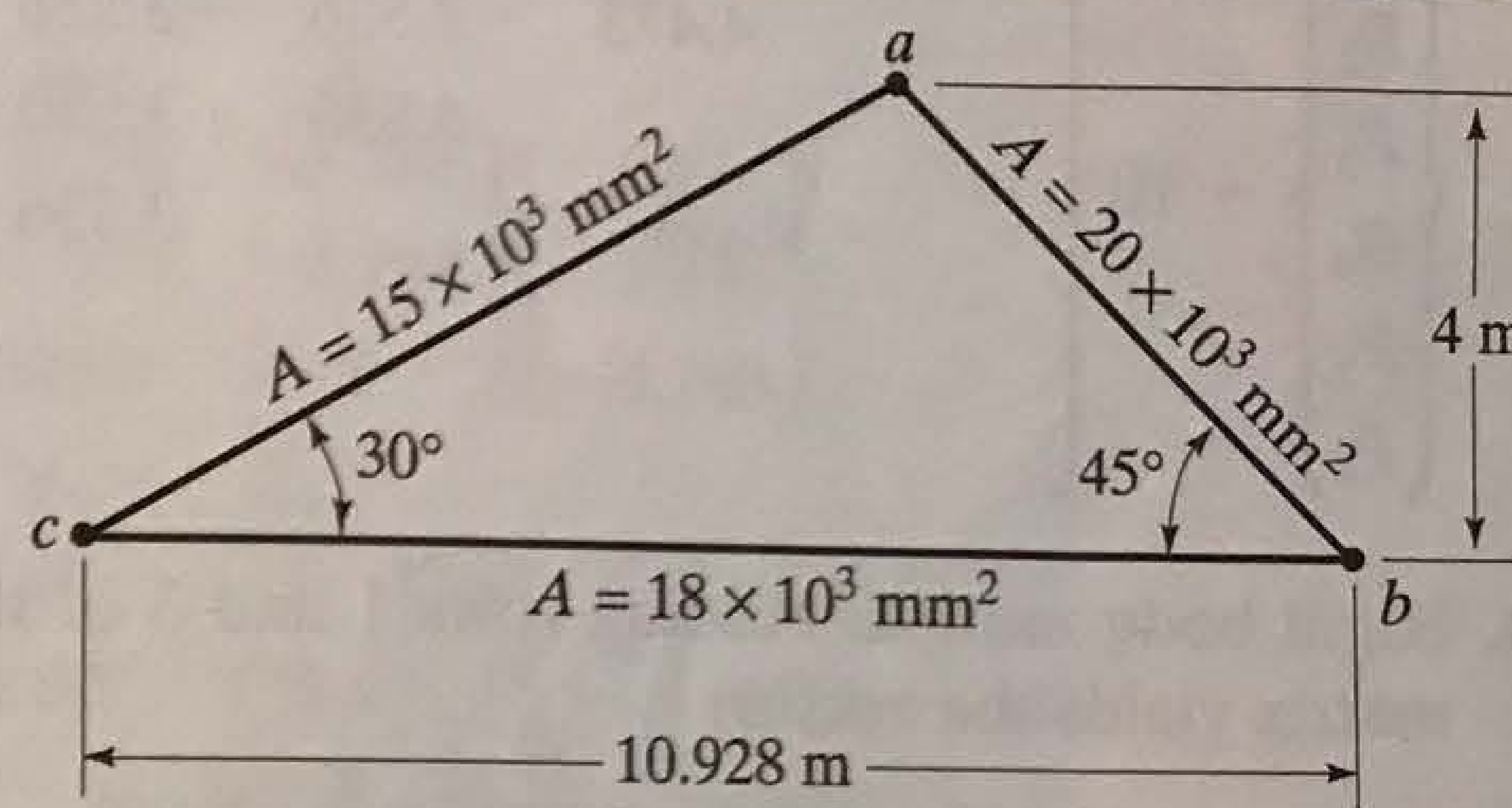
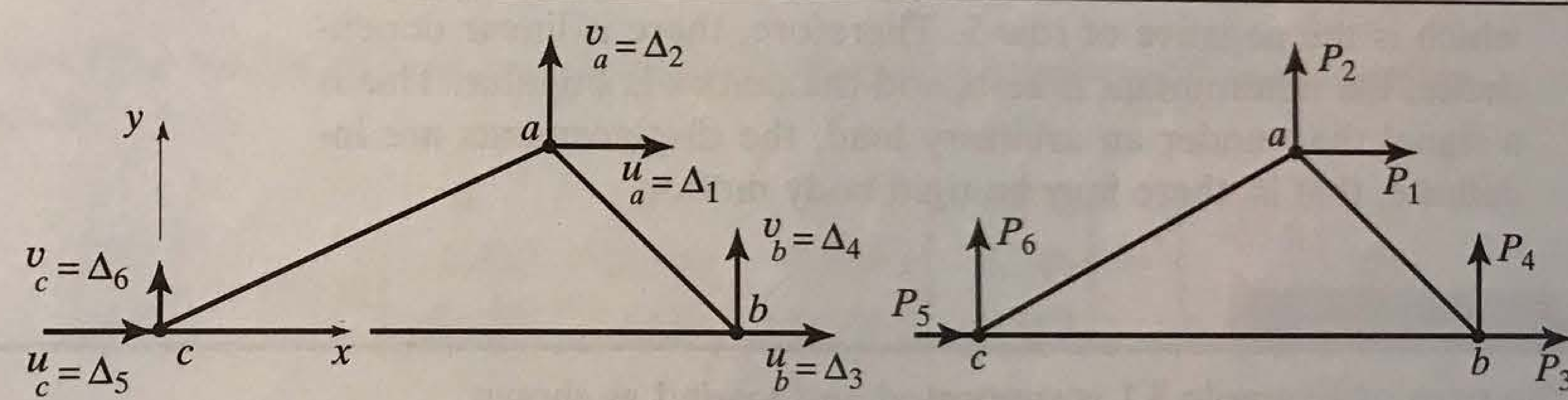
EXAMPLE 3.1

