Appendix for the Paper "A Robust State Transition Function for Multi-agent Epistemic Systems"

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1 State Transition in the Example Scenarios

In this section, we give the necessary background information about the state transition for the example scenarios in the introduction of the main text. For Example 1, we follow the definitions in [1, 2] for event update model based semantics of state transition.

Let $D = \langle \mathcal{AG}, \mathcal{F} \rangle$ be a multi-agent domain, $\mathcal{L}$ be the set of fluent literals and $\mathcal{L}_{\mathcal{AG}}$ denote the set of belief formulae.

Definition 1 (Event Update Model). *An* event update model $\mathbf{\Sigma}$ *is a tuple* $\langle \Sigma, R_1, \ldots, R_n, \textit{pre}, \textit{eff} \rangle$ *where*

(i) Σ *is a set of* events*;*
(ii) each R_i *is a binary relation on* Σ*;*
(iii) $\textit{pre} : \Sigma \rightarrow \mathcal{L}_{\mathcal{AG}}$ *is a function mapping each event* $e \in \Sigma$ *to a formula in* $\mathcal{L}_{\mathcal{AG}}$*; and*
(iv) $\textit{eff} : \Sigma \rightarrow 2^{\mathcal{L}}$ *is a function mapping each event* $e \in \Sigma$ *to a set of fluent literals* $\varphi \subseteq \mathcal{L}$*.*

An event update instance ω *is a pair* $(\mathbf{\Sigma}, e)$ *where* $\mathbf{\Sigma}$ *is an update model as defined above, and* $e \in \Sigma$ *is the designated or the actual event.*

Intuitively, an update model represents different views of an action occurrence depending on the observability of agents. Each view is represented by an event in Σ. The designated event is the one that agents who are aware of the action occurrence will observe. The relation R_i describes agent i's uncertainty on action execution—i.e., if $(\sigma, \tau) \in R_i$ and event σ is performed, then agent i may believe that event τ is executed instead. *pre* defines the action precondition and *eff* specifies the effect of the action.

Definition 2 (Updates by an Update Model). *Let* M *be a Kripke structure and* $\mathbf{\Sigma} = \langle \Sigma, R_1, \ldots, R_n, \textit{pre}, \textit{eff} \rangle$ *be an event update model. The* update *induced by* $\mathbf{\Sigma}$ *defines a Kripke structure* $M' = M \otimes \mathbf{\Sigma}$*, where:*

(i) $M'[S] = \{(u, \tau) \mid u \in M[S], \tau \in \Sigma, (M, u) \models \textit{pre}(\tau)\}$;
(ii) $((u, \tau), (v, \beta)) \in M'[i]$ *iff* $(u, \tau), (v, \beta) \in M'[S]$, $(u, v) \in M[i]$ *and* $(\tau, \beta) \in R_i$;
(iii) *For all* $(u, \tau) \in M'[S]$ *and* $\ell \in \mathcal{L}$, $M'[\pi]((u, \tau)) \models \ell$ *iff* $(M[\pi]((u)) \models \ell) \vee (\ell \in \textit{eff}(\tau))$.

The new Kripke structure M' is obtained from the component-wise cross-product of the old structure M and the event update model $\mathbf{\Sigma}$ for those worlds which satisfy precondition of the event. Valuation function of the worlds in the new structure is obtained by applying the effect of the event to the corresponding world in the original structure.

Definition 3 (State Update). *Let* (M, s) *be the initial state and* $(\mathbf{\Sigma}, e)$ *be an event update instance. The next state is the Kripke structure* (M', s') *where* $M' = M \otimes \mathbf{\Sigma}$ *and the new designated world is* $s' = (s, e)$*.*

Example 1. The initial Kripke structure in the first example, as depicted in Figure 1(a) is $(M,s)=\langle S,\pi,\mathcal{B}_A,\mathcal{B}_B\rangle$ where $S=\{s,u\}$, $\pi(s)=\{\neg on,\ sound,\ looking\_a,\ looking\_b\}$, $\pi(u)=\{\neg on,\ \neg sound,\ looking\_a,\ looking\_b\}$, $\mathcal{B}_A=\{(s,u),\ (u,u)\}$, $\mathcal{B}_B=\{(s,s),\ (u,u)\}$. The true world is s.

