Series and Parallel Addition of Matrices*

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AND

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Let A and B be Hermitian semi-definite matrices and let A^+ denote the Moore–Penrose generalized inverse. Then we define the parallel sum of A and B by the formula $A(A + B)^+B$ and denote it by $A : B$. If A and B are nonsingular this reduces to $A : B = (A^{-1} + B^{-1})^{-1}$ which is the well known electrical formula for addition of resistors in parallel. Then it is shown that the Hermitian semi-definite matrices form a commutative partially ordered semigroup under the parallel sum operation. Here the ordering $A \geqslant B$ means $A - B$ is semidefinite and the following inequality holds: $(A + B) : (C + D) \geqslant A : C + B : D$. If $R(A)$ denotes the range of A then it is found that $R(A : B) = R(A) \cap R(B)$. Moreover if A and B are orthogonal projection operators then $2 A : B$ is the orthogonal projection on $R(A) \cap R(B)$. The norms are found to satisfy the inequality $\| A : B \| \leqslant \| A \| : \| B \|$. Generalization to non-Hermitian operators are also developed.

I. Introduction

In this paper a new operation, called parallel addition, is defined for pairs of linear operators. Parallel addition originally arose in an attempt to generalize a network synthesis procedure of Duffin [4] and has already been studied in the scalar case by Erickson [5].

The connection of resistors in series and parallel is a familiar concept from elementary network theory. If two resistors having resistances A and B are connected in series the joint resistance is $S = A + B$, and if they are in

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parallel, the joint resistance is $P = (A^{-1} + B^{-1})^{-1} = AB/(A + B)$. These two methods of combining resistance are then called series and parallel addition.

It has been tacitly assumed in the above that A and B are positive numbers, since the formula for P is not necessarily defined otherwise. In the physical context the normal situation is for A and B to be positive, however, the case A and $B = 0$—a short circuit—can be handled by letting $P = 0$ if $A = 0$ and B is nonnegative. Thus the concept of parallel addition of nonnegative scalars is a well defined mathematical operation; it will be denoted here by $A : B$.

As is well known, positive semidefinite matrices are a generalization of nonnegative scalars. This suggests that the parallel addition of such matrices be defined by $A : B = A(A + B)^{-1} B$. If A and B are both nonsingular this is equivalent to $A : B = (A^{-1} + B^{-1})^{-1}$. The latter formula is not defined for singular A and B, however, the former will be defined if $A + B$ has an inverse. Even when $A + B$ is singular, the parallel addition may be defined by replacing $(A + B)^{-1}$ by $(A + B)^{+}$, the Moore-Penrose generalized inverse. The generalized inverse has a particularly simple form when $A + B$ is symmetric semidefinite. In fact, $A + B$ when restricted to its range is one to one, so that a genuine inverse may be defined there.

It proves just as simple to consider Hermitian semidefinite matrices rather than matrices which are necessarily real. However, in this paper, the operators are restricted to a finite dimensional vector space V.

Suppose that A and B are Hermitian semidefinite matrices. If they are both nonsingular it is obvious that parallel addition is commutative and associative. Moreover, the resulting matrix $A : B$ is also Hermitian semidefinite. However, if A and B are singular then these properties are not clear consequences of the definition. Nevertheless indirect arguments show that these properties hold in all situations.

Let $R(A)$ denote the range subspace of the matrix A. Then it is well known that $R(A + B) = R(A) + R(B)$ if A and B are Hermitian semidefinite matrices. It is found here that $R(A : B) = R(A) \cap R(B)$. The above two operations are just the operations used in defining the lattice of subspaces of the vector space V. As a consequence a sequence of series and parallel additions of matrices induces corresponding lattice operations. As is well known the lattice of subspaces is modular. This property is found to give identities relating certain series-parallel connections.

Orthogonal projection operators are, of course, Hermitian semidefinite and the theorem just mentioned has the following interesting application. If A and B are orthogonal projection operators then $2A : B$ is the orthogonal projection into the subspace $R(A) \cap R(B)$.

Semidefinite matrices form a partially ordered system if $A \geqslant B$ is taken

to mean $A - B$ is semidefinite. In terms of this system the following general inequality holds

$$(A + B) : (C + D) \geqslant A : C + B : D.$$

We term this the *series-parallel inequality* and it is a generalization of a scalar inequality of A. Lehman [11]. In particular if $A \geqslant B$ then it results that $A : C \geqslant B : C$. Thus we can say that the Hermitian semidefinite operators on a finite dimensional space form a partially ordered commutative semi-group with parallel addition as the group operation.

It is of interest to note that the boundary inequalities for parallel additions of matrices seem to be best expressed in terms of parallel addition of scalars. For example the following inequalities hold for the norm, trace, and determinant:

$$\| A : B \| \leqslant \| A \| : \| B \|,$$

$$\operatorname{tr}(A : B) \leqslant (\operatorname{tr} A) : (\operatorname{tr} B),$$

$$| A : B | \leqslant | A | : | B |.$$

These inequalities are best possible.

