Part #1- Relaxation Approach

①

q => is our solution. $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}$ $q = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ → this is the Solution label

$$\underset{q}{\operatorname{argmin}} \ \frac{1}{4} \sum (q_i - q_j)^2 w_{ij}$$

Finding an argument q that minimize the equation $\frac{1}{4} \sum (q_i - q_j)^2 w_{ij}$

one way → pick combination of solution. one at the time and then Calculate the relationship and once you have done that for every one of them you pick the smallest one.

but there are lots of Combination. Such that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ . . .

(2)

Relaxation approach

Second way → find the best q, mathematically.

So, q can be any Real number

$$\underset{q}{\arg\min}\ \frac{1}{4}\sum_{ij} w_{ij}(q_i - q_j)^2$$

$$\rightarrow \frac{1}{4}\sum_{ij} w_{ij}(q_i^2 - 2q_iq_j + q_j^2)$$

$$\rightarrow \underbrace{\frac{1}{4}\sum_{ij} w_{ij}q_i^2}_{A} - \underbrace{\frac{1}{2}\sum_{ij} w_{ij}q_iq_j}_{B} + \underbrace{\frac{1}{4}\sum_{ij} w_{ij}q_j^2}_{C}$$

part (A)

A & C are basically identical, q_i same a q_j

i,j both increases the same way

$$\sum_i \boxed{\sum_j w_{ij}q_i^2}$$

$$\rightarrow \sum_i \left(w_{i1}q_i^2 + w_{i2}q_i^2 + w_{i3}q_i^3 + \cdots\right)$$

$$\rightarrow \sum_i \left(w_{i1} + w_{i2} + w_{i3} + \cdots\right)q_i^2$$

adjacency matrix

$$\begin{array}{c} d_1 \\ d_2 \\ \vdots \end{array}\begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots \\ w_{21} & w_{22} & w_{23} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\rightarrow \sum_i d_i q_i^2$$

d_i degree of vertex

when we add up the entire row, its name is degree of vertex (d_i)

part (A) continue...

$\rightarrow \sum_i d_i q_i^2$

$\rightarrow \left(d_1 q_1^2 + d_2 q_2^2 + d_3 q_3^2 + \cdots \right)$

$\rightarrow q' D q$

D → degree matrix

q → solution lable vector

prev →

$[q_1 \; q_2] \underbrace{\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}}_{D} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

$[q_1 \; q_2] \begin{bmatrix} q_1 d_1 \\ q_2 d_2 \end{bmatrix}$

$d_1 q_1^2 + d_2 q_2^2 + \cdots$

(4)

part (B) $\sum_{ij} w_{ij} \, q_i q_j$

$$\rightarrow \sum_i \boxed{\sum_j w_{ij} \, q_i \, q_j}$$

For this part $\underline{\underline{i}}$ is Constant

I just move it outside.

$$\rightarrow \sum_i q_i \boxed{\sum_j w_{ij} \, q_j}$$

$$\rightarrow \sum_i q_i \left[w_{i1} q_1 + w_{12} q_2 + w_{i3} q_3 + \cdots \right]$$

$$\rightarrow \sum_i q_i \left[w_{i1} + w_{i2} + w_{i3} + \cdots \right] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

$$a_1 b_1 + a_2 b_2$$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

q → Solution label.

$$\rightarrow \sum_i q_i \left[w_{i1} + w_{i2} + w_{i3} + \cdots \right] q$$

$$\rightarrow \left[\overset{a_1}{q_1} (\overbrace{w_{11} + w_{12} + w_{13} \cdots}^{b_1}) + \overset{a_2}{q_2} (\overbrace{w_{21} \;\; w_{22} \;\; w_{33} \cdots}^{b_2}) + \cdots \right] q$$

$$\rightarrow \left[\begin{bmatrix} q_1 & q_2 & q_3 \cdots \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \cdots \\ w_{21} & w_{22} & w_{23} \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \right] q$$

