by Sophia Hassiotis at Stevens Institute of Technology

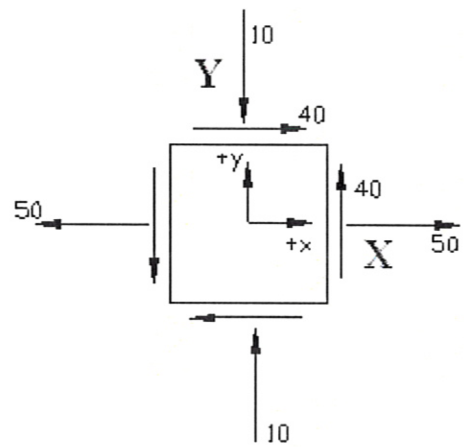
http://personal.stevens.edu/~shassiot/ModelingAndSimulation/Ch6EigenvalueAnalysis.pdf

CHAPTER 6: EIGENVALUE ANALYSIS

Modified March, 2011

6.1 Stress Tensor--Mohr's Circle for 2-D Stress

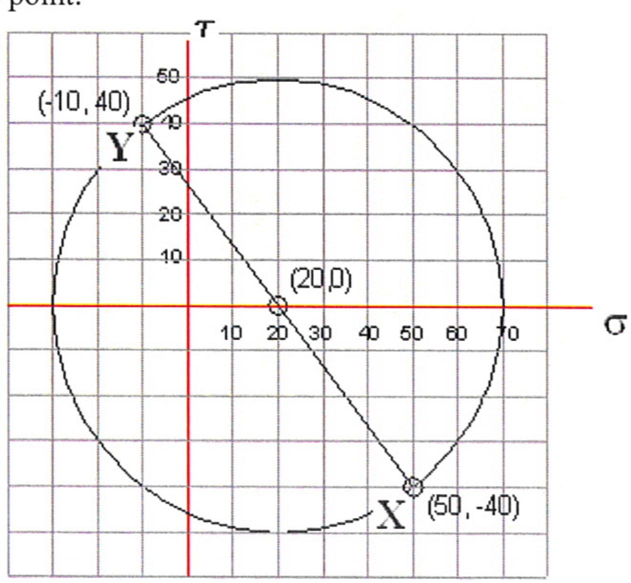
Let's study the stresses at a point using the Mohr's Circle.



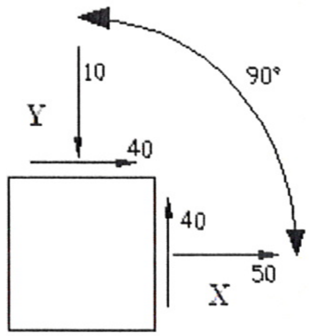
Notice the positive y and positive x directions on the point. Stresses on positive face acting towards a positive direction will be considered positive.

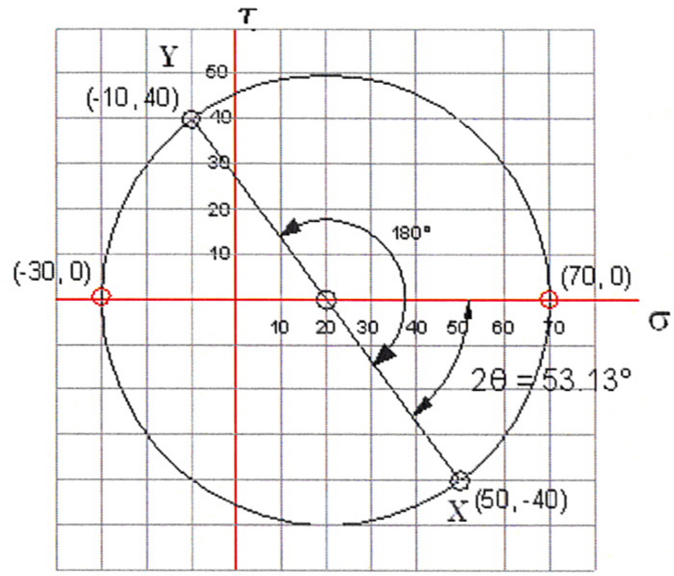
Plot the Mohr's circle as follows:

1) Set up a σ, τ axis
2) Place the center at $[\frac{1}{2}(\sigma_x + \sigma_y), 0]$. Here, the center is at ½(50-10) = 20
3) Plot face X at $(\sigma, -\tau)$. Here this point is at (50,-40). Connect X to the center and continue along the diameter until you intersect the circle at point Y. Those are the stresses on the Y face of the point.



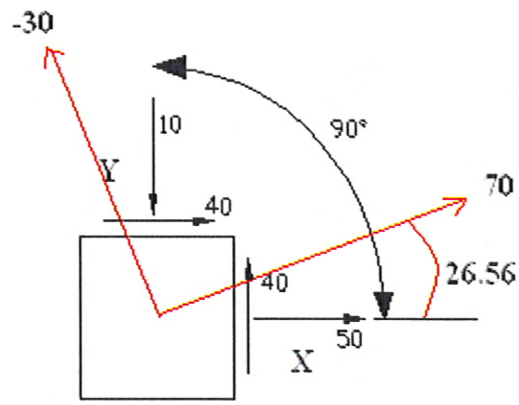
Note that 90 degrees on the point translate to 180 degrees on the circle. In general an angle ϑ on the point translates as an angle 2ϑ on the circle.





