

The probability functions of the GEV distribution and their derivatives

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December 2, 2022

Contents

1	Log-density $\log f$	1
1.1	Expression	1
1.2	Taylor expansion at $\xi = 0$	2
1.3	First-order derivatives: expression	2
1.4	First-order derivatives: Taylor expansions for $\xi \approx 0$	3
1.5	Second-order derivatives: expressions	3
1.6	Second-order derivatives: limits for $\xi \rightarrow 0$	4
2	Distribution function F	4
2.1	Expression	4
2.2	Taylor expansions for $\xi \approx 0$	5
2.3	First-order derivatives: expressions	5
2.4	First-order derivatives: Taylor expansions for $\xi \approx 0$	6
3	Quantile or return level	6
3.1	Expression	6
3.2	Taylor expansion for $\xi \approx 0$	6
3.3	First-order derivatives: expressions	6
3.4	First-order derivatives: Taylor expansion for $\xi \approx 0$	7
3.5	Second-order derivatives: expressions	7
3.6	Second-order derivatives: limit for $\xi \rightarrow 0$	7

1 Log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y - \mu}{\sigma}$$

and then V and U by

$$V := \xi z + 1, \quad U := \frac{1}{\sigma V} \left[1 + \xi - V^{-1/\xi} \right].$$

The log-likelihood is defined by

$$\log f = - \left(\frac{1}{\xi} + 1 \right) \log (\xi z + 1) - \frac{1}{(\xi z + 1)^{\frac{1}{\xi}}} - \log \sigma$$

1.2 Taylor expansion at $\xi = 0$

$$\log f = -\frac{e^z \log \sigma + e^z z + 1}{e^z} + \frac{((e^z - 1) z^2 - 2 e^z z) \xi}{2 e^z} - \frac{(3 z^4 + (8 e^z - 8) z^3 - 12 e^z z^2) \xi^2}{24 e^z} + \dots$$

which can be simplified as

$$\log f = -[\log \sigma + z + e^{-z}] + \frac{1}{2} [(1 - e^{-z})z - 2] z \xi - \frac{1}{24} [3z^2 e^{-z} + 8z(1 - e^{-z}) - 12] z^2 \xi^2$$

1.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z , so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z} \frac{\partial z}{\partial \mu} = -\frac{1}{\sigma} \frac{\partial \log f}{\partial z}$$

and up to the term $-\log \sigma$ in $\log f$, the same is true for σ so

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma} + \frac{\partial \log f}{\partial z} \frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} - \frac{z}{\sigma} \frac{\partial \log f}{\partial z}.$$

So it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ .

Raw expressions

Here are the raw expressions found by Maxima

$$\begin{aligned} \frac{\partial \log f}{\partial z} &= (\xi z + 1)^{-\frac{1}{\xi}-1} - \frac{\left(\frac{1}{\xi} + 1\right) \xi}{\xi z + 1} \\ \frac{\partial \log f}{\partial \xi} &= -\frac{\frac{\log(\xi z + 1)}{\xi^2} - \frac{z}{\xi(\xi z + 1)}}{(\xi z + 1)^{\frac{1}{\xi}}} + \frac{\log(\xi z + 1)}{\xi^2} - \frac{\left(\frac{1}{\xi} + 1\right) z}{\xi z + 1} \end{aligned}$$

Simplified expressions

We use the following simplifications

$$\begin{aligned} \frac{\partial \log f}{\partial z} &= -\sigma U, \\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2} [1 - V^{-1/\xi}] \log V - z U \frac{\sigma}{\xi}. \end{aligned}$$

so eventually

$$\begin{aligned} \frac{\partial \log f}{\partial \mu} &= U, \\ \frac{\partial \log f}{\partial \sigma} &= \frac{-1}{\sigma} + z U, \\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2} [1 - V^{-1/\xi}] \log V - z U \frac{\sigma}{\xi}. \end{aligned}$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$U = \frac{e^z - 1}{e^z \sigma} - \frac{(z^2 + (2e^z - 2)z - 2e^z)\xi}{2e^z \sigma} + \dots$$

