Probability Theory

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The lecture is based on the slides developed by Prof. Yu Zhang from ASU School of Computing and Augmented Intelligence

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Objectives

Objective

Define Probability Space and Conditional Probability

Objective

Discuss Bayes Rule

The Bayes Rule

- **Let (Ω, $\mathcal{B}$, P) be a probability space, and let $\{H_j\}$ be pairwise disjoint events in $\mathcal{B}$, with $\cup_{j=1,\ldots n} H_j = \Omega$ and $P(H_j)>0$, $\forall j$.**
  - Then $\forall B \in \mathcal{B}$ where $P(B) > 0$,

$$P(H_j \mid B) = \mathrm{P}(H_j)P(B \mid H_j) \,/\, \Sigma_{i=1,\ldots,n} \mathrm{P}(H_i)P(B \mid H_i), \ \forall j$$

Bayes' Theorem

- Given events A and B, where $P(B) \neq 0$,
  $P(A|B) = P(B \mid A)P(A) / P(B)$.

The Bayes Rule

- **Why is this at all helpful?**
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems (e.g., tracking, localization)

- **In the running for most important AI equation!**

Independence of Events

Let (Ω, $\mathscr{B}$, P) be a probability space, $\forall$ A,B $\in \mathscr{B}$, we say A and B are independent if $P(A,B) = P(A)P(B)$.

- Splits the joint distribution factors into a product of two simple ones.
- Usually, variables are not independent! But we can use independence as a **modelling assumption**:
  - Independence can be a simplifying assumption.
  - Empirical joint distributions: at best "close" to independent.

Conditional Independence

- P(Toothache, Cavity, Catch)
- If I have a cavity, the probability that the probe catches it does *not* depend on whether I have a toothache:
  - $P(+catch \mid +toothache, +cavity) = P(+catch \mid +cavity)$
- The same independence holds if I do not have a cavity:
  - $P(+catch \mid +toothache, -cavity) = P(+catch \mid -cavity)$
- "Catch" is conditionally independent of "Toothache" given "Cavity":
  - $P(Catch \mid Toothache, Cavity) = P(Catch \mid Cavity)$

Conditional Independence

- **Unconditional (absolute) independence very rare (why?).**
- **Conditional independence is our most basic and robust form of knowledge about uncertain environments.**
  - X is conditionally independent of Y given Z (i.e., $X \perp\!\!\!\perp Y|Z$)

    if and only if $\forall x,y,z$: $P(x,y \mid z) = P(x \mid z)P(y \mid z)$,

    or, equivalently, if and only if $\forall x,y,z$: $P(x \mid z,y) = P(x \mid z)P(y \mid z)$.