# Extended Locally Linear Embedding

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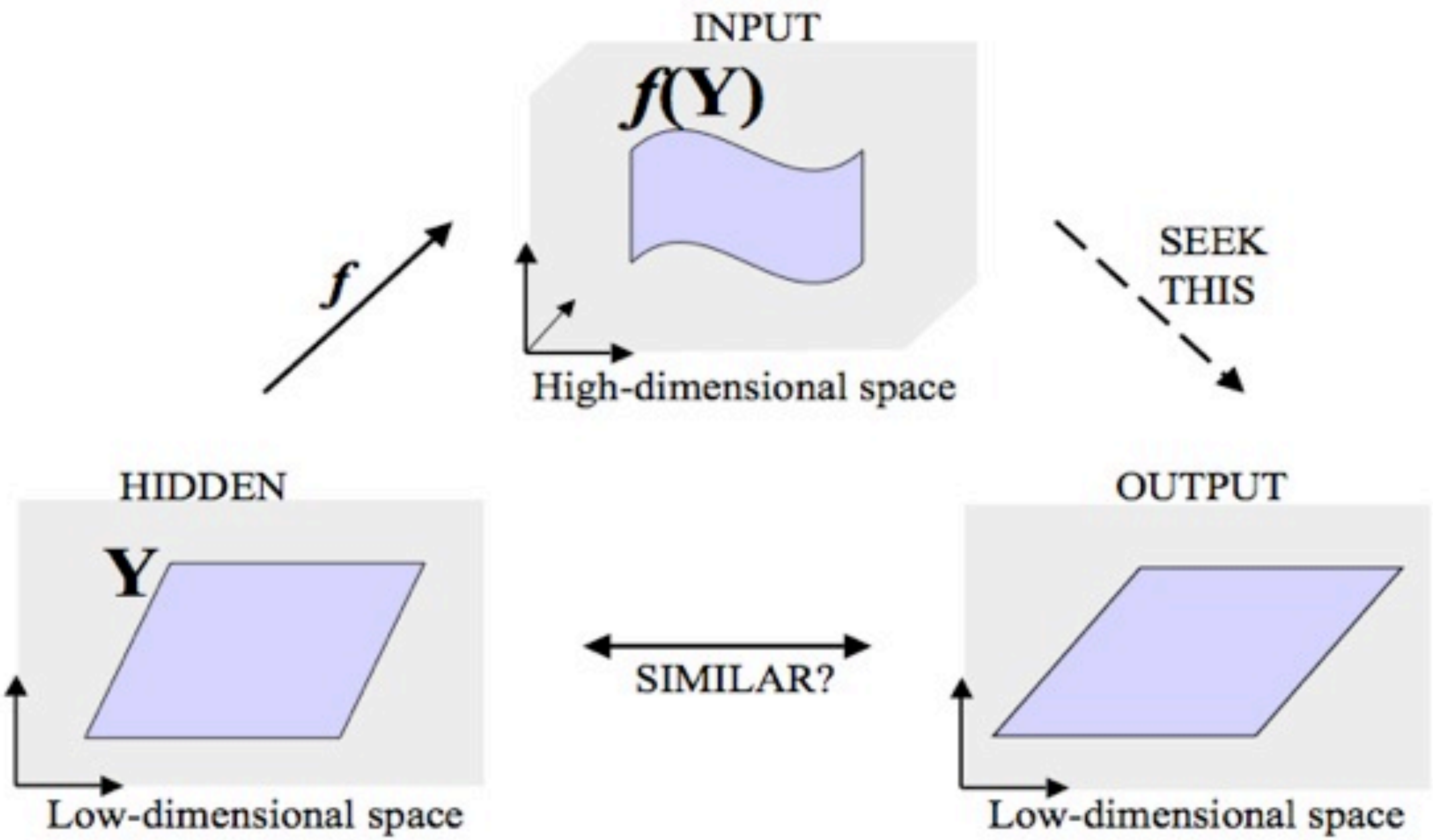
2014

# Generative Models in Manifold Learning



# Spectral Geometric Embedding

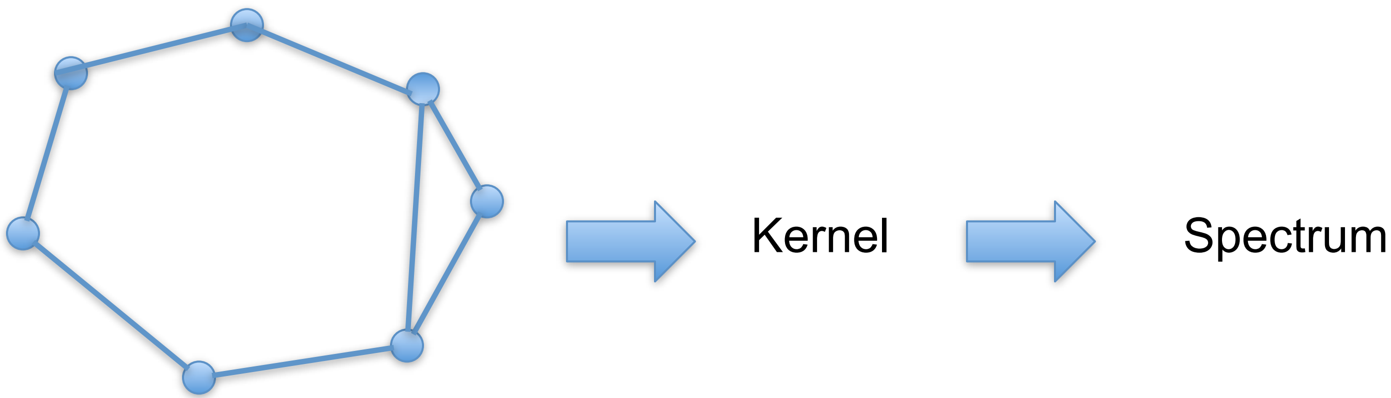
Given $x_1, \ldots, x_n \in \mathcal{M} \subset \mathbb{R}^N$,
Find $y_1, \ldots, y_n \in \mathbb{R}^d$ where $d << N$

