GVG Lab 2 - Solution

March 6, 2022

Task 1. *Find the image point* $[u, v]^\top$ *which is the projection of 3D point* $\vec{X}_\delta = [1, 2, 3]^\top$ *by the camera with the following image projection matrix*

$$\mathtt{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution: By the definition of the mathematical model of the projective camera we have

$$\eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta \neq 0$$

$$\eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

or, equivalently

$$\eta u = 1, \quad \eta v = 2, \quad \eta = 3$$

from which it follows that

$$u = \frac{1}{\eta} = \frac{1}{3}, \quad v = \frac{1}{\eta} = \frac{1}{2}.$$

$\square$

Task 2. *Find the coordinates of the camera projection center* $\vec{C}_\delta$ *of a camera with the following scaled image projection matrix*

$$\mathtt{Q} = \xi \mathtt{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: By defition,

$$\mathtt{P}_\beta = \left[\mathtt{A} \mid -\mathtt{A}\vec{C}_\delta\right]$$

where $\mathtt{A} = \mathtt{T}_{\delta \to \beta}$. Hence

$$\mathtt{Q} = \xi \mathtt{P}_\beta = \xi \left[\mathtt{A} \mid -\mathtt{A}\vec{C}_\delta\right] = \left[\xi\mathtt{A} \mid -\xi\mathtt{A}\vec{C}_\delta\right], \quad \xi \neq 0$$

The camera projection center can be computed in two ways.

1. The camera projection center $\vec{C}_\delta$ can be retrieved from the kernel of $\mathtt{Q}$, since

$$\mathtt{Q} \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \left[\xi\mathtt{A} \mid -\xi\mathtt{A}\vec{C}_\delta\right] \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \xi\mathtt{A}\vec{C}_\delta - \xi\mathtt{A}\vec{C}_\delta = \mathbf{0}$$

Computing the kernel of $\mathtt{Q}$ means solving the system of linear equations

$$\mathtt{Q}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top \tag{1}$$

We solve it by applying Gaussian elimination method:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

From the last row $x_3 + 2x_4 = 0$ we conclude that $x_3 = -2x_4$ and we let x_4 to be any real number. From the second row we get $x_2 = 0$. From the first row we obtain $x_1 = 0$. Thus, the solutions to (1) are

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ -2x_4 \\ x_4 \end{bmatrix} \;\middle|\; x_4 \in \mathbb{R} \right\}$$

We are interested in the representative of the kernel with last coordinate equal to 1. For this we take $x_4 = 1$ and get

$$\begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in S \Rightarrow \vec{C}_\delta = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Remark. *We know that the kernel of every scaled image projection matrix is one-dimensional, since rank of this matrix is always equal to* 3*. We also note that we will be always able to find the representative of the kernel of* $\mathtt{Q}$ *with the last coordinate equal to* 1 *since otherwise there would exist a nontrivial kernel to the invertible matrix* $\xi\mathtt{A}$.

2. Another way to compute the camera projection center $\vec{C}_\delta$ is the following:

$$\vec{C}_\delta = (-\xi\mathtt{A})^{-1}(-\xi\mathtt{A})\vec{C}_\delta = -(\xi\mathtt{A})^{-1}(-\xi\mathtt{A}\vec{C}_\delta) = -\mathtt{Q}_{1:3,1:3}^{-1}\mathtt{Q}_{1:3,4} =$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

□

Task 3. *Write down the coordinates of all three-dimensional points which project into image point* $[2, 1]^\top$ *by a camera with the following scaled image projection matrix*

$$\mathtt{Q} = \xi\mathtt{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: The equation which models the projection of the world point $\vec{X}_\delta$ which doesn't belong to the principal plane to the image point $[u, v]^\top$ has the form [1, Equation 6.12]:

$$\eta\vec{x}_\beta = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \vec{x}_\beta = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}, \eta \in \mathbb{R}, \eta \neq 0 \tag{2}$$

In other words, the world point $\vec{X}_\delta$ projects to the image point $[u, v]^\top$ if and only if there exists $\eta \neq 0$ such that Equation (2) is satisfied. Obviously, $\xi \neq 0$, since otherwise $\mathtt{Q}$ would be the zero matrix. Hence we can multiply (2) by ξ from both sides:

$$\xi\eta\vec{x}_\beta = \xi\mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta, \xi \neq 0$$

$$\xi\eta\vec{x}_\beta = \mathtt{Q} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \mathtt{Q}_{1:3,1:3}\vec{X}_\delta + \mathtt{Q}_{1:3,4}$$

The whole ray p of world points $\vec{X}_\delta$ which project to the image point $[u,v]^\top$ can be written as

$$p = \left\{ \mathtt{Q}_{1:3,1:3}^{-1}(\xi\eta\vec{x}_\beta - \mathtt{Q}_{1:3,4}) \;\middle|\; \eta \in \mathbb{R}, \eta \neq 0 \right\} =$$
$$= \left\{ \xi\eta\mathtt{Q}_{1:3,1:3}^{-1}\vec{x}_\beta - \mathtt{Q}_{1:3,1:3}^{-1}\mathtt{Q}_{1:3,4} \;\middle|\; \eta \in \mathbb{R}, \eta \neq 0 \right\}$$
$$= \left\{ \eta\mathtt{Q}_{1:3,1:3}^{-1}\vec{x}_\beta - \mathtt{Q}_{1:3,1:3}^{-1}\mathtt{Q}_{1:3,4} \;\middle|\; \eta \in \mathbb{R}, \eta \neq 0 \right\}$$

We compute

$$\mathtt{Q}_{1:3,1:3}^{-1}\vec{x}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$\mathtt{Q}_{1:3,1:3}^{-1}\mathtt{Q}_{1:3,4} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Hence the ray p has the form

$$p = \left\{ \eta \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; \eta \in \mathbb{R}, \eta \neq 0 \right\}$$

We can see that p is indeed a line in space with the camera projection center removed (**since this is the only point in the principal plane which belongs to the line connecting the camera projection center and the image point**). □

Task 4. *Denote the image coordinates by $[u,v]^\top$. Write down coordinates of all points in the three-dimensional space that projects on the line $v = 0$ by a camera with the following scaled image projection matrix*

$$\mathtt{Q} = \xi\mathtt{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: The equation which models the projection of the world point $\vec{X}_\delta$ which doesn't belong to the principal plane to the image point $[u,0]^\top$ has the form [1, Equation 6.12]:

$$\eta\vec{x}_\beta = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \vec{x}_\beta = \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix}, u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \tag{3}$$

We apply the same steps as in the previous task. The set σ of points $\vec{X}_\delta$ which satisfy Equation (3) can be expressed as

$$\sigma = \left\{ \eta\mathtt{Q}_{1:3,1:3}^{-1} \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} - \mathtt{Q}_{1:3,1:3}^{-1}\mathtt{Q}_{1:3,4} \;\middle|\; u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} =$$
$$= \left\{ \eta \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} =$$
$$= \left\{ \eta \begin{bmatrix} u \\ -u \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} =$$
$$= \left\{ \eta u \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \eta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} =$$
$$= \left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R}, b \neq 0 \right\}$$

We can see that σ is a plane in space with the line passing through the camera projection center removed (**since the intersection of the plane passing through the camera projection center and the image line with the principal plane is a line**). To get the parametric equation of this line we need to substitute $b = 0$ into the parametric equation of σ:

$$L = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}.$$

We can also see that the camera projection center belongs to L. $\square$

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla`