

Class XII

Set-1

Time Allowed: 3 hrs.

M.M: 100 Marks

Section A

Solution1:

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 & 1 & 1 + \cos \theta \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin \theta & 0 \\ 0 & 0 & \cos \theta \end{vmatrix}$$

$$= \sin \theta \cos \theta$$

$$= \frac{\sin 2\theta}{2}$$

We know that $-1 \leq \sin 2\theta \leq 1$.

$$\therefore \text{Maximum value of } \square = \frac{1}{2} \times 1 = \frac{1}{2}$$

Solution2: Given: $(A - I)^3 + (A + I)^3 - 7A$

$$= A^3 - I^3 - 3A^2I + 3AI^2 + A^3 + I^3 + 3A^2I + 3AI^2 - 7A$$

$$= 2A^3 + 6AI^2 - 7A$$

$$= 2A.A^2 + 6AI^2 - 7A$$

$$= 8A - 7A = A$$

Solution3: We have

$$A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix}$$

$$A' = \begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

We know that a matrix is symmetric if $A = A'$.

Thus,

$$A = \begin{vmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 3a \\ 2b & 1 & 3 \\ -2 & 3 & -1 \end{vmatrix}$$

Now,

$$2b=3$$

$$\Rightarrow b = \frac{3}{2}$$

$$\text{Also, } 3a = -2$$

$$\Rightarrow a = \frac{-2}{3}$$

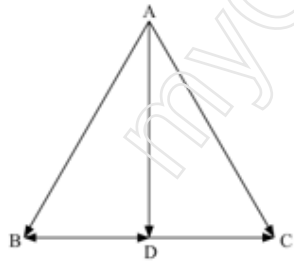
$$\text{Therefore, } a = \frac{-2}{3} \text{ and } b = \frac{3}{2}$$

Solution4: Let A and B be the points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$, respectively.

Also, let R divide AB externally in the ratio 2 : 1.

$$\therefore \text{Position vector of } R = \frac{2 \times (2\vec{a} + \vec{b}) - 1 \times (\vec{a} - 2\vec{b})}{2 - 1} = 3\vec{a} + 4\vec{b}$$

Solution5: In ΔABC ,



Using the triangle law of vector addition, we have

$$\vec{BC} = \vec{AC} - \vec{AB}$$

$$= (3\hat{i} - \hat{j} + 4\hat{k}) - (\hat{j} + \hat{k})$$

$$= 3\hat{i} - 2\hat{j} + 3\hat{k}$$

(Since AD is the median)

In $\triangle ABD$, using the triangle law of vector addition, we have

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$$

$$= (\hat{j} + \hat{k}) + \left(\frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k} \right)$$

$$= \frac{3}{2}\hat{i} + 0\hat{j} + \frac{5}{2}\hat{k}$$

$$\therefore AD = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2} = \frac{1}{2}\sqrt{34}$$

Hence, the length of the median through A is $\frac{1}{2}\sqrt{34}$ units.

Solution6: Given:

$$\text{Normal vector, } \hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

Perpendicular distance, $d = 5$ units

The vector equation of a plane that is at a distance of 5 units from the origin and has its normal vector $\hat{n} = 2\hat{i} - 3\hat{j} + 6\hat{k}$ is as follows:

$$\vec{r} \cdot \hat{n} = d$$

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$$

Section B

$$\text{Solution7: LHS: } \left(\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} \right) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \right)$$

$$= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right)$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left(\frac{A+B}{1-AB} \right) \right]$$

$$= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

OR

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1} \left(\frac{2\cos x}{1 - \cos^2 x} \right)$$

$$= \tan^{-1}(2\operatorname{cosec} x)$$

$$\left[\because 2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right]$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Solution8: Let the monthly incomes of Aryan and Babban be $3x$ and $4x$, respectively.

Suppose their monthly expenditures are $5y$ and $7y$, respectively.

Since each saves Rs 15,000 per month,

Monthly saving of Aryan: $3x - 5y = 15,000$

Monthly saving of Babban: $4x - 7y = 15,000$

The above system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

or,

$$AX = B, \text{ where } A = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 3 & -5 \\ 4 & -7 \end{vmatrix} = -21 - (-20) = -1$$

$$\text{Adj } A = \begin{bmatrix} -7 & -4 \\ 5 & 3 \end{bmatrix}^T = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|} \text{adj } A = -1 \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105000 - 75000 \\ 60000 - 45000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15000 \\ 15000 \end{bmatrix}$$

$$\Rightarrow x = 30,000 \text{ and } y = 15,000$$

Therefore,

Monthly income of Aryan = $3 \times \text{Rs } 30,000 = \text{Rs } 90,000$

Monthly income of Babbar = $4 \times \text{Rs } 30,000 = \text{Rs } 1,20,000$

From this problem, we are encouraged to understand the power of savings. We should save certain part of our monthly income for the future.

