On Functorial Lindelöfifiability

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Abstract. In the present paper, we prove that a topological space admits a functorial Lindelöfification if and only if its realcompactification is Lindelöf. To investigate the functorial Lindelöfifiability of a topological space, for each topological property P, we introduce the notion of "**functorial P-ification**" and give an explicit construction of the functorial P-ification. Moreover, for a discrete space X, we discuss the functorial $|X|$-Lindelöfifiability of X and study relationships with properties of the cardinal $|X|$. Finally, we apply our results concerning functorial κ-Lindelöfifiability (for some cardinal κ) to the space of ordinals and construct several functorial κ-Lindelöfifiable spaces.

Introduction

Throughout the present paper, we always suppose that **topological spaces are completely regular and Hausdorff**. Let X be a topological space. Our interest in the present paper is the property of the *functorial* Lindelöfification of X, where the functorial Lindelöfification of X is an extension space $X \to X'$ such that for any continuous map $f : X \to Y$, if Y is Lindelöf, then f can be extended uniquely to a continuous map $X' \to Y$. Although the Stone-Čech compactification always exists for any completely regular Hausdorff space of X, the functorial Lindelöfification may not exist. To investigate the functorial Lindelöfifiability of a topological space, for each topological property P, we introduce the notion of "functorial P-ification" and study the structure of the functorial P-ification of a topological space.

In [AHST18], F. Azarpanah, A. A. Hesari, A. R. Sarehi, and A. Taherifar constructed several Lindelöf-like extension spaces of X in the Stone-Čech compactification βX. They, in detail, studied their extension spaces from the point of view of the relationship between algebraic properties of the rings of continuous (resp. bounded continuous) functions $C(X)$ (resp. $C^*(X)$) on X and topological properties of X or βX. However, their Lindelöf-like extension spaces do not have a suitable functoriality. Therefore, by contrast, in the present paper, we discuss what properties can be concluded from the abstract functorial Lindelöfifiability. In particular, our interest is the existence of functorial P-ification and topological characterization of the functorial P-ifiability.

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Notation. We shall write Top for the category of **completely regular Hausdorff** topological spaces and continuous functions. We shall write $\mathbb{R}$ for the topological field of real numbers.

Let X be a set. We shall write $|X|$ for the cardinality of X. We shall write 2^X for the power set of X. For any family of sets $\mathcal{F}$, we shall write $\bigcup \mathcal{F}$ for the union of $\mathcal{F}$.

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Let X be a topological space. We shall write $C(X)$ (resp. $C^*(X)$) for the rings of continuous (resp. bounded continuous) functions on X. We shall write βX for the Stone-Čech compactification of X. Then, the natural restriction morphism of rings $C(\beta X) \xrightarrow{(-)|_X} C(X)$ induces an isomorphism of rings $C(\beta X) \xrightarrow[\sim]{(-)|_X} C^*(X)$.

§1. Functorial P-ification

Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory. In this section, we study the fundamental properties of the *functorial* P*-ification* and construct it as a subspace of the Stone-Čech compactification (cf. Definition 1.6).

Let us start to define the functorial P-ification, which is an analogue of the Stone-Čech compactification.

Definition 1.1. Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory.

(1) Let X be a topological space. We shall say that X **admits a functorial P-ification** $i_X : X \to \nu_{\mathsf{P}} X$ if $\nu_{\mathsf{P}} X$ belongs to P, and, moreover, for any object $Y \in \mathsf{P}$ and any continuous map $f : X \to Y$, there exists a unique continuous map $\tilde{f} : \nu_{\mathsf{P}} X \to Y$ such that $f = \tilde{f} \circ i$. If X admits a functorial P-ification $i_X : X \to \nu_{\mathsf{P}} X$, then we shall say that X is **functorial P-ifiable**.
(2) Let $f : X \to Y$ be a continuous map between topological spaces that admit functorial P-ifications. Write $\nu_{\mathsf{P}} f : \nu_{\mathsf{P}} X \to \nu_{\mathsf{P}} Y$ for the unique continuous map such that $i_Y \circ f = \nu_{\mathsf{P}} f \circ i_X$.
(3) Write $\mathsf{P}^\nu \subset \mathsf{Top}$ for the full subcategory determined by the topological spaces that admits a functorial P-ification.

Definition 1.2. We shall write $\mathsf{Cpt} \subset \mathsf{Top}$ for the full subcategory of compact Hausdorff topological spaces (where we note that, by Urysohn's lemma, a compact Hausdorff topological space is completely regular).

Remark 1.3. If X is a topological space that belongs to P, then the identity morphism $\mathrm{id}_X : X \to X = \nu_{\mathsf{P}} X$ satisfies property of the functorial P-ification. Hence, it holds that $\mathsf{P} \subset \mathsf{P}^\nu$. Moreover, the assignment $\nu_{\mathsf{P}} : \mathsf{P}^\nu \to \mathsf{P}$, $X \mapsto \nu_{\mathsf{P}} X$, $f \mapsto \nu_{\mathsf{P}} f$ is a left adjoin functor of the inclusion functor $\mathsf{P} \subset \mathsf{P}^\nu$ such that $\nu_{\mathsf{P}}|_{\mathsf{P}} = \mathrm{id}_{\mathsf{P}}$.

Remark 1.4. Assume that $\mathsf{P} = \mathsf{Cpt} \subset \mathsf{Top}$. By the uniqueness of a left adjoint functor of the inclusion functor $\mathsf{P} \subset \mathsf{P}^\nu$, then the functorial P-ification functor ν_{Cpt} is isomorphic to the Stone-Čech compactification functor $\beta : \mathsf{Top} \to \mathsf{Cpt}$.