For the system shown:

1. Write the member force-displacement relationships in global coordinates.
2. Assemble the global stiffness equations.
3. Show that the global stiffness equations contain rigid-body-motion terms. $E = 200{,}000$ MPa.



Define the coordinates, degrees of freedom, and external forces as follows:



1. Member force-displacement relationships (see Equation 2.5):
   Member *ab*

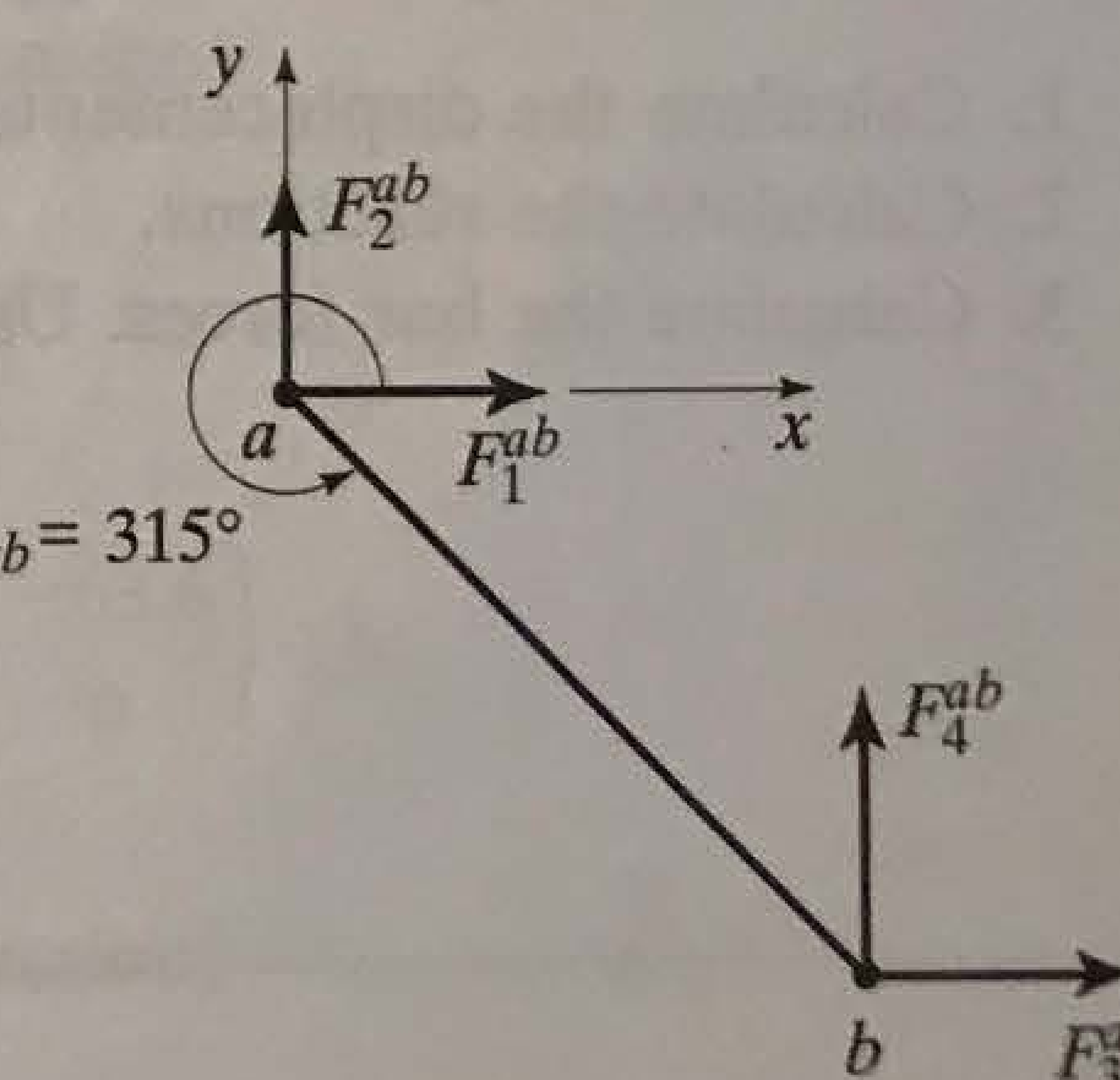
$$\frac{dF/\Delta A}{d\Delta/\Delta L} = \frac{dF\,\Delta L}{d\Delta\,\Delta A}$$

$$\left(\frac{EA}{L}\right)_{ab} = \frac{\overset{(\text{GPa}) \times \text{mm}^2}{200 \times 20 \times 10^3}}{4\sqrt{2} \times 10^3 \ \text{mm}} = 707.11 \text{ kN/mm}$$

$$\begin{Bmatrix} F_1^{ab} \\ F_2^{ab} \\ F_3^{ab} \\ F_4^{ab} \end{Bmatrix} = 707.11 \begin{bmatrix} 0.500 & -0.500 & -0.500 & 0.500 \\ & 0.500 & 0.500 & -0.500 \\ & & 0.500 & -0.500 \\ \text{Sym.} & & & 0.500 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix}$$