Solution9: $x = a \sin 2t (1 + \cos 2t)$

$$y = b \cos 2t (1 - \cos 2t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b - [-2 \sin 2t + 2 \cos 2t \sin 2t \times 2]}{a[2 \cos 2t + 2 \cos 4t]}$$

$$\frac{dy}{dx} = \frac{b}{a} \left[\frac{-2 \sin 2t + 2 \sin 4t}{2 \cos 2t + 2 \cos 4t} \right]$$

$$\left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = \frac{b}{a} \left[\frac{-2+0}{0-2} \right] = \frac{b}{a}$$

$$\text{and } \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{3}} = \frac{b}{a} \left[\frac{-2\sqrt{3}}{-2} \right] = \frac{\sqrt{3}b}{a}$$

OR

$$y = x^x$$

Applying logarithm,

$$\log y = x \log x$$

$$\frac{1}{y} \frac{dy}{dx} = \log x + x \times \frac{1}{x} = 1 + \log x$$

$$\frac{dy}{dx} = X^x [1 + \log x]$$

$$\frac{d^2 y}{dx^2} = \frac{d(x^x)}{dx} (1 + \log x) + x^x \left[\frac{d}{dx} (1 + \log x) \right]$$

$$= x^x (1 + \log x) (1 + \log x) + x^x \left[\frac{1}{x} \right]$$

$$= x^x (1 + \log x)^2 + x^{x-1}$$

$$\frac{d^2 y}{dx^2} - \frac{1}{y} \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} = x^x (1 + \log x)^2 + x^{x-1} - \frac{1}{x^2} (x^x (1 + \log x)^2) - \frac{x^x}{x}$$

$$= x^x (1 + \log x)^2 + x^{x-1} - x^x (1 + \log x)^2 - x^{x-1}$$

$$= 0$$

Hence proved.

Solution10: $f(x) = \begin{cases} \frac{1 - \sin^3 x}{3 \cos^2 x}, & \text{if } x < \frac{\pi}{2} \\ P, & \text{if } x = \frac{\pi}{2} \\ \frac{q(1 - \sin x)}{(\pi - 2x)^2}, & \text{if } x > \frac{\pi}{2} \end{cases}$

for continuity,

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1 - \sin^3 x}{3 \cos^2 x} \right) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \sin x)(1 + \sin^2 x + \sin x)}{3[1 - \sin^2 x]}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 + \sin^2 x + \sin x)}{3(1 + \sin x)} = \frac{1 + 1 + 1}{3(2)} = \frac{1}{2}$$

$$\text{Let } \frac{\pi}{2} - x = \theta \Rightarrow x = \frac{\pi}{2} - \theta$$

$$\lim_{x \rightarrow \frac{\pi}{2}} = \lim_{\theta \rightarrow 0} \frac{1 - \sin\left(\frac{\pi}{2} - \theta\right)}{(2\theta)^2} = \frac{q}{4} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$$

$$= \frac{q}{4} \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \frac{\theta}{2}}{\theta^2} = \frac{q}{2} \lim_{\theta \rightarrow 0} \frac{\sin^2 \frac{\theta}{2}}{4 \times (\theta/2)} = \frac{q}{8}$$

Now,

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{2} = p = \frac{q}{8}$$

$$\Rightarrow P = \frac{1}{2} \text{ and } q = \frac{8}{2} = 4$$

Solution11: Given:

$$\begin{aligned}
 x &= 3 \cos t - \cos^3 t & \text{Slope of the tangent, } \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 \cos t - 3 \sin^2 t \cos t}{-3 \sin t + 3 \cos^2 t \sin t} \\
 y &= 3 \sin t - \sin^3 t
 \end{aligned}$$

$$= \frac{3 \cos t - [\cos^2 t]}{-3 \sin t [\sin^2 t]}$$

$$\frac{dy}{dx} = \frac{-\cos^3 t}{\sin^3 t}$$

$$\therefore \text{Slope of the normal} = \frac{\sin^3 t}{\cos^3 t}$$

The equation of the normal is given by

$$\frac{y - (3 \sin t - \sin^3 t)}{x - (3 \cos t - \cos^3 t)} = \frac{\sin^3 t}{\cos^3 t}$$

$$\Rightarrow y \cos^3 t - 3 \sin t \cos^3 t + \sin^3 t \cos^3 t$$

$$= x \sin^3 t - 3 \cos t \sin^3 t + \sin^3 t \cos^3 t$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3(\sin t \cos t - \cos t \sin^3 t)$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3 \sin t \cos t (\cos^2 t - \sin^2 t)$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = \frac{3}{2} \sin 2t \cos 2t = \frac{3}{4} \sin 4t$$

$$\Rightarrow y \cos^3 t - x \sin^3 t = 3(\sin t \cos t - \cos t \sin^3 t)$$

$$\Rightarrow 4(y \cos^3 t - x \sin^3 t) = 3 \sin 4t$$

Hence proved.

