



SEARCH MATHEMATICA 8 DOCUMENTATION



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STRUCTURAL MECHANICS DOCUMENTATION

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## Chapter 6

### Analysis of Stress

#### 6.1 Introduction

With the *Structural Mechanics* functions, you can obtain a large spectrum of information about the stress state at a point in a deformable body. Mathematically, the state of stress at a point in an elastic body is determined by six independent stress components and is specified by a second-order symmetric Cartesian tensor, also known as the stress tensor. The values of these stress components change with the orientation of the coordinate system in which each stress component is defined. You can rotate the coordinate system so you can study some of the practical issues of the stress state at a point. For example, you can reduce certain stress components to zero in a particular orientation of the coordinate system. Such information is useful in calculating the stress concentrations in a stress analysis.

*Structural Mechanics* has the functions for computing the principal stress components and principal stress direction from a stress tensor. It includes the functions for computing maximum shear stress and its directions, and the graphical functions for plotting the principal stress planes and principal stress directions. You can even draw Mohr's circles directly from the state tensor. *Structural Mechanics* also provides a number of examples illustrating the use of these functions.

You usually study the state of stress to determine a single failure criterion from the stress tensor. From a number of proposed theories used to predict material failure, this chapter considers three of the most commonly used models: maximum normal stress, maximum shear stress, and distortion energy theory.

#### 6.2 State of Stress at a Point

At a given point on a surface, a stress vector depends on the orientation (unit normal) of the surface. For a different normal vector in the same stress field, the stress vector on the associated surface must also be different. The stress vector on three mutually perpendicular planes at a point specifies the state of stress at that point in a continuum. The components of these stress vectors form a tensor, the stress tensor. Mathematically, the stress tensor is a second-order Cartesian tensor with nine stress components. Three stress components that are perpendicular to the planes are called normal stresses. Those components acting tangent to these planes are called shear stresses. You assume the sign of a stress component is positive when its direction and the normal vector of the surface, on which the component of the stress tensor is acting, are of the same sign. Otherwise, assume the sign of the stress component is negative.

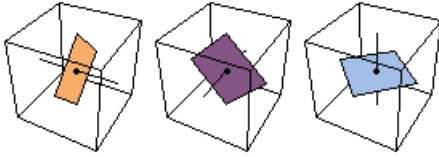
```
ShowPlane[{n1, n2, n3}]
```

produces a plot for the plane perpendicular to the vector specified by the arguments  $\{n_1, n_2, n_3\}$  passing at the origin.

Using the functions `ShowPlane` from *Structural Mechanics* and `DisplayTogetherArray` from the *Mathematica 4 Standard Add-on Packages*, you can illustrate three perpendicular planes.

```
In[1]:= << StructuralMechanics`
```

```
In[2]:= DisplayTogetherArray[{ShowPlane[{1, 0, 0}], ShowPlane[{0, 1, 0}], ShowPlane[{0, 0, 1}]}];
```



By using the function `PrincipalPlane`, you specify the origin of the direction vector, a scaling factor, and its direction.

```
PrincipalPlane[{x, y, z}, {n1, n2, n3}, mag]
```

gives a graphics object for a plane perpendicular to the vector specified by the arguments  $\{n_1, n_2, n_3\}$  and passing at the point  $\{x, y, z\}$  with the scaling factor  $mag$ .

First set some optional values to select a default font, frame the plot with a box, display grids on certain faces of the frame box, set the plot range, determine a viewpoint, and include the axes in the plot.

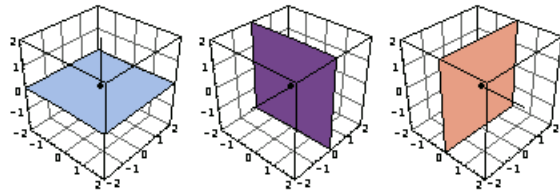
```
In[3]:= $DefaultFont = {"Courier", 8}
```

```
Out[3]= {Courier, 8}
```

```
In[4]:= opts = {Boxed -> True,
FaceGrids -> {{-1, 0, 0}, {0, 1, 0}, {0, 0, -1}},
PlotRange -> 2 {{-1, 1}, {-1, 1}, {-1, 1}},
ViewPoint -> {2.062, -2.275, 2.052},
Axes -> True};
```

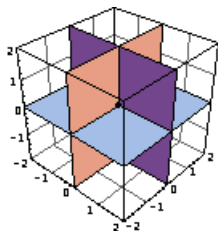
Here are the same planes with these options set. If you choose the scaling factor 4, the plane surfaces completely partition the plot box.

```
In[5]:= plns = DisplayTogetherArray[
Graphics3D[PrincipalPlane[{0, 0, 0}, {0, 0, 1}, 4], opts],
Graphics3D[PrincipalPlane[{0, 0, 0}, {0, 1, 0}, 4], opts],
Graphics3D[PrincipalPlane[{0, 0, 0}, {1, 0, 0}, 4], opts]];
```



Display these planes *plns* in a single plot with the `DisplayTogether` function from `Graphics`Graphics`` from the *Mathematica 4* Standard Add-on Packages.

```
In[6]:= DisplayTogether[
Graphics3D[PrincipalPlane[{0, 0, 0}, {0, 0, 1}, 4], opts],
Graphics3D[PrincipalPlane[{0, 0, 0}, {0, 1, 0}, 4], opts],
Graphics3D[PrincipalPlane[{0, 0, 0}, {1, 0, 0}, 4], opts]];
```



There are various ways to represent a stress tensor in *Structural Mechanics*. Now represent the stress components using the capital letter *S* and include the direction of a stress component as a subscript.

