

Principal Stresses & Invariants

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Introduction

This page covers principal stresses and stress invariants. Everything here applies regardless of the type of [stress tensor](#).

Coordinate transformations of 2nd rank tensors were discussed on this [coordinate transform page](#). The transform applies to any stress tensor, or strain tensor for that matter. It is written as

$$\sigma' = Q \cdot \sigma \cdot Q^T$$

Everything below follows from two facts: First, the tensors are symmetric. Second, the above coordinate transformation is used.

2-D Principal Stresses

In 2-D, the transformation equations are

$$\sigma'_{xx} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta$$

$$\sigma'_{yy} = \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta$$

$$\tau'_{xy} = (\sigma_{yy} - \sigma_{xx}) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

The full shear stress values are used, unlike strain transformations, which use half values for shear strain, i.e., $(\gamma/2)$.

[This page](#) performs full 3-D tensor [transforms](#), but can still be used for 2-D problems.. Enter values in

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the upper left 2x2 positions and rotate in the 1-2 plane to perform transforms in 2-D. The screenshot below shows a case of pure shear rotated 45° to obtain the principal stresses. Note also how the **Q** matrix transforms.

Tensor Transforms

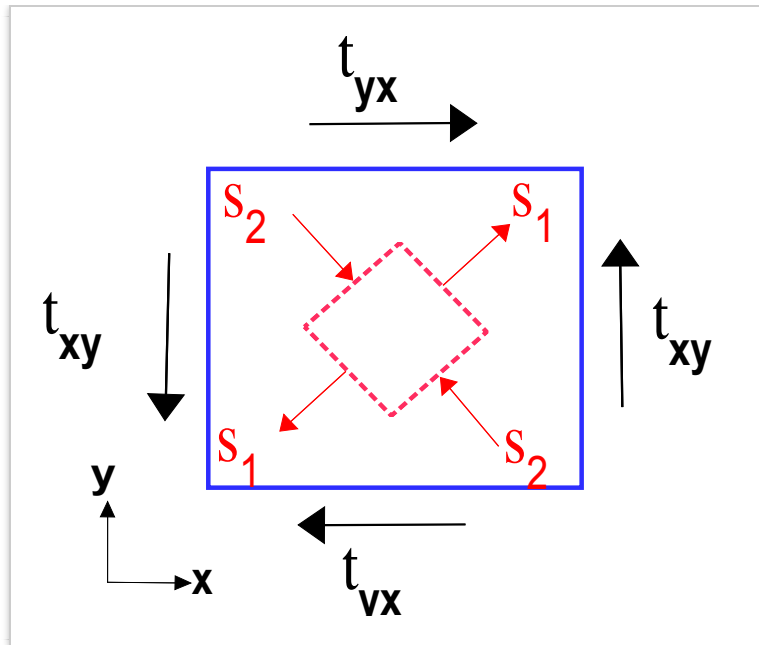
Tensor			Transform Matrix		
1	0	0	0.7071068	0.7071068	0
0	-1	0	-0.7071068	0.7071068	0
0	0	0	0	0	1

Rotate ☒ 1 - 2 ☐ 2 - 3 ☐ 3 - 1

following plane...

45 degrees

The figure below shows the stresses corresponding to the pure shear case in the tensor transform webpage example. The blue square aligned with the axes clearly undergoes shear. But the red square inscribed in the larger blue square only sees simple tension and compression. These are the principal values of the pure shear case in the global coordinate system.



