The probability functions of the GEV distribution and their derivatives

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Contents

1 Log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y - \mu}{\sigma}$$

and then V and U by

$$V := \xi\, z + 1, \qquad U := \frac{1}{\sigma V}\left[1 + \xi - V^{-1/\xi}\right].$$

The log-likelihood is defined by

$$\log f = -\left(\left(\frac{1}{\xi}+1\right)\,\log\left(\xi\,z+1\right)\right) - \frac{1}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}} - \log\sigma$$

1.2 Taylor expansion at $\xi = 0$

$$\log f = -\left(\frac{e^z\,\log\sigma + e^z\,z + 1}{e^z}\right) + \frac{\left(\left(e^z-1\right)\,z^2 - 2\,e^z\,z\right)\,\xi}{2\,e^z} - \frac{\left(3\,z^4 + \left(8\,e^z-8\right)\,z^3 - 12\,e^z\,z^2\right)\,\xi^2}{24\,e^z} + \cdots$$

which can be simplified as

$$\log f = -\left[\log\sigma + z + e^{-z}\right] + \frac{1}{2}\,\left[(1-e^{-z})z - 2\right]\,z\xi - \frac{1}{24}\,\left[3z^2e^{-z} + 8z(1-e^{-z}) - 12\right]z^2\xi^2$$

1.3 First-order derivatives: expression

Remark: scaling

In order to get quite simple expressions, we remark that $\log f$ depends on μ only through z, so

$$\frac{\partial \log f}{\partial \mu} = \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}\,\frac{\partial \log f}{\partial z}$$

and up to the term $-\log\sigma$ in $\log f$, the same is true for σ so

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma} + \frac{\partial \log f}{\partial z}\,\frac{\partial z}{\partial \sigma} = -\frac{1}{\sigma} - \frac{z}{\sigma}\,\frac{\partial \log f}{\partial z}.$$

So it is simpler to compute the derivative w.r.t. z and then find the derivatives w.r.t. μ and σ.

Raw expressions

Here are the raw expressions found by Maxima

$$\begin{aligned}
\frac{\partial \log f}{\partial z} &= \left(\xi\,z+1\right)^{-\left(\frac{1}{\xi}\right)-1} - \frac{\left(\frac{1}{\xi}+1\right)\,\xi}{\xi\,z+1}\\
\frac{\partial \log f}{\partial \xi} &= -\left(\frac{\frac{\log(\xi\,z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\,z+1)}}{\left(\xi\,z+1\right)^{\frac{1}{\xi}}}\right) + \frac{\log\left(\xi\,z+1\right)}{\xi^2} - \frac{\left(\frac{1}{\xi}+1\right)\,z}{\xi\,z+1}
\end{aligned}$$

Simplified expressions

We use the following simplifications

$$\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\sigma U,\\
\frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\,[1 - V^{-1/\xi}]\,\log V - z\,U\frac{\sigma}{\xi}.
\end{aligned}$$

so eventually

$$
\begin{aligned}
\frac{\partial \log f}{\partial \mu} &= U, \\
\frac{\partial \log f}{\partial \sigma} &= \frac{-1}{\sigma} + zU, \\
\frac{\partial \log f}{\partial \xi} &= \frac{1}{\xi^2}\,[1 - V^{-1/\xi}]\,\log V - z\,U\frac{\sigma}{\xi}.
\end{aligned}
$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

$$
U = \frac{e^z - 1}{e^z\,\sigma} - \frac{\left(z^2 + (2\,e^z - 2)\;z - 2\,e^z\right)\;\xi}{2\,e^z\,\sigma} + \cdots
$$

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\left(\frac{e^z - 1}{e^z}\right) + \frac{\left(z^2 + (2\,e^z - 2)\;z - 2\,e^z\right)\;\xi}{2\,e^z} + \cdots \\
\frac{\partial \log f}{\partial \xi} &= \frac{(e^z - 1)\;z^2 - 2\,e^z\,z}{2\,e^z} - \frac{\left(3\,z^4 + (8\,e^z - 8)\;z^3 - 12\,e^z\,z^2\right)\;\xi}{12\,e^z} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log f}{\partial z} &= -\left[1 - e^{-z}\right] - \frac{1}{2}\left[2 - 2(1 - e^{-z})z - e^{-z}z^2\right]\xi + o(\xi) \\
\frac{\partial \log f}{\partial \xi} &= -\frac{1}{2}\left[2z - (1 - e^{-z})z^2\right] + \frac{1}{12}\left[12 - 8(1 - e^{-z})z - 3z^2\right] z^2\xi + o(\xi)
\end{aligned}
$$

Limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial \log f}{\partial z} &= -\left(e^{-z}\;(e^z - 1)\right) \\
\lim_{\xi \to 0} \frac{\partial \log f}{\partial \xi} &= \frac{e^{-z}\;\left(z^2\;(e^z - 1) - 2\,z\,e^z\right)}{2}
\end{aligned}
$$

that is

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial \log f}{\partial z} &= 1 - e^{-z} \\
\lim_{\xi \to 0} \frac{\partial \log f}{\partial \xi} &= -\frac{z}{2}\left[2 - (1 - e^{-z})z\right]
\end{aligned}
$$

1.5 Second-order derivatives: expressions

Scaling

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial \mu^2} &= \frac{1}{\sigma^2} \, \frac{\partial^2 \log f}{\partial z^2} \\
\frac{\partial^2 \log f}{\partial \mu \partial \sigma} &= \frac{z}{\sigma^2} \, \frac{\partial^2 \log f}{\partial z^2} \\
\frac{\partial^2 \log f}{\partial \mu \partial \xi} &= \frac{-1}{\sigma} \, \frac{\partial^2 \log f}{\partial z \partial \xi} \\
\frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1}{\sigma^2} + \frac{z^2}{\sigma^2} \, \frac{\partial \log f}{\partial z} - \frac{z}{\sigma} \, \frac{\partial^2 \log f}{\partial z^2} \\
\frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= \frac{-z}{\sigma} \, \frac{\partial^2 \log f}{\partial z \partial \xi}
\end{aligned}
$$

Raw expressions

$$\frac{\partial^2 \log f}{\partial z^2} = \left(- \left(\frac{1}{\xi} \right) - 1 \right) \, \xi \, \left(\xi \, z + 1 \right)^{-\left(\frac{1}{\xi} \right) - 2} + \frac{\left(\frac{1}{\xi} + 1 \right) \, \xi^2}{\left(\xi \, z + 1 \right)^2}$$

$$\frac{\partial^2 \log f}{\partial z \partial \xi} = \left(\xi \, z + 1 \right)^{-\left(\frac{1}{\xi} \right) - 1} \, \left(\frac{\log \left(\xi \, z + 1 \right)}{\xi^2} + \frac{\left(- \left(\frac{1}{\xi} \right) - 1 \right) \, z}{\xi \, z + 1} \right) - \frac{\frac{1}{\xi} + 1}{\xi \, z + 1} + \frac{1}{\xi \, \left(\xi \, z + 1 \right)} + \frac{\left(\frac{1}{\xi} + 1 \right) \, \xi \, z}{\left(\xi \, z + 1 \right)^2}$$

$$\frac{\partial^2 \log f}{\partial \xi^2} = - \left(\frac{\left(\frac{\log(\xi \, z + 1)}{\xi^2} - \frac{z}{\xi \, (\xi \, z + 1)} \right)^2}{\left(\xi \, z + 1 \right)^{\frac{1}{\xi}}} \right) - \frac{- \left(\frac{2 \, \log(\xi \, z + 1)}{\xi^3} \right) + \frac{2 \, z}{\xi^2 \, (\xi \, z + 1)} + \frac{z^2}{\xi \, (\xi \, z + 1)^2}}{\left(\xi \, z + 1 \right)^{\frac{1}{\xi}}} - \frac{2 \, \log \left(\xi \, z + 1 \right)}{\xi^3} + \frac{2 \, z}{\xi^2 \, \left(\xi \, z + 1 \right)} + \frac{\left(\frac{1}{\xi} + 1 \right.}{\left(\xi \, z + \right.}$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial z^2} &= -(\xi + 1) \left[V^{-1/\xi} - \xi \right] \times \frac{1}{V^2} \\
\frac{\partial^2 \log f}{\partial z \partial \xi} &= V^{-1/\xi - 1} \left[\frac{1}{\xi^2} \, \log V - \frac{(\xi + 1)}{\xi} \, \frac{z}{V} \right] - \frac{1}{V} + (\xi + 1) \frac{z}{V^2} \\
\frac{\partial^2 \log f}{\partial \xi^2} &= - \left\{ \left[\frac{1}{\xi^2} \, \log V - \frac{1}{\xi} \, \frac{z}{V} \right]^2 + \left[-\frac{2}{\xi^2} \, \log V + \frac{2}{\xi} \, \frac{z}{V} + \frac{z^2}{V^2} \right] \times \frac{1}{\xi} \right\} \times V^{-1/\xi} - \frac{2}{\xi^3} \, \log V + \frac{2}{\xi^2} \, \frac{z}{V} + \frac{\xi + 1}{\xi} \, \frac{z^2}{V^2}
\end{aligned}
$$

