1

Stress in 3D

Mechanical Stress in 3D: Concept

Mechanical stress measures ***intensity level*** of ***internal forces*** in a material (solid or fluid body) idealized as a mathematical ***continuum***. The physical measure of stress is

Force per unit area e.g. N/mm^2 (MPa) or lbs/sq-in (psi)

This measure is convenient to assess the resistance of a material to permanent deformation (yield, creep, slip) and rupture (fracture, cracking). Comparing working and failure stress levels allows engineers to establish ***strength safety factors*** for structures.

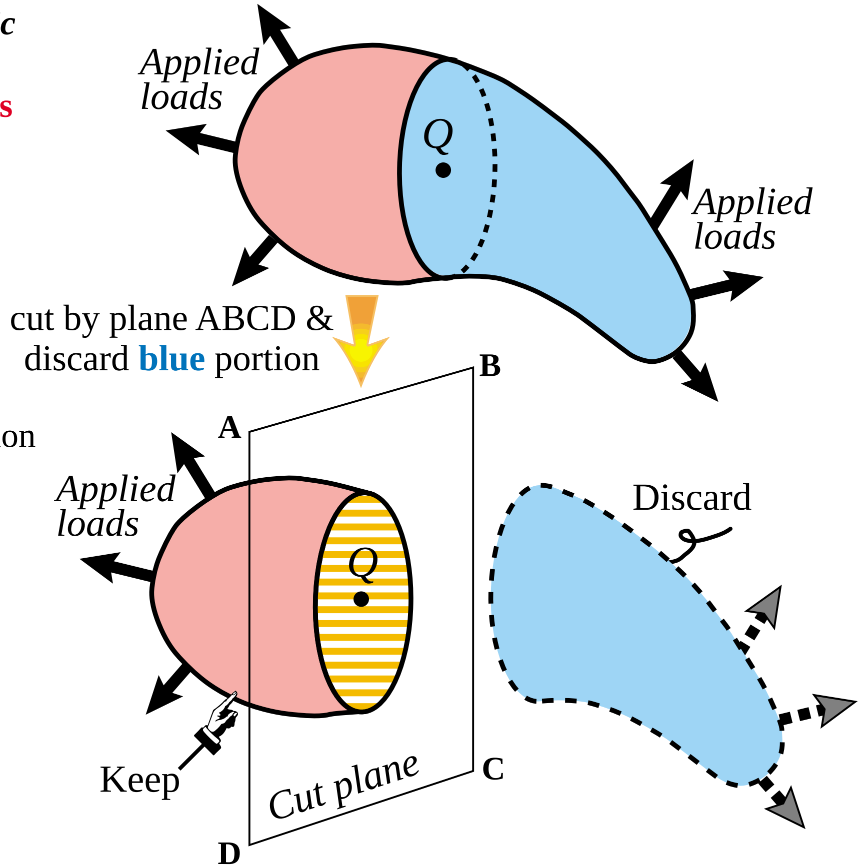
Stresses may vary from point to point. We next consider a solid body (could be a structure or part of one) in 3D.

Cutting a 3D Body

Consider a 3D solid body in ***static equilibrium*** under applied loads. We want to find the **state of stress** at an arbitrary point Q, which generally will be inside the body.

Cut the body by a ***plane*** ABCD that passes through Q as shown (How to ***orient*** the plane is discussed later.) The body is divided into two. Retain one portion (**red** in figure) and discard the other (**blue** in figure)

To ***restore equilibrium***, however, we must replace the discarded portion by the ***internal forces*** it had exerted on the kept portion.