$\cos\phi = \cos(2\pi - \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

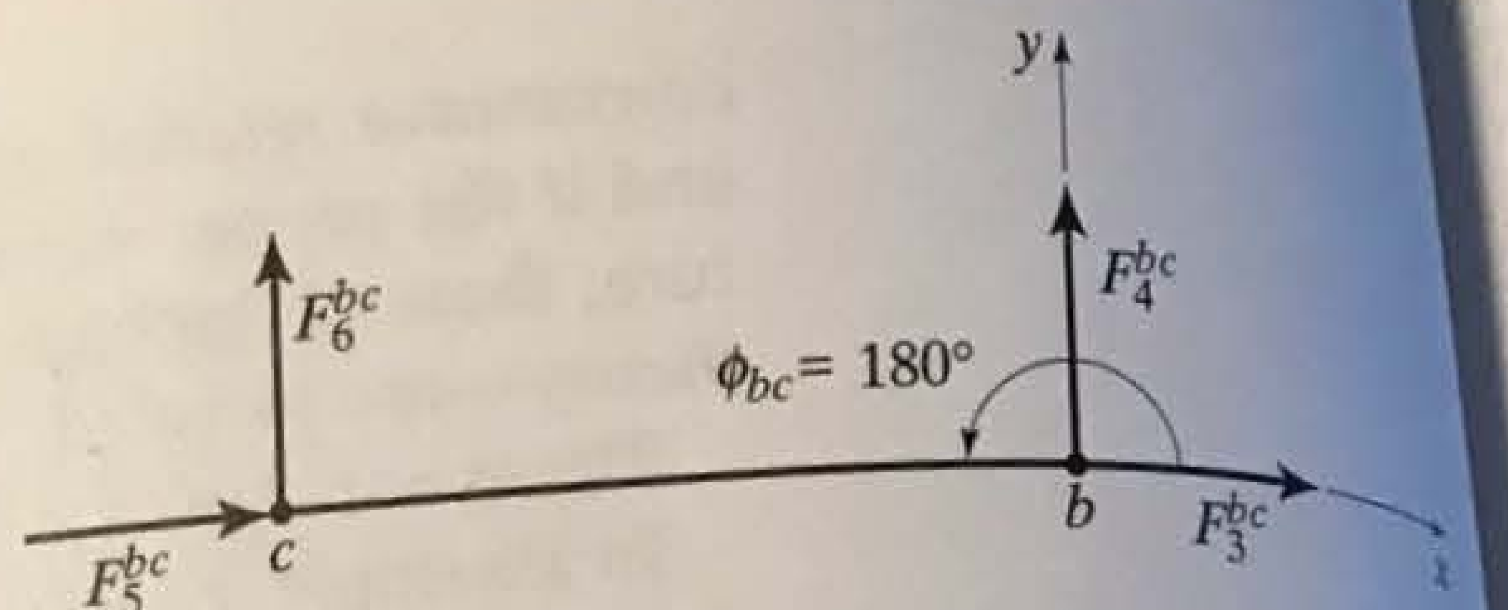
$\sin\phi = -\frac{\sqrt{2}}{2}$



Member bc

$$\left(\frac{EA}{L}\right)_{bc} = \frac{200 \times 18 \times 10^3}{10.928 \times 10^3} = 329.43 \text{ kN/mm}$$

