6

Plane Stress Transformations

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§6.1. Introduction

This Lecture deals with the *plate stress* problem. This is a two-dimensional stress state, briefly introduced in §1.5 of Lecture 1. It occurs frequently in two kinds of aerospace structural components:

1. Thin wall plates and shells; e.g., aircraft and rocket skins, and the pressure vessels of Lecture 3.
2. Shaft members that transmit torque. These will be studied in Lectures 7–9.

The material below focuses on thin *flat plates*, and works out the associated problem of plane stress transformations.

§6.2. Thin Plate in Plate Stress

In structural mechanics, a flat thin sheet of material is called a *plate*. The distance between the plate faces is the *thickness*, which is denoted by h. The *midplane* lies halfway between the two faces.

The direction normal to the midplane is the *transverse* direction. Directions parallel to the midplane are called *in-plane* directions. The global axis z is oriented along the transverse direction. Axes x and y are placed in the midplane, forming a right-handed Rectangular Cartesian Coordinate (RCC) system. Thus the equation of the midplane is $z = 0$. The $+z$ axis conventionally defines the *top surface* of the plate as the one that it intersects, whereas the opposite surface is called the *bottom surface*. See Figure 6.1.

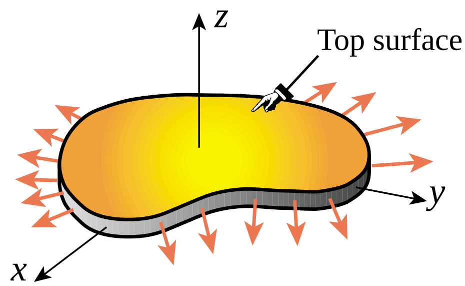

FIGURE 6.1. A plate structure in plane stress.

A plate loaded in its midplane is said to be in a state of *plane stress*, or a *membrane state*, if the following assumptions hold:

1. All loads applied to the plate act in the midplane direction, as pictured in Figure 6.1, and are symmetric with respect to the midplane.
2. All support conditions are symmetric about the midplane.
3. In-plane displacements, strains and stresses are taken to be uniform through the thickness.
4. The normal and shear stress components in the z direction are zero or negligible.

The last two assumptions are not necessarily consequences of the first two. For those to hold, the thickness h should be small, typically 10% or less, than the shortest in-plane dimension. If the plate thickness varies it should do so gradually. Finally, the plate fabrication must exhibit symmetry with respect to the midplane.

To these four assumptions we add an adscititious restriction:

5. The plate is fabricated of the same material through the thickness. Such plates are called *transversely homogeneous* or (in aerospace) *monocoque* plates.

The last assumption excludes wall constructions of importance in aerospace, in particular composite and honeycomb sandwich plates. The development of mathematical models for such configurations requires a more complicated integration over the thickness as well as the ability to handle coupled bending and stretching effects. Those topics fall outside the scope of the course.

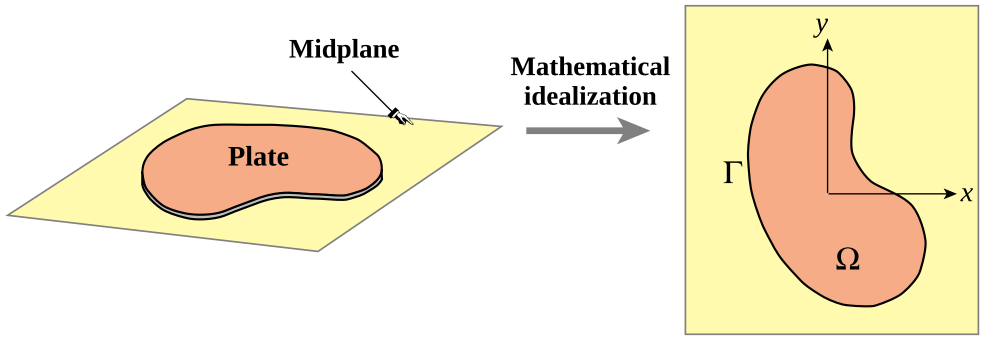

FIGURE 6.2. Mathematical model of plate in plane stress. (Symbols Ω and Γ, used to denote the plate interior and the boundary, respectively, are used in advanced courses.)

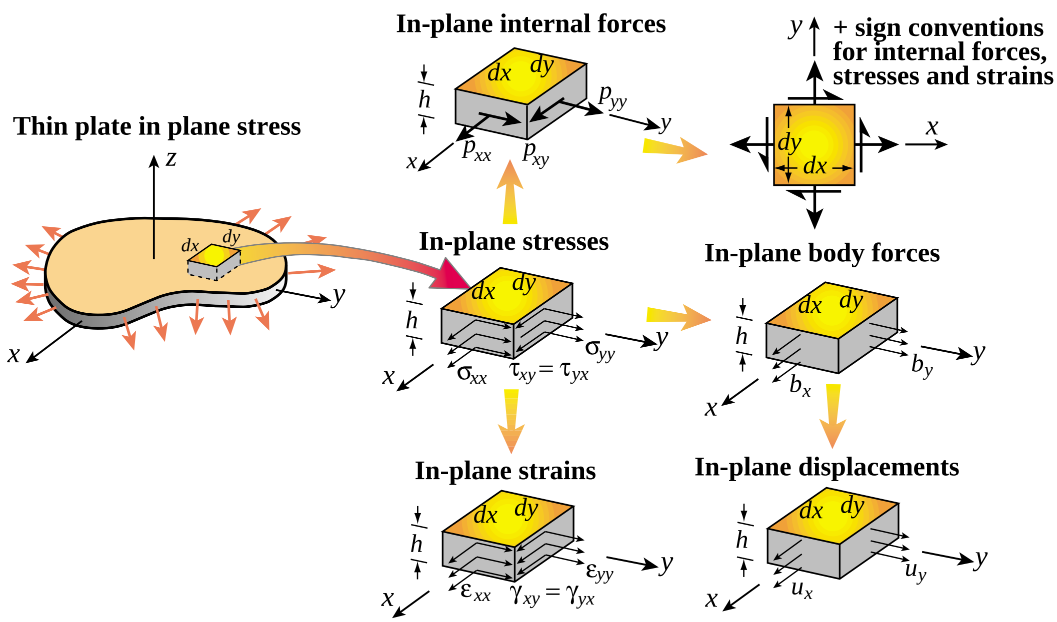

FIGURE 6.3. Notational conventions for in-plane stresses, strains, displacements and internal forces of a thin plate in plane stress.

Remark 6.1. Selective relaxation from assumption 4 leads to the so-called *generalized plane stress state*, in which nonzero σ_{zz} stresses are accepted; but these stresses do not vary with z. The *plane strain state* described in §5.6.2 of Lecture 5 is obtained if *strains* in the z direction are precluded: $\epsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$.

Remark 6.2. Transverse loading on a plate produces *plate bending*, which is associated with a more complex configuration of internal forces and deformations. This topic is studied in graduate-level courses.

Remark 6.3. The requirement of flatness can be relaxed to allow for a *curved* configuration, as long as the structure, or structure component, resists primarily *in-plane loads*. In that case the midplane becomes a *midsurface*. Examples are rocket and aircraft skins, ship and submarine hulls, open parachutes, boat sails and balloon walls. Such configurations are said to be in a *membrane* state. Another example are thin-wall members under torsion, which are covered in Lectures 8–9.

The plate in plane stress idealized as a two-dimensional problem is illustrated in Figure 6.2.

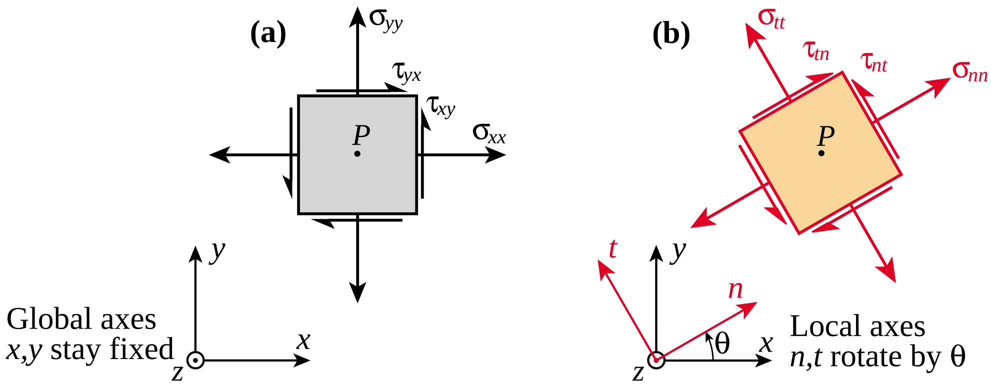

FIGURE 6.4. Plane stress system referred to global axes x, y (data) and to local rotated axes n, t. (Locations of point P in (a,b) coincide; they are drawn offset for visualization convenience.)

In this idealization the third dimension is represented as functions of x and y that are *integrated through the plate thickness*. Engineers often work with internal plate forces, which result from integrating the in-plane stresses through the thickness. See Figure 6.3.

In this Lecture we focus on the in-plane stresses σ_{xx}, σ_{yy} and τ_{xy} and their expressions with respect to an arbitrary system of axes

§6.3. 2D Stress Transformations

The stress transformation problem studied in this Lecture is illustrated in Figure 6.4. Stress components σ_{xx}, σ_{yy} and τ_{xy} at a midplane point P are given with respect to the *global axes* x and y, as shown in Figure 6.4(a). The material element about P is rotated by an angle θ that aligns it with axes n, t, as shown in Figure 6.4(b). (Note that location of point P in (a,b) coincide — they are drawn offset for visualization convenience.)

