The probability functions of the GEV distribution and their derivatives

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Contents

Note

The expressions computed by **Maxima** are reported in green. The expressions involving some manual calculations are reported in red. These are intended to be easier to implement as they make use of “auxiliary quantities” that may be reused in several expressions. Although they have been carfully checked, the expressions in red could still contain some errors.

1 Log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y-\mu}{\sigma}$$

and then V and U by

$$V := \xi\, z + 1, \qquad U := \frac{1}{\sigma V}\left[1+\xi - V^{-1/\xi}\right].$$

The log-likelihood is defined by

$$\log f = -\left(\left(\frac{1}{\xi}+1\right)\,\log\left(\xi\, z+1\right)\right) - \frac{1}{\left(\xi\, z+1\right)^{\frac{1}{\xi}}} - \log\sigma$$

1.2 Taylor expansion at $\xi = 0$

$$\log f = -\left(\frac{e^z\,\log\sigma + e^z\,z+1}{e^z}\right) + \frac{\left(\left(e^z-1\right)\,z^2-2\,e^z\,z\right)\,\xi}{2\,e^z} - \frac{\left(3\,z^4+\left(8\,e^z-8\right)\,z^3-12\,e^z\,z^2\right)\,\xi^2}{24\,e^z} + \cdots$$

which can be simplified as

$$\log f = -\left[\log\sigma + z + e^{-z}\right] + \frac{1}{2}\,\left[(1-e^{-z})z - 2\right]\,z\xi - \frac{1}{24}\,\left[3z^2e^{-z} + 8z(1-e^{-z}) - 12\right]z^2\xi^2$$

1.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z, so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}\,\frac{\partial \log f}{\partial z}$$

and up to the term $-\log\sigma$ in $\log f$, the same is true for σ so

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma} + \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} - \frac{z}{\sigma}\,\frac{\partial \log f}{\partial z}.$$

So it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ.

Raw expressions

Here are the raw expressions found by Maxima

$$\frac{\partial \log f}{\partial z} = \left(\xi\, z+1\right)^{-\left(\frac{1}{\xi}\right)-1} - \frac{\left(\frac{1}{\xi}+1\right)\,\xi}{\xi\, z+1}$$

$$\frac{\partial \log f}{\partial \xi} = -\left(\frac{\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}}{\left(\xi\, z+1\right)^{\frac{1}{\xi}}}\right) + \frac{\log\left(\xi\, z+1\right)}{\xi^2} - \frac{\left(\frac{1}{\xi}+1\right)\,z}{\xi\, z+1}$$

Simplified expressions

We use the following simplifications

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\sigma U,\\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right]\log V - z\,U\frac{\sigma}{\xi}.\end{aligned}$$

so eventually

$$\begin{aligned}\frac{\partial \log f}{\partial \mu} &= U,\\ \frac{\partial \log f}{\partial \sigma} &= \frac{-1}{\sigma} + zU,\\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right]\log V - z\,U\frac{\sigma}{\xi}.\end{aligned}$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$U = \frac{e^z - 1}{e^z\,\sigma} - \frac{\left(z^2 + (2\,e^z - 2)\;z - 2\,e^z\right)\,\xi}{2\,e^z\,\sigma} + \cdots$$

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\left(\frac{e^z - 1}{e^z}\right) + \frac{\left(z^2 + (2\,e^z - 2)\;z - 2\,e^z\right)\,\xi}{2\,e^z} + \cdots\\ \frac{\partial \log f}{\partial \xi} &= \frac{(e^z - 1)\;z^2 - 2\,e^z\,z}{2\,e^z} - \frac{\left(3\,z^4 + (8\,e^z - 8)\;z^3 - 12\,e^z\,z^2\right)\,\xi}{12\,e^z} + \cdots\end{aligned}$$

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\left[1 - e^{-z}\right] - \frac{1}{2}\left[2 - 2(1 - e^{-z})z - e^{-z}z^2\right]\xi + o(\xi)\\ \frac{\partial \log f}{\partial \xi} &= -\frac{1}{2}\left[2z - (1 - e^{-z})z^2\right] + \frac{1}{12}\left[12 - 8(1 - e^{-z})z - 3z^2\right] z^2\xi + o(\xi)\end{aligned}$$

Limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned}\lim_{\xi\to 0}\frac{\partial \log f}{\partial z} &= -\left(e^{-z}\;(e^z - 1)\right)\\ \lim_{\xi\to 0}\frac{\partial \log f}{\partial \xi} &= \frac{e^{-z}\;\left(z^2\;(e^z - 1) - 2\,z\,e^z\right)}{2}\end{aligned}$$

that is

$$\begin{aligned}\lim_{\xi\to 0}\frac{\partial \log f}{\partial z} &= 1 - e^{-z}\\ \lim_{\xi\to 0}\frac{\partial \log f}{\partial \xi} &= -\frac{z}{2}\left[2 - (1 - e^{-z})z\right]\end{aligned}$$

1.5 Second-order derivatives: expressions

Scaling

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial \mu^2} &= \frac{1}{\sigma^2}\,\frac{\partial^2 \log f}{\partial z^2}\\
\frac{\partial^2 \log f}{\partial \mu \partial \sigma} &= \frac{1}{\sigma^2}\left\{\frac{\partial \log f}{\partial z} + z\frac{\partial^2 \log f}{\partial z^2}\right\}\\
\frac{\partial^2 \log f}{\partial \mu \partial \xi} &= \frac{-1}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}\\
\frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1}{\sigma^2} + \frac{z}{\sigma^2}\left\{2\,\frac{\partial \log f}{\partial z} + z\,\frac{\partial^2 \log f}{\partial z^2}\right\}\\
\frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}
\end{aligned}
$$

Raw expressions

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial z^2} &= \left(-\left(\frac{1}{\xi}\right) - 1\right)\,\xi\,\left(\xi\,z+1\right)^{-\left(\frac{1}{\xi}\right)-2} + \frac{\left(\frac{1}{\xi}+1\right)\,\xi^2}{\left(\xi\,z+1\right)^2}\\
\frac{\partial^2 \log f}{\partial z \partial \xi} &= \left(\xi\,z+1\right)^{-\left(\frac{1}{\xi}\right)-1}\,\left(\frac{\log\left(\xi\,z+1\right)}{\xi^2} + \frac{\left(-\left(\frac{1}{\xi}\right)-1\right)\,z}{\xi\,z+1}\right) - \frac{\frac{1}{\xi}+1}{\xi\,z+1} + \frac{1}{\xi\,\left(\xi\,z+1\right)} + \frac{\left(\frac{1}{\xi}+1\right)\,\xi\,z}{\left(\xi\,z+1\right)^2}\\
\frac{\partial^2 \log f}{\partial \xi^2} &= -\left(\frac{\left(\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}\right)^2}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}}\right) - \frac{-\left(\frac{2\,\log(\xi\,z+1)}{\xi^3}\right) + \frac{2\,z}{\xi^2\,(\xi\,z+1)} + \frac{z^2}{\xi\,(\xi\,z+1)^2}}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}} - \frac{2\,\log\left(\xi\,z+1\right)}{\xi^3} + \frac{2\,z}{\xi^2\,\left(\xi\,z+1\right)} + \frac{\left(\frac{1}{\xi}+1\right.}{(\xi\,z+}
\end{aligned}
$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial z^2} &= -(\xi+1)\left[V^{-1/\xi} - \xi\right] \times \frac{1}{V^2}\\
\frac{\partial^2 \log f}{\partial z \partial \xi} &= V^{-1/\xi-1}\left[\frac{1}{\xi^2}\,\log V - \frac{(\xi+1)}{\xi}\,\frac{z}{V}\right] - \frac{1}{V} + (\xi+1)\frac{z}{V^2}\\
\frac{\partial^2 \log f}{\partial \xi^2} &= -\left\{\left[\frac{1}{\xi^2}\,\log V - \frac{1}{\xi}\,\frac{z}{V}\right]^2 + \left[-\frac{2}{\xi^2}\,\log V + \frac{2}{\xi}\,\frac{z}{V} + \frac{z^2}{V^2}\right] \times \frac{1}{\xi}\right\} \times V^{-1/\xi} - \frac{2}{\xi^3}\,\log V + \frac{2}{\xi^2}\,\frac{z}{V} + \frac{\xi+1}{\xi}\,\frac{z^2}{V^2}
\end{aligned}
$$