As described in Chapter 7, the `StressComponents` function generates the symbols representing the independent stress components in indicial notation.

```
In[7]:=strs=StressComponents[σ, Cartesian[x,y,z],Notation-> Indicial]
```

```
Out[7]= {σxx, σyy, σzz, σxy, σxz, σyz}
```

This is a vector representation of the symmetric stress tensor. Use the function `VectorToTensor` to convert this form into a symmetric tensor representation of a stress state at a point.

```
VectorToTensor[{Ixx, Iyy, Izz, Ixy, Ixz, Iyz}]

returns the symmetric tensor
{{Ixx, Ixy, Ixz}, {Ixy, Iyy, Iyz}, {Ixz, Iyz, Izz}}.
```

```
In[8]:=tns=VectorToTensor[strs]
```

```
Out[8]= {{σxx, σxy, σxz}, {σxy, σyy, σyz}, {σxz, σyz, σzz}}
```

```
In[9]:=MatrixForm[tns]
```

```
Out[9]/MatrixForm= 
$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}$$

```

In addition to this indicial notation, here you use a functional notation for the stress components.

```
In[10]:=VectorToTensor[StressComponents[σ, Cartesian[x,y,z]]]
```

```
Out[10]= {{σ[x,x][x,y,z], σ[x,y][x,y,z], σ[x,z][x,y,z]},
{σ[x,y][x,y,z], σ[y,y][x,y,z], σ[y,z][x,y,z]},
{σ[x,z][x,y,z], σ[y,z][x,y,z], σ[z,z][x,y,z]}}
```

```
In[11]:=MatrixForm[%]
```

```
Out[11]/MatrixForm= 
$$\begin{pmatrix} \sigma[x,x][x,y,z] & \sigma[x,y][x,y,z] & \sigma[x,z][x,y,z] \\ \sigma[x,y][x,y,z] & \sigma[y,y][x,y,z] & \sigma[y,z][x,y,z] \\ \sigma[x,z][x,y,z] & \sigma[y,z][x,y,z] & \sigma[z,z][x,y,z] \end{pmatrix}$$

```

## 6.3 Principal Stresses and Directions

When the normal vector of a surface and the stress vector acting on that surface are colinear, the direction of the normal vector is called a principal stress direction. The magnitude of the stress vector on the surface is called the principal stress value. You use the function `PrincipalStresses` to calculate the principal stresses from the stress state at a point.

```
PrincipalStresses[strs]

calculates the principal stress components for the stress components at a
point strs. The argument strs is represented as either the second-
order stress tensor or the stress state vector.
```

In the case where the values of all shear stresses are zero, the principal stresses are the normal stresses.

```
In[12]:=PrincipalStresses[{σxx, σyy, σzz, 0, 0, 0}]
```

```
Out[12]= {σzz, σxx, σxy}
```

Similarly, you use the function `PrincipalStressDirections` to determine the directions of the principal stress components.

```
PrincipalStressDirections[strs]

calculates principal stress direction for the stress components at a
point strs. The argument strs is represented as either the second-
order stress tensor or the stress state vector.
```

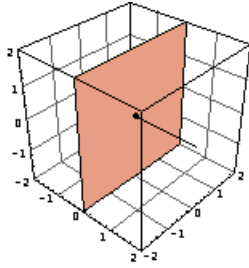
The principal stress directions are the unit vectors of the coordinate system in the case where no shear stress component is present.

```
In[13]:=PrincipalStressDirections[{σxx, σyy, σzz, 0, 0, 0}]
```

```
Out[13]= {{0, 0, 1}, {1, 0, 0}, {0, 1, 0}}
```

Visualize the first plane in the previously introduced graphics object *plns*, which originated at the origin of the coordinate system.

```
In[14]:=Show[Graphics3D[PrincipalPlane[{0,0,0},{1,0,0},4]],opts];
```



Compute the principal stresses with a nonzero shear stress value for  $\sigma_{xy}$ .

```
In[15]:=PrincipalStresses[{ $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\sigma_{xy}$ , 0, 0}]
```

$$\text{Out[15]} = \left\{ \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right), \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right), \sigma_{zz} \right\}$$