II. Preliminaries

In this paper we will consider operators on a finite dimensional complex inner product space. The range and null space of an operator A will be denoted by $R(A)$ and $N(A)$ respectively. The orthogonal projection onto $R(A)$ will be denoted by P_A. A Hermitian operator A will be said to be positive semidefinite if $(Ax, x) \geqslant 0$ for all x; the abbreviation HSD will be used. For an HSD operator A it is easy to prove that $(Ax, x) = 0$ iff $Ax = 0$, and that $R(A + B) = R(A) + R(B)$.

The Moore-Penrose generalized inverse [1], [13] of an operator A will be denoted by A^+. In the cases considered here A will be HSD. Then A when restricted to its range is one to one, and therefore invertible; A^+ is this inverse. It then follows that $AA^+ = A^+A = P_A$, and that A^+ is HSD iff A is HSD.

III. Algebraic Properties

DEFINITION. Let A and B be Hermitian semidefinite operators on the finite dimensional complex vector space V. The *series sum* of A and B is

defined to be the ordinary sum $A + B$. The *parallel sum* of A and B is defined by

$$A : B = A(A + B)^+ B. \tag{1}$$

Since $R(B) \subset R(A + B)$, for any x we have $Bx \in R(A + B)$. Then, since $A + B$ is invertible on its range, the complete definition of the generalized inverse is never used. In the later development of the theory it will turn out that reasonable results are obtained only when we can guarantee that $R(B) \subset R(A + B)$, as is the case here.

For scalars a, b we define $a : b = a(a + b)^+ b$, where $0^+ = 0$. This is the case studied by Erickson [5].

LEMMA 1. *If A and B are Hermitian semidefinite, then $A : B = B : A$.*

PROOF.

$$
\begin{aligned}
A : B &= A(A + B)^+ B \\
&= (A + B - B)(A + B)^+(A + B - A) \\
&= (A + B)(A + B)^+(A + B) - B(A + B)^+(A + B) \\
&\qquad\qquad\qquad - (A + B)(A + B)^+ A + B : A \\
&= A + B - BP_{(A+B)^+} - P_{A+B}A + B : A \\
&= B : A. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{QED}
\end{aligned}
$$

If the definition of $A : B$ is extended to general operators A and B, this proof would work if $A + B$ is nonsingular; if not, the lemma need not hold.

LEMMA 2. *If A and B are Hermitian semidefinite, then $A : B$ is Hermitian.*

PROOF.

$$
\begin{aligned}
(A : B)^* &= (A(A + B)^+ B)^* \\
&= B^*(A + B)^{*+} A^* = B(A + B)^+ A \\
&= B : A \\
&= A : B \qquad \text{by Lemma 1.} \qquad\qquad\qquad\text{QED}
\end{aligned}
$$

LEMMA 3. *If A and B are Hermitian semidefinite, then*

$$R(A : B) = R(A) \cap R(B).$$

PROOF.   Consider $x \in R(A) \cap R(B)$. Then

$$
\begin{aligned}
A : B(A^+ + B^+) x &= A(A + B)^+ BB^+x + B(A + B)^+ AA^+x \\
&= A(A + B)^+ x + B(A + B)^+ x \\
&= P_{A+B}x = x.
\end{aligned}
$$

Therefore $R(A : B) \supset R(A) \cap R(B)$. Since $R(A(A + B)^+ B) \subset R(A)$, the lemma then follows from Lemma 1.   QED

LEMMA 4. *If A and B are Hermitian semidefinite, then $A : B$ is semidefinite.*

PROOF.   For any z, let $A : Bz = x$. Then, as in Lemma 3,

$$A : B(A^+ + B^+) x = x$$

and

$$
\begin{aligned}
(A : Bz, z) = (x, z) &= (A : B(A^+ + B^+) x, z) \\
&= ((A^+ + B^+) x, A : Bz) \qquad \text{by Lemma 2} \\
&= ((A^+ + B^+) x, x) \\
&= (A^+x, x) + (B^+x, x) \geqslant 0. \qquad \text{QED}
\end{aligned}
$$

Although we cannot in general write $A : B = (A^{-1} + B^{-1})^{-1}$, and by letting

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we see that the "obvious" extension $A : B = (A^+ + B^+)^+$ will not work either, we do have an alternative definition of this type.

THEOREM 5. *If A and B are Hermitian semidefinite, then*

$$A : B = (P(A^+ + B^+) P)^+,$$

where P is the projection onto $R(A) \cap R(B)$.

