

The NHPP to Poisson-GP transformation and its derivatives

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July 6, 2022

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1 The PP to Poisson-GP transformation

1.1 Expression

With w being the reference block duration,

$$z_0 := \frac{u - \mu_0}{\sigma_0}, \quad C := 1 + \xi_0 z_0$$

The Poisson-GP parameter vector $\boldsymbol{\theta} := [\lambda, \sigma, \xi]$ relates to the vector $\boldsymbol{\theta}_0 := [\mu_0, \sigma_0, \xi_0]$ of PP parameters according to

$$\begin{aligned}\lambda &= \frac{1}{w \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1 \right)^{\frac{1}{\xi_0}}} \\ \sigma &= \sigma_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1 \right) \\ \xi &= \xi_0.\end{aligned}$$

In simpler form

$$\begin{aligned}\lambda &= C^{-1/\xi_0} w^{-1} \\ \sigma &= C \sigma_0 \\ \xi &= \xi_0\end{aligned}$$

Remind that these expressions can only be used when the threshold u is in the support of the GEV distribution with parameters $\boldsymbol{\theta}_0$.

1.2 Taylor approximation for $\xi \approx 0$

Raw expressions

$$\begin{aligned}\lambda &= \frac{1}{e^{\frac{u-\mu_0}{\sigma_0}} w} + \frac{(\mu_0^2 - 2u\mu_0 + u^2)\xi_0}{2e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^2 w} + \frac{((8\mu_0^3 - 24u\mu_0^2 + 24u^2\mu_0 - 8u^3)\sigma_0 + 3\mu_0^4 - 12u\mu_0^3 + 18u^2\mu_0^2 - 12u^3\mu_0 + 3u^4)\xi_0^2}{24e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^4 w} + \dots \\ \sigma &= \sigma_0 + (u - \mu_0)\xi_0 + \dots \\ \xi &= \xi_0 + \dots\end{aligned}$$

Simplified expressions

$$\begin{aligned}\lambda &= e^{-z_0} w^{-1} \left\{ 1 + z_0^2 \xi - \frac{z_0^3}{24} [8 - 3z_0] \xi_0^2 \right\} + o(\xi_0^2) \\ \sigma &= \sigma_0 [1 + z_0 \xi] + o(\xi_0^2) \\ \xi &= \xi_0\end{aligned}$$

2 First-order derivatives for λ : expressions

Raw expressions

$$\begin{aligned}
\frac{\partial \lambda}{\partial \mu_0} &= \frac{\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0 w} \\
\frac{\partial \lambda}{\partial \sigma_0} &= \frac{\frac{\log\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}{\xi_0^2} - \frac{u-\mu_0}{\sigma_0 \xi_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}}{w \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{\frac{1}{\xi_0}}} \\
\frac{\partial \lambda}{\partial \xi_0} &= \frac{\frac{\log\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}{\xi_0^2} - \frac{u-\mu_0}{\sigma_0 \xi_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}}{w \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{\frac{1}{\xi_0}}}
\end{aligned}$$

Simplified expressions

$$\begin{aligned}
\frac{\partial \lambda}{\partial \mu_0} &= \frac{\lambda}{\sigma_0 C} \\
\frac{\partial \lambda}{\partial \sigma_0} &= \frac{z \lambda}{\sigma_0 C} \\
\frac{\partial \lambda}{\partial \xi_0} &= \frac{\lambda}{\xi_0^2} \left[\log C - \xi_0 \frac{z_0}{C} \right]
\end{aligned}$$

3 First-order derivatives of λ : Taylor approximation

Raw expressions

$$\begin{aligned}
\frac{\partial \lambda}{\partial \mu_0} &= \frac{1}{e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0 w} + \frac{(u^2 + (-2\sigma_0 - 2\mu_0)u + 2\mu_0\sigma_0 + \mu_0^2)\xi_0}{2e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^3 w} + \dots \\
\frac{\partial \lambda}{\partial \sigma_0} &= \frac{u - \mu_0}{e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^2 w} + \frac{(u^3 + (-2\sigma_0 - 3\mu_0)u^2 + (4\mu_0\sigma_0 + 3\mu_0^2)u - 2\mu_0^2\sigma_0 - \mu_0^3)\xi_0}{2e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^4 w} + \dots \\
\frac{\partial \lambda}{\partial \xi_0} &= \frac{u^2 - 2\mu_0 u + \mu_0^2}{2e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^2 w} + \frac{(3u^4 + (-8\sigma_0 - 12\mu_0)u^3 + (24\mu_0\sigma_0 + 18\mu_0^2)u^2 + (-24\mu_0^2\sigma_0 - 12\mu_0^3)u + 8\mu_0^3\sigma_0 + 3\mu_0^4)\xi_0}{12e^{\frac{u-\mu_0}{\sigma_0}} \sigma_0^4 w} + \dots
\end{aligned}$$

Simplified expressions

$$\begin{aligned}\frac{\partial \lambda}{\partial \mu_0} &= \frac{1}{\sigma_0} e^{-z_0} w^{-1} \left\{ 1 + \frac{z_0}{2} [z_0 - 2] \xi_0 \right\} + o(\xi_0) \\ \frac{\partial \lambda}{\partial \sigma_0} &= \frac{z_0}{\sigma_0} e^{-z_0} w^{-1} \left\{ 1 + \frac{z_0}{2} [z_0 - 2] \xi_0 \right\} + o(\xi_0) \\ \frac{\partial \lambda}{\partial \xi_0} &= \frac{z_0^2}{2} e^{-z_0} w^{-1} \left\{ 1 + \frac{z_0}{12} [3z_0 - 8] \xi_0 \right\} + o(\xi_0)\end{aligned}$$

4 First-order derivatives for σ : expressions

Raw expressions

$$\begin{aligned}\frac{\partial \sigma}{\partial \mu_0} &= -\xi_0 \\ \frac{\partial \sigma}{\partial \sigma_0} &= 1 \\ \frac{\partial \sigma}{\partial \xi_0} &= u - \mu_0\end{aligned}$$

5 First-order derivatives for σ : Taylor approximation

Unneeded (see previous section).

