6

Plane Stress Transformations

Plane Stress State

Recall that in a body in plane stress, the general 3D stress state with 9 components (6 independent) reduces to 4 components (3 independent):

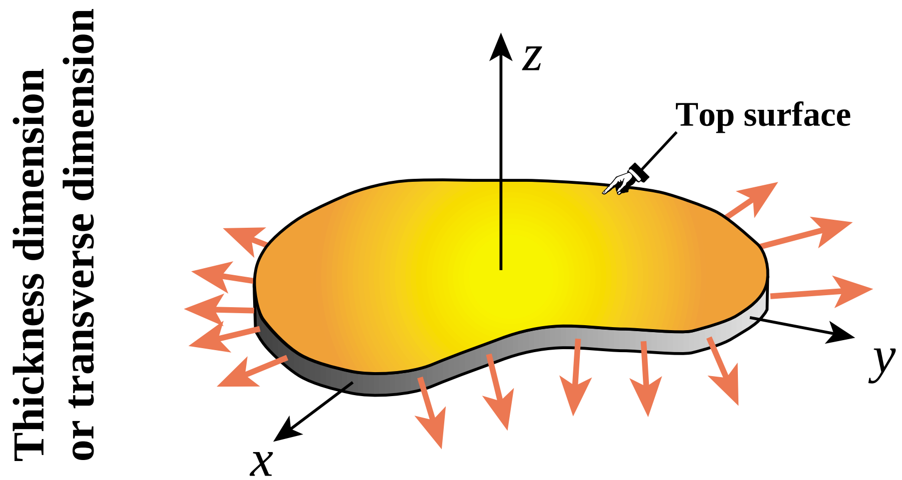
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \xrightarrow{\textbf{plane stress}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $\tau_{yx} = \tau_{xy}$

Plane stress occurs in thin plates and shells (e.g. aircraft & rocket skins, parachutes, balloon walls, boat sails, ...) as well as thin wall structural members in torsion.

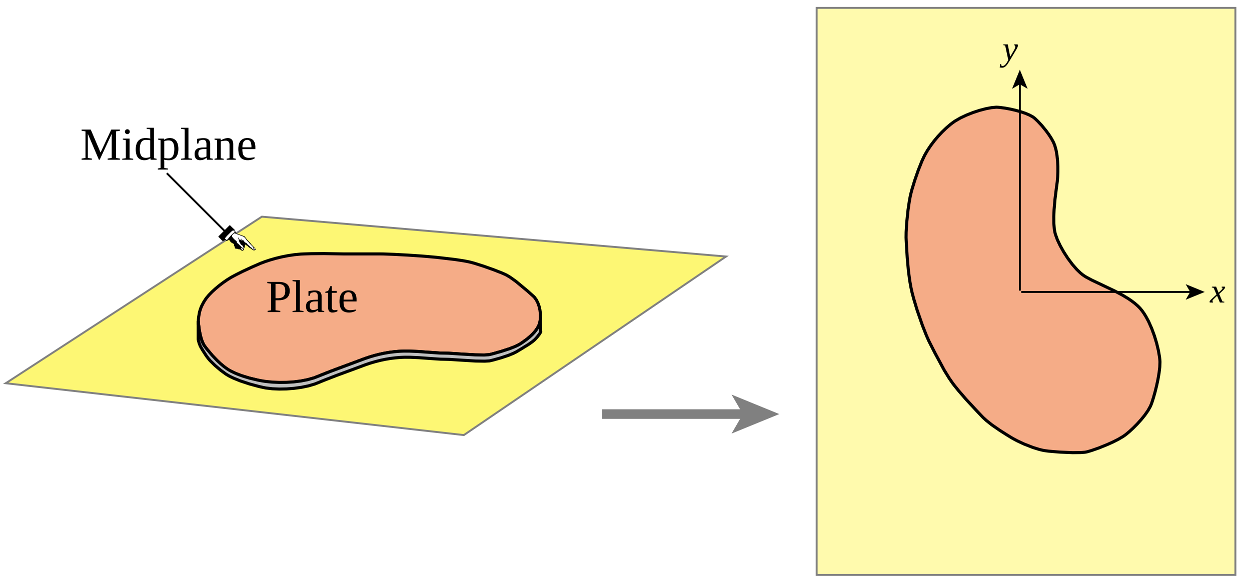
In this Lecture we will focus on thin flat plates and associated two-dimensional stress transformations

