ERIC BAILEY

# GRAPH THEORY EXERCISES

*Contents*

# *Graphs*

## *Exercises*

1. $\mathcal{P}(\{\,1,2,3\,\}) \coloneqq \{\,\emptyset, \{\,1\,\}, \{\,2\,\}, \{\,3\,\}, \{\,1,2\,\}, \{\,1,3\,\}, \{\,2,3\,\}, \{\,1,2,3\,\}\,\}$

2.

$$\dfrac{\dfrac{\dfrac{\emptyset \not\subseteq A}{\exists x \in \emptyset : x \notin A} \qquad \dfrac{}{\neg \exists x \in \emptyset}}{\bot}}{\emptyset \subseteq A}$$

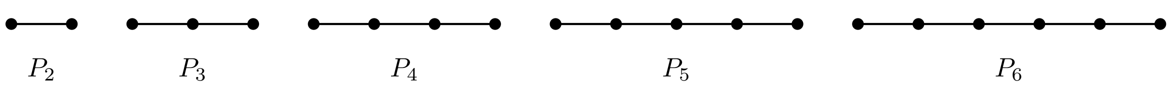
$\square$

3.

$$\dfrac{\dfrac{}{S(b) \coloneqq \{\, m \in V : m \notin S(m) \,\}} \qquad \dfrac{}{b \in V}}{b \in S(b) \iff b \notin S(b)} \equiv \dfrac{\dfrac{}{R = \{\, x \notin x \,\}}}{R \in R \iff R \notin R}$$

4. Let $S$ be the collection of all sets that can be described in an English sentence of twenty-five words or less. $S$ is not a set, because $S$ can be described in fewer than twenty-five words, and if it were a set, then $S$ would have to be a member of itself, which violates the axiomatic definition of a set.

5. The first five *path graphs*.




$$\begin{aligned} V(P_v) &= \{\,1,2,...,v\,\} \\ E(P_v) &= \{\,\{\,n-1,n\,\} : n \in \{\,2,...,v\,\}\,\} \\ |E(P_v)| &= v-1 \\ e &= v-1 \end{aligned}$$

The number of edges in a *path graph* $P_v$, where $v \geq 2$, is given by the formula $e = v - 1$.

$\square$

6.

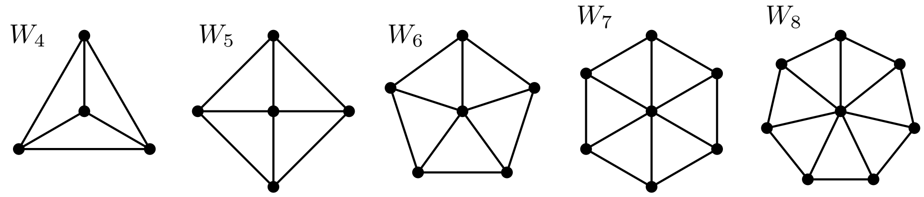



Figure 1: The first five *wheel graphs*.

$$
\begin{aligned}
V(W_v) &= \{\,1,2,3,...,v\,\} \\
E(W_v) &= \{\ \{\,1,2\,\},\{\,1,3\,\},...,\{\,1,v\,\}, \\
&\qquad \{\,2,3\,\},\{\,3,4\,\},...,\{\,v-1,v\,\}, \\
&\qquad \{\,v,2\,\}\ \} \\
&= \{\ \{\,\{\,1,n\,\} : n \in \{\,2,...,v\,\}\,\}, \\
&\qquad \{\,\{\,n-1,n\,\} : n \in \{\,3,...,v\,\}\,\}, \\
&\qquad \{\,v,2\,\}\ \} \\
|E(W_v)| &= (v-1)+(v-2)+1 \\
&= (v-1)+(v-1) \\
e &= 2(v-1)
\end{aligned}
$$

The number of edges in the *wheel graph* on $v$ vertices $W_v$, where $v \geq 4$, is given by the formula $e = 2(v-1)$.