Solution12: $I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - \cos^2 \theta - 4 \sin \theta} d\theta$

$$\Rightarrow I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - (1 - \sin^2 \theta) - 4 \sin \theta} d\theta$$

$$\Rightarrow I = \int \frac{(3 \sin \theta - 2) \cos \theta}{\sin^2 \theta - 4 \sin \theta + 4} d\theta$$

No, let $\sin \theta = t$

$$\Rightarrow \cos \theta d\theta = dt$$

$$\therefore I = \int \frac{(3t-2)dt}{t^2-4t+4}$$

$$\Rightarrow 3t-2 = A \frac{d}{dx}(t^2-4t+4) + B$$

$$\Rightarrow 3t-2 = A(2t-4) + B$$

$$\Rightarrow 3t-2 = (2A)t + B - 4A$$

Comparing the coefficients of the like powers of t, we get

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

and

$$B-4A = -2$$

$$\Rightarrow B - 4 \times \frac{3}{2} = -2$$

$$\Rightarrow B = -2 + 6 = 4$$

Substituting the values of A and B, we get

$$3t-2 = \frac{3}{2}(2t-4) + 4$$

$$\therefore I = \int \frac{(3t-2)dt}{t^2-4t+4}$$

$$= \int \left(\frac{\frac{3}{2}(2t-4) + 4}{t^2-4t+4} \right) dt$$

$$= \frac{3}{2} \int \left(\frac{2t-4}{t^2-4t+4} \right) dt + 4 \int \frac{dt}{t^2-4t+4}$$

$$= \frac{3}{2} I_1 + 4 I_2 \quad \dots\dots(1)$$

Here,

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4} \quad \text{and} \quad I_2 = \int \frac{dt}{t^2-4t+4}$$

Now

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4}$$

$$\text{Let } t^2 - 4t + 4 = p$$

$$\Rightarrow (2t-4)dt = dp$$

$$I_1 = \int \frac{(2t-4)dt}{t^2-4t+4}$$

$$= \int \frac{dp}{p} = \log|p| + C_1$$

$$= \log|t^2 - 4t + 4| + C_1 \quad \dots\dots\dots(2)$$

and

$$I_2 = \int \frac{dt}{t^2-4t+4}$$

$$= \int \frac{dt}{(t-2)^2}$$

$$= \int (t-2)^{-2} dt$$

$$= \frac{(t-2)^{-2+1}}{-2+1} + C_2$$

$$= \frac{-1}{t-2} + C_2 \quad \dots\dots(3)$$

From (1), (2) and (3), we get

$$I = \frac{3}{2} \log|t^2 - 4t + 4| + 4 \times \frac{-1}{t-2} + C_1 + C_2$$

$$= 32 \log |\sin^2 \theta - 4 \sin \theta + 4| + \frac{4}{2-t} + C \quad (\text{where } C = C_1 + C_2)$$

$$= \frac{3}{2} \log |(\sin \theta - 2)^2| + \frac{4}{2 - \sin \theta} + C$$

$$= \frac{3}{2} \times 2 \log |(\sin \theta - 2)| + \frac{4}{2 - \sin \theta} + C$$

$$= 3 \log |2 - \sin \theta| + \frac{4}{2 - \sin \theta} + C$$

OR

$$\text{Let } I = \int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx$$

Integrating by parts, we get

$$\Rightarrow I = \frac{1}{2} \left[e^{2x} \sin\left(\frac{\pi}{4} + x\right) \right]_0^{\pi} - \frac{1}{2} \left\{ \left[\frac{1}{2} e^{2x} \cos\left(\frac{\pi}{4} + x\right) \right]_0^{\pi} + \frac{1}{2} \int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x\right) dx \right\}$$

$$\Rightarrow I = \frac{1}{2} \left[e^{2x} \sin\left(\frac{\pi}{4} + x\right) \right]_0^{\pi} - \frac{1}{4} \left[e^{2x} \cos\left(\frac{\pi}{4} + x\right) \right]_0^{\pi} - \frac{1}{4} I$$

$$\Rightarrow \frac{5}{4} I = \frac{1}{2} \left[e^{2x} \sin\left(\pi + \frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \right] - \frac{1}{4} \left[e^{2\pi} \cos\left(\pi + \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) \right]$$

$$\Rightarrow \frac{5}{4} I = \frac{1}{2} \left[-e^{2\pi} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right] - \frac{1}{4} \left[e^{2\pi} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow I = -\frac{1}{5\sqrt{2}} (e^{2\pi} + 1)$$

Solution13: $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

$$\text{Let : } x^{\frac{3}{2}} = t$$

$$\Rightarrow \frac{3}{2} x^{\frac{1}{2}} dx = dt$$

$$x^{\frac{1}{2}} dx = \frac{2}{3} dt$$

Putting the values in I, we get

$$\begin{aligned}
 I &= \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx \\
 &= \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt
 \end{aligned}$$

Using the following formula of integration, we get

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\therefore \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt = \frac{2}{3} \sin^{-1} \left(\frac{t}{a^{\frac{3}{2}}} \right) + C$$

Again, putting the value of t, we get

$$\begin{aligned}
 \frac{2}{3} \int \frac{1}{\sqrt{a^3 - t^2}} dt &= \frac{2}{3} \sin^{-1} \left(\frac{t}{a^{\frac{3}{2}}} \right) + C \\
 &= \frac{2}{3} \sin^{-1} \left(\frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} \right) + C
 \end{aligned}$$

Here, C is constant of integration.