1.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 z^2} = -e^{-z}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} = \frac{e^{-z}\,\left(z\,\left(2\,e^z - 2\right) - 2\,e^z + z^2\right)}{2}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} = -\left(\frac{e^{-z}\,\left(z^3\,\left(8\,e^z - 8\right) - 12\,z^2\,e^z + 3\,z^4\right)}{12}\right)$$

$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 z^2} = -e^{-z}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} = \frac{1}{2}\left[-2 - 2(1 - e^{-z})z + e^{-z}z^2\right]$$
$$\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} = \frac{z^2}{12}\left[12 + 8(1 - e^{-z})z - 3e^{-z}z^2\right]$$

2 Distribution function F

2.1 Expression

Define

$$W := V^{-1/\xi} = [1 + \xi z]^{-1/\xi}, \qquad T := \frac{\log V}{\xi^2} - \frac{z}{\xi V}$$

Then

$$F := \exp\{-W\}.$$

For the distribution function the second-order derivatives has not yet been implemented.

2.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$F = \frac{1}{e^{\frac{1}{e^z}}} - \frac{z^2\,\xi}{2\,e^{\frac{1}{e^z}}\,e^z} - \frac{\left(\left(3\,e^z - 3\right)\,z^4 - 8\,e^z\,z^3\right)\,\xi^2}{24\,e^{\frac{1}{e^z}}\,\left(e^z\right)^2} + \cdots$$

Simplified expression

$$F = \exp\{-e^{-z}\}\left\{1 - e^{-z}\frac{z^2}{2}\xi + e^{-z}\frac{z^3}{24}\left[8 - 3(1 - e^{-z})z\right]\xi^2\right\} + o(\xi^2)$$

2.3 First-order derivatives: expressions

Scaling

$$\frac{\partial F}{\partial \mu} = \frac{-1}{\sigma}\,\frac{\partial F}{\partial z}$$
$$\frac{\partial F}{\partial \sigma} = \frac{-z}{\sigma}\,\frac{\partial F}{\partial z}$$

Raw expressions

$$\frac{\partial F}{\partial z} = (\xi\, z+1)^{-\left(\frac{1}{\xi}\right)-1}\, e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)}$$
$$\frac{\partial F}{\partial \xi} = -\left(\frac{e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)}\,\left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)$$

Simplified expressions

$$\frac{\partial F}{\partial z} = F W\,\frac{1}{V}$$
$$\frac{\partial F}{\partial \xi} = -F W T$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\frac{\partial F}{\partial z} = \frac{1}{e^{\frac{z\, e^z+1}{e^z}}} + \frac{\left((e^z-1)\, z^2 - 2\, e^z\, z\right)\,\xi}{2\, e^{\frac{z\, e^z+1}{e^z}}\, e^z} + \cdots$$
$$\frac{\partial F}{\partial \xi} = -\left(\frac{z^2}{2\, e^{\frac{z\, e^z+1}{e^z}}}\right) - \frac{\left((3\, e^z-3)\, z^4 - 8\, e^z\, z^3\right)\,\xi}{12\, e^{\frac{z\, e^z+1}{e^z}}\, e^z} + \cdots$$

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ (Gumbel distribution value at z),

$$\frac{\partial F}{\partial z} = F^\star \left\{1 + \frac{z}{2}\left[-2 + (1 - e^{-z})z\right]\xi\right\} + o(\xi)$$
$$\frac{\partial F}{\partial \xi} = F^\star \left\{-\frac{z^2}{2} + \frac{z^3}{12}\left[8 - 3\,(1 - e^{-z})z\right]\xi\right\} + o(\xi)$$

2.5 Second-order derivatives

CAUTION The second-order derivatives has not yet been implemented and the formula have not yet been fully checked.