PROOF.   Consider any $x \in V$. Since $R(A : B) = R(A) \cap R(B)$ and $A : B$ is Hermitian, $(A : B)^+ x = (A : B)^+ Px$. But, as in Lemma 3, $A : B(A^+ + B^+) Px = Px$. Then,

$$(A : B)^+ Px = P(A^+ + B^+) Px.$$

But $A^{++} = A$, therefore $A : B = (P(A^+ + B^+) P)^+$.   QED

LEMMA 6. *If A, B, and C are Hermitian semidefinite, then*

$$(A : B) : C = A : (B : C).$$

PROOF.   By Lemma 3, the range of both sides is $R(A) \cap R(B) \cap R(C)$.

Now, for $x \in R(A) \cap R(B) \cap R(C)$,

$$A : (B : C)\,(A^+ + B^+ + C^+)\,x = A(A + B : C)^+\,B : C(A^+ + B^+ + C^+)\,x$$
$$= A(A + B : C)^+\,B(B + C)^+\,CC^+x$$
$$+ A(A + B : C)^+\,C(B + C)^+\,BB^+x$$
$$+ B : C(A + B : C)^+\,AA^+x$$
$$= A(A + B : C)^+\,(B + C)\,(B + C)^+\,x$$
$$+ B : C(A + B : C)^+\,x$$
$$= (A + B : C)\,(A + B : C)^+\,x = x.$$

A similar computation holds for $(A : B) : C$. Then, as in the proof of Theorem 6, if P is the projection onto

$$R(A) \cap R(B) \cap R(C),\ A : (B : C) = (A : B) : C = (P(A^+ + B^+ + C^+)\,P)^+.$$

$$\text{QED}$$

THEOREM 7.   *The Hermitian semidefinite operators on V form a partially ordered commutative semigroup with the semigroup operation parallel addition.*

PROOF.   The semigroup property follows from Lemmas 1, 2, 3, and 7. The partial order property is proved in Corollary 21, to follow.

It has been proved that if A and B are HSD, then

$$R(A + B) = R(A) + R(B) \qquad \text{and} \qquad R(A : B) = R(A) \cap R(B).$$

These are just the operations used in defining the lattice of subspaces; it seems natural, therefore, to consider that lattice. One question in this regard may be easily answered.

THEOREM 8.   *If P and Q are projections, then the projection onto $R(P) \cap R(Q)$ is $2P : Q$.*

PROOF.   By Lemmas 2 and 3, $2P : Q$ is an Hermitian operator with the correct range. For $x \in R(P) \cap R(Q)$ we have

$$x = P : Q(P^+ + Q^+)\,x = P : Q(2x) = 2P : Qx. \qquad \text{QED.}$$

The formula of Theorem 8 answers Problem 96 of Halmos [8], for the finite dimensional case. In the published solution it is stated that the familiar algebraic operations are not likely to furnish a solution; Theorem 8 appears to meet the requirement.

Because the lattice of subspaces is modular, equalities may be between the range of various series-parallel combinations of operators. In any modular lattice $(a \wedge (b \vee c)) \wedge (b \wedge (c \vee a)) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ [2]; it follows

that for HSD operators A, B, C, the operators $A : (B + C) + B : (C + A)$ and $(A + B) : (B + C) : (C + A)$ have the same range. It appears that these facts about range spaces could be used to study switching circuits, possibly by using projections to represent their range spaces.

THEOREM 9. *If A and B are Hermitian semidefinite, and $Ax = ax$, $Bx = bx$, then $A : Bx = a : bx$.*

PROOF.

$$(A : B)^+ x = P_{A:B}(A^+ + B^+) P_{A:B}x$$
$$= P_{A:B}(A^+ + B^+) x$$
$$= (a^+ + b^+) x.$$

Then

$$A : Bx = (a^+ + b^+)^+ x = a : bx.$$

The parallel addition is not distributive with respect to series addition. There is, however, a weak distributive law with respect to multiplication.

THEOREM 10. *If A, B, and C are Hermitian semidefinite, and $AC = CA$, $BC = CB$, then $(AC) : (BC) = (A : B) C$. In particular, C might be CI, for some nonnegative scalar C.*

PROOF.

$$(AC) : (BC) = AC(AC + BC)^+ BC$$
$$= ACC^+(A + B)^+ BC$$
$$= A(A + B)^+ BCC^+C = (A : B) C. \qquad \text{QED}$$

This is the most general distributive law that could be expected, since if $AC \neq CA$, then AC is not Hermitian.

