

The Poisson-GP to NHPP transformation and its derivatives

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1 Poisson-GP to PP

1.1 Expression

Let $\boldsymbol{\theta} = [\lambda, \sigma, \xi]$ and $\boldsymbol{\theta}_0 = [\mu_0, \sigma_0, \xi_0]$ denote the Poisson-GP and PP parameter vectors. The threshold u being considered as fixed, the relation between the PP parameters $\boldsymbol{\theta}_0$ and the Poisson-GP parameters $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\mu_0 &= u + \frac{(\lambda w)^\xi - 1}{\xi} \sigma, \\ \sigma_0 &= (\lambda w)^\xi \sigma, \\ \xi_0 &= \xi.\end{aligned}$$

1.2 Taylor approximation for $\xi \approx 0$

Raw expressions

$$\begin{aligned}
\mu_0 &= \log(w\lambda) \sigma + u + \frac{(\log(w\lambda))^2 \sigma \xi}{2} + \frac{(\log(w\lambda))^3 \sigma \xi^2}{6} + \dots \\
\sigma_0 &= \sigma + \log(w\lambda) \sigma \xi + \frac{(\log(w\lambda))^2 \sigma \xi^2}{2} + \dots \\
\xi_0 &= +\xi + \dots
\end{aligned}$$

Simplified expressions

With $L := \log(\lambda w)$, we have

$$\begin{aligned}
\mu_0 &= u + \sigma \left\{ L + \frac{L^2}{2} \xi + \frac{L^3}{6} \xi^2 \right\} + o(\xi^2) \\
\sigma_0 &= \sigma \left\{ 1 + L \xi + \frac{L^2}{2} \xi^2 \right\} + o(\xi^2) \\
\xi_0 &= \xi + o(\xi^2)
\end{aligned}$$

2 First-order derivatives for μ_0 : expressions

Raw expressions

$$\begin{aligned}
\frac{\partial \mu_0}{\partial \lambda} &= \frac{\sigma (w\lambda)^\xi}{\lambda} \\
\frac{\partial \mu_0}{\partial \sigma} &= \frac{(w\lambda)^\xi - 1}{\xi} \\
\frac{\partial \mu_0}{\partial \xi} &= \frac{\sigma (w\lambda)^\xi \log(w\lambda)}{\xi} - \frac{\sigma \left((w\lambda)^\xi - 1 \right)}{\xi^2}
\end{aligned}$$

Simplified expressions

With $\nu := \lambda w$, we have

$$\begin{aligned}
\frac{\partial \mu_0}{\partial \lambda} &= \sigma \frac{\nu^\xi}{\lambda} \\
\frac{\partial \mu_0}{\partial \sigma} &= \frac{\nu^\xi - 1}{\xi} \\
\frac{\partial \mu_0}{\partial \xi} &= \frac{\sigma}{\xi} \left\{ \nu^\xi L - \frac{\nu^\xi - 1}{\xi} \right\} = \frac{\sigma}{\xi} \left\{ \nu^\xi L - \frac{\partial \mu_0}{\partial \sigma} \right\}
\end{aligned}$$

3 First-order derivatives for μ_0 : Taylor approximation

Raw expressions

$$\begin{aligned}\frac{\partial \mu_0}{\partial \lambda} &= \frac{\sigma}{\lambda} + \frac{\log(w\lambda) \sigma \xi}{\lambda} + \dots \\ \frac{\partial \mu_0}{\partial \sigma} &= \log(w\lambda) + \frac{(\log(w\lambda))^2 \xi}{2} + \dots \\ \frac{\partial \mu_0}{\partial \xi} &= \frac{\sigma (\log(w\lambda))^2}{2} + \frac{\sigma (\log(w\lambda))^3 \xi}{3} + \dots\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial \mu_0}{\partial \lambda} &= \frac{\sigma}{\lambda} \{1 + L\xi\} + o(\xi) \\ \frac{\partial \mu_0}{\partial \sigma} &= \frac{L}{2} \{2 + L\xi\} + o(\xi) \\ \frac{\partial \mu_0}{\partial \xi} &= \sigma \frac{L^2}{6} \{3 + 2L\xi\} + o(\xi)\end{aligned}$$

4 First-order derivatives for σ_0 : expressions

Raw expressions

$$\begin{aligned}\frac{\partial \sigma_0}{\partial \lambda} &= \frac{\sigma \xi (w\lambda)^\xi}{\lambda} \\ \frac{\partial \sigma_0}{\partial \sigma} &= (w\lambda)^\xi \\ \frac{\partial \sigma_0}{\partial \xi} &= \sigma (w\lambda)^\xi \log(w\lambda)\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial \sigma_0}{\partial \lambda} &= \sigma \xi \frac{\nu^\xi}{\lambda} = \xi \frac{\partial \mu_0}{\partial \lambda} \\ \frac{\partial \sigma_0}{\partial \sigma} &= \nu^\xi \\ \frac{\partial \sigma_0}{\partial \xi} &= \sigma \nu^\xi L\end{aligned}$$

5 First-order derivatives for ξ_0 : expressions

$$\begin{aligned}\frac{\partial \xi_0}{\partial \lambda} &= 0 \\ \frac{\partial \xi_0}{\partial \sigma} &= 0 \\ \frac{\partial \xi_0}{\partial \xi} &= 1\end{aligned}$$

6 First-order derivatives for σ_0 : Taylor approximation

Raw expressions

$$\begin{aligned}\frac{\partial \sigma_0}{\partial \lambda} &= +\frac{\sigma \xi}{\lambda} + \dots \\ \frac{\partial \sigma_0}{\partial \sigma} &= 1 + \log(w \lambda) \xi + \dots \\ \frac{\partial \sigma_0}{\partial \xi} &= \sigma \log(w \lambda) + \sigma (\log(w \lambda))^2 \xi + \dots\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial \sigma_0}{\partial \lambda} &= \frac{\sigma}{\lambda} \xi + o(\xi) \\ \frac{\partial \sigma_0}{\partial \sigma} &= 1 + L\xi + o(\xi) \\ \frac{\partial \sigma_0}{\partial \xi} &= \sigma L \{1 + L\xi\} + o(\xi)\end{aligned}$$