4

Strains

From Statics To Kinematics

Lectures 1-3 dealt with topics in Statics:

Applied Forces ⇒ Internal Forces ⇒ Stresses

This lecture takes us into Kinematics:

Stresses ⇒ Strains ⇒ Displacements ⇒ Size & Shape Changes

Specifically, we cover strains and their connections to displacements

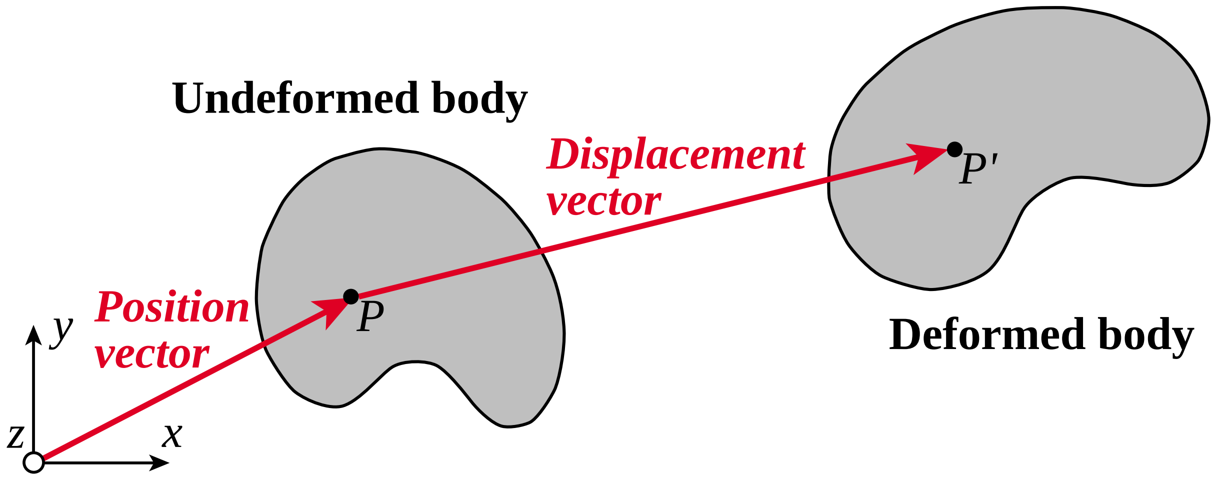
What is Strain (in Mechanics)?

A measure of deformation of a flexible body

All real materials deform under the action of stresses as well as temperature changes.

The connection between strains and stresses (and temperature changes) is done through constitutive equations that encode material properties. These are covered in Lecture 5.

Strains and Displacements



The displacement of a body particle (mathematically a point P in a continuum model of the body) is a vector that defines its motion, joining the initial position P to the final position P'. Displacements of all particles form a vector field.

Point strains are connected to partial derivatives of displacements with respect to the position coordinates $\{x,y,z\}$

These are called the strain-displacement equations

Strains Flavors (1)

Average vs. Point. Average strain is that taken over a finite portion of the body, for example using a strain gage or rosette. **Point strain** is obtained by a limit process in which the dimension(s) of the gaged portion is made to approach zero.

Normal vs. Shear. Normal strain (a.k.a. **extensional or dimensional strain)** measures changes in length along **one** specific direction. **Shear strain** measures changes in angles with respect to **two** specific directions.

Mechanical vs. Thermal. Mechanical strain is produced by stresses. **Thermal strain** is produced by temperature changes.