Note that the principal stress in the  $z$ -direction remains unchanged since the shear stress is applied in the  $x$ - $y$  plane only. Now compute the principal stress directions for this stress tensor.

```
In[16]:=PrincipalStressDirections[{ $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\sigma_{xy}$ , 0, 0}]
```

$$\text{Out[16]} = \left\{ \left\{ -\frac{\sigma_{xx} + \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2}}{2\sigma_{xy} \sqrt{1 + \frac{(\sigma_{xx} + \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2})^2}{4\sigma_{xy}^2}}}, \frac{1}{\sqrt{1 + \frac{(\sigma_{xx} + \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2})^2}{4\sigma_{xy}^2}}}}, 0 \right\}, \left\{ -\frac{\sigma_{xx} + \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2}}{2\sigma_{xy} \sqrt{1 + \frac{(\sigma_{xx} + \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2})^2}{4\sigma_{xy}^2}}}, \frac{1}{\sqrt{1 + \frac{(\sigma_{xx} + \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2})^2}{4\sigma_{xy}^2}}}}, 0 \right\}, \{0, 0, 1\} \right\}$$

As expected, since the  $\sigma_{xz}$ ,  $\sigma_{yz}$  components are zero, the positive direction of the  $z$  coordinate,  $\{0, 0, 1\}$  remains as a principal stress direction.

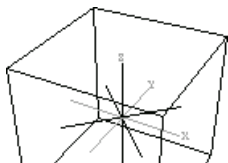
The function `PrincipalStressDirectionsPlot` plots the directions of principal stresses.

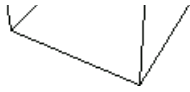
```
PlotPrincipalStressDirections[strs]
```

plots the principal stress directions for the stress components *strs* with respect to the reference coordinate system in gray. The argument *strs* is represented as either the second-order stress tensor or the stress state vector.

Show that the principal stress direction in the  $z$ -direction remains unchanged in the stress state  $\{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, 0, 0\}$ . Here the gray coordinates represent the reference coordinate system.

```
In[17]:=PrincipalStressDirectionsPlot[{1,1,1,.5,0,0}];
```





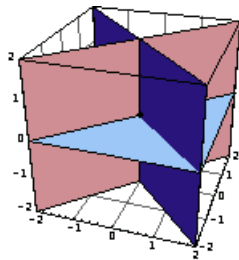
PrincipalPlane graphics objects allow you to visualize these planes. First, compute the normal directions of the principal plane.

```
In[18]:{d1,d2,d3}=PrincipalStressDirections[{1,1,1,1/2,0,0}]
```

```
Out[18]:{{{-1/sqrt(2), 1/sqrt(2), 0}, {0, 0, 1}, {1/sqrt(2), 1/sqrt(2), 0}}}
```

Here are the principal stress planes of this stress state.

```
In[19]:planes=DisplayTogether[
Graphics3D[PrincipalPlane[{0,0,0},d1,5],opts],
Graphics3D[PrincipalPlane[{0,0,0},d2,5],opts],
Graphics3D[PrincipalPlane[{0,0,0},d3,5],opts],
ViewPoint->{0.947,-2.830,1.595}];
```



You determine the principal stresses by calculating the stress state at the directions  $\{d_1, d_2, d_3\}$ .

```
In[20]:tns = VectorToTensor[{1, 1, 1, 1/2, 0, 0}];
```

```
In[21]:(newstr={d1,d2,d3}.tns.Transpose[{d1,d2,d3}])/MatrixForm
```

```
Out[21]/MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

```

Equivalently, the function PrincipalStresses calculates the same values for a stress state.

```
In[22]:PrincipalStresses[{1, 1, 1, 1/2, 0, 0}]
```

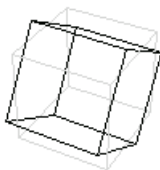
```
Out[22]:{1/2, 1, 3/2}
```

Another way to visualize these planes is to draw a picture of a rotated cube on whose surfaces the principal stresses are realized. You use the function UnitCubePlot for this purpose.

```
UnitCubePlot[stns]
```

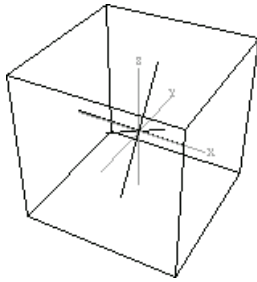
gives a plot with two unit cubes representing the planes of the coordinate system (in gray) and the principal planes (in black) for the stress state *stns*. The argument *stns* is represented as either the second-order stress tensor or the stress state vector.

```
In[23]:cube=UnitCubePlot[{5,4,7,0.01,0.03,2.91}];
```



The function `PrincipalStressDirectionsPlot` displays the principal stress direction in a different way.

```
In[24]:=dirs= PrincipalStressDirectionsPlot[{5,4,7,.01,.03,2.91}];
```



You extract the reference coordinate system from this graphics object `dirs`. Here you can generate a separate graphics object representing these directions.