- ISOMAP (Tenenbaum, et al, 00)
- LLE (Roweis, Saul, 00)
- Laplacian Eigenmaps (Belkin, Niyogi, 01)
- Local Tangent Space Alignment (Zhang, Zha, 02)
- Hessian Eigenmaps (Donoho, Grimes, 02)
- Diffusion Maps (Coifman, Lafon, et al, 04)

Related: Kernel PCA (Schoelkopf, et al, 98)

# Meta-Algorithm

- Construct a neighborhood graph
- Construct a positive semi-definite kernel
- Find the spectrum decomposition



# Recall: ISOMAP

1. Construct Neighborhood Graph.
2. Find shortest path (geodesic) distances.

$D_{ij}$ is $n \times n$

3. Embed using Multidimensional Scaling.

# Recall: LLE

- Construct a neighborhood Graph G=(V,E)
- Solve weights

$$\min_{\sum_j w_{ij}=1} \|X_i - \sum_{j\in\mathcal{N}(i)} w_{ij}\bar{X}_j\|^2, \quad \bar{X}_j = X_j - X_i.$$

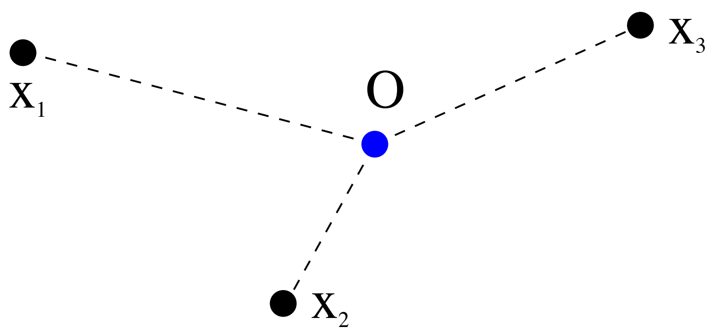
- Compute Embedding

$$\min_Y \sum_{i=1}^n \|Y_i - \sum_{j=1}^n W_{ij}Y_j\|^2 = \mathrm{trace}((I-W)Y^TY(I-W)^T).$$

$$W_{ij}^{n\times n} = \begin{cases} w_{ij} & j\in\mathcal{N}(i), \\ 0 & \text{other's.} \end{cases}$$

This is equivalent to find smallest eigenvectors of $K = (I-W)^T(I-W)$.

# Laplacian and LLE




$$\sum w_i x_i = 0$$

$$\sum w_i = 1$$

Hessian $H$. Taylor expansion :

$$f(x_i) = f(0) + x_i^t \nabla f + \frac{1}{2} x_i^t H x_i + o(\|x_i\|^2)$$

$$(I - W) f(0) = f(0) - \sum w_i f(x_i) \approx f(0) - \sum w_i f(0) - \sum_i w_i x_i^t \nabla f - \frac{1}{2} \sum_i x_i^t H x_i =$$

$$= -\frac{1}{2} \sum_i x_i^t H x_i \approx -trH = \Delta f$$

# Discrete Laplacian

Find $y_1, \ldots, y_n \in R$

$$\min \sum_{i,j} (y_i - y_j)^2 W_{ij}$$

Tries to preserve locality

# A Fundamental Identity

But

$$\frac{1}{2}\sum_{i,j}(y_i - y_j)^2 W_{ij} = \mathbf{y}^T L \mathbf{y}$$

$$\sum_{i,j}(y_i - y_j)^2 W_{ij} = \sum_{i,j}(y_i^2 + y_j^2 - 2y_i y_j) W_{ij}$$

$$= \sum_i y_i^2 D_{ii} + \sum_j y_j^2 D_{jj} - 2\sum_{i,j} y_i y_j W_{ij}$$

$$= 2\mathbf{y}^T L \mathbf{y}$$

# Embedding of Unnormalized Laplacian Eigenmap

$$\lambda = 0 \rightarrow \mathbf{y} = \mathbf{1}$$

$$\min_{\mathbf{y}^T\mathbf{1}=0} \mathbf{y}^T L \mathbf{y}$$

Let $Y = [\mathbf{y}_1 \mathbf{y}_2 \ldots \mathbf{y}_m]$

$$\sum_{i,j} ||Y_i - Y_j||^2 W_{ij} = \text{trace}(Y^T L Y)$$

subject to $Y^T Y = I$.

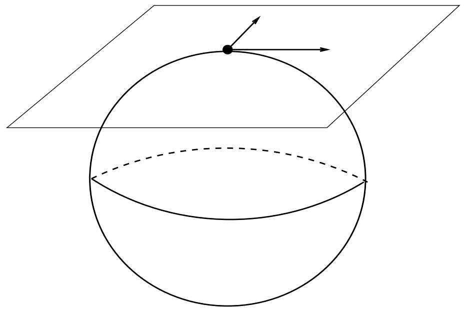
Use eigenvectors of $L$ to embed.

# Manifold Laplacian

Recall ordinary Laplacian in $\mathbb{R}^k$
This maps

$$f(x_1, \ldots, x_k) \to \left( -\sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i^2} \right)$$

Manifold Laplacian is the same on the tangent space.



# On the Manifold

smooth map $f : \mathcal{M} \to R$

$$\int_{\mathcal{M}} \|\nabla_{\mathcal{M}} f\|^2 \approx \sum_{i \sim j} W_{ij}(f_i - f_j)^2$$

Recall standard gradient in $\mathbb{R}^k$ of $f(z_1, \ldots, z_k)$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial z_1} \\ \frac{\partial f}{\partial z_2} \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial z_k} \end{bmatrix}$$

# Stokes Theorem

A Basic Fact

$$\int_{\mathcal{M}} \|\nabla_{\mathcal{M}} f\|^2 = \int f \cdot \Delta_{\mathcal{M}} f$$

This is like

$$\sum_{i,j} W_{ij}(f_i - f_j)^2 = \mathbf{f}^T \mathbf{L} \mathbf{f}$$

where

$\Delta_{\mathcal{M}} f$ is the manifold Laplacian

# Manifold Laplacian Eigenvectors

Eigensystem

$$\Delta_{\mathcal{M}} f = \lambda_i \phi_i$$

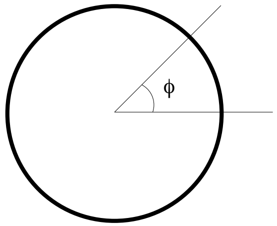
$\lambda_i \geq 0$ and $\lambda_i \to \infty$

$\{\phi_i\}$ form an orthonormal basis for $L^2(\mathcal{M})$

$$\int \|\nabla_{\mathcal{M}} \phi_i\|^2 = \lambda_i$$

Manifold Laplacian is non-compact!

# Example: Circle




$$-\frac{d^2u}{dt^2} = \lambda u \text{ where } u(0) = u(2\pi)$$

Eigenvalues are

$$\lambda_n = n^2$$

Eigenfunctions are

$$\sin(nt), \cos(nt)$$

Spherical Harmonics in high-D sphere!

# Spectral Growth

$$\lambda_1 \leq \lambda_2 \ldots \leq \lambda_j \leq \ldots$$

Then

$$A + \frac{2}{d}\log(j) \leq \log(\lambda_j) \leq B + \frac{2}{d}\log(j+1)$$

Example: on $S^1$

$$\lambda_j = j^2 \implies \log(\lambda_j) = \frac{2}{1}\log(j)$$

(Li and Yau; Weyl’s asymptotics)

# Solution of Heat Equations

Heat equation in $\mathbb{R}^n$:

$u(x,t)$ – heat distribution at time $t$.
$u(x,0) = f(x)$ – initial distribution. $x \in \mathbb{R}^n, t \in \mathbb{R}$.

$$\Delta_{\mathbb{R}^n} u(x,t) = \frac{du}{dt}(x,t)$$

Solution – convolution with the heat kernel:

$$u(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{\|x-y\|^2}{4t}} dy$$

# Discretization of Heat Eq.

Functional approximation:

Taking limit as $t \to 0$ and writing the derivative:

$$\Delta_{\mathbb{R}^n} f(x) = \frac{d}{dt}\left[(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{\|x-y\|^2}{4t}} dy\right]_0$$

$$\Delta_{\mathbb{R}^n} f(x) \approx -\frac{1}{t}(4\pi t)^{-\frac{n}{2}} \left( f(x) - \int_{\mathbb{R}^n} f(y) e^{-\frac{\|x-y\|^2}{4t}} dy \right)$$

Empirical approximation:

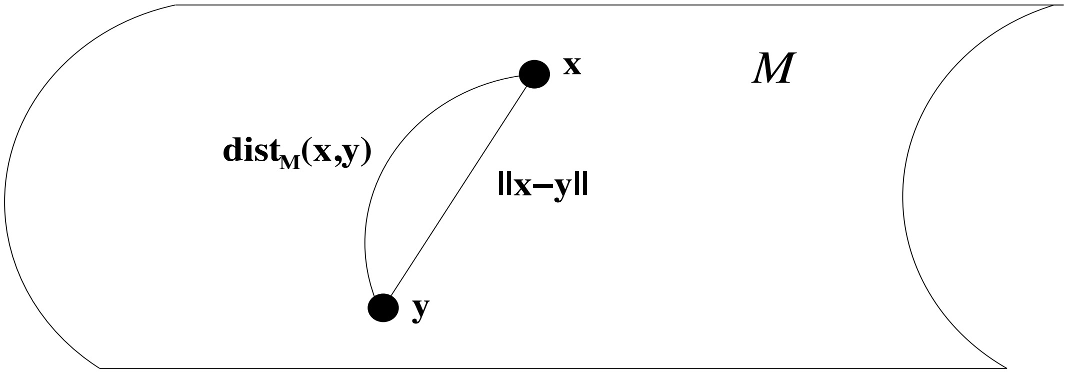
Integral can be estimated from empirical data.

$$\Delta_{\mathbb{R}^n} f(x) \approx -\frac{1}{t}(4\pi t)^{-\frac{n}{2}} \left( f(x) - \sum_{x_i} f(x_i) e^{-\frac{\|x-x_i\|^2}{4t}} \right)$$

# Some Difficulties for Manifolds

Some difficulties arise for manifolds:

- Do not know distances.
- Do not know the heat kernel.




