1

Stress in 3D

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§1.1. Introduction

This lecture introduces mechanical stresses in a 3D solid body. It covers definitions, notational and sign conventions, stress visualization, and reduction to two- and one-dimensional cases. Effect of coordinate axes transformation and the notion of tensors are briefly mentioned as advanced topics.

§1.2. Mechanical Stress: Concept

Mechanical stress in a solid body is a gross abstraction of the intensity of interatomic or intermolecular forces. If we look at a solid under increasing magnification, say through an electron microscope, we will see complicated features such as crystals, molecules and atoms appearing (and fading) at different scales. The detailed description of particle-interaction forces acting at such tiny scales is not only impractical but unnecessary for structural design and analysis. To make the idea tractable the body is viewed as a *continuum* of points in the mathematical sense, and a *stress state* is defined at each point by a force-over-area limit process.

Mechanical stress in a solid generalizes the simpler concept of *pressure* in a fluid. A fluid in static equilibrium (the so-called hydrostatic equilibrium in the case of a liquid) can support only a pressure state. (In a gas pressure may be tensile or compressive whereas in a liquid it must be compressive.) A solid body in static equilibrium can support a more general state of stress, which includes both *normal* and *shear* components. This generalization is of major interest to structural engineers because structures, for obvious reasons, are fabricated with solid materials.

§1.3. Mechanical Stress: Definition

This section goes over the process by which the stress state at an arbitrary point of a solid body is defined. The key ideas are based on Newton's third law of motion. Once the physical scenario is properly set up, mathematics takes over.

§1.3.1. What Does Stress Measure?

Mechanical stress measures the average intensity level of internal forces in a material (solid or fluid body) idealized as a *mathematical continuum*. The physical dimension of stress is

$$\boxed{\text{Force per unit area, e.g. N/mm}^2\text{ (MPa) or lbs/sq-in (psi)}} \tag{1.1}$$

This measure is convenient to assess the resistance of a material to permanent deformation (yield, creep, slip) as well as rupture (fracture, cracking). Comparing working and failure stress levels allows engineers to establish *strength safety factors* for structures. This topic (safety factor) is further covered in Lecture 2.

Stresses may vary from point to point in a body. In the following we consider a arbitrary solid body (which may be a structure) in a three dimensional (3D) setting. As previouly noted, this is actually its *idealization* as a mathematical continuum. Thus the term "body" actually refers to that idealization.

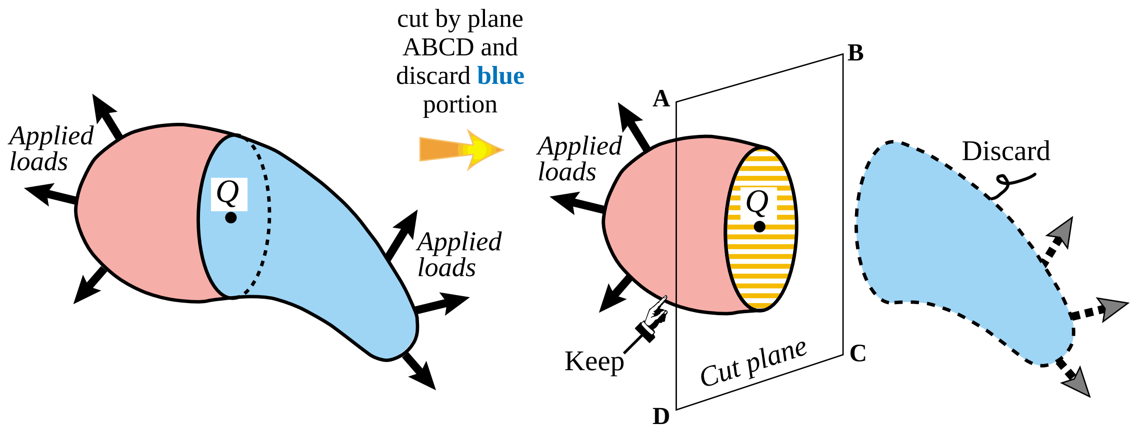


FIGURE 1.1. Cutting a 3D body through a plane passing by point Q.