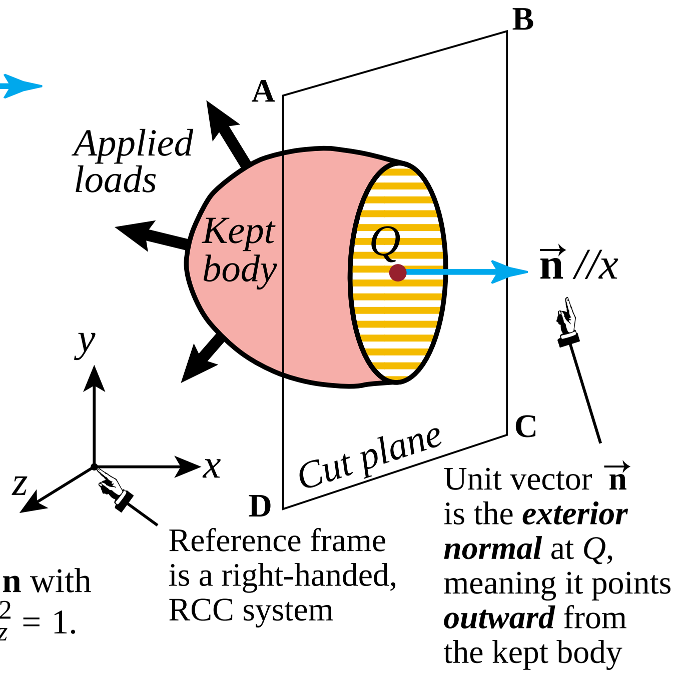
Orienting the Cut Plane

The cut plane ABCD is oriented by its unit normal direction vector **n**, or ***normal*** for short. By convention we will draw **n** as emerging from Q and pointing ***outward*** from the kept body. This direction identifies the ***exterior normal***.

With respect to the RCC system $\{x,y,z\}$, the normal vector has components

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

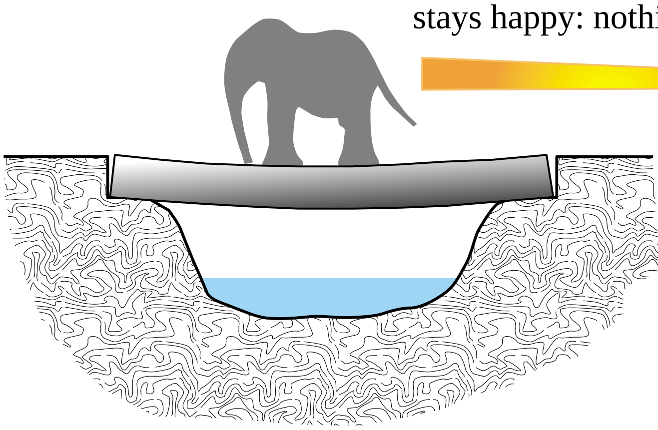
where $\{n_x, n_y, n_z\}$ are the direction cosines of **n** with respect to $\{x,y,z\}$. These satisfy $n_x^2 + n_y^2 + n_z^2 = 1$.

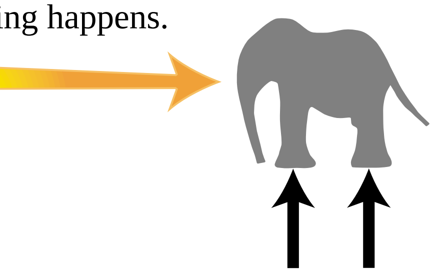


In the figure, the cut plane ABCD has been chosen with its exterior normal ***parallel*** to the $+x$ axis. Consequently $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

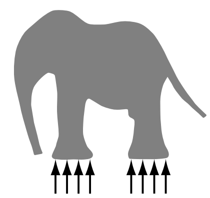
Digression: Action and Reaction
(Newton's 3rd Law, from Physics I)

Remove the "bridge" (log) and replace its effect on the elephant by ***reaction forces*** on the legs. The elephant stays happy: nothing happens.





Strictly speaking, reaction forces are ***distributed*** over the elephant leg contact areas. They are replaced above by equivalent point forces, a.k.a. ***resultants***, for visualization convenience