Careful analysis needed.

# The Heat Kernel Approximation

- $H_t(x,y) = \sum_i e^{-\lambda_i t}\phi_i(x)\phi_i(y)$
- in $\mathbb{R}^d$, closed form expression

$$H_t(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|x-y\|^2}{4t}}$$

- Goodness of approximation depends on the gap

$$\left| H_t(x,y) - \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|x-y\|^2}{4t}} \right|$$

- $H_t$ is a Mercer kernel intrinsically defined on manifold. Leads to SVMs on manifolds.

# Pointwise Convergence

$$f : \mathcal{M} \to \mathbb{R} \quad x \in \mathcal{M} \quad x_1, \ldots, x_n \in \mathcal{M}$$

Graph Laplacian:

$$L_n^t(f)(x) = f(x) \sum_j e^{-\frac{\|x - x_j\|^2}{t}} - \sum_j f(x_j) e^{-\frac{\|x - x_j\|^2}{t}}$$

**Theorem** [pointwise convergence] $t_n = n^{-\frac{1}{k+2+\alpha}}$

$$\lim_{n \to \infty} \frac{(4\pi t_n)^{-\frac{k+2}{2}}}{n} L_n^{t_n} f(x) = \Delta_{\mathcal{M}} f(x)$$

Belkin 03, Lafon Coifman 04, Belkin Niyogi 05, Hein et al 05

# Convergence of Eigenfunctions

**Theorem** [convergence of eigenfunctions]

$$\lim_{t\to 0, n\to\infty} Eig[L_n^{t_n}] \to Eig[\Delta_{\mathcal{M}}]$$

Belkin Niyogi 06

# Laplacian Eigenmaps (I) [Belkin-Niyogi]

**Step 1** [*Constructing the Graph*]

$$e_{ij} = 1 \Leftrightarrow \mathbf{x}_i \text{ "close to" } \mathbf{x}_j$$

1. $\epsilon$-neighborhoods. [*parameter* $\epsilon \in \mathbb{R}$] Nodes $i$ and $j$ are connected by an edge if

$$||\mathbf{x}_i - \mathbf{x}_j||^2 < \epsilon$$

2. $n$ nearest neighbors. [*parameter* $n \in \mathbb{N}$] Nodes $i$ and $j$ are connected by an edge if $i$ is among $n$ nearest neighbors of $j$ or $j$ is among $n$ nearest neighbors of $i$.

# Laplacian Eigenmaps (II)

**Step 2.** [*Choosing the weights*].

1. Heat kernel. [*parameter* $t \in \mathbb{R}$]. If nodes $i$ and $j$ are connected, put

$$W_{ij} = e^{-\frac{||\mathbf{x}_i - \mathbf{x}_j||^2}{t}}$$

2. Simple-minded. [*No parameters*]. $W_{ij} = 1$ if and only if vertices $i$ and $j$ are connected by an edge.

# Laplacian Eigenmaps (III)

**Step 3.** [*Eigenmaps*] Compute eigenvalues and eigenvectors for the generalized eigenvector problem:

$$Lf = \lambda D f$$

$D$ is diagonal matrix where

$$D_{ii} = \sum_j W_{ij}$$

$$L = D - W$$

Let $\mathbf{f}_0, \ldots, \mathbf{f}_{k-1}$ be eigenvectors.

Leave out the eigenvector $\mathbf{f}_0$ and use the next $m$ lowest eigenvectors for embedding in an $m$-dimensional Euclidean space.

# Connection to Markov Chain

- $L = D - W$
- $P = I - D^{-1}L = D^{-1}W$ is a markov matrix
- $v$ is generalized eigenvector of $L$: $L\,v = \lambda\, D\, v$
- $v$ is also a right eigenvector of $P$ with eigenvalue $1-\lambda$
- $P$ is lumpable iff v is piece-wise constant
- So Laplacian eigenmaps have Markov Chain interpretations (Diffusion Map)

# Another choice of eigenmaps

- Normalized positive semi-definite Laplacian

$$L_n = D^{-1/2}(D - W)D^{-1/2} = I - D^{-1/2}WD^{-1/2}$$

- $\phi_i$ is an eigenvector of $L_n$ with eigenvalue $\lambda_i$
- Laplacian eigenmap/Diffusion map:

$$Y = \begin{pmatrix} \lambda_1^{1/2}\phi_1 & \lambda_2^{1/2}\phi_2 & \ldots & \lambda_d^{1/2}\phi_d \end{pmatrix}$$

- $D^{1/2} v_i = \phi_i$
- Why not this eigenmap? Which eigenvector shall we use?

