CSE 579: Knowledge Representation & Reasoning

Module 1: Propositional Logic

Ali Altunkaya
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Outline

1. Propositional Logic (PL)
    1. Introduction
    2. Syntax (alphabet)
    3. Semantics (meaning)
    4. Notions in PL
        1. Satisfiability
        2. Tautology
        3. Equivalence
        4. Entailment
    5. Computing Propositional Logic (PL): DPLL algorithm

Module 1 Highlights

We will study 3 different formal languages during the course:

1-) Propositional Logic

2-) First Order Logic

3-) Answer Set Theory

Ultimate goal: Encode <u>natural language</u> into a <u>formal language</u> so we can do ….(reasoning)

Module 1 Highlights

Introduction to Propositional Logic (PL):

– Goal: How we can use PL to represent (encode) knowledge so we can manipulate it to derive new information.

– PL is a branch of logic, which studies ways of combining or altering statements/propositions to form more complicated statements/propositions.

– Study of propositions (declarative sentences or statements) about the world which can be given a truth value (True/False).

– PL uses components : not, and, or, if then.

– PL is compositional. You can combine propositions (F and G).

Module 1 Highlights

Introduction to Propositional Logic (PL):

An **atom** represents a proposition, which is either true or false.

- The sum of 3 and 5 equals to 8. (an atom)
- ~~Ready, steady, go. (not an atom)~~

Propositional connectives are used to compose the meaning.

- If a number is divisible by 4, then it is divisible by 2.
- $\text{divisibleBy4} \rightarrow \text{divisibleBy2}$

Reasoning example: If the train arrives late and there are no taxis at the station then John is late for his meeting.

- $\text{TrainLate} \land \neg\text{Taxi} \rightarrow \text{JohnLate}$
- $p \land \neg q \rightarrow r$

Module 1 Highlights

What is syntax?

What is semantics?

What is the relation or difference between them?

Module 1 Highlights

What is syntax, semantics, and the relation or difference between them?

<u>Syntax:</u> Grammar of a statement in a language (nouns, verbs etc).

<u>Semantics:</u> Meaning of a statement in the real-world.

Interpretation from syntax to semantics.

<u>Example:</u> “Colorless green ideas sleep furiously.”

Syntax: correct

Semantics: incorrect

Module 1 Highlights

Syntax: Alphabet of Propositional Logic (PL):

A propositional signature is a set of symbols called atoms:

- TrainLate, TaxiLate, JohnLate, p, q, r

Three types of propositional connectives :

- 2-place (binary): $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication)
- 1-place (unary): $\neg$ (negation or not)
- 0-place: $\perp$ (bottom, FALSE), T (top, TRUE)

The alphabet of propositional logic consists of:

- The atoms from the signature
- The propositional connectives
- Parentheses

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Syntax: Alphabet of Propositional Logic (PL):

A propositional formula of signature σ is defined recursively:

- Every atom is a formula.
- Both 0-place connectives ($\bot$ T) are formulas.
- If F is a formula then is ¬F a formula too.
- For any binary connective ◌ (∧, ∨, →), if F and G are formulas then (F◌G) is a formula too.

$$((\neg p \wedge q) \rightarrow \left(p \wedge (q \vee \neg r)\right))$$

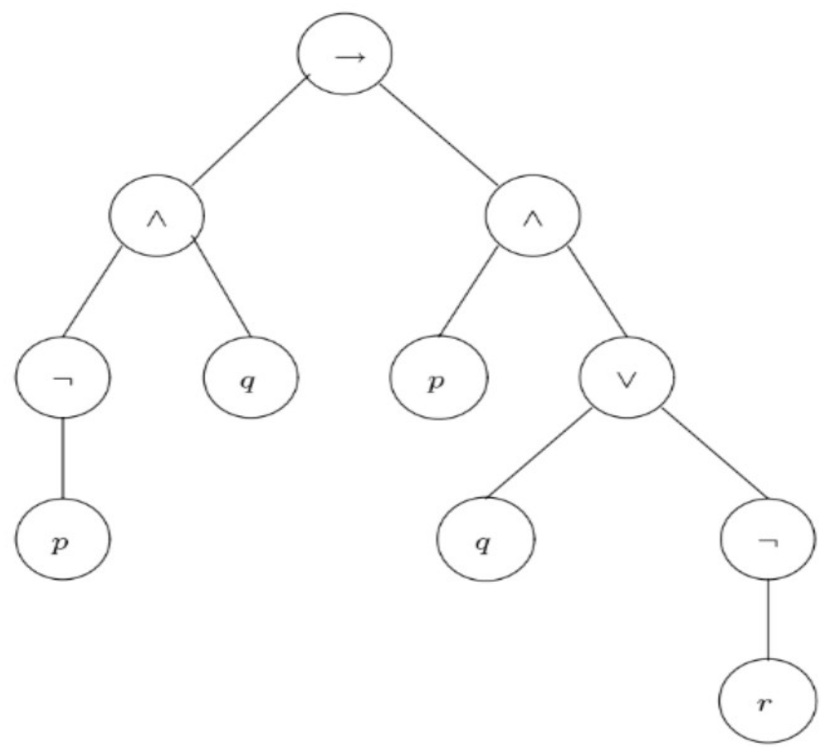
Module 1 Highlights

Syntax: Alphabet of Propositional Logic (PL):

Subformulas of are the formulas corresponding to the subtrees of the parse tree of:

$$((\neg p \wedge q) \rightarrow (p \wedge (q \vee \neg r)))$$

9 subformulas



Module 1 Highlights

Syntax: Alphabet of Propositional Logic (PL):

Binding precedence allows us to avoid many parentheses:

1) ¬
2) ∧,∨
3) →, ↔

$$((\neg p \wedge q) \rightarrow (p \wedge (q \vee \neg r)))$$

$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$

Module 1 Highlights

Semantics of Propositional Logic (PL):

– What is interpretation?

– How many interpretations are there for a given signature?

– What are truth values?

– What is truth table?

– Tables associated with propositional connectives

– Evaluation of a Formula

– Satisfaction

- see slides...

Module 1 Highlights

Semantics of Propositional Logic (PL):

Interpretation: is a function which maps signature symbols to truth values.

– How many interpretations are there for a given signature? 2^n

– What are truth values: True/False.

– Truth table

x	y	$\wedge(x,y)$	$\vee(x,y)$	$\rightarrow(x,y)$	$\leftrightarrow(x,y)$
f	f	f	f	t	t
f	t	f	t	t	f
t	f	f	t	f	f
t	t	t	t	t	t

– Tables associated with propositional connectives

– Evaluation of a Formula

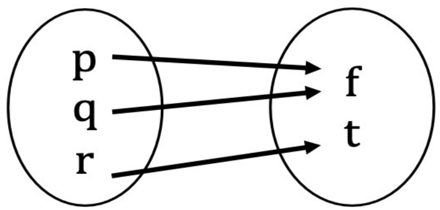
– <u>Interpretations</u> that satisfies a formula are called <u>models</u>.

Module 1 Highlights

Satisfiability: If a formula F is TRUE for some interpretation, then we say that the interpretation I satisfies F. Only one interpretation is sufficient. For a given formula F, is there any interpretation that satisfies it?

<u>NP-complete problem:</u> For n different atoms in the signature, there are 2^n possible interpretations. And we need to check all of them in worst-case scenario.

p	q	r
f	f	t



There are 10^{82} atoms in the universe.

$10^{82} =\sim 2^{300}$.

Formula: $((\neg p \wedge q) \rightarrow (p \wedge (q \vee \neg r))$

Q: The truth value of the formula

1. if I(p)=t, I(q)=t, I(r)=t?

t

2. if I(p)=f, I(q)=t, I(r)=f?

f

Q: How many interpretations for {p, q, r}?

p	q	r
f	f	f
f	f	t
f	t	f
f	t	t
t	f	f
t	f	t
t	t	f
t	t	t

Module 1 Highlights

Notions in Propositional Logic (PL):

– **Satisfiability**: only one interpretation is sufficient

– **Tautology**: all interpretations should satisfy it

– **Equivalence**: F is equivalent to G,

if for every interpretation I, $F^I = G^I$.

