The probability functions of the GEV distribution and their derivatives

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1 Log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y - \mu}{\sigma}$$

and then V and U by

$$V :=, \qquad U := \frac{1}{\sigma V}\left[1 + \xi - V^{-1/\xi}\right].$$

The log-likelihood is defined by

$$\log f =$$

1.2 Taylor expansion at $\xi = 0$

$$\log f =$$

which can be simplified as

$$\log f = -\left[\log\sigma + z + e^{-z}\right] + \frac{1}{2}\left[(1 - e^{-z})z - 2\right] z\xi - \frac{1}{24}\left[3z^2 e^{-z} + 8z(1 - e^{-z}) - 12\right] z^2\xi^2$$

1.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z, so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z}\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}\frac{\partial \log f}{\partial z}$$

and up to the term $-\log\sigma$ in $\log f$, the same is true for σ so

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma} + \frac{\partial \log f}{\partial z}\frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} - \frac{z}{\sigma}\frac{\partial \log f}{\partial z}.$$

So it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ.

Raw expressions

Here are the raw expressions found by Maxima

$$\frac{\partial \log f}{\partial z} =$$
$$\frac{\partial \log f}{\partial \xi} =$$

Simplified expressions

We use the following simplifications

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\sigma U,\\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right] \log V - z\, U \frac{\sigma}{\xi}.\end{aligned}$$

so eventually

$$\begin{aligned}\frac{\partial \log f}{\partial \mu} &= U,\\ \frac{\partial \log f}{\partial \sigma} &= \frac{-1}{\sigma} + zU,\\ \frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\left[1 - V^{-1/\xi}\right] \log V - z\, U \frac{\sigma}{\xi}.\end{aligned}$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$U =$$

$$\frac{\partial \log f}{\partial z} =$$
$$\frac{\partial \log f}{\partial \xi} =$$

$$\begin{aligned}\frac{\partial \log f}{\partial z} &= -\left[1 - e^{-z}\right] - \frac{1}{2}\left[2 - 2(1 - e^{-z})z - e^{-z}z^2\right]\xi + o(\xi)\\ \frac{\partial \log f}{\partial \xi} &= -\frac{1}{2}\left[2z - (1 - e^{-z})z^2\right] + \frac{1}{12}\left[12 - 8(1 - e^{-z})z - 3z^2\right] z^2\xi + o(\xi)\end{aligned}$$

Limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial \log f}{\partial z} =$$
$$\lim_{\xi \to 0} \frac{\partial \log f}{\partial \xi} =$$

that is

$$\lim_{\xi \to 0} \frac{\partial \log f}{\partial z} = 1 - e^{-z}$$
$$\lim_{\xi \to 0} \frac{\partial \log f}{\partial \xi} = -\frac{z}{2} \left[2 - (1 - e^{-z})z\right]$$

1.5 Second-order derivatives: expressions

Scaling

$$\frac{\partial^2 \log f}{\partial \mu^2} = \frac{1}{\sigma^2} \frac{\partial^2 \log f}{\partial z^2}$$
$$\frac{\partial^2 \log f}{\partial \mu \partial \sigma} = \frac{z}{\sigma^2} \frac{\partial^2 \log f}{\partial z^2}$$
$$\frac{\partial^2 \log f}{\partial \mu \partial \xi} = \frac{-1}{\sigma} \frac{\partial^2 \log f}{\partial z \partial \xi}$$
$$\frac{\partial^2 \log f}{\partial \sigma^2} = \frac{1}{\sigma^2} + \frac{z^2}{\sigma^2} \frac{\partial \log f}{\partial z} - \frac{z}{\sigma} \frac{\partial^2 \log f}{\partial z^2}$$
$$\frac{\partial^2 \log f}{\partial \sigma \partial \xi} = \frac{-z}{\sigma} \frac{\partial^2 \log f}{\partial z \partial \xi}$$

Raw expressions

$$\frac{\partial^2 \log f}{\partial z^2} =$$
$$\frac{\partial^2 \log f}{\partial z \partial \xi} =$$
$$\frac{\partial^2 \log f}{\partial \xi^2} =$$