In 2-D, the principal stress orientation, θ_P , can be computed by setting $\tau'_{xy} = 0$ in the above shear equation and solving for θ to get θ_P , the principal stress angle.

$$0 = (\sigma_{yy} - \sigma_{xx}) \sin \theta_P \cos \theta_P + \tau_{xy} (\cos^2 \theta_P - \sin^2 \theta_P)$$

This gives

$$\tan(2\theta_P) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

The transformation matrix, \mathbf{Q} , is

$$\mathbf{Q} = \begin{bmatrix} \cos \theta_P & \sin \theta_P \\ -\sin \theta_P & \cos \theta_P \end{bmatrix}$$

Inserting this value for θ_P back into the equations for the normal stresses gives the principal values. They are written as σ_{max} and σ_{min} , or alternatively as σ_1 and σ_2 .

$$\sigma_{max}, \sigma_{min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

They could also be obtained by using $\sigma' = \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^T$ with \mathbf{Q} based on θ_P .



Principal Stress Notation

Principal stresses can be written as σ_1 , σ_2 , and σ_3 . Only one subscript is usually used in this case to differentiate the principal stress values from the normal stress components: σ_{11} , σ_{22} , and σ_{33} .



2-D Principal Stress Example

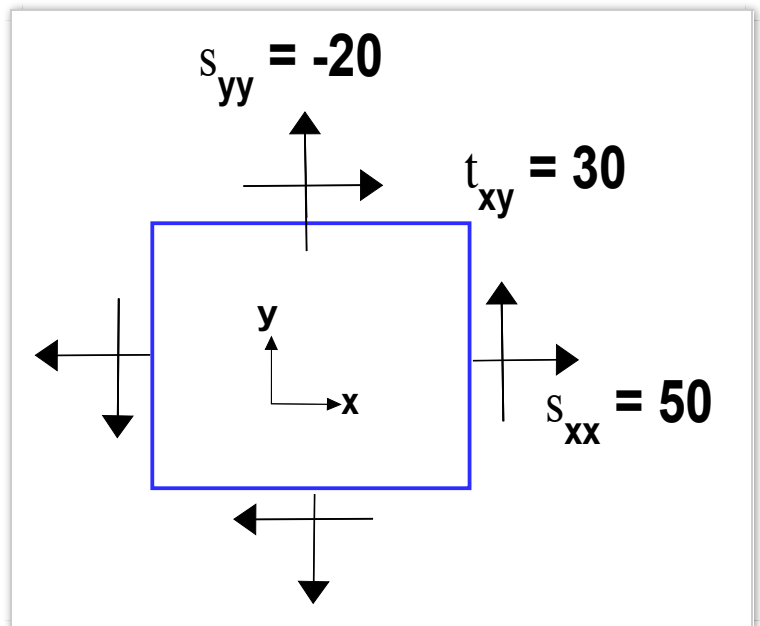
Start with the stress tensor

$$\sigma = \begin{bmatrix} 50 & 30 \\ 30 & -20 \end{bmatrix}$$

The principal orientation is

$$\tan(2\theta_P) = \frac{2 * 30}{50 - (-20)}$$

$$\theta_P = 20.3^\circ$$



The principal stresses are

$$\sigma_{max}, \sigma_{min} = \frac{50 - 20}{2} \pm \sqrt{\left(\frac{50 + 20}{2}\right)^2 + (30)^2}$$

$$\sigma_{max}, \sigma_{min} = 61.1, -31.1$$

The only limitation to using these equations for principal values is that it is not known which one applies to 20.3° and which applies to 110.3° . That's why an attractive alternative is

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} \cos(20.3^\circ) & \sin(20.3^\circ) \\ -\sin(20.3^\circ) & \cos(20.3^\circ) \end{bmatrix} \begin{bmatrix} 50 & 30 \\ 30 & -20 \end{bmatrix} \begin{bmatrix} \cos(20.3^\circ) & -\sin(20.3^\circ) \\ \sin(20.3^\circ) & \cos(20.3^\circ) \end{bmatrix}$$

$$= \begin{bmatrix} 61.1 & 0.0 \\ 0.0 & -31.1 \end{bmatrix}$$

This confirms that the 61.1 principal stress value in the σ_{11} slot is indeed 20.3° from the X-axis. The σ_{22} value is 90° from the first.

