Globally Optimal Learning for Structured Elliptical Losses

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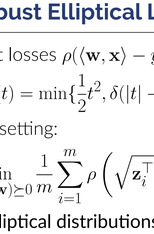
Motivation

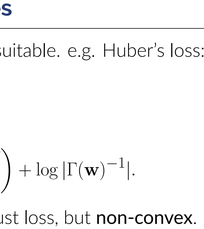
We are interested in regression problems where there are:

- Multiple correlated tasks, calls for **Structured Prediction**.
- Data that is heavy tailed or contaminated by outliers. Usually handled with **Robust Regression**.

For example, predicting river discharge in multiple locations:

- $\mathbf{x} = [☂(t), 🐟(t)]$
- $\mathbf{y} = 🐟(t+1) \in \mathbb{R}^n$





Structured Regression

For correlated regression tasks, it is common to use the **inverse covariance**:

$$\mathbf{z} = \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix},\ \Gamma^{-1} = \mathbb{E}_{\mathbf{z}}\left[\mathbf{z}\mathbf{z}^\top\right],\ \Gamma = \begin{bmatrix}\Gamma_{\mathbf{xx}} & \Gamma_{\mathbf{xy}}\\ \Gamma_{\mathbf{yx}} & \Gamma_{\mathbf{yy}}\end{bmatrix}$$

This matrix satisfies desirable properties:

- Optimal linear regressor obtained in closed form.

$$\mathbf{y}(\mathbf{x}) = \Gamma_{\mathbf{yy}}^{-1}\Gamma_{\mathbf{yx}}\mathbf{x}$$

- Conditional independence structures are manifested as sparsity patterns in Γ.

$$\Gamma(\mathbf{w}) = \begin{bmatrix} w_{11} & 0 & w_{13} & \cdots & 0\\ 0 & w_{22} & w_{23} & \cdots & w_{28}\\ & & \vdots & & \\ 0 & w_{28} & \cdots & & w_{88}\end{bmatrix}$$

Standard way to learn such matrices is maximum likelihood of a Gaussian Markov Random Field:

$$(GMRF)\quad \arg\min_{\mathbf{w}:\Gamma(\mathbf{w})\succeq 0} \frac{1}{m}\sum_{i=1}^{m}\mathbf{z}_i^\top\Gamma(\mathbf{w})\mathbf{z}_i + \log|\Gamma(\mathbf{w})^{-1}|. \tag{1}$$

Objective is **convex**, very efficient solutions. Minimizes **squared loss** if no structure imposed.
Caveat: not robust to heavy tailed and contaminated data, common in real-world applications.

Robust Elliptical Losses

In regression with single label, robust losses $\rho(\langle\mathbf{w},\mathbf{x}\rangle - y)$ are suitable. e.g. Huber's loss:

$$\rho(t) = \min\{\frac{1}{2}t^2, \delta(|t| - \frac{1}{2}\delta)\}$$

Suggestion for **structured multitask** setting:

$$(RMRF)\quad \arg\min_{\mathbf{w}:\Gamma(\mathbf{w})\succeq 0} \frac{1}{m}\sum_{i=1}^{m}\rho\left(\sqrt{\mathbf{z}_i^\top\Gamma(\mathbf{w})\mathbf{z}_i}\right) + \log|\Gamma(\mathbf{w})^{-1}|. \tag{2}$$

Based on maximizing likelihood of **elliptical distributions**. Robust loss, but **non-convex**.

Globally Optimal Learning

Main result: Formal guarantees on optimization, for most common losses in the literature

Gaussian	$\rho(t) = t^2$
Generalized Gaussian	$\rho(t) = t^{2\beta}, \beta \in (0,1)$
T distribution	$\rho(t) = \frac{n+\nu}{2}\log(1+\frac{t^2}{\nu}), \nu > 2$
Angular / Tyler	$\rho(t) = n\log(t^2)$
Huber	$\rho(t) = \min\{\frac{1}{2}t^2, \delta(\lvert t\rvert - \frac{1}{2}\delta)\}$

Assumptions:

- The loss $\rho(\sqrt{t})$ is twice differentiable and concave in t. Its derivative w.r.t t, denoted by ψ, satisfies $\psi(t) \geq -t\psi'(t)$ for all $t > 0$.
- Data comes from a **Spherically Invariant** distribution (close relatives of elliptical distributions), result applies at the **population limit**.
- Structure is linear. Includes graphical structures, but allows for many other structures. Matrices are defined by weights $\mathbf{w} \in \mathbb{R}^I$:

$$\Gamma(\mathbf{w}) = \sum_{\alpha\in I} w_\alpha \mathbf{G}_\alpha.$$

Theorem

Let $\mathbf{z}$ be an SIRV$(g, \Gamma^{-1}(\mathbf{w}^*))$ and ρ a function that satisfies our assumption. Consider the optimization problem:

$$\min_{\mathbf{w}:\Gamma(\mathbf{w})\succeq 0} \mathbb{E}_{\mathbf{z}}\left\{\rho\left(\sqrt{\mathbf{z}^\top\Gamma(\mathbf{w})\mathbf{z}}\right)\right\} + \log|\Gamma(\mathbf{w})^{-1}|. \tag{3}$$

If $\mathbf{w}$ is a stationary point of the loss, it holds that $\Gamma(\mathbf{w})$ equals $\Gamma(\mathbf{w}^*)$ up to a multiplicative constant.

Proof relies on the following useful result.
Lemma: Let $\mathbf{v}$ be an SIRV(g,Σ) with arbitrary texture g, and ρ a function that satisfies our assumption. Define the matrix:

$$\Sigma^\rho(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\left[\mathbf{v}\mathbf{v}^\top\psi(\|\mathbf{v}\|_2^2)\right].$$

Then $\Sigma^\rho(\mathbf{v})$ and Σ commute, and maintain the same order of eigenvalues.

Solution with Minimization-Majorization

Minimization-majorization with minimization steps based on Gaussian MRFs work much better than the alternatives in practice.

Algorithm 1 Minimization Majorization for Elliptical Markov Random Fields

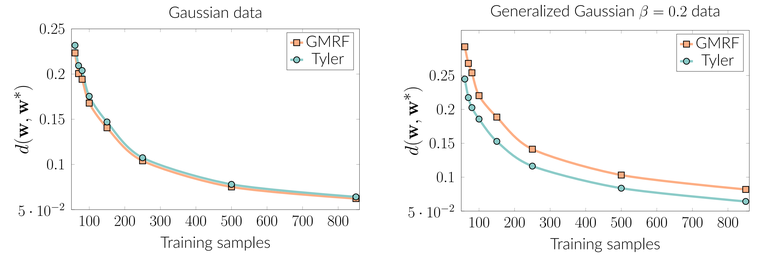
Require: $\rho : \mathbb{R}_{++} \to \mathbb{R}$, $\{\mathbf{z}_i\}_{i=1}^m$

1: Set $\Gamma_0 \leftarrow \boldsymbol{I}$
2: **for** $t = 0 \dots T$ **do**
3: Rescale data $\tilde{\mathbf{z}}_i = \psi(\sqrt{\mathbf{z}_i^\top\Gamma_t\mathbf{z}_i})^{\frac{1}{2}}\cdot\mathbf{z}_i \quad \forall i \in [m]$
4: Solve convex minimization in (1) with data $\{\tilde{\mathbf{z}}_i\}_{i=1}^m$ and set Γ_{t+1} with the solution
5: **end for**

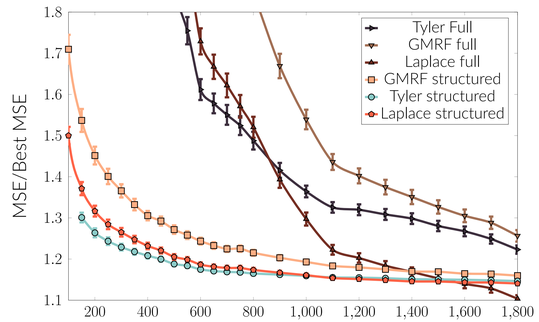
Experiments

We compare the GMRF and elliptical losses of RMRFs on synthetic and real-world data.

Synthetic: random structured covariances, robust losses significantly better for heavy-tailed data.



Kaggle "Huge Stock Market Dataset": Predict intra-day returns of 15 hidden stocks from 105 observed ones. Structure obtained with Graphical Lasso.



River Discharge dataset: Predict river discharge in 34 locations from past precipitation and discharge levels. Structure determined roughly by geographical proximity.