LEMMA 11. *If A and B are Hermitian semidefinite, and $Aw = Bx = u$, $Ay = Bz = v$, and $w + x = y + z$, then $u = v$.*

PROOF. By direct computation, $A : B(w + x) = u$, $A : B(y + z) = v$, and since $w + x = y + z$, it follows that $u = v$. \qquad QED

The impedance matrix of a resistive n-port network is semidefinite [10]. A series-parallel connection of 3-port networks is shown in Figure 1. If two n-port networks with nonsingular impedance matrices A and B are connected in parallel, then the impedance matric C of the parallel connection is given by $C = (A^{-1} + B^{-1})^{-1} = A : B$. If A or B is singular, then our formula for $A : B$ still gives the correct impedance. Lemma 11 is the crucial step in the proof, which will not be further discussed here.

a. A series parallel network (1-port)

b. Series parallel connection of 3-ports

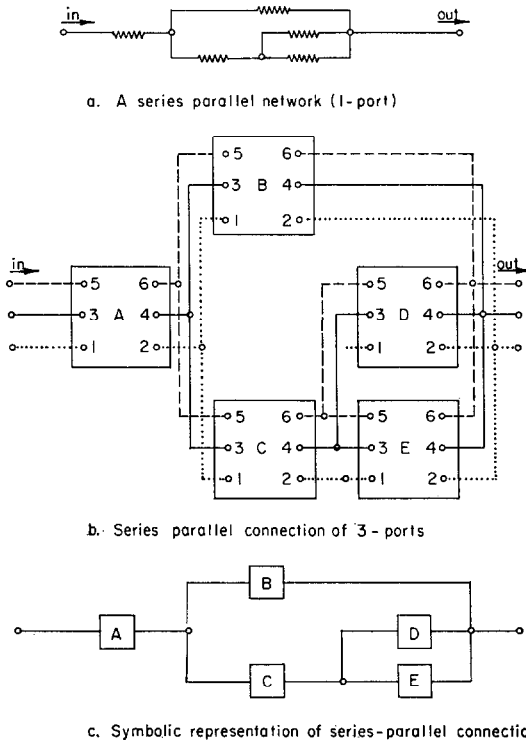c. Symbolic representation of series-parallel connection

FIG. 1

IV. THE MATRIX OF A PARALLEL SUM

If A and B are Hermitian, an orthonormal basis may be chosen such that $A + B$ is diagonal. Let $A + B = C$, $A : B = D$. Then $a_{ij} = -b_{ij}$ for $i \neq j$, and C^+ has the matrix $\mathrm{diag}(c_{ii}^+) = \mathrm{diag}((a_{ii} + b_{ii})^+)$. Then

$$d_{ij} = \sum_k a_{ik} \sum_m C_{km}^+ b_{mj}$$

$$= \sum_k a_{ik}(a_{kk} + b_{kk})^+ b_{kj}$$

since C is diagonal. If $a_{kk} = 0$, then since A is HSD, $a_{ik} = 0$ for all i, and we may then write with the convention $\frac{0}{0} = 0$.

$$d_{ij} = \sum_k \frac{a_{ik}}{a_{kk} + b_{kk}} b_{kj} \tag{2}$$

and, in particular,

$$d_{ii} = \frac{a_{ii}}{a_{ii} + b_{ii}} \, b_{ii} - \sum_{i \neq j} \frac{a_{ij}^2}{a_{jj} + b_{jj}} \, . \qquad (3)$$

LEMMA 12.   *Let* $a_1, ..., a_n$, $b_1, ..., b_n$ *be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i \right) : \left(\sum_{i=1}^{n} b_i \right) \geqslant \sum_{i=1}^{n} a_i : b_i \, .$$

PROOF.   This is merely Minkowski's inequality [9]. This inequality is more extensively discussed in Section V.

THEOREM 13.   *If A and B are Hermitian semidefinite, then*

$$\mathrm{tr} \, (A : B) \leqslant \mathrm{tr} \, (A) : \mathrm{tr} \, (B),$$

with equality iff $A = cB$, *for some scalar c (necessarily real).*

PROOF.   Since trace is invariant, we may choose a basis such that $A + B$ is diagonal. Then by (2),

$$\mathrm{tr}(A : B) = \sum_i \left(a_{ii} : b_{ii} - \sum_{i \neq j} \frac{a_{ij}^2}{a_{jj} + b_{jj}} \right)$$

$$\leqslant \sum_i a_{ii} : b_{ii}$$

$$\leqslant \left(\sum a_{ii} \right) : \left(\sum b_{ii} \right) = \mathrm{tr}(A) : \mathrm{tr}(B).$$

In the first inequality, equality holds iff all $a_{ij} = 0$, that is, iff $AB = BA$. In the second inequality, equality holds iff the a_{ii} and b_{ii} are proportional. Thus equality will hold in both iff $A = cB$.   QED

THEOREM 14.   *If A and B are Hermitian semidefinite, then*

$$| A : B | \leqslant | A | : | B | \, .$$

PROOF.   If either $| A | = 0$ or $| B | = 0$, then $| A : B | = 0$. If not, then A and B are invertible, and

$$| A^{-1} + B^{-1} | \geqslant | A^{-1} | + | B^{-1} | = | A |^{-1} + | B |^{-1}.$$

Then

$$| A : B | = | A^{-1} + B^{-1} |^{-1} \leqslant (| A |^{-1} + | B |^{-1})^{-1} = | A | : | B | \, .$$

                    QED

THEOREM 15. *If V is 2-dimensional, and $A + B$ is nonsingular, then*

$$A : B = \frac{A \mid B \mid + B \mid A \mid}{\mid A + B \mid}.$$

PROOF. The result follows from direct computation with matrix elements.   QED

No generalization to higher dimensions has been found.