Lemma 1.5. *Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathsf{Cpt} \subset \mathsf{P}$ and X a topological space that admits a functorial P-ification $i_X : X \to \nu_{\mathsf{P}} X$. Then the continuous map $\beta i_X : \beta X \to \beta \nu_{\mathsf{P}} X$ is a homeomorphism. In particular, the continuous map $i_X : X \to \nu_{\mathsf{P}} X$ may be regarded as the inclusion map $X \subset \nu_{\mathsf{P}}$ between subspaces of βX.*

Proof. Write $j_{(-)} : (-) \to \beta(-)$ for the natural embedding to the Stone-Čech compactification of $(-)$. Since $\beta X \in \mathsf{Cpt} \subset \mathsf{P}$, it follows from the universality of the functorial P-ification $\nu_{\mathsf{P}} X$ that there exists a unique morphism $j : \nu_{\mathsf{P}} X \to \beta X$ such that $j_X = j \circ i_X$. By the universality of the Stone-Čech compactification of $\nu_{\mathsf{P}} X$, there exists a morphism $g : \beta \nu_{\mathsf{P}} X \to \beta X$ such that $j = g \circ j_{\nu_{\mathsf{P}} X}$:

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "i_X"'] & \beta X \arrow[d, shift right, "\beta i_X"'] \\ \nu_{\mathsf{P}} X \arrow[r, "j_{\nu_{\mathsf{P}} X}"'] \arrow[ur, "j"] & \beta \nu_{\mathsf{P}} X. \arrow[u, shift right, "g"'] \end{tikzcd}$$

By the functoriality of the operation $\beta(-)$, it holds that $\beta i_X \circ j_X = j_{\nu_{\mathsf{P}} X} \circ i_X$. Hence, it holds that

$$g \circ \beta i_X \circ j_X = g \circ j_{\nu_{\mathsf{P}} X} \circ i_X = j \circ i_X = j_X.$$

By the universality of the Stone-Čech compactification of X, this implies that $g \circ \beta i_X = \mathrm{id}_{\beta X}$. Moreover, since $\beta i_X \circ j_X = j_{\nu_{\mathsf{P}} X} \circ i_X$, it holds that $(\beta i_X \circ j) \circ i_X = \beta i_X \circ j_X = j_{\nu_{\mathsf{P}} X} \circ i_X$.

Since $\beta\nu_{\mathsf{P}}X \in \mathsf{Cpt} \subset \mathsf{P}$, it follows from the universality of the functorial P-ification of X that $\beta i_X \circ j = j_{\nu_0 X}$. Hence, it holds that

$$\beta i_X \circ g \circ j_{\nu_{\mathsf{P}}X} = \beta i_X \circ j = j_{\nu_{\mathsf{P}}X}.$$

By the universality of the Stone-Čech compactification of $\nu_{\mathsf{P}}X$, this implies that $\beta i_X \circ g = \mathrm{id}_{\beta\nu_{\mathsf{P}}X}$. Thus, in particular, the natural continuous map βi_X is a homeomorphism. This completes the proof of Lemma 1.5. ☺

Next, we construct explicitly an extension of X in βX that represents the functorial P-ification.

Definition 1.6. Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathsf{Cpt} \subset \mathsf{P}$ and X a topological space. We shall write

$$\tilde{\nu}_{\mathsf{P}}X \stackrel{\mathrm{def}}{:=} \bigcap \big\{\beta f^{-1}(Y) \,\big|\, f : X \to Y \text{ is a continuous map such that } Y \in \mathsf{P}\big\} \subset \beta X.$$

Lemma 1.7. *Let $\mathsf{P} \subset \mathsf{Q} \subset \mathsf{Top}$ be full subcategories and X a topological space. Then, it holds that $\tilde{\nu}_{\mathsf{Q}}X \subset \tilde{\nu}_{\mathsf{P}}X$.*

Proof. Lemma 1.7 follows immediately from the definition of $\tilde{\nu}_{(-)}(?)$ and the inclusion relation $\mathsf{P} \subset \mathsf{Q}$. ☺

Lemma 1.8. *Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathsf{Cpt} \subset \mathsf{P}$ and X a functorial P-ifiable topological space. Then, it holds that $\nu_{\mathsf{P}}X = \tilde{\nu}_{\mathsf{P}}X$.*

Proof. If we write $i_X : X \hookrightarrow \nu_{\mathsf{P}}X$ for the inclusion map, then, since $\nu_{\mathsf{P}}X \in \mathsf{P}$, it follows from Lemma 1.5 that $\tilde{\nu}_{\mathsf{P}}X \subset \beta i_X^{-1}(\nu_{\mathsf{P}}X) = \nu_{\mathsf{P}}X$.

Next, we prove that $\nu_{\mathsf{P}}X \subset \tilde{\nu}_{\mathsf{P}}X$. Let $f : X \to Y$ be a continuous map such that $Y \in \mathsf{P}$. Then, by the universality of the functorial P-ification, there exists a unique morphism $\tilde{f} : \nu_{\mathsf{P}}X \to Y$ such that $\tilde{f}|_X = f$. By Lemma 1.5, it holds that $\beta f = \beta\tilde{f}$. Thus, it holds that $\nu_{\mathsf{P}}X \subset \beta\tilde{f}^{-1}(Y) = f^{-1}(Y)$. By allowing Y to vary over P, we conclude that $\nu_{\mathsf{P}}X \subset \tilde{\nu}_{\mathsf{P}}X$. This completes the proof of Lemma 1.8. ☺

Lemma 1.9. *Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathsf{Cpt} \subset \mathsf{P}$ and X a topological space. Then, X is functorial P-ifiable if and only if $\tilde{\nu}_{\mathsf{P}}X$ belongs to P.*

Proof. If X is functorial P-ifiable, then, by Lemma 1.8, it holds that $\tilde{\nu}_{\mathsf{P}}X = \nu_{\mathsf{P}}X \in \mathsf{P}$. Conversely, if $\tilde{\nu}_{\mathsf{P}}X \in \mathsf{P}$, then one can verify immediately that $\tilde{\nu}_{\mathsf{P}}X$ satisfies the required universality of the functorial P-ification of X. This completes the proof of Lemma 1.9. ☺

§2. Almost Compactification

In this section, we consider the case where any functorial P-ifiable space belongs to P.

Definition 2.1. Let X be a topological space. We shall say that X is **almost compact** if $\beta X \setminus X$ is of cardinality at most one. Write $\mathsf{AlmCpt} \subset \mathsf{Top}$ for the full subcategory determined by almost compact spaces.

Remark 2.2. For any topological space X and any subspace $X \subset Y \subset \beta X$, the natural morphism $\beta X \to \beta Y$ is a homeomorphism. Indeed, any bounded continuous function $f : Y \to [a, b] \subset \mathbb{R}$ can be extended uniquely to a continuous function $\beta f|_Y : \beta X \to [a, b] \subset \mathbb{R}$. Thus, in particular, for any $p \in \beta X \setminus X$, $\beta X \setminus \{p\}$ is almost compact.