Solution14: Let: $\int_{-1}^2 |x^3 - x| dx$.

$$f(x) = x^3 - x$$

$$f(x) = x^3 - x = x(x-1)(x+1)$$

The signs of f(x) for the different values are shown in the figure given below:



$$f(x) > 0 \text{ for all } x \in (-1, 0) \cup (1, 2)$$

$$f(x) < 0 \text{ for all } x \in (0, 1)$$

Therefore,

$$|x^3 - x| = \begin{cases} x^3 - x, & x \in (-1, 0) \cup (1, 2) \\ -(x^3 - x), & x \in (0, 1) \end{cases}$$

$$\therefore I = \int_{-1}^2 |x^3 - x| dx$$

$$\begin{aligned}
 &= \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx + \int_1^2 (x^3 - x) dx \\
 &= \int_{-1}^0 |x^3 - x| dx + \int_0^1 |x^3 - x| dx + \int_1^2 |x^3 - x| dx \\
 &= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 \\
 &= -\left(\frac{1}{4} - \frac{1}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) + \left(\frac{16}{4} - \frac{4}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) \\
 &= \frac{3}{4} + (4 - 2) = \frac{11}{4}
 \end{aligned}$$

Solution15: Given:

$$(1 - y^2)(1 + \log x) dx + 2xy dy = 0$$

$$\Rightarrow (1 - y^2)(1 + \log x) dx = -2xy dy$$

$$\Rightarrow \left(\frac{1 - \log x}{2x} \right) dx = - \left(\frac{y}{1 - y^2} \right) dy \quad \dots (1)$$

Let: $1 + \log x = t$

$$\text{and } (1 - y^2) = p \Rightarrow \frac{1}{x} dx = dt \text{ and } -2y dy = dp$$

Therefore, (1) becomes

$$\int \frac{t}{2} dt = \int \frac{1}{2p} dp$$

$$\Rightarrow \frac{t^2}{4} = \frac{\log p}{2} + C \quad \dots (2)$$

Substituting the values of t and p in (2), we get

$$\frac{(1 + \log x^2)}{4} = \frac{\log(1 - y^2)}{2} + C \quad \dots (3)$$

$$\text{At } x=1 \text{ and } y=0, (3) \text{ becomes } C = \frac{1}{4}$$

Substituting the value of C in (3), we get

$$\frac{(1 + \log x^2)}{4} = \frac{\log(1 - y^2)}{2} + \frac{1}{4}$$

$$\Rightarrow (1 + \log x^2) = 2 \log(1 - y^2) + 1$$

Or

$$(\log x)^2 + \log x^2 = \log(1 - y^2)^2$$

It is the required particular solution.

Solution16: Given:

$$(1 + y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$$

$$\text{Let } \tan^{-1}y = t$$

$$\Rightarrow y = \tan t$$

$$\Rightarrow \frac{dy}{dx} = \sec^2 t \frac{dt}{dx}$$

Therefore, the equation becomes

$$(1 + \tan^2 t) + (x - e^t) \sec^2 t \frac{dt}{dx} = 0$$

$$\Rightarrow \sec^2 t + (x - e^t) \left(\sec^2 t \right) \frac{dt}{dx} = 0$$

$$\Rightarrow 1 + (x - e^t) \frac{dt}{dx} = 0$$

$$\Rightarrow (x - e^t) \frac{dt}{dx} = -1$$

$$\Rightarrow x - e^t = \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} + 1 \cdot x = e^t$$

$$IF \int_1 e^t dt = e^t$$

$$\therefore e^t \cdot \left(\frac{dx}{dt} + 1 \cdot x \right) = e^t \cdot e^t$$

$$e \Rightarrow \frac{d}{dt}(xe^t) = e^{2t}$$

Integrating both the sides, we get

$$xe^t = \int e^{2t} dt$$

$$\Rightarrow xe^t = \frac{1}{2}e^{2t} + C \quad \dots(1) \quad \text{Substituting the value of t in (1), we get}$$

$$xe^{\tan^{-1}y} = \frac{1}{2}e^{2\tan^{-1}y} + C1$$

$$\Rightarrow e^{2\tan^{-1}y} = 2xe^{\tan^{-1}y} + C \quad \text{It is the required general solution.}$$

Solution17: It is given that $\vec{a}, \vec{b}, \vec{b} + \vec{c}$ and \vec{c} are coplanar.

Therefore, Scalar triple product = Volume of the parallelepiped = 0

$$(\vec{a} + \vec{b}) \cdot [(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})] = 0$$

$$\Rightarrow (\vec{a} + \vec{b}) \cdot [(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + 0 + (\vec{c} \times \vec{a})] = 0$$

$$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) = 0$$

$$\Rightarrow [abc] + 0 + 0 + 0 + 0 + [bca] = 0$$

$$\Rightarrow 2[abc] = 0$$

$$\Rightarrow [abc] = 0$$

Therefore, the vectors \vec{a}, \vec{b} and \vec{c} are coplanar.

Solution18: The equations of the given lines are

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k}) \quad \dots(1)$$

$$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}) \quad \dots(2)$$

Normal parallel to (1) is $\vec{n}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$.