3-D Principal Stresses

Coordinate transforms in 3-D are

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{12} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{13} & \sigma'_{23} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$$

The second **Q** matrix is once again the transpose of the first.

[This page](#) performs tensor transforms.

Tensor Transforms

Tensor			Transform Matrix		
0.51713	2.29799	2.42067	0.92303	0.32139	-0.21147
2.29799	5.14945	5.07393	-0.21147	0.88302	0.41898
2.42067	5.07393	5.33340	0.32139	-0.34202	0.88302

Rotate ☒ 1 - 2 20 degrees
 following ☐ 2 - 3
 plane... ☐ 3 - 1 Rotate

Back
 Reset
 Help

And [this page](#) calculates principal values (eigenvalues) and principal directions (eigenvectors).

Eigenvalues & Eigenvectors

Input			Eigenvectors		
4.420	1.35	3.509	0.7569918	0.2669156	0.5964222
1.35	3.832	0	-0.054428	0.9353492	-0.349513
3.509	0	3.207	-0.651153	0.2321164	0.7225794

Calculate

Eigenvalues

7.6606977
 3.753443
 0.0448594

Back
 Reset
 Help

It's important to remember that the inputs to both pages must be symmetric. In fact, both pages enforce

this.

The eigenvalues above can be written in matrix form as

$$\sigma = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 433 \end{bmatrix}$$



Maximum Shear Stress

The maximum shear stress at any point is easy to calculate from the principal stresses. It is simply

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

This applies in both 2-D and 3-D. The maximum shear always occurs in a coordinate system orientation that is rotated 45° from the principal coordinate system. For the principal stress tensor above

$$\sigma = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 125 & 0 \\ 0 & 0 & 433 \end{bmatrix}$$

The max and min principal stresses are in the σ_{33} and σ_{11} slots, respectively. So the max shear orientation is obtained by rotating the principal coordinate system by 45° in the (1 – 3) plane.

The max shear value itself is

$$\begin{aligned} \tau_{max} &= \frac{\sigma_{max} - \sigma_{min}}{2} \\ &= (433 - 24)/2 \\ &= 204 \end{aligned}$$

The manual way of computing principal stresses is to solve a cubic equation for the three principal values. The equation results from setting the following determinant equal to zero. The λ values, once computed, will equal the principal values of the stress tensor.

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

This can be expanded out to give

$$\begin{aligned} &(\sigma_{11} - \lambda) [(\sigma_{22} - \lambda)(\sigma_{33} - \lambda) - \sigma_{23}^2] - \\ &\sigma_{12} [\sigma_{12}(\sigma_{33} - \lambda) - \sigma_{23}\sigma_{13}] + \\ &\sigma_{13} [\sigma_{12}\sigma_{23} - (\sigma_{22} - \lambda)\sigma_{13}] = 0 \end{aligned}$$

Invariants

...and expanded out even further to give

$$\begin{aligned} \lambda^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\lambda^2 + (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2)\lambda - \\ (\sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{13}\sigma_{23}) = 0 \end{aligned}$$



"Now consider this..."

No matter what **coordinate transformation** you apply to the **stress** tensor, its principal stress had better be the same three values. And the only way for this to happen in the above equation is for the equation itself to always be the same, no matter the transformation. This means that the combinations of stress components, which serve as coefficients of the λ 's, must be *invariant* under coordinate transformations. Their values must not change. And that's why they are called *invariants*.

So the equation can be written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{13}\sigma_{23}$$

Although probably not initially apparent, the invariants are actually familiar quantities.

$$I_1 = \text{tr}(\boldsymbol{\sigma})$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}$$

$$I_3 = \det(\boldsymbol{\sigma})$$

They can be written in tensor notation as

$$I_1 = \sigma_{kk}$$

$$I_2 = \frac{1}{2} [(\sigma_{kk})^2 - \sigma_{ij}\sigma_{ij}]$$

$$I_3 = \epsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3}$$

I_3 is in **tensor notation**, but no one should actually calculate a determinant based on tensor notation rules because it is very inefficient.