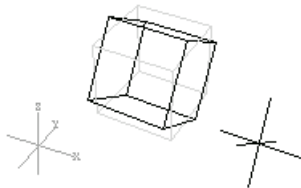
```
In[25]:=p0 = Graphics3D[dirs[[1, 1]], Boxed -> False];
```

Similarly, you can also extract the principal stress directions from the same graphics object `dirs`.

```
In[26]:=p1=Graphics3D[dirs[[1, 2]],Boxed->False];
```

Depicting these three figures `cube`, `p0`, and `p1` together provides a clear three-dimensional picture of the principal stress directions. You can arrange the relative positions of these graphics objects using the *Mathematica* graphics object `Rectangle`.

```
In[27]:=Show[Graphics[{Rectangle[{0,0},{1,1},p0],
Rectangle[{0,0},{2,2},cube],
Rectangle[{1,0},{2,1},p1]
}]];
```



Calculate the stress invariants of a stress tensor with the function `StressInvariants`.

`StressInvariants[strs]` gives the first, second, and third stress invariants of the stress components at a point *strs*. The argument *strs* is represented as either the second-order stress tensor or the stress state vector.

This calculates the stress invariants for a principal stress state.

```
In[28]:=StressInvariants[{σxx, σyy, σzz, 0, 0, 0}]
```

```
Out[28]= {σxx + σyy + σzz, σxx σyy + σxx σzz + σyy σzz, σxx σyy σzz}
```

If you add a shear term  $\epsilon$ , you can see how this shear stress affects the values of stress invariants.

```
In[29]:=StressInvariants[{σxx, σyy, σzz, ε, 0, 0}]
```

```
Out[29]= {σxx + σyy + σzz, -ε2 + σxx σyy + σxx σzz + σyy σzz, -ε2 σzz + σxx σyy σzz}
```

## 6.4 Maximum Shear Stresses and Directions

On a surface element in a continuum, you can always express the stress vector in terms of one normal component and one tangential component. A

certain orientation of the surface element can show that maximum shear stresses are realized. This section considers the maximum shear stresses and their directions for a given stress state. First, consider the function `MaximumShearStresses`.

`MaximumShearStresses[strs]`

calculates maximum shear stress components for the stress components at a point *strs*. The argument *strs* is represented as either the second-order stress tensor or the stress state vector.

`In[30]:=MaximumShearStresses[{ $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ , 0, 0, 0}]`

$$\text{Out[30]} = \left\{ \frac{1}{2} (-\sigma_{xx} + \sigma_{yy}), \frac{1}{2} (\sigma_{xx} - \sigma_{zz}), \frac{1}{2} (-\sigma_{yy} + \sigma_{zz}) \right\}$$

The expressions for the maximum shear stress components become complicated when you include the *xy* stress component in the stress tensor.

`In[31]:=MaximumShearStresses[{ $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{zz}$ ,  $\sigma_{xy}$ , 0, 0}]`

$$\begin{aligned} \text{Out[31]} = & \left\{ \frac{1}{2} \left( \frac{1}{2} \left( -\sigma_{xx} - \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right) + \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} + \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right) \right), \right. \\ & \frac{1}{2} \left( \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right) - \sigma_{zz} \right), \\ & \left. \frac{1}{2} \left( \frac{1}{2} \left( -\sigma_{xx} - \sigma_{yy} - \sqrt{\sigma_{xx}^2 + 4\sigma_{xy}^2 - 2\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2} \right) + \sigma_{zz} \right) \right\} \end{aligned}$$

Similar to the function `PrincipalStressDirections`, you use the function `MaximumShearDirections` to compute the values of the maximum shear direction for a given stress state at a point.

`MaximumShearDirections[strs]`

calculates maximum shear stress directions for the stress tensor or stress components at a point *strs*. The argument *strs* is represented as either the second-order stress tensor or the stress state vector.

In the principal stress directions, the shear components of the stress vector vanish. Thus, the shear stresses are zero in these directions.

Compute the maximum shear stresses for the stress vector  $\{1, 2, 3, 0, 0, 0\}$ .

`In[32]:=MaximumShearStresses[{1, 2, 3, 0, 0, 0}]`

$$\text{Out[32]} = \left\{ \frac{1}{2}, -1, \frac{1}{2} \right\}$$

The directions in which these values for the maximum shear stresses are realized are of practical interest. Calculate the maximum shear directions using the function `MaximumShearDirections`.

`In[33]:=MaximumShearDirections[{1, 2, 3, 0, 0, 0}]`

$$\text{Out[33]} = \left\{ \left\{ 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}, \left\{ \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\}, \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\} \right\}$$

From this output, the maximum stress value is -1, and it acts on the plane whose normal vector is  $\{1/\sqrt{2}, 0, 1/\sqrt{2}\}$ .

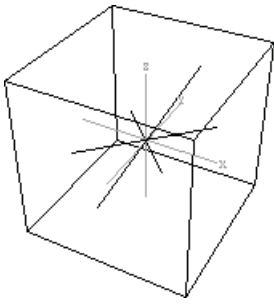
As in principal stress directions, you have the function `MaximumShearDirectionsPlot` to visualize the maximum shear stress direction for a given stress state.