Note that for any topological space X, it follows immediately that

$$X = \bigcap_{p \in \beta X \setminus X} (\beta X \setminus \{p\}).$$

By Remark 2.2, the equality displayed above leads us to the following proposition:

Proposition 2.3. *Let $\mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathsf{AlmCpt} \subset \mathsf{P}$ and X be a topological space. Then, X is functorial P-ifiable if and only if X belongs to P.*

Proof. Since $\mathsf{AlmCpt} \subset \mathsf{P}$, it follows from Remark 2.2 and the definition of $\tilde{\nu}_{\mathsf{P}}$ that

$$X \subset \tilde{\nu}_{\mathsf{P}} X \subset \bigcap_{p \in \beta X \setminus X} (\beta X \setminus \{p\}) = X.$$

Hence, it holds that $X = \tilde{\nu}_{\mathsf{P}} X$. Thus, Proposition 2.3 follows immediately from Lemma 1.9. ☺

Remark 2.4. One can easily prove that an almost compact topological space is pseudo-compact (cf. Definition 3.7) and locally compact. Hence, an almost compact space is Čech complete. In particular, by Proposition 2.3, if P is equal to the full subcategory determined by these topological properties, then $\mathsf{P} = \mathsf{P}^{\nu}$.

§3. Realcompactification

In the present section, we apply the theory developed in Section 1 to the case where the category $\mathsf{P} \subset \mathsf{Top}$ consists of realcompact topological spaces and characterize functorial Lindelöfifiability as a topological property of realcompactification (cf. Corollary 3.10).

Definition 3.1.

(1) We shall write $\mathbb{R}\text{-}\mathsf{Cpt} \subset \mathsf{Top}$ for the full subcategory determined by the realcompact topological spaces (cf. Remark 3.2).

(2) For any topological space X, we shall write υX for the realcompactification of X (cf. Remark 3.2 or [GJ60, Section 8.4]).

In the present section, we shall mainly be concerned with the situation that $\mathbb{R} \in \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$. Let us recall some fundamental relationships between realcompactifications and rings of continuous functions.

Remark 3.2. Let X be a topological space. Recall that X is **realcompact** if there exist a cardinal κ and a closed embedding $X \hookrightarrow \mathbb{R}^{\kappa}$. A point $x \in \beta X$ of the Stone-Čech compactification of X is **real** if any continuous map $X \to \mathbb{R}$ can be extended to a continuous map $X \cup \{x\} \to \mathbb{R}$. Here, we note that any point of X is a real point. Then, it is a well-known fact that X is realcompact if and only if any real point $x \in \beta X$ belongs to $X \subset \beta X$.

Note that the realcompactification υX is defined as the set of real points of X in βX. By the characterization of realcompactness mentioned as above, υX is automatically realcompact. Hence, it holds that

$$(\dagger) \qquad \upsilon X = \bigcap_{f \in C(X)} \beta f^{-1}(\mathbb{R}).$$

In particular, the realcompactification υX is the minimal realcompact subspace of βX. Thus, we conclude from these discussion that the natural restriction morphism $(-)|_X : C(\upsilon X) \to C(X)$ is an isomorphism of rings.

A real point $x \in \beta X$ of X can be characterized as a point $x \in \beta X$ such that the residue field of the ring $C(X)$ at the maximal ideal determined by $x \in \beta X$ is isomorphic to $\mathbb{R}$. Hence, the topological space υX may be reconstructed as the set of maximal ideals of $C(X)$ whose residue field is isomorphic to $\mathbb{R}$, together with the topology induced by the **Zariski topology** of the prime spectrum of the ring $C(X)$ (where we note that, by Urysohn's lemma, the topology induced by the Zariski topology coincides with the subspace topology of βX, cf., e.g., [AtiMac, Problem 1.26 (ii)]). Thus, if Y is an another topological space, and $f^* : C(Y) \to C(X)$ is a morphism of rings, then f^* induces a unique continuous map $\upsilon f : \upsilon X \to \upsilon Y$ such that the following diagram commutes:

$$(\ddagger) \qquad \begin{tikzcd} C(\upsilon Y) \arrow[r, "(-)|_Y", "\sim"'] \arrow[d, "(-)\circ \upsilon f"'] & C(Y) \arrow[d, "f^*"] \\ C(\upsilon X) \arrow[r, "(-)|_X", "\sim"'] & C(X). \end{tikzcd}$$

Remark 3.3 (Structure of the Realcompactification)**.** It is a well-known fact that a locally compact Lindelöf space X admits a **perfect map** (i.e., a continuous closed map whose fibers are compact) $X \to \mathbb{R}$ (cf., e.g., [Isk23, Proposition 2.4]). This follows easily from Frolík's theorem (cf. [Fro60], where we note that a Lindelöf locally compact space is Čech-complete and paracompact), but we can directly prove this as follows: Since X is locally compact, the one-point compactification αX of X is Hausdorff. Write $\infty \in \alpha X \setminus X$ for the unique point. By Urysohn's theorem, for any $x \in X$, there exists a continuous map $f_x : \alpha X \to [0,1]$ such that $f_x(\infty) = 0$, and $f_x(x) = 1$. Since X is Lindelöf, there exists a countable sequence $x_0, x_1, \cdots \in X$ such that $X = \bigcup_{n\in\mathbb{N}} f_{x_n}^{-1}((0,1])$. Then, the continuous map $f \overset{\mathrm{def}}{:=} \sum_{n\in\mathbb{N}} \frac{1}{2^n} f_{x_n}$ satisfies that $f^{-1}(0) = \{\infty\}$. This implies that the restriction $f|_X : X \to \mathbb{R}$ is perfect.

Thus, if a full subcategory $\{\mathbb{R}\} \cup \mathsf{Cpt} \subset \mathsf{P} \subset \mathsf{Top}$ is contained in the class "locally compact Lindelöf spaces", then for any subset $X \subset Z \subset \beta X$, the following assertions are equivalent:

(1) There exists a continuous map $f : X \to \mathbb{R}$ such that $Z = \beta f^{-1}(\mathbb{R})$.
(2) There exist an object $Y \in \mathsf{P}$ and a continuous map $f : X \to Y$ such that $Z = \beta f^{-1}(Y)$.
(3) Z is locally compact Lindelöf.

In particular, it holds that

$$\upsilon X = \bigcap \{ X \subset Y \subset \beta X \,|\, Y \text{ is Lindelöf locally compact} \} \subset \beta X.$$

This implies that for any perfect map $f : X \to Y$, if Y is realcompact, then X is also realcompact.

Lemma 3.4. *Let* $\mathsf{Cpt} \subset \mathsf{P} \subset \mathsf{Top}$ *be a full subcategory and* X *a topological space. Then, the following assertions hold:*

(1) *Assume that* $\mathbb{R} \in \mathsf{P}$*. Then, it holds that* $\tilde{\nu}_{\mathsf{P}} X \subset \upsilon X$*.*
(2) *Assume that* $\mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$*. Then, it holds that* $\upsilon X \subset \tilde{\nu}_{\mathsf{P}} X$*.*

In particular, if $\{\mathbb{R}\} \cup \mathsf{Cpt} \subset \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$*, then it holds that* $\upsilon X = \tilde{\nu}_{\mathsf{P}} X$*.*

Proof. Since $\mathbb{R} \in \mathsf{P}$, assertion (1) follows immediately from equation (†) in Remark 3.2 and the definition of $\tilde{\nu}_{\mathsf{P}} X$. Let $f : X \to Y$ be a continuous map such that Y is realcompact. Then, $\beta f^{-1}(Y)$ is also realcompact (cf. Remark 3.3). Thus, assertion (2) follows immediately from the fact that υX is the minimal realcompact subspace of βX such that $X \subset \upsilon X$ (cf. the sentence after the equation (†) in Remark 3.2). This completes the proof of Lemma 3.4. ☺

We then conclude our main result in the present paper.

Theorem 3.5. *Let* $\mathsf{Cpt} \subset \mathsf{P} \subset \mathsf{Top}$ *be a full subcategory such that* $\mathbb{R} \in \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$ *and* X *a topological space. Then, the following assertions are equivalent:*

(1) $X \in \mathsf{P}^{\nu}$.
(2) $\upsilon X \in \mathsf{P}$.
(3) *There exists an object* $Y \in \mathsf{P}$ *such that* $C(X)$ *is isomorphic as a ring to* $C(Y)$*.*

Moreover, if assertion (3) holds, then the functorial P*-ification of* X *coincides with* Y*.*

Proof. Equivalent "(1) ⇔ (2)" follows immediately from Lemma 1.9 and Lemma 3.4. Equivalent "(2) ⇔ (3)" follows immediately from the fact that the realcompactification of X can be reconstructed as a set of maximal ideals of $C(X)$ whose residue field is $\mathbb{R}$, together with the Zariski topology (cf. the above sentence of the diagram (‡) in Remark 3.2). The last assertion also follows formally from this fact. This completes the proof of Theorem 3.5. ☺

Corollary 3.6. *Let* $\mathsf{Cpt} \subset \mathsf{P} \subset \mathsf{Top}$ *be a full subcategory such that* $\mathbb{R} \in \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$*. Then, it holds that* $\mathsf{P}^{\nu} \cap \mathbb{R}\text{-}\mathsf{Cpt} = \mathsf{P}$*.*

Proof. Since $\mathsf{P} \subset \mathsf{P}^{\nu}$, it holds that $\mathsf{P} \subset \mathsf{P}^{\nu} \cap \mathbb{R}\text{-}\mathsf{Cpt}$. Let X be functorial P-ifiable realcompact topological space. Then, by the equivalence "(1) ⇔ (2)" in Theorem 3.5, it holds that $X = \upsilon X \in \mathsf{P}$. In particular, $\mathsf{P}^{\nu} \cap \mathbb{R}\text{-}\mathsf{Cpt} \subset \mathsf{P}$. This completes the proof of Corollary 3.6. ☺

Next, we prove that $\mathsf{P} \subsetneq \mathsf{P}^{\nu}$ if $\mathbb{R} \in \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$.

Definition 3.7. We shall write $\mathsf{PsCpt} \subset \mathsf{Top}$ for the full subcategory consisting of pseudocompact topological spaces. Recall that a topological space X is **pseudocompact** if any continuous map $X \to \mathbb{R}$ is bounded, i.e., $C^*(X) = C(X)$.

By the definition of the notion of a pseudocompact space, a topological space X is pseudocompact if and only if $\upsilon X = \beta X$. Thus, the following assertion holds:

Proposition 3.8. *Let $\mathsf{Cpt} \subset \mathsf{P} \subset \mathsf{Top}$ be a full subcategory such that $\mathbb{R} \in \mathsf{P} \subset \mathbb{R}\text{-}\mathsf{Cpt}$. Then, it holds that $\mathsf{PsCpt} \subset \mathsf{P}^\nu$. In particular, it holds that $\mathsf{P} \subsetneq \mathsf{P}^\nu$.*

Proof. Since $\mathsf{Cpt} \subset \mathsf{P}$, it follows from Theorem 3.5 and the definition of the notion of a pseudocompact space that $\mathsf{PsCpt} \subset \mathsf{P}^\nu$. Moreover, it follows immediately from the definition of the notion of a pseudocompact realcompact space that $\mathsf{PsCpt} \cap \mathbb{R}\text{-}\mathsf{Cpt} = \mathsf{Cpt}$. Since $\mathsf{Cpt} \subsetneq \mathsf{PsCpt}$, this implies that $\mathsf{P} \subsetneq \mathsf{P}^\nu$. This completes the proof of Proposition 3.8. ☺

At the end of the present section, we apply Theorem 3.5 to the category of Lindelöf spaces.

Definition 3.9. We shall write $\mathsf{Lind} \subset \mathsf{Top}$ for the full subcategory determined by the Lindelöf spaces.

Corollary 3.10. *Let X be a topological space. Then, the following assertions hold:*

(1) *X is functorial Lindelöfifiable.*
(2) *The realcompactification υX of X is Lindelöf.*
(3) *There exists a Lindelöf space Y such that $C(X)$ is isomorphic as a ring to $C(Y)$.*

Proof. Corollary 3.10 follows immediately from Theorem 3.5, together with the fact that $\{\mathbb{R}\} \cup \mathsf{Cpt} \subset \mathsf{Lind} \subset \mathbb{R}\text{-}\mathsf{Cpt}$. ☺

§4. Functorial κ-Lindelöfifiability of Discrete Spaces

In this section, we study a relationship between the functorial κ-Lindelöfifiability of discrete spaces and properties of the cardinality of discrete spaces.

Definition 4.1. Let κ and λ be cardinals, α an ordinal, and X a topological space.

(1) We shall use the notations $\alpha + 1$, ω_α, $\aleph_\alpha$, κ^+, and $\mathrm{cf}(\alpha)$ as they are defined in [Kun80, Chapter 1, 7.10, 7.18, 10.17, 10.18, 10.30]. Then, it is a well-known fact that $\alpha + 1$ is compact.
(2) We shall say that X is **κ-Lindelöf** if for any open covering $\mathcal{U}$ of X, there exists a subset $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \mathcal{V} = X$, and $|\mathcal{V}| < \kappa$.
(3) We shall write $\kappa\text{-}\mathsf{Lind} \subset \mathsf{Top}$ for the full subcategory determined by the κ-Lindelöf spaces.
(4) We shall write
$$\upsilon_\kappa X \overset{\mathrm{def}}{\coloneqq} \bigcap \{X \subset Y \subset \beta X \mid Y \in \kappa\text{-}\mathsf{Lind}\} \subset \beta X.$$

