MDS with Uncertainty: Sensor Network Localization (SNL)

Recall Classical MDS

Given d_{ij} pair distance $\forall (i,j)$ complete

? $y_i \in \mathbb{R}^k$ s.t.

$$\min_{Y\in\mathbb{R}^{n\times k}} \sum_{i,j} \left(\|y_i - y_j\|^2 - d_{ij}^2\right)^2$$

$$\Leftrightarrow \min_{Y} \|YY^T - B\|^2 \qquad B = -\frac{1}{2} H D H^T,\quad D = [d_{ij}^2]$$

$\Leftrightarrow$ eigen-decomp(B)

新的特点: SNL

$G = (V, E)$

V: sensor

① $(i,j) \in E$ iff d_{ij}

incomplete

② noise: $\tilde{d}_{ij} = d_{ij} + \epsilon_{ij}$ noise

③ anchor point $x_i = a_i$

partial

目标:

$$\|y_i - y_j\|^2 = d_{ij}^2 \qquad (i,j) \in E$$

$$\|a_i - y_j\|^2 = d_{ij}^2$$

? $\min\limits_{Y} \sum\limits_{(i,j)\in E} \left(\|y_i - y_j\|^2 - d_{ij}^2\right)^2$ $\Big\{$ gradient method, nonlinear opt.

不能 Eigen-decomp

练习: $\|y_i - y_j\|^2 = d_{ij}^2$ Quadratic equality

SD Relax $\Rightarrow$ "Linear"

$y_i \in \mathbb{R}^k$

$Y = [y_1 \cdots y_n]^{k \times n}$

$$\|y_i - y_j\|^2 = (y_i - y_j)^T (y_i - y_j)$$

$$= (e_i - e_j)^T [Y^T Y]^{n\times n} (e_i - e_j)$$

$e_i = (0 \cdots 0 \ \overset{i\text{-th}}{1} \ 0 \cdots 0)$

$y_i = Y \cdot e_i$

$y_i - y_j = Y(e_i - e_j)$

$$= (e_i - e_j)(e_i - e_j)^T \bullet [Y^T Y]$$

$X = Y^T Y \succeq 0$

$$= (e_i - e_j)(e_i - e_j)^T \bullet X$$

LMI. $X \succeq 0$ SD

Relaxation

$$X = Y^T Y \Longrightarrow X \succeq Y^T Y. \quad (X - Y^T Y \succeq 0)$$

$$\Longleftrightarrow \begin{bmatrix} I_k & Y \\ Y^T & X \end{bmatrix} \succeq 0, \quad X \succeq 0$$

Linear Matrix Inequality

Lemma $\|y_i - y_j\|^2 = d_{ij}^2, \quad (i,j) \in E$

SDR.

LMI

$$\begin{cases} Z = \begin{bmatrix} I & Y \\ Y^T & X \end{bmatrix} \succeq 0 \\ Z_{1:k,1:k} = I \\ (0; e_i - e_j)(0; e_i - e_j)^T \bullet Z = d_{ij}^2 \quad (i,j) \in E \end{cases}$$

Note:

① $d_{ij}^2(1-\epsilon_{ij}) \le \|y_i - y_j\|^2 \le d_{ij}^2(1+\epsilon_{ij}) \Longrightarrow$ LMI noise

② anchor point. $y_i = a_i, \ \|a_i - y_j\|^2 = d_{ij}^2 \Rightarrow (a_i; e_j)(a_i; e_j)^T \bullet Z = \tilde{d}_{ij}^2$

SPP approach → MDS

Matlab   Protein 3-D structure Reconstruction

CMDS   Schoenberg

SDP-MDS   Exact Recovery ?   Yinyu Ye group.

P. Biswas.

A. So.

Recall SPP

(SPP)   min $C \bullet X$ $C, X \in \mathbb{R}^{n\times n}$

s.t. $A_i \bullet X = b_i$ $i = 1, \dots, m$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

$X \succeq 0$

(SDD)   max $-b^T y$ $y \in \mathbb{R}^m$, $S \in \mathbb{R}^{n\times n}$

s.t. $S = C - \sum_{i=1}^{m} A_i y_i \succeq 0$

$F_p = \{X \succeq 0 : A_i X = b_i\}$

$F_D = \{(y, S) : S = C - \sum A_i y_i \succeq 0\}$

primal obj: $C \bullet X$ dual obj $b^T y$

Weak Duality $F_p \neq \phi$, $F_D \neq \phi$

$\Rightarrow \underbrace{C \bullet X - b^T y}_{\text{duality gap}} \geq 0$ $\forall X \in F_p$, $\forall (y, S) \in F_d$

