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INTRODUCTION TO GRAPH THEORY: EXERCISES

Contents

Graphs

Exercises

1. $\mathcal{P}(\{1,2,3\}) \coloneqq \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

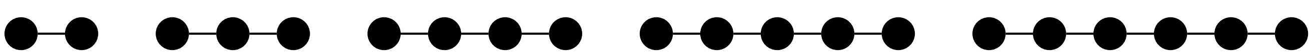
2. $$\dfrac{\dfrac{\dfrac{\emptyset \nsubseteq A}{\exists x \in \emptyset : x \notin A} \qquad \dfrac{}{\neg \exists x \in \emptyset}}{\bot}}{\emptyset \subseteq A}$$

$\square$

3. $$\dfrac{\dfrac{}{S(b) \coloneqq \{ m \in V : m \notin S(m) \}} \qquad \dfrac{}{b \in V}}{b \in S(b) \iff b \notin S(b)} \equiv \dfrac{\dfrac{}{R = \{ x \notin x \}}}{R \in R \iff R \notin R}$$

4. Let S be the collection of all sets that can be described in an English sentence of twenty-five words or less. S is not a set, because S can be described in fewer than twenty-five words, and if it were a set, then S would have to be a member of itself, which violates the axiomatic definition of a set.

5. The first five *path graphs*.



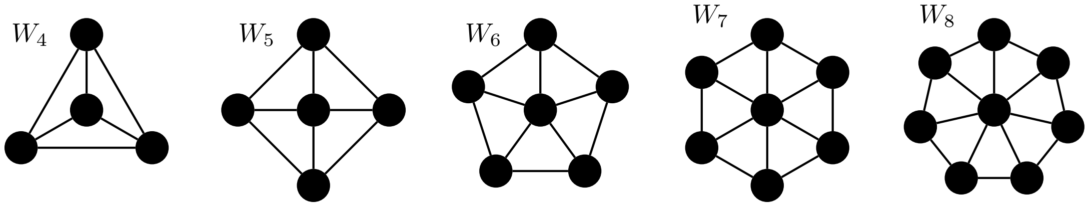
P_2 P_3 P_4 P_5 P_6

$$\begin{aligned} V(P_v) &= \{1, 2, ..., v\} \\ E(P_v) &= \{\{n-1, n\} : n \in \{2, ..., v\}\} \\ |E(P_v)| &= v - 1 \\ e &= v - 1 \end{aligned}$$

The number of edges in a *path graph* P_v, where $v \geq 2$, is given by the formula $e = v - 1$.

$\square$

6. The first five *wheel graphs*.



$$
\begin{aligned}
V(W_v) &= \{\, 1, 2, 3, ..., v \,\} \\
E(W_v) &= \{\, \{\, 1, 2 \,\}, \{\, 1, 3 \,\}, ..., \{\, 1, v \,\}, \\
&\qquad \{\, 2, 3 \,\}, \{\, 3, 4 \,\}, ..., \{\, v-1, v \,\}, \\
&\qquad \{\, v, 2 \,\} \,\} \\
&= \{\, \{\, \{\, 1, n \,\} : n \in \{\, 2, ..., v \,\} \,\}, \\
&\qquad \{\, \{\, n-1, n \,\} : n \in \{\, 3, ..., v \,\} \,\}, \\
&\qquad \{\, v, 2 \,\} \,\} \\
|E(W_v)| &= (v-1) + (v-2) + 1 \\
&= (v-1) + (v-1) \\
e &= 2(v-1)
\end{aligned}
$$

The number of edges in the *wheel graph* on v vertices W_v, where $v \geq 4$, is given by the formula $e = 2(v-1)$.

□

7.

$$
\begin{aligned}
1 + 2 + ... + (v-1) &= (1/2)v(v-1) \\
&= E(K_v) && \text{(T2)} \\
&= (v-1) + (v-2) + ... + (v-(v-1)) \\
&= 1 + 2 + ...(v-1)
\end{aligned}
$$

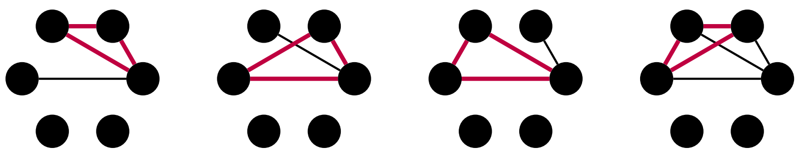
Imagine drawing K_v by joining vertex 1 to vertices 2 through v, creating $v-1$ edges; then joining vertex 2 to vertices 3 through v, creating $v-2$ edges; and so on, i.e. $(v-1) + (v-2) + ... + (v-(v-1))$, or equivalently, $1 + 2 + ... + (v-1)$.

□

8. Let G be a graph with v vertices and e edges. In terms of v and e, $\overline{G}$ has $(1/2)v(v-1) - e$ edges.

9. If a graph G has $v = 6$ then G or $\overline{G}$ (possibly both) has a subgraph isomorphic to K_3.

   In the graph G or $\overline{G}$ there exists a vertex a of degree three or more. Let there be three adjacent vertices b, c, and d. If any of the edges $\{ b, c \}$, $\{ b, d \}$, or $\{ c, d \}$ are present in the graph, then the graph contains a subgraph isomorphic to K_3. If none of those edges is present, then they are all present in the other graph, which thus contains a subgraph isomorphic to K_3.



10. From the proof in Exercise 9, it follows that in any graph of degree 6 there exists a subgraph H, such that either H or $\overline{H}$ is isomorphic to K_3. Equivalently, in any gathering of six people (a graph G with $v = 6$) there are either three people who are mutually acquainted (a subgraph isomorphic to K_3) or three people who are mutually unacquainted (a subgraph whose complement is isomorphic to K_3).

11.

$$
\begin{aligned}
2|\emptyset| &= \sum_{v \in V} deg(v) = 0 && \text{(BC)} \\
G(V, E) \rightarrow 2|E| &= \sum_{v \in V} deg(v) && \text{(IH)} \\
H(V', E'), |E'| &= |E| + 1 && (1) \\
\exists\ H_0(V', E'_0), |E'_0| &= |E| && (2) \\
2|E'_0| &= \sum_{v \in V'} deg(v) && \text{(by IH)} \\
H &\cong (V', E'_0 \cup \{ e \}), e \notin E'_0 && (3) \\
\sum_{v \in V'} deg(v) + 2 &= 2|E| && (4) \\
2|E| + 2 = 2(|E| + 1) &= \sum_{v \in V'} deg(v) && (5)
\end{aligned}
$$

□

12. a)

$$
\begin{aligned}
(4 \times 3) + (2 \times 5) + (2 \times 6) + (1 \times 8) &= 2e \\
12 + 10 + 12 + 8 &= 2e \\
21 &= e
\end{aligned}
$$

b) $(7 \times 3) = 2e \rightarrow e \notin \mathbb{Q}$

13. p57

Glossary

path graph If v is a an integer greater than or equal to 2, the *path graph* on v vertices, denoted “P_v”, is the graph having the vertex set $\{1, 2, 3, ..., v\}$ and edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, ..., \{v - 1, v\}\}$. 5

wheel graph . 6

define