The *transformation problem* consists of expressing σ_{nn}, σ_{tt}, and $\tau_{nt} = \tau_{tn}$ in terms of the stress data σ_{xx}, σ_{yy} and τ_{xy}, and of the angle θ. Two methods, one analytical and one graphical, will be described here. Before that is done, it is useful to motivate what are the main uses of these transformations.

§6.3.1. Why Are Stress Transformations Important?

The transformation problem has two major uses in structural analysis and design.

- **Find stresses along a given skew direction**. Here θ is given as data. This has several applications. Two examples:
    1. Analysis of fiber reinforced composites if the direction of the fibers is not aligned with the $\{x, y\}$ axes. A tensile normal stress perpendicular to the fibers may cause delamination, and a compressive one may trigger buckling.
    2. Oblique joints that may fail by shear parallel to the joint. For example, welded joints.
- **Find max/min normal stresses, max in-plane shear and overall max shear**. This may be important for strength and safety assessment. Here finding the angle θ is part of the problem.

Both cases are covered in the following subsections.

§6.3.2. Method of Equations

The derivation given on pages 524-525 of Vable or in §7.2 of Beer-Johnston-DeWolf is based on the *wedge method*. This will be omitted here for brevity. The final result is

$$\boxed{\begin{aligned} \sigma_{nn} &= \sigma_{xx}\ \cos^2\theta + \sigma_{yy}\ \sin^2\theta + 2\,\tau_{xy}\ \sin\theta\ \cos\theta, \\ \sigma_{tt} &= \sigma_{xx}\ \sin^2\theta + \sigma_{yy}\ \cos^2\theta - 2\,\tau_{xy}\ \sin\theta\ \cos\theta, \\ \tau_{nt} &= -(\sigma_{xx} - \sigma_{yy})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta). \end{aligned}} \tag{6.1}$$

Couple of checks are useful to verify these equations. If $\theta = 0^\circ$,

$$\sigma_{nn} = \sigma_{xx}, \quad \sigma_{tt} = \sigma_{yy}, \quad \tau_{nt} = \tau_{xy}, \tag{6.2}$$

as expected. If $\theta = 90^\circ$,

$$\sigma_{nn} = \sigma_{yy}, \quad \sigma_{tt} = \sigma_{xx}, \quad \tau_{nt} = -\tau_{xy}. \tag{6.3}$$

which is also OK (can you guess why τ_{nt} at $\theta = 90^\circ$ is $-\,\tau_{xy}$?). Note that

$$\sigma_{xx} + \sigma_{yy} = \sigma_{nn} + \sigma_{tt}. \tag{6.4}$$

This sum is independent of θ, and is called a *stress invariant*. (Mathematically, it is the trace of the stress tensor.) Consequently, if σ_{nn} is computed, the fastest way to get σ_{tt} is as $\sigma_{xx} + \sigma_{yy} - \sigma_{nn}$, which does not require trig functions.

§6.3.3. Double Angle Version

For many developments it is convenient to express the transformation equations (6.1) in terms of the "double angle" 2θ by using the well known trigonometric relations $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\ \sin\theta\ \cos\theta$, in addition to $\sin^2\theta + \cos^2\theta = 1$. The result is

$$\boxed{\begin{aligned} \sigma_{nn} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\ \cos 2\theta + \tau_{xy}\ \sin 2\theta, \\ \tau_{nt} &= -\frac{\sigma_{xx} - \sigma_{yy}}{2}\ \sin 2\theta + \tau_{xy}\ \cos 2\theta. \end{aligned}} \tag{6.5}$$

Here σ_{tt} is omitted since, as previously noted, it can be quickly computed as $\sigma_{tt} = \sigma_{xx} + \sigma_{yy} - \sigma_{nn}$.

§6.3.4. Principal Stresses, Planes, Directions, Angles

The maximum and minimum values attained by the normal stress σ_{nn}, considered as a function of θ, are called *principal stresses*. (A more precise name is *in-plane principal normal stresses*, but the qualifiers "in-plane" and "normal" are often dropped for brevity.) Those values occur if the derivative $d\sigma_{nn}/d\theta$ vanishes. Differentiating the first of (6.1) and passing to double angles gives

$$\begin{aligned} \frac{d\sigma_{nn}}{d\theta} &= 2(\sigma_{yy} - \sigma_{xx})\ \sin\theta\ \cos\theta + 2\tau_{xy}\ (\cos^2\theta - \sin^2\theta) \\ &= (\sigma_{yy} - \sigma_{xx})\ \sin 2\theta + 2\tau_{xy}\ \cos 2\theta = 0. \end{aligned} \tag{6.6}$$