Remark 4.2. Note that the notion of an $\aleph_0$-Lindelöf space (resp. an $\aleph_1$-Lindelöf space) is equivalent to the notion of a compact space (resp. a Lindelöf space). In particular, it holds that $\beta X = \upsilon_{\aleph_0} X$ and that $\upsilon X = \upsilon_{\aleph_1} X$. Moreover, it holds that
$$\beta X = \upsilon_{\aleph_0} X \supset \upsilon X = \upsilon_{\aleph_1} X \supset \cdots \supset \upsilon_{|X|^+} X = X.$$

Lemma 4.3. *Let κ, λ be infinite cardinals and $f : X \to Y$ a perfect map (cf. Remark 3.3). Then, if Y is κ-Lindelöf, then X is also κ-Lindelöf.*

Proof. Assume that Y is κ-Lindelöf. Let $\mathcal{U}$ be an open covering of X and $y \in Y$ a point. Since $f^{-1}(y)$ is compact, there exists a finite subset $\mathcal{U}_y \subset \mathcal{U}$ such that $f^{-1}(y) \subset \bigcup \mathcal{U}_y$. For any $y \in \mathrm{Im}(f)$, write $V_y \overset{\mathrm{def}}{\coloneqq} \bigcap_{U \in \mathcal{U}_y} (Y \setminus f(X \setminus U))$. Since f is closed, V_y is open. Moreover, since $y \in V_y$, the family $\mathcal{V} \overset{\mathrm{def}}{\coloneqq} \{V_y \mid y \in \mathrm{Im}(f)\}$ is an open covering of Y. Since Y is κ-Lindelöf, there exists a subset $\mathcal{V}_0 \subset \mathcal{V}$ such that $Y = \bigcup \mathcal{V}_0$, and, moreover, $|\mathcal{V}_0| < \kappa$. Then, it follows immediately that the subset $\mathcal{U}_0 \overset{\mathrm{def}}{\coloneqq} \bigcup \{\mathcal{U}_y \mid y \in Y, V_y \in \mathcal{V}_0\}$ is an open covering of X such that $|\mathcal{U}_0| < \kappa$. This implies that X is κ-Lindelöf. This completes the proof of Lemma 4.3. ☺

Corollary 4.4. *Let X be a topological space and κ an infinite cardinal. Then, X is functorial κ-Lindelöfifiable if and only if $\upsilon_\kappa X$ is κ-Lindelöf.*

Proof. By Lemma 4.3, it holds that $\tilde{\nu}_{\kappa\text{-}\mathsf{Lind}} X = \upsilon_\kappa X$. Thus, Corollary 4.4 follows immediately from Lemma 1.8. ☺

Next, we study the functorial κ-Lindelöfifiability of a discrete space.

Remark 4.5. Let κ is a cardinal. Recall that a filter U on a set X is κ-complete if for any family $U_0 \subset U$ such that $|U_0| < \kappa$, it holds that $\bigcap U_0 \in U$. Note that a $\{0,1\}$-valued (σ-additive) measure μ on a set X corresponds to an $\aleph_0$-complete ultrafilter $\mathcal{F}_\mu$ as follows: for any $U \subset X$, $\mu(U) = 1$ if and only if $U \in \mathcal{F}_\mu$.

Recall that a cardinal κ is **measurable** if κ is uncountable, and, moreover, there exists a κ-complete nonprincipal ultrafilter on κ (cf. [Jech, Definition 10.3]). Note that this definition is different to the definition of the measurablility in [GJ60, Chapter 12]. By [Jech, Lemma 10.2], the least cardinal that carries a non-trivial $\{0,1\}$-valued (σ-additive) measure is measurable. Conversely, if κ is measurable, then the $\{0,1\}$-valued measure on κ constructed from a κ-complete ultrafilter is (σ-additive and) non-trivial. Hence, the least cardinal that carries a non-trivial $\{0,1\}$-valued measure is equal to the least measurable cardinal.

If κ admits a non-trivial $\{0,1\}$-valued (σ-additive) measure, then, for any $\lambda > \kappa$, λ admits a non-trivial $\{0,1\}$-valued measure. Thus, by [GJ60, Theorem 12.2], a discrete space $|X|$ is realcompact if and only if any cardinal less than $|X|$ is non-measurable.

In the remainder of the present section, we use the following notation:

Definition 4.6. Let X be a discrete space.

(1) Let $U \subset X$ be a subset. We shall write
$$O(U) \overset{\text{def}}{:=} \{\mathcal{F} \,|\, \mathcal{F} \text{ is an ultrafilter over } X, \text{ and, moreover, } U \in \mathcal{F}\}.$$

(2) We shall regard βX as the set of ultrafilters over X equipped with the topology generated by the family $\{O(U) \,|\, U \subset X\}$. Then, we identify a point of X with a principal ultrafilter over X.

Theorem 4.7. *Let κ be an infinite cardinal, X a discrete space, and $\mathcal{F} \in \beta X \setminus X$ a point. Then, $\mathcal{F} \in \upsilon_\kappa X$ if and only if $\mathcal{F}$ is κ-complete.*

Proof. First, we prove necessity. Assume that there exists $\mathcal{F}_0 \subset \mathcal{F}$ such that $|\mathcal{F}_0| < \kappa$ and that $\bigcap \mathcal{F}_0 \notin \mathcal{F}$. Write
$$Y \overset{\text{def}}{:=} \bigcup \left\{ \overline{X \setminus F} \subset \beta X \,\middle|\, F \in \mathcal{F}_0 \right\},$$
where the closure is taken as a subset of βX. Since $\mathcal{F}$ is an ultrafilter over X, for any $F \in \mathcal{F}_0$, it holds that $X \setminus F \notin \mathcal{F}$. This implies that for any $F \in \mathcal{F}_0$, $\mathcal{F} \notin \overline{X \setminus F}$. In particular, it holds that $\mathcal{F} \notin Y$. Since for each $F \in \mathcal{F}_0$, $\overline{X \setminus F} \subset \beta X$ is compact, and $|\mathcal{F}_0| < \kappa$, we conclude that Y is κ-Lindelöf. Thus, it holds that $\mathcal{F} \notin \upsilon_\kappa X$. This completes the proof of necessity.