V. PARTIAL ORDERING AND PARALLEL ADDITION

DEFINITION. If A and B are linear operators, then $A \geqslant B$ is defined to mean that $A - B$ is positive semidefinite.

The results of this section will be motivated by electrical networks. The proofs, however, will be purely algebraic. Consider the diagram below, which shows nine resistors in a series-parallel network. If the dotted connections are not present, the joint impedance between terminals x and y is

$$Z = (A + B + C) : (D + E + F) : (G + H + J), \tag{4}$$

and if the dotted connections are present

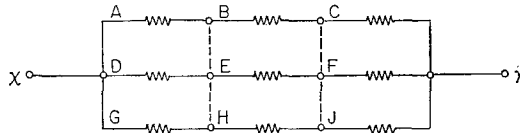$$Z' = A : D : G + B : E : H + C : F : J. \tag{5}$$

FIG. 2

It was observed by Lehman [11] that $Z \geqslant Z'$ because the current takes the path of least resistance, and in the second case more paths are available. The general case would then be

THEOREM 16. *If $r_{ij} > 0$, $i = 1,...,m$, $j = 1,...,n$, then*

$$\left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} r_{ij} \right)^{-1} \right)^{-1} \geqslant \sum_{j=1}^{n} \left(\sum_{i=1}^{m} r_{ij} \right)^{-1} \right)^{-1}. \tag{6}$$

PROOF. This is Minkowski's inequality [9]. In the present notation we may rewrite this.

COROLLARY 17. *If $r_{ij} \geqslant 0$, $i = 1,..., m$, $j = 1,..., n$, then*

$$\prod_{i=1}^{m} : \sum_{j=1}^{n} r_{ij} \geqslant \sum_{j=1}^{n} \prod_{i=1}^{m} : r_{ij} \tag{7}$$

where $\prod : r_{ij} = r_{1j} : r_{2j} : \cdots : r_{mj}$.

PROOF. For $r_{ij} > 0$ this is Theorem 16 above; the case $r_{ij} = 0$ is easily obtained by continuity.

It seems that the same physical reasoning must apply to n-port networks, so that Corollary 17 will hold for the joint impedance matrices. The proof below follows a proof of Theorem 16 given by Reza [11].

LEMMA 18. *If A and B are Hermitian semidefinite, then for any x, y, z such that $x + y = z$*

$$(A : Bz, z) \leqslant (Ax, x) + (By, y). \tag{8}$$

Moreover, if $z \in R(A) + R(B)$, then if $x_0 = (A + B)^{+} Bz$ and $y_0 = (A + B)^{+} Az$, $x_0 + y_0 = z$ and equality holds in (8).

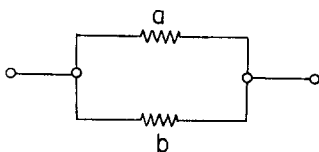We first give a heuristic argument. Consider the network of Figure 3

FIG. 3

For a given current input z, the current will divide $z = x + y$ in such a fashion that the power dissipated $ax^2 + by^2$ is minimum. That is $a : bz^2 \leqslant ax^2 + by^2$ which is Lemma 18 in the scalar case. The same argument may be used for n-ports with impedance matrices.

PROOF. For x_0, y_0 as above,

$$Ax_0 = By_0 = A : Bz, \quad \text{and} \quad x_0 + y_0 = (A + B)^{+} (A + B) z = z.$$

Then

$$(A : Bz, z) = (A : Bz, x_0) + (A : Bz, y_0) = (Ax_0, x_0) + (By_0, y_0)$$

so that equality holds.

For any z, and $x + y = z$, let $x^1 = P_{A+B}x$, $y^1 = P_{A+B}y$, and $z^1 = P_{A+B}z$.

Then let
$$x_0{}^1 = (A + B)^+ Bz^1, \qquad y_0{}^1 = (A + B)^+ Az^1.$$

Since $x_0{}^1 + y_0{}^1 = z^1 = x^1 + y^1$, we may write $x^1 = x_0 + t$, $y^1 = y_0 - t$. Then
$$(Ax^1, x^1) = (Ax_0{}^1, x_0{}^1) + 2\mathrm{Re}(Ax_0{}^1, t) + (At, t)$$
$$(By^1, y^1) = (By_0{}^1, y_0{}^1) - 2\mathrm{Re}(By_0{}^1, t) + (Bt, t).$$

Since $Ax_0{}^1 = By_0{}^1$ it follows that
$$(Ax^1, x^1) + (By^1, y^1) = (Ax_0{}^1, x_0{}^1) + (By_0{}^1, y_0{}^1) + (At, t) + (Bt, t)$$
$$\geqslant (A : Bz^1, z^1). \tag{9}$$