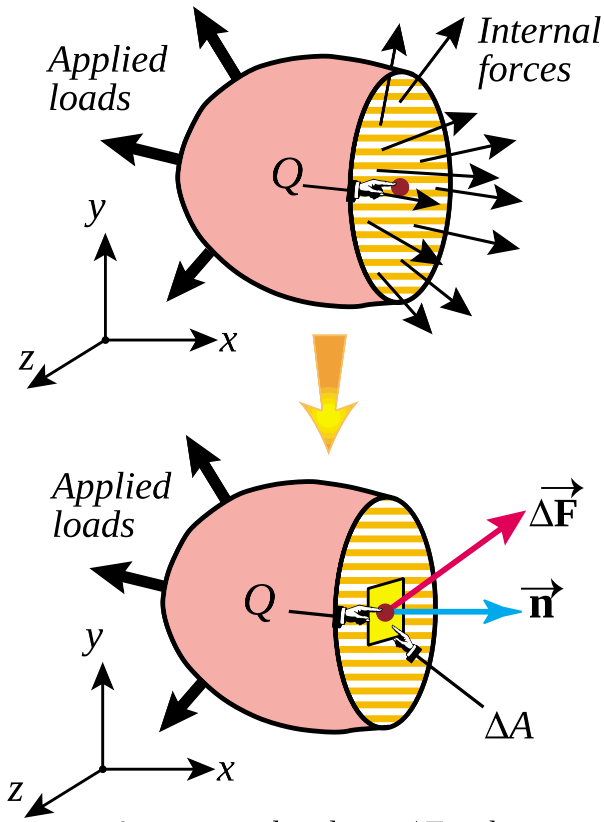


Internal Forces on Cut Plane

Those internal forces generally will form a system of ***distributed forces per unit of area***, which, being vectors, generally will vary in magnitude and direction as we move from point to point of the cut plane, as pictured.

Next, we focus our attention on point Q. Pick an ***elemental area*** ΔA around Q that lies on the cut plane. Call $\Delta\mathbf{F}$ the ***resultant*** of the internal forces that act on ΔA. Draw that vector with origin at Q, as pictured. Don't forget the normal

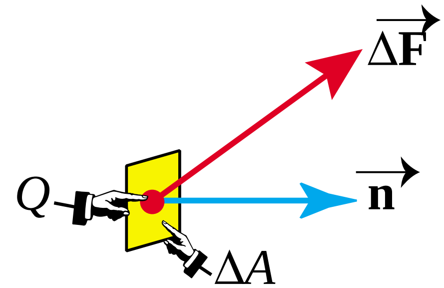
The use of the increment symbol Δ suggests a pass to the limit. This will be done later to define the stresses at Q



Arrows are placed over $\Delta\mathbf{F}$ and $\mathbf{n}$ to remind you that they are ***vectors***

Internal Force Components

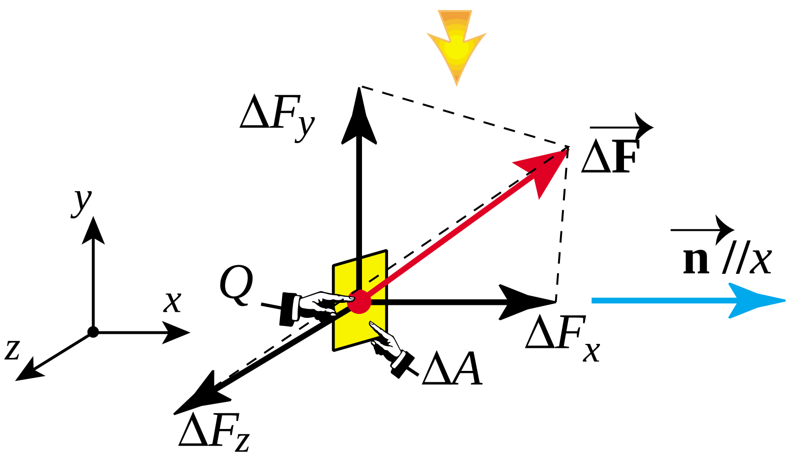
Zoom on the elemental area about Q, omitting the kept-body and applied loads for clarity:



Project vector $\Delta\mathbf{F}$ on axes x, y and z to get its components ΔF_x, ΔF_y and ΔF_z, respectively. See bottom figure.

Note that component ΔF_x is aligned with the cut-plane normal, because **n** is parallel to x. It is called the ***normal internal force*** component.

