$Y_i \sim N(0, \Sigma) \quad i = 1, \dots, n$

$\Sigma = \sigma_x^2 u u^T + \sigma_\varepsilon^2 I_p, \quad \sigma_\varepsilon^2 = 1 \quad$ SNR $R = \frac{\sigma_x^2}{\sigma_\varepsilon^2} = \sigma_x^2$

$\hat{\Sigma}_n = \frac{1}{n} Y Y^T \quad Y = [Y_1, \dots, Y_n]^{p \times n}$

Phase Transition

1° $\hat{\lambda}_{max}(\hat{\Sigma}_n) \to \begin{cases} b & R \le \sqrt{\gamma} \\ f(\sigma_x^2, \gamma), & R \ge \sqrt{\gamma} \end{cases}$

$= (1+\sigma_x^2)(1+\frac{\gamma}{\sigma_x^2}) = (1+R)(1+\frac{\gamma}{R})\sigma_\varepsilon^2$

2° $|\langle \hat{v}_{max}, u \rangle| \to \begin{cases} 0 & R \le \sqrt{\gamma} \\ \left(\dfrac{1-\frac{\gamma}{R^2}}{1+\gamma+\frac{2\gamma}{R}}\right)^{\frac{1}{2}} < 1, & R \ge \sqrt{\gamma} \end{cases}$

$\text{eig}(\hat{\Sigma}_n) = \frac{1}{n} Y Y^T \quad Y_i = \Sigma^{\frac{1}{2}} Z_i \quad Z_i \sim N(0, I_p)$

$= \frac{1}{n} \Sigma^{\frac{1}{2}} Z Z^T \Sigma^{\frac{1}{2}} = \Sigma^{\frac{1}{2}} S_n \Sigma^{\frac{1}{2}}, \quad S_n = \frac{1}{n} Z Z^T \sim$ MP dist, Wishart

$\sim \Sigma^{\frac{1}{2}} (\Sigma^{\frac{1}{2}} S_n \Sigma^{\frac{1}{2}}) \Sigma^{-\frac{1}{2}} = (S_n \Sigma)$ eigval

(λ, v) $S_n \Sigma$ eigenpair

(*) $S_n \Sigma v = \lambda v \quad \|v\|_2 = 1$

$\hat{v} = c(\Sigma^{\frac{1}{2}} v) \quad \|\hat{v}\|_2 = 1 \quad \hat{\Sigma}_n = \Sigma^{\frac{1}{2}} S_n \Sigma^{\frac{1}{2}}$

$\hat{\Sigma}_n \hat{v} = \lambda \hat{v}$

$1 = \hat{v}^T \hat{v} = c^2 (\Sigma^{\frac{1}{2}} v)^T (\Sigma^{\frac{1}{2}} v) = c^2 v^T \Sigma v = c^2 v^T (\sigma_x^2 u u^T + I_p) v$

$= c^2 \sigma_x^2 (u^T v)^2 + c^2 \Rightarrow c^2 = \frac{1}{\sigma_x^2 (u^T v)^2 + 1}$ σ_ε^2

$\Sigma = \sigma_x^2 u u^T + I_p$

$S_n(\sigma_x^2 u u^T + \sigma_\varepsilon^2 I_p) v = \lambda v$

$(\lambda I_p - \sigma_\varepsilon^2 I_p) v = \sigma_x^2 S_n u (u^T v)$

(**) $$v = (\lambda I_p - \sigma_\varepsilon^2 S_n)^{-1} \cdot \sigma_x^2 S_n u \, \underline{(u^T v)}$$

$$\underline{(u^T v)} = u^T (\quad Z \quad)^{-1} \cdot \sigma_x^2 S_n u \, \underline{(u^T v)}$$

Assume $u^T v \neq 0$

$$1 = \sigma_x^2 \, u^T (\lambda I_p - \sigma_\varepsilon^2 \underline{\underline{S_n}})^{-1} \underline{\underline{S_n}} u$$

Wishart

$S_n = W \Lambda W^T \qquad \Lambda = \mathrm{diag}(\lambda_i)_{i=1,\dots,p} \sim$ MP law

$\alpha_i = W_i^T u \qquad W^T W = W \cdot W^T = I_p$.

$$1 = \sigma_x^2 \underbrace{\sum_{i=1}^{p} \frac{\lambda_i}{\lambda - \sigma_\varepsilon^2 \lambda_i} \alpha_i^2} \qquad \lambda_i \sim MP$$

Assume rank-1 u. uniformly on S^{p-1}, $\mathbb{E}(\alpha_i^2) \sim \frac{1}{p}$

$$\sum_{i=1}^{p} \alpha_i^2 = 1$$

Monte-Carlo Sum.

$$\sim \sigma_x^2 \frac{1}{p} \sum_{i=1}^{p} \frac{\lambda_i}{\lambda - \sigma_\varepsilon^2 \lambda_i} \sim \sigma_x^2 \int \frac{t}{\lambda - \sigma_\varepsilon^2 t} \, d\mu^{MP}(t)$$

$\sigma_\varepsilon^2 = 1$

Stieltjes Transform (Terry Tao 2011, RMT 网络)

$$s(z) = \int_{-\infty}^{+\infty} \frac{1}{t - z} \, d\mu^{MP}(t), \quad z \in \mathbb{C}$$

Hilbert transform

Lemma

(A) $$\int \frac{t}{\lambda - t}\, d\mu^{MP}(t) = -\lambda s(\lambda) - 1 \qquad \lambda \in \mathbb{R}^+$$