Physical Interpretation of Invariants

The physical interpretation of the invariants depends on what tensor the invariants are

computed from. For any **stress** or **strain** tensor, I_1 is directly related to the **hydrostatic component** of that tensor. This is universal.

I_2 tends to be related more to the **deviatoric** aspects of **stress** and **strain**. For stress tensors, it is closely related to the **von Mises stress**.

Finally, I_3 does not seem to have any physical significance as the determinant of a stress or strain tensor. But it does when applied to the **deformation gradient**. In that case, $I_3 = V_F/V_O$, the ratio of deformed to initial volume, which is 1 for incompressible materials like rubber.



Invariant Example

This example will start with a random stress tensor and demonstrate that the invariants are indeed invariant under **coordinate transformations**. Start with the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 30 & 20 \\ 30 & -20 & -10 \\ 20 & -10 & 10 \end{bmatrix}$$

Its invariants are

$$I_1 = 50 + (-20) + 10$$

$$= 40$$

$$I_2 = (50)(-20) + (-20)(10) + (10)(50) - (30)^2 - (20)^2 - (-10)^2$$

$$= -21$$

$$I_3 = \det(\boldsymbol{\sigma})$$

$$= -2.8$$

Now **rotate the coordinate system** by some random amount shown in the screenshot.

Tensor Transforms

Tensor			Transform Matrix		
30.8453	-10.6435	27.3252	0.8624982	0.4507153	-0.2230711
-10.6435	-34.65246	3.5643	-0.218967	0.7358681	0.6410419
27.3252	3.5643	43.7634	0.4547269	-0.505392	0.7330073

Rotate ☐ 1 - 2 degrees
 following ☒ 2 - 3
 plane... ☐ 3 - 1

The transformed stress tensor is now

$$\sigma = \begin{bmatrix} 30.8 & -10.6 & 27.3 \\ -10.6 & -34.6 & 03.5 \\ 27.3 & 3.5 & 43.7 \end{bmatrix}$$

And the "new" invariants are

$$\begin{aligned} I_1 &= 30.8 + (-34.6) + 43.7 \\ &= 40 \end{aligned}$$

$$\begin{aligned} I_2 &= (30.8)(-34.6) + (-34.6)(43.7) + (43.7)(30.8) - (-10.6)^2 - (27.3)^2 - (3.5)^2 \\ &= -21 \end{aligned}$$

$$\begin{aligned} I_3 &= \det(\sigma) \\ &= -2.8 \end{aligned}$$

So they are indeed invariant.

Just for grins... Let's compute the principal stresses and then recompute the invariants to

demonstrate again that they are invariant.

The principal stresses are

Eigenvalues & Eigenvectors

Input			Eigenvectors		
50	30	20	0.9161354	0.2888504	0.2779593
30	-20	-10	-0.3754704	0.8611946	0.3425871
20	-10	10	-0.1404201	-0.4182211	0.8974256
<input type="button" value="Calculate"/>					

Eigenvalues

65.526852	-37.057681	11.530828
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Eigenvalues & Eigenvectors

Input			Eigenvectors		
30.8	-10.6	27.3	0.7551059	-0.220196	-0.617518
-10.6	-34.6	3.5	0.1956547	0.9746741	-0.108303
27.3	3.5	43.7	0.625727	-0.039039	0.7790646
<input type="button" value="Calculate"/>					

Eigenvalues

11.56539	-37.11674	65.451350
----------	-----------	-----------

$$\sigma' = \begin{bmatrix} 65.5 & 0 & 0 \\ 0 & -37.1 & 0 \\ 0 & 0 & 11.5 \end{bmatrix}$$

Curiously, the two different starting points lead to the principal values being listed in different orders. This is typical and has no special meaning. The final, physical answer is the same. The **Q** will accommodate the different listing orders.