Flat Plate in Plane Stress

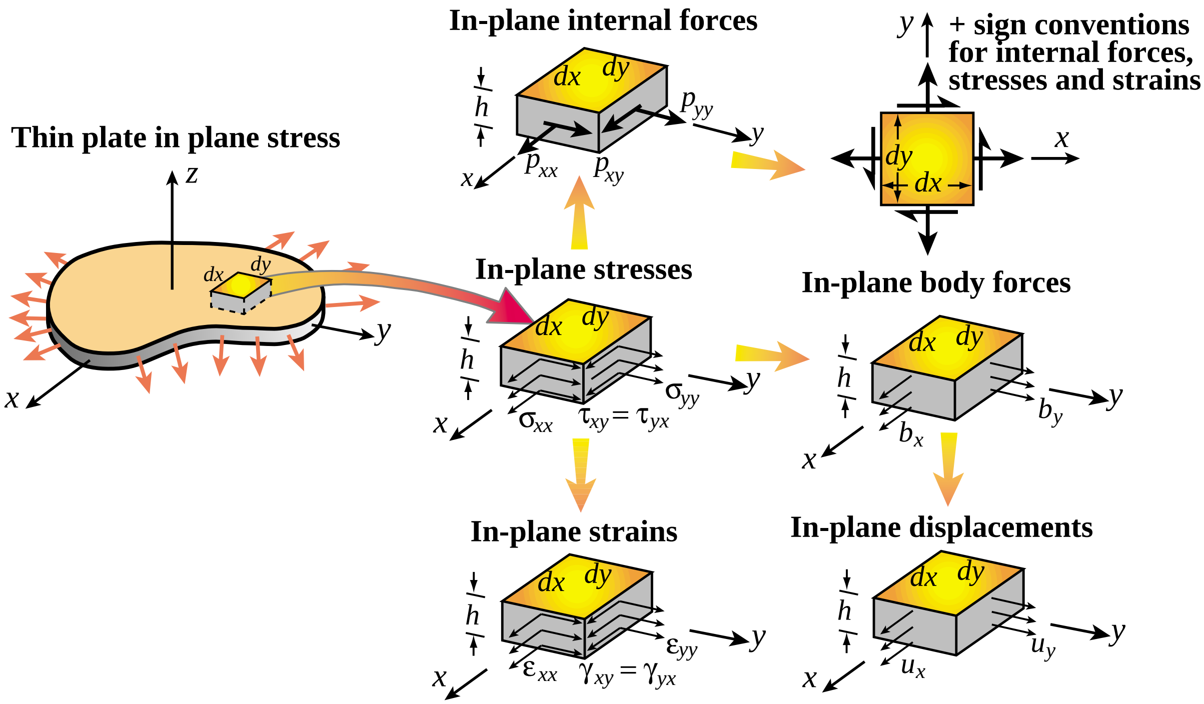


Inplane dimensions: in *x,y* **plane**

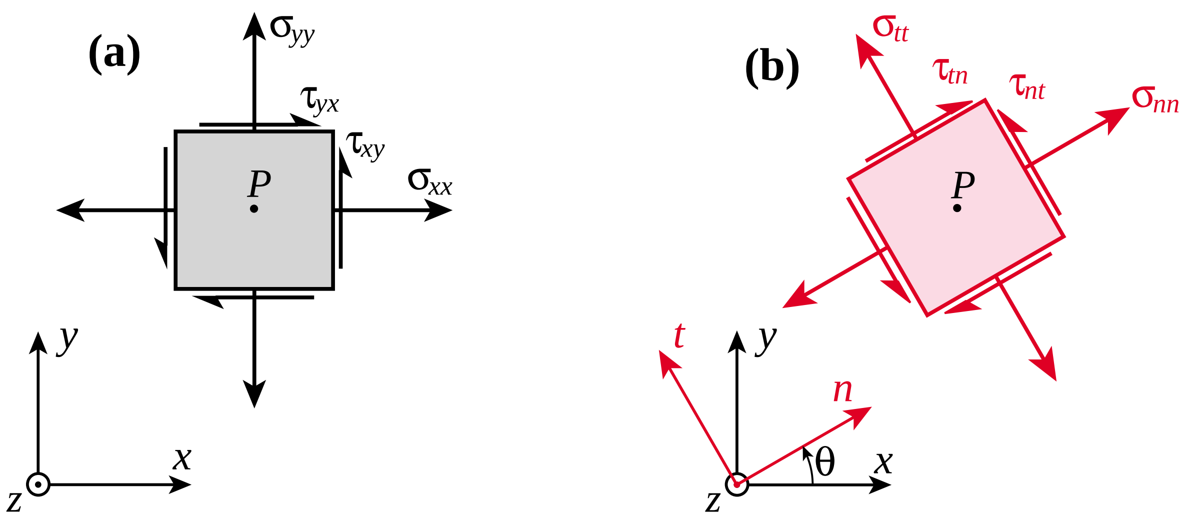
Mathematical Idealization as a Two Dimensional Problem



Internal Forces, Stresses, Strains



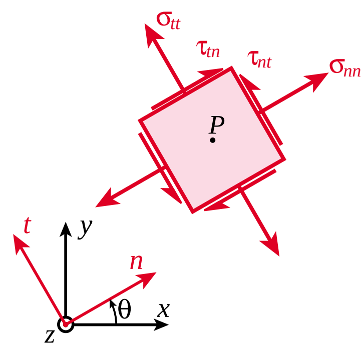
Stress Transformation in 2D



Global axes
x,y stay fixed

Local axes n,t
rotate by θ with
respect to x,y

Problem Statement



Plane stress transformation problem:
given σ_{xx} , σ_{yy} , τ_{xy} **and angle** θ
express σ_{nn}, σ_{tt} and τ_{nt} **in terms of the data**

This transformation has two major uses:

Find stresses along a given skew direction
Here angle θ is given as data

Find max/min normal stresses, max in-plane shear and overall max shear
Here finding angle θ is part of the problem

Analytical Solution

This is also called **method of equations** in Mechanics of Materials books. A derivation using the **wedge method** gives

$$\begin{aligned} \sigma_{nn} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\,\tau_{xy} \sin\theta \cos\theta \\ \sigma_{tt} &= \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\,\tau_{xy} \sin\theta \cos\theta \\ \tau_{nt} &= -(\sigma_{xx} - \sigma_{yy}) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta) \end{aligned}$$

For quick checks when θ is 0^o or 90^o, see Notes. The sum of the two transformed normal stresses

$$\sigma_{nn} + \sigma_{tt} = \sigma_{xx} + \sigma_{yy}$$

is *independent* of the angle θ: it is called a **stress invariant** (mathematically, this is the trace of the stress tensor). A geometric interpretation using the Mohr's circle is immediate.

Double Angle Version

Using double-angle trig relations such as $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2 \sin\theta \cos\theta$, the transformation equations may be rewritten as

$$\sigma_{nn} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$
$$\tau_{nt} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Here σ_{tt} is omitted since it may be easily recovered as $\sigma_{xx} + \sigma_{yy} - \sigma_{nn}$

Principal Stresses: Terminology

The **max and min values** taken by the **in-plane normal stress** σ_{nn} when viewed as a function of the angle θ are called **principal stresses** (more precisely, **principal in-plane normal stresses**, but qualifiers "in-plane" and "normal" are often omitted).

The **planes on which those stresses act** are the **principal planes**.

The **normals to the principal planes** are contained in the *x*,*y* plane. They are called the **principal directions**.

The θ **angles formed by the principal directions and the *x* axis** are called the **principal angles**.