This is satisfied for $\theta = \theta_p$ if

$$\boxed{\tan 2\theta_p = \frac{2\,\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}} \tag{6.7}$$

There are two double-angle solutions $2\theta_1$ and $2\theta_2$ given by (6.7) in the range $[0 \le \theta \le 360^\circ]$ or $[-180^\circ \le \theta \le 180^\circ]$ that are 180° apart. (Which θ range is used depends on the textbook; here we use the first one.) Upon dividing those values by 2, the *principal angles* θ_1 and θ_2 define the *principal planes*, which are 90° apart. The normals to the principal planes define the *principal stress directions*. Since they differ by a 90° angle of rotation about z, it follows that the *principal stress directions are orthogonal*.

As previously noted, the normal stresses that act on the principal planes are called the *in-plane principal normal stresses*, or simply *principal stresses*. They are denoted by σ_1 and σ_2, respectively. Using (6.7) and trigonometric relations it can be shown that their values are given by

$$\boxed{\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}.} \tag{6.8}$$

To evaluate (6.8) it is convenient to go through the following staged sequence:

1. Compute

$$\sigma_{av} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \quad \text{and} \quad R = +\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}. \tag{6.9}$$

   Meaning of these values: σ_{av} is the *average normal stress* at P (recall that $\sigma_{xx} + \sigma_{yy}$ is an invariant and so is σ_{av}), whereas R is the *radius of the Mohr's circle*, as described in §6.3.7 below, thus the symbol. Furthermore, R represents the *maximum in-plane shear stress* value, as discussed in §6.3.5 below.

2. The principal stress values are

$$\sigma_1 = \sigma_{av} + R, \qquad \sigma_2 = \sigma_{av} - R. \tag{6.10}$$

3. Note that the *a priori* computation of the principal angles is not needed to get the principal stresses if one follows the foregoing steps. If finding those angles is of interest, use (6.7).

Comparing (6.6) with the second of (6.5) shows that $d\sigma_{nn}/d\theta = 2\tau_{nt}$. Since $d\sigma_{nn}/d\theta$ vanishes for a principal angle, so does τ_{nt}. Hence the *principal planes are shear stress free*.

§6.3.5. Maximum Shear Stresses

Planes on which the maximum shear stresses act can be found by setting $d\tau_{nt}/d\theta = 0$. A study of this equation shows that the maximum shear planes are located at $\pm 45^\circ$ from the principal planes, and that the maximum and minimum values of τ_{nt} are $\pm R$. See for example, §7.3 of the Beer-Johnston-DeWolf textbook.

This result can be obtained graphically on inspection of Mohr's circle, covered later.

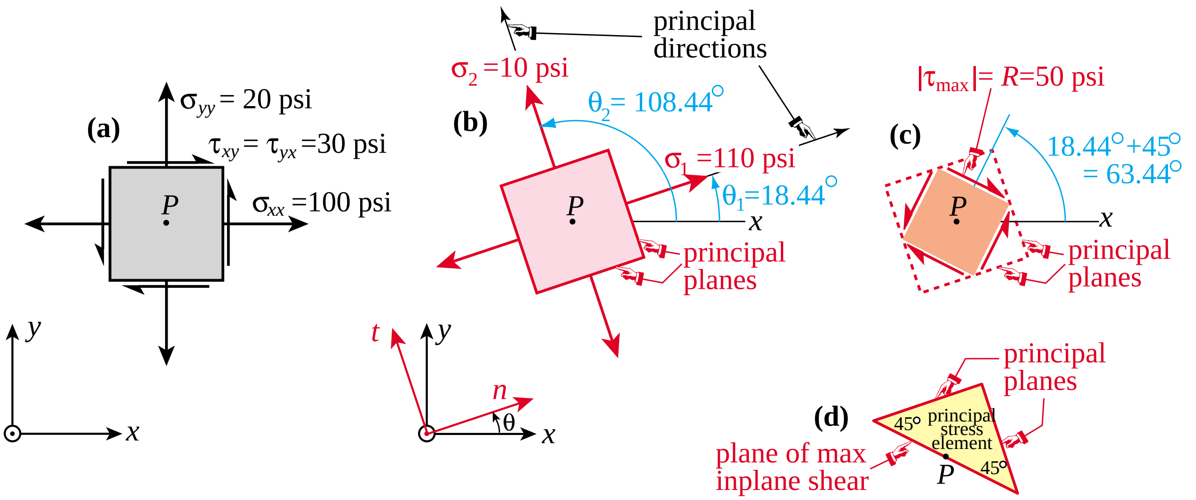

FIGURE 6.5. Plane stress example: (a) given data: stress components σ_{xx}, σ_{yy} and τ_{xy}; (b) principal stresses and angles; (c) maximum shear planes; (d) a principal stress element (actually four PSE can be drawn, that shown is one of them). Note: locations of point P in (a) through (d) coincide; they are drawn offset for visualization convenience.

