Leveraging Koopman operator and Deep Neural Networks for Parameter Estimation and Future Prediction of Duffing oscillators.

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Outline

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- The Koopman Operator
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- Encoder and Normalization
- Linear layer –Stage II

3.Results
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- Future state prediction
- $\mathcal{K}$ Eigenvalues

Introduction: Dynamical Systems Modeling

Approaches for encountering the dynamical system

- Newton –Analytical view
- Poincare –Phase Plane

Prior knowledge of system is required

- Koopman –Infinite dimensional
  - ✓ Bifurcation data-driven
  - ✓ Robustness

Introduction: Literature review

1931 –Bernard O. Koopman –"Hamiltonian Systems and Transformations in Hilbert Space."

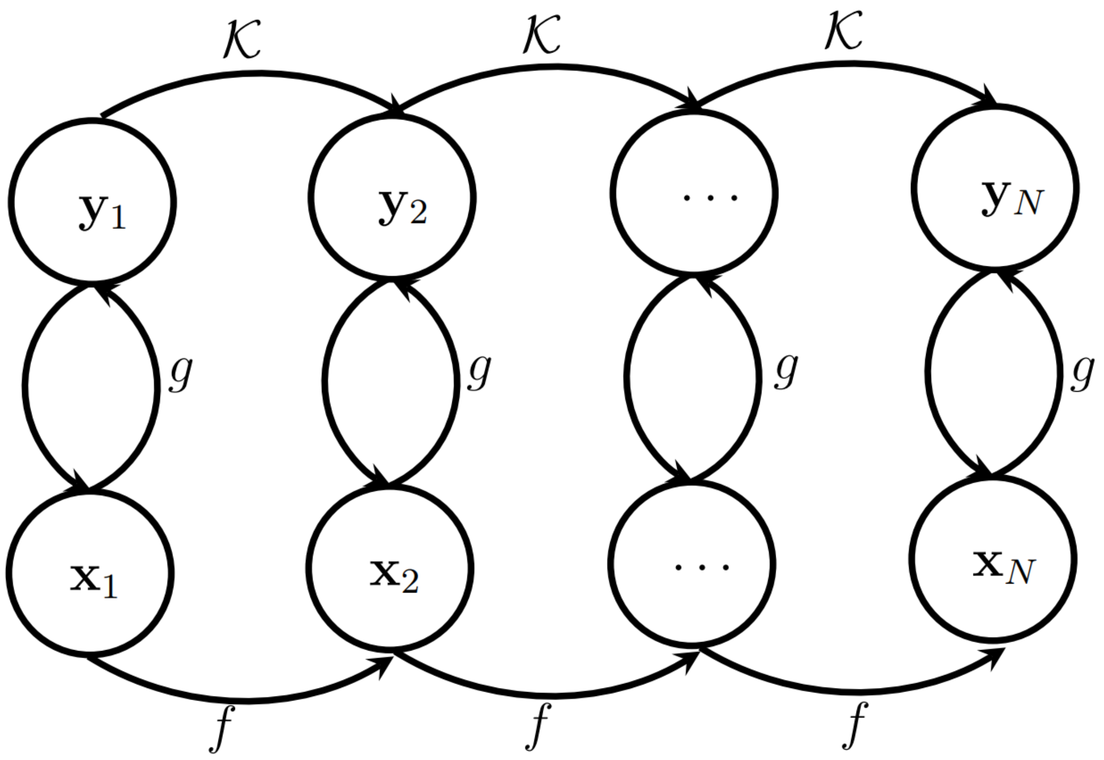


Figure 1. The Koopman Operator.

$$
\begin{aligned}
y_n &= \mathcal{G}(x_n) \\
y_{n+1} &= \mathcal{K}(y_n) \text{ in Hilbert space} \\
x_{n+1} &= \mathcal{G}^{-1}(y_n)
\end{aligned}
$$

Koopman Operator Linear properties

$$
\begin{aligned}
y_{n+m} &= \underbrace{\mathcal{K}\left(\mathcal{K}\left(\mathcal{K}(\cdots\mathcal{K}(y_n))\right)\right)}_{m\ times} \\
&= A^m\, y_n
\end{aligned}
$$

$x_{n+1} = f(x_n)$ in Euclidian space

Introduction: Literature review

2009 –Peter J. Schmid –"Dynamic mode decomposition of numerical and experimental data."