According to [2, 4], the event update instance of the *start_machine* action is computed based on observability of agents at the true world and shown in Figure 1(a). Formally the event update instance is $(\mathbf{\Sigma},\sigma)$ where $\mathbf{\Sigma}=\langle\Sigma,R_A,R_B,pre,eff\rangle$ where $\Sigma=\{\sigma\}$, $R_A=R_B=\{(\sigma,\sigma)\}$, $pre(\sigma)=\neg on\wedge sound$, $eff(\sigma)=\{on\}$. The true event is σ.

The next state is (M',s') where $M'=M\otimes\mathbf{\Sigma}$ and $s'=(s,\sigma)$. Note that $(M,s)\vDash pre(\sigma)$, $(M,u)\nvDash pre(\sigma)$, so the set of worlds at the next state are $M'[S]=\{(s,\sigma)\}$. The accessibility relations at the next state are $M'[A]=\emptyset$ and $M'[B]=\{((s,\sigma),(s,\sigma))\}$. Agent A does not have any accessibility relation at the true world (s,σ) and believes in every belief formula. The initial state, the event update instance and the next state is shown in Figure 1(a) in the main text. Since A considers only σ event and $(M,u)\nvDash pre(\sigma)$, the accessibility relation of agent A from world s to u is removed. Therefore, at the next state agent A does not have any accessibility relation at the true world (s,σ) and believes in every belief formula.

1.1 State Transition Function of [3]

[3] has constructed a model of state transition with two different operators $\mathbf{B}_i$, $\mathbf{K}_i$ for belief and knowledge, respectively. They assume that $\mathbf{B}_i$ satisfies $KD45_n$ and $\mathbf{K}_i$ satisfies $S5_n$ properties. In addition they assume both $\mathbf{B}_i$, $\mathbf{K}_i$ satisfy *KB1* and *KB2* properties. In their setting, a Kripke structure is defined as a tuple $\langle S,\pi,\mathcal{B}_1,\ldots,\mathcal{B}_n,\mathcal{K}_1,\ldots,\mathcal{K}_n\rangle$ and contains two different accessibility relations for each agent corresponding to the belief and knowledge operators. A pointed state is (M,s) where M is a Kripke structure as defined above and s is the designated (actual) world. Different from our setup, ontic actions are unconditional in [3]. Let ψ be the precondition and φ be the effect of an ontic action a. For a world $u\in M[S]$ which satisfy the precondition of a (i.e. $(M,u)\vDash\psi$), the new world $u^+\in M'[S]$ in the next state created by applying the effect of the action has the valuation $M'[\pi](u^+)=(M[\pi](u)\setminus(\overline{\varphi}))\cup\varphi$.

[3] defines full observer, partial observer and oblivious agents for ontic, sensing, announcement actions. For ontic actions partial observability is the same as full observability. Observability of the agents are computed at the designated world s and are fixed across all worlds. Hence the set F, O of full observer and oblivious agents for an ontic action a are

$$F=\{i\in\mathcal{AG}\mid(M,s)\vDash\delta_{i,\mathsf{a}}\},$$
$$O=\{i\in\mathcal{AG}\mid(M,s)\nvDash\delta_{i,\mathsf{a}}\}.$$

In the setting of [3], full observer agents correct their beliefs about precondition of an action, but not about observability. To construct the next state, they define the temporary relations $\mathcal{R}_i$ representing agents beliefs after they have possibly been reset (updated with information from $\mathbf{K}_i$) due to observing an action whose precondition contradicts the agents belief. For an agent $i\in\mathcal{AG}$,

$$\mathcal{R}_i=\{(u,v)\mid(u,v)\in\mathcal{B}_i\ \vee\ ((u,v)\in\mathcal{K}_i\wedge(M,u)\vDash\mathbf{B}_i\neg\psi)\}$$