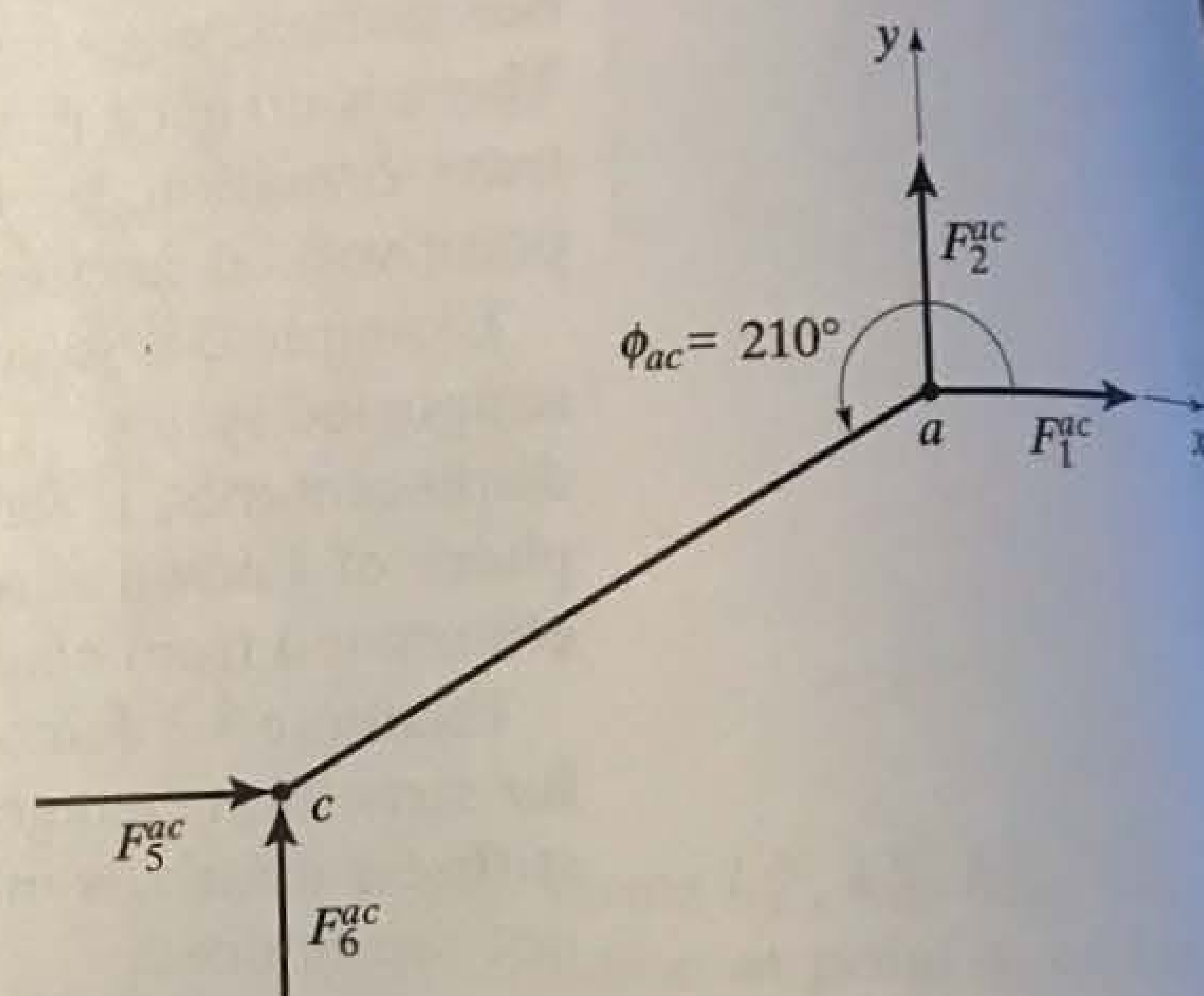
$$\begin{Bmatrix} F_3^{bc} \\ F_4^{bc} \\ F_5^{bc} \\ F_6^{bc} \end{Bmatrix} = 329.43 \begin{bmatrix} 1.000 & 0 & -1.000 & 0 \\ & 0 & 0 & 0 \\ & & 1.000 & 0 \\ \text{Sym.} & & & 0 \end{bmatrix} \begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$



Member ac

$$\left(\frac{EA}{L}\right)_{ac} = \frac{200 \times 15 \times 10^3}{8 \times 10^3} = 375.00 \text{ kN/mm}$$

$$\begin{Bmatrix} F_1^{ac} \\ F_2^{ac} \\ F_5^{ac} \\ F_6^{ac} \end{Bmatrix} = 375.00 \begin{bmatrix} 0.750 & 0.433 & -0.750 & -0.433 \\ & 0.250 & -0.433 & -0.250 \\ & & 0.750 & 0.433 \\ \text{Sym.} & & & 0.250 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$



2. Global stiffness equations in matrix form (see Equation 3.5):

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = 10^2 \begin{bmatrix} 6.348 & -1.912 & -3.536 & 3.536 & -2.812 & -1.624 \\ & 4.473 & 3.536 & -3.536 & -1.624 & -0.938 \\ & & 6.830 & -3.536 & -3.294 & 0 \\ & \text{Sym.} & & 3.536 & 0 & 0 \\ & & & & 6.107 & 1.624 \\ & & & & & 0.938 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$