Define principal stresses as the maximum and minimum normal stresses on the point. They always occur at a plane in which the shear stress in zero and that plane is called the principal plane. Here, the principal stresses are 70 and -30 ksi.

The angle between the X-face and the principal plane, ϑp, is ½ (53.13°) = 26.56°.



6.2 Definition of Eigenvalues and Eigenvectors

The standard form of an eigenvalue problem is: Solve for λ and x such that:

$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$.

A is an nxn matrix, **x** is an unknown column of n elements and λ is an unknown scalar. We need to find the **x** (called the eigenvector) and λ (called the eigenvalue) that satisfy this equation.

We can rewrite:

$(\mathbf{A}-\lambda\,\mathbf{I})\,\mathbf{x}=\mathbf{0}$

The values of λ that satisfy this equation are given by solving det|**Ax**- λ I|=0.

Example:

$$\mathbf{A}= \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

Then det|**A**- λ **I**|=0 becomes: determinant of $\begin{bmatrix} 2-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = 0$

or: $(2-\lambda)(1-\lambda)-9=0$ OR $\lambda^2-3\lambda-7=0$

This is the **characteristic equation.** Solving for λ we get: λ=4.54 and –1.54

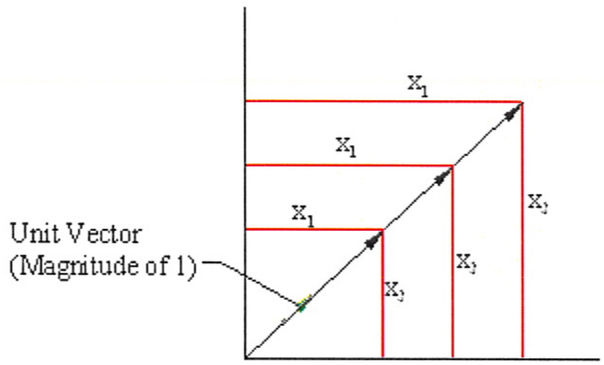
```
In Matlab:
» A=[2 3; 3 1];
» eig(A)
ans =    4.5414
        -1.5414
```

Having obtained the eigenvalues, we can go back, substitute and get the eigenvectors: $(\mathbf{A}-\lambda_i \mathrm{I})\,x=0$

Each eigenvalue has its own eigenvector. For the first eigenvalue λ=4.54 the eigenvector is:

$$\begin{bmatrix} 2-4.54 & 3 \\ 3 & 1-4.54 \end{bmatrix}\begin{Bmatrix} x1 \\ x2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \text{OR} \qquad \begin{bmatrix} -2.54 & 3 \\ 3 & -3.54 \end{bmatrix}\begin{Bmatrix} x1 \\ x2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Note that these are rank-deficient and have no unique solution. One way to solve them is to assume, say x1=1. Then solve for x2. In this problem x2=0.8467. The vector that solves this problem is (x1, x2)=(1, 0.8467). If we divide it by its magnitude, the unit vector is (0.763, 0.6464).



We repeat for the second eigenvalue λ=-1.5414 to solve for the eigenvector that corresponds to this eigenvalue. Please verify that an eigenvector of (0.646, -0.763) solves the problem.

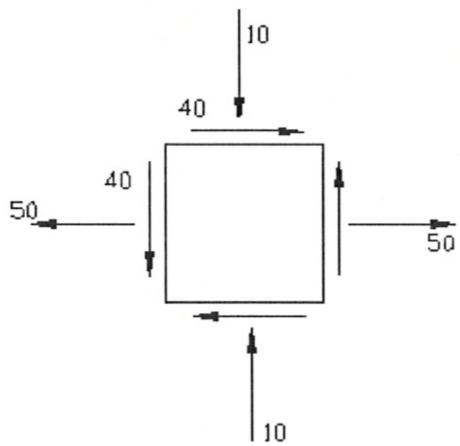
In MATLAB type:

```
» [u,lambda]=eig(a)

u =

  -0.7630    0.6464          Note that the two eigenvectors are the two columns
  -0.6464   -0.7630


lambda =

   4.5414         0          Note that the two eigenvalues are in the diagonal
        0   -1.5414
```

6.3 Eigenvalue Analysis and the Stress Tensor

A second-order tensor is a special quantity whose values depend on the plane that we choose to study it. For example, stress is a second-order tensor. When you know the stresses on a cubic element, you can use transformation of axis to find the stresses at any plane of that cube. The Mohr's circle gives you a graphical way to transform the stresses and arrive at the principal stresses and planes. A theoretical way to do the same is to find the eigenvalues and eigenvectors of your stress tensor.

Let's use eigenvalue analysis to find the principal stresses and planes of the point below.