# Generalized Heat Kernels

- Find Gaussian kernel $K_\varepsilon(x,y) = \exp\left(-\frac{\|x-y\|^2}{\varepsilon^2}\right)$
- Normalize kernel

$$K^{(\alpha)}(x,y) = \frac{K_\varepsilon(x,y)}{p^\alpha(x)p^\alpha(y)} \quad \textit{where} \quad p(x) = \int K_\varepsilon(x,y)d\mu(y)$$

- Renormalized kernel

$$A_\varepsilon(x,y) = \frac{K^{(\alpha)}(x,y)}{\sqrt{d^{(\alpha)}(x)}\sqrt{d^{(\alpha)}(y)}} \quad \textit{where} \quad d^{(\alpha)}(x) = \int K^{(\alpha)}(x,y)d\mu(y)$$

  - α=1, Laplacian-Beltrami operator, separate geometry from density
  - α=0, classical normalized graph Laplacian

# General Diffusion Map

- P.S.D. Radial basis kernel $$K_\varepsilon(x,y) = h\left(\frac{\|x-y\|^2}{\varepsilon^2}\right)$$
- Normalize kernel

$$K^{(\alpha)}(x,y) = \frac{K_\varepsilon(x,y)}{p^\alpha(x)p^\alpha(y)} \quad where \quad p(x) = \int K_\varepsilon(x,y)d\mu(y)$$

- Markov kernel

$$a_\varepsilon^{(\alpha)}(x,y) = \frac{K^{(\alpha)}(x,y)}{d^{(\alpha)}(x)} \quad where \quad d^{(\alpha)}(x) = \int K^{(\alpha)}(x,y)d\mu(y)$$

- Diffusion Operator:

$$A_\varepsilon^{(\alpha)} f(x) = \int a_\varepsilon^{(\alpha)}(x,y) f(y) p(y) dy, \qquad p(x) = \frac{\exp(-U(x))}{Z}$$

$$\Delta_\varepsilon^{(\alpha)} = \frac{I - A_\varepsilon^{(\alpha)}}{\varepsilon}$$

# Convergence of Diffusion Map [Coifman-Lafon 2005]

- Uniform sampling: Laplacian eigenmap converges to Laplacian-Beltrami operators [Belkin-Niyogi]
- Nonuniform sampling with $p(x)$
  - α=1: $\Delta_{\varepsilon}^{(1)} = \frac{I - A_{\varepsilon}^{(1)}}{\varepsilon} = \Delta_0 + O(\varepsilon^{1/2})$ where $\Delta_0$ is Laplacian-Beltrami operator on Riemannian manifolds
  - α=1/2: backward Fokkar-Planck operator
  - α=0: classical normalized graph laplacian

# Convergence Theorem [Coifman-Lafon 2006]

**Theorem 7.1.** Let $\mathcal{M} \in \mathbb{R}^p$ be a compact smooth submanifold, $q(x)$ be a probability density on $\mathcal{M}$, and $\Delta_{\mathcal{M}}$ be the Laplacian-Beltrami operator on $\mathcal{M}$.

(67)
$$\lim_{t \to 0} L_{t,\alpha} = \frac{\Delta_{\mathcal{M}}(f q^{1-\alpha})}{q^{1-\alpha}} - \frac{\Delta_{\mathcal{M}}(q^{1-\alpha}))}{q^{1-\alpha}}.$$

This suggests that

- for $\alpha = 1$, it converges to the Laplacian-Beltrami operator $\lim_{t\to 0} L_{t,1} = \Delta_{\mathcal{M}}$;
- for $\alpha = 1/2$, it converges to a Schrödinger operator whose conjugation leads to a forward Fokker-Planck equation;
- for $\alpha = 0$, it is the normalized graph Laplacian.

# Hessian LLE

- Laplacian LLE

$$f^T L f = \sum_{i \geq j} w_{ij}(f_i - f_j)^2 \geq 0 \sim \int \|\nabla_M f\|^2 = \int (\mathrm{trace}(f^T \mathcal{H} f))^2$$

where $\mathcal{H} = [\partial^2/\partial_i \partial_j] \in \mathbb{R}^{d \times d}$ is the Hessian matrix.

- Hessian LLE

$$\min \int \|\mathcal{H} f\|^2, \quad \|f\| = 1$$

- Laplacian kernel: const + linear + bilinear
- Hessian kernel: const + linear functions

Note that: $\Delta(f) = trace\big(H(f)\big)$

# Hessian LLE Algorithm (I)

**Algorithm 3:** Hessian LLE Algorithm

**Input**: A weighted undirected graph $G = (V, E, d)$ such that

$V = \{x_i \in \mathbb{R}^p : i = 1, \dots, n\}$

$E = \{(i, j) :$ if $j$ is a neighbor of $i$, i.e. $j \in \mathcal{N}_i\}$, e.g. $k$-nearest neighbors

**Output**: Euclidean $k$-dimensional coordinates $Y = [y_i] \in \mathbb{R}^{k \times n}$ of data.

***Step 1***: Compute local PCA on neighborhood of $x_i$, for,

$$\tilde{X}^{(i)} = [x_{i_1} - \mu_i, ..., x_{i_k} - \mu_i]^{p \times k} = \tilde{U}^{(i)} \tilde{\Sigma} (\tilde{V}^{(i)})^T, \quad x_{i_j} \in \mathcal{N}(x_i),$$

where $\mu_i = \sum_{j=1}^{k} x_{i_j} = \frac{1}{k} X_i \mathbf{1}$, $\tilde{U}^{(i)} = [\tilde{U}_1^{(i)}, ..., \tilde{U}_k^{(i)}]$ is an approximate tangent space at $x_i$;

Continued...