§1.3.2. Cutting a Body

The 3D solid body depicted on the left of Figure 1.1 is asumed to be in *static equilibrium* under the applied loads acting on it. We want to find the state of stress at an arbitrary point Q. This point will generally be located inside the body, but it could also lie on its surface.

Cut the body by a plane ABCD that passes through Q (How to *orient* the plane is discussed in the next subsection.) The body is divided into two. Retain one portion (red in figure) and discard the other (blue in figure), as shown on the right of Figure 1.1.

To restore equilibrium, however, we must *replace the discarded portion by the internal forces it had exerted on the kept portion*. This is a consequence of Newton's third law: action and reaction. If you are rusty on that topic please skim the Addendum at the end of this Lecture.

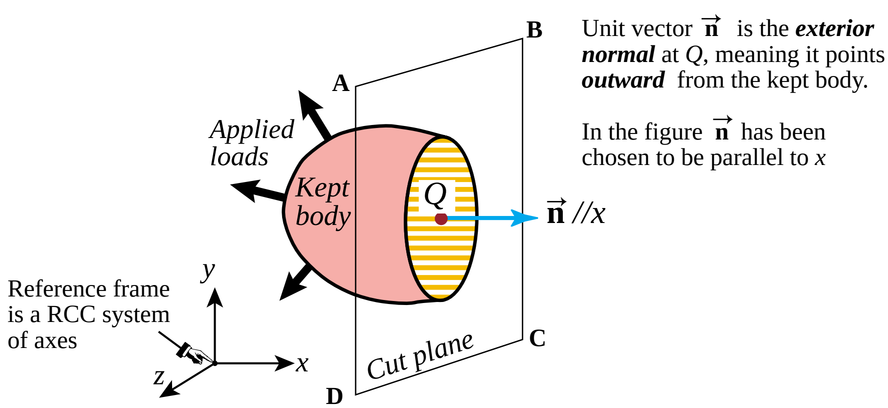


FIGURE 1.2. Orienting the cut plane ABCD by its exterior unit normal vector $\vec{\mathbf{n}}$.

§1.3.3. Orienting the Cut Plane

The cut plane ABCD is oriented by its unit normal direction vector $\vec{\mathbf{n}}$, or *normal* for short. See Figure 1.2. By convention we will draw $\vec{\mathbf{n}}$ as emerging from Q and pointing *outward* from the kept body, as shown in Figure 1.2. This direction identifies the so-called *exterior* normal.

At this point we refer the body to a Rectangular Cartesian Coordinate (RCC) system of axes $\{x, y, z\}$. This reference frame obeys the right-hand orientation rule. The coordinates of point Q, denoted by $\{x_Q, y_Q, z_Q\}$, are called its *position coordinates* or *position components*. The *position vector* of Q is denoted by

$$\vec{\mathbf{x}}_Q = \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix} \tag{1.2}$$

but we will not need to use this vector here.

With respect to $\{x, y, z\}$ the normal vector has components

$$\vec{\mathbf{n}} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{1.3}$$

in which $\{n_x, n_y, n_z\}$ are the direction cosines of $\vec{\mathbf{n}}$ with respect to $\{x, y, z\}$, respectively. Since $\vec{\mathbf{n}}$ is a unit vector, those components must verify the unit length condition

$$n_x^2 + n_y^2 + n_z^2 = 1. \tag{1.4}$$

In Figure 1.2, the cut plane ABCD has been chosen with its exterior normal parallel to the $+x$ axis. Consequently (1.3) reduces to

$$\vec{\mathbf{n}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.5}$$

§1.3.4. Internal Forces on Elemental Area

Recall that the action of the discarded (blue) portion of the body on the kept (red) portion is replaced by a system of *internal forces* that restores static equilibrium. This replacement is illustrated on the right of Figure 1.3. Those internal forces generally will form a system of *distributed forces* per unit of area, which, being vectors, generally will vary in magnitude and direction as we move from point to point of the cut plane, as pictured over there.

Next we focus our attention on point Q. Pick an elemental area ΔA around Q *that lies on the cut plane*. Call $\Delta\vec{\mathbf{F}}$ the *resultant* of the internal forces that act on ΔA. Draw that vector with origin at Q, as pictured on the right of Figure 1.3. Do not forget to draw also the unit normal vector $\vec{\mathbf{n}}$.