Placing these stresses in a tensor matrix we obtain:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 50 & 40 \\ 40 & -10 \end{bmatrix}$$

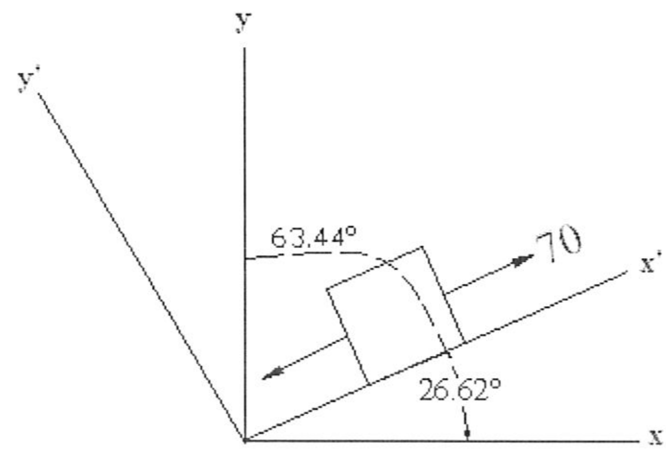
<u>Principal Stresses and planes</u>

For every stress condition, there exists a particular plane on which the shear stress, τ, is zero and the normal stress, σ, achieves its maximum value. This maximum stress is called the principal stress and can be found by the eigenvalues of the stress tensor. The planes on which the principal stresses act are the principal planes and they can be found by the eigenvectors of the stress tensor.

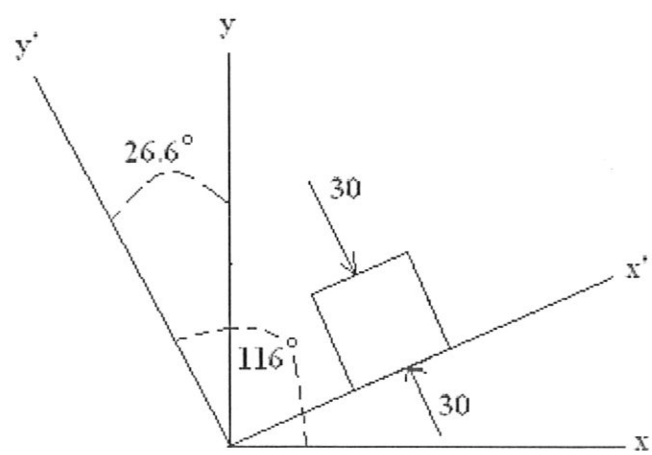
Example

For the stress tensor above, the first eigenvalue is 70 and it is the maximum principal stress. It corresponds to an eigenvecror of $\begin{Bmatrix} 0.894 \\ 0.447 \end{Bmatrix}$, which indicates the direction of the principal stress as follows:

$\begin{Bmatrix} 0.894 \\ 0.447 \end{Bmatrix} = \begin{Bmatrix} \cos^{-1}\theta x \\ \cos^{-1}\theta y \end{Bmatrix}$ from which we solve for $\begin{Bmatrix} \theta x \\ \theta y \end{Bmatrix} = \begin{Bmatrix} 26.62° \\ 63.44° \end{Bmatrix}$. This is the distance (in degrees) of the principal stress (70) to the x and y axes respectively.

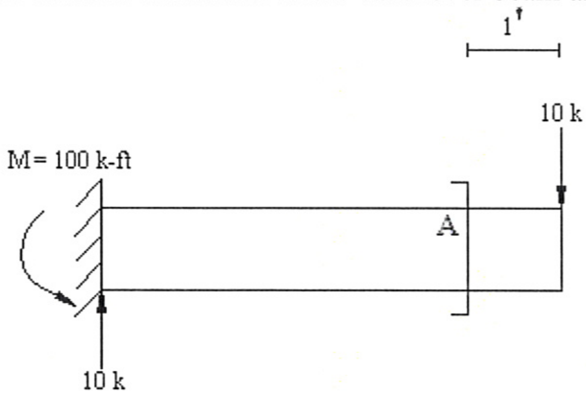


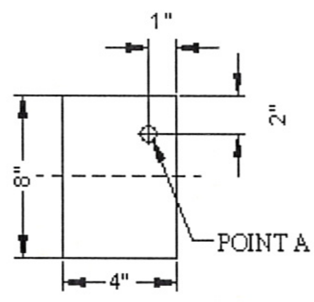
The second eigenvalue is -30 and it is the minimum principal stress. It corresponds to an eigenvecror of $\begin{Bmatrix} -0.447 \\ 0.894 \end{Bmatrix}$, which indicates the direction of the principal stress as follows: $\begin{Bmatrix} -0.447 \\ 0.894 \end{Bmatrix} = \begin{Bmatrix} \cos^{-1}\theta x \\ \cos^{-1}\theta y \end{Bmatrix}$ from which we solve for $\begin{Bmatrix} \theta x \\ \theta y \end{Bmatrix} = \begin{Bmatrix} 116.° \\ 26.6° \end{Bmatrix}$. This is the distance (in degrees) of the principal stress (30) to the x and y axes respectively.



6.4 Stresses on a Beam in Bending

The stresses at Point A of the cantilever beam are found with the use of mechanics.

