The probability functions of the GP2 distribution and their derivatives

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Contents

1 The log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y}{\sigma}. \tag{1}$$

Define A and B by

$$A :=, \qquad B := \tag{2}$$

then the log-density is given by

$$\log f = .$$

1.2 Taylor expansion for $\xi \approx 0$

Raw expression

The Taylor approximation of $\log f$ for $\xi \approx 0$ is

$$\log f =$$

Simplified expression

$$\log f = -\left[\log \sigma + z\right] + \frac{z(z-2)}{2}\,\xi - \frac{z^2(2z-3)}{6}\,\xi^2 + o(\xi^2)$$

1.3 First-order derivatives: expressions

Raw expressions

$$\frac{\partial \log f}{\partial \sigma} =$$
$$\frac{\partial \log f}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma}\,[1 - (\xi + 1)B]$$
$$\frac{\partial \log f}{\partial \xi} = \frac{1}{\xi^2}\,[A - \xi(\xi + 1)B]$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

For $\xi \approx 0$ we have

$$\frac{\partial \log f}{\partial \sigma} =$$
$$\frac{\partial \log f}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial \log f}{\partial \sigma} = \frac{z-1}{\sigma} - \frac{z(z-1)}{\sigma}\,\xi + o(\xi)$$
$$\frac{\partial \log f}{\partial \xi} = \frac{z(z-2)}{2} - \frac{z^2(2z-3)}{3}\,\xi + o(\xi)$$

1.5 Second-order derivatives: expressions

Raw expressions

$$\frac{\partial^2}{\partial \sigma^2}\,\log f =$$
$$\frac{\partial^2}{\partial \sigma \partial \xi}\,\log f =$$
$$\frac{\partial^2}{\partial \xi^2}\,\log f =$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial^2}{\partial\sigma^2}\,\log f &= \frac{1}{\sigma^2}\,\left[1 - 2(\xi+1)B + \xi(\xi+1)B^2\right]\\
\frac{\partial^2}{\partial\sigma\partial\xi}\,\log f &= \frac{1}{\sigma}\,\left[B - (\xi+1)B^2\right]\\
\frac{\partial^2}{\partial\xi^2}\,\log f &= \frac{1}{\xi^3}\,\left[-2A + 2\xi B + \xi^2(\xi+1)B^2\right]
\end{aligned}
$$

1.6 Second-order derivatives: limits for $\xi \to 0$

Raw expressions

Here are the limits for $\xi \to 0$

$$
\begin{aligned}
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\sigma^2} &=\\
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\sigma\partial\xi} &=\\
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\xi^2} &=
\end{aligned}
$$

Simplified expressions

The corresponding simplified expressions are

$$
\begin{aligned}
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\sigma^2} &= \frac{1-2z}{\sigma^2}\\
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\sigma\partial\xi} &= -\frac{z(z-1)}{\sigma}\\
\lim_{\xi\to 0}\frac{\partial^2\log f}{\partial\xi^2} &= -\frac{z^2(2z-3)}{3}.
\end{aligned}
$$

2 Cumulated hazard (log-survival) $H = \log S$

2.1 Expression

The cumulated hazard $H(y) = \log S(y)$

$$H :=$$

2.2 Taylor approximation for $\xi \approx 0$

Raw expression

$$H =$$

Simplified expression

$$H = z - \frac{z^2}{2}\,\xi + \frac{z^3}{3}\,\xi^2 + o(\xi^2) \tag{3}$$

2.3 First-order derivatives: expressions

Raw expressions

$$\frac{\partial H}{\partial \sigma} =$$
$$\frac{\partial H}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial H}{\partial \sigma} = -\frac{1}{\sigma}\,B$$
$$\frac{\partial H}{\partial \xi} = -\frac{1}{\xi^2}\,[A - \xi B]$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

Here are the limits for $\xi \approx 0$

$$\frac{\partial H}{\partial \sigma} =$$
$$\frac{\partial H}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial H}{\partial \sigma} = -\frac{z}{\sigma} + \frac{z^2}{\sigma}\,\xi + o(\xi)$$
$$\frac{\partial H}{\partial \xi} = -\frac{z^2}{2} + \frac{2z^3}{3}\,\xi + o(\xi)$$

2.5 Second-order derivatives: expressions

Raw expressions

$$\frac{\partial^2}{\partial\sigma^2} H =$$
$$\frac{\partial^2}{\partial\sigma\partial\xi} H =$$
$$\frac{\partial^2}{\partial\xi^2} H =$$

Simplified expressions

$$\frac{\partial^2}{\partial\sigma^2} H = \frac{1}{\sigma^2} \left[2B - \xi B^2\right]$$
$$\frac{\partial^2}{\partial\sigma\partial\xi} H = \frac{1}{\sigma} B^2$$
$$\frac{\partial^2}{\partial\xi^2} H = \frac{1}{\xi^3} \left[2A - 2\xi B - \xi^2 B^2\right]$$

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma^2} =$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma\partial\xi} =$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\xi^2} =$$

The corresponding simplified expresions are

$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma^2} = \frac{2z}{\sigma^2}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma\partial\xi} = \frac{z^2}{\sigma}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\xi^2} = \frac{2z^3}{3}.$$

3 Quantile or return period

3.1 Expression

The quantile corresponding to an exceedance probability $q := 1 - p$ for $\xi \neq 0$

$$\rho =$$

3.2 Taylor approximation for $\xi \approx 0$

$$ \tag{4}$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\frac{\partial \rho}{\partial \sigma} =$$
$$\frac{\partial \rho}{\partial \xi} =$$

3.4 First-order derivatives: Taylor approximation

$$\frac{\partial \rho}{\partial \sigma} =$$
$$\frac{\partial \rho}{\partial \xi} =$$

3.5 Second-order derivatives: expressions

Raw expressions

$$\frac{\partial^2 \rho}{\partial \sigma^2} =$$
$$\frac{\partial^2 \rho}{\partial \sigma \partial \xi} =$$
$$\frac{\partial^2 \rho}{\partial \xi^2} =$$

Simplified expressions

Define $V := [q^{-\xi} - 1]/\xi$ so that $\rho = \sigma V$, and

$$W := \frac{\partial V}{\partial \xi} = -\frac{1}{\xi}\left\{V + q^{-\xi}\log q\right\}.$$

$$\begin{aligned}\frac{\partial^2 \rho}{\partial \sigma^2} &= 0\\ \frac{\partial^2 \rho}{\partial \sigma \partial \xi} &= W\\ \frac{\partial^2 \rho}{\partial \xi^2} &= \frac{\sigma}{\xi^2}\left\{2V + q^{-\xi}\log q\,[2 + \xi \log q]\right\}\end{aligned}$$

3.6 Second-order derivatives: limit for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \sigma^2} =$$

$$\lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \sigma \partial \xi} =$$

$$\lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \xi^2} =$$