The use of the increment symbol Δ suggests a pass to the limit. And indeed this will be done in equations (1.6) below, to define three stress components at Q.

§1.3.5. Projecting the Internal Force Resultant

Zoom now on the elemental area about Q, omitting both the kept-body and applied loads for clarity, as pictured on the left of Figure 1.4.

Project the internal force resultant $\Delta\vec{\mathbf{F}}$ on the reference axes $\{x, y, z\}$. This produces three components: ΔF_x, ΔF_y and ΔF_z, as shown on the right of Figure 1.4.

Component ΔF_x is aligned with the cut-plane normal, because $\vec{\mathbf{n}}$ has been taken to be parallel to x. This is called the *normal internal force component* or simply *normal force*. On the other hand, components ΔF_y and ΔF_z *lie on the cut plane*. These are called *tangential internal force components* or simply *tangential forces*.

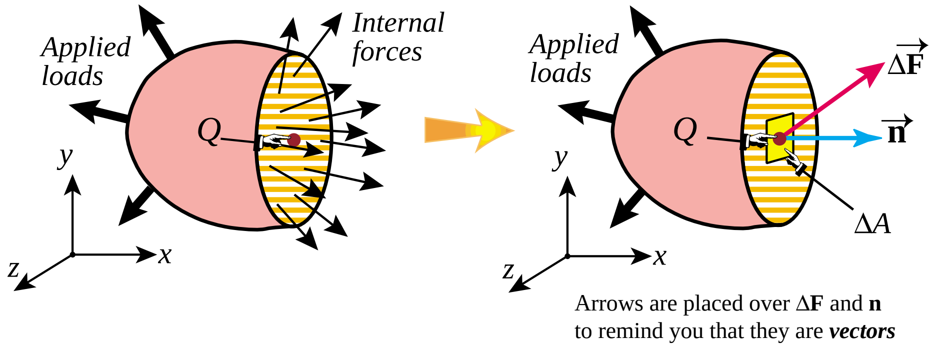


FIGURE 1.3. Internal force resultant $\Delta\vec{\mathbf{F}}$ over elemental area about point Q.

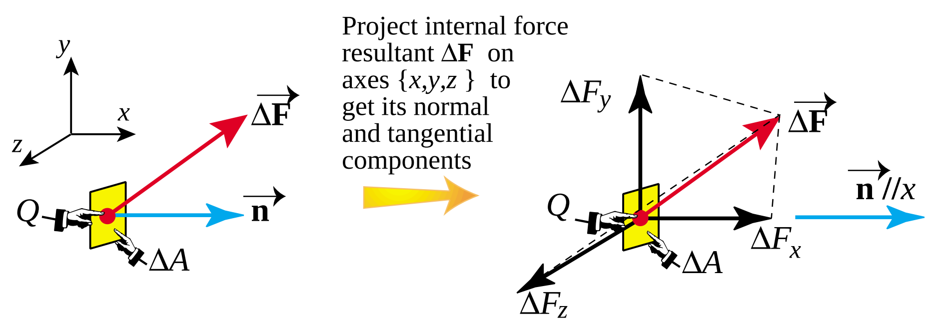


FIGURE 1.4. Projecting the internal force resultant onto normal and tangential components.

§1.3.6. Defining Three Stress Components

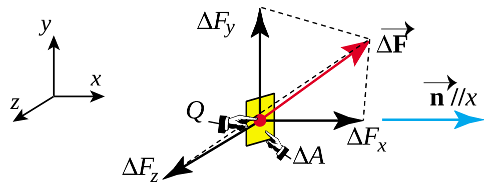
We define the three x-stress components at point Q by taking the limits of internal-force-over-elemental-area ratios as that area shrinks to zero:

$$\boxed{\begin{array}{lll} \sigma_{xx} \stackrel{\text{def}}{=} \lim\limits_{\Delta A\to 0} \dfrac{\Delta F_x}{\Delta A} & \text{normal stress component} \\ \tau_{xy} \stackrel{\text{def}}{=} \lim\limits_{\Delta A\to 0} \dfrac{\Delta F_y}{\Delta A} & \text{shear stress component} \\ \tau_{xz} \stackrel{\text{def}}{=} \lim\limits_{\Delta A\to 0} \dfrac{\Delta F_z}{\Delta A} & \text{shear stress component} \end{array}} \tag{1.6}$$

σ_{xx} is called a *normal stress*, whereas τ_{xy} and τ_{xz} are *shear stresses*. See Figure 1.5.