3. Rigid body motion. Adding rows 1 and 3 of the global stiffness matrix yields the vector:

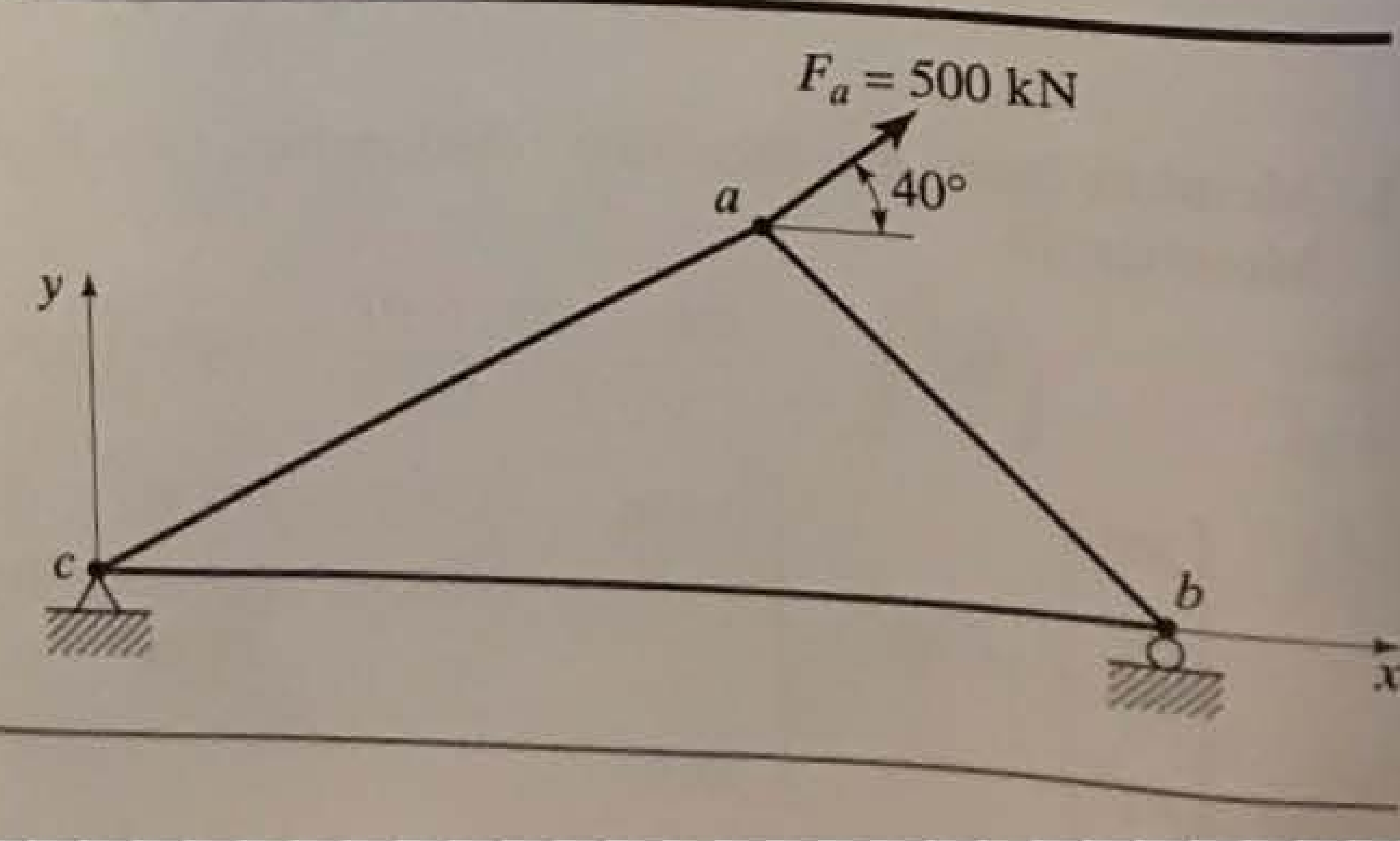
$$\lfloor 2.812 \quad 1.624 \quad 3.294 \quad 0 \quad -6.107 \quad -1.624 \rfloor$$

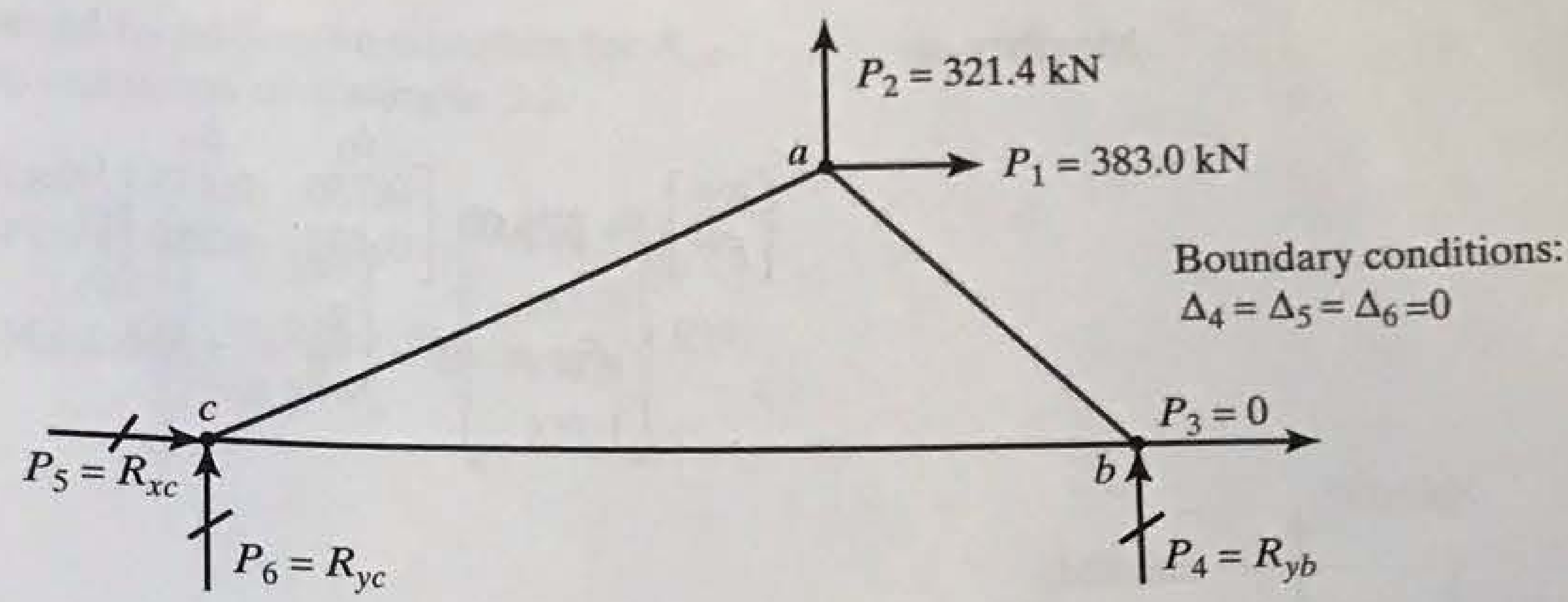
which is the negative of row 5. Therefore, there is linear dependence, the determinant is zero, and the matrix is singular. This is a signal that, under an arbitrary load, the displacements are indefinite; that is, there may be rigid body motion.