Normal parallel to (2) is $\vec{n}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$.

The required line is perpendicular to the given lines. So, the normal \vec{n} parallel to the required line is perpendicular to \vec{n}_1 and \vec{n}_2 .

$$\begin{aligned}\therefore \vec{n} &= \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}\end{aligned}$$

Thus, the vector equation of the required line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \gamma(24\hat{i} + 36\hat{j} + 72\hat{k})$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + k(2\hat{i} + 3\hat{j} + 6\hat{k}) \quad (\text{Where } k = 12\gamma)$$

Also, the Cartesian equation of the required line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Solution19: Let E_1, E_2 and E_3 be the events denoting the selection of A, B and C as managers, respectively.

$$P(E_1) = \text{Probability of selection of A} = \frac{1}{7}$$

$$P(E_2) = \text{Probability of selection of B} = \frac{2}{7}$$

$$P(E_3) = \text{Probability of selection of C} = \frac{4}{7}$$

Let A be the event denoting the change not taking place.

$$P\left(\frac{A}{E_1}\right) = \text{Probability that A does not introduce change} = 0.2$$

$$P\left(\frac{A}{E_2}\right) = \text{Probability that B does not introduce change} = 0.5$$

$$P\left(\frac{A}{E_3}\right) = \text{Probability that C does not introduce change} = 0.7$$

$$\therefore \text{required probability} = P\left(\frac{E_3}{A}\right)$$

By Bayes' theorem, we have

$$P\left(\frac{E_3}{A}\right) = \frac{P(E_3)P\left(\frac{A}{E_3}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$

$$\begin{aligned}
 &= \frac{\frac{4}{7} \times 0.7}{\frac{1}{7} \times 0.2 + \frac{2}{7} \times 0.5 + \frac{4}{7} \times 0.7} \\
 &= \frac{2.8}{0.2 + 1 + 2.8} = \frac{2.8}{4} = 0.7
 \end{aligned}$$

OR

Total of 7 on the dice can be obtained in the following ways:

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

$$\text{Probability of getting a total of 7} = \frac{3}{36} = \frac{1}{6}$$

$$\text{Probability of not getting a total of 10} = 1 - \frac{1}{12} = \frac{11}{12}$$

Total of 10 on the dice can be obtained in the following ways:

(4, 6), (6, 4), (5, 5)

$$\text{Probability of getting a total of 10} = \frac{3}{36} = \frac{1}{12}$$

$$\text{Probability of not getting a total of 10} = 1 - \frac{1}{12} = \frac{11}{12}$$

Let E and F be the two events, defined as follows:

E = Getting a total of 7 in a single throw of a dice

F = Getting a total of 10 in a single throw of a dice

$$P(E) = \frac{1}{6}, P(\bar{E}) = \frac{5}{6}, P(F) = \frac{1}{12}, P(\bar{F}) = \frac{11}{12}$$

A wins if he gets a total of 7 in 1st, 3rd or 5th ... throws.

$$\text{Probability of A getting a total of 7 in the 1st throw} = \frac{1}{6}$$

A will get the 3rd throw if he fails in the 1st throw and B fails in the 2nd throw.

$$\text{Probability of A getting a total of 7 in the 3rd throw} = P(\bar{E})P(\bar{F})P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6}$$

Similarly, probability of getting a total of 7 in the 5th throw =

$$P(\bar{E})P(\bar{F})P(\bar{E})P(\bar{F})P(E) = \frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \text{ and so on}$$

$$\begin{aligned}
 \text{Probability of winning of A} &= \frac{1}{6} + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \right) + \left(\frac{5}{6} \times \frac{11}{12} \times \frac{5}{6} \times \frac{11}{12} \times \frac{1}{6} \right) + \dots \\
 &= \frac{\frac{1}{6}}{1 - \frac{5}{6} \times \frac{11}{12}} = \frac{12}{17}
 \end{aligned}$$

$$\therefore \text{Probability of winning of B} = 1 - \text{Probability of winning of A} = 1 - \frac{12}{17} = \frac{5}{17}$$

SECTION C

Solution20: Given: $f(x) = 9x^2 + 6x - 5$

$$\text{Let } y = 9x^2 + 6x - 5$$

$$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$

$$\Rightarrow (3x+1)^2 = y+6$$

$$\Rightarrow 3x+1 = \sqrt{y+6}$$

$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3} \text{ as } x \in N$$

$$\Rightarrow \sqrt{y+6}-1 > 0$$

$$\Rightarrow y+6 > 1$$

$$\Rightarrow y > -5 \text{ and } y \in N$$

So, the function is invertible if the range of the function $f(x)$ is $\{1, 2, 3, \dots\}$.

Therefore, the inverse of the function $f(x)$ is $f^{-1}(y)$, i.e.x.