But $(Ax^1, x^1) = (AP_{A+B}x, P_{A+B}x) = (Ax, x)$ and similarly for y^1, z^1. The lemma then follows from (9).   QED

COROLLARY 19.   *If A and B are Hermitian semidefinite, then*
$$(A : Bz, z) \leqslant (Az, z) : (Bz, z).$$

PROOF.   If $(Az, z) + (Bz, z) = 0$ then the result is clear. If not let
$$x = \frac{(Bz, z)}{((A + B) z, z)} z \qquad \text{and} \qquad y = \frac{(Az, z)}{((A + B) z, z)} z.$$

Then $x + y = z$ and by Lemma 17
$$(A : Bz, z) \leqslant (Ax, x) + (By, y) = (A : Bz, z).   \qquad \text{QED}$$

LEMMA 20.   *Let A, B, C, and D be Hermitian semidefinite, then*
$$(A + B) : (C + D) \geqslant A : C + B : D.$$

PROOF.   It suffices to consider $z \in R(A + B + C + D)$, then by Lemma 18, for suitable x_0, y_0
$$((A + B) : (C + D) z, z) = ((A + B) x_0, x_0) + ((C + D) y_0, y_0)$$
$$= (Ax_0, x_0) + (Bx_0, x_0) + (Cy_0, y_0) + (Dy_0, y_0)$$
$$\geqslant (A : Cz, z) + (B : Dz, z).   \qquad \text{QED}$$

COROLLARY 21.   *If A, B, and C are Hermitian semidefinite, then $A \geqslant B$ implies $A : C \geqslant B : C$.*

PROOF.   Let $A - B = D$. Then D is HSD and
$$A : C = (B + D) : (C + 0) \geqslant B : C + D : 0 = B : C.   \qquad \text{QED}$$

This completes the proof of Theorem 7. The property of Corollary 21 is the necessary relation between the semigroup operation and the partial order [6].

We may now prove the series-parallel inequality for operators.

THEOREM 22. *Let A_{ij}, $i = 1,..., m$, $j = 1,..., n$ be Hermitian semidefinite.* *Then*

$$\prod_{i=1}^{m} : \sum_{j=1}^{n} A_{ij} \geqslant \sum_{j=1}^{n} \prod_{i=1}^{m} : A_{ij}. \tag{10}$$

PROOF.   The proof is by induction. Lemma 20 is the case $m = 2$, $n = 2$. Now assume that (10) is true for m, n and all smaller, then for m, $n + 1$

$$\prod_{i=1}^{m} : \sum_{j=1}^{n+1} A_{ij} = \prod_{i=1}^{m} : \left(\sum_{j=1}^{n} A_{ij} + A_{i,n+1} \right)$$

$$\geqslant \prod_{i=1}^{m} : \sum_{j=1}^{n} A_{ij} + \prod_{i=1}^{m} : A_{i,n+1} \qquad \text{by the case } m, 2$$

$$\geqslant \sum_{j=1}^{n} \prod_{i=1}^{m} : A_{ij} + \prod_{i=1}^{m} : A_{i,n+1} \qquad \text{by the case } m, n$$

$$= \sum_{j=1}^{n+1} \prod_{i=1}^{m} : A_{ij}.$$

And for $m + 1$, n

$$\prod_{i=1}^{m+1} : \sum_{j=1}^{n} A_{ij} = \left(\prod_{i=1}^{m} : \sum_{j=1}^{n} A_{ij} \right) : \sum_{j=1}^{n} A_{m+1,j}$$

$$\geqslant \left(\sum_{j=1}^{n} \prod_{i=1}^{m} : A_{ij} \right) : \sum_{j=1}^{n} A_{m+1,j} \qquad \text{by the case } m, n$$

$$\geqslant \sum_{j=1}^{n} \left(\prod_{i=1}^{m} : A_{ij} \right) : A_{m+1,j} \qquad \text{by the case } 2, n$$

$$= \sum_{j=1}^{n} \prod_{i=1}^{m+1} : A_{ij}. \qquad\qquad \text{QED}$$

COROLLARY 23. *Let $A_1, ..., A_n$ be Hermitian semidefinite. Then*

$$\sum_{i=1}^{n} A_i \geqslant n^2 \prod_{i=1}^{n} : A_i .$$

PROOF. This follows from Theorem 22 with $m = n$. For example, if $n = 3$

$$(A + B + C) : (B + C + A) : (C + A + B)$$
$$\geqslant A : B : C + B : C : A + C : A : B$$
$$\tfrac{1}{3}(A + B + C) \geqslant 3A : B : C. \qquad \text{QED}$$

Corollary 23 is a generalization of the classical inequality between the arithmetic and harmonic means [9]. This inequality is sometimes proved by a convexity argument. In the present case, we have

THEOREM 24. *Parallel addition is a concave operation for Hermitian semidefinite operators.*