Next, we prove sufficiency. Assume that there exists a κ-Lindelöf subspace $X \subset Y \subset \beta X$ such that $\mathcal{F} \notin Y$. For any element $\mathcal{G} \in Y$, there exist open subsets $\mathcal{F} \in U_\mathcal{G} \subset \beta X$ and $\mathcal{G} \in V_\mathcal{G} \subset \beta X$ such that $U_\mathcal{G} \cap V_\mathcal{G} = \varnothing$. Then, it holds that $X \cap V_\mathcal{G} \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter over X, it holds that $X \setminus V_\mathcal{G} \in \mathcal{F}$. Since Y is κ-Lindelöf, there exists a subset $Y_0 \subset Y$ such that $|Y_0| < \kappa$, and $Y \subset \bigcup_{\mathcal{G} \in Y_0} V_\mathcal{G}$. Then, it holds that
$$\bigcap_{\mathcal{G} \in Y_0} X \setminus V_\mathcal{G} = X \setminus \bigcup_{\mathcal{G} \in Y_0} V_\mathcal{G} \subset X \setminus Y = \varnothing.$$
This implies that $\mathcal{F}$ is not κ-complete. This completes the proof of Theorem 4.7. ☺

Corollary 4.8. *Let κ be an uncountable cardinal and X a discrete space. Then the following assertions hold:*

(1) *Assume that any cardinal less than or equal to $|X|$ is non-measurable. Then, it holds that $\upsilon_\kappa X = X$. In particular, X is functorial κ-Lindelöfifiable if and only if $|X| < \kappa$.*

(2) *Assume that any cardinal less than or equal to κ is non-measurable. Then, it holds that $\upsilon_\kappa X = \upsilon X$. In particular, X is functorial κ-Lindelöfifiable if and only if υX is κ-Lindelöf.*

Proof. Assertion (1) follows immediately from [GJ60, Theorem 12.2] and Remark 4.2. Assertion (2) follows immediately from [Jech, Lemma 10.2], Theorem 4.7, and the fact that any κ-complete ultrafilter over a subset of X induces a κ-complete ultrafilter over X. ☺

Corollary 4.9. *If a discrete space X admits a functorial $|X|$-Lindelöfification, then $|X|$ is measurable.*

Proof. Since X is not $|X|$-Lindelöf, and X admits a functorial $|X|$-Lindelöfification, it holds that $X \subsetneq \upsilon_{|X|}X$. Thus, by Theorem 4.7, there exists a non-trivial $|X|$-complete ultrafilter over X. In particular, $|X|$ is measurable. This completes the proof of Corollary 4.9. ☺

Next, we consider the converse implication of Corollary 4.9.

Definition 4.10. We shall say that a cardinal κ is **strongly compact** if for any set S, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S (cf. [THI, Chapter 1, Proposition 4.1]).

Theorem 4.11. *Let κ be a cardinal. Then, the following assertions are equivalent:*

(1) *κ is strongly compact.*
(2) *Any discrete space admits a functorial κ-Lindelöfification.*

Proof. First, we prove the implication "(1) $\Rightarrow$ (2)". Assume that κ is strongly compact. Let X be a discrete space. By Corollary 4.4, to prove that X admits a functorial κ-Lindelöfification, it sufficies to prove that $\upsilon_\kappa X$ is κ-Lindelöf. For each family of subsets $\mathcal{A}$ of X, consider the following condition

($\dagger_{\mathcal{A}}$) For any κ-complete ultrafilter $\mathcal{F}$ over X, there exists an element $U \in \mathcal{A}$ such that $U \in \mathcal{F}$.

Then, by Theorem 4.7, $\mathcal{A}$ satisfies condition ($\dagger_{\mathcal{A}}$) if and only if $\upsilon_\kappa X \subset \bigcup_{A\in\mathcal{A}} O(A)$. Hence, to prove that $\upsilon_\kappa X$ is κ-Lindelöf, it sufficies to prove that for any family of subsets $\mathcal{U}$ of X that satisfies condition ($\dagger_{\mathcal{U}}$), there exists a subset $\mathcal{U}' \subset \mathcal{U}$ of cardinality less than κ such that $\mathcal{U}'$ satisfies condition ($\dagger_{\mathcal{U}_0}$).

Let $\mathcal{U}$ be a family of subsets of X that satisfies condition ($\dagger_{\mathcal{U}}$). Write

$$\mathcal{F} \overset{\text{def}}{:=} \left\{\bigcap_{U\in\mathcal{U}'} X \setminus U \,\middle|\, \mathcal{U}' \subset \mathcal{U}, |\mathcal{U}'| < \kappa\right\}.$$

Assume that any subset $\mathcal{U}' \subset \mathcal{U}$ of cardinality less than κ does not satisfy condition ($\dagger_{\mathcal{U}'}$). Let $\mathcal{U}' \subset \mathcal{U}$ be a subset such that $|\mathcal{U}'| < \kappa$. Since $\mathcal{U}'$ does not satisfy condition ($\dagger_{\mathcal{U}'}$), it follows from Theorem 4.7 that there exists a κ-complete ultrafilter $\mathcal{F}_{\mathcal{U}'} \in \upsilon_\kappa X \setminus \bigcup_{U\in\mathcal{U}'} O(U)$ over X. Then, it holds that $\{X \setminus U \,|\, U \in \mathcal{U}'\} \subset \mathcal{F}_{\mathcal{U}'}$. Since $|\mathcal{U}'| < \kappa$, and $\mathcal{F}_{\mathcal{U}'}$ is κ-complete, this implies that $\bigcap_{U\in\mathcal{U}'} X \setminus U \neq \varnothing$. In particular, $\mathcal{F}$ is a κ-complete filter base over X. Since κ is strongly compact, there exists a κ-complete ultrafilter $\mathcal{F}^\dagger$ such that $\mathcal{F} \subset \mathcal{F}^\dagger$. Since $\mathcal{U}$ satisfies condition ($\dagger_{\mathcal{U}}$), this implies that

$$\mathcal{F}^\dagger \in \beta X \setminus \bigcup\{O(U) \,|\, U \in \mathcal{U}\} \subset \beta X \setminus \upsilon_\kappa X,$$

in contradiction to the fact that $\mathcal{F}^\dagger$ is κ-complete. Thus, there exists a subset $\mathcal{U}_0 \subset \mathcal{U}$ of cardinality less than κ such that $\mathcal{U}_0$ satisfies condition ($\dagger_{\mathcal{U}_0}$). This completes the proof of the implication "(1) $\Rightarrow$ (2)".

Next, we prove the implication "(2) $\Rightarrow$ (1)". Assume that any discrete space admits a functorial κ-Lindelöfification. Let X be a set and $\mathcal{F}$ a κ-complete filter over X. We regard X as a discrete topological space. Then, X admits a functorial κ-Lindelöfification. Hence, by Corollary 4.4, $\upsilon_\kappa X$ is κ-Lindelöf.