□

7.

$$
\begin{aligned}
1+2+...+(v-1) &= (1/2)v(v-1) \\
&= E(K_v) \qquad \text{(T2)} \\
&= (v-1)+(v-2)+...+(v-(v-1)) \\
&= 1+2+...(v-1)
\end{aligned}
$$

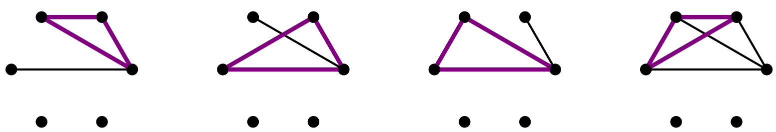
Imagine drawing $K_v$ by joining vertex 1 to vertices 2 through $v$, creating $v-1$ edges; then joining vertex 2 to vertices 3 through $v$, creating $v-2$ edges; and so on, i.e. $(v-1)+(v-2)+...+(v-(v-1))$, or equivalently, $1+2+...+(v-1)$.

□

8. Let $G$ be a graph with $v$ vertices and $e$ edges. In terms of $v$ and $e$, $\overline{G}$ has $(1/2)v(v-1)-e$ edges.

9. If a graph $G$ has $v = 6$ then $G$ or $\overline{G}$ (possibly both) has a subgraph isomorphic to $K_3$.

   In the graph $G$ or $\overline{G}$ there exists a vertex $a$ of degree three or more. Let there be three adjacent vertices $b$, $c$, and $d$. If any of the edges $\{ b, c \}$, $\{ b, d \}$, or $\{ c, d \}$ are present in the graph, then the graph contains a subgraph isomorphic to $K_3$. If none of those edges is present, then they are all present in the other graph, which thus contains a subgraph isomorphic to $K_3$.



10. From the proof in Exercise 9, it follows that in any graph of degree 6 there exists a subgraph $H$, such that either $H$ or $\overline{H}$ is isomorphic to $K_3$. Equivalently, in any gathering of six people (a graph $G$ with $v = 6$) there are either three people who are mutually acquainted (a subgraph isomorphic to $K_3$) or three people who are mutually unacquainted (a subgraph whose complement is isomorphic to $K_3$).

11.

$$2|\emptyset| = \sum_{v \in V} deg(v) = 0 \tag{BC}$$

$$G(V, E) \to 2|E| = \sum_{v \in V} deg(v) \tag{IH}$$

$$H(V', E'), |E'| = |E| + 1 \tag{1}$$

$$\exists\ H_0(V', E_0'), |E_0'| = |E| \tag{2}$$

$$2|E_0'| = \sum_{v \in V'} deg(v) \tag{by IH}$$

$$H \cong (V', E_0' \cup \{ e \}), e \notin E_0' \tag{3}$$

$$\sum_{v \in V'} deg(v) + 2 = 2|E| \tag{4}$$

$$2|E| + 2 = 2(|E| + 1) = \sum_{v \in V'} deg(v) \tag{5}$$

$\square$

12. a)

$$\begin{aligned}(4 \times 3) + (2 \times 5) + (2 \times 6) + (1 \times 8) &= 2e \\ 12 + 10 + 12 + 8 &= 2e \\ 21 &= e\end{aligned}$$

b) $(7 \times 3) = 2e \to e \notin \mathbb{Q}$

13. If $C_n \cong \overline{C_n}$, then it must be the case that $|E(C_n)| + \left|E(\overline{C_n})\right| = |E(K_n)|$, i.e. $n + n = \frac{n(n-1)}{2}$.

$$\begin{aligned} n + n &= \frac{n(n-1)}{2} \\ 2n &= \frac{n(n-1)}{2} \\ 4n &= n(n-1) \\ 4n &= n^2 - n \\ 0 &= n^2 - 5n \\ n &= \{\, 0, 5 \,\} \end{aligned}$$

Figure 2: Solving for $n$ in Exercise 13

Since $n = 5$ is a solution, $|E(C_5)| + \left|E(\overline{C_5})\right| = |E(K_5)|$, and thus $C_5 \cong \overline{C_5}$.

$\square$

A graph with no edges ($n = 0$) has no cycles. Therefore the only self-complementary cycle graph is $C_5$.

$\square$

14. If $G \cong \overline{G}$ for a graph $G$ of order $v$, then $|E(G)| + \left|E(\overline{G})\right| = |E(K_v)|$, i.e. $e + e = 2e = \frac{v(v-1)}{2}$. So the number of edges $e$ in $G$ is $\frac{v(v-1)}{4}$, and $4 \mid v(v-1)$. Therefore, for any self-complementary graph $G$ of order $v$, $4 \mid v$ or $4 \mid v - 1$.