The accessibility relations in the next state (M',s') are computed as follows. Full observers correct their beliefs about action precondition and observe the effect of the action. Namely, the belief relation of a full observer agent i in M' is

$$\mathcal{B}'_i=\mathcal{B}_i\ \cup\ \{(u^+,v^+)\mid(M,u)\vDash\psi,\ (M,v)\vDash\psi,\ (u,v)\in\mathcal{R}_i\}$$

The accessibility relation for the knowledge operator of a full observer agent i at the next state becomes

$$\mathcal{K}'_i=\mathcal{K}_i\ \cup\ \{(u^+,v^+)\mid(M,u)\vDash\psi,\ (M,v)\vDash\psi,\ (u,v)\in\mathcal{K}_i\}$$

The beliefs of oblivious agents do not change, they continue to consider the old state. However an oblivious agent updates his knowledge relation such that he considers possible that the action might have happened (recall that knowledge relation satisfies $S5_n$ property).

$$\mathcal{B}'_i = \mathcal{B}_i \cup \{(u^+, v) \mid (M,u) \models \psi,\ (u,v) \in \mathcal{B}_i\}$$
$$\mathcal{K}'_i = \mathcal{K}_i \cup \{(u^+, v^+) \mid (M,u) \models \psi,\ (M,v) \models \psi,\ (u,v) \in \mathcal{K}_i\} \cup \{(u^+, v),\ (v, u^+) \mid (M,u) \models \psi,\ (u,v) \in \mathcal{K}_i\}$$

Example 2. As shown in Figure 1(b) in the main text, the initial Kripke structure in the second example is $(M,s) = \langle S, \pi, \mathcal{B}_A, \mathcal{B}_B, \mathcal{K}_A, \mathcal{K}_B \rangle$ where

$S = \{s, u, v\}$,
$\pi(s) = \{\neg open,\ looking\_a,\ looking\_b,\ looking\_c,\ haskey\_a\}$,
$\pi(u) = \{\neg open,\ looking\_a,\ \neg looking\_b,\ looking\_c,\ haskey\_a\}$,
$\pi(v) = \{\neg open,\ \neg looking\_a,\ \neg looking\_b,\ \neg looking\_c,\ haskey\_a\}$,
$\mathcal{B}_A = \{(s,s)\}$, $\mathcal{B}_B = \{(s,u)\}$, $\mathcal{B}_C = \{(s,s),\ (u,v)\}$,
$\mathcal{K}_A = \{(s,s)\}$, $\mathcal{K}_B = \{(s,s),\ (s,u),\ (u,u),\ (u,s)\}$,
$\mathcal{K}_C = \{(s,s),\ (u,u),\ (u,v),\ (v,v),\ (v,u)\}$.

The true world is s. Observability of agents are computed at s and thus all agents are full observers. Namely, $F = \{A, B, C\}$ and $O = \emptyset$.

The precondition of the action is $\psi = haskey\_a$ and its effect is $\varphi = \{open\}$. All worlds satisfy the precondition of the *open_box* action i.e. $(M,s) \models \psi$, $(M,u) \models \psi$, $(M,v) \models \psi$. To compute the next state according to state transition function of [3], we first compute the temporary relation $\mathcal{R}_i$. In this example, the agents do not have false belief about the action precondition at any world, thus the relation $\mathcal{R}_i$ is identical to $\mathcal{B}_i$. Then the accessibility relations in the next state (M', s') are
$\mathcal{B}'_A = \{(s^+, s^+)\}$, $\mathcal{B}'_B = \{(s^+, u^+)\}$, $\mathcal{B}'_C = \{(s^+, s^+),\ (u^+, v^+)\}$,
$\mathcal{K}'_A = \{(s^+, s^+)\}$, $\mathcal{K}'_B = \{(s^+, s^+),\ (s^+, u^+),\ (u^+, u^+),\ (u^+, s^+)\}$,
$\mathcal{K}'_C = \{(s^+, s^+),\ (u^+, u^+),\ (u^+, v^+),\ (v^+, v^+),\ (v^+, u^+)\}$.