# Hessian LLE Algorithm (II)

4 ***Step 2***: Hessian estimation, assumed $d$-dimension: define

$$M = [1, \tilde{V}_1, ..., \tilde{V}_k, \tilde{V}_1\tilde{V}_2, ..., \tilde{V}_{d-1}\tilde{V}_d] \in \mathbb{R}^{k\times(1+d+\binom{d}{2})}$$

where $\tilde{V}_i\tilde{V}_j = [\tilde{V}_{ik}\tilde{V}_{jk}]^T \in \mathbb{R}^k$ denotes the elementwise product (Hadamard product) between vector $\tilde{V}_i$ and $\tilde{V}_j$. Now we perform a Gram-Schmidt Orthogonalization procedure on $M$, get

$$\tilde{M} = [1, \hat{v}_1, ..., \hat{v}_k, \hat{w}_1, \hat{w}_2, ..., \hat{w}_{\binom{d}{2}-1}] \in \mathbb{R}^{k\times(1+d+\binom{d}{2})}$$

Define Hessian by

$$[H^{(i)}]^T = [last \quad \binom{d}{2} \quad columns \quad of \quad \tilde{M}]_{k\times\binom{d}{2}}$$

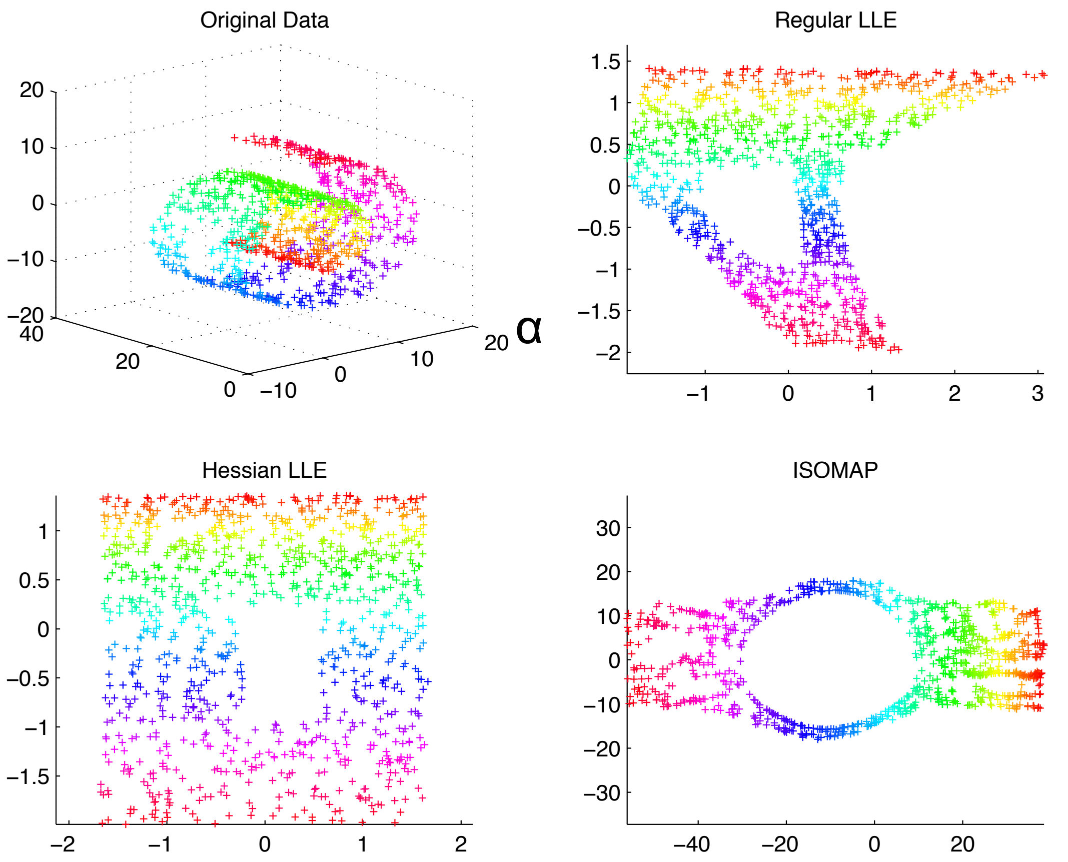
as the first $d+1$ columns of $\tilde{M}$ consists an orthonormal basis for the kernel of Hessian.

5 ***Step 3***: Define

$$K = \sum_{i=1}^{n} S^{(i)} H^{(i)T} H^{(i)} S^{(i)T} \in \mathbb{R}^{n\times n}, \quad [x_1, .., x_n]S^{(i)} = [x_{i_1}, ..., x_{i_k}],$$

find smallest $d+1$ eigenvectors of $K$ and drop the smallest eigenvector, the remaining $d$ eigenvectors will give rise to a $d$-embedding.

# Comparisons on Swiss Roll with holes



# Two Assumptions on ISOMAP

**(ISO1)** *Isometry.* The mapping $\psi$ preserves geodesic distances. That is, define a distance between two points $m$ and $m'$ on the manifold according to the distance travelled by a bug walking along the manifold $M$ according to the shortest path between $m$ and $m'$. Then the isometry assumption says that

$$G(m, m') = |\theta - \theta'|, \qquad \forall m \leftrightarrow \theta, m' \leftrightarrow \theta',$$

where $|\cdot|$ denotes Euclidean distance in $\mathbb{R}^d$.