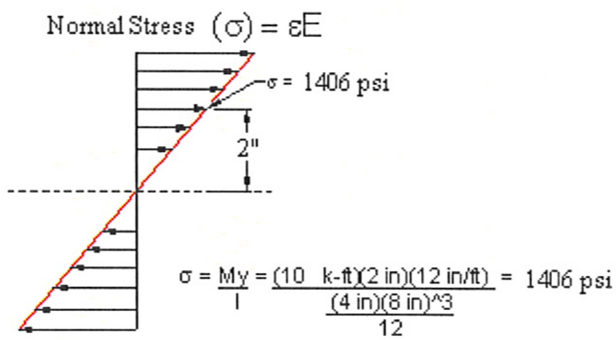




Moment on Plane: M=10 k-ft
Shear on Plane: V=10 k

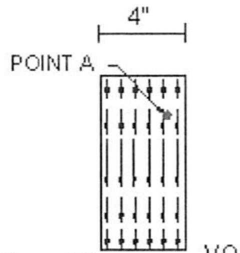
Normal Stress

The applied moment on the beam creates normal stress throughout the cross-section of the beam that varies linearly from top to bottom. The normal stress is greatest at the top and bottom fibers. In this case, due to the direction of the moment, the normal stress is positive on top and negative on the bottom fibers. The stresses at the neutral axis are zero, and the normal stress at Point A is found by σ=My/I.

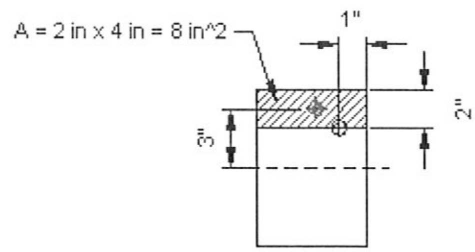


Shear Stress

For this given cross-sectional shape, shear stress is greatest in the center and decreases to zero at the top and bottom fibers. The shear stress at Point A is found by τ=VQ/It.



Shear Stress (τ) = $\frac{VQ}{It}$

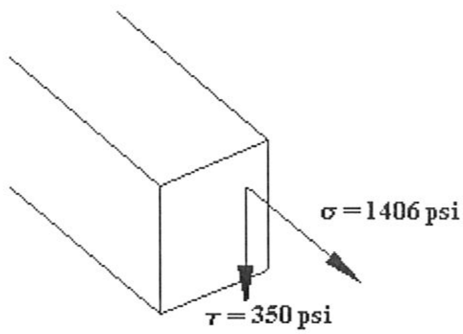


Q = Ad = (8 in^2)(3 in) = 24 in^2

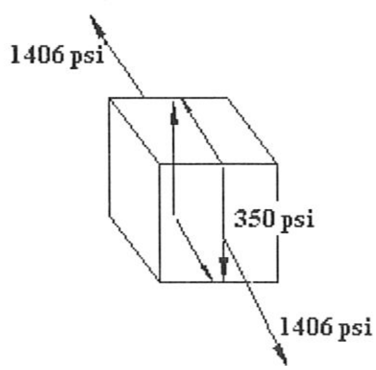
$$\tau = \frac{VQ}{It} = \frac{(10\text{ k})(24\text{ in}^3)}{\frac{(4\text{ in})(8\text{ in})^3}{12}(4\text{ in})} = 350\text{ psi}$$

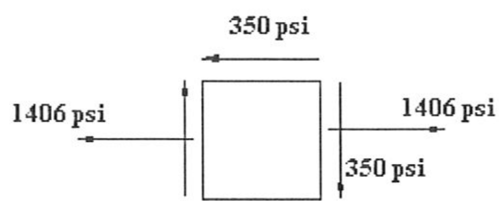
Stresses at a point

The stresses on Point A located at the face of the beam are shown below.



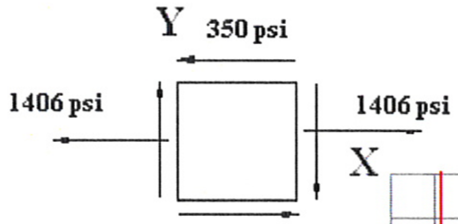
To help analyze the stresses at a Point A, we take an infinitesimally small cube around that point. For equilibrium, the normal and shear stresses are as shown below. Note that this is a 2-Dimentional state of stress.

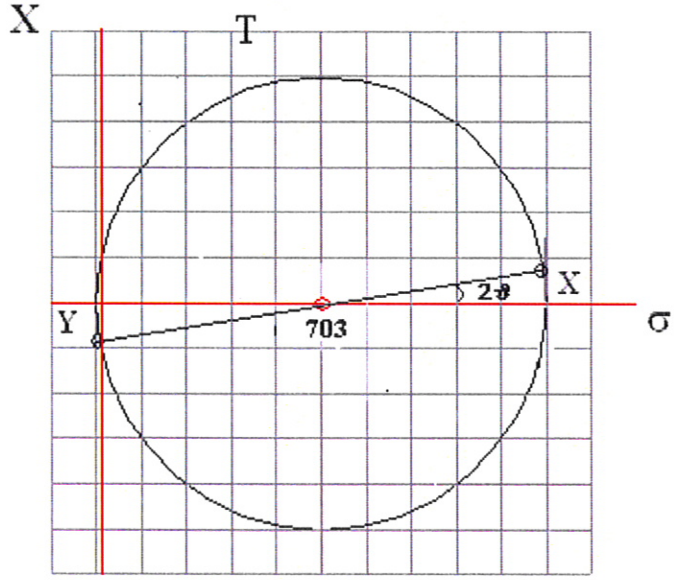




In addition, this is the state of stress at a particular plane on the beam. The stresses would be different if we chose to take a FBD at different cut along the beam. As the angle ϑ changes, the normal and shear stresses on the plane will change. In order to find the variation of stress along different planes and to find the principal stresses and planes, we can use the Mohr's circle.

