ECE 365 **Fall 2019**

September 26, 2019

Quiz 4

Name:

NetID:

1. **[Eigenvalues and Eigenvectors (10 points)]**
   Find the eigenvalues and orthonormal eigenvectors of the matrix

$$A = \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}.$$

Solution: The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 8-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} \\ &= (8-\lambda)^2 - 4 = \lambda^2 - 16\lambda + 60 = (\lambda - 10)(\lambda - 6). \end{aligned}$$

Setting the above polynomial to zero, we get $\lambda_1 = 10, \lambda_2 = 6$.

To find the eigenvector corresponding to the eigenvalue $\lambda_1 = 10$, we solve $(A - 10I)\boldsymbol{u}_1 = 0$, i.e.,

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \mathbf{0}.$$

This gives us $u_{11} = -u_{12}$, i.e., which can be scaled to unit norm to obtain

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

To find the eigenvector corresponding to the eigenvalue $\lambda_2 = 6$, we solve $(A - 6I)\boldsymbol{u}_2 = 0$. i.e.,

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \mathbf{0}.$$

This gives us $u_{21} = u_{22}$, i.e.,

$$\boldsymbol{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. **[SVD (10 points)]**
   Suppose the matrix A can be written as

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & -1 & 4 \end{bmatrix}.$$

   (a) Find A.
   Solution: Multiplying the three matrices we obtain:

$$A = \begin{bmatrix} 18 & 12 & 12 \\ 12 & 18 & -12 \end{bmatrix}$$

(b) Find the compact form SVD of A.

Solution: It is easy to see that the above expansion is an SVD in compact form if we normalize the columns in the left matrix to have unit norm, with the normalizing constant of $\frac{1}{\sqrt{2}}$ for both columns, and normalize the rows in the right matrices to have unit norm, with the normalizing constant of $\frac{1}{3\sqrt{2}}$ for both rows. Scaling the middle diagonal matrix by 6 to make up for the normalization, we obtain:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 30 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} \end{bmatrix}.$$

3. **[SVD and Image Compression (10 points)]**
Explain how the SVD can be used for image compression.

Solution: If the image is an $m \times n$ matrix of pixel values, the number of values in the image equals mn. If we use a rank-k (with $k < \min\{m, n\}$) SVD approximation of the image, we require to store $(m + n + 1)k$ values, which can be much smaller than mn if k is small.