$$\mathbf{X} = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t_1) & \mathbf{x}(t_2) & \cdots & \mathbf{x}(t_m) \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X}' = \begin{bmatrix} | & | & & | \\ \mathbf{x}(t_1') & \mathbf{x}(t_2') & \cdots & \mathbf{x}(t_m') \\ | & | & & | \end{bmatrix}$$

$$\mathbf{X}' \approx \mathbf{A}\mathbf{X}.$$

$$\mathbf{A} = \underset{\mathbf{A}}{\operatorname{argmin}} \|\mathbf{X}' - \mathbf{A}\mathbf{X}\|_F = \mathbf{X}'\mathbf{X}^\dagger$$

2015 –Matthew O. Williams – "A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition."

$$\mathbf{z} = \mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_p(\mathbf{x}) \end{bmatrix}$$

$$\mathbf{A_Z} = \underset{\mathbf{A_Z}}{\operatorname{argmin}} \|\mathbf{Z}' - \mathbf{A_Z}\mathbf{Z}\|_F = \mathbf{Z}'\mathbf{Z}^\dagger$$

2019 – Enio Vasconcelos Filho –"A Dynamic Mode Decomposition Approach With Hankel Blocks to Forecast Multi-Channel Temporal Series"

$$\mathbf{H} = \begin{bmatrix} x(t_1) & x(t_2) & \cdots & x(t_p) \\ x(t_2) & x(t_3) & \cdots & x(t_{p+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x(t_q) & x(t_{q+1}) & \cdots & x(t_m) \end{bmatrix}$$

Introduction: Numerical data (dataset)

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t),$$

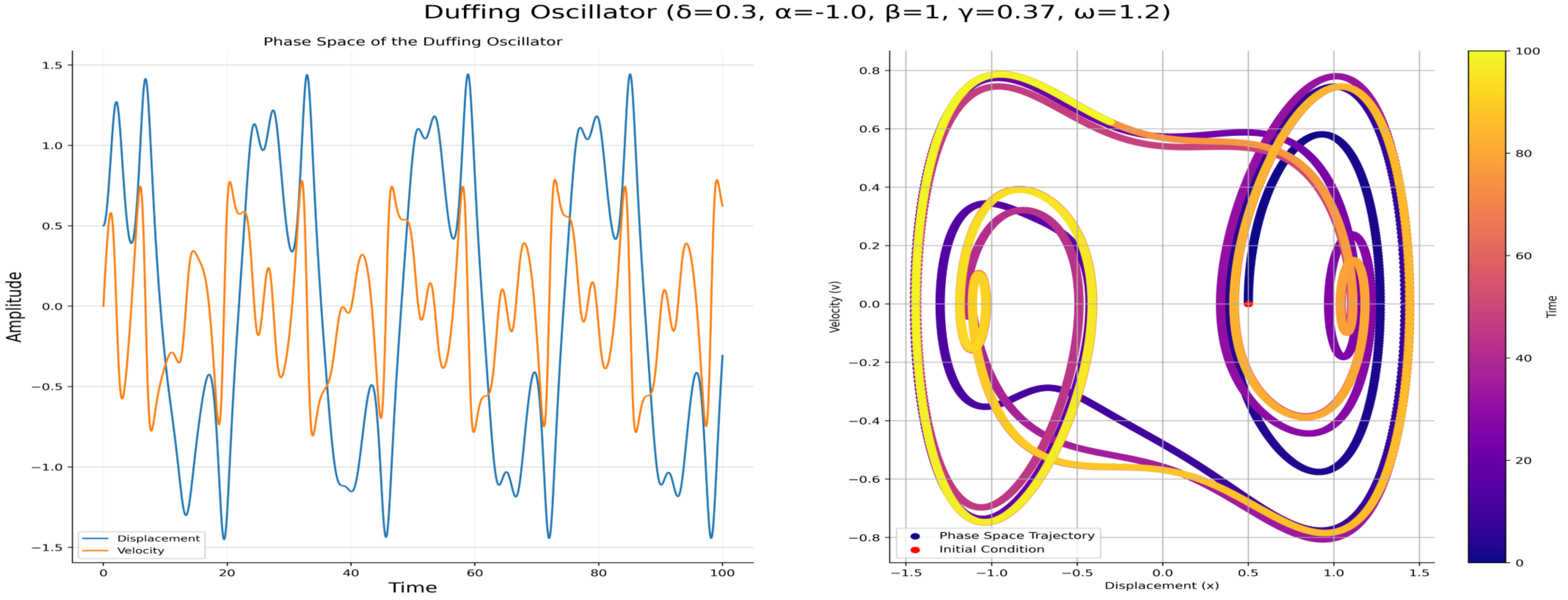


Figure 2. Numerical result of Duffing oscillator.