Solving for principal stresses using the Mohr's Circle

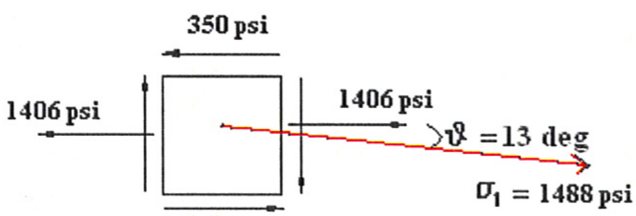


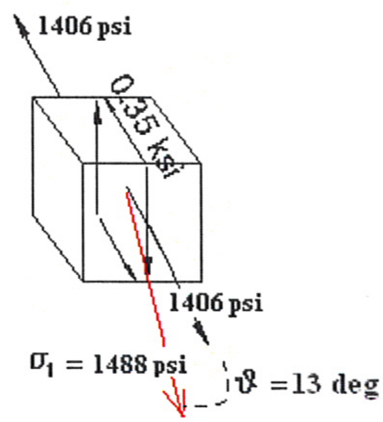


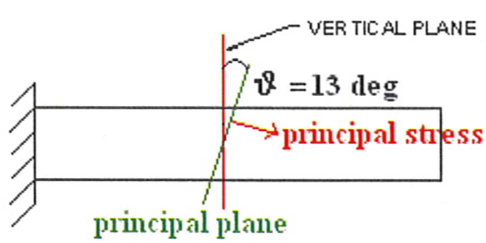
Center: (1406+0)/2=703
Point X at (1406,350)
Point Y at (0,-350)

Radius: 785
σ_1 **= 1488 psi**
σ_2 **= --82 psi**

2ϑ =26 deg.







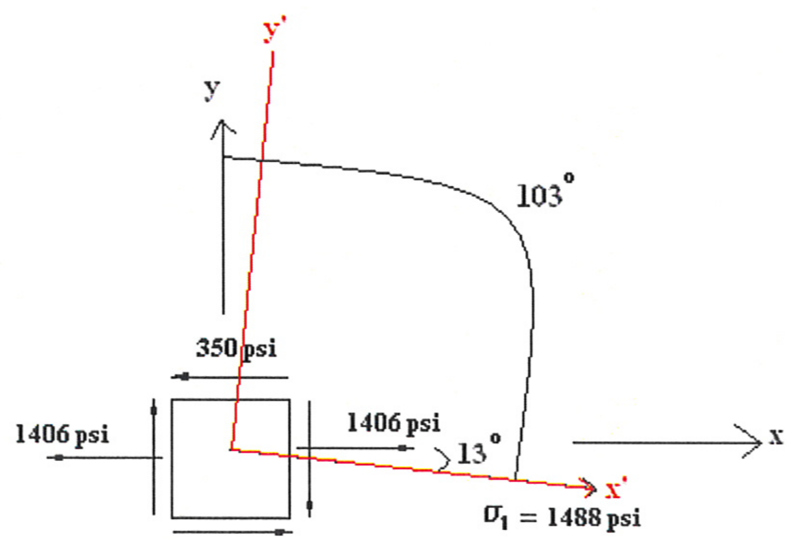
Solving for principal stresses using eigenvalue analysis

The stress tensor is

$$\begin{bmatrix} 1406 & -350 \\ -350 & 0 \end{bmatrix}$$

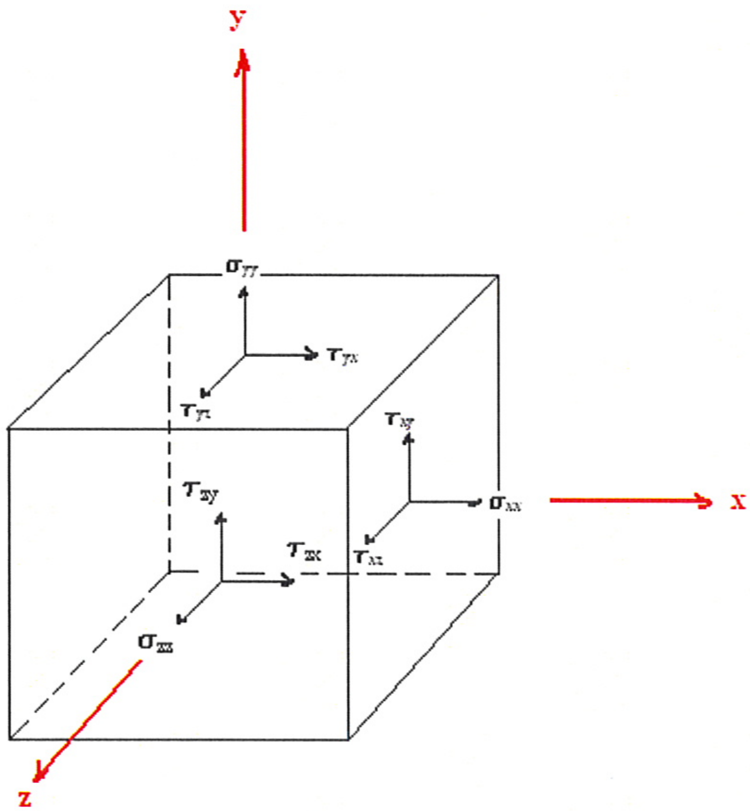
```
Using MATLAB
a =
        1406      -350
        -350         0
>> [e1 e2]=eig(a)
e1 =
   -0.2289    0.9734
   -0.9734   -0.2289
e2 =
         -82.3    1488
```

