Introduction to KRR

Notions in Propositional Logic

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Objectives

Objective

Explain the concepts of satisfiability, tautology, equivalence, entailment in propositional logic

Objective

Explain how these concepts are related to each other

Satisfiability, Tautology, Equivalence, Entailment

Satisfiability

A propositional formula F is **satisfiable** if some interpretation satisfies F

Q: Which one is satisfiable? Choose all

1. $(p \rightarrow (q \rightarrow p))$
2. $(p \rightarrow (p \rightarrow q))$
3. $(p \rightarrow q) \wedge (p \wedge \neg q)$

p	q	3
f	f	f
f	t	f
t	f	f
t	t	f

A set of propositional formulas is **satisfiable** if some interpretation satisfies all formulas in the set.

{ p∨q, ¬p∨¬q }

p	q
t	f

Tautology

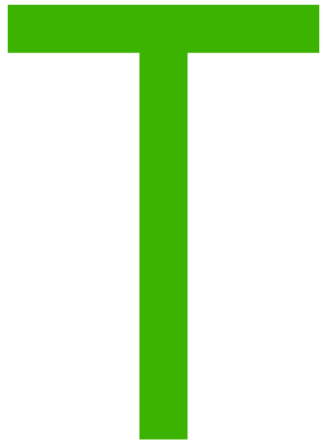
A propositional formula F is a **tautology** if every interpretation satisfies F

Q: Which one is a tautology? Choose all

1. $(p \rightarrow q) \rightarrow (\neg p \vee q)$

2. $(p \rightarrow (q \rightarrow p))$

3. $(p \rightarrow (p \rightarrow q))$



Equivalence

F is equivalent to G (symbolically, $F \Leftrightarrow G$) if, for every interpretation I, $F^I = G^I$

- In other words, $F \leftrightarrow G$ is a tautology

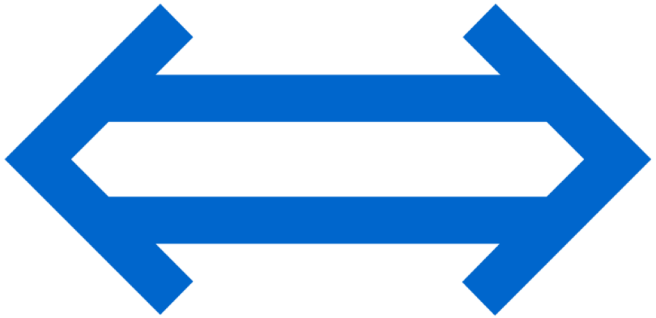
Q: Which formulas are equivalent to each other?

1. $(p \rightarrow (q \rightarrow p))$ and $(p \vee \neg p)$
2. $(p \rightarrow (p \rightarrow q)$ and p
3. $(p \rightarrow q)$ and $(q \rightarrow p)$
4. $(p \rightarrow q) \rightarrow (p \wedge \neg q)$ and $\bot$

p	q
t	f
t	t

More examples

- $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$
- $p \rightarrow q \Leftrightarrow \neg p \vee q$
- $\neg(p \rightarrow q) \Leftrightarrow (p \wedge \neg q)$
- $(p \wedge q) \rightarrow r \Leftrightarrow p \rightarrow (q \rightarrow r)$