The next state is shown in Figure 1(b) bottom. The true world is $s' = s^+$. The state transition function of [3] handles correcting agents' beliefs about precondition, but not about observability. Actually, in this example correcting beliefs about precondition is not relevant as all agents have correct belief about precondition. However, correction for observability is necessary because the next state computed by [3] is counter-intuitive. B believes that the box is open but B believes that he is oblivious i.e. B believes that he is not looking at the box. Ideally, since B has observed the effect of the action, B must have realized that he is full observer of the action. There is also anomaly about higher order beliefs about agents' observability: A believes that B is looking at the box and A believes that B believes that the box is open but A believes that B believes that B is not looking. Furthermore, B believes that C is looking and B believes that C believes that the box is open however B believes that C believes that C is not looking. This outcome is not realistic.

2 Proof of the Theorems

As stated in the original text, the theorems assume that the action a is executable at the true state i.e. $(M,s) \models \psi$, "a **causes** φ **if** μ" belongs to domain D, $i \in \mathcal{AG}$, $\ell \in \varphi$, and η is a belief formula.

Theorem 1. *If $(M,s) \models \delta_{i,\mathsf{a}}$ and $(M,s) \models \mathbf{B}_i\, \mu$ then $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_i\, \ell$.*

Proof: Assume that $(M,s) \models \delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_i\, \mu$ hold. We examine the worlds $u \in M[S]$ such that $(s,u) \in M[i]$. Since $(M,s) \models \mathbf{B}_i\, \mu$, we know $(M,u) \models \mu$ for every u. There are two cases to consider:

(i) $(M,s) \models \mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then $(s', u_i) \in M'[i]$. Since $(M,u) \models \mu$ and $M'[\pi](u_i) = \phi(\mathsf{a}, \lambda_i(u))$, we obtain $(M', u_i) \models \ell$, for $\ell \in \varphi$. Therefore, $(M', s') \models \mathbf{B}_i\ell$, for $\ell \in \varphi$.
(ii) $(M,s) \models \neg\mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then by construction, $(s', u') \in M'[i]$ for those u such that $(M,u) \models \psi \wedge \delta_{i,\mathsf{a}}$. Since $u' = \phi(\mathsf{a}, u)$ and $(M,u) \models \mu$, we obtain $(M', u') \models \ell$, for $\ell \in \varphi$. Therefore, $(M', s') \models \mathbf{B}_i\ell$ holds, $\ell \in \varphi$. In both cases we have shown $(M', s') \models \mathbf{B}_i\ell$, for $\ell \in \varphi$ hence the result is established.

Theorem 2. *If* $(M,s) \models \neg\delta_{i,\mathsf{a}}$ *then* $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_i\eta$ *iff* $(M,s) \models \mathbf{B}_i\eta$.

Proof: Assume that $(M,s) \models \neg\delta_{i,\mathsf{a}}$ and $(M,s) \models \mathbf{B}_i\eta$. Then $(M,u) \models \eta$ for $u \in M[S]$ such that $(s,u) \in M[i]$. Since $(M,s) \models \neg\delta_{i,\mathsf{a}}$, by construction, $(s', u) \in M'[i]$ if and only if $(s,u) \in M[i]$. Note that all worlds and accessibility relations in M are maintained in M'. There is no accessibility relation from a world in $M[S]$ to a world in $M'[S] \setminus M[S]$. Therefore $(M', u) \models \eta$ if and only if $(M,u) \models \eta$ for a world $u \in M[S]$. Consequently, we obtain $(M', s') \models \mathbf{B}_i\,\eta$ if and only if $(M,s) \models \mathbf{B}_i\,\eta$.

Theorem 3. *Suppose that* $(M,s) \models \delta_{i_1,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\delta_{i_2,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\mathbf{B}_{i_2}\,\delta_{i_3,\mathsf{a}}, \ldots, (M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_{k-1}}\,\delta_{i_k,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\delta_{i_{k+1},\mathsf{a}}$ *hold for some* $k \geq 0$.
If $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\mu$ *then* $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\ell$.

