Leveraging Information Contained in Theory Presentations

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Supervisors:

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    - One library to formalize all of Mathematics

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  - Organizational Structures
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Current libraries of mathematics are full of **redundancy**

Monoid: One theory, Multiple Representations

Lean

```
class monoid (M : Type u)
 extends semigroup M, has_one M :=
  (one_mul : ∀ a : M, 1 * a = a)
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```

MMT

```
theory Monoid : ?NatDed =
 includes ?Semigroup
 unit : tm u # e
 unit_axiom : ⊢ ∀ [x] = x * e = x

theory Semigroup : ?NatDed =
 u : sort
 comp : tm u → tm u → tm u
  # 1 * 2 prec 40
 assoc : ⊢ ∀ [x, y, z]
  (x * y) * z = x * (y * z)
 assocLeftToRight :
  {x,y,z} ⊢ (x * y) * z
          = x * (y * z)
  = [x,y,z]
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 assocRightToLeft :
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  = [x,y,z] sym assocLR
```

Haskell

```
class Semigroup a => Monoid a where
  mempty :: a
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Coq

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Alternative Definition:

```
Record monoid := {
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Agda

```
record Monoid c ℓ : Set (suc (c ⊔ ℓ)) where
 infixl 7 _•_
 infix 4 _≈_
 field
  Carrier : Set c
  _≈_ : Rel Carrier ℓ
  _•_ : Op₂ Carrier
  isMonoid : IsMonoid _≈_ _•_ ε

record IsMonoid (• : Op₂) (ε : A)
: Set (a ⊔ ℓ) where
  field
   isSemigroup : IsSemigroup •
   identity : Identity ε

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   identityˡ : LeftIdentity ε •
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Can we abstract over these design decisions?

Monoid: One theory, Many Constructions

```
theory Homomorphism {
  M1, M2 : Monoid
  hom : M1.A → M2.A
  pres-e : hom (M1.e) = M2.e
  pres-op : (x y : M1.A) →
      hom (M1.op x y) = M2.op (hom x) (hom y)
}
theory Isomorphism {
  M1, M2 : Monoid
  f : Homomorphism M1 M2
  g : M2.A → M1.A
  id_1 : {x : M1.A} → (g ∘ f.hom) x = x
  id_2 : {x : M2.A} → (f.hom ∘ g) x = x
}
theory Endomorphism {
  M : Monoid
  Homomorphism M M
}
theory Automorphism {
  M1, M2 : Monoid
  Isomorphism M1 M2
}
```

```
theory Product {
  M1, M2 : Monoid
  e : M1.A × M2.A
  op : M1.A × M2.A  → M1.A × M2.A  → M1.A × M2.A
  lunit : {x : M1.A × M2.A} → op e x = x
  runit : {x : M1.A × M2.A} → op x e = x
  assoc : {x y z : M1.A × M2.A} →
          op x (op y z) = op (op x y) z
}
theory Submonoid {
  M : Monoid
  subset : Set → Set
  A_s : subset M.A
  e_s : A_s
  op_s : A_s → A_s → A_s
}
type Expr :=
  e    : Expr
  op   : Expr → Expr → Expr
type OpenExpr :=
  vars : {n : Nat} → Fin n → OpenExpr
  e    : OpenExpr
  op   : OpenExpr → OpenExpr → OpenExpr
```

signature, trivial sub-theory, monomorphisms, epimorphisms, kernel of a homomorphism, composition of morphisms, quotient algebra, staged term language, induction principle, evaluation of terms, simplification of terms, equivelance of terms, printers, ...