Principal Angles

To find the **principal angles**, set the derivative of σ_{nn} with respect to θ to zero. Using the double-angle version,

$$\frac{d\,\sigma_{nn}}{d\,\theta} = (\sigma_{yy} - \sigma_{xx}) \sin 2\theta + 2\tau \cos 2\theta = 0$$

This is satisfied for $\theta = \theta_p$ if

$$\boxed{\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}} \qquad (*)$$

It can be shown that (*) provides two principal double angles, $2\theta_{p1}$ and $2\theta_{p2}$, within the range of interest, which is [0, 360^o] or [–180^o,180^o] (range conventions vary between textbooks).
The **two values differ by 180^o**. On dividing by 2 we get the principal angles θ_{p1} and θ_{p2} **that differ by 90^o**. Consequently the **two principal directions are orthogonal**.

Principal Stress Values

Replacing the principal angles given by (*) of the previous slide into the expression for σ_{nn} and using trig identities, we get

$$\boxed{\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}}$$

in which $\sigma_{1,2}$ denote the principal normal stresses. Subscripts 1 and 2 correspond to taking the + and – signs, respectively, of the square root.

A staged procedure to compute these values is described in the next slide.

Staged Procedure To Get Principal Stresses

1. Compute

$$\sigma_{av} = \frac{\sigma_{xx} + \sigma_{yy}}{2}, \qquad R = +\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Meaning: σ_{av} is the **average normal stress** (recall that $\sigma_{xx} + \sigma_{yy}$ is an invariant and so is σ_{av}), whereas R is the **radius of Mohr's circle** described later. This R also represents the **maximum in-plane shear** value, as discussed in the Lecture notes.

2. The principal stresses are

$$\sigma_1 = \sigma_{av} + R\,, \qquad \sigma_2 = \sigma_{av} - R$$

3. The above procedure **bypasses the computation of principal angles**. Should these be required to find principal directions, use equation (*) of the **Principal Angles** slide.

Additional Properties

1. The in-plane shear stresses on the principal planes **vanish**

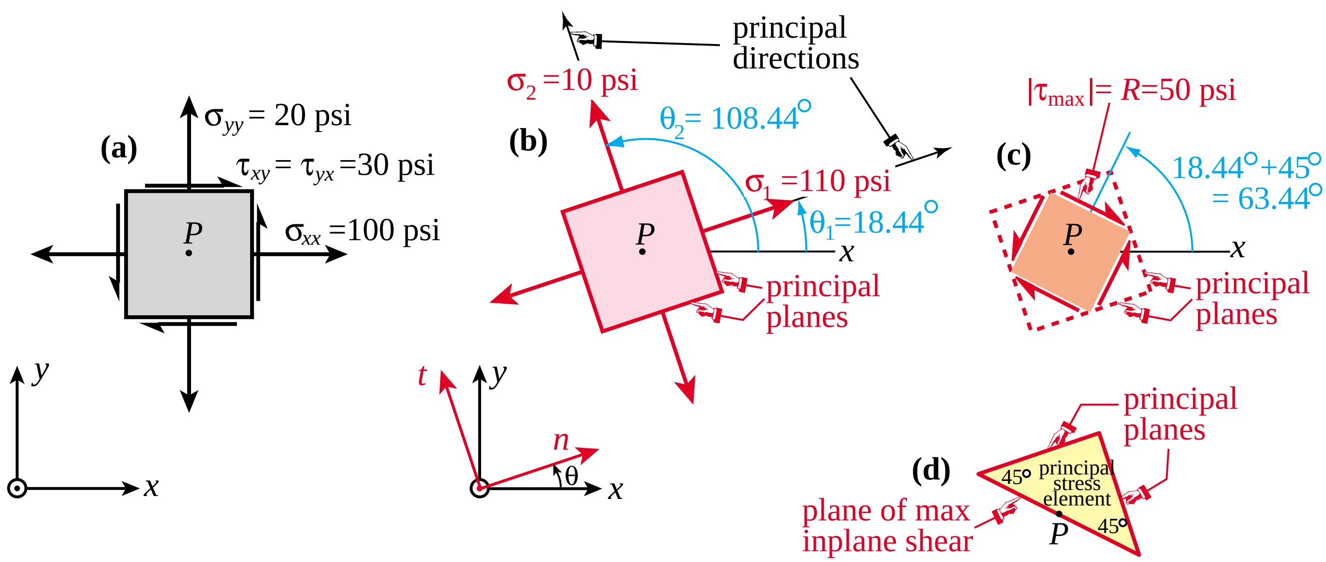
2. The **maximum and minimum in-plane shears** are $+R$ and $-R$, respectively