Scaling

$$\frac{\partial^2 F}{\partial \mu^2} = \frac{1}{\sigma^2}\,\frac{\partial^2 F}{\partial z^2}$$
$$\frac{\partial^2 F}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2}\left\{\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}\right\}$$
$$\frac{\partial^2 F}{\partial \mu \partial \xi} = \frac{-1}{\sigma}\,\frac{\partial^2 F}{\partial z \partial \xi}$$
$$\frac{\partial^2 F}{\partial \sigma^2} = \frac{z}{\sigma^2}\left\{2\,\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2}\right\}$$
$$\frac{\partial^2 F}{\partial \sigma \partial \xi} = \frac{-z}{\sigma}\,\frac{\partial^2 F}{\partial z \partial \xi}$$

Raw expressions

$$\frac{\partial^2 F}{\partial z^2} = \left(-\left(\frac{1}{\xi}\right) - 1\right)\,\xi\,(\xi\,z+1)^{-\left(\frac{1}{\xi}\right)-2}\,e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)} + (\xi\,z+1)^{-\left(\frac{2}{\xi}\right)-2}\,e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}$$

$$\frac{\partial^2 F}{\partial z \partial \xi} = (\xi\,z+1)^{-\left(\frac{1}{\xi}\right)-1}\,e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}\left(\frac{\log(\xi\,z+1)}{\xi^2} + \frac{\left(-\left(\frac{1}{\xi}\right)-1\right)\,z}{\xi\,z+1}\right) - (\xi\,z+1)^{-\left(\frac{2}{\xi}\right)-1}\,e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}\left(\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}\right)$$

$$\frac{\partial^2 F}{\partial \xi^2} = -\left(\frac{e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}\left(\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}\right)^2}{(\xi\,z+1)^{\frac{1}{\xi}}}\right) + \frac{e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}\left(\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}\right)^2}{(\xi\,z+1)^{\frac{2}{\xi}}} - \frac{e^{-\left(\frac{1}{(\xi\,z+1)^{\frac{1}{\xi}}}\right)}\left(-\left(\frac{2\,\log(\xi\,z+1)}{\xi^3}\right) + \frac{2\,z}{\xi^2\,(\xi\,z+1)} + \frac{z^2}{\xi\,(\xi\,z+1)^2}\right)}{(\xi\,z+1)^{\frac{1}{\xi}}}$$

Simplified expressions

$$\frac{\partial^2 F}{\partial z^2} = FW^{1+2\xi}\,\{-[1+\xi] + W\}$$
$$\frac{\partial^2 F}{\partial z \partial \xi} = FW^{1+\xi}\left\{T\,[1-W] - \frac{z}{V}\right\}$$
$$\frac{\partial^2 F}{\partial \xi^2} = FW\left\{-T^2\,[1-W] + \frac{2}{\xi}\,T - \frac{z^2}{\xi V^2}\right\}$$

Note that $W^{1+2\xi} = W/V^2$ and $W^{1+\xi} = W/V$, which may be faster to evaluate.

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi\to 0} \frac{\partial^2 F}{\partial z^2} = e^{-e^{-z}-2\,z} - e^{-e^{-z}-z}$$
$$\lim_{\xi\to 0} \frac{\partial^2 F}{\partial z \partial \xi} = \frac{\left(z^2\,\left(e^z-1\right) - 2\,z\,e^z\right)\,e^{-\left(e^{-z}\,(2\,z\,e^z+1)\right)}}{2}$$
$$\lim_{\xi\to 0} \frac{\partial^2 F}{\partial^2 \xi^2} = -\frac{\left(z^4\,\left(3\,e^{e^{-z}\,(2\,z\,e^z+1)} - 3\,e^{e^{-z}\,(z\,e^z+1)}\right) - 8\,z^3\,e^{e^{-z}\,(2\,z\,e^z+1)}\right)\,e^{-e^{-z}\,(2\,z\,e^z+1)-e^{-z}\,(z\,e^z+1)}}{12}$$

It seems that Maxima can not find the limit of $\partial^2 F/\partial\xi^2$ in a non interactive mode so the value above was obtained by a paste-and-copy of the result obtained in an interactive session.