Strong Duality of SDP

(1) $F_p \neq \phi$, $F_d \neq \phi$

(2) F_p or F_d has an interior solution

$\Rightarrow$ $X^* \in F_p$, $(y^*, S^*) \in F_D$ is optimal soln.

iff $C \bullet X^* = b^T y^*$ duality gap is zero

$X^* S^* = 0$. Complementary

存在内点 Soln

X^*, $\underline{(y^*, S^*)}$ $X^* \cdot S^* = 0$ opt.

"witness" primal-dual pair

(*) $\operatorname{rank}(X^*) + \operatorname{rank}(S^*) \leq n$

SDP-MDS.

$$Z = \begin{bmatrix} I_k & Y \\ Y^T & X \end{bmatrix} \succeq 0$$

假设 $d_{ij} = \|x_i - x_j\|$ $x_i \in R^k$

$$\exists \; Z^* = \begin{bmatrix} I_k & Y^* \\ Y^{*T} & X^* \end{bmatrix} \succeq 0$$ interior point soln

$$\left.\begin{aligned} &\operatorname{rank}(Z^*) \geq k \\ &\operatorname{rank}(X^*) + \operatorname{rank}(S^*) \leq n \end{aligned}\right\} \Rightarrow \operatorname{rank}(Z^*) + \operatorname{rank}(S^*) \leq k + n$$

$\} \Rightarrow \operatorname{rank}(Z^*) \geq k$

$\operatorname{rank}(S^*) \leq n$

$X^* = Y^T Y \quad Y \in R^{k \times n}$

$\Rightarrow \quad rank(Z^*) = k \quad (\geq k)$ minimal rank

代数 $\quad rank(S^*) = n \quad (\leq n)$ maximal rank.

几何: "Universal Rigidity (UR)"

there is a unique embedding $y_i \in R^k \hookrightarrow R^l \quad l \geq k$

$(\underbrace{y_i}_{k}, \underbrace{0, \cdots 0}_{l-k})$ s.t. $\|y_i - y_j\| = d_{ij}, (i,j) \in E$

minimal dim embedding $k : R^k$

Schoenberg '1938 : G is complete $\xRightarrow{\text{spectrum}}$ UR.

G is incomplete

[So-Ye '2007] $\quad$ G general.

UR $\Leftrightarrow$ SDP maximal rank sol'n

rank$(S^*) = n$. rank$(Z^*) = k$

<u>Theorem</u> Equivalent statements

- (几何) G is UR or has a unique embedding in R^k
- (代数) SDP has a max-rank feasible sol'n rank$(Z^*) = k$ (rank$(S^*) = n$)
- $X^* = Y^T Y$ or trace$(X - Y^T Y) = 0$

  eigen decomp of X^*.

<u>UR</u> vs. polynomial $(n, k, \log(\frac{1}{\varepsilon}))$

Ye, ICCM '2010. ; Fields '2011

Maximum Variance Unfolding (Manifold Learning)

$$X = Y^TY \Rightarrow X \succeq Y^TY$$

MVU

$K = Y^TY$ 　　变量　　 $d_{ij}^2 = K_{ii} + K_{jj} - 2K_{ij}$

$K_{ij} = \langle Y_i, Y_j \rangle$

"SDP"

$$\max \ \mathrm{trace}(K)$$

$$\text{s.t. } K_{ii} + K_{jj} - 2K_{ij} = d_{ij}^2$$

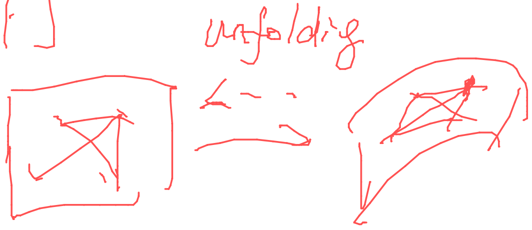
$$\sum_j K_{ij} = 0$$

$$K \succeq 0$$

$K \Rightarrow Y^TY$

Why max tr(K)? 　　不能 work.

[So-Ye 2007]



SDP.

k-manifold

(k+1)-lateration graph

"Unfold manifold"