Example 6.1. This example is pictured in Figure 6.5. Given: $\sigma_{xx} = 100$ psi, $\sigma_{yy} = 20$ psi and $\tau_{xy} = 30$ psi, as shown in Figure 6.5(a), find the principal stresses and their directions. Following the recommended sequence (6.9)–(6.10), we compute first

$$\sigma_{av} = \frac{100+20}{2} = 60 \text{ psi}, \qquad R = +\sqrt{\left(\frac{100-20}{2}\right)^2 + 30^2} = 50 \text{ psi}, \tag{6.11}$$

from which the principal stresses are obtained as

$$\sigma_1 = 60 + 50 = 110 \text{ psi}, \quad \sigma_2 = 60 - 50 = 10 \text{ psi}. \tag{6.12}$$

To find the angles formed by the principal directions, use (6.7):

$$\begin{aligned} &\tan 2\theta_p = \frac{2\times 30}{100-20} = \frac{3}{4} = 0.75, \text{ with solutions } 2\theta_1 = 36.87^\circ,\ 2\theta_2 = 2\theta_1 + 180^\circ = 216.87^\circ, \\ &\text{whence } \theta_1 = 18.44^\circ, \quad \theta_2 = \theta_1 + 90^\circ = 108.44^\circ. \end{aligned} \tag{6.13}$$

These values are shown in Figure 6.5(b). As regards maximum shear stresses, we have $|\tau_{max}| = R = 50$ psi. The planes on which these act are located at $\pm 45^\circ$ from the principal planes, as illustrated in Figure 6.5(c).

§6.3.6. Principal Stress Element

Some authors, such as Vable, introduce here the so-called *principal stress element* or PSE. This is a wedge formed by the two principal planes and the plane of maximum in-plane shear stress. Its projection on the $\{x, y\}$ plane is an isosceles triangle with one right angle and two 45° angles. For the foregoing example, Figure 6.5(d) shows a PSE.

There are actually 4 ways to draw a PSE, since one can join point P to the opposite corners of the square aligned with the principal planes in two ways along diagonals, and each diagonal splits the square into two triangles. The 4 images may be sequentially produced by applying sucessive rotations of 90°. Figure 6.5(d) shows one of the 4 possible PSEs for the example displayed in that Figure. The PSE is primarily used for the visualization of material failure surfaces in fracture and yield, a topic only covered superficially in this course.

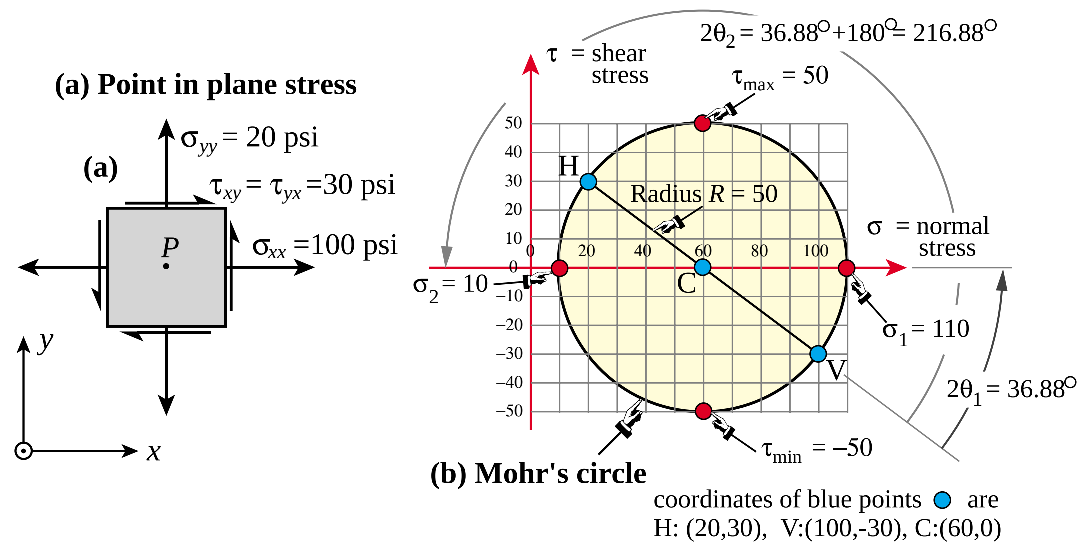

FIGURE 6.6. Mohr's circle for plane stress example of Figure 6.5.