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ (Gumbel distribution value at z),

$$\frac{\partial^2 F}{\partial z^2} = F^\star\left[e^{-z} - 1\right] + o(1)$$
$$\frac{\partial^2 F}{\partial z \partial \xi} = \frac{1}{2}\,F^\star\left[z^2\left(1-e^{-z}\right) - 2z\right] + o(1)$$
$$\frac{\partial^2 F}{\partial \xi^2} = \frac{1}{12}\,F^\star z^3\left[8 - 3\left(1-e^{-z}\right)z\right] + o(1)$$

3 Quantile or return level

3.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\mathrm{GEV}}(p)$ is given by

$$\rho = \mu - \frac{\left(1-\frac{1}{A^\xi}\right)\,\sigma}{\xi}$$

3.2 Taylor expansion for $\xi \approx 0$

$$\rho = \mu - \log A\,\sigma + \frac{(\log A)^2\ \sigma\,\xi}{2} - \frac{(\log A)^3\ \sigma\,\xi^2}{6} + \cdots$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\begin{aligned}
\frac{\partial\rho}{\partial\mu} &= 1 \\
\frac{\partial\rho}{\partial\sigma} &= -\left(\frac{1 - \frac{1}{A^\xi}}{\xi}\right) \\
\frac{\partial\rho}{\partial\xi} &= \frac{\left(1 - \frac{1}{A^\xi}\right)\ \sigma}{\xi^2} - \frac{\log A\,\sigma}{A^\xi\,\xi}
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\partial\rho}{\partial\mu} &= 1 \\
\lim_{\xi\to 0} \frac{\partial\rho}{\partial\sigma} &= -\log A \\
\lim_{\xi\to 0} \frac{\partial\rho}{\partial\xi} &= \frac{(\log A)^2\ \sigma}{2}
\end{aligned}$$

3.4 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\begin{aligned}
\frac{\partial\rho}{\partial\mu} &= +1 + \cdots \\
\frac{\partial\rho}{\partial\sigma} &= -\log A + \frac{(\log A)^2\ \xi}{2} + \cdots \\
\frac{\partial\rho}{\partial\xi} &= \frac{(\log A)^2\ \sigma}{2} - \frac{(\log A)^3\ \sigma\,\xi}{3} + \cdots
\end{aligned}$$

3.5 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ, we consider only the standardized return level

$$\rho^\star = \frac{\frac{1}{A^\xi} - 1}{\xi}$$

so that $\rho = \mu + \sigma\rho^\star$.

Raw expressions

$$\frac{\partial \rho^\star}{\partial \xi} = -\left(\frac{\log A}{A^\xi\, \xi}\right) - \frac{\frac{1}{A^\xi} - 1}{\xi^2}$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{(\log A)^2}{A^\xi\, \xi} + \frac{2\, \log A}{A^\xi\, \xi^2} + \frac{2\, \left(\frac{1}{A^\xi} - 1\right)}{\xi^3}$$

Simplified expression

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{1}{\xi}\, [\rho^\star + \log A] - \rho^\star\, \log A$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{1}{\xi^2}\, [\rho^\star + \log A] - \frac{\partial \rho^\star}{\partial \xi}\left[\log A + \frac{1}{\xi}\right]$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^\star}{\partial \xi} \\ 0 & \frac{\partial \rho^\star}{\partial \xi} & \sigma \frac{\partial^2 \rho^\star}{\partial \xi^2} \end{bmatrix}$$

3.6 Second-order derivatives: limit for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial \rho^\star}{\partial \xi} = \frac{(\log A)^2}{2}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \rho^\star}{\partial \xi^2} = -\left(\frac{(\log A)^3}{3}\right)$$