$\square$

15. 

Find two other self-complementary graphs (Trudeau, *Graph Theory* 57).

16. 

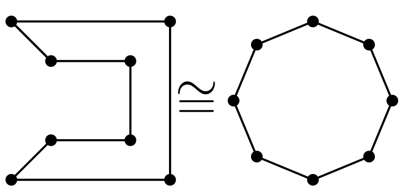
Find a self-complementary graph with $v = 8$.

17. $P_4$ is the self-complementary graph with $v = 4$.

18. Between the two set of 10 elements, $\{\, a, b, ..., j \,\}$ and $\{\, 1, 2, ..., 10 \,\}$, there exist $10!$, or $3,628,800$, one-to-one correspondences.

19. $f : \mathbb{Z}^+ \to 2\mathbb{Z}^+$, $f(x) = 2n$ is bijective.

20. The first graph in Figure 35 contains a subgraph



which is isomorphic to $C_8$, whereas the second graph contains no subgraph isomorphic to $C_8$. Therefore, the two graphs are not isomorphic.

21.

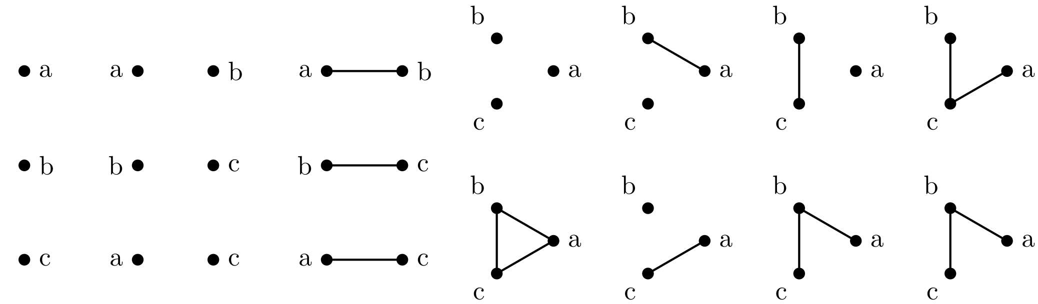


Figure 3: The 17 unequal subgraphs of $K_3$

22.

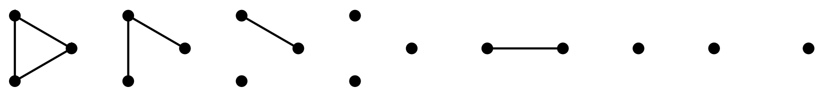
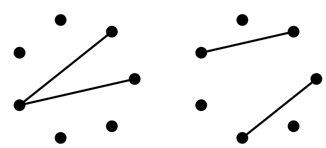


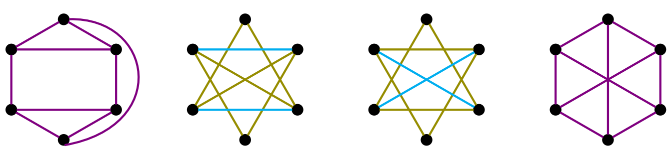
Figure 4: The seven *nonisomorphic* subgraphs of $K_3$

23. $C_3$ is a subgraph of the first graph, but not the second. Therefore, the two graphs are not isomorphic.

24. The graphs of Figure 31 are not isomorphic because their complements have different degree distributions: the first has one vertex of degree 2, two of degree 1, and four of degree 0, whereas the second has four of degree 1, and three of degree 0.



25. The graphs of Figure 46 are not isomorphic beacuse their complements are nonisomorphic, as shown below. Notably, the cyan edges in one complement are missing in the other.