§6.3.7. Mohr's Circle

Mohr's circle is a graphical representation of the plane stress state at a point. Instead of using the methods of equations, a circle is drawn on the $\{\sigma, \tau\}$ plane. The normal stress $\sigma(\theta)$ and the share stress $\tau(\theta)$ are plotted along the horizontal and vertical axes, respectively, with θ as a parameter. All stress states obtained as the angle θ is varied fall on a circle called Mohr's circle.[1]

This representation was more important for engineers before computers and calculators appeared. But it still retains some appealing features, notably the clear visualization of principal stresses and maximum shear. It also remains important in theories of damage, fracture and plasticity that have a "failure surface".

To explain the method we will construct the circle corresponding to Example 6.1, which is reproduced in Figure 6.6(a) for convenience. Draw horizontal axis $\sigma = \sigma_{nn}(\theta)$ to record normal stresses, and vertical axis $\tau = \tau_{nt}(\theta)$ to record shear stresses. Mark two points: V (for "vertical cut", meaning a plane with exterior normal parallel to x) at $(\sigma_{xx}, -\tau_{xy}) = \{100, -30\}$ and H (for "horizontal cut", meaning a plane with exterior normal parallel to y) at $(\sigma_{yy}, \tau_{xy}) = (20, 30)$.

The midpoint between H and V is C, the circle center, which is located at $(\frac{1}{2}(\sigma_{xx} + \sigma_{yy}), 0) = (\sigma_{av}, 0) = (60, 0)$. Now draw the circle. It may be verified that its radius is R as given by (6.9); which for Example 6.1 is $R = 50$. See Figure 6.6(b).

The circle intersects the σ axis at two points with normal stresses $\sigma_{avg} + R = 110 = \sigma_1$ and $\sigma_{av} - R = 10 = \sigma_2$. Those are the principal stresses. Why? At those two points the shear stress τ_{nt} vanishes, which as we have seen characterizes the principal planes. The maximum in-plane shear occurs when $\tau = \tau_{nt}$ is maximum or minimum, which happens at the highest and lowest points of the circle. Obviously $\tau_{max} = R = 50$ and $\tau_{min} = -R = -50$. What is the normal stress when the shear is maximum or minimum? By inspection it is $\sigma_{av} = 60$ because those points lie on a vertical line that passes through the circle center C.

[1] Introduced by Christian Otto Mohr (a civil engineer and professor at Dresden) in 1882. Other important personal contributions were the concept of statically indeterminate structures and theories of material failure by shear.

Other features, such as the correlation between the angle θ on the physical plane and the rotation angle 2θ traversed around the circle will be explained in class if there is time. If not, one can find those details in Chapter 7 of Beer-Johnston-DeWolf textbook, which covers Mohr's circle well.

§6.4. What Happens in 3D?

Despite the common use of simplified 1D and 2D structural models, the world is actually three-dimensional. Stresses and strains actually "live" in 3D. When the extra(s) space dimensions are accounted for, some paradoxes are resolved. In this final section we take a quick look at principal stresses in 3D, stating the major properties as recipes.

§6.4.1. Including the Plane Stress Thickness Dimension

To fix the ideas, 3D stress results will be linked to the plane stress case studied in Example 6.1. and pictured in Figure 6.5. Its 3D state of stress in $\{x, y, z\}$ coordinates is defined by the 3×3 stress matrix

$$\mathbf{S} = \begin{bmatrix} 100 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.14}$$

It may be verified that the eigenvalues of this matrix, arranged in descending order, are

$$\sigma_1 = 110, \quad \sigma_2 = 10, \quad \sigma_3 = 0. \tag{6.15}$$

Now in 3D there are *three principal stresses*, which act on three mutually orthogonal *principal planes* that are *shear stress free*. For the stress matrix (6.14) the principal stress values are 110, 10 and 0. But those are precisely the eigenvalues in (6.15). This result is general:

The principal stresses in 3D are the eigenvalues of the 3D stress matrix

The stress matrix is *symmetric*. A linear algebra theorem says that a real symmetric matrix has a full set of eigenvalues and eigenvectors, and that both eigenvalues an eigenvectors are guaranteed to be *real*. The normals to the principal planes (the so-called *principal directions*) are defined by the three orthonormalized eigenvectors, but this topic will not be pursued further.

In plane stress, *one of the eigenvalues is always zero* because the last row and column of **S** are null; thus one principal stress is zero. The associated principal plane is normal to the transverse axis z, as can be physically expected. Consequently $\sigma_{zz} = 0$ is a principal stress. The other two principal stresses are the *in-plane principal stresses*, which were those studied in §6.3.4. It may be verified that their values are given by (6.8), and that their principal directions lie in the $\{x, y\}$ plane.