Network: Structure

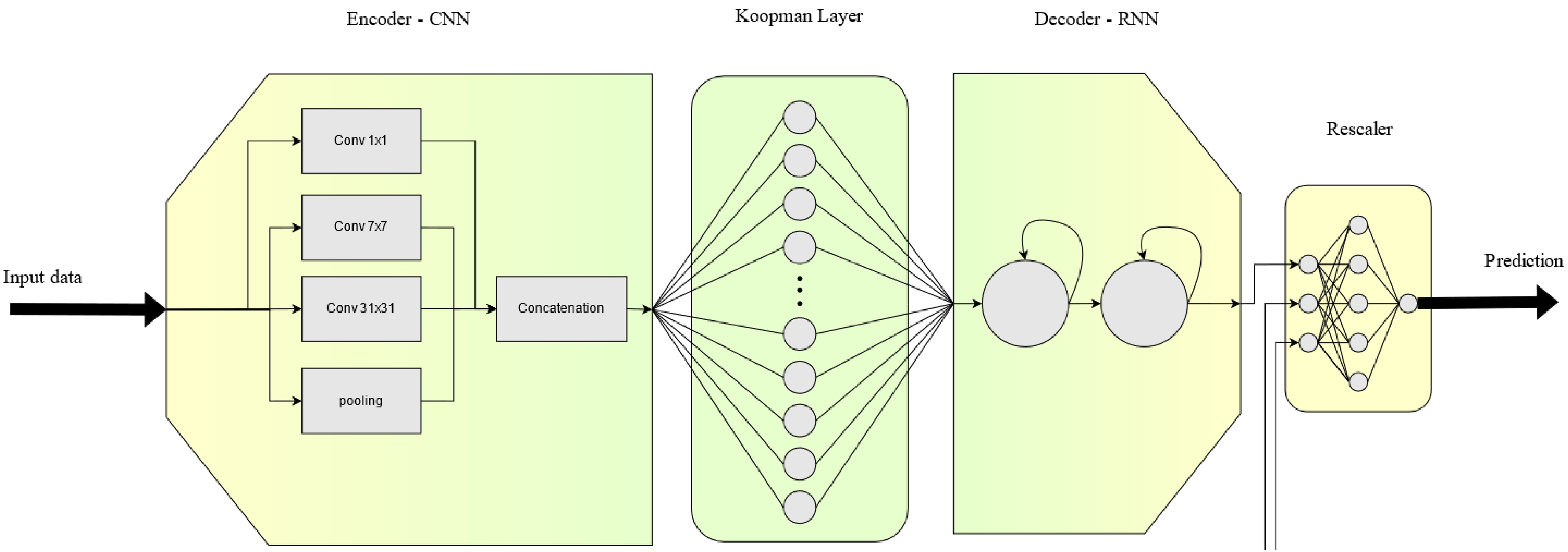


Figure 3. Network’s Diagram.