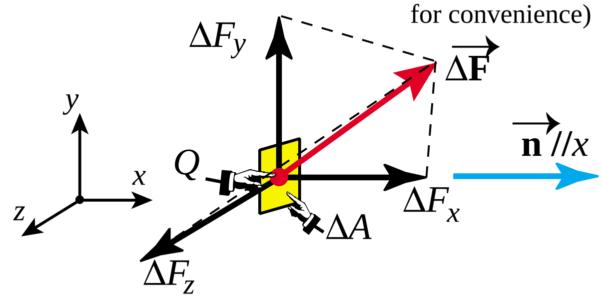
Components ΔF_y and ΔF_z lie on the cut plane. They are called ***tangential internal force*** components.



We are now ready to define stresses.

Stress Components at a Point: *x* Cut

(Reproduced from previous slide for convenience)



Define the ***x-stress components*** at point Q by taking the limits of internal-force-over-area ratios as the elemental area shrinks to zero:

$$\sigma_{xx} \stackrel{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \qquad \tau_{xy} \stackrel{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \qquad \tau_{xz} \stackrel{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}$$

σ_{xx} is called a ***normal stress,*** whereas τ_{xy} and τ_{xz} are ***shear stresses***.

Stress Components at a Point: *y* and *z* Cuts

It turns out we need ***nine*** stress components in 3D to fully characterize the stress state at a point. So far we got only three. Six more are obtained by repeating the same take-the-limit procedure with ***two other cut planes.*** The obvious choice is to pick planes normal to the other two axes: y and z.

Taking **n**//y we get three more components, one normal and two shear:

$$\boxed{\sigma_{yy} \quad \tau_{yx} \quad \tau_{yz}}$$

These are called ***y-stress components***.

Taking **n**//z we get three more components, one normal and two shear:

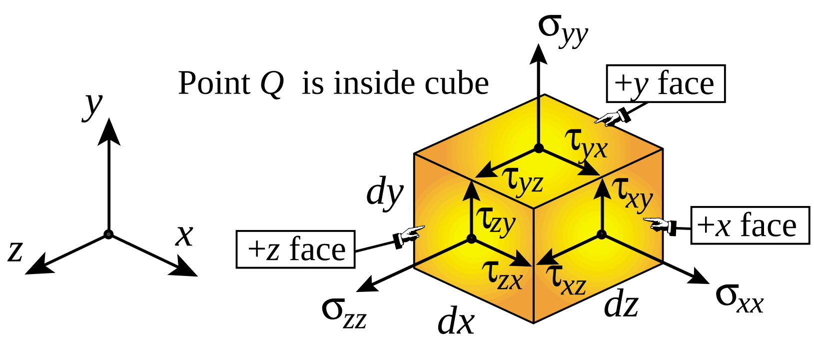
$$\boxed{\sigma_{zz} \quad \tau_{zx} \quad \tau_{zy}}$$

These are called ***z-stress components***.

Together with the three ***y-stress components*** found before, this makes up a total of nine, as required.

Visualization on "Stress Cube"

The foregoing nine stress components may be conveniently visualized on a ***"stress cube"*** as follows. Cut an ***infinitesimal cube*** *dx dy dz* around *Q* with sides parallel to the RCC axes {*x*,*y*,*z*}. Draw the components on the ***positive cube faces*** (defined below) as



Note that stresses are ***forces per unit area***, **not** forces, although they look like forces in the picture.

Strictly speaking, this is a "cube" only if $dx=dy=dz$, else it should be called a parallelepided; but that is difficult to pronounce.

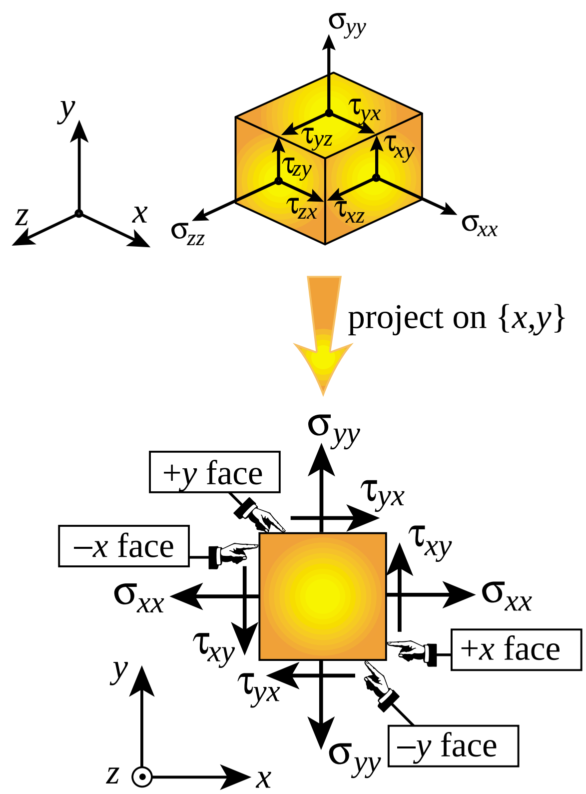
The three positive cube faces are those with exterior (outward) normals aligned with +*x*, +*y* and +*z*, respectively. Positive (+) values for stress components on those faces are as shown. More on sign conventions later.

What Happens On The Negative Faces?

The stress cube has three ***positive*** (+) ***faces***. The three opposite ones are ***negative*** (–) ***faces***. Outward normals at – faces point along –x, –y and –z. What do stresses on those faces look like? To maintain static equilibrium, stress components must be ***reversed***.

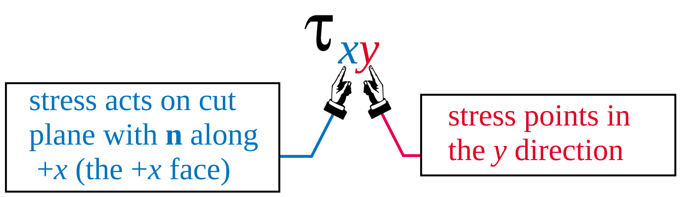
For example, a positive σ_{xx} points along +x on the +x face, but along –x on the –x face. A positive τ_{xy} points along +y on the +x face but along –y on the –x face.

To visualize the reversal, it is convenient to project the stress cube onto the {x,y} plane by looking at it from the +z direction. The resulting 2D figure clearly displays the "reversal recipe" given above.



Notational & Sign Conventions

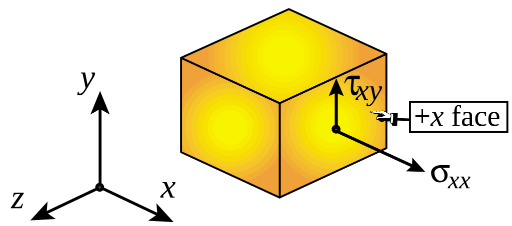
Shear stress components have two indices. The first one identifies the cut plane on which it acts, through the normal to that plane. The second index identifies component direction. For example:



Stress sign conventions are as follows.

For a ***normal stress***: positive (negative) if it produces tension (compression) in the material.

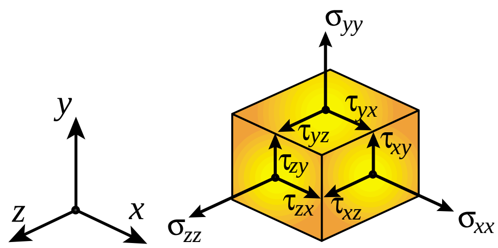
For a ***shear stress***: positive if, when acting on the + face identified by the first index, it points in the + direction identified by the second index. Example: τ_{xy} is + if on the +x face it points in the +y direction; see figure. The sign of a shear stress has no physical meaning; it is entirely conventional.



Both σ_{xx} and τ_{xy} are + as drawn above

Matrix Representation of Stress

The nine components of stress referred to the $\{x,y,z\}$ axes may be arranged as a 3 x 3 matrix, which is configured as



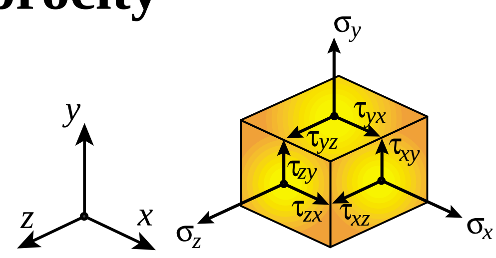
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Note that normal stresses are placed in the ***diagonal*** of this square matrix.