W → adjacancy matrix

$$\Longrightarrow q' W q$$

(5) go back to the original one.

$$\rightarrow \frac{1}{4} \underbrace{\sum_{ij} w_{ij} q_i^2}_{A} - \underbrace{\frac{1}{2} \sum_{ij} w_{ij} q_i q_j}_{B} + \frac{1}{4} \underbrace{\sum_{ij} w_{ij} q_j^2}_{C}$$

A & C are same value

$$\Rightarrow \frac{1}{2} \sum_{ij} w_{ij} q_i^2 - \frac{1}{2} \sum_{ij} w_{ij} q_i q_j$$

$$\Rightarrow \frac{1}{2} q' D q - \frac{1}{2} q' w q$$

$$\Rightarrow \frac{1}{2} q' [D - w] q$$

this is called laplacian $\underline{\underline{L}}$

$$\Rightarrow \frac{1}{2} q' L q$$

$$\Longrightarrow \underset{q}{\operatorname{argmin}} \ \frac{1}{4} \sum_{ij} w_{ij} (q_i - q_j)^2 =$$

$$\Longleftrightarrow$$

$$\underset{q}{\operatorname{argmin}} \ 2 q' L q$$

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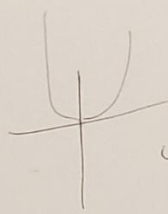
$$\underset{q}{\operatorname{argmin}} \ \frac{1}{4} \sum_{ij} w_{ij} (q_i - q_j)^2 = \underset{q}{\operatorname{argmin}} \ 2q'Lq$$

⇒ we know that when minimizing the constant doesn't matter.

$y = x$

$x = 0$



$y = 2x$

$x = 0$

So, we can remove $\frac{1}{2}$ at the front

$\boxed{Lq = \lambda_2 q}$

Remember

eigen value

eigen vector

→ matrix

$$\Rightarrow \underset{q}{\operatorname{argmin}} \ q'Lq$$

vector

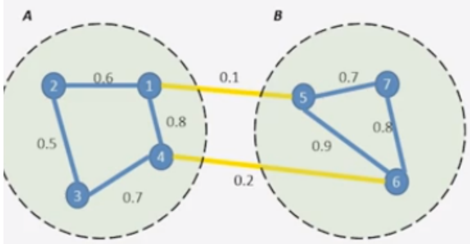
To find q,

L → Second smallest eigen value. Return minimum eigen vectors → find the minimum L

L → biggest eigen value. return maximum eigen vectors → find maximum L

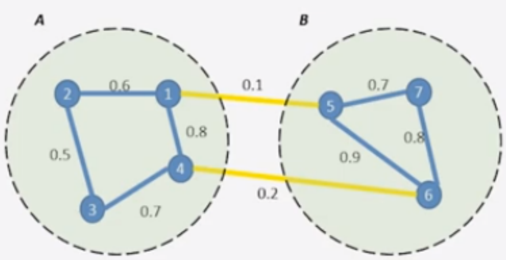
Spectral clustering example in video lecturer "Going Beyond MinCut".

Graph and Similarity Matrix



	x1	x2	x3	x4	x5	x6	x7
x1	0	0.6	0	0.8	0.1	0	0
x2	0.6	0	0.5	0	0	0	0
x3	0	0.5	0	0.7	0	0	0
x4	0.8	0	0.7	0	0	0.2	0
x5	0.1	0	0	0	0	0.9	0.7
x6	0	0	0	0.2	0.9	0	0.8
x7	0	0	0	0	0.7	0.8	0

Graph and Laplacian Matrix



	x1	x2	x3	x4	x5	x6	x7
x1	1.5	-0.6	0	-0.8	-0.1	0	0
x2	-0.6	1.1	-0.5	0	0	0	0
x3	0	-0.5	1.2	-0.7	0	0	0
x4	-0.8	0	-0.7	1.7	0	-0.2	0
x5	-0.1	0	0	0	1.7	-0.9	-0.7
x6	0	0	0	-0.2	-0.9	1.9	-0.8
x7	0	0	0	0	-0.7	-0.8	1.5

Graph & Similarity Matrix.