Remark 1.1. We tacitly assumed that the limits (1.6) exist and are finite. This assumption is part of the axioms of continuum mechanics.

Remark 1.2. The use of two different letters: σ and τ for normal and shear stresses, respectively, is traditional in American undergraduate education. It follows the influential textbooks by Timoshenko (a key contributor to the development of Enginering Mechanics education in the US) that appeared in the 1930-40s. The main reason for carrying along two symbols instead of one was to emphasize their distinct physical effects on structural materials. It has the disadvantage, however, of poor fit with the tensorial formulation used in more advanced (graduate level) courses in continuum mechanics. In those courses a more unified notation, such as σ_{ij} or τ_{ij} for *all* stress components, is used.



(reproduced from previous figure for convenience)

$$\boxed{\sigma_{xx} \overset{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \qquad \tau_{xy} \overset{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A} \qquad \tau_{xz} \overset{\text{def}}{=} \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A}}$$

FIGURE 1.5. Defining stress components σ_{xx}, τ_{xy} and τ_{xz} as force-over-area mathematical limits.

§1.3.7. Six More Components

Are we done? No. It turns out we need *nine* stress components in 3D to fully characterize the stress state at a point. So far we got only three. Six more are obtained by repeating the same procedure: find internal force resultant, project onto axes, divide by elemental area and take-the-limit, using *two more cut planes*. The obvious choice is to pick planes normal to the other two axes: y and z.

Taking $\vec{\mathbf{n}}$ parallel to y and going through the motions we get three more components, one normal and two shear:

$$\boxed{\sigma_{yy}, \quad \tau_{yx}, \quad \tau_{yz}.} \tag{1.7}$$

These are called the y-stress components. Finally, taking $\vec{\mathbf{n}}$ parallel to z we get three more components, one normal and two shear:

$$\boxed{\sigma_{zz}, \quad \tau_{zx}, \quad \tau_{zy}.} \tag{1.8}$$

These are called the z-stress components. On grouping (1.6) , (1.7) and (1.8) we arrive at a total of nine components, as required for full characterization of the stress at a point. Now we are done.

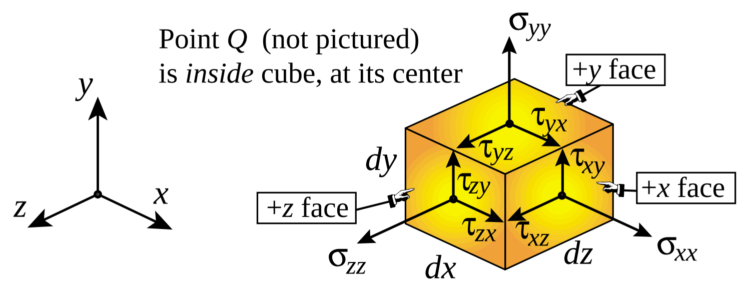
§1.3.8. Visualization on Stress Cube

The foregoing nine stress components may be conveniently visualized on a ”stress cube” as follows. Cut an infinitesimal cube about Q with sides parallel to the RCC axes $\{x, y, z\}$, and dimensioned dx, dy and dz, respectively. Draw the components on the positive cube faces (“positive face” is defined below) as depicted in Figure 1.6

The three positive cube faces are those with exterior (outward) normals aligned with $+x$, $+y$ and $+z$, respectively. Positive $(+)$ values for stress components on those faces are as drawn in Figure 1.6. More on sign conventions later.

§1.3.9. What Happens on the Negative Faces?

The stress cube has three positive $(+)$ faces. The three opposite ones are negative $(-)$ faces. Outward normals at $-$ faces point along $-x$, $-y$ and $-z$, respectively. What do stresses on those faces look like? To maintain static equilibrium, stress components must be *reversed*.



Note that stresses are ***forces per unit area***, **not** forces, although they look like forces in the picture.

Strictly speaking, this is a "cube" only if $dx=dy=dz$, else it should be called a parallelepided; but that is difficult to pronounce.
(The correct math term is ``rectangular prism")

FIGURE 1.6. Visualizing stresses on faces of "stress cube" aligned with reference frame axes. Stress components are positive as drawn.

