The probability functions of the GEV distribution and their derivatives

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Contents

Note

In this report

- The expressions computed by **Maxima** are reported in green.
- The expressions involving some manual calculations are reported in red. These are intended to be easier to implement as they make use of "auxiliary quantities" that may be reused in several expressions.

The simplifications are not needed in the section devoted to the quantile but they help much for the log-density and the distribution. Although they have been carfully checked, the expressions in red could still contain some errors. Please report any error at `https://github.com/yvesdeville/nieve/issues/`.

1 Auxiliary variables

In order to evaluate the the log-density $\log f(y)$ or the distribution function $F(y)$ for some y, the following auxiliary variables will be used

$$\begin{cases} z := \dfrac{y-\mu}{\sigma}, \\ V := 1 + \xi z, \\ W := V^{-1/\xi}, \\ T := \dfrac{\log V}{\xi^2} - \dfrac{z}{\xi V}, \\ R := \dfrac{1}{\xi}\left[2T - \dfrac{z^2}{V^2}\right]. \end{cases} \tag{1}$$

In practice z should take quite moderate value, say between -30 and 30 and if this is not the case, the wanted results can be anticipated. It is worth noting that for fixed values of y, of the location μ and the scale σ these quantities all have a finite limit and Taylor expansion for $\xi \approx 0$

$$\begin{aligned} W &= e^{-z} + \frac{z^2 e^{-z}}{2}\,\xi - \frac{z^3 e^{-z}[3z-8]}{24}\,\xi^2 + o(\xi^2), \\ T &= \frac{z^2}{2} - \frac{2z^3}{3}\xi + \frac{3z^4}{4}\xi^2 + o(\xi^2), \\ R &= -\frac{z^3}{3} + \frac{z^4}{2}\,\xi - \frac{3z^5}{5}\,\xi^2 + o(\xi^2). \end{aligned}$$

So although their are not defined when $\xi = 0$, the quantities W, T and R can be prolongated by continuity at $\xi = 0$. The log-density, the distribution functions and their derivatives can be expressed with this auxiliary variables. Although W and T require the use of trancendental functions: non-fractional power and log, the functions are subsequently obtained multiplications and divisions of these quantities. So if the auxiliary variables of (1) are safely evaluated in the wanted range of values for y and the GEV parameters, then all the results of interest will be evaluated safely and efficiently.

2 Log-density $\log f$

2.1 Expression

The log-likelihood is defined by

$$\log f = -\left(\frac{1}{\xi} + 1\right) \log\left(\xi\, z + 1\right) - \frac{1}{\left(\xi\, z + 1\right)^{\frac{1}{\xi}}} - \log\sigma$$

$$\log f := -\log\sigma - (\xi + 1)\left[\xi T + \frac{z}{V}\right] - W \tag{2}$$

2.2 Taylor expansion at $\xi = 0$

$$\log f = -\frac{e^z\, \log\sigma + e^z\, z + 1}{e^z} + \frac{\left(\left(e^z - 1\right)\, z^2 - 2\, e^z\, z\right)\, \xi}{2\, e^z} - \frac{\left(3\, z^4 + \left(8\, e^z - 8\right)\, z^3 - 12\, e^z\, z^2\right)\, \xi^2}{24\, e^z} + \cdots$$

which can be simplified as

$$\log f = -\left[\log\sigma + z + e^{-z}\right] + \frac{1}{2}\, \left[(1 - e^{-z})z - 2\right]\, z\xi - \frac{1}{24}\, \left[3z^2 e^{-z} + 8z(1 - e^{-z}) - 12\right] z^2\xi^2 \tag{3}$$

2.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z, so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}\,\frac{\partial \log f}{\partial z}$$

and up to the term $-\log\sigma$ in $\log f$, the same is true for σ so it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ using

$$\begin{aligned} \frac{\partial \log f}{\partial \mu} &= -\frac{1}{\sigma}\,\frac{\partial \log f}{\partial z} \\ \frac{\partial \log f}{\partial \sigma} &= -\frac{1}{\sigma} - \frac{z}{\sigma}\,\frac{\partial \log f}{\partial z}. \end{aligned} \tag{4}$$