PROOF. Let $\theta(A, B) = A : B$. Then, if a is a scalar, $\theta(aA, aB) = a\theta(A, B)$. Then if $a_1, a_2 \geqslant 0$ and $a_1 + a_2 = 1$, using Theorem 22

$$\theta(a_1A_1 + a_2A_2 , a_1B_1 + a_2B_2) \geqslant a_1\theta(A_1 , B_1) + a_2\theta(A_2 , B_2).$$
$$\text{QED}$$

VI. CONTINUITY

THEOREM 25. *If A and B are Hermitian semidefinite, then*

$$\| A : B \| \leqslant \| A \| : \| B \| .$$

PROOF. Since A is HSD, for any x

$$(Ax, Ax) \leqslant \| A \| (Ax, x) \qquad (11)$$

and in fact, if $A \neq 0$, for any ϵ there is an x_0 such that

$$\frac{(Ax_0 , Ax_0)}{(Ax_0 , x_0)} \geqslant \| A \| - \epsilon. \qquad (12)$$

Now let $y = A : Bx$, where

$$\frac{(A : Bx, A : Bx)}{(A : Bx, x)} \geqslant \| A : B \| - \epsilon.$$

Then $A : B(A^+ + B^+) y = y$ as in Lemma 3, and

$$(A : Bx, x) = (y, x) = (A : B(A^+ + B^+) y, x)$$
$$= ((A^+ + B^+) y, A : Bx) = (A^+y, y) + (B^+y, y). \qquad (13)$$

Since $y \in R(A : B)$, let $y = Au = Bv$. Then $(A^+y, y) = (u, Au)$ and $(B^+y, y) = (v, Bv)$. Then from (11) and (12)

$$\frac{(u, Au)}{(Au, Au)} + \frac{(v, Bv)}{(Bv, Bv)} = \frac{(A : Bx, x)}{(A : Bx, A : Bx)}$$

and then from (11)

$$\frac{1}{\| A \|} + \frac{1}{\| B \|} \leqslant \frac{1}{\| A : B \| - \epsilon} \quad \text{or} \quad \| A \| : \| B \| \geqslant \| A : B \| - \epsilon$$

but ϵ was arbitrary.   QED

By Theorem 9, equality will hold if there is a y such that $Ay = \| A \| y$ and $By = \| B \| y$.

Theorem 25 expresses continuity of parallel addition about 0; the next theorem applies at other points. We thus consider $(A + X) : (B + Y) - A : B$ and obtain bounds for the error in terms of the partial order of operators.

LEMMA 26. *If A and B are Hermitian semidefinite operators such that $B \geqslant A$, then*

$$(B^+ - A^+) P_A = B^+(B - A) A^+.\tag{14}$$

PROOF.

$$B^+(B - A) A^+ = B^+BA^+ - B^+AA^+ = P_BA^+ - B^+P_A = A^+ - B^+P_A$$

since $R(B) \supset R(A)$.   QED

LEMMA 27. *If A and B are Hermitian semidefinite operators and $C = A + B$ then $P_C(I - C^+B) = C^+A$.*

PROOF. $C^+A + C^+B = C^+C = P_C$. Then

$$P_C(I - C^+B) = P_C - C^+B = C^+A.\qquad\text{QED}$$

THEOREM 28. *Let A, B and X be Hermitian semidefinite and*

$$G = A : (B + X) - A : B.$$

Then G is HSD and if $C = A + B$,

$$G = AC^+(C : X) C^+A\tag{15}$$

and

$$\| G \| \leqslant \| C^+A \|^2 \| X \|.\tag{16}$$

PROOF.

$$G = A(C + X)^+ (B + X) - AC^+B$$

$$= A[(C + X)^+ - C^+] B + A(C + X)^+ X.$$

Since $P_C B = B$ Lemma 24 applies and then

$$G = - A(C + X)^+ XC^+B + A(C + X)^+ X. \qquad (17)$$

By the definition of G, $P_C G = G$, and since G is Hermitian $GP_C = G$. Therefore

$$G = GP_C = - A(C + X)^+ XC^+BP_C + A(C + X)^+ XP_C$$

$$= A(C + X)^+ XP_C(I - C^+B).$$

Then Lemma 27 applies and we have

$$G = A(C + X)^+ XC^+A = AC^+C(C + X)^+ XC^+A$$

$$= AC^+(C : X) C^+A. \qquad (18)$$

Since $C : X$ is HSD and (18) is congruent to $C : X$, G is HSD. Then using Theorem 25