– **Entailment**: A set of formulas θ, entails a formula F,

if every interpretation that satisfies all formulas in θ, also satisfies F.

also called logical consequence.

– Some Useful Equivalence formulas (e.g. DeMorgan rules), used for reductions.

– **Reductions between problems**: all these 4 notions are strongly related with each other.

See slides....

Module 1 Highlights

Propositional Logic (PL) & Knowledge Representation (KR):

Syntax:

In Natural Languages, there are words and sentences (or statements).

In Propositional Logic, there are atoms and formulas.

Semantics:

In Natural Languages, meaning can be any object in Real-World.

In Propositional Logic, meaning can be only TRUE or FALSE.

Module 1 Highlights

Computing Propositional Logic (PL):

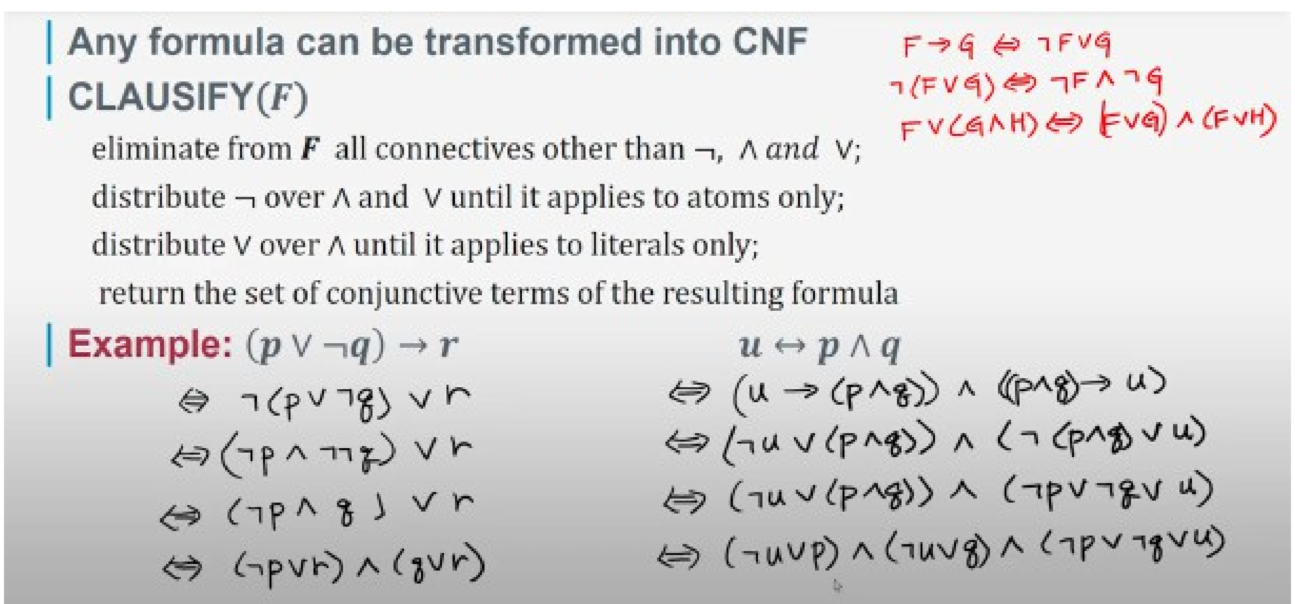
Satisfiability (SAT) problem: We want to make sure, there is no contradiction between our proposition formulas.

– Most SAT solvers accept Conjunctive Normal Form (CNF) as input.

– Any formula can be transformed into CNF, using CLAUSIFY algorithm and equivalence transformations and De Morgan's laws.