Monoid: Multiple Theories, Same Constructions

```
theory Monoid {
  A  : type
  e  : A
  op : A ⟶ A ⟶ A
  lunit : {x : A} ⟶ op e x = x
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```
theory MonoidHom {
  M1, M2 : Monoid
  hom : M1.A ⟶ M2.A
  pres-e : hom (M1.e) = M2.e
  pres-op : (x y : M1.A) ⟶
      hom (M1.op x y) = M2.op (hom x) (hom y)
}
```

```
type MonoidExpr :=
  e    : MonoidExpr
  op   : MonoidExpr ⟶ MonoidExpr ⟶ MonoidExpr
```

```
theory Group {
  A  : type
  e  : A
  op : A ⟶ A ⟶ A
  inv : A ⟶ A
  lunit : {x : A} ⟶ op e x = x
  runit : {x : A} ⟶ op x e = x
  linverse : {x : A} ⟶ op x (inv x) == e
  rinverse : {x : A} ⟶ op (inv x) x == e
  assoc : {x y z : A} ⟶ op x (op y z) = op (op x y) z
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theory GroupHom {
  G1, G2 : Group
  hom : G1.A ⟶ G2.A
  pres-e : hom (G1.e) = G2.e
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      hom (G1.op x y) = G2.op (hom x) (hom y)
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Can we make use of this uniformity?

Universal Algebra

A theory:

$$\Gamma = (\mathcal{S}, \mathcal{F}, \mathcal{E})$$

- $\mathcal{S}$: a sort
- $\mathcal{F}$: set of function symbols
- $\mathcal{E}$: set of axioms

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- $\mathcal{S}$: a sort

- $\mathcal{F}$: set of function symbols

- $\mathcal{E}$: set of axioms

- A homomorphism between two Γ-algebras:
  - hom : $\mathcal{S}_1 \rightarrow \mathcal{S}_2$
  - For every `op` $\in \mathcal{F}$:

    $\texttt{hom (op}_1\texttt{ x}_1\texttt{ ... x}_n\texttt{) = op}_2\texttt{ (hom x}_1\texttt{) ... (hom x}_n\texttt{)}$

- The closed term language L induced by Γ is the set of:
  - All constants of Γ
  - For every `op` $\in \mathcal{F}$, with arity > 0:

    $\texttt{t}_{\texttt{op}}\ \texttt{t}_1 \cdots \texttt{t}_n$, such that $\texttt{t}_1 \cdots \texttt{t}_n$ are closed terms of L.

Redundancies in Libraries

Agda

Construction	Number of Occurrences
Signatures	7
Homomorphisms	7
Monomorphisms	7
Isomorphisms	7
Products	10
Products of Signatues	3
Term Language	3
Evaluation Function	3
Total	47

Lean

Construction	Number of Occurrences
Homomorphisms (Bundled)	3
Homomorphisms (Unbundled)	8
Products	22
Subtheory	5
Total	38

- $>$ 20 algebraic structures in each library.

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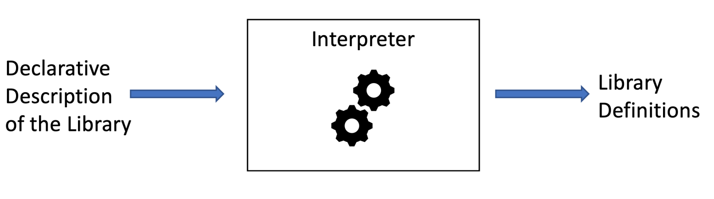
- $>$ 20 algebraic structures in each library.
- $>$ 200 algebraic structures in our library.
- $>$ 300 algebraic structures collected by Peter Jipsen.[1]

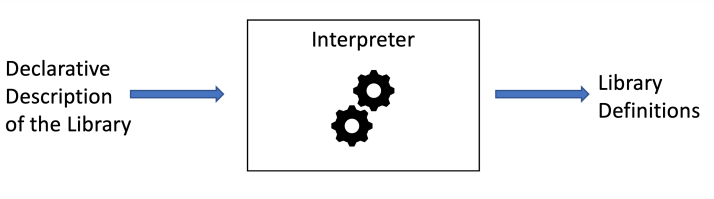
[1] source: `http://math.chapman.edu/~jipsen/structures/doku.php`

Can the abstractions and uniformity provided by universal algebra be captured by meta-programs that generate parts of algebra libraries?

Generative Approach to Library Building



Generative Approach to Library Building



- Inspiration: Haskell