Proof: Suppose that $(M,s) \models \delta_{i_1,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\delta_{i_2,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\mathbf{B}_{i_2}\,\delta_{i_3,\mathsf{a}}$, ..., $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\delta_{i_{k+1},\mathsf{a}}$ and $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\mu$ hold. This implies that if $(s,t) \in M[i_1]$, $(t,v) \in M[i_2]$, ..., $(w,y) \in M[i_k]$, $(y,z) \in M[i_{k+1}]$, then $(M,t) \models \delta_{i_2,\mathsf{a}}$, $(M,v) \models \delta_{i_3,\mathsf{a}}$, ..., $(M,y) \models \delta_{i_{k+1},\mathsf{a}}$, $(M,z) \models \mu$ hold.

There are two possible cases for the transition function: (i) $(M,s) \models \mathbf{B}_{i_1}(\neg\psi \vee \neg\delta_{i_1,\mathsf{a}})$. Then $(s', t_{i_1}) \in M'[i_1]$. (ii) $(M,s) \models \neg\mathbf{B}_{i_1}(\neg\psi \vee \neg\delta_{i_1,\mathsf{a}})$. Then $(s', t') \in M'[i_1]$ if $(M,t) \models \psi \wedge \delta_{i_1,\mathsf{a}}$. Thus if $(s', \hat{t}) \in M'[i_1]$, then $\hat{t} \in \{t', t_{i_1}\}$. Since $(M,t) \models \delta_{i_2,\mathsf{a}}$, in the next state we have $(\hat{t}, \hat{v}) \in M'[i_2]$ where $\hat{v} \in \{v', v_{i_2}\}$. Using a similar argument, we obtain $(\hat{v}, \hat{u}) \in M'[i_3]$, ..., $(\hat{w}, \hat{y}) \in M'[i_k]$, $(\hat{y}, \hat{z}) \in M'[i_{k+1}]$ where $\hat{u} \in \{u', u_{i_3}\}$, ..., $\hat{y} \in \{y', y_{i_k}\}$, $\hat{z} \in \{z', z_{i_{k+1}}\}$.

Now, we consider agent i_{k+1}. By assumption $(M,z) \models \mu$. In the first case suppose $\hat{z} = z_{i_{k+1}}$. Since $\lambda_{i_{k+1}}(z) \models \mu$ and $M'[\pi](z_{i_{k+1}}) = \phi(\mathsf{a}, \lambda_{i_{k+1}}(z))$ we have $(M', z_{i_{k+1}}) \models \ell$. In the second case suppose $\hat{z} = z'$. Since $(M,z) \models \mu$, by definition $(M', z') \models \ell$ for $\ell \in \varphi$. Consequently, $(M', \hat{z}) \models \ell$ for $\ell \in \varphi$. Thus in the next state, $(s', \hat{t}) \in M'[i_1]$, $(\hat{t}, \hat{v}) \in M'[i_2]$, ..., $(\hat{w}, \hat{y}) \in M'[i_k]$, $(\hat{y}, \hat{z}) \in M'[i_{k+1}]$ where $(M', \hat{z}) \models \ell$, for $\ell \in \varphi$. Consequently, $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\ell$ holds, for $\ell \in \varphi$.

Theorem 4. *Suppose that* $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\neg\delta_{i_{k+1},\mathsf{a}}$ *holds where* $i_k \neq i_{k+1}$ *and* $k \geq 0$. *If* $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$ *then* $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$.