Strains Flavors (2)

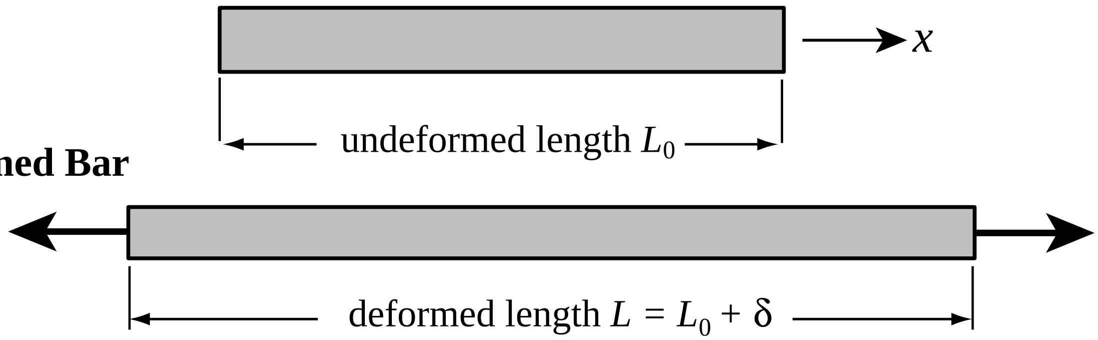
Finite vs. Infinitesimal. **Finite strains** are obtained using exact measures of the changes in dimensions or angles.
Infinitesimal strains (a.k.a. linearized or small strains) are obtained by linearizing a finite strain measure with respect to displacement gradients.

Strain Measures. For **finite strains,** several mathematical measures are in use, often identified with a person name in front. For example Lagrangian strains, Eulerian strains, Hencky strains, Almansi strains, Murnaghan strains, Biot strains, etc.
They have one common feature: as strains get small, meaning <<1, all measures coalesce into the infinitesimal (linearized) version. A brief discussion of Lagrangian versus Eulerian in provided when defining 1D normal strains later.

Average Normal Strain in 1D

(a) Undeformed Bar

(b) Deformed Bar



$$\boxed{\epsilon_{av} \overset{\text{def}}{=} \frac{L - L_0}{L_{ref}} = \frac{\delta}{L_{ref}}}$$

Two common choices for the reference length are

$L_{ref} = L_0$, **the initial gage length: Lagrangian strain**

$L_{ref} = L$, **the final gage length: Eulerian strain**

Strain Measures: Lagrangian vs. Eulerian

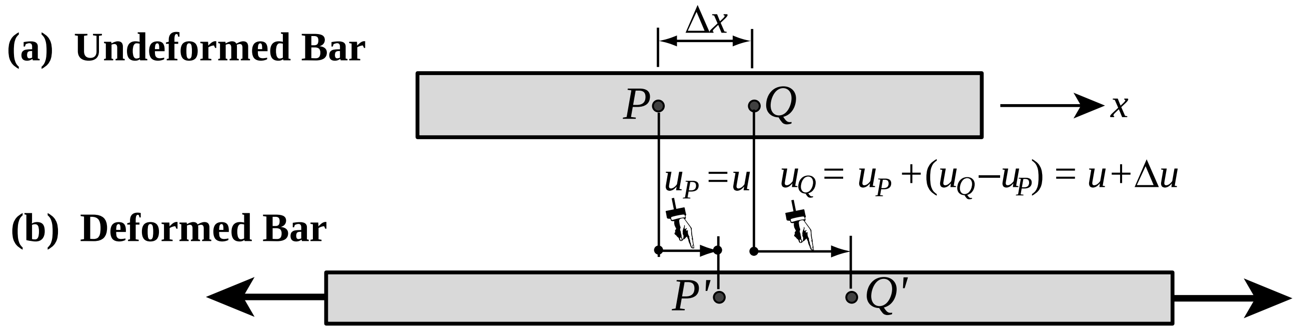
Lagrangian strains are preferred in **solid and structural mechanics**

Eulerian strains are preferred in **fluid mechanics**

For **large strains** the difference between these two measures can be huge, as will be seen in Problem 3 of Recitation #2.

If the strain is small, say <1%, the difference becomes insignificant, as the example in the Notes shows.