For any $F \in \mathcal{F}$, write $\tilde{F} \overset{\text{def}}{:=} \overline{F} \cap \upsilon_\kappa X \subset \upsilon_\kappa X$, where the closure $\overline{F}$ is taken as a subset of βX. Since $\mathcal{F}$ is κ-complete, for any $\mathcal{F}_0 \subset \mathcal{F}$ such that $|\mathcal{F}_0| < \kappa$, it holds that $\bigcap_{F\in\mathcal{F}_0} \tilde{F} \supset \bigcap \mathcal{F}_0 \in \mathcal{F}$, i.e., $\bigcap_{F\in\mathcal{F}_0} \tilde{F} \neq \varnothing$. Since $\upsilon_\kappa X$ is κ-Lindelöf, it holds that $\bigcap_{F\in\mathcal{F}} \tilde{F} \neq \varnothing$. Let $\mathcal{F}^\dagger \in \bigcap_{F\in\mathcal{F}} \tilde{F}$ be an element. Then, it holds that $\mathcal{F} \subset \mathcal{F}^\dagger$. Moreover, by Theorem 4.7,

$\mathcal{F}^{\dagger}$ is a κ-complete ultrafilter over X. Thus, we conclude that κ is strongly compact. This completes the proof of Theorem 4.11. ☺

Corollary 4.12. *If the cardinality of a discrete space X is strongly compact, then X admits a functorial $|X|$-Lindelöfification.*

Proof. Corollary 4.12 follows immediately from Theorem 4.11. ☺

§5. Examples

In this final section, we give some examples of functorial κ-Lindelöfifiable spaces. For any topological space X and any closed subspace $A \subset X$, we shall write X/A for the quotient topological space of X by the equivalent relation $(A \times A) \cup \Delta_X \subset X \times X$, where $\Delta_X \subset X \times X$ is the diagonal subset. Since we assume that X is completely regular, X/A is Hausdorff.

We shall introduce the following cardinal functions:

Definition 5.1. Let X be a topological space.

(1) The **Lindelöf degree** of X, denoted $L(X)$, is defined as the smallest infinite cardinal κ such that every open cover of X has a subcollection of cardinality $\leq \kappa$ which covers X (cf. [KJ84, Chapter 1, § 3]).

(2) We shall write

$$oL(X) \stackrel{\text{def}}{:=} \min\{\kappa \,|\, X = \upsilon_{\kappa^+}X\} + \omega_0.$$

We shall way that $oL(X)$ is the **outer Lindelöf degree** of X.

Remark 5.2. Let X be a topological space.

(1) One can verify easily that

$$L(X) = \min\left\{\kappa \,\middle|\, X \text{ is } \kappa^+\text{-Lindelöf}\right\} + \omega_0.$$

In particular, for a cardinal κ, X is κ-Lindelöf if and only if $L(X) < \kappa$.

(2) It follows immediately that $\aleph_0 \leq oL(X) \leq L(X) \leq \max\{|X|, \aleph_0\}$.

(3) By Remark 3.3, X is realcompact if and only if $oL(X) = \aleph_0$. Moreover, by Remark 4.5, if X is discrete, then $|X|$ is non-measurable if and only if $oL(X) = \aleph_0$.

(4) For any discrete space X such that $2^{\aleph_0} < |X|$, if $|X|$ is non-measurable, then it holds that $2^{\aleph_0} = 2^{oL(X)\chi(X)} < |X|$, where $\chi(X)$ is the *character* of X (cf. [KJ84, Chapter 1, §3]). Hence, the Arhangel'skiĭ-type inequality for $oL(-)$ and $\chi(-)$ does not hold.

For any linearly ordered set $(L, \leq)$ and elements $a, b \in L$ such that $a < b$, we shall write $[a, b] \stackrel{\text{def}}{:=} \{x \in L \,|\, a \leq x \leq b\} \subset L$. We regard L as a topoplogical space whose topology is generated by $\{[a, b] \setminus \{a, b\} \,|\, a, b \in L\}$.

Lemma 5.3. *Let κ be an infinite cardinal and α an infinite ordinal. Then, the following assertions hold:*

(1) *It holds that $L(\alpha) = \mathrm{cf}(\alpha) + \omega_0$. Moreover, for any open covering $\mathcal{U}$ of α, if $|\mathcal{U}| < \mathrm{cf}(\alpha)$, then there exists a subset $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| < \aleph_0$.*

(2) *If $\mathrm{cf}(\alpha) > \omega_0$, then $\beta\alpha = \alpha + 1$. In particular, it holds that $oL(\alpha) = \mathrm{cf}(\alpha) + \omega_0$.*

(3) *For any subset $A \subset \kappa + 1$, if $\kappa \in A$, then $L(A) < \kappa$.*

Proof. First, we prove assertion (1). Let $f : \mathrm{cf}(\alpha) \to \alpha$ be a map such that $\mathrm{Im}(f) \subset \alpha$ is unbounded and $\mathcal{U}$ an open covering of α. Since $[0, -] \subset \alpha$ is compact, for any $\gamma < \mathrm{cf}(\alpha)$, there exists a finite subcover $\mathcal{V}_\gamma \subset \mathcal{U}$ such that $[0, f(\gamma)] \subset \bigcup \mathcal{V}_\gamma$. Then, $\bigcup_{\gamma < \mathrm{cf}(\alpha)} \mathcal{V}_\gamma \subset \mathcal{U}$ is a subcover of $\mathcal{U}$ whose cardinality is less than or equal to $\mathrm{cf}(\alpha)$. This imples that α is $\mathrm{cf}(\alpha)^+$-Lindelöf.

Assume that there exists a subset $\mathcal{V}_0$ of the open covering $\{[0, \gamma] \,|\, \gamma < \alpha\}$ of α such that $\kappa_0 \stackrel{\text{def}}{:=} |\mathcal{V}_0| < \mathrm{cf}(\alpha)$ and that $\alpha = \bigcup \mathcal{V}_0$. Let $f_0 : \kappa_0 \xrightarrow{\sim} \mathcal{V}_0$ be a bijection. Then, since $\alpha = \bigcup \mathcal{V}_0$, the map $\kappa_0 \to \alpha, \gamma \mapsto \max f_0(\gamma)$ is unbounded. This contradicts to our assumption that $\kappa_0 < \mathrm{cf}(\alpha)$. Thus, α is not $\mathrm{cf}(\alpha)$-Lindelöf. In particular, it holds that $L(\alpha) = \mathrm{cf}(\alpha)$.