Finally, calculating the invariants one last time using the principal values gives

$$I_1 = 65.5 + (-37.1) + 11.5$$

$$= 40$$

$$I_2 = (65.5)(-37.1) + (-37.1)(11.5) + (11.5)(65.5)$$

$$= -21$$

$$I_3 = \det(\boldsymbol{\sigma}') = (65.5)(-37.1)(11.5)$$

$$= -2.8$$

And so it works again. Note how the invariant calculations are rather simple when all the off-diagonal terms are zero. The determinant is especially easy.



2-D Invariants

In the 2-D world, there are only two invariants.

$$I_1 = \text{tr}(\boldsymbol{\sigma}) = \sigma_{11} + \sigma_{22}$$

$$I_2 = \det(\boldsymbol{\sigma}) = \sigma_{11} * \sigma_{22} - \sigma_{12}^2$$

In 2-D, things are just simple enough to directly prove the *invariance of invariants*. We'll prove it for I_1 here. Recall the transformation equations are

$$\sigma'_{xx} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

$$\sigma'_{yy} = \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta$$

$$\tau'_{xy} = (\sigma_{yy} - \sigma_{xx}) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

So $\sigma'_{xx} + \sigma'_{yy}$ is

$$\begin{aligned}\sigma'_{xx} + \sigma'_{yy} &= \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta + \\ &\quad \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta\end{aligned}$$

and the shear terms cancel, while the normal stress terms combine to give

$$\begin{aligned}\sigma'_{xx} + \sigma'_{yy} &= \sigma_{xx}(\cos^2 \theta + \sin^2 \theta) + \sigma_{yy}(\sin^2 \theta + \cos^2 \theta) \\ &= \sigma_{xx} + \sigma_{yy}\end{aligned}$$

So it works again.



Calculating Principal Values Manually in 3-D

Here's how to compute the roots of a cubic equation... applied to principal values. Start with

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

Recall that the resulting λ 's will be the principal values of the stress or strain tensor... and the invariants will need to be calculated up front.

The first step is to calculate two intermediate quantities, Q and R .

$$Q = \frac{3I_2 - I_1^2}{9} \quad R = \frac{2I_1^3 - 9I_1 I_2 + 27I_3}{54}$$

Then calculate another intermediate quantity, θ .

$$\theta = \cos^{-1} \left(\frac{R}{\sqrt{-Q^3}} \right)$$

And finally, the principal values are

$$\epsilon_1 = 2 \sqrt{-Q} \cos\left(\frac{\theta}{3}\right) + \frac{1}{3}I_1$$

$$\epsilon_2 = 2 \sqrt{-Q} \cos\left(\frac{\theta + 2\pi}{3}\right) + \frac{1}{3}I_1$$

$$\epsilon_3 = 2 \sqrt{-Q} \cos\left(\frac{\theta + 4\pi}{3}\right) + \frac{1}{3}I_1$$

Note that minus signs are easily confused here because they are present in the original cubic invariant equation being solved.

Ref: <http://mathworld.wolfram.com/CubicFormula.html>

Summary

This principle of invariant quantities under [coordinate transformations](#) is in fact universal across all [matrices](#) that are symmetric and being transformed according to

$$\mathbf{A}' = \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T$$

where \mathbf{A} is "any symmetric matrix."

And recall that the product of any [matrix](#) with its transpose is always a symmetric result, so this result would qualify. This is especially relevant to $\mathbf{F}^T \cdot \mathbf{F}$, whose invariants are used in the [Mooney-Rivlin Law](#) of rubber behavior. Mooney-Rivlin's Law and coefficients will be discussed on [this page](#). As an added teaser, we will see that the 3rd invariant of $\mathbf{F}^T \cdot \mathbf{F}$ for rubber always equals 1 because rubber is incompressible. So not only is it a constant, independent of coordinate transformations, but it is even a constant value, always equal to 1, independent of [coordinate transformations](#) and the state of deformation.



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