Some terminology is needed. Suppose that the three principal stresses are ordered, as usually done, by decreasing algebraic value:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \tag{6.16}$$

Then σ_1 is called the *maximum* principal stress, σ_3 the *minimum* principal stress, and σ_2 the *intermediate* principal stress. (Note that this ordering is by *algebraic*, rather than by absolute, value.) In the plane stress example (6.14) the maximum, intermediate and minimum normal stresses are 110, 10 and 0, respectively. The first two are the in-plane principal stresses.

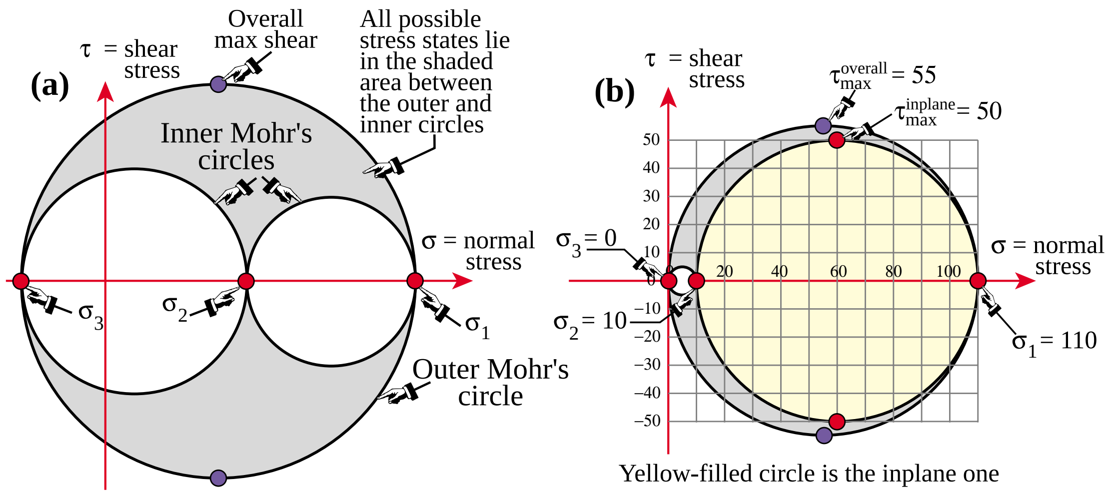

FIGURE 6.7. Mohr's circles for a 3D stress state: (a) general case; (b) plane stress example of Figure 6.5. In (b) the Mohr circle of Figure 6.6 is the rightmost inner circle. Actual stress states lie on the grey shaded areas.

§6.4.2. 3D Mohr Circles

More surprises (pun intended): in 3D there are actually *three Mohr circles*, not just one. To draw the circles, start by getting the eigenvalues σ_1, σ_2 and σ_3 of the stress matrix, for example through the `eig` function of Matlab. Suppose they are ordered as per the convention (6.16). Mark their values on the σ axis of the σ vs. τ plane. Draw the 3 circles with diameters $\{\sigma_1, \sigma_2\}$, $\{\sigma_2, \sigma_3\}$ and $\{\sigma_1, \sigma_3\}$, as sketched in Figure 6.7(a).[2] For the principal values (6.15) the three Mohr circles are drawn in Figure 6.7(b), using a scale similar to that of Figure 6.6.

Of the three circles the wider one goes from σ_1 (maximum normal stress) to σ_3 (minimum normal stress). This is known as the *outer* or "big" circle, while the two others are called the *inner* circles. It can be shown (this is proven in advanced courses in continuum mechanics) that all *actual* stress states at the material point lie *between* the outer circle and the two inner ones. Those are marked as grey shaded areas in Figure 6.7.

§6.4.3. Overall Maximum Shear

For ductile materials such as metal alloys, which yield under shear, the 3D Mohr-circles diagram is quite useful for visualizing the *overall maximum shear stress* at a point, and hence establish the factor of safety against that failure condition. Looking at the diagram of Figure 6.7(a), clearly the overall maximum shear, called $\tau_{max}^{overall}$, is given by the highest and lowest point of the *outer* Mohr's circle, marked as a blue dot in that figure. (Note that only its absolute value is of importance for safety checks; the sign has no importance.) But that is also equal to the radius R_{outer} of that circle. If the three principal stresses are algebraically ordered as in (6.16), then

$$\tau_{max}^{overall} = R_{outer} = \frac{\sigma_3 - \sigma_1}{2} \tag{6.17}$$

Note that the intermediate principal stress σ_2 does *not* appear in this formula.

[2] In the general 3D case there is no simple geometric construction of the circles starting from the six x, y, z independent stress components, as done in §6.3.7 for plane stress.