Raw expressions

Here are the raw expressions found by Maxima

$$\frac{\partial \log f}{\partial z} = \left(\xi\, z + 1\right)^{-\frac{1}{\xi} - 1} - \frac{\left(\frac{1}{\xi} + 1\right)\, \xi}{\xi\, z + 1}$$

$$\frac{\partial \log f}{\partial \xi} = -\frac{\frac{\log(\xi\, z + 1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z + 1)}}{\left(\xi\, z + 1\right)^{\frac{1}{\xi}}} + \frac{\log\left(\xi\, z + 1\right)}{\xi^2} - \frac{\left(\frac{1}{\xi} + 1\right)\, z}{\xi\, z + 1}$$

Simplified expressions

We use the following simplifications

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= \frac{1}{V}\,[W-\xi-1]\\ \frac{\partial \log f}{\partial \xi} &= T\,[1-W]-\frac{z}{V}\end{aligned} \tag{5}$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\frac{e^z-1}{e^z}+\frac{\left(z^2+(2\,e^z-2)\;z-2\,e^z\right)\;\xi}{2\,e^z}+\cdots\\ \frac{\partial \log f}{\partial \xi} &= \frac{(e^z-1)\;z^2-2\,e^z\,z}{2\,e^z}-\frac{\left(3\,z^4+(8\,e^z-8)\;z^3-12\,e^z\,z^2\right)\;\xi}{12\,e^z}+\cdots\end{aligned}$$

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\left[1-e^{-z}\right]-\frac{1}{2}\,\left[2-2(1-e^{-z})z-e^{-z}z^2\right]\,\xi+o(\xi)\\ \frac{\partial \log f}{\partial \xi} &= -\frac{1}{2}\,\left[2z-(1-e^{-z})z^2\right]+\frac{1}{12}\,\left[12-8(1-e^{-z})z-3z^2\right]\,z^2\xi+o(\xi)\end{aligned} \tag{6}$$

Limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned}\lim_{\xi\to 0}\frac{\partial \log f}{\partial z} &= -e^{-z}\,\left(e^z-1\right)\\ \lim_{\xi\to 0}\frac{\partial \log f}{\partial \xi} &= \frac{e^{-z}\,\left(z^2\,\left(e^z-1\right)-2\,z\,e^z\right)}{2}\end{aligned}$$

that is

$$\begin{aligned}\lim_{\xi\to 0}\frac{\partial \log f}{\partial z} &= 1-e^{-z}\\ \lim_{\xi\to 0}\frac{\partial \log f}{\partial \xi} &= -\frac{z}{2}\,\left[2-(1-e^{-z})z\right]\end{aligned} \tag{7}$$

2.5 Second-order derivatives: expressions

Scaling

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial \mu^2} &= \frac{1}{\sigma^2}\,\frac{\partial^2 \log f}{\partial z^2}\\
\frac{\partial^2 \log f}{\partial \mu \partial \sigma} &= \frac{1}{\sigma^2}\left\{\frac{\partial \log f}{\partial z} + z\frac{\partial^2 \log f}{\partial z^2}\right\}\\
\frac{\partial^2 \log f}{\partial \mu \partial \xi} &= \frac{-1}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}\\
\frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1}{\sigma^2} + \frac{z}{\sigma^2}\left\{2\,\frac{\partial \log f}{\partial z} + z\,\frac{\partial^2 \log f}{\partial z^2}\right\}\\
\frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}
\end{aligned} \tag{8}$$

Raw expressions

$$\frac{\partial^2 \log f}{\partial z^2} = \left(-\frac{1}{\xi} - 1\right)\,\xi\,(\xi\,z+1)^{-\frac{1}{\xi}-2} + \frac{\left(\frac{1}{\xi}+1\right)\,\xi^2}{(\xi\,z+1)^2}$$

$$\frac{\partial^2 \log f}{\partial z \partial \xi} = (\xi\,z+1)^{-\frac{1}{\xi}-1}\left(\frac{\log(\xi\,z+1)}{\xi^2} + \frac{\left(-\frac{1}{\xi}-1\right)\,z}{\xi\,z+1}\right) - \frac{\frac{1}{\xi}+1}{\xi\,z+1} + \frac{1}{\xi\,(\xi\,z+1)} + \frac{\left(\frac{1}{\xi}+1\right)\,\xi\,z}{(\xi\,z+1)^2}$$

$$\frac{\partial^2 \log f}{\partial \xi^2} = -\frac{\left(\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}\right)^2}{(\xi\,z+1)^{\frac{1}{\xi}}} - \frac{-\frac{2\,\log(\xi\,z+1)}{\xi^3} + \frac{2\,z}{\xi^2\,(\xi\,z+1)} + \frac{z^2}{\xi\,(\xi\,z+1)^2}}{(\xi\,z+1)^{\frac{1}{\xi}}} - \frac{2\,\log(\xi\,z+1)}{\xi^3} + \frac{2\,z}{\xi^2\,(\xi\,z+1)} + \frac{\left(\frac{1}{\xi}+1\right)\,z^2}{(\xi\,z+1)^2}$$

