The probability functions of the GEV distribution and their derivatives

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Contents

1 Scope

We consider a "quantity" related to the GEV distribution: the log-density $\log f(x; \boldsymbol{\theta})$, the distribution $F(q; \boldsymbol{\theta})$ and or quantile $q(p; \boldsymbol{\theta})$. The vector $\boldsymbol{\theta}$ is formed by the trhee elements: location μ, scale σ and shape ξ. We want to have a closed-form expression for each quantity and its first-order derivative w.r.t. $\boldsymbol{\theta}$ and possibly the second-order derivative.

As is weel-known, the expressions defining the quantities take depending on $\xi = 0$ or $\xi \neq 0$. The expressions for $\xi \neq 0$ are usable for very small ξ, say $\xi \approx 1e - 7$, but will show some inacuracy when ξ is very small. We will use specific expression for $|\xi| < \epsilon$ where ϵ is a very small positive real. These expressions are derived from Taylor expansions at $\xi = 0$. More precisely we use

- Second order Taylor approximation for the quantity,

- First order Taylor approximation for the derivative of the quantity,
- Zero order Taylor approximation for the 2-nd order derivative of the quantity.

So the derivatives up to order two will be continuous and moreover they will be consistent. For instance the approximation of the 1-st order derivative is the derivative of the approximation at $\xi = 0$.

As a rule, we implement the formulas given in red but we give the formulas given by **maxima** in green.

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2 Log-density $\log f$

2.1 Expression

Define z by

$$z := \frac{y-\mu}{\sigma}$$

and then V and U by

$$V := \xi\, z + 1, \qquad U := \frac{1}{\sigma V}\left[1 + \xi - V^{-1/\xi}\right].$$

The log-likelihood is defined by

$$\log f = -\left(\frac{1}{\xi}+1\right)\log\left(\xi\, z+1\right) - \frac{1}{\left(\xi\, z+1\right)^{\frac{1}{\xi}}} - \log\sigma$$

2.2 Taylor expansion at $\xi = 0$

$$\log f = -\frac{e^{z}\,\log\sigma + e^{z}\,z + 1}{e^{z}} + \frac{\left(\left(e^{z}-1\right)\,z^{2} - 2\,e^{z}\,z\right)\,\xi}{2\,e^{z}} - \frac{\left(3\,z^{4} + \left(8\,e^{z}-8\right)\,z^{3} - 12\,e^{z}\,z^{2}\right)\,\xi^{2}}{24\,e^{z}} + \cdots$$

which can be simplified as

$$\log f = -\left[\log\sigma + z + e^{-z}\right] + \frac{1}{2}\left[(1-e^{-z})z - 2\right] z\xi - \frac{1}{24}\left[3z^2 e^{-z} + 8z(1-e^{-z}) - 12\right] z^2\xi^2$$

2.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z, so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}\,\frac{\partial \log f}{\partial z}$$

and up to the term $-\log\sigma$ in $\log f$, the same is true for σ so

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma} + \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} - \frac{z}{\sigma}\,\frac{\partial \log f}{\partial z}.$$

So it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ.

Raw expressions

Here are the raw expressions found by Maxima

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= (\xi\, z+1)^{-\frac{1}{\xi}-1} - \frac{\left(\frac{1}{\xi}+1\right)\,\xi}{\xi\, z+1} \\
\frac{\partial \log f}{\partial \xi} &= -\frac{\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}}{(\xi\, z+1)^{\frac{1}{\xi}}} + \frac{\log\left(\xi\, z+1\right)}{\xi^2} - \frac{\left(\frac{1}{\xi}+1\right)\, z}{\xi\, z+1}
\end{aligned}
$$

Simplified expressions

We use the following simplifications

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\sigma U, \\
\frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right]\log V - z\, U \frac{\sigma}{\xi}.
\end{aligned}
$$

so eventually

$$
\begin{aligned}
\frac{\partial \log f}{\partial \mu} &= U, \\
\frac{\partial \log f}{\partial \sigma} &= \frac{-1}{\sigma} + zU, \\
\frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right]\log V - z\, U \frac{\sigma}{\xi}.
\end{aligned}
$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$
U = \frac{e^z - 1}{e^z\,\sigma} - \frac{\left(z^2 + (2\,e^z - 2)\, z - 2\,e^z\right)\,\xi}{2\,e^z\,\sigma} + \cdots
$$

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\frac{e^z - 1}{e^z} + \frac{\left(z^2 + (2\,e^z - 2)\, z - 2\,e^z\right)\,\xi}{2\,e^z} + \cdots \\
\frac{\partial \log f}{\partial \xi} &= \frac{(e^z - 1)\, z^2 - 2\,e^z\, z}{2\,e^z} - \frac{\left(3\,z^4 + (8\,e^z - 8)\, z^3 - 12\,e^z\, z^2\right)\,\xi}{12\,e^z} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\left[1 - e^{-z}\right] - \frac{1}{2}\left[2 - 2(1 - e^{-z})z - e^{-z}z^2\right]\xi + o(\xi) \\
\frac{\partial \log f}{\partial \xi} &= -\frac{1}{2}\left[2z - (1 - e^{-z})z^2\right] + \frac{1}{12}\left[12 - 8(1 - e^{-z})z - 3z^2\right] z^2\xi + o(\xi)
\end{aligned}
$$

Limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\partial \log f}{\partial z} &= -e^{-z}\,\left(e^{z}-1\right)\\
\lim_{\xi\to 0} \frac{\partial \log f}{\partial \xi} &= \frac{e^{-z}\,\left(z^2\,\left(e^{z}-1\right)-2\,z\,e^{z}\right)}{2}
\end{aligned}$$

that is

$$\begin{aligned}
\lim_{\xi\to 0} \frac{\partial \log f}{\partial z} &= 1-e^{-z}\\
\lim_{\xi\to 0} \frac{\partial \log f}{\partial \xi} &= -\frac{z}{2}\,\left[2-(1-e^{-z})z\right]
\end{aligned}$$

2.5 Second-order derivatives: expressions

Scaling

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial \mu^2} &= \frac{1}{\sigma^2}\,\frac{\partial^2 \log f}{\partial z^2}\\
\frac{\partial^2 \log f}{\partial \mu \partial \sigma} &= \frac{z}{\sigma^2}\,\frac{\partial^2 \log f}{\partial z^2}\\
\frac{\partial^2 \log f}{\partial \mu \partial \xi} &= \frac{-1}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}\\
\frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1}{\sigma^2}+\frac{z^2}{\sigma^2}\,\frac{\partial \log f}{\partial z}-\frac{z}{\sigma}\,\frac{\partial^2 \log f}{\partial z^2}\\
\frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma}\,\frac{\partial^2 \log f}{\partial z \partial \xi}
\end{aligned}$$

Raw expressions

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial z^2} &= \left(-\frac{1}{\xi}-1\right)\,\xi\,\left(\xi\,z+1\right)^{-\frac{1}{\xi}-2}+\frac{\left(\frac{1}{\xi}+1\right)\,\xi^2}{\left(\xi\,z+1\right)^2}\\
\frac{\partial^2 \log f}{\partial z \partial \xi} &= \left(\xi\,z+1\right)^{-\frac{1}{\xi}-1}\,\left(\frac{\log\left(\xi\,z+1\right)}{\xi^2}+\frac{\left(-\frac{1}{\xi}-1\right)\,z}{\xi\,z+1}\right)-\frac{\frac{1}{\xi}+1}{\xi\,z+1}+\frac{1}{\xi\,\left(\xi\,z+1\right)}+\frac{\left(\frac{1}{\xi}+1\right)\,\xi\,z}{\left(\xi\,z+1\right)^2}\\
\frac{\partial^2 \log f}{\partial \xi^2} &= -\frac{\left(\frac{\log(\xi\,z+1)}{\xi^2}-\frac{z}{\xi\,(\xi\,z+1)}\right)^2}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}}-\frac{-\frac{2\,\log(\xi\,z+1)}{\xi^3}+\frac{2\,z}{\xi^2\,(\xi\,z+1)}+\frac{z^2}{\xi\,(\xi\,z+1)^2}}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}}-\frac{2\,\log\left(\xi\,z+1\right)}{\xi^3}+\frac{2\,z}{\xi^2\,\left(\xi\,z+1\right)}+\frac{\left(\frac{1}{\xi}+1\right)\,z^2}{\left(\xi\,z+1\right)^2}
\end{aligned}$$

Simplified expressions

$$\frac{\partial^2 \log f}{\partial z^2} = -(\xi+1)\left[V^{-1/\xi} - \xi\right] \times \frac{1}{V^2}$$
$$\frac{\partial^2 \log f}{\partial z \partial \xi} = V^{-1/\xi - 1}\left[\frac{1}{\xi^2}\,\log V - \frac{(\xi+1)}{\xi}\,\frac{z}{V}\right] - \frac{1}{V} + (\xi+1)\frac{z}{V^2}$$
$$\frac{\partial^2 \log f}{\partial \xi^2} = -\left\{\left[\frac{1}{\xi^2}\,\log V - \frac{1}{\xi}\,\frac{z}{V}\right]^2 + \left[-\frac{2}{\xi^2}\,\log V + \frac{2}{\xi}\,\frac{z}{V} + \frac{z^2}{V^2}\right] \times \frac{1}{\xi}\right\} \times V^{-1/\xi} - \frac{2}{\xi^3}\,\log V + \frac{2}{\xi^2}\,\frac{z}{V} + \frac{\xi+1}{\xi}\,\frac{z^2}{V^2}$$

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} = -e^{-z}$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} = \frac{e^{-z}\,\left(z\,\left(2\,e^z - 2\right) - 2\,e^z + z^2\right)}{2}$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} = -\frac{e^{-z}\,\left(z^3\,\left(8\,e^z - 8\right) - 12\,z^2\,e^z + 3\,z^4\right)}{12}$$

$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} = -e^{-z}$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} = \frac{1}{2}\,\left[-2 - 2(1-e^{-z})z + e^{-z}z^2\right]$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} = \frac{z^2}{12}\,\left[12 + 8(1-e^{-z})z - 3e^{-z}z^2\right]$$

3 Distribution function F

3.1 Expression

Define

$$W := V^{-1/\xi} = [1+\xi z]^{-1/\xi}.$$

Then

$$F := \exp\{-W\}.$$

For the distribution function we will only consider the first-order derivatives since the second-order derivatives are not needed in most applications.

3.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$F = \frac{1}{e^{\frac{1}{e^z}}} - \frac{z^2\,\xi}{2\,e^{\frac{1}{e^z}}\,e^z} - \frac{\left(\left(3\,e^z - 3\right)\,z^4 - 8\,e^z\,z^3\right)\,\xi^2}{24\,e^{\frac{1}{e^z}}\,\left(e^z\right)^2} + \cdots$$

Simplified expression

$$F = \exp\{-e^{-z}\}\left\{1 - e^{-z}\,\frac{z^2}{2}\xi + e^{-z}\frac{z^3}{24}\,\left[8 - 3(1-e^{-z})z\right]\xi^2\right\} + o(\xi^2)$$

3.3 First-order derivatives: expressions

Scaling

$$\frac{\partial F}{\partial \mu} = \frac{-1}{\sigma}\,\frac{\partial F}{\partial z}$$
$$\frac{\partial F}{\partial \sigma} = \frac{z}{\sigma}\,\frac{\partial F}{\partial z}$$

Raw expressions

$$\frac{\partial F}{\partial z} = (\xi\, z+1)^{-\frac{1}{\xi}-1}\, e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}}$$
$$\frac{\partial F}{\partial \xi} = -\frac{e^{-\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}}\,\left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)}{(\xi\, z+1)^{\frac{1}{\xi}}}$$

Simplified expressions

$$\frac{\partial F}{\partial z} = FW\,\frac{1}{V}$$
$$\frac{\partial F}{\partial \xi} = -FW\left\{\frac{\log V}{\xi^2} - \frac{z}{\xi V}\right\}$$

3.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\frac{\partial F}{\partial z} = \frac{1}{e^{\frac{z\,e^z+1}{e^z}}} + \frac{\left((e^z-1)\,z^2 - 2\,e^z\,z\right)\,\xi}{2\,e^{\frac{z\,e^z+1}{e^z}}\,e^z} + \cdots$$
$$\frac{\partial F}{\partial \xi} = -\frac{z^2}{2\,e^{\frac{z\,e^z+1}{e^z}}} - \frac{\left((3\,e^z-3)\,z^4 - 8\,e^z\,z^3\right)\,\xi}{12\,e^{\frac{z\,e^z+1}{e^z}}\,e^z} + \cdots$$

Simplified expressions

$$\frac{\partial F}{\partial z} = \exp\{-z - e^{-z}\}\left\{1 + \frac{z}{2}\left[-2 + (1 - e^{-z})z\right]\xi\right\} + o(\xi^2)$$
$$\frac{\partial F}{\partial \xi} = \exp\{-z - e^{-z}\}\left\{-\frac{z^2}{2} + \frac{z^3}{12}\left[8 - 3(1 - e^{-z})z\right]\xi\right\} + o(\xi^2)$$

4 Quantile or return level

4.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\text{GEV}}(p)$ is given by

$$\rho = \mu - \frac{\left(1 - \frac{1}{A^\xi}\right)\sigma}{\xi}$$

4.2 Taylor expansion for $\xi \approx 0$

$$\rho = \mu - \log A\,\sigma + \frac{(\log A)^2\,\sigma\,\xi}{2} - \frac{(\log A)^3\,\sigma\,\xi^2}{6} + \cdots$$

4.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\frac{\partial \rho}{\partial \mu} = 1$$
$$\frac{\partial \rho}{\partial \sigma} = -\frac{1 - \frac{1}{A^\xi}}{\xi}$$
$$\frac{\partial \rho}{\partial \xi} = \frac{\left(1 - \frac{1}{A^\xi}\right)\sigma}{\xi^2} - \frac{\log A\,\sigma}{A^\xi\,\xi}$$

$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \mu} = 1$$
$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \sigma} = -\log A$$
$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \xi} = \frac{(\log A)^2\,\sigma}{2}$$

4.4 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\frac{\partial \rho}{\partial \mu} = +1 + \cdots$$
$$\frac{\partial \rho}{\partial \sigma} = -\log A + \frac{(\log A)^2\, \xi}{2} + \cdots$$
$$\frac{\partial \rho}{\partial \xi} = \frac{(\log A)^2\, \sigma}{2} - \frac{(\log A)^3\, \sigma\, \xi}{3} + \cdots$$

4.5 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ, we consider only the standardised return level

$$\rho^\star = \frac{\frac{1}{A^\xi} - 1}{\xi}$$

so that $\rho = \mu + \sigma \rho^\star$.

Raw expressions

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{\log A}{A^\xi\, \xi} - \frac{\frac{1}{A^\xi} - 1}{\xi^2}$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{(\log A)^2}{A^\xi\, \xi} + \frac{2\, \log A}{A^\xi\, \xi^2} + \frac{2\, \left(\frac{1}{A^\xi} - 1\right)}{\xi^3}$$

Simplified expression

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{1}{\xi}\, [\rho^\star + \log A] - \rho^\star\, \log A$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{1}{\xi^2}\, [\rho^\star + \log A] - \frac{\partial \rho^\star}{\partial \xi} \left[\log A + \frac{1}{\xi}\right]$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^\star}{\partial \xi} \\ 0 & \frac{\partial \rho^\star}{\partial \xi} & \sigma \frac{\partial^2 \rho^\star}{\partial \xi^2} \end{bmatrix}$$

4.6 Second-order derivatives: limit for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial \rho^\star}{\partial \xi} = \frac{(\log A)^2}{2}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \rho^\star}{\partial \xi^2} = -\frac{(\log A)^3}{3}$$