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INTRODUCTION TO GRAPH THEORY: EXERCISES

Contents

Graphs

Exercises

1. $\mathcal{P}(\{1,2,3\}) \coloneqq \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

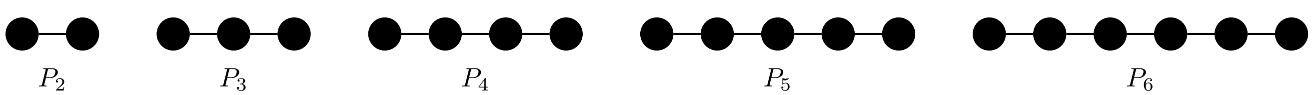
2.
$$\dfrac{\dfrac{\dfrac{\emptyset \nsubseteq A}{\exists x \in \emptyset : x \notin A} \qquad \dfrac{}{\neg \exists x \in \emptyset}}{\bot}}{\emptyset \subseteq A}$$

$\square$

3.
$$\dfrac{\dfrac{}{S(b) \coloneqq \{ m \in V : m \notin S(m) \}} \qquad \dfrac{}{b \in V}}{b \in S(b) \iff b \notin S(b)} \equiv \dfrac{\dfrac{}{R = \{ x \notin x \}}}{R \in R \iff R \notin R}$$

4. Let S be the collection of all sets that can be described in an English sentence of twenty-five words or less. S is not a set, because S can be described in fewer than twenty-five words, and if it were a set, then S would have to be a member of itself, which violates the axiomatic definition of a set.

5. The first five *path graphs*.

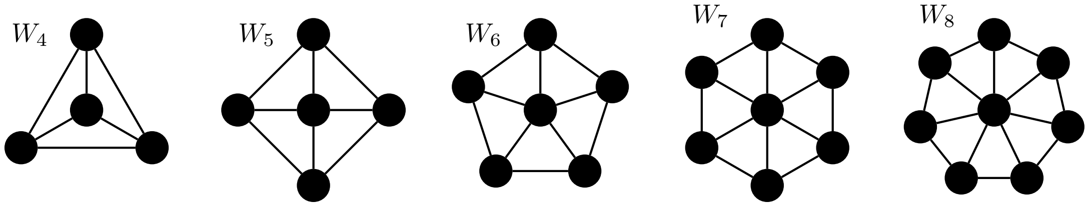


$$\begin{aligned} V(P_v) &= \{1, 2, ..., v\} \\ E(P_v) &= \{\{n-1, n\} : n \in \{2, ..., v\}\} \\ |E(P_v)| &= v - 1 \\ e &= v - 1 \end{aligned}$$

The number of edges in a *path graph* P_v, where $v \geq 2$, is given by the formula $e = v - 1$.

$\square$

6. The first five *wheel graphs*.



$$
\begin{aligned}
V(W_v) &= \{\,1, 2, 3, ..., v\,\} \\
E(W_v) &= \{\,\{\,1, 2\,\}, \{\,1, 3\,\}, ..., \{\,1, v\,\}, \\
&\qquad \{\,2, 3\,\}, \{\,3, 4\,\}, ..., \{\,v-1, v\,\}, \\
&\qquad \{\,v, 2\,\}\,\} \\
&= \{\,\{\,\{\,1, n\,\} : n \in \{\,2, ..., v\,\}\,\}, \\
&\qquad \{\,\{\,n-1, n\,\} : n \in \{\,3, ..., v\,\}\,\}, \\
&\qquad \{\,v, 2\,\}\,\} \\
|E(W_v)| &= (v-1) + (v-2) + 1 \\
&= (v-1) + (v-1) \\
e &= 2(v-1)
\end{aligned}
$$

The number of edges in the *wheel graph* on v vertices W_v, where $v \geq 4$, is given by the formula $e = 2(v-1)$.

□

7.

$$
\begin{aligned}
1 + 2 + ... + (v-1) &= (1/2)v(v-1) \\
&= E(K_v) \qquad \text{(T2)} \\
&= (v-1) + (v-2) + ... + (v-(v-1)) \\
&= 1 + 2 + ...(v-1)
\end{aligned}
$$

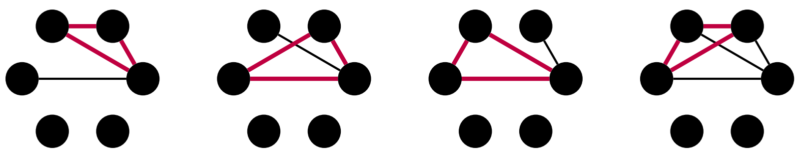
Imagine drawing K_v by joining vertex 1 to vertices 2 through v, creating $v-1$ edges; then joining vertex 2 to vertices 3 through v, creating $v-2$ edges; and so on, i.e. $(v-1)+(v-2)+...+(v-(v-1))$, or equivalently, $1+2+...+(v-1)$.