Let $\mathcal{W}$ be an open covering of α such that $|\mathcal{W}| < \mathrm{cf}(\alpha)$. If for any $W \in \mathcal{W}$, $\alpha \setminus W \subset \alpha$ is unbounded, then, by [Kun80, Chapter 2, Lemma 6.8 (a)], $\bigcap_{W \in \mathcal{W}}(\alpha \setminus W) \subset \alpha$ is also unbounded. This contradicts to our assumption that $\alpha = \bigcup \mathcal{W}$. This implies that there exists $W \in \mathcal{W}$ such that $\alpha \setminus W$ is bounded. Since $[0, \sup(\alpha \setminus W)] \subset \bigcup \mathcal{W}$, and $[0, \sup(\alpha \setminus W)]$ is compact, there exists a finite subset $\mathcal{W}_0 \subset \mathcal{W}$ such that $[0, \sup(\alpha \setminus W)] \subset \bigcup \mathcal{W}_0$. Thus, it holds that $\alpha = W \cup \bigcup \mathcal{W}_0$. This completes the proof of assertion (1).

Next, we prove assertion (2). By assertion (1), α is countably compact. Hence, α is pseudocompact. Let $f : \alpha \to \mathbb{R}$ be a continuous map. Then, f is bounded. To prove that $\beta\alpha = \alpha + 1$, it sufficies to prove that there exists $\gamma < \alpha$ such that for any $\gamma < \gamma_0 < \alpha$, $f(\gamma) = f(\gamma_0)$. Hence, we may assume without loss of generality that $\mathrm{Im}(f) \subset [0, 1]$. Since $\mathrm{cf}(\alpha) > \aleph_0$, it follows from [Kun80, Chapter 2, Lemma 6.8 (a)] that for any $n \in \mathbb{N}$, there exists a unique $0 \le k(n) < 2^n$ such that $f^{-1}([k(n)/2^n, (k(n) + 1)/2^n)) \subset \alpha$ is unbounded. Write a for the unique element $\bigcap_{n\in\mathbb{N}}[k(n)/2^n, (k(n)+1)/2^n)$. Since $\mathrm{cf}(\alpha) > \aleph_0$, there exists an ordinal $\gamma < \alpha$ such that $f([\gamma, \alpha)) = \{a\}$. This implies that $\beta\alpha = \alpha + 1$. Moreover, by assertion (1), for any subspace $\alpha \subset X \subset \beta\alpha$, if $L(X) < \mathrm{cf}(\alpha)$, then $X = \beta\alpha$. Hence, by assertion (1), it holds that $\mathrm{cf}(\alpha) \le oL(\alpha) \le L(\alpha) = \mathrm{cf}(\alpha)$. This completes the proof of assertion (2).

Next, we prove assertion (3). Let $\mathcal{U}$ be an open covering of A. Then, there exists an element $U \in \mathcal{U}$ such that $\kappa \in U$. Since $|\kappa \setminus U| < \kappa$, there exists a subset $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| < \kappa$, and $A \setminus U \subset \bigcup \mathcal{V}$. Then, $\mathcal{V} \cup \{U\} \subset \mathcal{U}$ is a subset such that $A \subset \bigcup(\mathcal{V} \cup \{U\})$, and $|\mathcal{V} \cup \{U\}| < \kappa$. This completes the proof of Lemma 5.3. ☺

Definition 5.4.

(1) For any set X, we shall write X_d for the discrete topological space obtained by the set X.

(2) For any topological spaces X and Y, we shall write $X \sqcup Y$ for the disjoint union of X and Y.

Example 5.5. Let κ and λ be (infinite) non-measurable cardinals. Then, the following assertions hold:

(1) Assume that $\aleph_0 < \mathrm{cf}(\kappa) = \kappa < \lambda$. Then, it holds that $oL(\kappa \sqcup \lambda_d) = \kappa < L(\kappa \sqcup \lambda_d) = \lambda$.

(2) Assume that $\kappa < \mathrm{cf}(\lambda) = \lambda$. Then, it holds that $oL(\kappa_d \sqcup \lambda) = L(\kappa_d \sqcup \lambda) = \lambda$, and, moreover, for any $\kappa < \mu \le \lambda$, $\kappa_d \sqcup \lambda$ is functorial μ-Lindelöfifiable but not functorial κ-Lindelöfifiable.

Proof. Since $\beta(\kappa \sqcup \lambda_d) = (\kappa+1) \sqcup \beta\lambda_d$, assertion (1) and the equality $oL(\kappa_d \sqcup \lambda) = L(\kappa_d \sqcup \lambda) = \lambda$ follow immediately from Corollary 4.8 (1) and Lemma 5.3 (1) (2). Moreover, for any $\aleph_0 < \mu \le \lambda$, it holds that $\upsilon_\mu(\kappa_d \sqcup \lambda) = \kappa_d \sqcup (\lambda + 1)$. Thus, the last assertion of assertion (2) follows. This completes the each assertions in Example 5.5. ☺

Next, we consider classes smaller than Lind.

Example 5.6. Write $\mathsf{LCLind} \subset \mathsf{Top}$ for the full subcategory determined by the locally compact Lindelöf spaces. The space of rational numbers $\mathbb{Q} \subset \mathbb{R}$ (whose topology is induced from $\mathbb{R}$) is σ-compact but not locally compact. Thus, $\mathbb{Q}$ is not functorial LCLind-ifiable.

Finally, we give an example of functorial Lindelöfifiable space that is not functorial σ-compactifiable.

Example 5.7. Write $A_0 \subset \omega_1$ for the set of countable successor ordinals and $A \overset{\mathrm{def}}{:=} A_0 \cup \{\omega_1\} \subset \omega_1 + 1$. Then, since A_0 is discrete, any compact subset of A is finite. In particular, A is Lindelöf (cf. Lemma 5.3 (3)) but not σ-compact. Write

$$X \overset{\mathrm{def}}{:=} (\omega_0 \times \omega_1) \cup (\{\omega_0\} \times A) \subset (\omega_0 + 1) \times (\omega_1 + 1).$$

Let $n < \omega_0$ be an integer and $f : \{n\} \times \omega_1 \to \mathbb{R}$ a continuous function. Since $\beta\omega_1 = \omega_1 + 1$, there exists a continuous extention $f_1 : \{n\} \times (\omega_1 + 1) \to \mathbb{R}$ of f. By applying Tieze's extention theorem to $(\omega_0 + 1) \times (\omega_1 + 1)$, there exists a continuous extension $\tilde{f} : X \to \mathbb{R}$ of f_1. This implies that $\omega_0 \times (\omega_1 + 1) \subset \upsilon X$. Since $\omega_0 \times (\omega_1 + 1)$ is Lindelöf, and A is Lindelöf, it holds that

$$\upsilon X = (\omega_0 \times (\omega_1 + 1)) \cup (\{\omega_0\} \times A).$$

Since A is not σ-compact, this implies that X is functorial Lindelöfifiable but not functorial σ-compactifiable. ☺

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