - |x| & , \text{if } x \neq 1 \\ -1 & , \text{if } x = 1 \end{cases}$ be a real valued function. Find the points where f (x) is not differentiable.

5. Show that the function $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right) & ; \quad x > 0 \\ 0 & ; \quad x = 0 \end{cases}$ is,

(i) differentiable at x = 0, if m > 1.

(ii) continuous but not differentiable at x = 0, if 0 < m < 1.

(iii) neither continuous nor differentiable, if $m \le 0$.

6. Draw a graph of the function, y = [x] + |1 – x| $-1 \le x \le 3$. Determine the points, if any, where this function is not differentiable, where [·] denotes the greatest integer function.

7. If f′(2) = 4 then Evaluate $\lim\limits_{x\to 0} \dfrac{f(1+\cos x) - f(2)}{\tan^2 x}$.

8. Let a function f : R → R be given by f(x + y) = f(x) f(y) for all x, y ∈ R and f(x) ≠ 0 for any x ∈ R. If the function f (x) is differentiable at x = 0, show that f′(x) = f′(0) f(x) for all x ∈ R. Also, determine f(x).

9. Discuss the continuity & differentiability of the function f (x) = |sin x| + sin |x|, x ∈ R. Draw a rough sketch of the graph of f (x) . Also comment on periodicity of function f(x).

10. Given $f(x) = \cos^{-1}\left(\operatorname{sgn}\left(\dfrac{2[x]}{3x - [x]}\right)\right)$ where sgn ( ) denotes the signum function & [ ] denotes the greatest integer function. Discuss the continuity & differentiability of f (x) at x = ± 1.

11. If f (x) = x^2 – 2 |x| then test the derivability of g (x) in the interval [– 2, 3], where

$$g(x) = \begin{cases} \min\{f(t);\ -2 \le t \le x\} & -2 \le x < 0 \\ \max\{f(t); 0 \le t \le x\} & 0 \le x \le 3 \end{cases}$$

12. Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.

$f(x) = x\sin\dfrac{1}{x}\sin\dfrac{1}{x\sin\frac{1}{x}}$ where $x \ne 0$, $x \ne \dfrac{1}{r\pi}$ & f(0) = f (1/rπ) = 0, r = 1, 2, 3,.......

13. Let R be the set of real numbers and f: R→R be such that for all x & y in R $|f(x) - f(y)| \le |x - y|^3$. Prove that f(x) is constant.

14. Let f: R → (– π, π) be a derivable function such that $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$, xy < 1.

If $f(1) = \dfrac{\pi}{2}$ & $\text{Limit}_{x\to 0} \dfrac{f(x)}{x} = 2$, find f(x).

15. The function f is defined by y = f(x). Where x = 2t – | t |, y = t^2 + t | t |, t ∈ R. Draw the graph of f for the interval $-1 \le x \le 1$. Also discuss its continuity & differentiability at x = 0.

16. Discuss the continuity and differentiability of f(x) = [x] + $\{x\}^2$ and also draw its graph. Where [.] and {.} denotes the greatest integer function and fractional part function respectively.

17. If f(x) = –1 + |x – 1|, $-1 \le x \le 3$; g(x) = 2 – |x + 1|, $-2 \le x \le 2$, then calculate (fog) (x) & (gof) (x). Draw their graph. Discuss the continuity of (fog) (x) at x = –1 & the differentiability of (gof) (x) at x = 1.

Answers

Exercise # 1

1. C **2.** A **3.** D **4.** A **5.** A **6.** B **7.** A

8. C **9.** A **10.** B **11.** C **12.** D **13.** C **14.** C

15. D **16.** D **17.** C **18.** B **19.** B **20.** D

21. AC

Exercise # 2

1. not differentiable at $x = 0$ **2.** not diff. at $x = 0$

3. f is continuous but not differentiable at $x = 1$

4. f (x) is differentiable except at $x = 0$

6. f is not derivable at all integral values in $-1 < x \leq 3$

7. -2 **8.** $f(x) = e^{xf'(0)}\ \forall\ x \in R$

9. f (x) is continuous but not differentiable at x = 0, f(x) is not periodic.

10. f is discontinuous at x = 2 and continuous at all other point f is not differentiable at x = 1, 3/2 & 2 and differentiable at all other points.

11. not derivable at x = 0 and 2

12. continuous in $0 \leq x \leq 1$ & not differentiable at $x = 0$

14. $f(x) = 2 \tan^{-1} x$

15. $f(x) = 2x^2$ for $0 \leq x \leq 1$ & $f(x) = 0$ for $-1 \leq x < 0$, f is differentiable & hence continuous at $x = 0$

16. Continuous everywhere but not differentiable at integral points.

17. $(fog)(x) = x+1$ for $-2 \leq x \leq -1$, $= -(x+1)$ for $-1 < x \leq 0$ & $= x-1$ for $0 < x \leq 2$.
(fog)(x) is continuous at $x = -1$,
$(gof)(x) = x+1$ for $-1 \leq x \leq 1$ & $3-x$ for $1 < x \leq 3$. (gof) (x) is not differentiable at $x = 1$