```
data List a = Nil | Cons a (List a)
          deriving (Eq, Show, Ord, Read,
              -- by enabling some extensions
                  Functor, Generic, Data,
                  Foldable,Traversable, Lift)


data Point = Point { _x :: Double, _y :: Double }
makeLenses ''Point
```

Requirements

1. A small **language** to represent theories.

2. Some **meta programs** to manipulate these theories.

3. A **type checker** for the theories and constructions.

4. A large **library** of theories.

Tog: Language and TypeChecker

- Dependently typed language
    - Martin-Löf type theory.
- Experimental language, in the style of Agda

```
record Monoid (A : Set) : Set where
  constructor monoid
  field
   e  : A
   op : A -> A -> A
   lunit : {x : A} -> (op e x) == x
   runit : {x : A} -> (op x e) == x
   assoc : {x y z : A} ->
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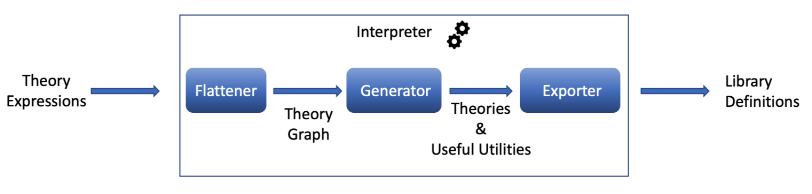
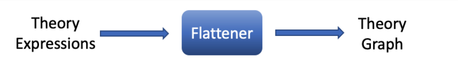
Approach: Three-Phase Interpreter



1. The Flattener

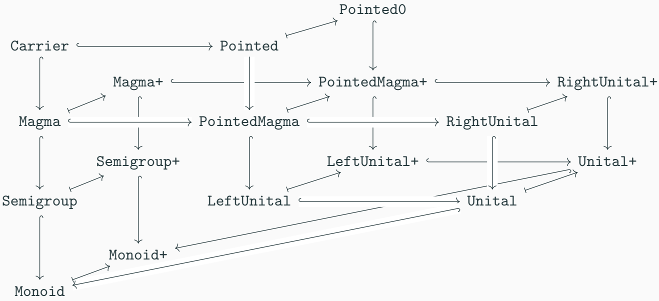


Theory Expressions

Flattener

Theory Graph

Theory Graph



Pointed0

Carrier

Pointed

Magma+

PointedMagma+

RightUnital+

Magma

PointedMagma

RightUnital

Semigroup+

LeftUnital+

Unital+

Semigroup

LeftUnital

Unital

Monoid+

Monoid

1. The Flattener: Combinators

Theory Expressions

1. Extension

```
Semigroup  = extend Magma {assoc: ...}
```

$$\begin{tikzcd}\texttt{Magma} \arrow[r, hook] & \texttt{Semigroup}\end{tikzcd}$$

1. The Flattener: Combinators

Theory Expressions

$$\begin{tikzcd}
\texttt{Magma} \arrow[r, hook] \arrow[d, mapsto] & \texttt{Semigroup} \\
\texttt{AdditiveMagma} &
\end{tikzcd}$$

1. Extension

```
Semigroup  = extend Magma {assoc: ...}
```

2. Rename

```
AdditiveMagma = rename Magma {op to +}
```

1. The Flattener: Combinators

Theory Expressions

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\texttt{Magma} \arrow[r, hook] \arrow[d, mapsto] \arrow[dr] & \texttt{Semigroup} \arrow[d, mapsto] \\
\texttt{AdditiveMagma} \arrow[r, hook] & \texttt{AdditiveSemigroup}
\end{tikzcd}$$

1. Extension

```
Semigroup  = extend Magma {assoc: ...}
```

2. Rename

```
AdditiveMagma = rename Magma {op to +}
```

3. Combine

```
AdditiveSemigroup =
  combine Semigroup {op to +} AdditiveMagma {}
```

1. The Flattener: Computing Pushouts

Pushouts are a 5-ary operations:

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d] & \Delta \\
\Phi &
\end{tikzcd}$$

- 3 theories.
- 2 arrows.