Now,

$$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

$$f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = 4$$

Solution21:
$$\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} xy + yz + zx - x^2 - y^2 - z^2 & zx - y^2 & xy - z^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & xy - z^2 & yz - x^2 \\ xy + yz + zx - x^2 - y^2 - z^2 & yz - x^2 & zx - y^2 \end{vmatrix}$$

$$\Rightarrow \Delta = (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 1 & xy - z^2 & yz - x^2 \\ 1 & yz - x^2 & zx - y^2 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = (xy + yz + zx - x^2 - y^2 - z^2) \begin{vmatrix} 1 & zx - y^2 & xy - z^2 \\ 0 & (x + y + z)(y - z) & (x + y + z)(z - x) \\ 0 & (x + y + z)(y - x) & (x + y + z)(z - y) \end{vmatrix}$$

$$\Rightarrow \Delta = (x + y + z)^2 (xy + yz + zx - x^2 - y^2 - z^2) [(y - z)(z - y) - (z - x)(y - x) - 0 + 0]$$

$$\Rightarrow \Delta = (x + y + z)^2 (xy + yz + zx - x^2 - y^2 - z^2)^2$$

So, Δ is divisible by $(x + y + z)$.

The quotient when Δ is divisible by $(x + y + z)$ is $(x + y + z)(xy + yz + zx - x^2 - y^2 - z^2)$.

OR

$$A = IA$$

i.e.

$$\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 8 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 - 8R_1$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \rightarrow \frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -12 & -13 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & 0 & -8 \end{bmatrix} A$$

Applying $R_3 \rightarrow R_3 + 12R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ 1 & -4 & 0 \end{bmatrix} A$$

Applying $R_3 \rightarrow -R_3$ and $R_2 \rightarrow R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} A$$

Thus, we have

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix}$$

The given system of equations is

$$8x + 4y + 3z = 19$$

$$2x + y + z = 5$$

$$x + 2y + 2z = 7$$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B$$

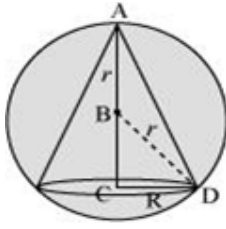
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{13}{3} & \frac{2}{3} \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 19 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 + \frac{10}{3} - \frac{7}{3} \\ 19 - \frac{65}{3} + \frac{14}{3} \\ -19 + 20 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore x = 1, y = 2 \text{ and } z = 1$$

Solution22: A sphere of fixed radius (r) is given.

Let R and h be the radius and the height of the cone, respectively.



The volume (V) of the cone is given by,

$$V = \frac{1}{3} \pi R^2 h$$

Now, from the right triangle BCD , we have:

$$BC = \sqrt{r^2 - R^2}$$

$$\therefore h = r + \sqrt{r^2 - R^2}$$

$$\therefore V = \frac{1}{3} \pi R^2 (r + \sqrt{r^2 - R^2}) = \frac{1}{3} \pi R^2 r + \frac{1}{3} \pi R^2 \sqrt{r^2 - R^2}$$

$$\begin{aligned} \therefore \frac{dV}{dR} &= \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} + \frac{\pi R^2}{3} \cdot \frac{(-2R)}{2\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2}{3} \pi R \sqrt{r^2 - R^2} - \frac{\pi R^3}{3\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2\pi R (r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}} \\ &= \frac{2}{3} \pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}} \end{aligned}$$

$$\text{Now, } \frac{dV}{dR^2} = 0$$

$$\Rightarrow \frac{2\pi R r}{3} = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2 (r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 4r^4 - 4r^2 R^2 = 9R^4 + 4r^4 - 12R^2 r^2$$

$$\Rightarrow 9R^4 - 8r^2 R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\begin{aligned}
 \text{Now, } \frac{d^2V}{dR^2} &= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi Rr^2 - 3\pi R^3)(-6R)}{9(r^2 - R^2)} \cdot \frac{1}{2\sqrt{r^2 - R^2}} \\
 &= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) + (2\pi Rr^2 - 3\pi R^3)(3R)}{9(r^2 - R^2)} \cdot \frac{1}{2\sqrt{r^2 - R^2}}
 \end{aligned}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

\therefore The volume is the maximum when $R^2 = \frac{8r^2}{9}$.

When $R^2 = \frac{8r^2}{9}$, height of the cone = $r + \sqrt{r^2 - \frac{8r^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$.

Hence, it can be seen that the altitude of a right circular cone of maximum volume that

can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Let volume of the sphere be $V_s = \frac{4}{3}\pi r^3$.

$$r = 3\sqrt{\frac{3V_s}{4\pi}}$$

\therefore Volume of cone, $V = \frac{1}{3}\pi R^2 h$

$$\Rightarrow R = \frac{2\sqrt{2}}{3}r$$

$$V = \frac{1}{3}\pi \left(\frac{2\sqrt{2}}{3}r \right)^2 \times \frac{4r}{3}$$

$$\Rightarrow V = \frac{1}{3}\pi \frac{8r^3}{9} \times \frac{4r}{3}$$

$$V = \frac{32\pi r^3}{81} = \frac{32}{81}\pi \left[\frac{3V_s}{4\pi} \right]$$

\therefore Volume of cone in terms of sphere = $\frac{8V_s}{27}$

OR

Consider the function $f(x) = \sin 3x - \cos 3x$.