3. The max/min in-plane shears act on planes located at $+45^\circ$ and -45° from the principal planes. These are the **principal shear planes**

4. A **principal stress element** (used in some textbooks) is obtained by drawing a triangle with two sides parallel to the principal planes and one side parallel to a principal shear plane

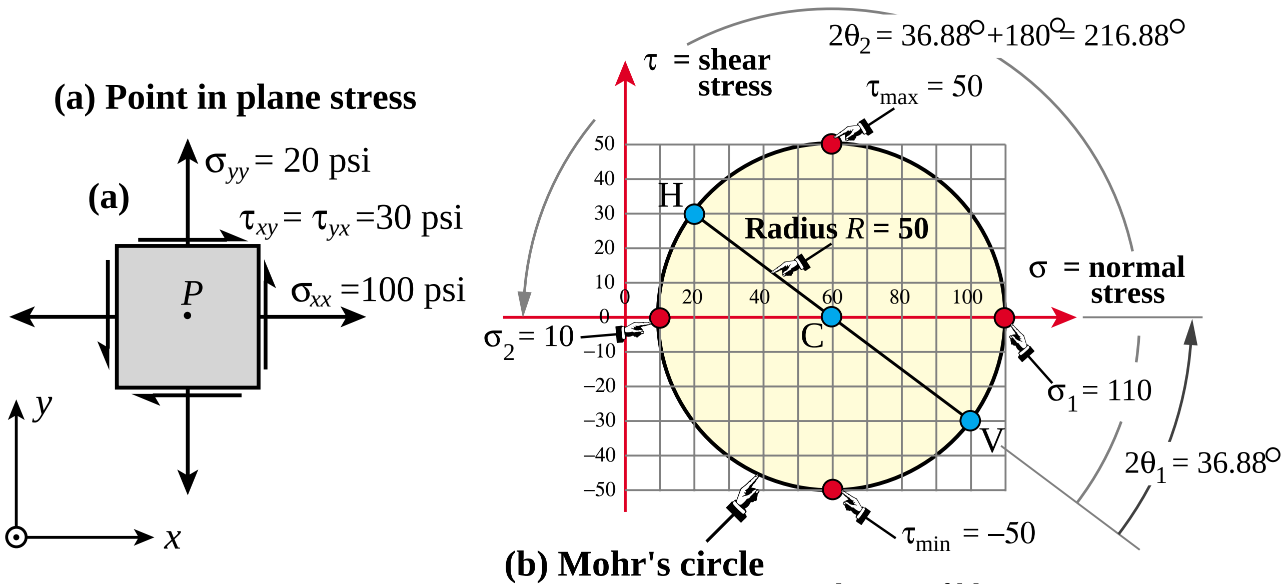
For further details, see Lecture notes. Some of these properties can be visualized more easily using the **Mohr's circle**, which provides a **graphical solution** to the plane stress transformation problem

Numeric Example



For computation details see Lecture notes

Graphical Solution of Example Using Mohr's Circle



coordinates of blue points are
H: (20,30), V:(100,-30), C:(60,0)

What Happens in 3D?

This topic be briefly covered in class if time allows, using the following slides.

If not enough time, ask students to read Lecture notes (Sec 7.3), with particular emphasis on the computation of the overall maximum shear

General 3D Stress State

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

There are three (3) principal stresses, identified as

$$\sigma_1 \; , \;\; \sigma_2 \;\; , \;\; \sigma_3$$

Principal Stresses in 3D (2)

The σ_i turn out to be the eigenvalues of the stress matrix. They are the roots of a cubic polynomial (the so-called characteristic polynomial)

$$C(\sigma) = \det \begin{bmatrix} \sigma_{xx}-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma \end{bmatrix}$$