Simplified expressions

Using the auxiliary variables (1)

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial z^2} &= -\frac{\xi+1}{V^2}\,[W-\xi]\\
\frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{W}{V}\,\left[T - \frac{z}{V}\right] - \frac{1}{V} + (\xi+1)\frac{z}{V^2}\\
\frac{\partial^2 \log f}{\partial \xi^2} &= -W\left\{T^2 - R\right\} - R + \frac{z^2}{V^2}
\end{aligned} \tag{9}$$

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z}\\
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{e^{-z}\,\left(z\,(2\,e^z-2) - 2\,e^z + z^2\right)}{2}\\
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} &= -\frac{e^{-z}\,\left(z^3\,(8\,e^z-8) - 12\,z^2\,e^z + 3\,z^4\right)}{12}
\end{aligned}$$

$$
\begin{aligned}
\lim_{\xi\to 0}\frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z}\\
\lim_{\xi\to 0}\frac{\partial^2 \log f}{\partial z\partial\xi} &= \frac{1}{2}\left[-2-2\,(1-e^{-z})\,z+e^{-z}z^2\right]\\
\lim_{\xi\to 0}\frac{\partial^2 \log f}{\partial^2\xi^2} &= \frac{z^2}{12}\left[12+8\,(1-e^{-z})\,z-3\,e^{-z}z^2\right]
\end{aligned}
$$

3 Distribution function F

3.1 Expression

Using the auxiliary variables (1)

$$F := \exp\{-W\}. \tag{10}$$

3.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$F = \frac{1}{e^{\frac{1}{e^z}}} - \frac{z^2\,\xi}{2\,e^{\frac{1}{e^z}}\,e^z} - \frac{\left((3\,e^z-3)\,z^4-8\,e^z\,z^3\right)\,\xi^2}{24\,e^{\frac{1}{e^z}}\,(e^z)^2} + \cdots$$

Simplified expression

$$F = \exp\{-e^{-z}\}\left\{1-e^{-z}\frac{z^2}{2}\xi+e^{-z}\frac{z^3}{24}\left[8-3(1-e^{-z})z\right]\xi^2\right\}+o(\xi^2) \tag{11}$$

3.3 First-order derivatives: expressions

Scaling

$$
\begin{aligned}
\frac{\partial F}{\partial\mu} &= \frac{-1}{\sigma}\,\frac{\partial F}{\partial z}\\
\frac{\partial F}{\partial\sigma} &= \frac{-z}{\sigma}\,\frac{\partial F}{\partial z}
\end{aligned}
\tag{12}
$$

Raw expressions

$$\frac{\partial F}{\partial z} = (\xi\, z+1)^{-\frac{1}{\xi}-1}\; e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}}$$

$$\frac{\partial F}{\partial \xi} = -\frac{e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}}\left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)}{(\xi\, z+1)^{\frac{1}{\xi}}}$$

Simplified expressions

$$\begin{aligned}\frac{\partial F}{\partial z} &= FW\,\frac{1}{V}\\ \frac{\partial F}{\partial \xi} &= -FWT\end{aligned} \tag{13}$$

3.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\frac{\partial F}{\partial z} = \frac{1}{e^{\frac{z\,e^z+1}{e^z}}} + \frac{\left((e^z-1)\; z^2 - 2\,e^z\, z\right)\,\xi}{2\,e^{\frac{z\,e^z+1}{e^z}}\; e^z} + \cdots$$

$$\frac{\partial F}{\partial \xi} = -\frac{z^2}{2\,e^{\frac{z\,e^z+1}{e^z}}} - \frac{\left((3\,e^z-3)\; z^4 - 8\,e^z\, z^3\right)\,\xi}{12\,e^{\frac{z\,e^z+1}{e^z}}\; e^z} + \cdots$$

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ which is the value of the standard Gumbel distribution function at z,

$$\begin{aligned}\frac{\partial F}{\partial z} &= F^\star\left\{1 + \frac{z}{2}\left[-2 + (1-e^{-z})z\right]\xi\right\} + o(\xi)\\ \frac{\partial F}{\partial \xi} &= F^\star\left\{-\frac{z^2}{2} + \frac{z^3}{12}\left[8 - 3\,(1-e^{-z})z\right]\xi\right\} + o(\xi)\end{aligned} \tag{14}$$

3.5 Second-order derivatives

CAUTION The second-order derivatives has not yet been implemented and the formula have not yet been fully checked.