□

8. Let G be a graph with v vertices and e edges. In terms of v and e, $\overline{G}$ has $(1/2)v(v-1) - e$ edges.

9. If a graph G has $v = 6$ then G or $\overline{G}$ (possibly both) has a subgraph isomorphic to K_3.

   In the graph G or $\overline{G}$ there exists a vertex a of degree three or more. Let there be three adjacent vertices b, c, and d. If any of the edges $\{ b, c \}$, $\{ b, d \}$, or $\{ c, d \}$ are present in the graph, then the graph contains a subgraph isomorphic to K_3. If none of those edges is present, then they are all present in the other graph, which thus contains a subgraph isomorphic to K_3.



10. From the proof in Exercise 9, it follows that in any graph of degree 6 there exists a subgraph H, such that either H or $\overline{H}$ is isomorphic to K_3. Equivalently, in any gathering of six people (a graph G with $v = 6$) there are either three people who are mutually acquainted (a subgraph isomorphic to K_3) or three people who are mutually unacquainted (a subgraph whose complement is isomorphic to K_3).

11.

$$2|\emptyset| = \sum_{v \in V} deg(v) = 0 \quad \text{(BC)}$$

$$G(V, E) \to 2|E| = \sum_{v \in V} deg(v) \quad \text{(IH)}$$

$$H(V', E'), |E'| = |E| + 1 \quad (1)$$

$$\exists\ H_0(V', E'_0), |E'_0| = |E| \quad (2)$$

$$2|E'_0| = \sum_{v \in V'} deg(v) \quad \text{(by IH)}$$

$$H \cong (V', E'_0 \cup \{ e \}), e \notin E'_0 \quad (3)$$

$$\sum_{v \in V'} deg(v) + 2 = 2|E| \quad (4)$$

$$2|E| + 2 = 2(|E| + 1) = \sum_{v \in V'} deg(v) \quad (5)$$

$\square$

12. a)

$$(4 \times 3) + (2 \times 5) + (2 \times 6) + (1 \times 8) = 2e$$
$$12 + 10 + 12 + 8 = 2e$$
$$21 = e$$

b) $(7 \times 3) = 2e \to e \notin \mathbb{Q}$

13. If $C_n \cong \overline{C_n}$, then it must be the case that $|E(C_n)| + \left|E(\overline{C_n})\right| = |E(K_n)|$, i.e. $n + n = \frac{n(n-1)}{2}$.

$$
\begin{aligned}
n + n &= \frac{n(n-1)}{2} \\
2n &= \frac{n(n-1)}{2} \\
4n &= n(n-1) \\
4n &= n^2 - n \\
0 &= n^2 - 5n \\
n &= \{\, 0, 5 \,\}
\end{aligned}
$$

Since $n = 5$ is a solution, $|E(C_5)| + \left|E(\overline{C_5})\right| = |E(K_5)|$, and thus $C_5 \cong \overline{C_5}$.

□

A graph with no edges ($n = 0$) has no cycles. Therefore the only self-complementary cycle graph is C_5.

□

14. If $G \cong \overline{G}$ for a graph G of order v, then $|E(G)| + \left|E(\overline{G})\right| = |E(K_v)|$, i.e. $e + e = 2e = \frac{v(v-1)}{2}$. So the number of edges e in G is $\frac{v(v-1)}{4}$, and $4 \mid v(v-1)$. Therefore, for any self-complementary graph G of order v, $4 \mid v$ or $4 \mid v - 1$.

□

15. .

Find two other self-complementary graphs (Trudeau, *Graph Theory* 57)

Glossary

path graph If v is a an integer greater than or equal to 2, the *path graph* on v vertices, denoted "P_v", is the graph having the vertex set $\{\, 1, 2, 3, ..., v \,\}$ and edge set $\{\, \{\, 1, 2 \,\}, \{\, 2, 3 \,\}, \{\, 3, 4 \,\}, ..., \{\, v - 1, v \,\} \,\}$. 5

wheel graph . 6

define