$$\begin{aligned} \frac{\partial \log f}{\partial z} &= -\frac{e^z - 1}{e^z} + \frac{(z^2 + (2e^z - 2)z - 2e^z)\xi}{2e^z} + \dots \\ \frac{\partial \log f}{\partial \xi} &= \frac{(e^z - 1)z^2 - 2e^z z}{2e^z} - \frac{(3z^4 + (8e^z - 8)z^3 - 12e^z z^2)\xi}{12e^z} + \dots \end{aligned}$$

$$\begin{aligned} \frac{\partial \log f}{\partial z} &= -[1 - e^{-z}] - \frac{1}{2} [2 - 2(1 - e^{-z})z - e^{-z}z^2] \xi + o(\xi) \\ \frac{\partial \log f}{\partial \xi} &= -\frac{1}{2} [2z - (1 - e^{-z})z^2] + \frac{1}{12} [12 - 8(1 - e^{-z})z - 3z^2] z^2 \xi + o(\xi) \end{aligned}$$

Limits for $\xi \rightarrow 0$

Here are the limits for $\xi \rightarrow 0$

$$\begin{aligned} \lim_{\xi \rightarrow 0} \frac{\partial \log f}{\partial z} &= -e^{-z} (e^z - 1) \\ \lim_{\xi \rightarrow 0} \frac{\partial \log f}{\partial \xi} &= \frac{e^{-z} (z^2 (e^z - 1) - 2ze^z)}{2} \end{aligned}$$

that is

$$\begin{aligned} \lim_{\xi \rightarrow 0} \frac{\partial \log f}{\partial z} &= 1 - e^{-z} \\ \lim_{\xi \rightarrow 0} \frac{\partial \log f}{\partial \xi} &= -\frac{z}{2} [2 - (1 - e^{-z})z] \end{aligned}$$

1.5 Second-order derivatives: expressions

Scaling

$$\begin{aligned} \frac{\partial^2 \log f}{\partial \mu^2} &= \frac{1}{\sigma^2} \frac{\partial^2 \log f}{\partial z^2} \\ \frac{\partial^2 \log f}{\partial \mu \partial \sigma} &= \frac{z}{\sigma^2} \frac{\partial^2 \log f}{\partial z^2} \\ \frac{\partial^2 \log f}{\partial \mu \partial \xi} &= \frac{-1}{\sigma} \frac{\partial^2 \log f}{\partial z \partial \xi} \\ \frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1}{\sigma^2} + \frac{z^2}{\sigma^2} \frac{\partial \log f}{\partial z} - \frac{z}{\sigma} \frac{\partial^2 \log f}{\partial z^2} \\ \frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma} \frac{\partial^2 \log f}{\partial z \partial \xi} \end{aligned}$$

Raw expressions

$$\begin{aligned}\frac{\partial^2 \log f}{\partial z^2} &= \left(-\frac{1}{\xi} - 1\right) \xi (\xi z + 1)^{-\frac{1}{\xi}-2} + \frac{\left(\frac{1}{\xi} + 1\right) \xi^2}{(\xi z + 1)^2} \\ \frac{\partial^2 \log f}{\partial z \partial \xi} &= (\xi z + 1)^{-\frac{1}{\xi}-1} \left(\frac{\log(\xi z + 1)}{\xi^2} + \frac{\left(-\frac{1}{\xi} - 1\right) z}{\xi z + 1} \right) - \frac{\frac{1}{\xi} + 1}{\xi z + 1} + \frac{1}{\xi (\xi z + 1)} + \frac{\left(\frac{1}{\xi} + 1\right) \xi z}{(\xi z + 1)^2} \\ \frac{\partial^2 \log f}{\partial \xi^2} &= -\frac{\left(\frac{\log(\xi z + 1)}{\xi^2} - \frac{z}{\xi (\xi z + 1)}\right)^2}{(\xi z + 1)^{\frac{1}{\xi}}} - \frac{-\frac{2 \log(\xi z + 1)}{\xi^3} + \frac{2 z}{\xi^2 (\xi z + 1)} + \frac{z^2}{\xi (\xi z + 1)^2}}{(\xi z + 1)^{\frac{1}{\xi}}} - \frac{2 \log(\xi z + 1)}{\xi^3} + \frac{2 z}{\xi^2 (\xi z + 1)} + \frac{\left(\frac{1}{\xi} + 1\right) z^2}{(\xi z + 1)^2}\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial^2 \log f}{\partial z^2} &= -(\xi + 1) \left[V^{-1/\xi} - \xi \right] \times \frac{1}{V^2} \\ \frac{\partial^2 \log f}{\partial z \partial \xi} &= V^{-1/\xi-1} \left[\frac{1}{\xi^2} \log V - \frac{(\xi + 1)}{\xi} \frac{z}{V} \right] - \frac{1}{V} + (\xi + 1) \frac{z}{V^2} \\ \frac{\partial^2 \log f}{\partial \xi^2} &= -\left\{ \left[\frac{1}{\xi^2} \log V - \frac{1}{\xi} \frac{z}{V} \right]^2 + \left[-\frac{2}{\xi^2} \log V + \frac{2}{\xi} \frac{z}{V} + \frac{z^2}{V^2} \right] \times \frac{1}{\xi} \right\} \times V^{-1/\xi} - \frac{2}{\xi^3} \log V + \frac{2}{\xi^2} \frac{z}{V} + \frac{\xi + 1}{\xi} \frac{z^2}{V^2}\end{aligned}$$