6 First-order derivatives for ξ : expressions

Simplified expressions

$$\begin{aligned}\frac{\partial \xi}{\partial \mu_0} &= 0 \\ \frac{\partial \xi}{\partial \sigma_0} &= 0 \\ \frac{\partial \xi}{\partial \xi_0} &= 1\end{aligned}$$

7 Second-order derivatives for λ : expressions

Raw expressions

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial \mu_0^2} &= - \frac{\left(-\frac{1}{\xi_0} - 1\right) \xi_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^2 w} \\
\frac{\partial^2 \lambda}{\partial \mu_0 \partial \sigma_0} &= - \frac{\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0^2 w} - \frac{(u-\mu_0) \left(-\frac{1}{\xi_0} - 1\right) \xi_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^3 w} \\
\frac{\partial^2 \lambda}{\partial \mu_0 \partial \xi_0} &= \frac{\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-1} \left(\frac{\log\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}{\xi_0^2} + \frac{(u-\mu_0) \left(-\frac{1}{\xi_0} - 1\right)}{\sigma_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)} \right)}{\sigma_0 w}
\end{aligned}$$

Simplified expressions

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial \mu_0^2} &= -[1 + \xi_0] \frac{\lambda}{C^2 \sigma_0^2} \\
\frac{\partial^2 \lambda}{\partial \mu_0 \partial \sigma_0} &= -\frac{\lambda}{C \sigma_0^2} \left\{ 1 - [1 + \xi_0] \frac{z_0}{C} \right\} \\
\frac{\partial^2 \lambda}{\partial \mu_0 \partial \xi_0} &= \frac{\lambda}{C \sigma_0 \xi_0} \left\{ -[1 + \xi_0] \frac{z_0}{C} + \frac{\log C}{\xi_0} \right\}
\end{aligned}$$

Raw expressions

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial \sigma_0^2} &= - \frac{2 (u-\mu_0) \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-1}}{\sigma_0^3 w} - \frac{(u-\mu_0)^2 \left(-\frac{1}{\xi_0} - 1\right) \xi_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-2}}{\sigma_0^4 w} \\
\frac{\partial^2 \lambda}{\partial \sigma_0 \partial \xi_0} &= \frac{(u-\mu_0) \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)^{-\frac{1}{\xi_0}-1} \left(\frac{\log\left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)}{\xi_0^2} + \frac{(u-\mu_0) \left(-\frac{1}{\xi_0} - 1\right)}{\sigma_0 \left(\frac{(u-\mu_0)\xi_0}{\sigma_0} + 1\right)} \right)}{\sigma_0^2 w}
\end{aligned}$$

Simplified expressions

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial \sigma_0^2} &= -\frac{\lambda z_0}{C \sigma_0^2} \left\{ 2 - [1 + \xi_0] \frac{z_0}{C} \right\} \\
\frac{\partial^2 \lambda}{\partial \sigma_0 \partial \xi_0} &= \frac{\lambda z_0}{C \sigma_0 \xi_0} \left\{ -[1 + \xi_0] \frac{z_0}{C} + \frac{\log C}{\xi_0} \right\} = z_0 \frac{\partial^2 \lambda}{\partial \mu_0 \partial \xi_0}
\end{aligned}$$

8 Second-order derivatives for λ : limit for $\xi \rightarrow 0$

$$\begin{aligned}
\lim_{\xi \rightarrow 0} \frac{\partial^2 \lambda}{\partial \mu_0^2} &= \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 w} \\
\lim_{\xi \rightarrow 0} \frac{\partial^2 \lambda}{\partial \mu_0 \partial \sigma_0} &= \frac{u e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w} - \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 w} - \frac{\mu_0 e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w} \\
\lim_{\xi \rightarrow 0} \frac{\partial^2 \lambda}{\partial \mu_0 \partial \xi_0} &= \frac{(u^2 + (-2\sigma_0 - 2\mu_0)u + 2\mu_0\sigma_0 + \mu_0^2) e^{-\frac{u-\mu_0}{\sigma_0}}}{2\sigma_0^3 w} \\
\lim_{\xi \rightarrow 0} \frac{\partial^2 \lambda}{\partial \sigma_0^2} &= -\frac{2u e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w} + \frac{2\mu_0 e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w} + \frac{\left(e^{\frac{\mu_0}{\sigma_0}} u^2 - 2\mu_0 e^{\frac{\mu_0}{\sigma_0}} u + \mu_0^2 e^{\frac{\mu_0}{\sigma_0}}\right) e^{-\frac{u}{\sigma_0}}}{\sigma_0^4 w} \\
\lim_{\xi \rightarrow 0} \frac{\partial^2 \lambda}{\partial \sigma_0 \partial \xi_0} &= \frac{u e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w} - \frac{e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^2 w} - \frac{\mu_0 e^{\frac{\mu_0}{\sigma_0} - \frac{u}{\sigma_0}}}{\sigma_0^3 w}
\end{aligned}$$

9 Second-order derivatives for σ : expressions

$$\begin{aligned}
\frac{\partial^2 \sigma}{\partial \mu_0^2} &= 0 \\
\frac{\partial^2 \sigma}{\partial \mu_0 \partial \sigma_0} &= 0 \\
\frac{\partial^2 \sigma}{\partial \mu_0 \partial \xi_0} &= -1
\end{aligned}$$