Network: Encoder

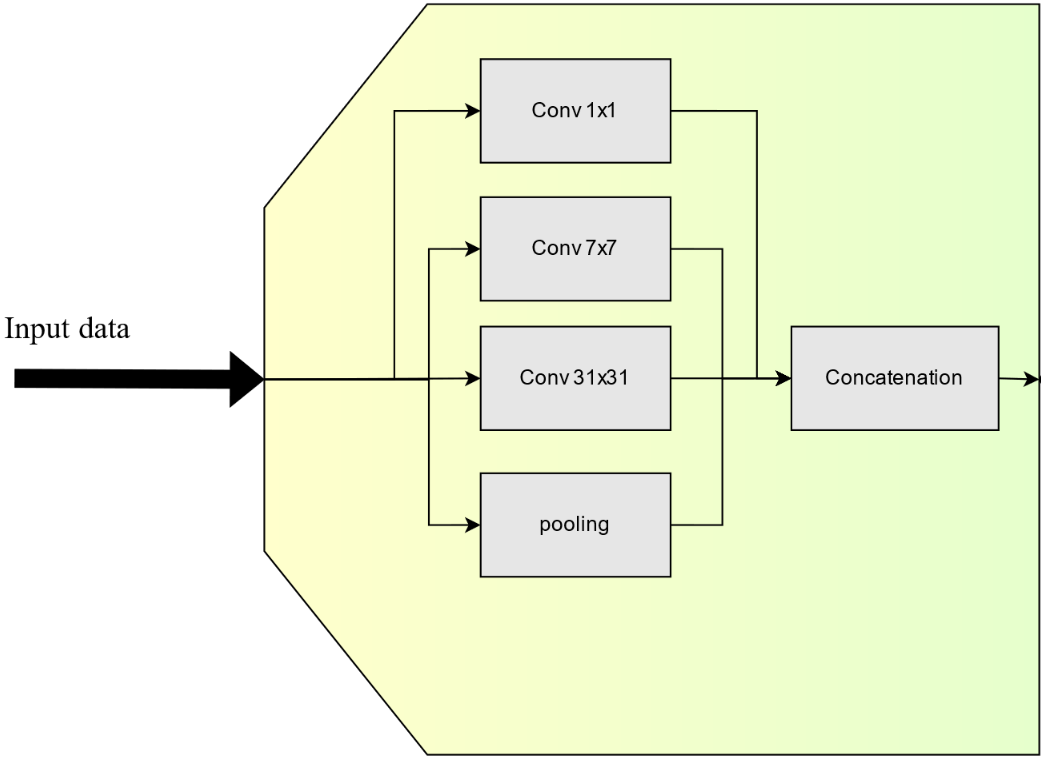


Figure 4. Network’s Diagram.

CNN

- Similar to Frequency transformations
- Considering different Time scales with different Convolutional filter size.

Inception

- Considering different Time scales with different Convolutional filter size.
- Not dependent to input size
- Improving Back propagation Flow

- Autoregressive generation produces the output sequentially, while encoding occurs only once.

Network: The Koopman Evolution Matrix

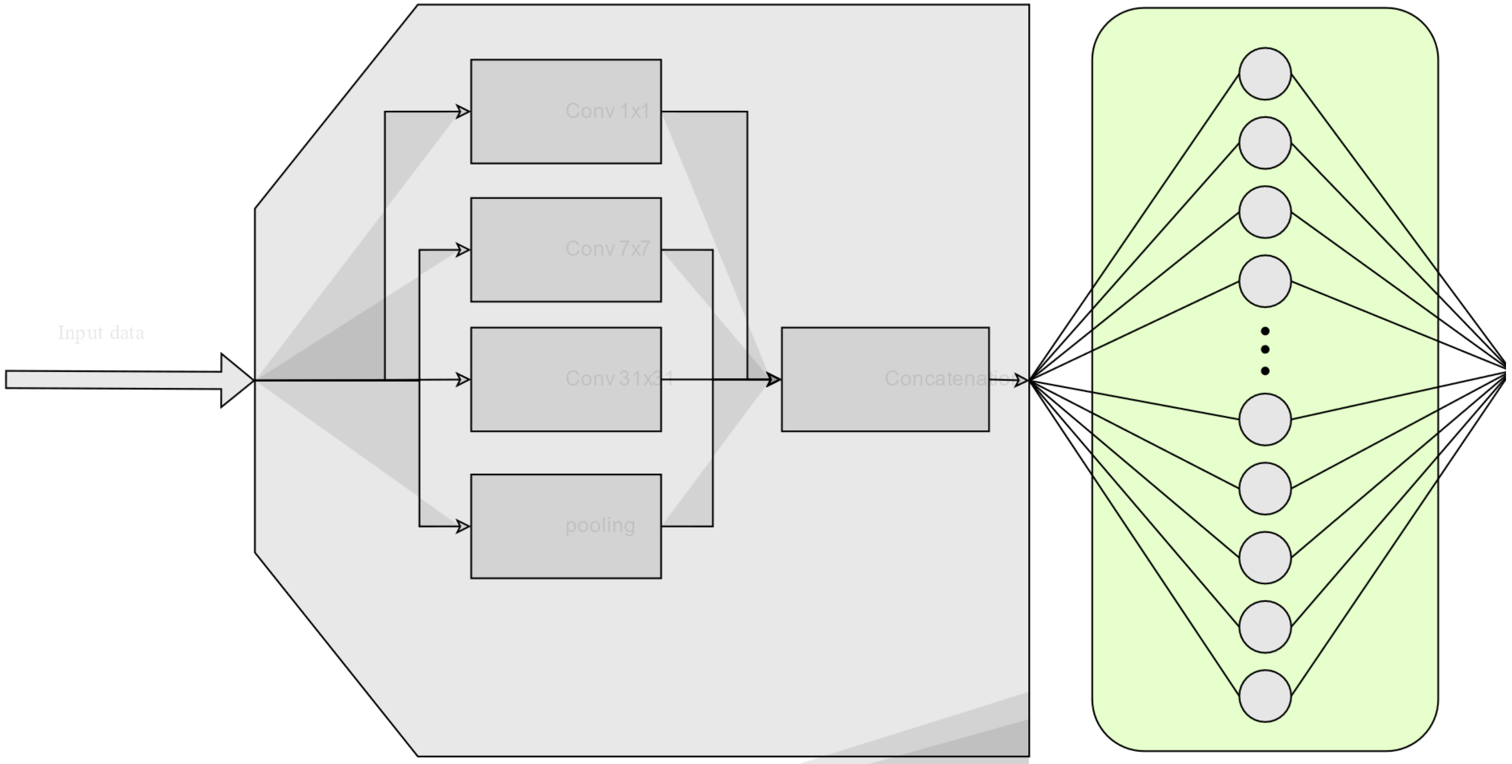


Figure 5. Network’s Koopman Linear Layer.