For the stress tensor above, the first eigenvalue is 1488 and it is the maximum principal stress. It corresponds to an eigenvecror of $\begin{Bmatrix} 0.9734 \\ -0.2289 \end{Bmatrix}$, which indicates the direction of the principal stress as follows:
$\begin{Bmatrix} 0.9734 \\ -0.2289 \end{Bmatrix} = \begin{Bmatrix} \cos^{-1}\theta x \\ \cos^{-1}\theta y \end{Bmatrix}$ from which we solve for $\begin{Bmatrix} \theta x \\ \theta y \end{Bmatrix} = \begin{Bmatrix} 13° \\ 103° \end{Bmatrix}$. This is the distance (in degrees) of the principal stress (1488) to the x and y axes respectively.



6.5 3D Stress and Eignevalue analysis

The stress at a point can be completely defined if we know the normal and shear stresses at three perpendicular planes as shown in the figure. The normal stresses are shown as σ_{xx}, σ_{yy}, and σ_{zz}. The shear stresses are shown as τ_{ij}, where *i* is the plane of action and *j* is the axis of direction of the stress.



The stress tensor for the point is

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

The stress vector **T** that acts on a plane defined by a unit direction vector **n** is $\mathbf{T}=\sigma\mathbf{n}$. In the 3-D case, there exist three mutually perpendicular planes of zero shear stress (principal planes). On these planes, the normal stresses have maximum and minimum values (principal stresses). By convention $\sigma_1 > \sigma_2 > \sigma_3$.

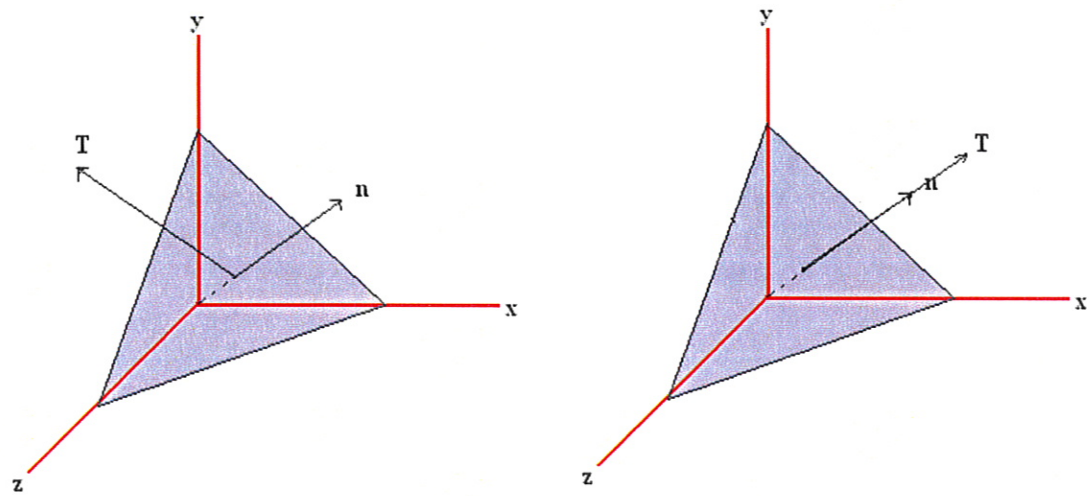
Let's ask ourselves, which plane is the principal plane? Or, where is

$$\sigma\mathbf{n} = \sigma_p\mathbf{n} \qquad \text{or}$$

$$(\sigma\mathbf{n} - \sigma_p\mathbf{n}) = \mathbf{0} \quad \text{or}$$

$$(\sigma - \sigma_p)\,\mathbf{n} = \mathbf{0}$$

Note that this is the definition of an eigenvalue problem, in which σ_p is the eigenvalue and **n** is the eigenvector.