$$f'(x) = 3\cos 3x + 3\sin 3x$$

$$= 3(\sin 3x + \cos 3x)$$

$$= 3\sqrt{2} \left\{ \sin 3x \cos\left(\frac{\pi}{4}\right) + \cos 3x \sin\left(\frac{\pi}{4}\right) \right\}$$

$$= 3\sqrt{2} \left\{ \sin\left(3x + \frac{\pi}{4}\right) \right\}$$

For the increasing interval $f'(x) > 0$.

$$3\sqrt{2} \left\{ \sin\left(3x + \frac{\pi}{4}\right) \right\} > 0$$

$$\sin\left(3x + \frac{\pi}{4}\right) > 0$$

$$\Rightarrow 0 < 3x + \frac{\pi}{4} < \pi$$

$$\Rightarrow 0 < 3x + \frac{3\pi}{4}$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

Also,

$$\sin\left(3x + \frac{\pi}{4}\right) > 0$$

$$\text{when, } 2\pi < 3x + \frac{\pi}{4} < 3\pi$$

$$\Rightarrow \frac{7\pi}{4} < 3x < \frac{11\pi}{4}$$

$$\Rightarrow \frac{7\pi}{12} < x < \frac{11\pi}{12}$$

Therefore, intervals in which function is strictly increasing in

$$0 < x < \frac{\pi}{4} \text{ and } \frac{7\pi}{12} < x < \frac{11\pi}{12}.$$

Similarly, for the decreasing interval $f'(x) < 0$.

$$3\sqrt{2} \left\{ \sin \left(3x + \frac{\pi}{4} \right) \right\} < 0$$

$$\sin \left(3x + \frac{\pi}{4} \right) < 0$$

$$\Rightarrow \pi < 3x + \frac{\pi}{4} < 2\pi$$

$$\Rightarrow \frac{3\pi}{4} < 3x < \frac{7\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} < x < \frac{7\pi}{12}$$

Also,

$$\sin \left(3x + \frac{\pi}{4} \right) < 0$$

$$\text{When } 3\pi < 3x + \frac{\pi}{4} < 4\pi,$$

$$\Rightarrow \frac{11\pi}{4} < 3x < \frac{15\pi}{4}$$

$$\Rightarrow \frac{11\pi}{12} < x < \frac{15\pi}{12}$$

The function is strictly decreasing in $\frac{\pi}{4} < x < \frac{7\pi}{12}$ and $\frac{11\pi}{12} < x < \pi$.

Solution23: Given: $x^2 + y^2 \leq 2ax$, $y^2 \geq ax$, $x, y \geq 0$

$$\Rightarrow x^2 + y^2 - 2ax \leq 0, y^2 \geq ax, x, y \geq 0$$

$$\Rightarrow x^2 + y^2 - 2ax + a^2 - a^2 \leq 0, y^2 \geq ax, x, y \geq 0$$

$$\Rightarrow (x-a)^2 + y^2 \leq a^2, y^2 \geq ax, x, y \geq 0$$

To find the points of intersection of the circle $[(x-a)^2 + y^2 = a^2]$

and the parabola $[y^2 = ax]$, we will substitute $y^2 = ax$ in $(x-a)^2 + y^2 = a^2$.

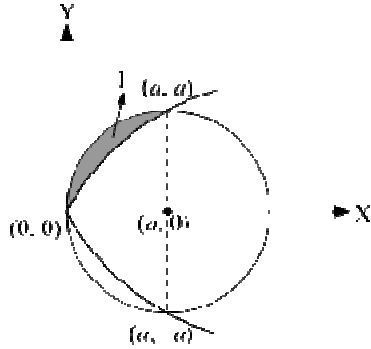
$$(x-a)^2 + ax = a^2$$

$$\Rightarrow x^2 + a - 2ax + ax = a^2$$

$$\Rightarrow x(x - a) = 0$$

$$\Rightarrow x = 0, a$$

Therefore, the points of intersection are $(0, 0)$, (a, a) and $(a, -a)$.



Now,

Area of the shaded region = I

Area of I from $x=0$ to $x=a$

$$= \left[\int_0^a \left(\sqrt{a^2 - (x-a)^2} \right) dx - \int_0^a \sqrt{ax} dx \right] \text{ Let } x-a=t \text{ for the first part of the integral}$$

$$= \int_0^a \left(\sqrt{a^2 - (x-a)^2} \right) dx.$$

$$\Rightarrow dx = dt$$

$$\therefore A_I = \int_{-a}^0 \sqrt{a^2 - t^2} dt - 2 \frac{\sqrt{a}}{3} \Big| x^{\frac{3}{2}} \Big|_0^a$$

$$= \left[\frac{t}{2} \sqrt{a^2 - t^2} + \frac{1}{2} a^2 \sin^{-1} \frac{t}{a} \right]_{-a}^0 - \frac{2a^2}{3}$$

$$= 0 - \left(-\frac{\pi a^2}{4} \right) - \frac{2a^2}{3}$$

$$A_I = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2$$

$$\therefore \text{Area of the shaded region} = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2 \text{ square units}$$

Solution24: The equation of the plane passing through three given points can be given by

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ x-3 & y-0 & z-1 \\ x-4 & y+1 & z-0 \end{vmatrix} = 0$$

performing elementary row operations $R_2 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_1 - R_3$, we get

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 0 \\ 4-2 & -1-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

Solving the above determinant, we get

$$\Rightarrow (x-2)(2-0) - (y-2)(-1-0) + (z-1)(-3+4) = 0$$

$$\Rightarrow (2x-4) + (y-2) + (z-1) = 0$$

$$\Rightarrow 2x + y + z - 7 = 0$$

Therefore, the equation of the plane is $2x + y + z - 7 = 0$.

