The NHPP to Poisson-GP transformation and its derivatives

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Contents

1 The PP to Poisson-GP transformation

1.1 Expression

With w being the reference block duration,

$$z_0 := \frac{u - \mu_0}{\sigma_0}, \qquad C := 1 + \xi_0 z_0$$

The Poisson-GP parameter vector $\boldsymbol{\theta} := [\lambda,\, \sigma,\, \xi]$ relates to the vector $\boldsymbol{\theta}_0 := [\mu_0,\, \sigma_0,\, \xi_0]$ of PP parameters according to

$$\begin{aligned}
\lambda &= \frac{1}{w\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{\frac{1}{\xi_0}}}\\
\sigma &= \sigma_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)\\
\xi &= \xi_0.
\end{aligned}$$

In simpler form

$$\begin{aligned}
\lambda &= C^{-1/\xi_0} w^{-1}\\
\sigma &= C\sigma_0\\
\xi &= \xi_0
\end{aligned}$$

Remind that these expressions can only be used when the threshold u is in the support of the GEV distribution with parameters $\boldsymbol{\theta}_0$.

1.2 Taylor approximation for $\xi \approx 0$

Raw expressions

$$\begin{aligned}
\lambda &= \frac{1}{e^{\frac{u-\mu_0}{\sigma_0}}\,w}+\frac{\left(\mu_0^2-2\,u\,\mu_0+u^2\right)\,\xi_0}{2\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^2\,w}+\frac{\left(\left(8\,\mu_0^3-24\,u\,\mu_0^2+24\,u^2\,\mu_0-8\,u^3\right)\,\sigma_0+3\,\mu_0^4-12\,u\,\mu_0^3+18\,u^2\,\mu_0^2-12\,u^3\,\mu_0+3\,u^4\right)\,\xi_0^2}{24\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^4\,w}+\cdots\\
\sigma &= \sigma_0+(u-\mu_0)\,\xi_0+\cdots\\
\xi &= +\xi_0+\cdots
\end{aligned}$$

Simplified expressions

$$\begin{aligned}
\lambda &= e^{-z_0}w^{-1}\left\{1+z_0^2\,\xi-\frac{z_0^3}{24}\,[8-3z_0]\,\xi_0^2\right\}+o(\xi_0^2)\\
\sigma &= \sigma_0\,[1+z_0\,\xi]+o(\xi_0^2)\\
\xi &= \xi_0
\end{aligned}$$

2 First-order derivatives for λ: expressions

Raw expressions

$$\frac{\partial\lambda}{\partial\mu_0} = \frac{\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0\,w}$$

$$\frac{\partial\lambda}{\partial\sigma_0} = \frac{\frac{\log\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}{\xi_0^2} - \frac{u-\mu_0}{\sigma_0\,\xi_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}}{w\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{\frac{1}{\xi_0}}}$$

$$\frac{\partial\lambda}{\partial\xi_0} = \frac{\frac{\log\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}{\xi_0^2} - \frac{u-\mu_0}{\sigma_0\,\xi_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}}{w\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{\frac{1}{\xi_0}}}$$

Simplified expressions

$$\frac{\partial\lambda}{\partial\mu_0} = \frac{\lambda}{\sigma^\star C}$$

$$\frac{\partial\lambda}{\partial\sigma_0} = \frac{z\lambda}{\sigma^\star C}$$

$$\frac{\partial\lambda}{\partial\xi_0} = \frac{\lambda}{\xi_0^2}\,\left[\log C - \xi_0\frac{z_0}{C}\right]$$

3 First-order derivatives of λ: Taylor approximation

Raw expressions

$$\frac{\partial\lambda}{\partial\mu_0} = \frac{1}{e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0\,w} + \frac{\left(u^2+\left(-2\,\sigma_0-2\,\mu_0\right)\,u+2\,\mu_0\,\sigma_0+\mu_0^2\right)\,\xi_0}{2\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^3\,w} + \cdots$$

$$\frac{\partial\lambda}{\partial\sigma_0} = \frac{u-\mu_0}{e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^2\,w} + \frac{\left(u^3+\left(-2\,\sigma_0-3\,\mu_0\right)\,u^2+\left(4\,\mu_0\,\sigma_0+3\,\mu_0^2\right)\,u-2\,\mu_0^2\,\sigma_0-\mu_0^3\right)\,\xi_0}{2\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^4\,w} + \cdots$$

$$\frac{\partial\lambda}{\partial\xi_0} = \frac{u^2-2\,\mu_0\,u+\mu_0^2}{2\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^2\,w} + \frac{\left(3\,u^4+\left(-8\,\sigma_0-12\,\mu_0\right)\,u^3+\left(24\,\mu_0\,\sigma_0+18\,\mu_0^2\right)\,u^2+\left(-24\,\mu_0^2\,\sigma_0-12\,\mu_0^3\right)\,u+8\,\mu_0^3\,\sigma_0+3\,\mu_0^4\right)\,\xi_0}{12\,e^{\frac{u-\mu_0}{\sigma_0}}\,\sigma_0^4\,w} + \cdots$$

Simplified expressions

$$\frac{\partial\lambda}{\partial\mu_0} = \frac{1}{\sigma_0}\, e^{-z_0} w^{-1} \left\{1 + \frac{z_0}{2}\, [z_0 - 2]\, \xi_0\right\} + o(\xi_0)$$
$$\frac{\partial\lambda}{\partial\sigma_0} = \frac{z_0}{\sigma_0}\, e^{-z_0} w^{-1} \left\{1 + \frac{z_0}{2}\, [z_0 - 2]\, \xi_0\right\} + o(\xi_0)$$
$$\frac{\partial\lambda}{\partial\xi_0} = \frac{z_0^2}{2}\, e^{-z_0} w^{-1} \left\{1 + \frac{z_0}{12}\, [3z_0 - 8]\, \xi_0\right\} + o(\xi_0)$$

4 First-order derivatives for σ: expressions

Raw expressions

$$\frac{\partial\sigma}{\partial\mu_0} = -\xi_0$$
$$\frac{\partial\sigma}{\partial\sigma_0} = 1$$
$$\frac{\partial\sigma}{\partial\xi_0} = u - \mu_0$$

5 First-order derivatives for σ: Taylor approximation

Unneeded (see previous section).