**(ISO2)** *Convexity.* The parameter space $\Theta$ is a convex subset of $\mathbb{R}^d$. That is, if $\theta, \theta'$ is a pair of points in $\Theta$, then the entire line segment $\{(1-t)\theta + t\theta' : t \in (0,1)\}$ lies in $\Theta$.

Convexity is hard to meet: consider two balls in an image which never intersect, whose center coordinate space $(x_1,y_1,x_2,y_2)$ must have a hole.

# Relaxations (Donoho-Grimes'2003)

**(LocISO1)** *Local Isometry.* In a small enough neighborhood of each point $m$, geodesic distances to nearby points $m'$ in $M$ are identical to Euclidean distances between the corresponding parameter points $\theta$ and $\theta'$.

**(LocISO2)** *Connectedness.* The parameter space $\Theta$ is a open connected subset of $\mathbb{R}^d$.

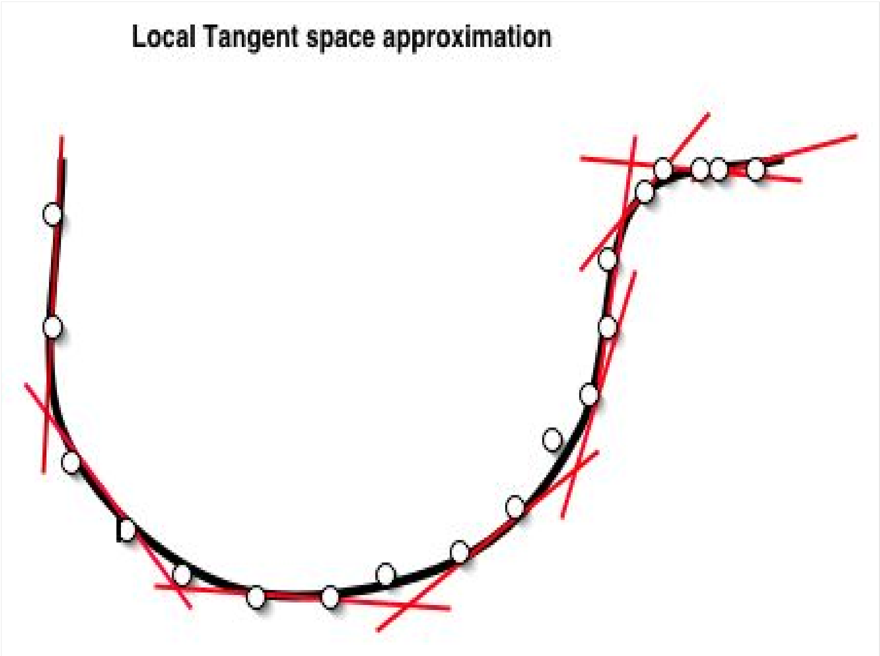
# Convergence of Hessian LLE (Donoho-Grimes)

**Theorem 1** *Suppose $M = \psi(\Theta)$ where $\Theta$ is an open connected subset of $\mathbb{R}^d$, and $\psi$ is a locally isometric embedding of $\Theta$ into $\mathbb{R}^n$. Then $\mathcal{H}(f)$ has a $d+1$ dimensional nullspace, consisting of the constant function and a d-dimensional space of functions spanned by the original isometric coordinates.*

We give the proof in Appendix A.

**Corollary 2** *Under the same assumptions as Theorem 1, the original isometric coordinates $\theta$ can be recovered, up to a rigid motion, by identifying a suitable basis for the null space of $\mathcal{H}(f)$.*

# Local Tangent Space Alignment




$$\min_Y \sum_{i\sim j} \|y_i - U_i U_j^T y_j\|^2$$

where $U_i$ is a local PCA basis for tangent space at point $x_i \in \mathbb{R}^p$.

# LTSA Algorithm (Zha-Zhang'05)

**Algorithm 4:** LTSA Algorithm

**Input**: A weighted undirected graph $G = (V, E, d)$ such that

$V = \{x_i \in \mathbb{R}^p : i = 1, \ldots, n\}$

$E = \{(i, j) :$ if $j$ is a neighbor of $i$, i.e. $j \in \mathcal{N}_i\}$, e.g. $k$-nearest neighbors

**Output**: Euclidean $k$-dimensional coordinates $Y = [y_i] \in \mathbb{R}^{k \times n}$ of data.

***Step 1***: Compute local PCA on neighborhood of $x_i$, $x_{i_j} \in \mathcal{N}(x_i)$,

$$\tilde{X}^{(i)} = [x_{i_1} - \mu_i, ..., x_{i_k} - \mu_i]^{p \times k} = \tilde{U}^{(i)} \tilde{\Sigma} (\tilde{V}^{(i)})^T,$$

where $\mu_i = \sum_{j=1}^k x_{i_j} = \frac{1}{k} X_i \mathbf{1}$, $\tilde{U}^{(i)} = [\tilde{U}_1^{(i)}, ..., \tilde{U}_k^{(i)}]$ is an approximate tangent space at $x_i$. Define

$$G_i = [1/\sqrt{k}, \tilde{V_1}^{(i)}, ..., \tilde{V_d}^{(i)}]^{k \times (d+1)};$$

***Step 2***: Alignment (kernel) matrix

$$K^{n \times n} = \sum_{i=1}^n S_i W_i W_i^T S_i^T, \quad W_i^{k \times k} = I - G_i G_i^T,$$

where selection matrix $S_i^{n \times k} : [x_{i_1}, ..., x_{i_k}] = [x_1, ..., x_n] S_i^{n \times k}$;

***Step 3***: Find smallest $d + 1$ eigenvectors of $K$ and drop the smallest eigenvector, the remaining $d$ eigenvectors will give rise to a $d$-embedding.

