Review of Mathematical Foundations – Part 2

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Objectives

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Define Probability Space

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Discuss Conditional Probability and Bayes Rule

Probability Space (1/2)

A probability space is a triplet $(\Omega, \mathcal{B}, P)$ that is used to model a process or an experiment with random outcomes.

- The **sample space** Ω is the set of all possible outcomes of an experiment
  - Consider two different experiments

    (1) Tossing a coin;   (2) Tossing a die

Probability Space (2/2)

- $\mathcal{B}$ *:* a sigma algebra (or Borel field), or informally, a collection of subsets of Ω, subject to some constraints (like containing the empty set, being closed under complements and countable union)

- *P:* a measure called **probability** defined on $\mathcal{B}$, that satisfies
  - $P(A) \geq 0$ for all $A \in \mathcal{B}$
  - $P(\Omega) = 1$
  - If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint then $P(\cup A_i) = \sum P(A_i)$ (i.e., $A_j A_k = \varnothing, \forall j \neq k$)

Conditional Probability

- Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let $H \in \mathcal{B}$ with $P(H)>0$. For any $B \in \mathcal{B}$, we define

$$P(B|H) = P(BH) / P(H)$$

and call $P(B|H)$ the **conditional probability** of B, given H.

The Total Probability Rule

- Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let $\{H_j\}$ be pairwise disjoint events in $\mathcal{B}$ (i.e., $H_jH_k = \varnothing, \forall j \neq k$) and $\bigcup_{j=1,\ldots,\infty} H_j = \Omega$. Suppose $P(H_j)>0, \forall j$, then

$$P(B) = \sum_{j=1,\ldots,\infty} P(H_j)P(B|H_j)$$

-- Such $\{H_j\}$ is called a partition of Ω.

The Bayes Rule

- Let $(\Omega, \mathcal{B}, P)$ be a probability space, and let $\{H_j\}$ be pairwise disjoint events in $\mathcal{B}$ with $\cup_{j=1,\ldots,\infty} H_j = \Omega$, and $P(H_j)>0$, $\forall j$. We have, $\forall B\in \mathcal{B}$ and $P(B)>0$,

$$P(H_j|B) = \frac{P(H_j)\, P(B\,|H_j)}{\sum_{i=1,\ldots,\infty} P(H_i)P(B|H_i)}, \qquad \forall\, j$$

Independence of Events

- Let $(\Omega, \mathcal{B}, P)$ be a probability space, $\forall A, B \in \mathcal{B}$, we say A and B are independent if $P(AB) = P(A)P(B)$.