– While transforming a formula into CNF, too many clauses can be generated 2^n. We can solve it by using auxiliary atoms.

$$\boxed{\begin{array}{c}\text{F}\\ (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r}\end{array}} \xrightarrow{\text{Tr}} \boxed{\begin{array}{c}\Gamma\\ \boldsymbol{u} \leftrightarrow \boldsymbol{p} \vee \boldsymbol{q}\\ \boldsymbol{u} \wedge \boldsymbol{r}\end{array}}$$



Any formula can be transformed into CNF

CLAUSIFY(F)

eliminate from F all connectives other than $\neg$, $\wedge$ *and* $\vee$;

distribute $\neg$ over $\wedge$ and $\vee$ until it applies to atoms only;

distribute $\vee$ over $\wedge$ until it applies to literals only;

return the set of conjunctive terms of the resulting formula

$F \rightarrow G \Leftrightarrow \neg F \vee G$

$\neg(F \vee G) \Leftrightarrow \neg F \wedge \neg G$

$F \vee (G \wedge H) \Leftrightarrow (F \vee G) \wedge (F \vee H)$

Example: $(p \vee \neg q) \rightarrow r$

$\Leftrightarrow \neg(p \vee \neg q) \vee r$

$\Leftrightarrow (\neg p \wedge \neg\neg q) \vee r$

$\Leftrightarrow (\neg p \wedge q) \vee r$

$\Leftrightarrow (\neg p \vee r) \wedge (q \vee r)$

$u \leftrightarrow p \wedge q$

$\Leftrightarrow (u \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow u)$

$\Leftrightarrow (\neg u \vee (p \wedge q)) \wedge (\neg (p \wedge q) \vee u)$

$\Leftrightarrow (\neg u \vee (p \wedge q)) \wedge (\neg p \vee \neg q \vee u)$

$\Leftrightarrow (\neg u \vee p) \wedge (\neg u \vee q) \wedge (\neg p \vee \neg q \vee u)$

Module 1 Highlights

Computing Propositional Logic (PL):

Unit propagation: Assume formulas are in CNF.

- If there is a unit clause (single literal), we can simplify the formula by assuming these two options: Either it is TRUE or FALSE.
- Then we can easily check the satisfiability.
- If there is no unit clause, just choose one atom and guess (T, F)

DPLL algorithm is actually just application of Unit-propagation in a recursive algorithm.

$$
\begin{array}{l}
\mathrm{DPLL}(F, U) \\
\quad \textsc{Unit-propagate}(F, U); \\
\quad \textbf{if } F \text{ contains the empty clause } \textbf{then} \text{ return}; \\
\quad \textbf{if } F = \top \textbf{ then} \text{ exit with a model of } U; \\
\quad L \leftarrow \text{a literal containing an atom from } F; \\
\quad \mathrm{DPLL}(F|_L, U \cup \{L\}); \\
\quad \mathrm{DPLL}(F|_{\overline{L}}, U \cup \{\overline{L}\})
\end{array}
$$

Module 1 Highlights

Computing Propositional Logic (PL):

Q: Apply DPLL to $(\neg p \vee q) \wedge (\neg p \vee r) \wedge (q \vee r) \wedge (\neg q \vee \neg r)$

$DPLL((\neg p \vee q) \wedge (\neg p \vee r) \wedge (q \vee r) \wedge (\neg q \vee \neg r),\ \emptyset)$

$UP($ " $,\ \emptyset)$

$L := p$

$DPLL(q \wedge r \wedge (q \vee r) \wedge (\neg q \vee \neg r),\ \{p\})$

$UP($ " $,\ \{p\})$

$\{p, q\})$

$F := r \wedge \neg r$ $\quad \{p, q, r\})$

$F := T \wedge \neg T$

$DPLL((q \vee r) \wedge (\neg q \vee \neg r),\ \{\neg p\})$

$L := q$

$DPLL(\neg r$ $\quad \{\neg p, q\})$

$UP(\neg r$ $\quad \{\neg p, q\})$

$F := T$ $\quad \boxed{\{\neg p, q, \neg r\})}$

Thanks

&

Questions