`MaximumShearDirectionsPlot[strs]`

plots the maximum shear stress directions for the stress components at a point *strs*. The argument *strs* is represented as either the second-order stress tensor or the stress state vector.

This shows the maximum shear directions for the stress vector  $\{1, 1, 1, 0.1, 0, 0\}$ .

```
In[34]:=MaximumShearDirectionsPlot[{1,1,1,0.1,0,0}];
```



## 6.5 Mohr's Circles

Mohr's circles provide a planar representation of a three-dimensional stress state. Using Mohr's circles, you can calculate the stress state in a given direction graphically. The technique has been in use for over 100 years. Although current computer technology makes calculation of stress states easier, *Structural Mechanics* includes the method for its instructional and historical importance.

The function `MohrsCircles` generates the Mohr's circles for a given stress state.

`MohrsCircles[stvs]` plots Mohr's circles for the stress components *stvs*. The argument *stvs* is represented as either the second-order stress tensor or the stress state vector.

The gray areas in the Mohr's circles represent the stress states that can be realized by changing the orientation of the coordinate system for a given stress state. Here generate the Mohr's circles for the stress vector  $\{1, 2, 4, 0, 0, 0\}$ . The  $x$  axis (labeled with  $\sigma$ ) is the normal stress component, and the  $y$  axis (labeled with  $\tau$ ) is the tangent (shear) component.

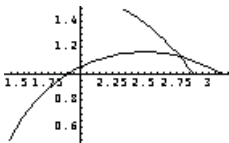
Compute the traction in a plane for which the normal vector is given by using a graphical method based on the technique used in Mohr's circles.

The function `MohrsCirclesTractions` generates a pair of arches.

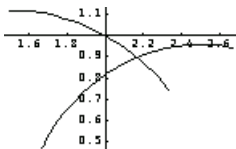
`MohrsCirclesTractions[stvs, { $n_1$ ,  $n_2$ ,  $n_3$ }]` produces the crossing curves to compute the tractions on a surface defined by a normal vector  $\{n_1, n_2, n_3\}$  for the stress components *stvs*. The argument *stvs* is represented as either the second-order stress tensor or the stress state vector.

The crossing point of these two curves from `MohrsCirclesTractions` gives the normal and tangential components of the stress on the surface element whose direction is  $\{n_1, n_2, n_3\}$ .

```
In[35]:=p1=MohrsCirclesTraction[{1,2,4,0,0,0},{1,2,2}];
```



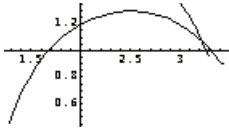
```
In[36]:=p2=MohrsCirclesTraction[{1,2,4,0,0,0},{1,2,1}];
```



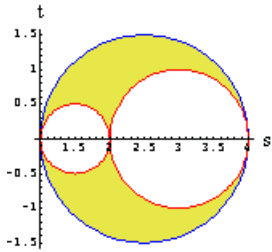
```
In[37]:=p3=MohrsCirclesTraction[{1,2,4,0,0,0},{1,2,3}];
```

```
1.4{ \
```





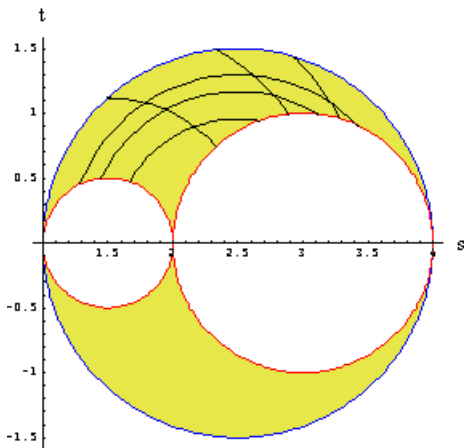
```
In[38]:= p4 = MohrCircles[{1, 2, 4, 0, 0, 0}];
```



The approximate normal and tangential components for this state are  $\{3.205263, 1.07034\}$ . You can find the instructions for reading coordinates of a point on a two-dimensional plot by choosing the menu item **Get Graphics Coordinates** from the **Input** menu on the tool bar.

Superimpose the Mohr's circles for this stress state and the traction curves to illustrate where the crossing point of the two curves is located in the circle plot.

```
In[39]:= Show[p4, p1, p2, p3, PlotRange -> All];
```



The function **NormalShearStresses** calculates the normal stress and shear components for given principal stresses and direction.

```
NormalShearStresses[{σ1, σ2, σ3}, {n1, n2, n3}]
```

calculates the normal stress and shear components for the principal stresses  $\{\sigma_1, \sigma_2, \sigma_3\}$  on a surface element with the normal direction  $\{n_1, n_2, n_3\}$ .