If the principal stresses are *not* ordered, it is necessary to use the max function in a more complicated formula that picks up the largest of the 3 radii:

$$\tau_{max}^{overall} = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right| \right) \tag{6.18}$$

Taking absolute values in (6.18) is important because the max function picks up the largest algebraic value. For example, writing $\tau_{max}^{overall} = \max(30, -50, 20) = 30$ picks up the wrong value. On the other hand $\tau_{max}^{overall} \max(|30|, |-50|, |20|) = 50$ is correct.

§6.4.4. Plane Stress Revisited

Going back to plane stress, how do overall maximum shear and in-plane maximum shear compare? Recall that one of the principal stresses is zero. The ordering (6.16) of the principal stresses is assumed, and one of those three is zero. The following cases may be considered:

(A) The zero stress is the intermediate one: $\sigma_2 = 0$. If so, the in-plane Mohr circle is the outer one and the two shear maxima coincide:

$$\tau_{max}^{overall} = \tau_{max}^{inplane} = \frac{\sigma_1 - \sigma_3}{2} \tag{6.19}$$

(B) The zero stress is either the largest one or the smallest one. Two subcases:

$$\begin{aligned} &\text{(B1)} \quad \text{If } \sigma_1 \ge \sigma_2 \ge 0 \text{ and } \sigma_3 = 0: \quad \tau_{max}^{overall} = \frac{\sigma_1}{2}, \\ &\text{(B2)} \quad \text{If } \sigma_3 \le \sigma_2 \le 0 \text{ and } \sigma_1 = 0: \quad \tau_{max}^{overall} = -\frac{\sigma_3}{2}. \end{aligned} \tag{6.20}$$

In the plane stress example of Figure 6.5, the principal stresses are given by (6.12). Since both in-plane principal stresses (110 psi and 10 psi) are positive, the zero principal stress is the smallest one. We are in case (B), subcase (B1). Consequently $\tau_{max}^{overall} = \frac{1}{2}\sigma_1 = 55$ psi, whereas $\tau_{max}^{inplane} = \frac{1}{2}(\sigma_1 - \sigma_2) = 50$ psi.

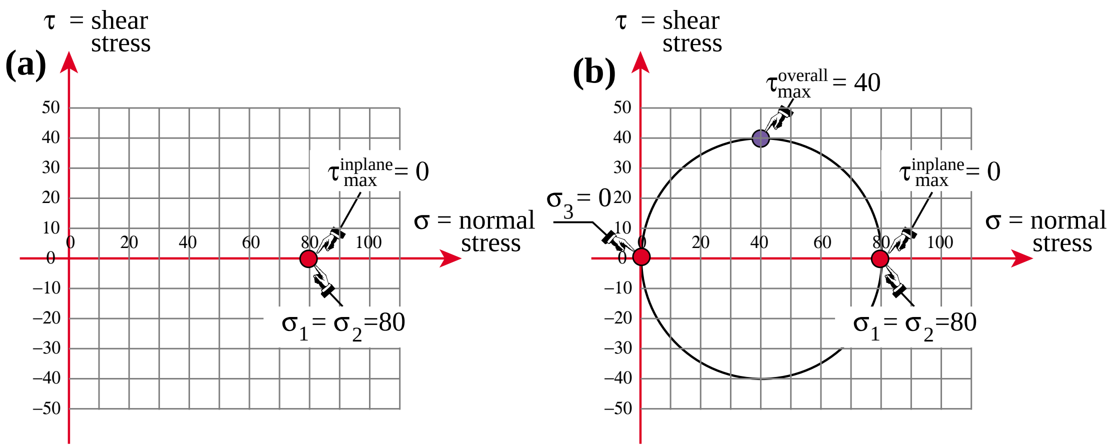

FIGURE 6.8. The sphere paradox: (a) Mohr's in-plane circle reduced to a point, whence $\tau^{inplane}_{max} = 0$; (b) drawing the 3D circles shows that $\tau^{overall}_{max} = 40$.

§6.4.5. The Sphere Paradox

Taking the third dimension into account clarifies some puzzles such as the "sphere paradox." Consider a thin spherical pressure vessel with $p = 160$ psi, $R = 10$ in and $t = 0.1$ in. In Lecture 3, the wall stress in spherical coordinates was found to be $\sigma = p\,R/(2\,t) = 80$ ksi, same in all directions. The stress matrix at any point in the wall, taking z as the normal to the sphere at that point, is

$$\begin{bmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.21}$$

The Mohr circle reduces to a point, as illustrated in Figure 6.8(a), and the maximum in-plane shear stress is zero. And remains zero for any p. If the sphere is fabricated of a ductile material, it should *never break* regardless of pressure. Plainly we have a paradox.

The paradox is resolved by considering the other two Mohr circles. These additional circles are pictured in Figure 6.8(b). In this case the outer circle and the other inner circle coalesce. The overall maximum shear stress is 40 ksi, which is nonzero. Therefore, increasing the pressure will eventually produce yield, and the paradox is solved.