1.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z} \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{e^{-z} \, \left(z \, \left(2 \, e^z - 2 \right) - 2 \, e^z + z^2 \right)}{2} \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} &= - \left(\frac{e^{-z} \, \left(z^3 \, \left(8 \, e^z - 8 \right) - 12 \, z^2 \, e^z + 3 \, z^4 \right)}{12} \right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 z^2} &= -e^{-z} \\
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial z \partial \xi} &= \frac{1}{2}\left[-2 - 2(1-e^{-z})z + e^{-z}z^2\right] \\
\lim_{\xi\to 0} \frac{\partial^2 \log f}{\partial^2 \xi^2} &= \frac{z^2}{12}\left[12 + 8(1-e^{-z})z - 3e^{-z}z^2\right]
\end{aligned}
$$

2 Distribution function F

2.1 Expression

Define

$$
W := V^{-1/\xi} = [1+\xi z]^{-1/\xi}, \qquad T := \frac{\log V}{\xi^2} - \frac{z}{\xi V}
$$

Then

$$
F := \exp\{-W\}.
$$

For the distribution function the second-order derivatives has not yet been implemented.

2.2 Taylor expansions for $\xi \approx 0$

Raw expression

$$
F = \frac{1}{e^{\frac{1}{e^z}}} - \frac{z^2\,\xi}{2\,e^{\frac{1}{e^z}}\,e^z} - \frac{\left((3\,e^z-3)\,z^4 - 8\,e^z\,z^3\right)\,\xi^2}{24\,e^{\frac{1}{e^z}}\,(e^z)^2} + \cdots
$$

Simplified expression

$$
F = \exp\{-e^{-z}\}\left\{1 - e^{-z}\frac{z^2}{2}\xi + e^{-z}\frac{z^3}{24}\left[8 - 3(1-e^{-z})z\right]\xi^2\right\} + o(\xi^2)
$$

2.3 First-order derivatives: expressions

Scaling

$$
\begin{aligned}
\frac{\partial F}{\partial \mu} &= \frac{-1}{\sigma}\,\frac{\partial F}{\partial z} \\
\frac{\partial F}{\partial \sigma} &= \frac{z}{\sigma}\,\frac{\partial F}{\partial z}
\end{aligned}
$$

Raw expressions

$$\frac{\partial F}{\partial z} = (\xi\, z+1)^{-\left(\frac{1}{\xi}\right)-1}\; e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)}$$

$$\frac{\partial F}{\partial \xi} = -\left(\frac{e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)}\left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)$$

Simplified expressions

$$\frac{\partial F}{\partial z} = FW\,\frac{1}{V}$$

$$\frac{\partial F}{\partial \xi} = -FWT$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

$$\frac{\partial F}{\partial z} = \frac{1}{e^{\frac{z\, e^z+1}{e^z}}} + \frac{\left((e^z-1)\; z^2 - 2\, e^z\, z\right)\, \xi}{2\, e^{\frac{z\, e^z+1}{e^z}}\; e^z} + \cdots$$

$$\frac{\partial F}{\partial \xi} = -\left(\frac{z^2}{2\, e^{\frac{z\, e^z+1}{e^z}}}\right) - \frac{\left((3\, e^z-3)\; z^4 - 8\, e^z\, z^3\right)\, \xi}{12\, e^{\frac{z\, e^z+1}{e^z}}\; e^z} + \cdots$$

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ (Gumbel distribution value at z),

$$\frac{\partial F}{\partial z} = F^\star \left\{1 + \frac{z}{2}\left[-2 + (1-e^{-z})z\right]\xi\right\} + o(\xi)$$

$$\frac{\partial F}{\partial \xi} = F^\star \left\{-\frac{z^2}{2} + \frac{z^3}{12}\left[8 - 3\,(1-e^{-z})z\right]\xi\right\} + o(\xi)$$

2.5 Second-order derivatives

CAUTION The second-order derivatives has not yet been implemented and the formula have not yet been fully checked.