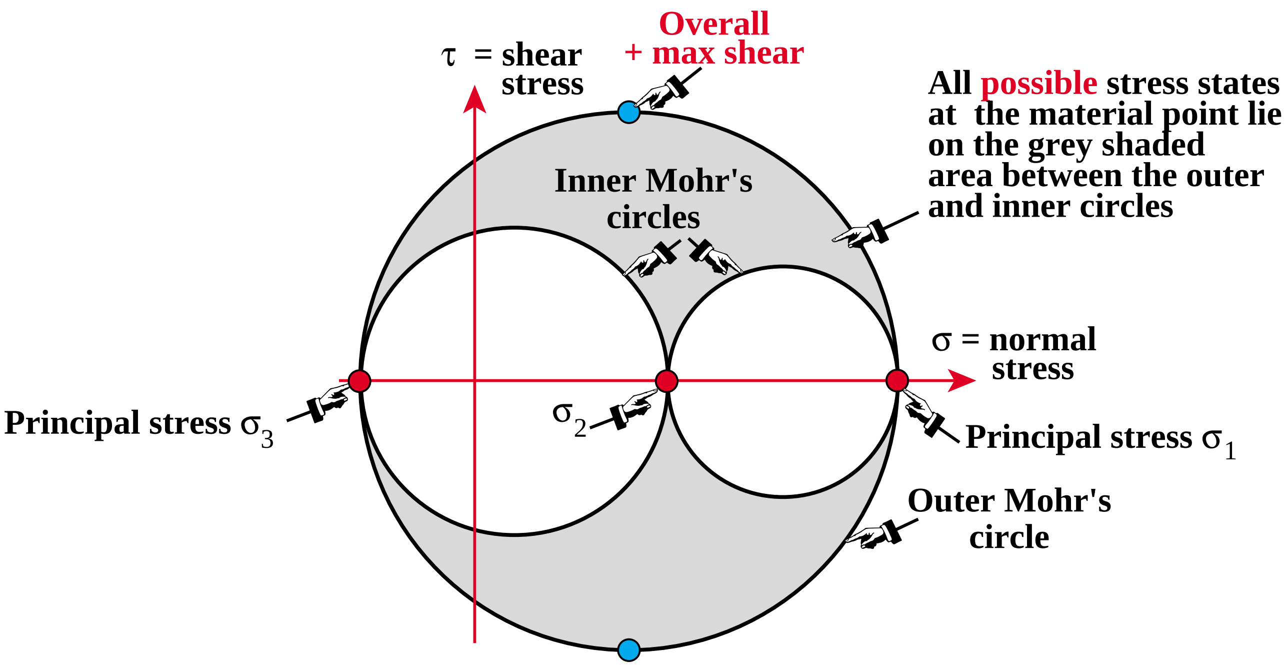
$$= -\sigma^3 + I_1\,\sigma^2 - I_2\,\sigma + I_3 = 0$$

The principal directions are given by the eigenvectors of the stress matrix.

Both eigenvalues and eigenvectors can be numerically computed by the Matlab function eig(.)

3D Mohr Circles

(Yes, There Is More Than One)



The overall maximum shear, which is the radius of the outer Mohr's circle, is important for assessing strength safety of ductile materials

The Overall Maximum Shear is the Radius of the Outer Mohr's Circle

If the principal stresses are algebraically ordered as

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

then

$$\tau_{max}^{overall} = R_{outer} = \frac{\sigma_1 - \sigma_3}{2}$$

Note that the intermediate principal stress σ_2 does not appear.

If they are not ordered it is necessary to use the max function in a more complicated formula that picks up the largest of the three radii:

$$\tau_{max}^{overall} = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right| \right)$$

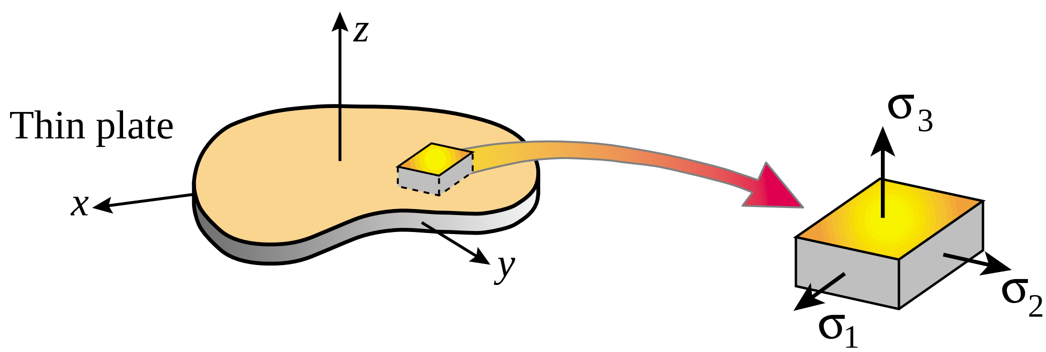
Plane Stress in 3D: The 3rd Principal Stress