- This layer designed to serve as the Koopman operator evolution matrix.

- NO Bias.
- NO Activation Function !!!

Network: Encoder and Normalization

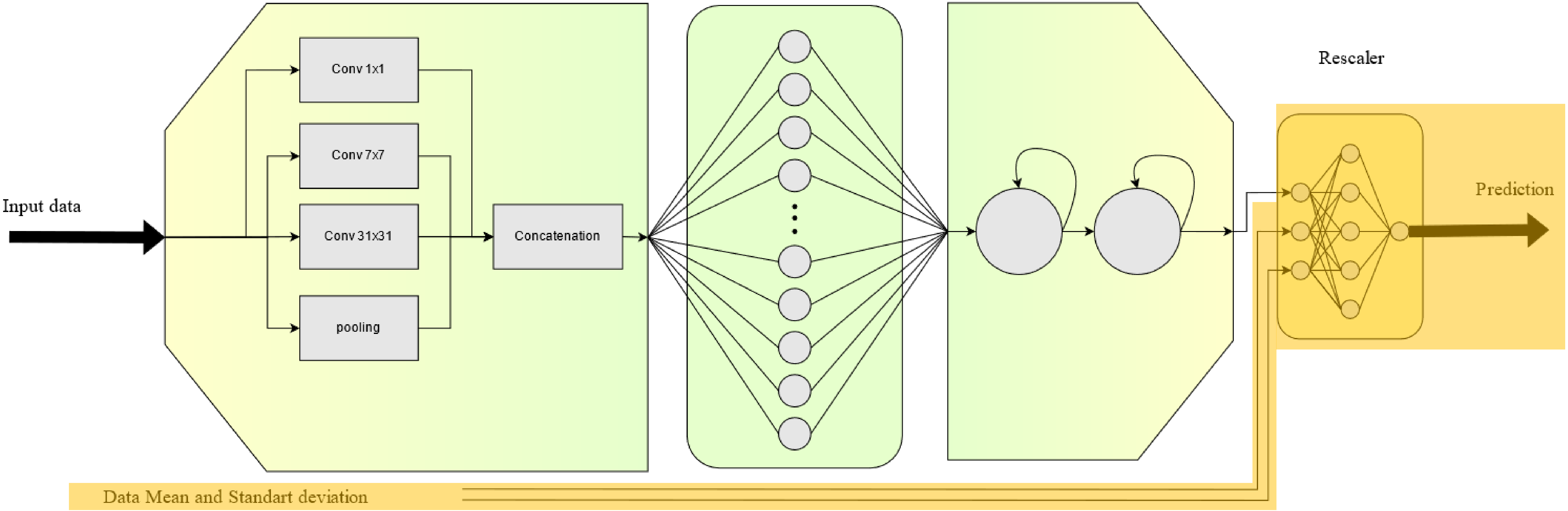


Figure 6. Network's structure –Rescaler.