For example, a positive σ_{xx} points along the $+x$ direction on the $+x$ face, but along $-x$ on the opposite $-x$ face. A positive τ_{xy} points along $+y$ on the $+x$ face but along $-y$ on the $-x$ face.

To better visualize stress reversal, it is convenient to project the stress cube onto the $\{x, y\}$ plane by looking at it from the $+z$ direction. The resulting 2D diagram, shown in Figure 1.7, clearly displays the rule given above.

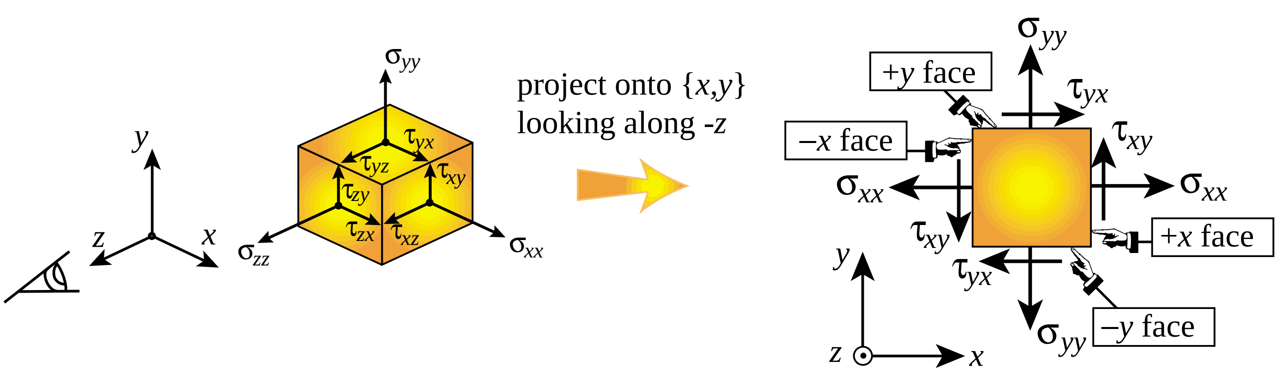


FIGURE 1.7. Projecting onto $\{x, y\}$ to display component stress reversals on going from + to − faces.

§1.4. Notational Conventions

§1.4.1. Sign and Subscripting Conventions

Stress component sign conventions are as follows.

- For a normal stress: positive (negative) if it produces tension (compression) in the material.
- For a shear stress: positive if, when acting on the + face identified by the first index, it points in the + direction identified by the second index. Example: τ_{xy} is + if on the $+x$ face it points in the $+y$ direction.

These conventions are illustrated for σ_{xx} and τ_{xy} on the left of Figure 1.8.

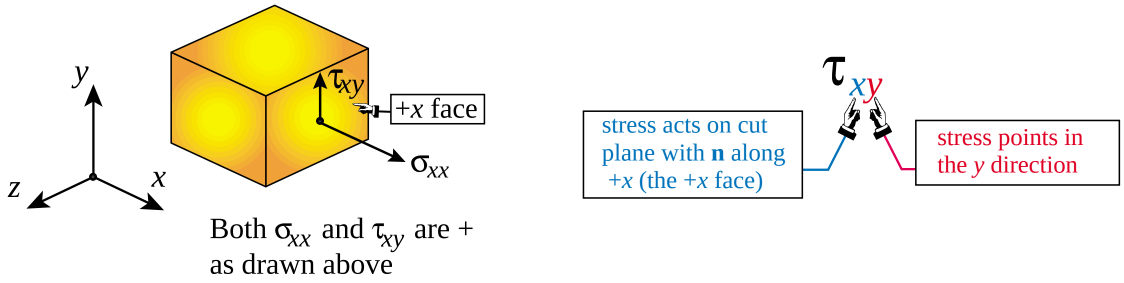


FIGURE 1.8. Sign and subscripting conventions for stress components.

Shear stress components have two different subscript indices. The first one identifies the cut plane on which it acts as defined by the unit exterior *normal* to that plane. The second index identifies component *direction*. That convention is illustrated for the shear stress component τ_{xy} on the right of Figure 1.8.