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```
combine AdditiveMonoid {} Group { ... } over Monoid
combine AdditiveMonoid {} MultMonoid {} over Carrier
```

$$\begin{tikzcd}\texttt{Monoid} \arrow[r, hook] \arrow[d, mapsto] & \texttt{Group} \\ \texttt{AdditiveMonoid} & \end{tikzcd}$$

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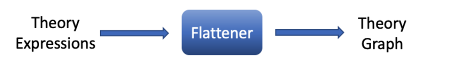
1. The Flattener

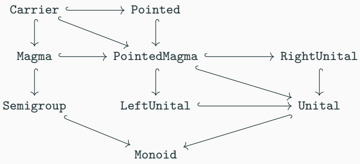


Theory Expressions

Flattener

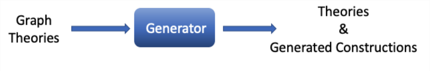
Theory Graph

```
Pointed = extend Carrier {e : A}
Magma   =
  extend Carrier {op : A -> A -> A}
Semigroup    =
  extend Magma {assoc: ...}
PointedMagma =
  combine Pointed {} Magma {} over Carrier
LeftUnital   =
  extend PointedMagma { lunit_e : ... }
RightUnital  =
  extend PointedMagma { runit_e : ... }
Unital = combine LeftUnital {} RightUnital {}
         over PointedMagma
Monoid = combine Unital {} Semigroup {} over Magma
```



Empty, Carrier, Pointed, UnaryOperation, PointedUnarySystem, FixedPoint, Magma, AdditiveMagma, MultMagma, PointedMagma, Involution, UnaryDistributes, UnaryAntiDistribution, IdempotentUnary, InvolutiveMagma, LeftInverseMagma, RightInverseMagma, IdempotentMagma, IdempotentAdditiveMagma, Pointed0Magma, PointedPlusMagma, AdditivePointedMagma, Pointed1Magma, PointedTimesMagma, MultPointedMagma, CommutativeMagma, CommutativeAdditiveMagma, CommutativePointedMagma, AntiAbsorbent, SteinerMagma, Squag, PointedSteinerMagma, Sloop, LeftDistributiveMagma, RightDistributiveMagma, Unital, LeftBiMagma, RightBiMagma, QuasiGroup, MoufangLaw, MoufangQuasiGroup, Loop, MoufangIdentity, MoufangLoop, Shelf, LeftBinaryInverse, RightBinaryInverse, BinaryInverse, Rack, Spindle, Quandle, RightSelfInverse, Semigroup, AdditiveSemigroup, CommutativeSemigroup, MultCommutativeSemigroup, CancellativeSemigroup, InvolutivePointedSemigroup, Band, MiddleAbsorption, MiddleCommute, RectangularBand, NormalBand, RightMonoid, LeftMonoid, PointedSemigroup, AdditivePointedSemigroup, AdditiveUnital, MultPointedSemigroup, Monoid, AdditiveMonoid, DoubleMonoid, CommutativeMonoid, CancellativeMonoid, CancellativeCommutativeMonoid, Zero, AdditiveCommutativeMonoid, BooleanGroup, InverseUnaryOperation, Inverse, PseudoInverse, PseudoInvolution, RegularSemigroup, QuasiInverse, Group, AdditiveGroup, CommutativeGroup, MultGroup, AbelianGroup, AbelianAdditiveGroup, RingoidSig, LeftRingoid, RightRingoid, Ringoid, NonassociativeRing, AndDeMorgan, OrDeMorgran, DualDeMorgan, AssociativeLeftRingoid, LeftPreSemiring, AssociativeRightRingoid, RightPreSemiring, PreSemiring, AssocPlusRingoid, AssocTimesRingoid, NearSemiring, NearRing, SemiRng, Rng, SemiRngWithUnit, Semiring, Ring, CommutativeRing, BooleanRing, IdempotentSemiRng, IdempotentSemiring, InvolutiveFixes, InvolutiveFixedPoint, InvolutiveRingoid, InvolutiveRing, JacobianIdentity, AntiCommutativeRing, LieRing, MeetSemilattice, MultMeetSemilattice, BoundedMeetSemilattice, JoinSemilattice, BoundedJoinSemilattice, DualSemilattices, LeftAbsorption, LeftAbsorptionOp, Absorption, Lattice, Modularity, ModularLattice, DistributiveLattice, BoundedJoinLattice, BoundedMeetLattice, BoundedLattice, BoundedModularLattice, BoundedDistributiveLattice,

2. The Generator

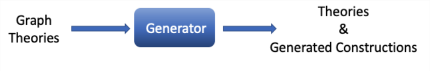


Graph Theories

Generator

Theories & Generated Constructions

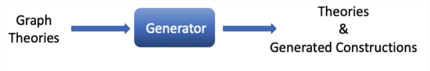
2. The Generator



- Uni-sorted equational theory: $\Gamma = (\mathcal{S}, \mathcal{F}, \mathcal{E})$

