Homework 4 - Problem 1:

$F(x) = x - 0.2 - 0.8 \sin x = 0$ in $I = [0, \pi/2]$

Solve using fixed-pt iteration, prove the convergence of iteration scheme: $x_{n+1} = 0.2 + 0.8 \sin x_n$ for $x_0 \in I$.

$x - 0.2 - 0.8 \sin x = 0$

$x = 0.2 + 0.8 \sin x = g(x)$

① $g : I \to I$

② $|g(x_2) - g(x_1)| \leq L |x_2 - x_1| \ \forall \ x_1, x_2 \in I$ for some $L < 1$

$\Longleftrightarrow \max |g'(x)| = L < 1$ w/ $x \in I$.

① $g : I \to I$ given $\min_{x \in I} g(x) = 0.2$, $\max_{x \in I} g(x) = 1$ ✓

② $|g'| = |0.8 \cos x| \leq 0.8 < 1$

Hence, by the Contractive Mapping Theorem, $\{x_n\} \to x_*$

Homework 4 - Problem 2:

$f(x,y) = xy^2 + xy - x^4 - 1 = 0$

$g(x,y) = x^2 + y - 2 = 0$

Check that $\underset{\sim}{x}_* = (1,1)^T$ is one of the sol. to $F(x) = \underline{0}$ where

$$F(x) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

$$\nabla F = F'(\underset{\sim}{x}_*) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{(x,y)=(1,1)}$$

$$= \begin{pmatrix} y^2+y-4x^3 & 2xy+x \\ 2x & 1 \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix} = -8 \neq 0 \quad \therefore \quad \underline{\text{Invertible}}.$$

F, F', F'' are continuous in a neighborhood of $\underset{\sim}{x}_*$ since f & g are polynom.

Newton's Method: $\underset{\sim}{x}_{n+1} = \underset{\sim}{x}_n - f'^{-1}(\underset{\sim}{x}_n)\, f(\underset{\sim}{x}_n)$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \nabla F^{-1}(x_n, y_n)\, F(x_n, y_n)$$

$$= \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} y_n^2 + y_n - 4x_n & 2x_n y_n + x_n \\ 2x_n & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_n y_n^2 + x_n y_n - x_n^4 - 1 \\ x^2 + y_n - 2 \end{pmatrix}$$

Newton method will converge to x_* for all init. guesses close to x_*.