Scaling

$$\frac{\partial^2 F}{\partial \mu^2} = \frac{1}{\sigma^2} \frac{\partial^2 F}{\partial z^2}$$
$$\frac{\partial^2 F}{\partial \mu \partial \sigma} = \frac{z}{\sigma^2} \frac{\partial^2 F}{\partial z^2}$$
$$\frac{\partial^2 F}{\partial \mu \partial \xi} = \frac{-1}{\sigma} \frac{\partial^2 F}{\partial z \partial \xi}$$
$$\frac{\partial^2 F}{\partial \sigma^2} = \frac{z^2}{\sigma^2} \frac{\partial F}{\partial z} - \frac{z}{\sigma} \frac{\partial^2 F}{\partial z^2}$$
$$\frac{\partial^2 F}{\partial \sigma \partial \xi} = \frac{-z}{\sigma} \frac{\partial^2 F}{\partial z \partial \xi}$$

Raw expressions

$$\frac{\partial^2 F}{\partial z^2} = \left(-\left(\frac{1}{\xi}\right) - 1\right) \xi \, (\xi\, z+1)^{-\left(\frac{1}{\xi}\right)-2} \, e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} + (\xi\, z+1)^{-\left(\frac{2}{\xi}\right)-2} \, e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)}$$

$$\frac{\partial^2 F}{\partial z \partial \xi} = (\xi\, z+1)^{-\left(\frac{1}{\xi}\right)-1} \, e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} \left(\frac{\log(\xi\, z+1)}{\xi^2} + \frac{\left(-\left(\frac{1}{\xi}\right)-1\right) z}{\xi\, z+1}\right) - (\xi\, z+1)^{-\left(\frac{2}{\xi}\right)-1} \, e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} \left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)$$

$$\frac{\partial^2 F}{\partial \xi^2} = -\left(\frac{e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} \left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)^2}{(\xi\, z+1)^{\frac{1}{\xi}}}\right) + \frac{e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} \left(\frac{\log(\xi\, z+1)}{\xi^2} - \frac{z}{\xi\,(\xi\, z+1)}\right)^2}{(\xi\, z+1)^{\frac{2}{\xi}}} - \frac{e^{-\left(\frac{1}{(\xi\, z+1)^{\frac{1}{\xi}}}\right)} \left(-\left(\frac{2\,\log(\xi\, z+1)}{\xi^3}\right) + \frac{2\,z}{\xi^2\,(\xi\, z+1)} + \frac{z^2}{\xi\,(\xi\, z+1)^2}\right)}{(\xi\, z+1)^{\frac{1}{\xi}}}$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial^2 F}{\partial z^2} &= F W^{1+2\xi}\, \{-[1+\xi] + W\} \\
\frac{\partial^2 F}{\partial z \partial \xi} &= F W^{1+\xi} \left\{ T\,[1-W] - \frac{z}{V} \right\} \\
\frac{\partial^2 F}{\partial \xi^2} &= F W \left\{ -T^2\,[1-W] + \frac{2}{\xi} T - \frac{z^2}{\xi V^2} \right\}
\end{aligned}
$$

Note that $W^{1+2\xi} = W/V^2$ and $W^{1+\xi} = W/V$, which may be faster to evaluate.

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial^2 F}{\partial z^2} &= e^{-e^{-z} - 2\,z} - e^{-e^{-z} - z} \\
\lim_{\xi \to 0} \frac{\partial^2 F}{\partial z \partial \xi} &= \frac{\left(z^2\,\left(e^z - 1\right) - 2\,z\,e^z\right)\, e^{-\left(e^{-z}\,(2\,z\,e^z + 1)\right)}}{2} \\
\lim_{\xi \to 0} \frac{\partial^2 F}{\partial^2 \xi^2} &= -\frac{\left(z^4\,\left(3\,e^{e^{-z}\,(2\,z\,e^z+1)} - 3\,e^{e^{-z}\,(z\,e^z+1)}\right) - 8\,z^3\,e^{e^{-z}\,(2\,z\,e^z+1)}\right)\, e^{-e^{-z}\,(2\,z\,e^z+1) - e^{-z}\,(z\,e^z+1)}}{12}
\end{aligned}
$$

It seems that Maxima can not find the limit of $\partial^2 F/\partial \xi^2$ in a non interactive mode so the value above was obtained by a paste-and-copy of the result obtained in an interactive session.