EXAMPLE 3.2

The truss of Example 3.1 is supported and loaded as shown.

1. Calculate the displacements at a and b.
2. Calculate the reactions.
3. Calculate the bar forces. Use equations of Example 3.1.





1. Displacements. The upper three global stiffness equations can be written as follows:

$$\begin{Bmatrix} 383.0 \\ 321.4 \\ 0 \end{Bmatrix} = 10^2 \begin{bmatrix} 6.348 & -1.912 & -3.536 \\ & 4.473 & 3.536 \\ \text{Sym.} & & 6.830 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} + 10^2 \begin{bmatrix} 3.536 & -2.812 & -1.624 \\ -3.536 & -1.624 & -0.938 \\ -3.536 & -3.294 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Inverting the first matrix and solving for the displacements yields

$$\lfloor \Delta_1 \quad \Delta_2 \quad \Delta_3 \rfloor = \lfloor 0.871 \quad 1.244 \quad -0.193 \rfloor \text{ mm}$$

2. Reactions. The lower three stiffness equations now yield the reactions:

$$\begin{Bmatrix} R_{yb} \\ R_{xc} \\ R_{yc} \end{Bmatrix} = 10^2 \begin{array}{c} \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} 3.536 & -3.536 & -3.536 \\ -2.812 & -1.624 & -3.294 \\ -1.624 & -0.938 & 0 \end{bmatrix} \end{array} \begin{Bmatrix} 0.871 \\ 1.244 \\ -0.193 \end{Bmatrix}$$

$$+ 10^2 \begin{array}{c} \begin{array}{ccc} \Delta_4 & \Delta_5 & \Delta_6 \end{array} \\ \begin{bmatrix} 3.536 & 0 & 0 \\ & 6.107 & 1.624 \\ \text{Sym.} & & 0.938 \end{bmatrix} \end{array} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -63.6 \\ -383.4 \\ -258.1 \end{Bmatrix} \text{ kN}$$

3. Bar forces. The bar forces may now be obtained from the member stiffness equations:

Member *ab*

$$\begin{Bmatrix} F_1^{ab} \\ F_2^{ab} \end{Bmatrix} = 707.11 \begin{array}{c} \begin{array}{ccc} \Delta_1 & \Delta_2 & \Delta_3 \end{array} \\ \begin{bmatrix} 0.500 & -0.500 & -0.500 \\ -0.500 & 0.500 & 0.500 \end{bmatrix} \end{array} \begin{Bmatrix} 0.871 \\ 1.244 \\ -0.193 \end{Bmatrix} = \begin{Bmatrix} -63.6 \\ 63.6 \end{Bmatrix} \text{ kN}$$

global → local

$$F_{ab} = F_2^{ab} \cdot \sqrt{2} = +90.0 \text{ kN (tension)}$$

Member *bc*

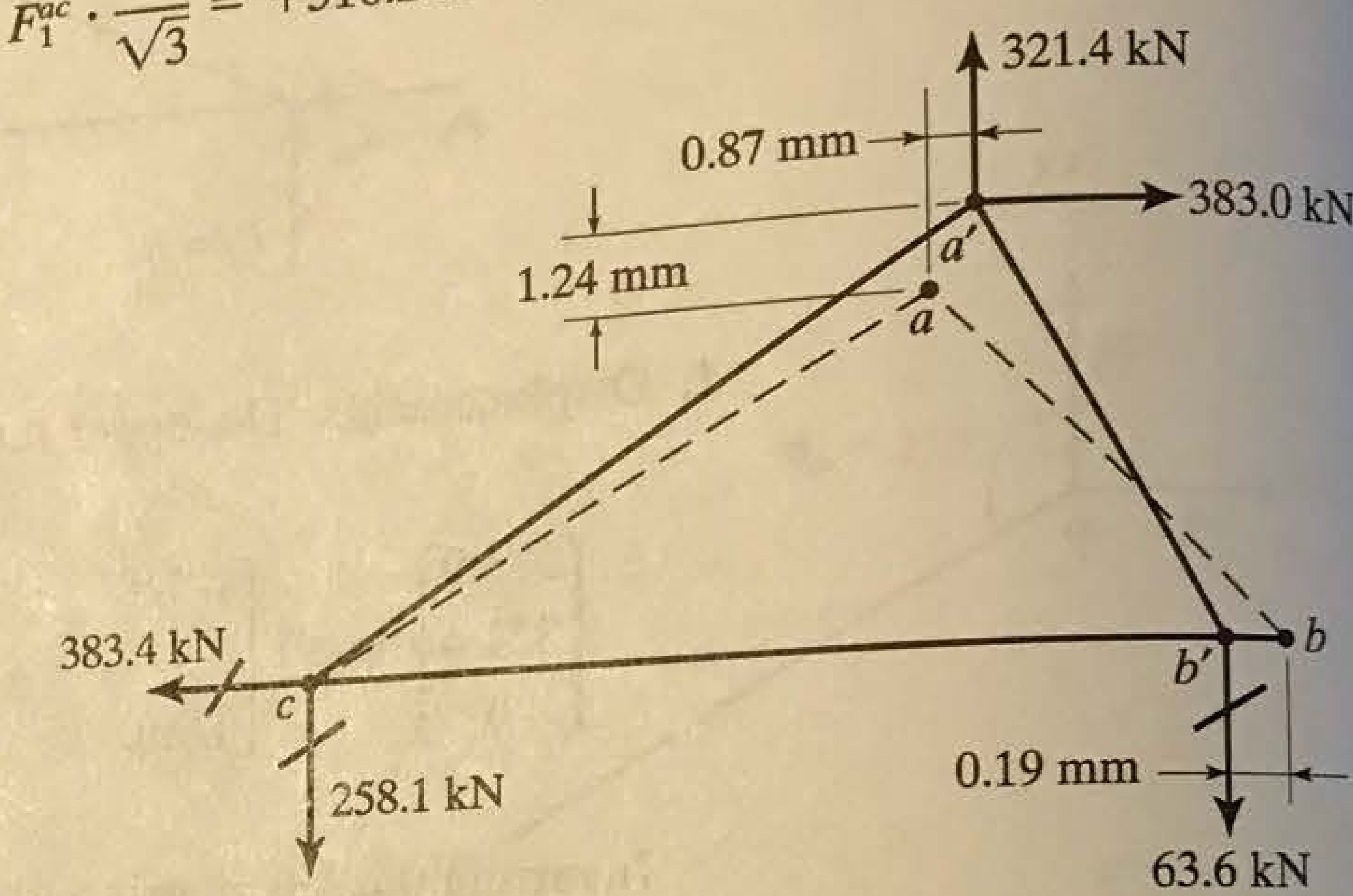
$$\begin{Bmatrix} F_3^{bc} \\ F_4^{bc} \end{Bmatrix} = 329.43 \begin{array}{c} \Delta_3 \\ \begin{bmatrix} 1.00 \\ 0 \end{bmatrix} \end{array} \{-0.193\} = \begin{Bmatrix} -63.6 \\ 0 \end{Bmatrix} \text{ kN}$$