Point Normal Strain in 1D



Take the limit as the distance between *P* and *Q* goes to zero:

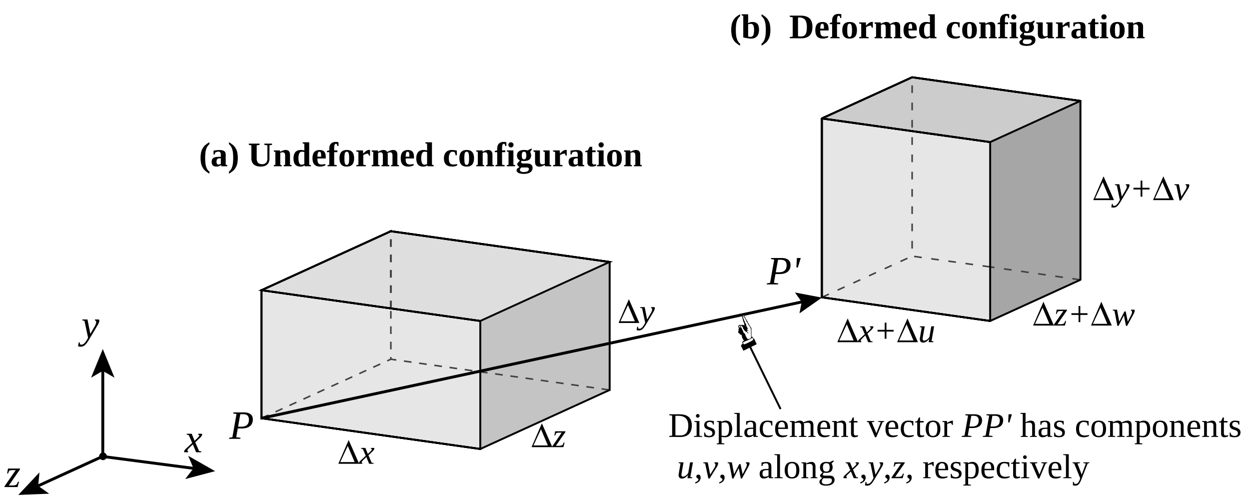
$$\varepsilon_P \stackrel{\text{def}}{=} \lim_{\Delta x \to 0} \frac{(u + \Delta u) - u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \qquad (*)$$

In 3D (*) generalizes to

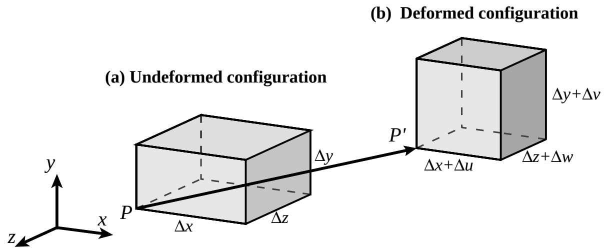
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

with involves three changes: (1) point label P is dropped as point becomes "generic", (2) subscript *xx* is appended to identify normal component, and (3) the ordinary derivative becomes a partial derivative.

Normal Strains in 3D (1)



Normal Strains in 3D (2)



Average normal strains:

$$\epsilon_{xx,av} \overset{\text{def}}{=} \frac{u+\Delta u - u}{\Delta x} = \frac{\Delta u}{\Delta x} \qquad \epsilon_{yy,av} \overset{\text{def}}{=} \frac{v+\Delta v - v}{\Delta y} = \frac{\Delta v}{\Delta y}$$