Network: Linear layer –Stage II

According to Koopman Operator theory, the evolution matrix is a

$$\mathcal{L}\left(A^{50} y_n, \mathcal{G}\ (y_{n+50})\right)$$

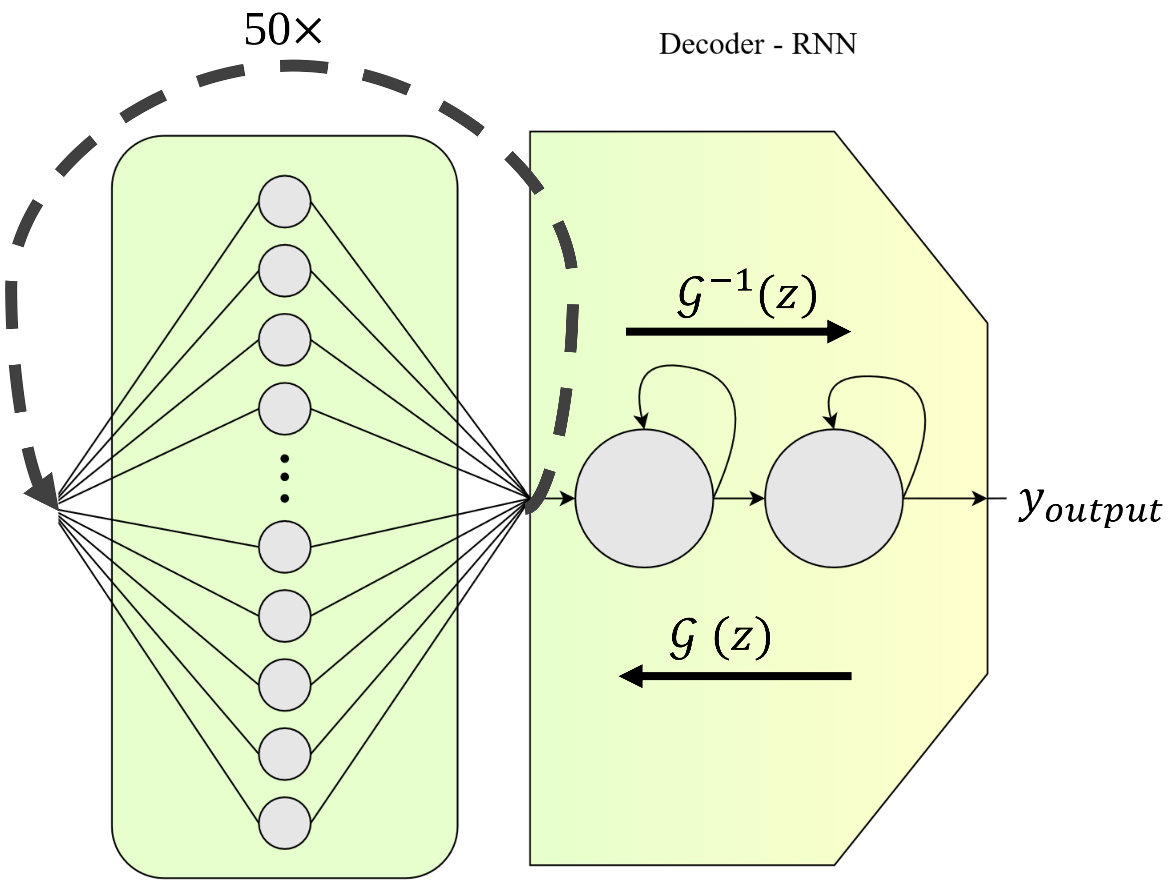
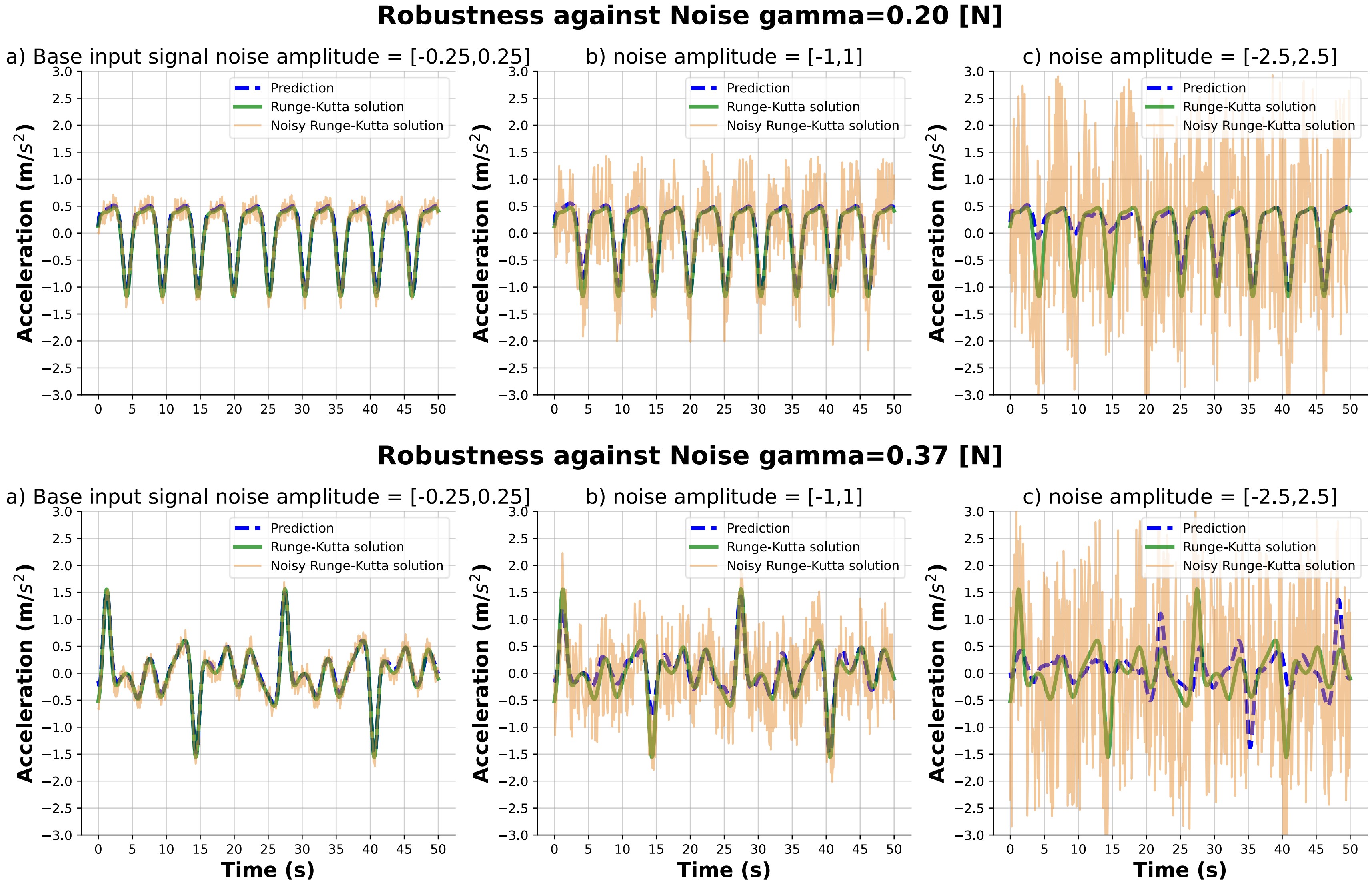