Scaling

$$
\begin{aligned}
\frac{\partial^2 F}{\partial \mu^2} &= \frac{1}{\sigma^2}\,\frac{\partial^2 F}{\partial z^2} \\
\frac{\partial^2 F}{\partial \mu \partial \sigma} &= \frac{1}{\sigma^2}\,\left\{\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}\right\} \\
\frac{\partial^2 F}{\partial \mu \partial \xi} &= \frac{-1}{\sigma}\,\frac{\partial^2 F}{\partial z \partial \xi} \\
\frac{\partial^2 F}{\partial \sigma^2} &= \frac{z}{\sigma^2}\,\left\{2\,\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}\right\} \\
\frac{\partial^2 F}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma}\,\frac{\partial^2 F}{\partial z \partial \xi}
\end{aligned}
\tag{15}
$$

Raw expressions

$$\frac{\partial^2 F}{\partial z^2} = \left(-\frac{1}{\xi} - 1\right) \xi \, (\xi\, z + 1)^{-\frac{1}{\xi} - 2} \, e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} + (\xi\, z + 1)^{-\frac{2}{\xi} - 2} \, e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}}$$

$$\frac{\partial^2 F}{\partial z \partial \xi} = (\xi\, z + 1)^{-\frac{1}{\xi} - 1} \, e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} \left(\frac{\log(\xi\, z + 1)}{\xi^2} + \frac{\left(-\frac{1}{\xi} - 1\right) z}{\xi\, z + 1}\right) - (\xi\, z + 1)^{-\frac{2}{\xi} - 1} \, e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} \left(\frac{\log(\xi\, z + 1)}{\xi^2} - \frac{z}{\xi\, (\xi\, z + 1)}\right)$$

$$\frac{\partial^2 F}{\partial \xi^2} = -\frac{e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} \left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\, (\xi\, z+1)}\right)^2}{(\xi\, z + 1)^{\frac{1}{\xi}}} + \frac{e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} \left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\, (\xi\, z+1)}\right)^2}{(\xi\, z + 1)^{\frac{2}{\xi}}} - \frac{e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}} \left(-\frac{2\, \log(\xi\, z+1)}{\xi^3} + \frac{2\, z}{\xi^2\, (\xi\, z+1)} + \frac{z^2}{\xi\, (\xi\, z+1)^2}\right)}{(\xi\, z + 1)^{\frac{1}{\xi}}}$$

Simplified expressions

Using the auxiliary variables of (1)

$$\begin{aligned}
\frac{\partial^2 F}{\partial z^2} &= \frac{FW}{V^2}\,\{-[1+\xi]+W\}\\
\frac{\partial^2 F}{\partial z \partial \xi} &= \frac{FW}{V}\,\left\{T\,[1-W]-\frac{z}{V}\right\}\\
\frac{\partial^2 F}{\partial \xi^2} &= FW\,\left\{-T^2\,[1-W]+R\right\}
\end{aligned} \tag{16}$$

Note that $W^{1+2\xi} = W/V^2$ and the second form is faster to evaluate.

3.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned}
\lim_{\xi\to 0}\frac{\partial^2 F}{\partial z^2} &= e^{-e^{-z}-2\,z} - e^{-e^{-z}-z}\\
\lim_{\xi\to 0}\frac{\partial^2 F}{\partial z\partial \xi} &= \frac{\left(z^2\,\left(e^{e^{-z}\,(2\,z\,e^z+1)} - e^{e^{-z}\,(z\,e^z+1)}\right) - 2\,z\,e^{e^{-z}\,(2\,z\,e^z+1)}\right)\,e^{-e^{-z}\,(2\,z\,e^z+1)-e^{-z}\,(z\,e^z+1)}}{2}\\
\lim_{\xi\to 0}\frac{\partial^2 F}{\partial^2 \xi^2} &= -\frac{\left(z^4\,\left(3\,e^{e^{-z}\,(2\,z\,e^z+1)} - 3\,e^{e^{-z}\,(z\,e^z+1)}\right) - 8\,z^3\,e^{e^{-z}\,(2\,z\,e^z+1)}\right)\,e^{-e^{-z}\,(2\,z\,e^z+1)-e^{-z}\,(z\,e^z+1)}}{12}
\end{aligned}$$

It seems that Maxima can not find the limit of $\partial^2 F/\partial\xi^2$ in a non interactive mode so the value above was obtained by a paste-and-copy of the result obtained in an interactive session.