1.6 Second-order derivatives: limits for $\xi \rightarrow 0$

Here are the limits for $\xi \rightarrow 0$

$$\begin{aligned}\lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z} \\ \lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{e^{-z} (z (2e^z - 2) - 2e^z + z^2)}{2} \\ \lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} &= -\frac{e^{-z} (z^3 (8e^z - 8) - 12z^2 e^z + 3z^4)}{12}\end{aligned}$$

$$\begin{aligned}\lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z} \\ \lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{1}{2} [-2 - 2(1 - e^{-z})z + e^{-z}z^2] \\ \lim_{\xi \rightarrow 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} &= \frac{z^2}{12} [12 + 8(1 - e^{-z})z - 3e^{-z}z^2]\end{aligned}$$

2 Distribution function F

2.1 Expression

Define

$$W := V^{-1/\xi} = [1 + \xi z]^{-1/\xi}.$$

Then

$$F := \exp\{-W\}.$$

For the distribution function we will only consider the first-order derivatives since the second-order derivatives are not needed in most applications.

2.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$F = \frac{1}{e^{\frac{1}{\varepsilon^z}}} - \frac{z^2 \xi}{2 e^{\frac{1}{\varepsilon^z}} e^z} - \frac{((3 e^z - 3) z^4 - 8 e^z z^3) \xi^2}{24 e^{\frac{1}{\varepsilon^z}} (e^z)^2} + \dots$$

Simplified expression

$$F = \exp\{-e^{-z}\} \left\{ 1 - e^{-z} \frac{z^2}{2} \xi + e^{-z} \frac{z^3}{24} [8 - 3(1 - e^{-z})z] \xi^2 \right\} + o(\xi^2)$$

2.3 First-order derivatives: expressions

Scaling

$$\begin{aligned} \frac{\partial F}{\partial \mu} &= \frac{-1}{\sigma} \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial \sigma} &= \frac{z}{\sigma} \frac{\partial F}{\partial z} \end{aligned}$$

Raw expressions

$$\begin{aligned} \frac{\partial F}{\partial z} &= (\xi z + 1)^{-\frac{1}{\xi} - 1} e^{-\frac{1}{(\xi z + 1)^{\frac{1}{\xi}}}} \\ \frac{\partial F}{\partial \xi} &= -\frac{e^{-\frac{1}{(\xi z + 1)^{\frac{1}{\xi}}}} \left(\frac{\log(\xi z + 1)}{\xi^2} - \frac{z}{\xi(\xi z + 1)} \right)}{(\xi z + 1)^{\frac{1}{\xi}}} \end{aligned}$$