```
data EqTheory = EqTheory  {
  name         :: Name_   ,
  sort         :: Constr  ,  -- the carrier 𝒮
  funcTypes    :: [Constr],  -- function symbols ℱ
  axioms       :: [Constr],  -- equations ℰ
  waist        :: Int        -- the number of parameters
}
```

2. The Generator



- Uni-sorted equational theory: $\Gamma = (\mathcal{S}, \mathcal{F}, \mathcal{E})$

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  funcTypes    :: [Constr],  -- function symbols ℱ
  axioms       :: [Constr],  -- equations ℰ
  waist        :: Int        -- the number of parameters
}
```

- Instance of a theory:

```
type EqInstance = (Name_,[Binding],Expr)
```

Example:

$$\texttt{\{A : Set\}} \rightarrow \texttt{M : Monoid A}$$

Constructions for Free!

```
record Monoid (A : Set)
    : Set where
  e    : A
  op   : A -> A -> A
  lunit : ...
  runit : ...
  assoc : ...
```

```
record Hom {A1 A2 : Set}
       (M1 : Monoid A1) (M2 : Monoid A2)
       : Set where
  hom  : A1 -> A2
  pres-e  : hom (e M1) = e M2
  pres-op : {x1 x2 : A1} ->
    hom ((op M1) x1 x2) = (op M2) (hom x1) (hom x2)
```

```
homomorphism :: Eq.EqTheory -> Decl
homomorphism thry =
  let nm = "Hom"
     i1@(n1,b1,e1) = Eq.eqInstance thry (Just 1)
      i2@(n2,b2,e2) = Eq.eqInstance thry (Just 2)
      fnc    = homFunc thry i1 i2
      axioms = map (presAxiom thry i1 i2 fnc) (thry ^. Eq.funcTypes)
  in Record (mkName nm)
   (mkParams $ b1 ++ b2 ++
               map (\(n,e) -> Bind [mkArg n] e) [(n1,e1),(n2,e2)])}
   (RecordDeclDef setType (mkName $ nm ++ "C") (mkField $ fnc : axioms))
```

Constructions for Free!

```
record Monoid (A : Set)
      : Set where
  e   : A
  op  : A -> A -> A
  lunit : ...
  runit : ...
  assoc : ...
```

```
record Hom {A1 A2 : Set}
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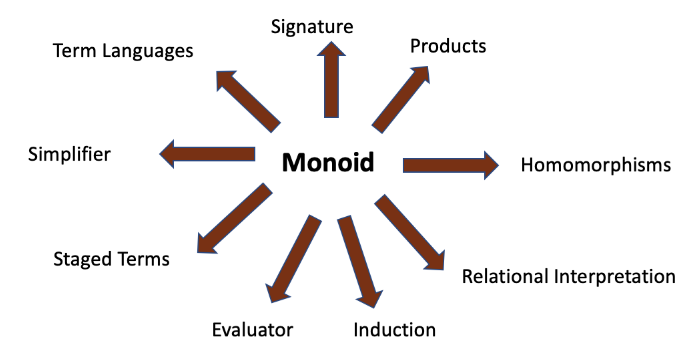
  pres-op : {x1 x2 : A1} ->
    hom ((op M1) x1 x2) = (op M2) (hom x1) (hom x2)
```