Here, we need to find the σ_p and the **n** that satisfy

$$\left(\begin{bmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix} - \sigma_p\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}\right)\begin{Bmatrix}x1\\x2\\x3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Since σ is symmetric positive definite, there will be three real eigenvalues σ_p.
For a non-trivial solution,

$$\det\begin{vmatrix}\sigma_{xx}-\sigma_p & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy}-\sigma_p & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}-\sigma_p\end{vmatrix}=0$$

This results in a characteristic equation, which is independent of the coordinate system:

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

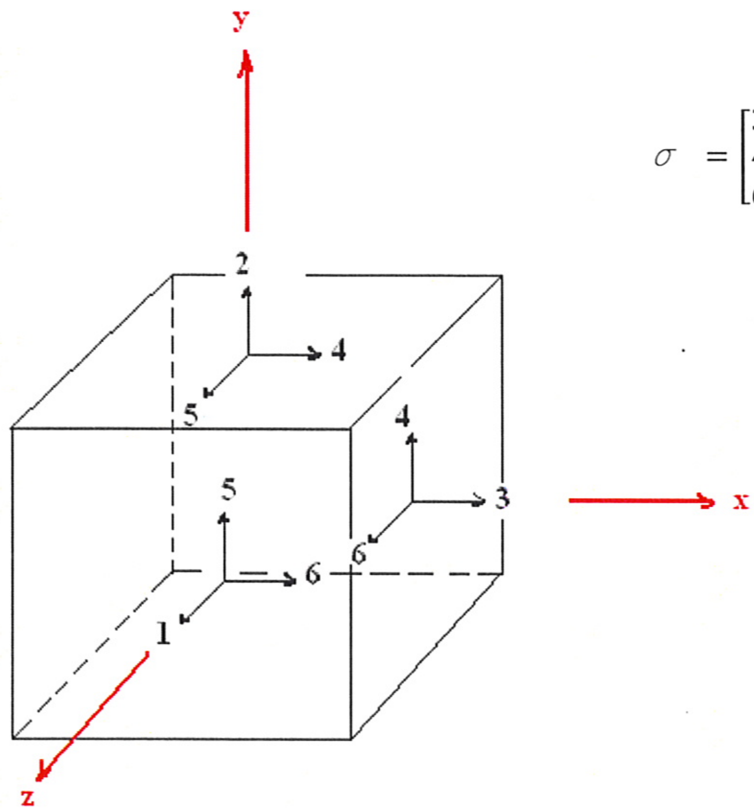
I_1 is the trace of $\sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$

$$I_2 = \begin{vmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{yx} & \sigma_{yy}\end{vmatrix} + \begin{vmatrix}\sigma_{xx} & \tau_{xz}\\ \tau_{zx} & \sigma_{zz}\end{vmatrix} + \begin{vmatrix}\sigma_{yy} & \tau_{yz}\\ \tau_{zy} & \sigma_{zz}\end{vmatrix}$$

I_3 is the determinate of σ, $\det\begin{vmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{vmatrix}$

Example

$$\sigma = \begin{bmatrix} 3 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 1 \end{bmatrix}$$



a) To find the Principal Stresses, we solve for the eigenvalues.

$$\det \begin{vmatrix} 3-\sigma_p & 4 & 6 \\ 4 & 2-\sigma_p & 5\tau_{yz} \\ 6 & 5 & 1-\sigma_p \end{vmatrix} = 0$$

The characteristic equation is $\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$, where

$I_1 = 3+2+1 = 6$

$$I_2 = \begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 6 \\ 6 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 5 & 1 \end{vmatrix} = -66$$

I_3 is the determinate of σ, $\det \begin{vmatrix} 3 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 1 \end{vmatrix} = 83$

Thus, solving for the roots of $\sigma_p^3 - 6\sigma_p^2 - 66\sigma_p - 83 = 0$, we get $\sigma_1=12.049$ $\sigma_2=-4.528$, and $\sigma_3=-1.521$.

b) To find the principal directions, we solve for the eigenvectors.
For $\sigma_1=12.049$

$$\begin{bmatrix} 3-12.049 & 4 & 6 \\ 4 & 2-12.049 & 5 \\ 6 & 5 & 1-12.040 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This reduces to:

$$\begin{bmatrix} -9.049 & 4 & 6 \\ 4 & -10.049 & 5 \\ 6 & 5 & -11.049 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

There is no unique solution for eigenvectors. Here assume x1=1 and solve for x2 and x3:

-9.049(1) + 4(x2) + 6(x3) = 0
4(1) - 10.049(x2) + 5(x3) = 0 from which x2=0.862 and x3=0.933.

Thus the first eigenvector, **n1**, is $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.862 \\ 0.933 \end{Bmatrix}$ which has a magnitude of $\sqrt{1^2 + 0.862^2 + 0.933^2}$=1.27

Divide the eigenvector by its magnitude to get $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 0.533 \\ 0.577 \end{Bmatrix} = \begin{Bmatrix} \cos\vartheta\, x \\ \cos\vartheta\, y \\ \cos\vartheta\, z \end{Bmatrix}$.

For σ_2=-4.528

$$\begin{bmatrix} 3+4.528 & 4 & 6 \\ 4 & 2+4.528 & 5 \\ 6 & 5 & 1+4.528 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This reduces to:

$$\begin{bmatrix} 7.528 & 4 & 6 \\ 4 & 6.528 & 5 \\ 6 & 5 & -5.528 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Again, assume x1=1 and solve for x2 and x3 to find the second eigenvector, **n2**, as $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.7116 \\ -1.729 \end{Bmatrix}$ which has a magnitude of $\sqrt{1^2 + 0.7116^2 + 1.729^2}$=2.12

Divide the eigenvector by its magnitude to get $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0.4716 \\ 0.3356 \\ -0.8154 \end{Bmatrix}$.