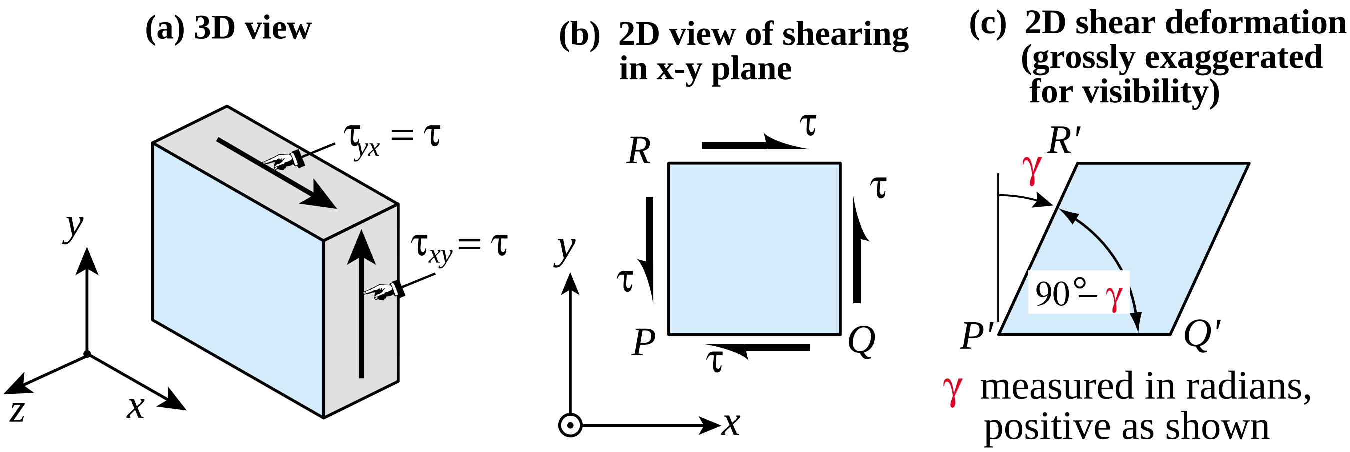
$$\epsilon_{zz\,,av} \overset{\text{def}}{=} \frac{w+\Delta w - w}{\Delta z} = \frac{\Delta w}{\Delta z}$$

Point strains at P:

$$\epsilon_{xx} \overset{\text{def}}{=} \lim_{\Delta x\to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} \overset{\text{def}}{=} \lim_{\Delta y\to 0} \frac{\Delta v}{\Delta y} = \frac{\partial v}{\partial y}$$

$$\epsilon_{zz} \overset{\text{def}}{=} \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z}$$