We will call this a 3D ***stress matrix,*** although in more advanced courses this is the representation of a second-order tensor.

Shear Stress Reciprocity

From moment equilibrium conditions on the stress cube it can be shown that



$$\boxed{\tau_{xy} = \tau_{yx}\,, \quad \tau_{xz} = \tau_{zx}\,, \quad \tau_{yz} = \tau_{zy}}$$

in ***magnitude***. In other words: switching shear stress indices does not change its value. Note, however, that stresses point in different directions: τ_{xy}, say, points along y whereas τ_{yx} points along x.

This property is known as ***shear stress reciprocity***. It follows that the stress matrix is ***symmetric:***

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} = \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} = \tau_{xz} & \tau_{zy} = \tau_{yz} & \sigma_{zz} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy} & \tau_{yz} \\ \mathit{symm} & & \sigma_{zz} \end{bmatrix}$$

Consequently the 3D stress state depends on only ***six (6) independent parameters***: three normal stresses and three shear stresses.

Simplifications: 2D and 1D Stress States

For certain structural configurations such as thin plates, all stress components with a *z* subscript may be considered negligible, and set to zero. The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

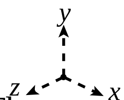
This 2D simplification is called ***plane stress state.*** Since $\tau_{xy} = \tau_{yx}$, this state is characterized by ***three*** independent stress components: σ_{xx}, σ_{yy} and $\tau_{xy} = \tau_{yx}$.

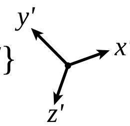
A further simplification occurs in structures such as bars or beams, in which all stress components except σ_{xx} may be considered negligible and set to zero, whence the stress matrix reduces to

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

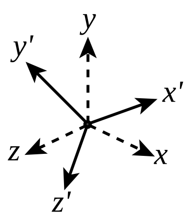
This is called a ***one-dimensional stress state***. There is only ***one*** independent stress component left: σ_{xx}

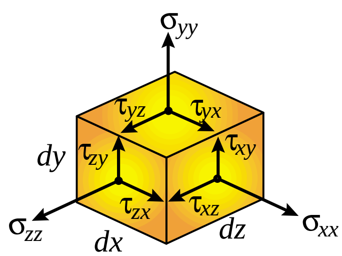
Changing Coordinate Axes (1)

Suppose we change axes $\{x,y,z\}$ 

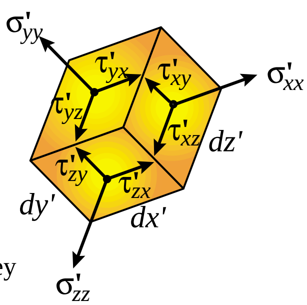
to another set $\{x',y',z'\}$ 

that also forms a RCC system. The stress cube centered at Q is rotated to realign with $\{x',y',z'\}$ as pictured below.





Note that Q is *inside both cubes*; they are drawn offset for clarity



The stress components change accordingly, as shown in matrix form:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \xrightarrow{\text{becomes}} \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix}$$

Can the primed components be expressed in terms of the original ones?

Changing Coordinate Axes (2)

The answer is ***yes***. All primed stress components can be expressed in terms of the unprimed ones and of the direction cosines of $\{x',y',z'\}$ with respect to $\{x,y,z\}$. This operation is called a ***stress transformation.***

For a general 3D state this operation is quite complicated because there are three direction cosines. In this introductory course we will cover only transformations for the 2D ***plane stress*** state. These are simpler because changing axes in 2D depends on only one direction cosine or, equivalently, the rotation angle about the z axis.

Why do we bother to look at stress transformations? Well, material failure may depend on the ***maximum normal tensile stress*** (for brittle materials) or the ***maximum absolute shear stress*** (for ductile materials). To find those we generally have to look at ***parametric rotations*** of the coordinate system, as in the skew-cut bar example studied later. Once such dangerous stress maxima are found for a given structure, the engineer can determine ***strength safety factors***.