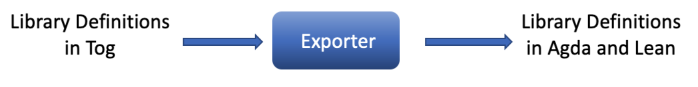
```
homFunc :: Eq.EqTheory -> Eq.EqInstance -> Eq.EqInstance -> Constr
homFunc thry i1 i2 =
  let carrier = thry ^. Eq.sort
  in Constr (mkName homFuncName) $
       Fun (Eq.projectConstr thry i1 carrier)
           (Eq.projectConstr thry i2 carrier)
```

Constructions for Free!

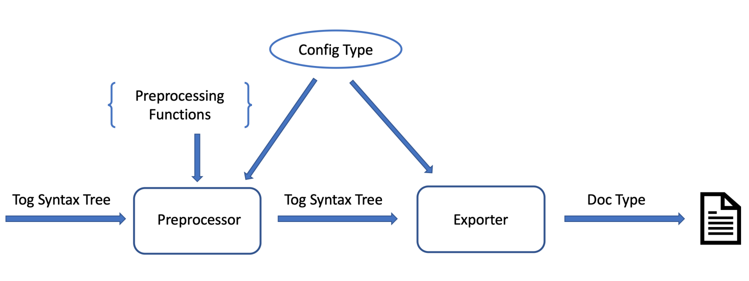


monomorphism, epimorphism, endomorphism, isomorphism, automorphism, kernel of a morphism, composition of morphisms, quotient algebra, sub-theory, trivial sub-theory, parse trees, equivalence of terms, ...

3. The Exporter


Library Definitions in Tog
Exporter
Library Definitions in Agda and Lean

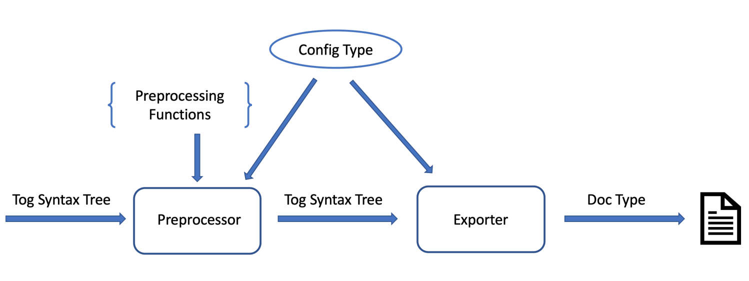
3. The Exporter



Preprocessor

- Universes
- Prelude definitions
- Non-linear pattern matching
- Functions as constructors
- Names misalignment

3. The Exporter



Exporter

```
class Export a where
  export :: Config -> a -> Doc
```

Results

Starting with **227** theory expressions:

- **5092** library definitions.
- **32,459** lines of code.
- Exported to **Lean**, **Agda** (flat and predicate style theories).

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Starting with **227** theory expressions:

- **5092** library definitions.
- **32,459** lines of code.
- Exported to **Lean**, **Agda** (flat and predicate style theories).
- Average time:

Flattener	5.17 s
Generator	2.7 s
Exporter	9.1 us
Type-checking	28 mins

Future Work

- Generalizing the approach to **generalized algebraic theories**.

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- Proof assistants as **program families**.
    - better understanding how design decisions affect theory presentations
    - staged exporter to multiple proof assistants

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- Generalizing the approach to **generalized algebraic theories**.
- Proof assistants as **program families**.
  - better understanding how design decisions affect theory presentations
  - staged exporter to multiple proof assistants
- a **DSL** for library development.

```
Monoid = combine Unital and Semigroup over Magma
         generate Homomorphism, OpenTerms, Simplifier
         using (waist=1,eq="=")
         export_to lean
```

Conclusion

Summary of Contributions:

- Highlighted the redundancy in libraries formalizing the algebraic hierarchy.
- Built a library of 227 theories describing the algebraic hierarchy using theory combinators.
- Compiled a list of structures that can be generated from theory presentations.
- Generated some of these constructions in Tog, a small implementation of a dependently typed language.
- Exported the library to Agda and Lean.