$$F_{bc} = F_3^{bc} = -63.6 \text{ kN (compression)}$$

Member *ac*

$$\begin{Bmatrix} F_1^{ac} \\ F_2^{ac} \end{Bmatrix} = 375.00 \begin{array}{cc} \Delta_1 & \Delta_2 \\ \end{array} \begin{bmatrix} 0.750 & 0.433 \\ 0.433 & 0.250 \end{bmatrix} \begin{Bmatrix} 0.871 \\ 1.244 \end{Bmatrix} = \begin{Bmatrix} 447.0 \\ 258.0 \end{Bmatrix} \text{ kN}$$

$$F_{ac} = F_1^{ac} \cdot \frac{2}{\sqrt{3}} = +516.2 \text{ kN (tension)}$$

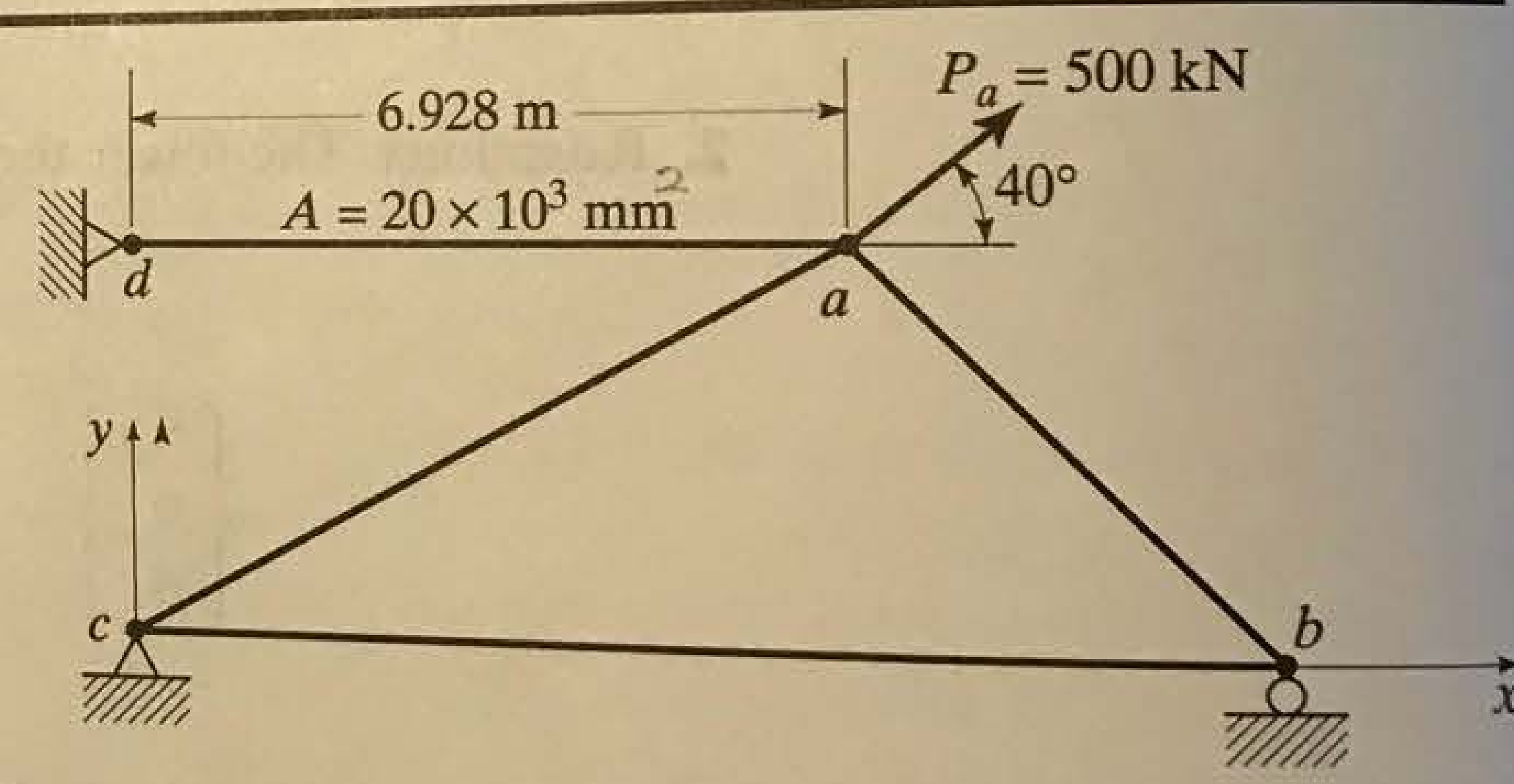


EXAMPLE 3.3

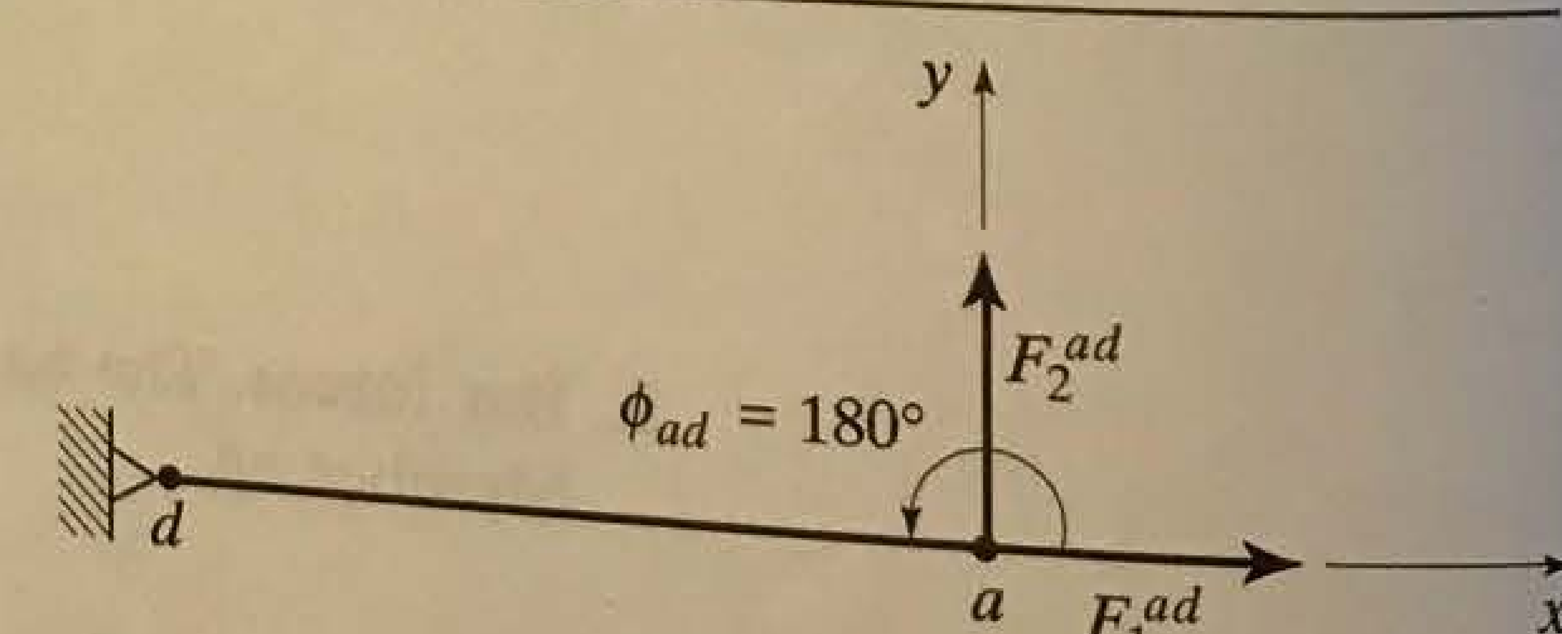
The truss shown is the same as in Example 3.2 except for the addition of the horizontal tie *ad*.

1. Calculate the displacements at *a* and *b*.
2. Calculate the reactions.

E = 200,000 MPa
= 200 GPA



Member *ad*. Member force-displacement relationships at node *a* (see Equation 2.5):



$$F_1^{ad} = \frac{\overset{\text{GPA}}{200} \times \overset{\text{mm}}{20 \times 10^3}}{6.928 \times 10^3_{\text{mm}}}(1.000)\ \Delta_1 = 577.37\Delta_1 \text{ kN}$$

$$F_2^{ad} = 0$$

1. Displacements. No nonzero degrees of freedom have been added. The stiffness coefficient of *ad* can be added to the stiffness matrix of Example 3.2, resulting in the following:

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = 10^{-2} \begin{bmatrix} \begin{pmatrix} 6.348 \\ +5.774 \end{pmatrix} & -1.912 & -3.536 \\ & 4.473 & 3.536 \\ \text{Sym.} & & 6.830 \end{bmatrix}^{-1} \begin{Bmatrix} 383.0 \\ 321.4 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.383 \\ 1.228 \\ -0.437 \end{Bmatrix} \text{ mm}$$