Simplified expressions

$$\begin{aligned} \frac{\partial F}{\partial z} &= FW \frac{1}{V} \\ \frac{\partial F}{\partial \xi} &= -FW \left\{ \frac{\log V}{\xi^2} - \frac{z}{\xi V} \right\} \end{aligned}$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{1}{e^{\frac{z e^z + 1}{e^z}}} + \frac{((e^z - 1) z^2 - 2 e^z z) \xi}{2 e^{\frac{z e^z + 1}{e^z}} e^z} + \dots \\ \frac{\partial F}{\partial \xi} &= -\frac{z^2}{2 e^{\frac{z e^z + 1}{e^z}}} - \frac{((3 e^z - 3) z^4 - 8 e^z z^3) \xi}{12 e^{\frac{z e^z + 1}{e^z}} e^z} + \dots\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial F}{\partial z} &= \exp\{-z - e^{-z}\} \left\{ 1 + \frac{z}{2} [-2 + (1 - e^{-z})z] \xi \right\} + o(\xi) \\ \frac{\partial F}{\partial \xi} &= \exp\{-z - e^{-z}\} \left\{ -\frac{z^2}{2} + \frac{z^3}{12} [8 - 3(1 - e^{-z})z] \xi \right\} + o(\xi)\end{aligned}$$

3 Quantile or return level

3.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\text{GEV}}(p)$ is given by

$$\rho = \mu - \frac{\left(1 - \frac{1}{A^\xi}\right) \sigma}{\xi}$$

3.2 Taylor expansion for $\xi \approx 0$

$$\rho = \mu - \log A \sigma + \frac{(\log A)^2 \sigma \xi}{2} - \frac{(\log A)^3 \sigma \xi^2}{6} + \dots$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\begin{aligned}\frac{\partial \rho}{\partial \mu} &= 1 \\ \frac{\partial \rho}{\partial \sigma} &= -\frac{1 - \frac{1}{A^\xi}}{\xi} \\ \frac{\partial \rho}{\partial \xi} &= \frac{\left(1 - \frac{1}{A^\xi}\right) \sigma}{\xi^2} - \frac{\log A \sigma}{A^\xi \xi}\end{aligned}$$

$$\begin{aligned}\lim_{\xi \rightarrow 0} \frac{\partial \rho}{\partial \mu} &= 1 \\ \lim_{\xi \rightarrow 0} \frac{\partial \rho}{\partial \sigma} &= -\log A \\ \lim_{\xi \rightarrow 0} \frac{\partial \rho}{\partial \xi} &= \frac{(\log A)^2 \sigma}{2}\end{aligned}$$

3.4 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\begin{aligned}\frac{\partial \rho}{\partial \mu} &= +1 + \dots \\ \frac{\partial \rho}{\partial \sigma} &= -\log A + \frac{(\log A)^2 \xi}{2} + \dots \\ \frac{\partial \rho}{\partial \xi} &= \frac{(\log A)^2 \sigma}{2} - \frac{(\log A)^3 \sigma \xi}{3} + \dots\end{aligned}$$

3.5 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ , we consider only the standardized return level

$$\rho^* = \frac{\frac{1}{A^\xi} - 1}{\xi}$$

so that $\rho = \mu + \sigma \rho^*$.

Raw expressions

$$\begin{aligned}\frac{\partial \rho^*}{\partial \xi} &= -\frac{\log A}{A^\xi \xi} - \frac{\frac{1}{A^\xi} - 1}{\xi^2} \\ \frac{\partial^2 \rho^*}{\partial \xi^2} &= \frac{(\log A)^2}{A^\xi \xi} + \frac{2 \log A}{A^\xi \xi^2} + \frac{2 \left(\frac{1}{A^\xi} - 1 \right)}{\xi^3}\end{aligned}$$

Simplified expression

$$\begin{aligned}\frac{\partial \rho^*}{\partial \xi} &= -\frac{1}{\xi} [\rho^* + \log A] - \rho^* \log A \\ \frac{\partial^2 \rho^*}{\partial \xi^2} &= \frac{1}{\xi^2} [\rho^* + \log A] - \frac{\partial \rho^*}{\partial \xi} \left[\log A + \frac{1}{\xi} \right]\end{aligned}$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^*}{\partial \xi} \\ 0 & \frac{\partial \rho^*}{\partial \xi} & \sigma \frac{\partial^2 \rho^*}{\partial \xi^2} \end{bmatrix}$$

3.6 Second-order derivatives: limit for $\xi \rightarrow 0$

Here are the limits for $\xi \rightarrow 0$

$$\lim_{\xi \rightarrow 0} \frac{\partial \rho^*}{\partial \xi} = \frac{(\log A)^2}{2}$$
$$\lim_{\xi \rightarrow 0} \frac{\partial^2 \rho^*}{\partial \xi^2} = -\frac{(\log A)^3}{3}$$