Here use exact values for the stress state vector  $\{1, 2, 4\}$  and their direction vector.

```
In[40]:= NormalShearStresses[{1, 2, 4}, {1, 2, 3}/Sqrt[1+4+9]]
```

```
{ 4/5, -sqrt[229]/5 }
```

$$\text{Out[40]} = \left\{ \frac{--}{14}, \frac{--}{14} \right\}$$

As expected, this result is exact. In order to compare it with the graphical result obtained using `MohrsCirclesTraction` earlier,  $\{3.205263, 1.07034\}$ , calculate the numerical values for this result using the function `N`.

```
In[41]:=N[%]
```

```
Out[41]= {3.21429, 1.08091}
```

Now consider a more complicated example with shear stresses present.

```
In[42]:=strs={10.0,2.0,7.0,1.0,0.5,1.5}
```

```
Out[42]= {10., 2., 7., 1., 0.5, 1.5}
```

This is the stress state in tensor form.

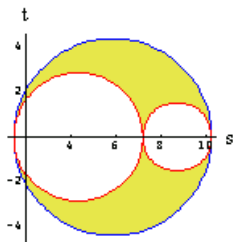
```
In[43]:=VectorToTensor[strs]//MatrixForm
```

```
Out[43]//MatrixForm= 
$$\begin{pmatrix} 10. & 1. & 0.5 \\ 1. & 2. & 1.5 \\ 0.5 & 1.5 & 7. \end{pmatrix}$$

```

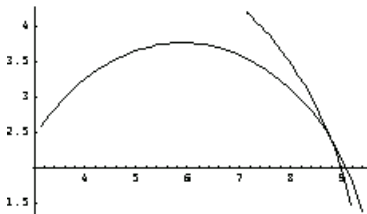
Produce the Mohr's circles for this stress state.

```
In[44]:=p1=MohrsCircles[strs,PlotRange->All];
```



Graphically compute the normal stress and shear components in the direction  $(1, 2, 3)$ .

```
In[45]:=p2=MohrsCirclesTraction[strs,{1,2,3}/Sqrt[14]];
```



You can also compute these values by using `NormalShearStresses`. First compute the principal stresses.

```
In[46]:=pss=PrincipalStresses[strs]
```

```
Out[46]= {10.2754, 7.22214, 1.50249}
```

Calculate the normal and tangent components of the stresses in this direction.

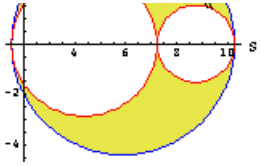
```
In[47]:=NormalShearStresses[Sort[pss],{1,2,3}/Sqrt[1+4+9]]
```

```
Out[47]= {8.77638, 2.43185}
```

As shown in the previous Mohr's circles examples, you can graphically compute the principal stresses for this stress state with `PrincipalStresses`.

```
In[48]:=Show[p1,p2,PlotRange->All,AspectRatio->1];
```





Using the functions `PrincipalStresses` and `NormalShearStresses`, you compute the normal and tangential components for a given stress state. Now, consider the inverse problem; namely, the directions are computed from given principal stresses and the normal and shear stress components. Use the function `NormalShearDirections` to calculate the direction cosines of the plane in which a given normal stress and shear stresses are realized from the principal stress components.

```
NormalShearDirections[{ $\sigma_1, \sigma_2, \sigma_3$ }, { $\sigma_N, \sigma_S$ }]
```

calculates the direction cosines for the principal stresses  
 $\{\sigma_1, \sigma_2, \sigma_3\}$  and the normal stress and shear components  $\{\sigma_N, \sigma_S\}$ .

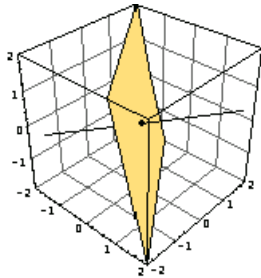
Now compute the direction in which the given normal and shear components are  $\{8.7, 2.4\}$  for the principal stress values  $\{10.275, 7.22, 1.50\}$ .

```
In[49]:=nsd=NormalShearDirections[{10.275, 7.22, 1.50},{8.7,2.4}]
```

```
Out[49]={0.782536, 0.565085, 0.211026}
```

This is the plane on which these values of the normal stress and shear stresses are realized.

```
In[50]:=Show[Graphics3D[PrincipalPlane[{0,0,0},nsd,5]],opts];
```



You can verify these values using the function `NormalShearStresses`.

```
In[51]:=NormalShearStresses[{10.275, 7.22, 1.50},nsd]
```

```
Out[51]={8.87542, 2.14629}
```

## 6.6 Failure Theories

This section uses the functionality in *Structural Mechanics* to consider three fundamental failure criteria:

- maximum normal stress theory
- maximum shear stress theory
- distortion energy theory

### 6.6.1 The Maximum Normal Stress Theory

This simple theory is included only for its historical interest. Its predictions sometimes result in unsafe designs, and they do not agree with experimental data (e.g., the torsional analysis of bars). Thus, it is not recommended for use in modern designs. The maximum normal stress theory states that material failure occurs whenever the largest principal stress equals the strength of the material.