Remark 1.3. The foregoing subscripting convention plainly applies also to normal stresses, in which case the direction of the cut plane and the direction of the component merge. Because of this coalescence some authors (for instance, Beer, Johnston and DeWolf in their *Mechanics of Materials* book) drop one of the subscripts and denote σ_{xx}, say, simply by σ_x.

Remark 1.4. The sign of a normal component is physically meaningful since some structural materials, for example concrete, respond differently to tension and compression. On the other hand, the sign of a shear stress has no physical meaning; it is entirely conventional.

§1.4.2. Matrix Representation of Stress

The nine components of stress referred to the x,y,z axes may be arranged as a 3 x 3 matrix, which is configured as

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.9}$$

Note that normal stresses are placed in the *diagonal* of this square matrix.

We will call this a 3D *stress matrix*, although in more advanced courses this is the representation of a second-order tensor called — as may be expected — the stress tensor.

§1.4.3. Shear Stress Reciprocity

From moment equilibrium conditions on the infinitesimal stress cube it may be shown that

$$\boxed{\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{xy} = \tau_{zy},} \tag{1.10}$$

in *magnitude*. In other words: *switching shear stress indices does not change its value*. Note, however, that index-switched shear stresses point in different directions: τ_{xy}, say, points along y whereas τ_{yx} points along x.

For a proof of (1.10) see, for example, pp. 26–27 of Beer-Johnston-DeWolf 5th ed.

Property (1.10) is known as *shear stress reciprocity*. It follows that the stress matrix is *symmetric*:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} = \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} = \tau_{xz} & \tau_{zy} = \tau_{yz} & \sigma_{zz} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma_{yy} & \tau_{yz} \\ symm & & \sigma_{zz} \end{bmatrix} \tag{1.11}$$

Consequently the 3D stress state depends on only six (6) *independent* components: three normal stresses and three shear stresses.

§1.5. Simplifications: 2D and 1D Stress States

For certain structural configurations such as thin plates, all stress components with a z subscript may be considered negligible, and set to zero. The stress matrix of (1.9) becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.12}$$

This two-dimensional simplification is called a *plane stress* state. Since $\tau_{xy} = \tau_{yx}$, plane stress is fully characterized by just *three* independent stress components: the two normal stresses σ_{xx} and σ_{yy}, and the shear stress τ_{xy}.

A further simplification occurs in structures such as bars or beams, in which all stress components except σ_{xx} may be considered negligible and set to zero, whence the stress matrix reduces to

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.13}$$

This is called a *one-dimensional stress state*. There is only *one* independent stress component left: the normal stress σ_{xx}.

§1.6. Advanced Topics

We mention a couple of advanced topics that either fall outside the scope of the course, or will be later covered for special cases.

§1.6.1. Changing Coordinate Axes

Suppose we change axes $\{x, y, z\}$ to another set $\{x', y', z'\}$ that also forms a RCC system. The stress cube centered at Q is rotated to realign with $\{x', y', z'\}$ as illustrated in Figure 1.9. The stress components change accordingly, as compactly shown in matrix form:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix} \tag{1.14}$$

Can the primed components be expressed in terms of the original ones? The answer is *yes*. All primed stress components can be expressed in terms of the unprimed ones and of the direction cosines of $\{x', y', z'\}$ with respect to $\{x, y, z\}$. This operation is called a *stress transformation*.

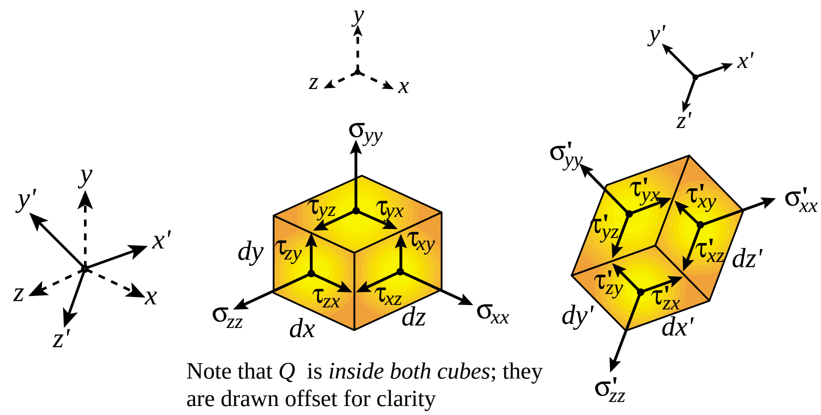


FIGURE 1.9. Transforming stress components in 3D when reference axes change.