Figure 7. Network’s Koopman Linear Layer.

Results: Future state prediction



Accuracy and loss criteria

MSELoss: 0.0023

Results: $\mathcal{K}$ Eigenvalues Pt.1

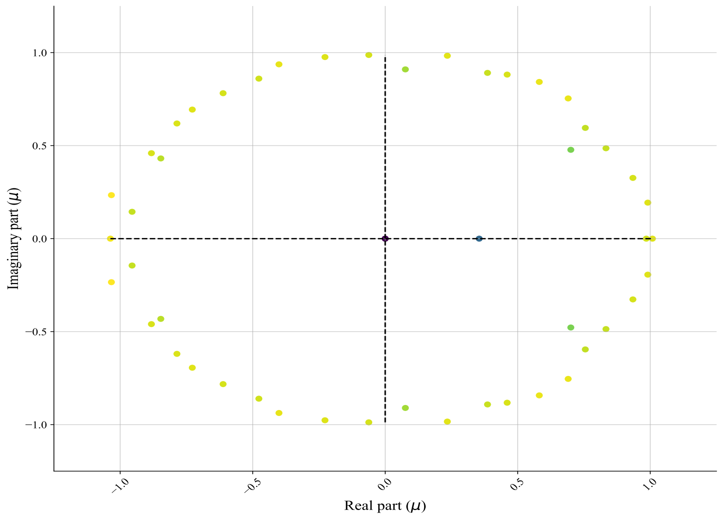


Figure 9. The eigen values of the HankelDMD.

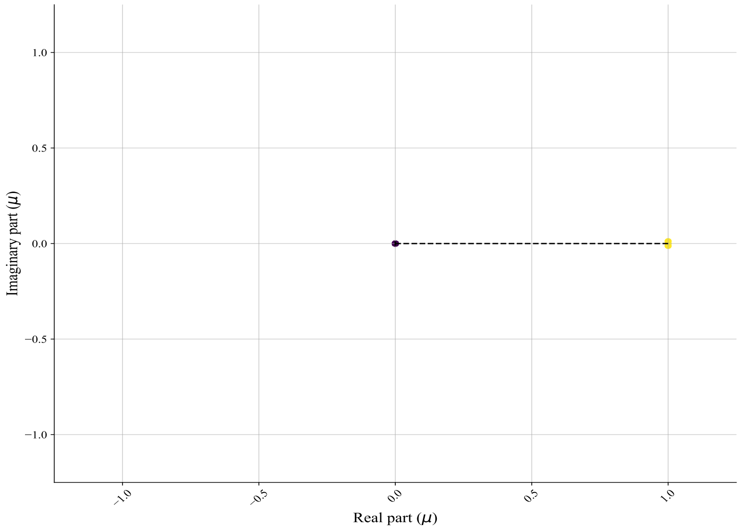


Figure 10. The eigen values of the DMD.