(B) $$\int \frac{t^2}{(\lambda-t)^2}\, d\mu^{MP}(t) = \lambda^2 s'(\lambda) + 2\lambda s(\lambda) + 1$$

Proof: (A) $$T(\lambda) = \int \frac{t}{\lambda - t}\, d\mu^{MP}(t)$$

$$1 + T(\lambda) = \int \frac{\lambda - t + t}{\lambda - t}\, d\mu^{MP}(t) = \lambda \underbrace{\int \frac{1}{\lambda - t}\, d\mu^{MP}(t)}_{= -\lambda s(\lambda)}$$

$$T(\lambda) = -\lambda s(\lambda) - 1 \qquad \boxed{\text{证}}$$

(B) $$\int \frac{t^2}{(\lambda - t)^2}\, d\mu^{MP}(t) = -T(\lambda) - \lambda T'(\lambda) \qquad \boxed{\text{证}}$$

S-transform

$$S(z) = \frac{(1-\gamma) - z + \sqrt{(z-1+\gamma)^2 - 4\gamma z}}{2\gamma z}, \qquad z \in \mathbb{C},\ \neq 0$$

Note: $$S^{-1}(z) + z = \frac{1}{1+\gamma S(z)} \Rightarrow \text{poly}(S, z) = 0 \qquad \Uparrow \text{ largest root}$$

Lemma (A) $\qquad \lambda \geq b$

$$(**) \Rightarrow 1 = \frac{\sigma_x^2}{4\gamma}\left[2\lambda - (a+b) - 2\sqrt{(\lambda - a)(\lambda - b)}\right]$$

$$1 = \frac{\sigma_x^2}{4\gamma}\left[2\lambda - (a+b) - 2\sqrt{(\lambda-a)(\lambda-b)}\right]$$

$\lambda \geq b$ holds.

$\lambda = b$ noise spectrum upper bnd.

$$a = (1-\sqrt{\gamma})^2$$
$$b = (1+\sqrt{\gamma})^2$$
$$b - a = 4\sqrt{\gamma}$$

$$1 = \frac{\sigma_x^2}{4\gamma}[b-a] = \frac{\sigma_x^2}{\sqrt{\gamma}}$$

$$\sigma_x^2 = \sqrt{\gamma} \quad (\text{SNR}: R = \sqrt{\gamma})$$

Phase - Transition boundary

$$\sigma_x^2 \geq \sqrt{\gamma}. \qquad \lambda = (1+\sigma_x^2)\left(1+\frac{\gamma}{\sigma_x^2}\right) = f(\sigma_x^2; \gamma)$$

$$\left(= (1+R)\left(1+\frac{\gamma}{R}\right)\sigma_\varepsilon^2\right)$$

biased of signal strength

(**) 左右同时 $V^T\cdot$(**)

$$1 = V^TV = \|(u^Tv)\sigma_X^2(\lambda I_p - \sigma_\varepsilon^2 S_n)^{-1} S_n u\|_2^2$$

$$= (u^Tv)^2\sigma_X^4 \quad u^TS_n(\lambda I_p - \sigma_\varepsilon^2 S_n)^{-2} S_n u.$$

$$(u^Tv)^{-2} = \sigma_X^4 \quad \underline{u^TS_n(\lambda I_p - \sigma_\varepsilon^2 S_n)^{-2} S_n u}$$

$$S_n = W\Lambda W^T \qquad \alpha_i = W_i^T u$$

$$\underline{\quad} = \sigma_X^4 \sum_{i=1}^{p} \frac{\lambda_i^2}{(\lambda - \sigma_\varepsilon^2\lambda_i)^2}\alpha_i^2 \leftarrow \frac{1}{p}$$

$$\underline{\sim \sigma_X^4 \int \frac{t^2}{(\lambda - \sigma_\varepsilon^2 t)^2} d\mu^{MP}(t)}$$

Lemma (B)

$\sigma_\varepsilon^2 = 1$ $\Rightarrow$ $$= \frac{\sigma_X^4}{\gamma\gamma}\left[-4\lambda + (a+b) + 2\sqrt{(\lambda-a)(\lambda-b)} + \frac{\lambda(2\lambda-(a+b))}{\sqrt{(\lambda-a)(\lambda-b)}}\right]$$

$$u^Tv^2 = \frac{1-\frac{\gamma}{R^2}}{1+\gamma+\frac{2\gamma}{R}}$$

$$\Rightarrow \quad u^T\hat{v}^2 = u^T(c\Sigma^{\frac{1}{2}}v) = \frac{1+R}{1+R(u^Tv)^2}(u^Tv)^2$$

$$= \frac{g(R,\gamma) < 1}{\frac{1-\frac{\gamma}{\sigma_X^4}}{1+\frac{\gamma}{\sigma_X^2}}}, \quad R > \sqrt{\gamma}$$

RMT

MP law for Wishart

$$\Sigma = \sigma_x^2 u u^T + \sigma_\varepsilon^2 I_p$$

$$\hat{\lambda}_{max}(\hat{\Sigma}_n) \neq \sigma_x^2$$

$$= \begin{cases} f(\sigma_x^2), & R \geq \sqrt{\gamma} \\ b, & R \leq \sqrt{\gamma} \end{cases}$$

PC, $\hat{v}_{max}$ in conic neighbor of u

$g(R, \gamma)$

Open. $\sigma_\varepsilon^2 I_p$ homogeous.

$\mathrm{diag}(\sigma_{\varepsilon_i}^2)$ heterogenous. ? Owen Wang

Rank-Sparsity: Robust PCA

SDP uu^T + sparse

条件? Compressed Sensing

Candes, Ma, Wright, Recht, Joel Tropp