Some Useful Equivalence

$F \rightarrow G \Leftrightarrow \neg F \vee G$

$F \leftrightarrow G \Leftrightarrow (F \rightarrow G) \wedge (G \rightarrow F)$

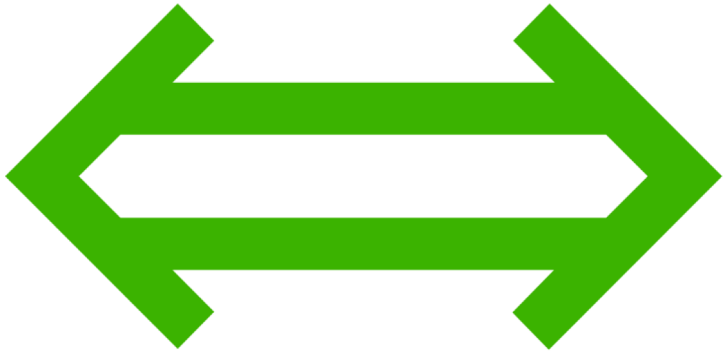
$\neg\neg F \Leftrightarrow F$

$\neg(F \wedge G) \Leftrightarrow \neg F \vee \neg G$

$\neg(F \vee G) \Leftrightarrow \neg F \wedge \neg G$

$(F \vee (G \wedge H)) \Leftrightarrow (F \vee G) \wedge (F \vee H)$

$(F \wedge (G \vee H)) \Leftrightarrow (F \wedge G) \vee (F \wedge H)$



Entailment

A set Γ of formulas entails a formula F (symbolically, $\Gamma \models F$) if, every interpretation that satisfies all formulas in Γ satisfies F also.

- c.f. Entailment uses the same symbol as satisfaction, the difference being what appears on the left of $\models$.

Q: True or false?

- $\{A, B\} \models A \wedge B$ ○
- $\{A, A \rightarrow B\} \models B$ ○
- $\{A\} \models A \vee B$ ○
- $\{A\} \models A \wedge B$ ×
- $\emptyset \models B$ ×
- $\{\bot\} \models B$ ○

A	B
T	F

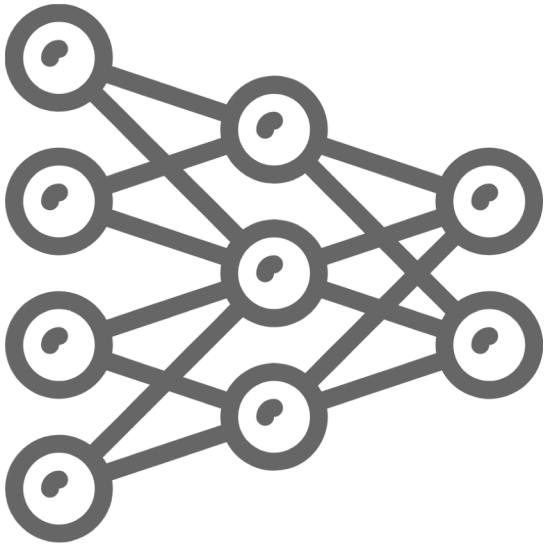
The formulas entailed by Γ are also called the logical consequences of Γ.

Algorithm for Entailment Checking

To check a set Γ of formulas entails a formula F

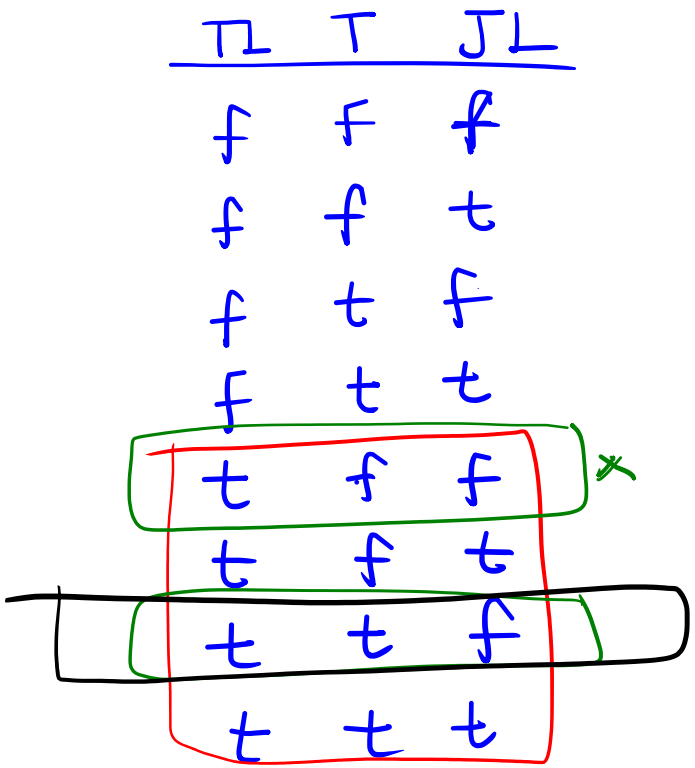
For each interpretation I,

- For each formula G in Γ, check if I satisfies G:
  - If no, continue to next interpretation
  - If yes: check if I satisfies F
    - If yes: continue to next interpretation
    - If no: exit and report “Not Entailed”
- (When all checking all interpretations are done)
  - Report “Entailed”