7 Simple strategy to partitioning

$$A = \{ i \mid q_i < 0 \} \quad , \quad B = \{ i \mid q_i \geq 0 \}$$

	x_1	x_2	x_3	x_4	...
x_1	0	0.6	0	0.8	d_1
x_2	0.6	0	.5	0	d_2
x_3	0	.5	0	0.7	d_3
x_4	0.8	0	0.7	0	d_4
⋮					

W

adjancancy Matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 & \\ 0 & d_2 & 0 & 0 & \cdots \\ 0 & 0 & d_3 & 0 & \\ 0 & 0 & 0 & d_4 & \\ & \vdots & & & \end{bmatrix} \Rightarrow$$

Degree matrix

$$L = D - W \Rightarrow \begin{bmatrix} 1.5 & -0.6 & 0 & -0.8 & \cdots \\ -0.6 & 1.1 & -0.5 & 0 & \cdots \\ 0 & -0.5 & 1.2 & -0.7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

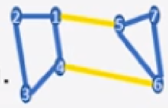
according to eigen value/ eigen vector steps.

Need to find the eigen values and eigen vectors on Laplacian Matrix.

Solve Eigen Problem

Pre-processing

- Build Laplacian matrix L of the graph.



$\Lambda =$

0
0.1588
1.2705
1.3692
2.2751
2.6238
2.9027

$\mathbf{x} =$

0.378	-0.2962	0.3027	-0.6041	0.0429	0.3638	-0.4226
0.378	-0.3805	0.6392	0.4487	-0.0125	-0.233	0.2192
0.378	-0.3608	-0.5812	0.4834	-0.0221	0.2736	-0.2832
0.378	-0.2649	-0.398	-0.4373	0.0429	-0.3899	0.5323
0.378	0.4298	0.0443	0.0159	-0.6004	0.4291	0.3544
0.378	0.406	-0.0317	0.0012	-0.2174	-0.6116	-0.5196
0.378	0.4665	0.0247	0.0923	0.7667	0.1681	0.1195

Find

- Eigenvalues Λ and eigenvectors **x** of matrix **L**.
- Map vertices to the corresponding components of the 2nd eigenvector.

x1	-0.2962
x2	-0.3805
x3	-0.3608
x4	-0.2649
x5	0.4298
x6	0.406
x7	0.4665

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we can find the value of λ

according to $n \times n$ matrix

we have λ_n, eigen value.

→ each eigen value (A) will preduce

a eigen vector (x)

eign value her

we will find the eigen vectors based on

the second smallest eign value (λ_2)

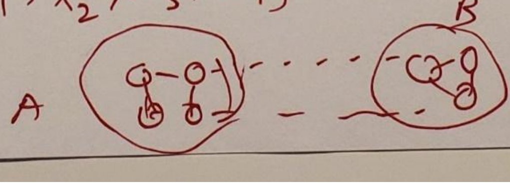
So this is the solution lable vector (q)

= then split the vector into

two pants.

$A = \{ i \mid q_i < 0 \}$ $\quad$ $B = \{ i \mid q_i \geq 0 \}$

$A = \{ X_1, X_2, X_3, X_4 \}$ $\quad$ $B = \{ X_5, X_6, X_7 \}$



Spectral Clustering

x1	-0.2962
x2	-0.3805
x3	-0.3608
x4	-0.2649
x5	0.4298
x6	0.406
x7	0.4665

Split at value 0
Cluster A: Negative points
Cluster B: Positive Points

x1	-0.2962
x2	-0.3805
x3	-0.3608
x4	-0.2649

X5	0.4298
X6	0.406
X7	0.4665