According to this theory, for the principal stresses,  $\sigma_1 > \sigma_2 > \sigma_3$ , when the criterion of failure is the yielding stress, the failure occurs whenever  $\sigma_1$  is equal to the tensile yield strength, or  $-\sigma_3$  is equal to the negative value of the compressive yield strength. The yielding failure criterion was

$\sigma_1$  is equal to the tensile yield strength, or  $\sigma_3$  is equal to the negative value of the compressive yield strength. The yielding failure criterion was used mostly for ductile materials.

For brittle materials, the ultimate strength is used as the criterion of failure, and this theory predicts that the failure occurs whenever  $\sigma_1$  is equal to the ultimate tensile strength, or  $\sigma_3$  is equal to the negative value of the ultimate compressive strength. The following example demonstrates how the functions introduced in this chapter can be adopted in a design problem.

Assume that the stress analysis results in a stress tensor (in MPa) as follows.

```
In[52]:=VectorToTensor[{300,450,-300,10,50,x}]/MatrixForm
```

```
Out[52]/MatrixForm=

$$\begin{pmatrix} 300 & 10 & 50 \\ 10 & 450 & x \\ 50 & x & -300 \end{pmatrix}$$

```

Here  $\sigma_{xx} = x$  is the stress component, which varies with the magnitude of the applied forces. Your task is to find the maximum value of  $x$  for a material whose tensile and compressive yield strengths are 600 MPa and 400 MPa, respectively.

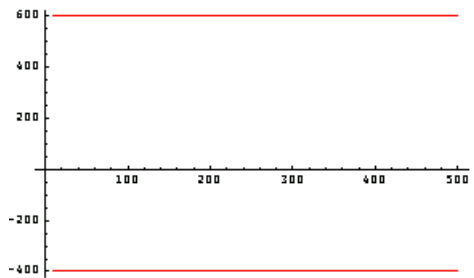
For  $x = 0$ , no failure occurs since the principal stresses components are between 600 MPa and -400 MPa.

```
In[53]:=PrincipalStresses[{300.,450.,-300.,10.,50.,0.}]
```

```
Out[53]={450.679, -304.139, 303.46}
```

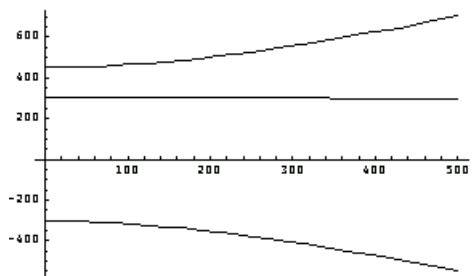
To solve this problem graphically, first mark the safety limits of the design.

```
In[54]:=limits=Plot[{600,-400},{x,10,500},PlotStyle->{RGBColor[1,0,0]}];
```



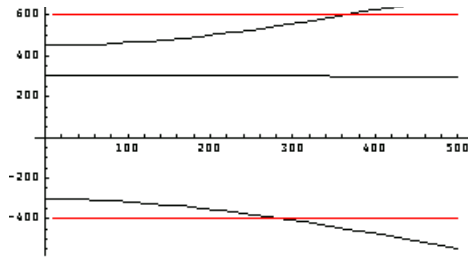
Plot the values of these three principal stresses by changing the value of  $x$ .

```
In[55]:=maxnormal=Plot[Evaluate[PrincipalStresses[{300,450,-300,10,50,x}]],{x,0,500}];
```



By superimposing these two plots, you can observe that two of the principal stresses cross the safety limits of the material at two points.

```
In[56]:=Show[{maxnormal,limits}];
```



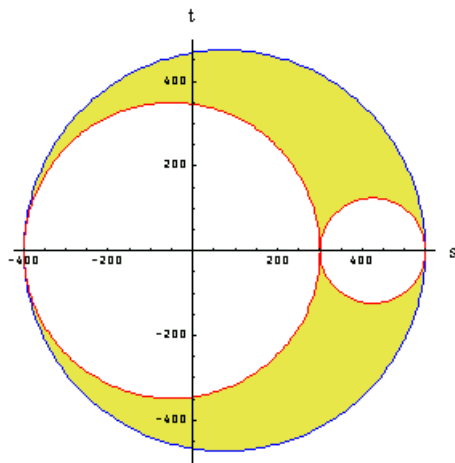
Thus,  $\sigma_{\max} = 286.98 \text{ MPa}$ .

## 6.6.2 The Maximum Shear Stress Theory

The maximum shear stress theory is based on yielding, thus you can only use it for ductile materials. The theory states that yielding begins whenever the maximum shear stress in a mechanical element reaches the experimental maximum shear stress of the material of the element. According to this theory, the yield strength in shear is half of the normal yield strength.