Proof: First consider a special case where $(M,s) \models \delta_{i_1,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\delta_{i_2,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\,\mathbf{B}_{i_2}\,\delta_{i_3,\mathsf{a}}$, ..., $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_{k-1}}\,\delta_{i_k,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\neg\delta_{i_{k+1},\mathsf{a}}$. Assume that $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$ holds for a belief formula η. Suppose that $(s,t) \in M[i_1]$, $(t,v) \in M[i_2]$, ..., $(w,y) \in M[i_k]$, $(y,z) \in M[i_{k+1}]$. By assumption, $(M,y) \models \neg\delta_{i_{k+1},\mathsf{a}}$ and $(M,z) \models \eta$. In the proof of part (3), we have established that $(M', s') \models \mathbf{B}_{i_1}\delta_{i_2}$, ..., $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\delta_{i_{k+1},\mathsf{a}}$. This implies that if $(s', \hat{t}) \in M'[i_1]$, $(\hat{t}, \hat{v}) \in M'[i_2]$, ..., $(\hat{w}, \hat{y}) \in M'[i_k]$, then $(M', \hat{y}) \models \delta_{i_k,a}$. Note that $\hat{t} \in \{t', t_{i_1}\}$, ... , $\hat{y} \in \{y', y_{i_k}\}$. Since agent i_{k+1} is oblivious at y; we have $(\hat{y}, z) \in M'[i_{k+1}]$. Recall that all worlds and accessibility relations in M are maintained in M'. In the next state M', there is no accessibility relation from a world in $M[S]$ to a world in $M'[S] \setminus M[S]$. Therefore $(M', z) \models \eta$ if and only if $(M,z) \models \eta$ for a world $z \in M[S]$. Thus, we obtain $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$.

Now consider a general case where $(M,s) \models \delta_{i_1}$, $(M,s) \models \mathbf{B}_{i_1}\,\delta_{i_2}$, $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_{m-1}}\,\delta_{i_m}$, $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_m}\,\neg\delta_{i_{m+1}}$, ..., $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\neg\delta_{i_{k+1}}$ where $0 \leq m \leq k$. Let $\hat{\eta}$ be a belief formula in the form $\mathbf{B}_{i_{m+1}} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$. We have assumed that $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$ therefore $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_m}\,\hat{\eta}$ holds. By the above analysis, since $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_m}\,\neg\delta_{i_{m+1}}$, we obtain $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_m}\,\hat{\eta}$ which is equivalent to stating $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_m}\,\mathbf{B}_{i_{m+1}} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$. Hence in the general case, $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\,\mathbf{B}_{i_{k+1}}\,\eta$ holds.

Theorem 5. *Suppose that* $(M,s) \models \delta_{i,\mathsf{a}}$ *and* $(M,s) \models \mathbf{C}_G\,\delta_{i,\mathsf{a}}$ *for all* $i \in G$ *where* $G \subseteq \mathcal{AG}$. *If* $(M,s) \models \mathbf{C}_G\,\mathbf{B}_j\,\mu$ *for* $j \in G$ *then* $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{C}_G\,\mathbf{B}_j\,\ell$.

Proof: Suppose that $(M,s) \models \delta_{i,\mathsf{a}}$ and $(M,s) \models C_G\, \delta_{i,\mathsf{a}}$ for all $i \in G$. Suppose also that $(M,s) \models \mathbf{C}_G\, \mathbf{B}_j\, \mu$ holds for an agent $j \in G$. Then we have $(M,s) \models \delta_{i_1,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\, \delta_{i_2,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1}\, \mathbf{B}_{i_2}\, \delta_{i_3,\mathsf{a}}$,..., $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_{k-1}}\, \delta_{i_k,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\, \delta_{j,\mathsf{a}}$ for an arbitrary $i_1, \ldots, i_k, j \in G$, $k \geq 0$. Since $(M,s) \models \mathbf{C}_G\, \mathbf{B}_j\, \mu$, we know $(M,s) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\, \mathbf{B}_j\, \mu$ holds. From part (3), we obtain $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{i_k}\, \mathbf{B}_j\, \ell$ holds, for $\ell \in \varphi$. Since $i_1, \ldots, i_k$ are arbitrary agents in G, we reach $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{C}_G\, \mathbf{B}_j\, \ell$ holds, for $\ell \in \varphi$.