Entailment: Example

$$TL \wedge \neg T \rightarrow JL, \quad TL, \quad \neg JL \quad \models T$$



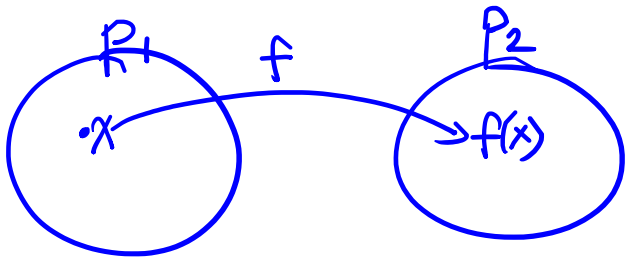
Reductions between Problems

Starting Points

Intuitively, these problems are strongly related

A reduction from problem P_1 to P_2 is a function f such that

- For each input x to P_1, the answer of P_1 for input x coincides with the answer of P_2 for input $f(x)$,
- Given x, the input $f(x)$ can be efficiently computed.



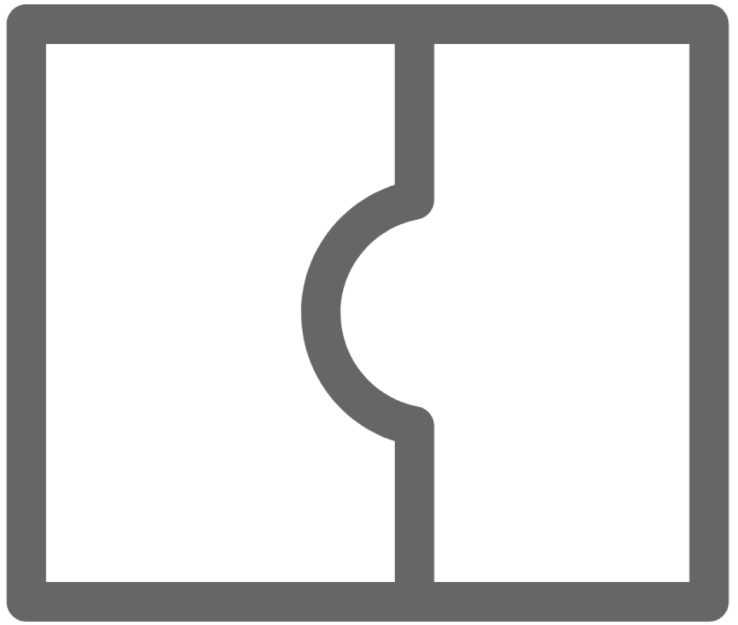
These (and many other) problems can be reduced to (un)satisfiability

Satisfiability solvers are a useful tool for KR

How are Tautology and Satisfiability related?

- F is a tautology iff $\neg F$ is unsatisfiable

- **Example:** $p \vee \neg p$ is a tautology iff $\neg(p \vee \neg p)$ is unsatisfiable



How are Equivalence and Tautology related?

- F is equivalent to G iff $F \leftrightarrow G$ is a tautology

- **Example:** $p \rightarrow q$ is equivalent to $\neg p \vee q$ iff $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is a tautology

How are Equivalence and Entailment related?

- F is equivalent to G iff
  - F entails G and
  - G entails F

- Example: $p \rightarrow q$ is equivalent to $\neg p \vee q$ iff
  - $p \rightarrow q$ entails $\neg p \vee q$ and
  - $\neg p \vee q$ entails $p \rightarrow q$

How are Entailment and Tautology related?

- **$\{F_1, \ldots, F_n\} \models G$ iff $(F_1 \wedge \cdots \wedge F_n) \rightarrow G$ is a tautology**
  - $\{p \vee q, \neg p \vee q\} \models q$ iff $(p \vee q) \wedge (\neg p \vee q) \rightarrow q$ is a tautology

- **$\emptyset \models G$ iff G is a tautology**

How are Entailment and Satisfiability related?

- $F \models G$ iff $F \wedge \neg G$ is unsatisfiable
  - $\{p \vee q, \neg p \vee q\} \models q$ iff $\{p \vee q, \neg p \vee q, \neg q\}$ is unsatisfiable

Wrap-Up