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ (Gumbel distribution value at z),

$$\begin{aligned}
\frac{\partial^2 F}{\partial z^2} &= F^\star\left[e^{-z}-1\right] + o(1)\\
\frac{\partial^2 F}{\partial z\partial \xi} &= \frac{1}{2}\,F^\star\left[z^2\left(1-e^{-z}\right)-2z\right] + o(1)\\
\frac{\partial^2 F}{\partial \xi^2} &= \frac{1}{12}\,F^\star z^3\left[8-3\left(1-e^{-z}\right)z\right] + o(1)
\end{aligned} \tag{17}$$

4 Quantile or return level

4.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\text{GEV}}(p)$ is given by

$$\rho = \mu - \frac{\left(1 - \frac{1}{A^{\xi}}\right)\,\sigma}{\xi} \tag{18}$$

4.2 Scaling and auxiliary variables

$$\rho = \mu + \sigma\rho^{\star}$$

$$\rho^{\star} := \frac{A^{-\xi} - 1}{\xi} \tag{19}$$

4.3 Taylor expansion for $\xi \approx 0$

$$\rho = \mu - \log A\,\sigma + \frac{(\log A)^2\,\sigma\,\xi}{2} - \frac{(\log A)^3\,\sigma\,\xi^2}{6} + \cdots$$

4.4 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\begin{aligned}
\frac{\partial\rho}{\partial\mu} &= 1\\
\frac{\partial\rho}{\partial\sigma} &= -\frac{1 - \frac{1}{A^{\xi}}}{\xi}\\
\frac{\partial\rho}{\partial\xi} &= \frac{\left(1 - \frac{1}{A^{\xi}}\right)\,\sigma}{\xi^2} - \frac{\log A\,\sigma}{A^{\xi}\,\xi}
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi\to 0}\frac{\partial\rho}{\partial\mu} &= 1\\
\lim_{\xi\to 0}\frac{\partial\rho}{\partial\sigma} &= -\log A\\
\lim_{\xi\to 0}\frac{\partial\rho}{\partial\xi} &= \frac{(\log A)^2\,\sigma}{2}
\end{aligned}$$

4.5 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\begin{aligned}
\frac{\partial\rho}{\partial\mu} &= +1 + \cdots\\
\frac{\partial\rho}{\partial\sigma} &= -\log A + \frac{(\log A)^2\,\xi}{2} + \cdots\\
\frac{\partial\rho}{\partial\xi} &= \frac{(\log A)^2\,\sigma}{2} - \frac{(\log A)^3\,\sigma\,\xi}{3} + \cdots
\end{aligned}$$

4.6 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ, we consider only the standardized return level

$$\rho^\star = \frac{\frac{1}{A^\xi} - 1}{\xi} \tag{20}$$

so that $\rho = \mu + \sigma \rho^\star$.

Raw expressions

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{\log A}{A^\xi\, \xi} - \frac{\frac{1}{A^\xi} - 1}{\xi^2}$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{(\log A)^2}{A^\xi\, \xi} + \frac{2\, \log A}{A^\xi\, \xi^2} + \frac{2\, \left(\frac{1}{A^\xi} - 1\right)}{\xi^3}$$

Simplified expression

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{1}{\xi}\, [\rho^\star + \log A] - \rho^\star\, \log A \tag{21}$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{1}{\xi^2}\, [\rho^\star + \log A] - \frac{\partial \rho^\star}{\partial \xi} \left[\log A + \frac{1}{\xi}\right] \tag{22}$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^\star}{\partial \xi} \\ 0 & \frac{\partial \rho^\star}{\partial \xi} & \sigma \frac{\partial^2 \rho^\star}{\partial \xi^2} \end{bmatrix}$$

4.7 Second-order derivatives: limit for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned} \lim_{\xi \to 0} \frac{\partial \rho^\star}{\partial \xi} &= \frac{(\log A)^2}{2} \\ \lim_{\xi \to 0} \frac{\partial^2 \rho^\star}{\partial \xi^2} &= -\frac{(\log A)^3}{3} \end{aligned} \tag{23}$$