6 First-order derivatives for ξ: expressions

Simplified expressions

$$\frac{\partial\xi}{\partial\mu_0} = 0$$
$$\frac{\partial\xi}{\partial\sigma_0} = 0$$
$$\frac{\partial\xi}{\partial\xi_0} = 1$$

7 Second-order derivatives for λ: expressions

Raw expressions

$$\frac{\partial^2\lambda}{\partial\mu_0^2} = -\frac{\left(-\frac{1}{\xi_0}-1\right)\,\xi_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^2\,w}$$

$$\frac{\partial^2\lambda}{\partial\mu_0\partial\sigma_0} = -\frac{\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0^2\,w} - \frac{(u-\mu_0)\,\left(-\frac{1}{\xi_0}-1\right)\,\xi_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^3\,w}$$

$$\frac{\partial^2\lambda}{\partial\mu_0\partial\xi_0} = \frac{\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-1}\left(\frac{\log\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}{\xi_0^2}+\frac{(u-\mu_0)\,\left(-\frac{1}{\xi_0}-1\right)}{\sigma_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}\right)}{\sigma_0\,w}$$

Simplified expressions

$$\frac{\partial^2\lambda}{\partial\mu_0^2} = -\left[1+\xi_0\right]\,\frac{\lambda}{C^2\sigma_0^2}$$

$$\frac{\partial^2\lambda}{\partial\mu_0\partial\sigma_0} = -\frac{\lambda}{C\sigma_0^2}\,\left\{1-\left[1+\xi_0\right]\,\frac{z_0}{C}\right\}$$

$$\frac{\partial^2\lambda}{\partial\mu_0\partial\xi_0} = \frac{\lambda}{C\sigma_0\xi_0}\,\left\{-\left[1+\xi_0\right]\,\frac{z_0}{C}+\frac{\log C}{\xi_0}\right\}$$

Raw expressions

$$\frac{\partial^2\lambda}{\partial\sigma_0^2} = -\frac{2\,(u-\mu_0)\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0^3\,w} - \frac{(u-\mu_0)^2\,\left(-\frac{1}{\xi_0}-1\right)\,\xi_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^4\,w}$$

$$\frac{\partial^2\lambda}{\partial\sigma_0\partial\xi_0} = \frac{(u-\mu_0)\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)^{-\frac{1}{\xi_0}-1}\left(\frac{\log\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}{\xi_0^2}+\frac{(u-\mu_0)\,\left(-\frac{1}{\xi_0}-1\right)}{\sigma_0\,\left(\frac{(u-\mu_0)\,\xi_0}{\sigma_0}+1\right)}\right)}{\sigma_0^2\,w}$$

Simplified expressions

$$\frac{\partial^2\lambda}{\partial\sigma_0^2} = -\frac{\lambda z_0}{C\sigma_0^2}\,\left\{2-\left[1+\xi_0\right]\,\frac{z_0}{C}\right\}$$

$$\frac{\partial^2\lambda}{\partial\sigma_0\partial\xi_0} = \frac{\lambda z_0}{C\sigma_0\xi_0}\,\left\{-\left[1+\xi_0\right]\,\frac{z_0}{C}+\frac{\log C}{\xi_0}\right\} = z_0\,\frac{\partial^2\lambda}{\partial\mu_0\partial\xi_0}$$

8 Second-order derivatives for λ: limit for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial^2 \lambda}{\partial \mu_0^2} = \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 \, w}$$

$$\lim_{\xi \to 0} \frac{\partial^2 \lambda}{\partial \mu_0 \partial \sigma_0} = \frac{u \, e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w} - \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 \, w} - \frac{\mu_0 \, e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w}$$

$$\lim_{\xi \to 0} \frac{\partial^2 \lambda}{\partial \mu_0 \partial \xi_0} = \frac{\left(u^2 + \left(-2\,\sigma_0 - 2\,\mu_0\right) \, u + 2\,\mu_0\,\sigma_0 + \mu_0^2\right) \, e^{-\frac{u-\mu_0}{\sigma_0}}}{2\,\sigma_0^3 \, w}$$

$$\lim_{\xi \to 0} \frac{\partial^2 \lambda}{\partial \sigma_0^2} = -\frac{2\,u\,e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w} + \frac{2\,\mu_0\,e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w} + \frac{\left(e^{\frac{\mu_0}{\sigma_0}} \, u^2 - 2\,\mu_0\,e^{\frac{\mu_0}{\sigma_0}} \, u + \mu_0^2\,e^{\frac{\mu_0}{\sigma_0}}\right) \, e^{-\frac{u}{\sigma_0}}}{\sigma_0^4 \, w}$$

$$\lim_{\xi \to 0} \frac{\partial^2 \lambda}{\partial \sigma_0 \partial \xi_0} = \frac{u \, e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w} - \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 \, w} - \frac{\mu_0 \, e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 \, w}$$

9 Second-order derivatives for σ: expressions

$$\frac{\partial^2 \sigma}{\partial \mu_0^2} = 0$$

$$\frac{\partial^2 \sigma}{\partial \mu_0 \partial \sigma_0} = 0$$

$$\frac{\partial^2 \sigma}{\partial \mu_0 \partial \xi_0} = -1$$