$$\| G \| \leqslant \| AC^+ \| \| C : X \| \| C^+A \|$$

$$\leqslant \| AC^+ \|^2 \| X \| . \qquad \text{QED}$$

LEMMA 29. *If $A, B,$ and X are* HSD, *then*

$$2(A + X) : (B + X) + (A + B) : (2X)$$

$$= 2(A + B + X) : X + A : (B + 2X) + (A + 2X) : B.$$

PROOF. Let $A + B + 2X = D$, then by computation both sides equal

$$2AD^+B + 4AD^+X + 2XD^+B + 2BD^+X + 2XD^+X.$$

LEMMA 30. *If A, B and X are* HSD, *and*

$$H = (A + X) : (B + X) - A : B - X : X,$$

then H is HSD *and for $C = A + B$*

$$2H = AC^+(C : 2X) C^+A + BC^+(C : 2X) C^+B - \tfrac{1}{2} C : 2X \qquad (19)$$

and

$$\| H \| \leqslant 2(\| C^+A \|^2 + \| C^+B \|^2) \| X \| . \qquad (20)$$

PROOF. By Lemma 29,

$$2H = A : (B + 2X) - A : B + B : (A + 2X) - B : A$$

$$+ 2X : (A + B + X) - 2X : X - (A + B) : (2X),$$

then using Theorem 28

$$2H = AC^+(C:2X)\,C^+A + BC^+(C:2X)\,C^+B$$
$$+ 2X(2X)^+\,(C:2X)\,(2X)^+\,X - C:(2X)$$

which simplifies to (19). By Theorem 22 H is HSD. Therefore

$$2H \leqslant AC^+(C:2X)\,C^+A + BC^+(C:2X)\,C^+B$$

and, as in Theorem 28

$$\|\,H\,\| \leqslant \|\,AC^+\,\|^2\,\|\,2X\,\| + \|\,BC^+\,\|^2\,\|\,2X\,\|$$

which is (20).   QED

THEOREM 31. *If A, B, X and Y are Hermitian semidefinite, then*

$$\|\,(A+X):(B+Y) - A:B\,\|$$
$$\leqslant 2(\|\,(A+B)^+\,A\,\|^2 + \|\,(A+B)^+\,B\,\|^2 + \tfrac{1}{4})\,\|\,X+Y\,\|. \qquad (21)$$

PROOF. It follows from Corollary 21 that

$$(A+X):(B+Y) - A:B \leqslant (A+X+Y):(B+X+Y) - A:B,$$

and using Lemma 30

$$\|\,(A+X+Y):(B+X+Y) - A:B\,\|$$
$$\leqslant \|\,(A+X+Y):(B+X+Y) - A:B - (X+Y):(X+Y)\,\|$$
$$+ \|\,(X+Y):(X+Y)\,\|$$
$$\leqslant 2(\|\,(A+B)^+\,A\,\|^2 + \|\,(A+B)^+\,B\,\|^2)\,\|\,(X+Y)\,\| + \|\,\tfrac{1}{2}(X+Y)\,\|$$
$$= 2(\|\,(A+B)^+\,A\,\|^2 + \|\,(A+B)^+\,B\,\|^2 + \tfrac{1}{4})\,\|\,X+Y\,\|. \qquad \text{QED}$$

VII. GENERALIZATION

The definition of parallel addition $A:B = A(A+B)^+\,B$ may be applied to any pair of linear operators, since the generalized inverse is always defined. However, without suitable restrictions, very little of the preceding theory will hold. Four rather natural extensions have been considered, and will be developed in another paper. These extensions are the following:

(I) The networks which motivated the theory here were resistive. A natural extension is to consider networks with reactive elements. In that case the impedance matrices will be "positive real" [4], and will in general be non-Hermitian. Certain parts of the series-parallel algebra will extend.

(II) A linear operator A on a real vector space is said to be almost positive definite if $(Ax, x) \geqslant 0$ for all x and $(Ax, x) = 0$ only if $Ax = 0$ [12]; alternatively, if $A = H + S$ with H Hermitian semidefinite, S skew, and $R(S) \subset R(H)$. The latter definition is related to positive real matrices, where a similar range relation holds between the real and imaginary parts. The algebra of Section III holds for almost positive definite matrices; however, Lemma 18 is not true in this case, and the remainder of the theory will not follow.

(III) If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a partitioned matrix with a square, then the gyration of A, $\Gamma(A)$, is defined by

$$\Gamma(A) = \begin{bmatrix} a^{-1} & -a^{-1}b \\ ca^{-1} & d - ca^{-1}b \end{bmatrix}.$$

This operation has been studied by Tucker [14], and is the basis of the network synthesis method Duffin, Hazony, and Morrison [4]. A hybrid addition of operators A and B may then be defined by

$$A : B = \Gamma(\Gamma(A) + \Gamma(B)).$$

For nonsingular A and B this includes series and parallel addition as special cases, depending on the partition. Even in the singular case hybrid addition may be made the basis for an algebra similar to that developed here.

(IV) The question of extending the definition of parallel sum to operators in Hilbert space suggests itself. Although $(A + B)^+$ is defined [1], it need not be bounded, and thus $A(A + B)^+ B$ may not be defined on the whole space. It would appear that a different definition of parallel addition is necessary.

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