Theorem 6. *Suppose that $(M,s) \models \mathbf{C}_G\, \neg\delta_{i,\mathsf{a}}$ holds where $i \in \mathcal{AG}$, $G \subseteq \mathcal{AG}$.
If $(M,s) \models \mathbf{C}_G\, \mathbf{B}_i\, \eta$ then $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{C}_G\, \mathbf{B}_i\, \eta$.*

Proof: Suppose that $(M,s) \models \mathbf{C}_G\, \neg\delta_{i,\mathsf{a}}$ and $(M,s) \models \mathbf{C}_G\, \mathbf{B}_i\, \eta$ hold where $i \in \mathcal{AG}$, $G \subseteq \mathcal{AG}$. Then we have $(M,s) \models \mathbf{B}_{j_1}\, \neg\delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{j_1}\, \mathbf{B}_{j_2}\, \neg\delta_{i,\mathsf{a}}$,..., $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_{k-1}}\, \neg\delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_k}\, \neg\delta_{i,\mathsf{a}}$ for an arbitrary $j_1, \ldots, j_k \in G$, $k \geq 0$. Moreover since $(M,s) \models \mathbf{C}_G\, \mathbf{B}_i\, \eta$, we know that $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_k}\, B_i\, \eta$ holds. First assume that $j_k \neq i$. Then from part (4), we obtain $(M', s') \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_k}\, \mathbf{B}_i\, \eta$. Now consider the case $j_k = i$. In this case, there can be a sequence of agents $j_{m+1} = \ldots = j_k = i$ at the end, where $0 \leq m \leq k$ and $j_m \neq i$. Namely, $(M,s) \models \mathbf{B}_{j_1}\, \neg\delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{j_1}\, \mathbf{B}_{j_2}\, \neg\delta_{i,\mathsf{a}}$,..., $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_m}\, \neg\delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_{m+1}}\, \neg\delta_{i,\mathsf{a}}$, ..., $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_{k-1}}\, \neg\delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_k}\, \neg\delta_{i,\mathsf{a}}$ where $j_{m+1} = \ldots = j_k = i$. Let $\hat{\eta}$ be a belief formula in the form $\mathbf{B}_{j_{m+1}} \ldots \mathbf{B}_{j_k}\, \mathbf{B}_i\, \eta$. Recall that $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_k}\, \mathbf{B}_i\, \eta$ by assumption. This is equivalent to $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_m}\, \mathbf{B}_i\, \hat{\eta}$ as $j_{m+1} = i$. Since $(M,s) \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_m}\, \neg\delta_{i,\mathsf{a}}$ and $j_m \neq i$; from part (4), we get $(M', s') \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_m}\, \mathbf{B}_i\, \hat{\eta}$. Since $j_{m+1} = i$, this is equivalent to stating $(M', s') \models \mathbf{B}_{j_1} \ldots \mathbf{B}_{j_m}\, \mathbf{B}_{j_{m+1}}\, \mathbf{B}_{j_{m+2}} \ldots \mathbf{B}_{j_k}\, \mathbf{B}_i\, \eta$.
In both cases we have reached $(M', s') \models \mathbf{B}_{i_1} \ldots \mathbf{B}_{j_k}\, \mathbf{B}_i\, \eta$ for an arbitrary $j_1, \ldots, j_k \in G$, $k \geq 0$. Therefore $(M', s') \models \mathbf{C}_G\, \mathbf{B}_i\, \eta$.

Theorem 7. *Suppose that the conditional effects of the action φ do not involve any fluent in $\delta_{i,\mathsf{a}}$ for any $i \in \mathcal{AG}$. For $i, j \in \mathcal{AG}$, if $(M,s) \models \delta_{i,\mathsf{a}}$ then $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_i \delta_{i,\mathsf{a}}$. If $(M,s) \models \delta_{i,\mathsf{a}}$ and $(M,s) \models \mathbf{B}_i \delta_{j,\mathsf{a}}$ then $\Phi_D(\mathsf{a}, (M,s)) \models \mathbf{B}_i \mathbf{B}_j \delta_{j,\mathsf{a}}$.*