Consider plane stress but now account for the third dimension. One of the principal stresses, call it for the moment σ_3 , is zero:

$$\sigma_1 \;,\quad \sigma_2 \;,\quad \sigma_3 = 0$$

where σ_1 and σ_2 are the inplane principal stresses obtained as described earlier in Lecture 6.

The zero principal stress σ_3 is aligned with the *z* axis (the thickness direction) while σ_1 and σ_2 act in the *x*,*y* plane:



Plane Stress in 3D: Overall Max Shear

Let us now (re)order the principal stresses by algebraic value as

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

To compute the overall maximum shear 2 cases are considered:

(A) Inplane principal stresses have opposite signs. Then the zero stress is the intermediate one: σ_2 , and

$$\tau_{max}^{overall} = \tau_{max}^{inplane} = \frac{\sigma_1 - \sigma_3}{2}$$

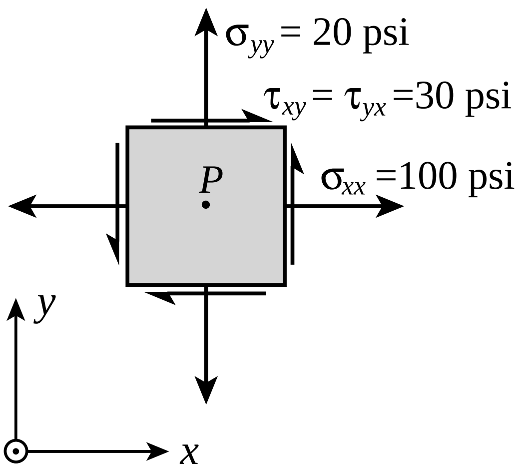
(B) Inplane principal stresses have the same sign. Then

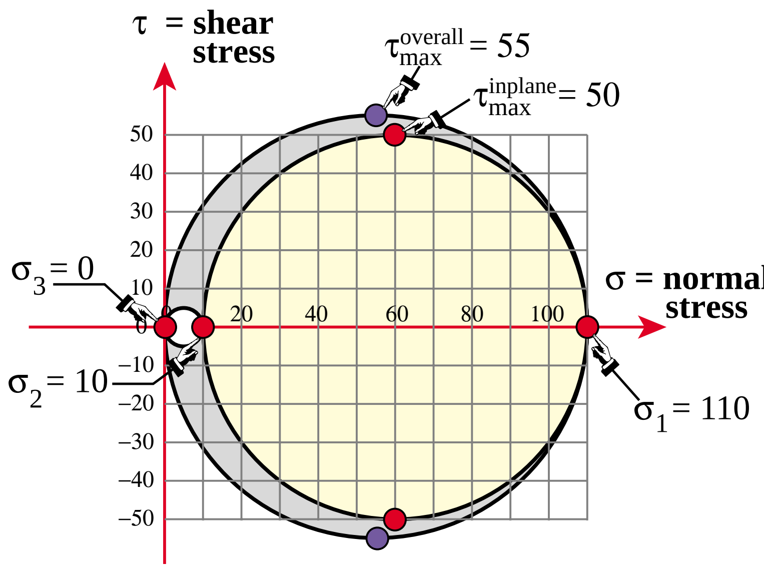
If $\sigma_1 \geq \sigma_2 \geq 0$ and $\sigma_3 = 0$, $\tau_{max}^{overall} = \dfrac{\sigma_1}{2}$

If $\sigma_3 \geq \sigma_2 \geq 0$ and $\sigma_1 = 0$, $\tau_{max}^{overall} = -\dfrac{\sigma_3}{2}$

Plane Stress in 3D: Example

Plane stress example treated earlier:





Yellow-filled circle is the in-plane Mohr's circle