# From LTSA to Connection Laplacian

LTSA (Zhang-Zha'05):

$$\min_Y \sum_{i\sim j} \|y_i - U_i U_j^T y_j\|^2$$

where $U_i$ is a local PCA basis for tangent space at point $x_i \in \mathbb{R}^p$.

Vector Connection Laplacian (Singer-Wu'11):

$$\min_Y \sum_{i\sim j} \|y_i - O_{ij} y_j\|^2, \quad O_{ij} = \arg\min_O \|U_i - O_{ij} U_j\|^2$$

where $U_i$ is a local PCA basis for tangent space at point $x_i \in \mathbb{R}^p$.

# Comparisons of Manifold Learning Techniques

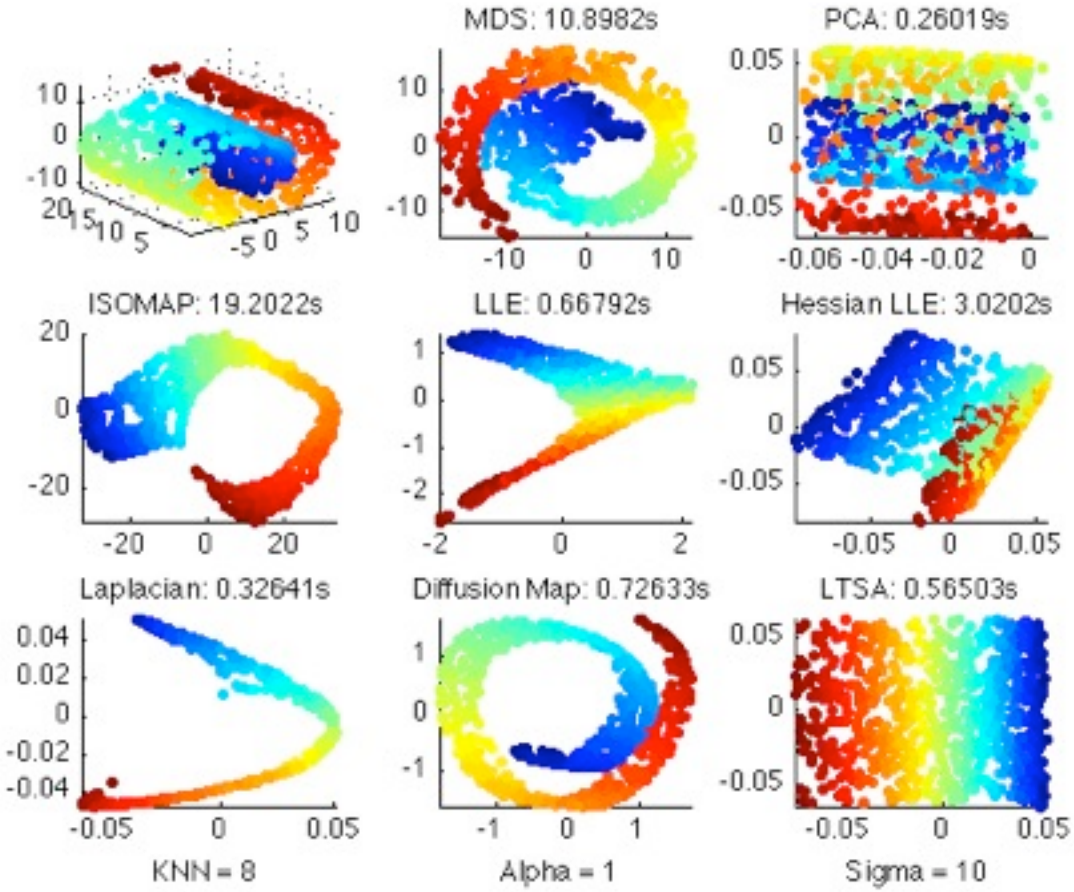
- MDS
- PCA
- ISOMAP
- LLE
- Hessian LLE
- Laplacian LLE
- Diffusion Map
- Local Tangent Space Alignment
- Matlab codes: mani.m

Courtesy of Todd Wittman

# How To Compare

- Speed
- Manifold Geometry
- Non-convexity
- Curvature
- Corners
- Noise
- Non-uniform Sampling
- Sparse Data
- Clustering
- High-Dimensional Data: *Can the method process image manifolds?*
- Sensitivity to Parameters
  - K Nearest Neighbors: *Isomap, LLE, Hessian, Laplacian, KNN Diffusion*
  - Sigma: *Diffusion Map, KNN Diffusion*

# Speed on Swiss Roll




Now go to Todd Wittman's slides, page 13th…

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