For σ_3=-1.521

$$\begin{bmatrix} 3+1.521 & 4 & 6 \\ 4 & 2+1.521 & 5 \\ 6 & 5 & 1+1.521 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This reduces to:

$$\begin{bmatrix} 4.521 & 4 & 6 \\ 4 & 3.521 & 5 \\ 6 & 5 & 2.521 \end{bmatrix} \begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Again, assume x1=1 and solve for x2 and x3 to find the third eigenvector, **n3**, as $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1.235 \\ 0.07 \end{Bmatrix}$ which has a magnitude of $\sqrt{1^2 + 1.235^2 + 0.07^2}$=1.59

Divide the eigenvector by its magnitude to get $\begin{Bmatrix} x1 \\ x2 \\ x3 \end{Bmatrix} = \begin{Bmatrix} 0.628 \\ -0.776 \\ 0.044 \end{Bmatrix}$.

Note that the eigenvectors are orthogonal to each other. As such their dot product is zero:

$$\boldsymbol{n1} \cdot \boldsymbol{n2} = \boldsymbol{n1} \cdot \boldsymbol{n3} = \boldsymbol{n2} \cdot \boldsymbol{n3} = 0$$

The Rankine Failure Criterion

The Rankine, or maximum principal stress criterion, asserts that yielding begins at a point in a member where the maximum principal stress reaches a value equal to the tensile (or compressive) yield strength of the material.

HOMEWORK 9

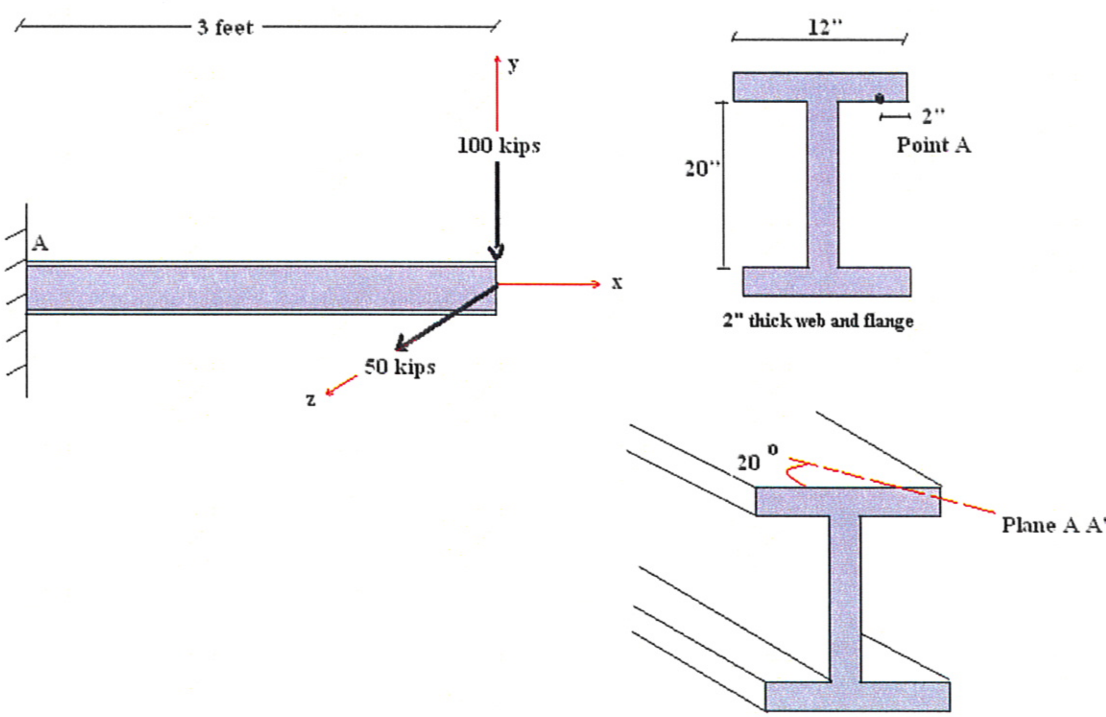
- Solve for the stresses at Point A (normal and shear)
- Clearly draw a cube and show the stresses.
- Draw a square and show the stresses
- Write out the stress tensor

HOMEWORK 10

- Find principal stresses and directions at point A using Mohr's Circle
- Find stresses at plane AA' using Mohr's Circle

HOMEWORK 11

- Find principal stresses and directions using eigenvalue analysis
- Check your eigenvalues using MATLAB
- Explain the meaning of the eigenvectors by drawing the stress directions



Beam for HWK 9, 10, and 11.

HOMEWORK 12

Given stress at a point: σ_{xx} **=100**, σ_{yy}**= - 400** , σ_{zz}**= 500,** τ_{xy}**=1000,** τ_{xz}**= - 800**, and τ_{yz}= 300, find the principal stresses and directions.