Proof: Suppose that $(M,s) \models \delta_{i,\mathsf{a}}$ holds and $(s,u) \in M[i]$. In the first case, $(M,s) \models \mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then $(s', u_i) \in M'[i]$. Since $M'[\pi](u_i) = \phi(\mathsf{a}, \lambda_i(u))$ and φ is disjoint from $\delta_{i,\mathsf{a}}$, we have $(M', u_i) \models \delta_{i,\mathsf{a}}$. Consequently, $(M', s') \models \mathbf{B}_i \delta_{i,\mathsf{a}}$. In the second case, $(M,s) \not\models \mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then $(s', u') \in M'[i]$ if $(M,u) \models \psi \wedge \delta_{i,\mathsf{a}}$. Since φ is disjoint from $\delta_{i,\mathsf{a}}$, we have $(M', u') \models \delta_{i,\mathsf{a}}$ for all u' such that $(s', u') \in M'[i]$. Thus $(M', s') \models \mathbf{B}_i \delta_{i,\mathsf{a}}$. In both cases we obtain $(M', s') \models \mathbf{B}_i \delta_{i,\mathsf{a}}$ hence the first part of the theorem is established.

For the second part, suppose that $(M,s) \models \delta_{i,\mathsf{a}}$, $(M,s) \models \mathbf{B}_i \delta_{j,\mathsf{a}}$ holds and $(s,u) \in M[i]$, $(u,v) \in M[j]$. By assumption, $(M,u) \models \delta_{j,\mathsf{a}}$. In the first case, $(M,s) \models \mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then $(s', u_i) \in M'[i]$ where $(M', u_i) \models \delta_{i,\mathsf{a}}$. By assumption, $(M', u_i) \models \delta_{j,\mathsf{a}}$. In the first subcase, $(M, u_i) \models \mathbf{B}_j(\neg\psi \vee \neg\delta_{j,\mathsf{a}})$. Then $(u_i, v_j) \in M'[j]$ where $(M', v_j) \models \psi \wedge \delta_{j,\mathsf{a}}$. Hence, in this subcase $(M', u_i) \models \mathbf{B}_j \delta_{j,\mathsf{a}}$. In the second subcase, $(M, u_i) \not\models \mathbf{B}_j(\neg\psi \vee \neg\delta_{j,\mathsf{a}})$. Then $(u_i, v') \in M'[j]$ if $(M,v) \models \psi \wedge \delta_{j,\mathsf{a}}$. Hence, in this subcase $(M', u_i) \models \mathbf{B}_j \delta_{j,\mathsf{a}}$. In both subcases we have $(M', u_i) \models \mathbf{B}_j \delta_{j,\mathsf{a}}$ thus $(M', s') \models \mathbf{B}_i \mathbf{B}_j \delta_{j,\mathsf{a}}$ holds.
In the second case, $(M,s) \models \neg\mathbf{B}_i(\neg\psi \vee \neg\delta_{i,\mathsf{a}})$. Then $(s', u') \in M'[i]$ if $(M,u) \models \psi \wedge \delta_{i,\mathsf{a}}$. Note that by assumption $(M', u') \models \delta_{j,\mathsf{a}}$. In the first subcase, $(M, u') \models \mathbf{B}_j(\neg\psi \vee \neg\delta_{j,\mathsf{a}})$. Then $(u', v_j) \in M'[j]$ where $(M', v_j) \models \psi \wedge \delta_{j,\mathsf{a}}$. Hence, in this subcase $(M', u') \models \mathbf{B}_j \delta_{j,\mathsf{a}}$. In the second subcase, $(M, u') \not\models \mathbf{B}_j(\neg\psi \vee \neg\delta_{j,\mathsf{a}})$. Then $(u', v') \in M'[j]$ if $(M,v) \models \psi \wedge \delta_{j,\mathsf{a}}$. Hence, in this subcase $(M', u') \models \mathbf{B}_j \delta_{j,\mathsf{a}}$. In both subcases we have $(M', u') \models \mathbf{B}_j \delta_{j,\mathsf{a}}$ thus $(M', s') \models \mathbf{B}_i \mathbf{B}_j \delta_{j,\mathsf{a}}$ is established.

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