Now, the equation of the line passing through two given points is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$

$$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$

$$\Rightarrow x = (-\lambda + 3), y = (\lambda - 4), z = (6\lambda - 5)$$

At the point of intersection, these points satisfy the equation of the plane

$$2x + y + z - 7 = 0$$

Putting the values of x, y and z in the equation of the plane, we get the value of λ .

$$2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$$

$$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$$

$$\Rightarrow 5\lambda = 10$$

$$\Rightarrow \lambda = 2$$

Thus, the point of intersection is P(1, -2, 7).

Now, let P divide the line AB in the ratio m : n.

By the section formula, we have

$$\begin{aligned}
 1 &= \frac{2m+3n}{m+n} \\
 \Rightarrow m+2n &= 0 \\
 \Rightarrow m &= -2n \\
 \Rightarrow mn &= \frac{-2}{1}
 \end{aligned}$$

Hence, P externally divides the line segment AB in the ratio 2 : 1.

Solution25: Let X denote the total number of red balls when four balls are drawn one by one with replacement.

P (getting a red ball in one draw) = $\frac{2}{3}$

P (getting a white ball in one draw) = $\frac{1}{3}$

X	0	1	2	3	4
P(X)	$\left(\frac{1}{3}\right)^4$	$\frac{2}{3}\left(\frac{1}{3}\right)^3 \cdot {}^4C_1$	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) \cdot {}^4C_2$	$\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \cdot {}^4C_3$	$\left(\frac{2}{3}\right)^4$
	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

Using the formula for mean, we have

$$\begin{aligned}
 \bar{X} &= \sum P_i X_i \\
 \text{Mean}(\bar{X}) &= \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 2\left(\frac{24}{81}\right) + 3\left(\frac{32}{81}\right) + 4\left(\frac{16}{81}\right) \\
 &= \frac{1}{81}(8 + 48 + 96 + 64) \\
 &= \frac{216}{81} = \frac{8}{3}
 \end{aligned}$$

Using the formula for variance, we have

$$\begin{aligned}
 \text{Var}(X) &= \sum P_i X_i^2 - \left(\sum P_i X_i\right)^2 \\
 \text{Var}(X) &= \left\{ \left(0 \times \frac{1}{81}\right) + 1\left(\frac{8}{81}\right) + 4\left(\frac{24}{81}\right) + 9\left(\frac{32}{81}\right) + 16\left(\frac{16}{81}\right) \right\} - \left(\frac{8}{3}\right)^2 \\
 &= \frac{64}{81} - \frac{64}{9} = \frac{8}{9}
 \end{aligned}$$

Hence, the mean of the distribution is $\frac{8}{3}$ and the variance of the distribution is $\frac{8}{9}$.

Solution26: Let the numbers of units of products A and B to be produced be x and y , respectively.

Product	Machine	
	I (h)	II (h)
A	3	3
B	2	1

Total profit : $Z = 7x + 4y$

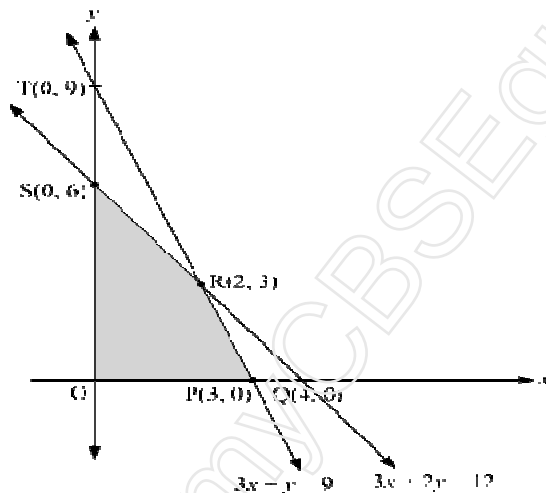
We have to maximise $Z = 7x + 4y$, which is subject to constraints.

$3x + 2y \leq 12$ (Constraint on machine I)

$3x + y \leq 9$ (Constraint on machine II)

$\Rightarrow x \geq 0$ and $y \geq 0$

The given information can be graphically expressed as follows:



Values of $Z = 7x + 4y$ at the corner points are as follows:

Corner Point	$Z = 7x + 4y$
(0, 6)	24
(2, 3)	26
(3, 0)	21

Therefore, the manufacturer has to produce 2 units of product A and 3 units of product B for the maximum profit of Rs 26.