Average Shear Strain in *x,y* Plane

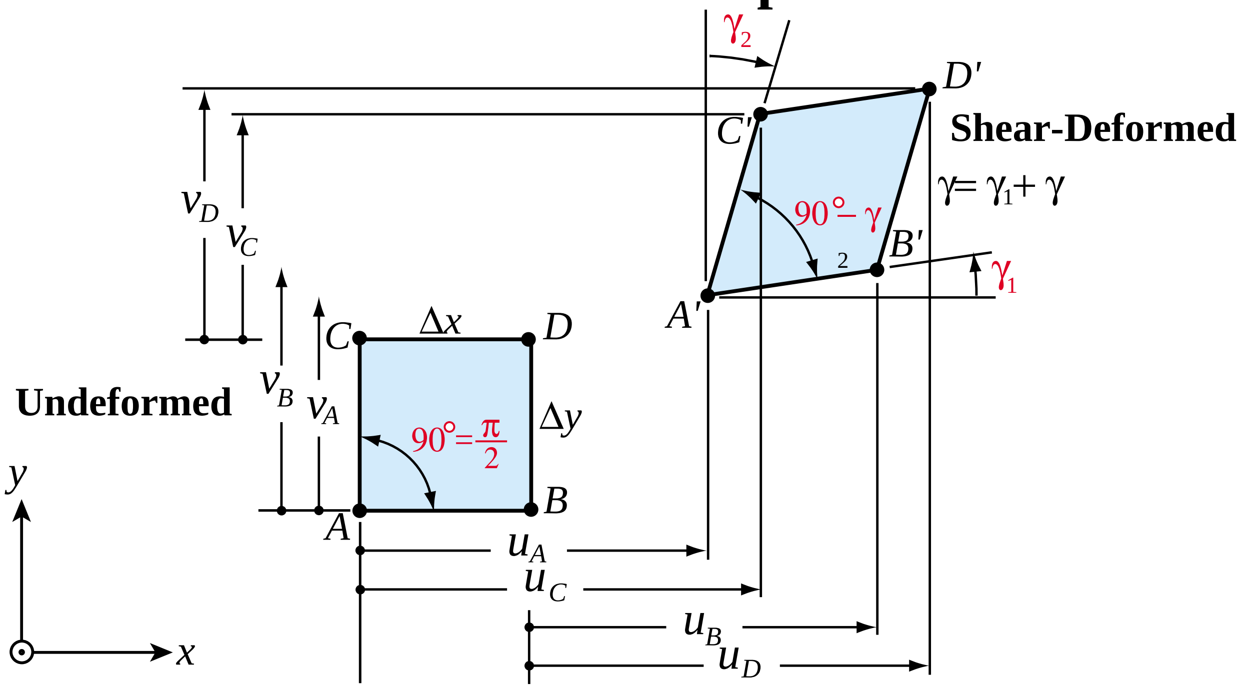


The average shear strain is

$$\gamma_{xy,av} \overset{\text{def}}{=} \gamma.$$

Positive if original right angle **decreases** by γ, as shown

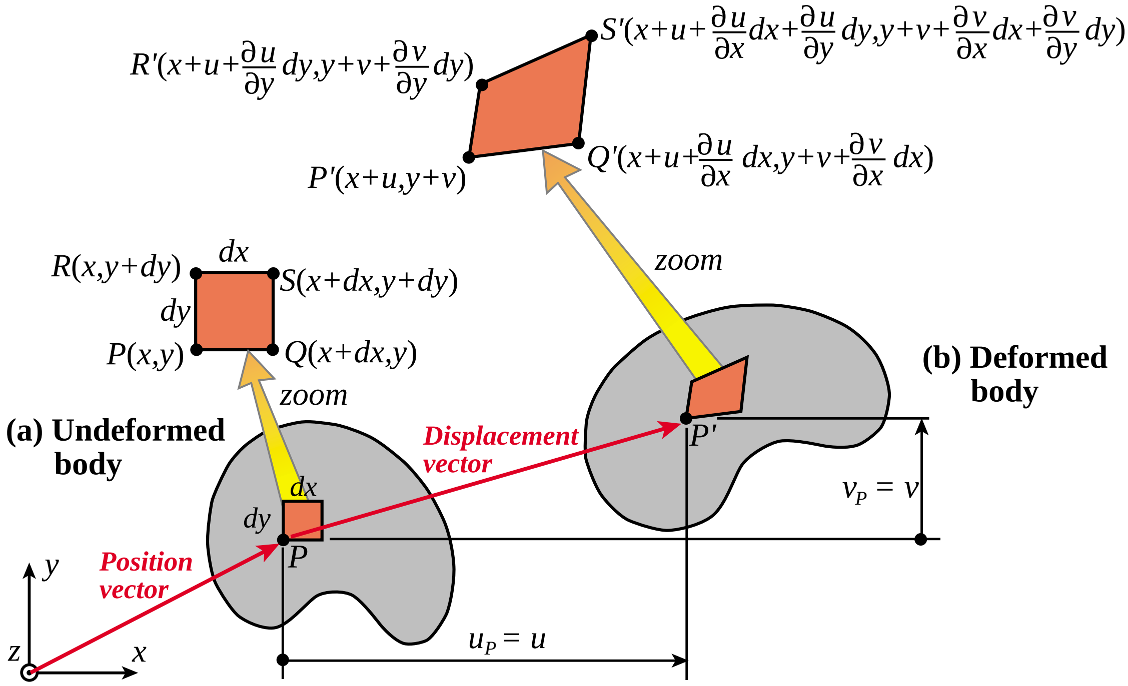
Average Shear Strain in *x,y* Plane in Terms of Corner Displacements



$$\gamma_{xy,av} = \gamma = \gamma_1 + \gamma_2 \approx \frac{\Delta v_{BA}}{\Delta x} + \frac{\Delta u_{CA}}{\Delta y} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y}.$$