Simplified expressions

With $F^\star := \exp\{-e^{-z} - z\}$ (Gumbel distribution value at z),

$$
\begin{aligned}
\frac{\partial^2 F}{\partial z^2} &= F^\star \left[e^{-z} - 1\right] + o(1) \\
\frac{\partial^2 F}{\partial z \partial \xi} &= \frac{1}{2} F^\star \left[z^2 \left(1 - e^{-z}\right) - 2z\right] + o(1) \\
\frac{\partial^2 F}{\partial \xi^2} &= \frac{1}{12} F^\star z^3 \left[8 - 3\left(1 - e^{-z}\right) z\right] + o(1)
\end{aligned}
$$

3 Quantile or return level

3.1 Expression

With $A := -\log p$, the quantile $\rho := q_{\text{GEV}}(p)$ is given by

$$
\rho = \mu - \frac{\left(1 - \frac{1}{A^\xi}\right)\,\sigma}{\xi}
$$

3.2 Taylor expansion for $\xi \approx 0$

$$\rho = \mu - \log A\,\sigma + \frac{(\log A)^2\ \sigma\,\xi}{2} - \frac{(\log A)^3\ \sigma\,\xi^2}{6} + \cdots$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\begin{aligned}
\frac{\partial \rho}{\partial \mu} &= 1 \\
\frac{\partial \rho}{\partial \sigma} &= -\left(\frac{1 - \frac{1}{A^\xi}}{\xi}\right) \\
\frac{\partial \rho}{\partial \xi} &= \frac{\left(1 - \frac{1}{A^\xi}\right)\ \sigma}{\xi^2} - \frac{\log A\,\sigma}{A^\xi\,\xi}
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi \to 0} \frac{\partial \rho}{\partial \mu} &= 1 \\
\lim_{\xi \to 0} \frac{\partial \rho}{\partial \sigma} &= -\log A \\
\lim_{\xi \to 0} \frac{\partial \rho}{\partial \xi} &= \frac{(\log A)^2\ \sigma}{2}
\end{aligned}$$

3.4 First-order derivatives: Taylor expansion for $\xi \approx 0$

$$\begin{aligned}
\frac{\partial \rho}{\partial \mu} &= +1 + \cdots \\
\frac{\partial \rho}{\partial \sigma} &= -\log A + \frac{(\log A)^2\ \xi}{2} + \cdots \\
\frac{\partial \rho}{\partial \xi} &= \frac{(\log A)^2\ \sigma}{2} - \frac{(\log A)^3\ \sigma\,\xi}{3} + \cdots
\end{aligned}$$

3.5 Second-order derivatives: expressions

Scaling

Since ρ is a linear function of μ and σ, we consider only the standardized return level

$$\rho^\star = \frac{\frac{1}{A^\xi} - 1}{\xi}$$

so that $\rho = \mu + \sigma\rho^\star$.

Raw expressions

$$\frac{\partial \rho^\star}{\partial \xi} = -\left(\frac{\log A}{A^\xi\,\xi}\right) - \frac{\frac{1}{A^\xi} - 1}{\xi^2}$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{(\log A)^2}{A^\xi\,\xi} + \frac{2\,\log A}{A^\xi\,\xi^2} + \frac{2\,\left(\frac{1}{A^\xi} - 1\right)}{\xi^3}$$

Simplified expression

$$\frac{\partial \rho^\star}{\partial \xi} = -\frac{1}{\xi}\,\left[\rho^\star + \log A\right] - \rho^\star\,\log A$$

$$\frac{\partial^2 \rho^\star}{\partial \xi^2} = \frac{1}{\xi^2}\,\left[\rho^\star + \log A\right] - \frac{\partial \rho^\star}{\partial \xi}\left[\log A + \frac{1}{\xi}\right]$$

$$\frac{\partial \rho}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \rho^\star}{\partial \xi} \\ 0 & \frac{\partial \rho^\star}{\partial \xi} & \sigma \frac{\partial^2 \rho^\star}{\partial \xi^2} \end{bmatrix}$$

3.6 Second-order derivatives: limit for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi \to 0} \frac{\partial \rho^\star}{\partial \xi} = \frac{(\log A)^2}{2}$$
$$\lim_{\xi \to 0} \frac{\partial^2 \rho^\star}{\partial \xi^2} = -\left(\frac{(\log A)^3}{3}\right)$$