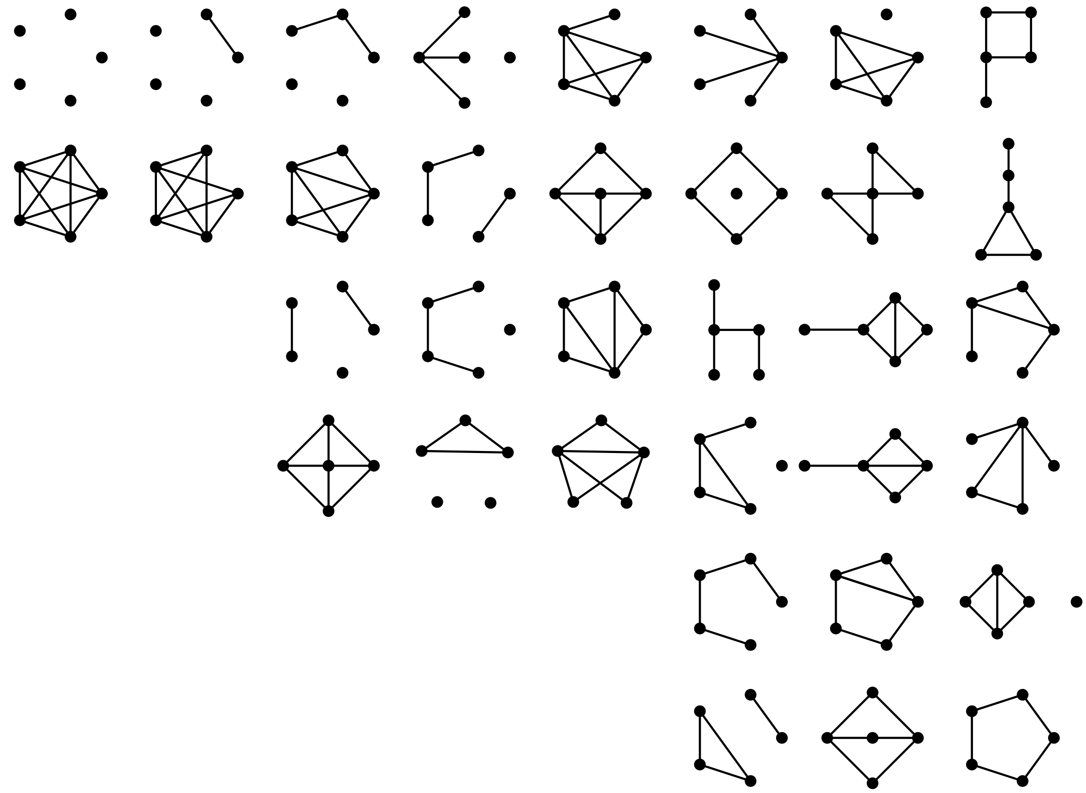
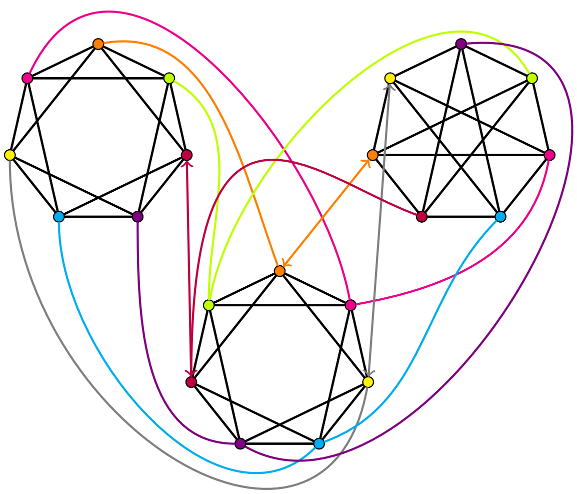
26. 

Figure 5: All 34 graphs having $v = 5$

27. There are an odd number of nonisomorphic graphs with $v = 4$. Is this special? Are there other values of $v$ for which the number of nonisomorphic graphs is also odd? (Trudeau, *Graph Theory* 59). https://math.stackexchange.com/questions/86790/is-there-a-reason-why-the-number-of-non-isomorphic-graphs-with-v-4-is-odd

28.

Figure 6: A pair of isomorphic graphs and their shared canonization



# *Glossary*

*nonisomorphic* . 9 Define nonisomorphic

*path graph* If $v$ is a an integer greater than or equal to 2, the *path graph* on $v$ vertices, denoted “$P_v$”, is the graph having the vertex set $\{1, 2, 3, ..., v\}$ and edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, ..., \{v - 1, v\}\}$. 5

*wheel graph* Given a vertex set of $\{1, 2, 3, ..., v\}$, where $v \geq 4$, the edge set of the *wheel graph* $W_v$ is $\{\{1, 2\}, \{1, 3\}, ..., \{1, v\}, \{2, 3\}, \{3, 4\}, ..., \{v - 1, v\}, \{v, 2\}\}$. 6