See Notes for derivation details

Arbitrary Body in 3D - see Notes



3D (Point) Strain-Displacement Equations

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

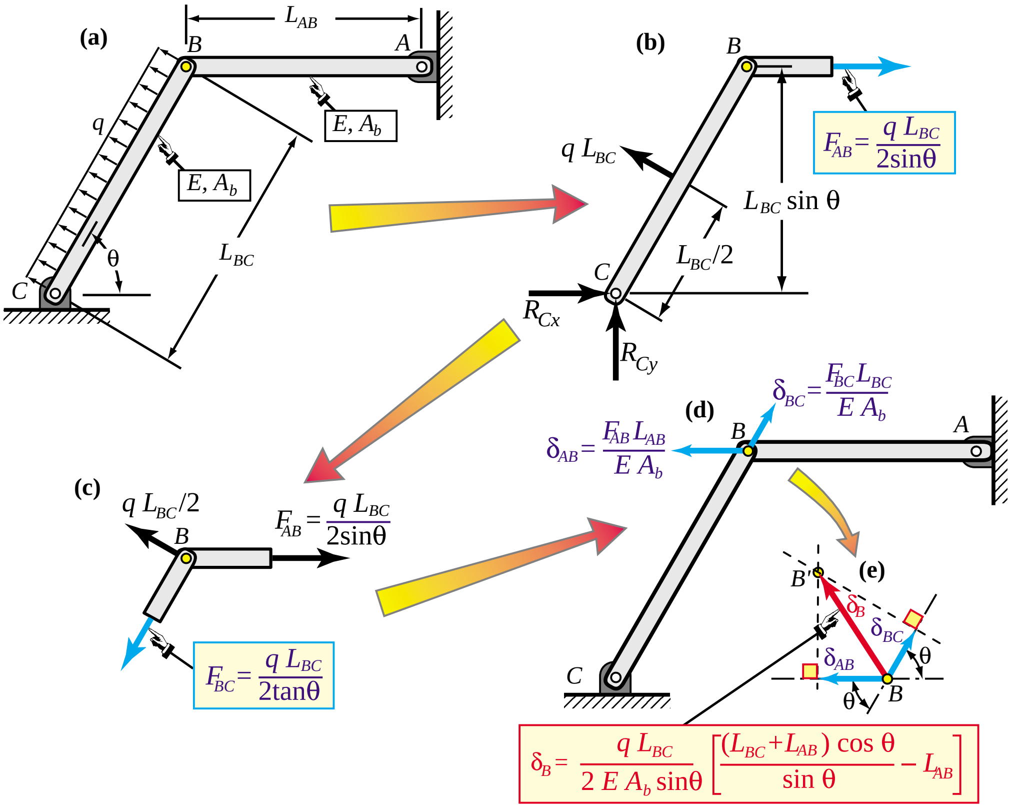
Strain matrix:

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

in which the shear strain components verify reciprocity:

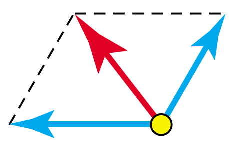
$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{yz} = \gamma_{zy} \qquad \gamma_{zx} = \gamma_{xz}$$

Displacement Calculations for Truss

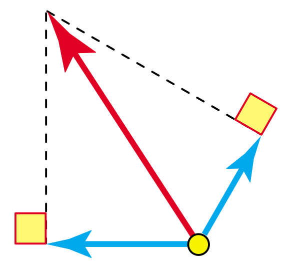


See Section 4.5 of Notes for the worked out example

Displacement Vector Composition Differs From That of Force Vectors



(a) force composition



(b) displacement composition (assuming small deformations)

The physical significance of the diagram on the right will be illustrated with Problem 1 of Recitation #2