Results : $\mathcal{K}$ Eigenvalues Pt.2

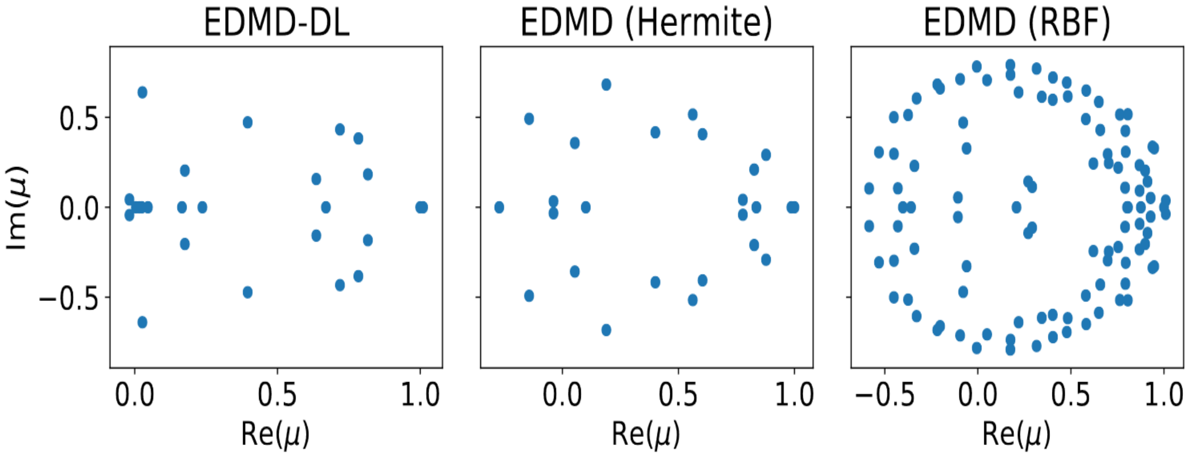


Figure 11. Eigen values of the Evolution Matrix (Q. Li, F. Dietrich, E. M. Bollt, and I. G. Kevrekidis)*

*"Extended dynamic mode decomposition with dictionary learning: A data-driven adaptive spectral decomposition of the Koopman operator,"

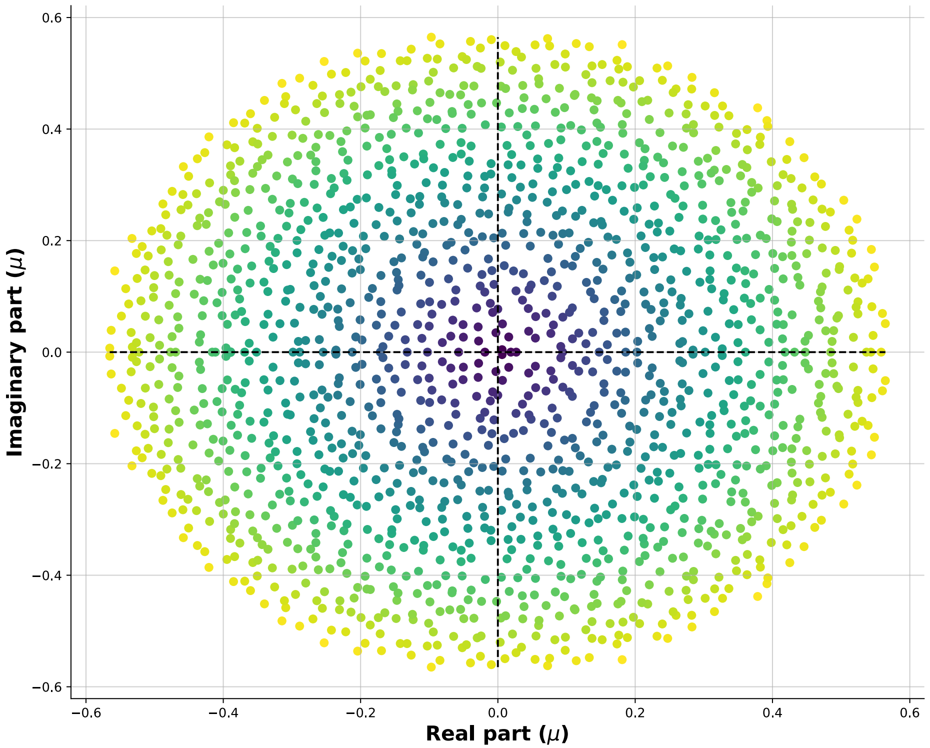


Figure 12. Eigen values of the Evolution Matrix (Image by Author).

Fin

Thank You

Any Question? 🧐