Simplified expressions

$$\frac{\partial^2 \log f}{\partial z^2} = -(\xi + 1) \left[V^{-1/\xi} - \xi\right] \times \frac{1}{V^2}$$
$$\frac{\partial^2 \log f}{\partial z \partial \xi} = V^{-1/\xi - 1} \left[\frac{1}{\xi^2} \log V - \frac{(\xi + 1)}{\xi} \frac{z}{V}\right] - \frac{1}{V} + (\xi + 1) \frac{z}{V^2}$$
$$\frac{\partial^2 \log f}{\partial \xi^2} = -\left\{\left[\frac{1}{\xi^2} \log V - \frac{1}{\xi} \frac{z}{V}\right]^2 + \left[-\frac{2}{\xi^2} \log V + \frac{2}{\xi} \frac{z}{V} + \frac{z^2}{V^2}\right] \times \frac{1}{\xi}\right\} \times V^{-1/\xi} - \frac{2}{\xi^3} \log V + \frac{2}{\xi^2} \frac{z}{V} + \frac{\xi + 1}{\xi} \frac{z^2}{V^2}$$

1.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} =$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} =$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} =$$

$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} = -e^{-z}$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} = \frac{1}{2}\left[-2 - 2(1 - e^{-z})z + e^{-z}z^2\right]$$
$$\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} = \frac{z^2}{12}\left[12 + 8(1 - e^{-z})z - 3e^{-z}z^2\right]$$

2 Distribution function F

2.1 Expression

Define

$$W := V^{-1/\xi} = [1 + \xi z]^{-1/\xi}.$$

Then

$$F := \exp\{-W\}.$$

For the distribution function we will only consider the first-order derivatives since the second-order derivatives are not needed in most applications.

2.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$F =$$

Simplified expression

$$F = \exp\{-e^{-z}\}\left\{1 - e^{-z}\frac{z^2}{2}\xi + e^{-z}\frac{z^3}{24}\left[8 - 3(1 - e^{-z})z\right]\xi^2\right\} + o(\xi^2)$$

2.3 First-order derivatives: expressions

Scaling

$$\frac{\partial F}{\partial \mu} = \frac{-1}{\sigma} \frac{\partial F}{\partial z}$$
$$\frac{\partial F}{\partial \sigma} = \frac{z}{\sigma} \frac{\partial F}{\partial z}$$

Raw expressions

$$\frac{\partial F}{\partial z} =$$
$$\frac{\partial F}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial F}{\partial z} = FW \frac{1}{V}$$
$$\frac{\partial F}{\partial \xi} = -FW \left\{ \frac{\log V}{\xi^2} - \frac{z}{\xi V} \right\}$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\frac{\partial F}{\partial z} =$$
$$\frac{\partial F}{\partial \xi} =$$

Simplified expressions

$$\frac{\partial F}{\partial z} = \exp\{-z - e^{-z}\} \left\{ 1 + \frac{z}{2} \left[-2 + (1 - e^{-z})z \right] \xi \right\} + o(\xi)$$
$$\frac{\partial F}{\partial \xi} = \exp\{-z - e^{-z}\} \left\{ -\frac{z^2}{2} + \frac{z^3}{12} \left[8 - 3(1 - e^{-z})z \right] \xi \right\} + o(\xi)$$

3 Quantile or return level

3.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\text{GEV}}(p)$ is given by

$$\rho =$$

3.2 Taylor expansion for $\xi \approx 0$

$$\rho =$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\frac{\partial \rho}{\partial \mu} =$$
$$\frac{\partial \rho}{\partial \sigma} =$$
$$\frac{\partial \rho}{\partial \xi} =$$

$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \mu} =$$
$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \sigma} =$$
$$\lim_{\xi \to 0} \frac{\partial \rho}{\partial \xi} =$$

3.4 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\frac{\partial \rho}{\partial \mu} =$$
$$\frac{\partial \rho}{\partial \sigma} =$$
$$\frac{\partial \rho}{\partial \xi} =$$

3.5 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ, we consider only the standardized return level

$$\rho^\star =$$

so that $\rho = \mu + \sigma\rho^\star$.

Raw expressions

$$\frac{\partial \rho^\star}{\partial \xi} =$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} =$$

Simplified expression

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{1}{\xi}\,[\rho^\star + \log A] - \rho^\star\,\log A$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{1}{\xi^2}\,[\rho^\star + \log A] - \frac{\partial \rho^\star}{\partial \xi}\left[\log A + \frac{1}{\xi}\right]$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^\star}{\partial \xi} \\ 0 & \frac{\partial \rho^\star}{\partial \xi} & \sigma \frac{\partial^2 \rho^\star}{\partial \xi^2} \end{bmatrix}$$

3.6 Second-order derivatives: limit for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial \rho^\star}{\partial \xi} =$$
$$\lim_{\xi \to 0} \frac{\partial^2 \rho^\star}{\partial \xi^2} =$$