The maximum shear stress is  $(\sigma_1 - \sigma_3) / 2$  where  $\sigma_1 > \sigma_2 > \sigma_3$  are the principal stresses. This point is located at the uppermost point of Mohr's circles.

```
In[57]:=MohrCircles[{300,450,-300,10,50,285}];
```



```
In[58]:={g3,g2,g1}=Sort[PrincipalStresses[{300.,450,-300,10,50,1}]]
```

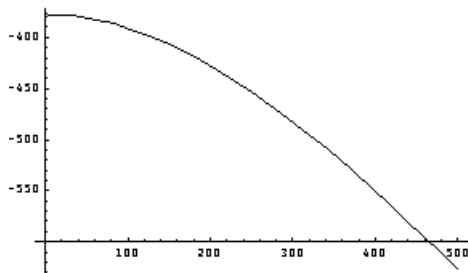
```
Out[58]:{-304.138,303.449,450.689}
```

```
In[59]:=(g1-g3)/2
```

```
Out[59]:377.414
```

Plot the maximum shear stress as a function of  $x$ .

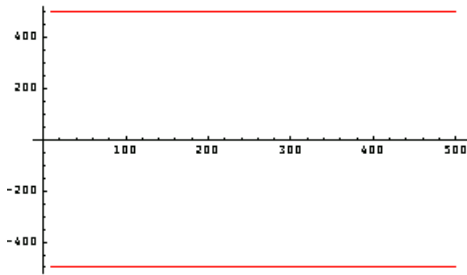
```
In[60]:=maxshear=Plot[gs=Evaluate[Sort[PrincipalStresses[{300.,450,-300,10,50,x}]]];.5(gs[[1]]-gs[[3]]),{x,0,500}];
```



According to this theory, for a material with a yield strength of 1000 MPa, the limits of the yield strength in shear are  $-1000/2 = -500 \text{ MPa}$  and

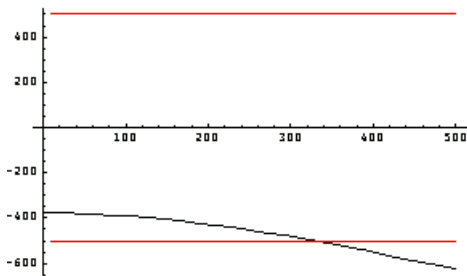
$1000/2 = 500 \text{ MPa}$ .

```
In[61]:=limits=Plot[{1000/2,-1000/2},{x,10,500},PlotStyle->{RGBColor[1,0,0]}];
```



Superimposing this plot and the safety limit lines shows the location of the limit value for  $x$ .

```
In[62]:=Show[maxshear,limits];
```



The maximum shear theory is more conservative than the maximum normal stress theory. The results from the maximum shear theory are always on the "safe side." The coordinates of the approximate point where the yielding begins are  $\{323.86, -498.98\}$ :  $\sigma_{xy_{max}} = 323.86 \text{ MPa}$ .

## 6.6.3 The Distortion Energy Theory

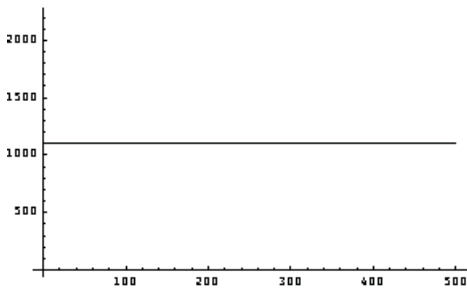
The distortion energy theory is also known as both the shear energy theory and the von Mises-Hencky theory. Like the maximum shear theory, this theory is only for ductile materials. The distortion energy theory produces the most conservative stress levels among the three methods considered in this chapter.

Using this theory, here is the criterion for the beginning of the yield.

$$2 \sigma_y^2 = \frac{1}{2} (\sigma_1 - \sigma_2)^2 + \frac{1}{2} (\sigma_2 - \sigma_3)^2 + \frac{1}{2} (\sigma_1 - \sigma_3)^2$$

In this expression,  $\sigma_y$  is the material yield strength, and  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are the principal stresses. Solve this equality for  $\sigma_y$  and plot  $\sigma_y$  with respect to  $x$  as in the previous example for the same stress state.

```
In[63]:=maxshear=Plot[gs=Evaluate[Sort[PrincipalStresses[{300.,450,-300,10,50,x}]]];Sqrt[.5((gs[[1]]-gs[[2]])^2+(gs[[2]]-gs[[3]])^2+(gs[[3]]-gs[[1]])^2)],{x,0,500}];
```



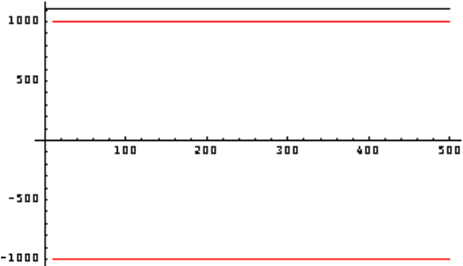
Draw the limiting lines for the same material used in the previous example.

```
In[64]:=limits=Plot[{1000,-1000},{x,10,500},
PlotStyle->RGBColor[1,0,0],
DisplayFunction->Identity];
```

J ,

Superimposing these two plots shows for what value of  $x$  yielding begins.

```
In[65]:=Show[maxshear,limits];
```



The coordinates for the approximate point where yielding begins are {416.27, 979.28}.

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