Preservation Axioms

```
record Monoid (A : Set)
    : Set where
  e  : A
  op : A —> A —> A
  lunit : ...
  runit : ...
  assoc : ...
```

```
record Hom {A1 A2 : Set}
      (M1 : Monoid A1) (M2 : Monoid A2)
      : Set where
  hom : A1 $—>$ A2
  pres—e : hom (e M1) = e M2
  pres—op : {x1 x2 : A1} —>
    hom ((op M1) x1 x2) =
      (op M2) (hom x1) (hom x2)
```

```
equation :: Eq.EqTheory -> Eq.EqInstance -> Eq.EqInstance ->
            Constr -> Constr -> Expr
equation thry i1 i2 hom constr =
  let (bind1,expr1) = Eq.applyProjConstr thry i1 constr Nothing
      (_    ,expr2) = Eq.applyProjConstr thry i2 constr Nothing
  in if bind1 == [] then Eq (lhs hom expr1) (rhs hom expr2)
     else Pi (Tel bind1) $ Eq (lhs hom expr1) (rhs hom expr2)
```

```
lhs :: Constr -> Expr -> Expr
lhs (Constr n _) fexpr =
    App [mkArg (n ^. name),Arg fexpr]
```

```
rhs :: Constr -> Expr -> Expr
rhs (Constr n _) fexpr =
    functor (n ^. name) fexpr
```

Sagemath:

```
class Monoid_class(Parent):
  def __init__(self, names):
    from sage.categories.monoids import Monoids
    category = Monoids().FinitelyGeneratedAsMagma()
    Parent.__init__(self, base=self, names=names, category=category)

-- sage.categories.monoids
class Monoids(CategoryWithAxiom)
  _base_category_class_and_axiom = (Semigroups, "Unital")
```

Isabelle:

```
locale monoid =
  fixes G (structure)
  assumes m_closed [intro, simp]:
         ⟦x ∈ carrier G; y ∈ carrier G⟧ ⇒ x ⊕ y  ∈ carrier G
      and m_assoc:
         ⟦x ∈ carrier G; y ∈ carrier G; z ∈ carrier G⟧
          ⇒ (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)
      and one_closed [intro, simp]: 1 ∈ carrier G
      and l_one [simp]: x ∈ carrier G ⇒ 1 ⊕ x = x
      and r_one [simp]: x ∈ carrier G ⇒ x ⊕ 1 = x
```

Combine renames

$$\begin{tikzcd}
\texttt{Carrier} \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] & \texttt{Pointed} \arrow[d, hook] \\
\texttt{Magma} \arrow[r, mapsto] \arrow[d, hook] \arrow[dr, dotted] & \texttt{AdditiveMagma=(A,+)} \arrow[d, dotted] \\
\texttt{Semigroup=(A,op,assoc)} \arrow[r, dotted] & \texttt{??}
\end{tikzcd}$$

```
AdditiveSemigroup =
  combine AdditiveMagma {} Semigroup {op to +}
  over Magma
```

Distinguished Arrows

$$\begin{tikzcd}
\texttt{Carrier} \arrow[r, hook, blue] \arrow[d, hook, blue] \arrow[dr, hook, blue] & \texttt{UnaryOperation} \arrow[r, hook, blue] \arrow[d, hook] & \texttt{Involution} \\
\texttt{Magma} \arrow[r, hook] & \texttt{InvolutiveMagmaSig} \arrow[d, hook, blue] & \\
& \texttt{UnaryAntiDistribution} &
\end{tikzcd}$$

Inductive Types in Agda and Lean

```
data MonTerm  (n : Nat) (A : Set) : Set where
  v : Fin n → MonTerm n A
  sing : A → MonTerm n A
  op : MonTerm n A → MonTerm n A → MonTerm n A
  e : MonTerm n A
```

```
inductive MonTerm  (n : Nat) (A : Type) : Type
  | v : Fin n → MonTerm
  | sing : A → MonTerm
  | op : MonTerm → MonTerm → MonTerm
  | e  : MonTerm
open MonTerm
```