For a general 3D state this operation is complicated because there are three direction cosines. In this introductory course we will cover only transformations for the *2D plane stress* state. The transformations are simpler (and more explicit) since changing axes in 2D depends on only one direction cosine or, equivalently, the rotation angle about the z axis.

Why bother to look at stress transformations? One important reason: material failure may depend on the maximum normal tensile stress (for brittle materials) or the maximum absolute shear stress (for ductile materials). To find those we generally have to look at parametric rotations of the coordinate system, as in the skew-cut bar example studied in the first Recitation. Once such dangerous stress maxima are found for critical points of a given structure, the engineer can determine strength safety factors.

§1.6.2. A Word on Tensors

The state of stress at a point is not a scalar or a vector. It is a more complicated mathematical object, called a *tensor* (more precisely, a second-order tensor). Tensors are not covered in undergraduate engineering courses as mathematical entities. Accordingly we will deal with stresses (and strains, which are also tensors) using a physical approach complemented with recipes. Nonetheless for those of you interested in the deeper mathematical aspect before reaching graduate school, here is a short list that "extrapolates" tensors from two entities you already (should have) encountered in Calculus and Physics courses.

- **Scalars** are defined by *magnitude*. Examples: temperature, pressure, density, charge.
- **Vectors** are defined by *magnitude* and *direction*. Examples: force, displacement, velocity, acceleration, electric current.
- **Second-order tensors** are defined by *magnitude* and *two directions*. Examples: stress, strain, electromagnetic field strength, space curvature in GRT.

For mechanical stress, the two directions are: the orientation of the cut plane (as defined by its outward normal) and the orientation of the internal force component.

All three kinds of objects (scalars, vectors, and second-order tensors) may vary from point to point. This means that they are expressed as functions of the position coordinates. In mathematical physics such functions are called *fields*.

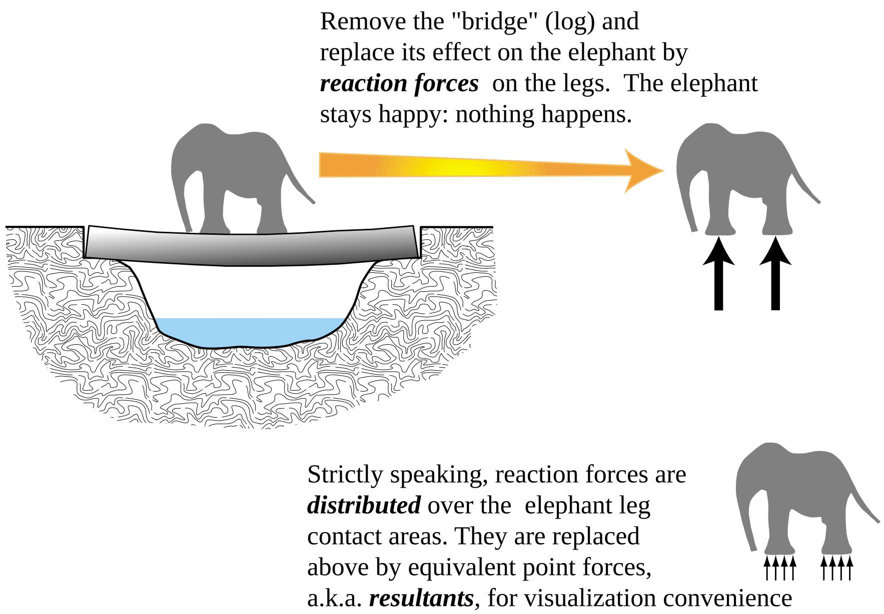


FIGURE 1.10. How to keep a bridge-crossing elephant happy.

§1.7. Addendum: Action vs. Reaction Reminder

This is a digression about Newton's Third Law of motion, which you should have encountered in Physics 1. See cartoon in Figure 1.10 for a reminder of how to replace the effect of a removed physical body by that of its reaction forces on the kept body.