The probability functions of the GP2 distribution and their derivatives

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Contents

1 The log-density $\log f$

1.1 Expression

Define z by

$$z := \frac{y}{\sigma}. \tag{1}$$

Define A and B by

$$A := \log\left(\frac{\xi\, y}{\sigma} + 1\right), \qquad B := \frac{y}{\sigma\,\left(\frac{\xi\, y}{\sigma} + 1\right)} \tag{2}$$

then the log-density is given by

$$\log f = -\left(\left(\frac{1}{\xi} + 1\right)\,\log\left(\frac{\xi\, y}{\sigma} + 1\right)\right) - \log\sigma.$$

1.2 Taylor expansion for $\xi \approx 0$

Raw expression

The Taylor approximation of $\log f$ for $\xi \approx 0$ is

$$\log f = -\left(\frac{\sigma\,\log\sigma + y}{\sigma}\right) - \frac{\left(2\,y\,\sigma - y^2\right)\,\xi}{2\,\sigma^2} + \frac{\left(3\,y^2\,\sigma - 2\,y^3\right)\,\xi^2}{6\,\sigma^3} + \cdots$$

Simplified expression

$$\log f = -\left[\log\sigma + z\right] + \frac{z(z-2)}{2}\,\xi - \frac{z^2(2z-3)}{6}\,\xi^2 + o(\xi^2)$$

1.3 First-order derivatives: expressions

Raw expressions

$$\frac{\partial \log f}{\partial \sigma} = \frac{\left(\frac{1}{\xi} + 1\right)\,\xi\, y}{\sigma^2\,\left(\frac{\xi\, y}{\sigma} + 1\right)} - \frac{1}{\sigma}$$

$$\frac{\partial \log f}{\partial \xi} = \frac{\log\left(\frac{\xi\, y}{\sigma} + 1\right)}{\xi^2} - \frac{\left(\frac{1}{\xi} + 1\right)\, y}{\sigma\,\left(\frac{\xi\, y}{\sigma} + 1\right)}$$

Simplified expressions

$$\frac{\partial \log f}{\partial \sigma} = -\frac{1}{\sigma}\,\left[1 - (\xi + 1)B\right]$$

$$\frac{\partial \log f}{\partial \xi} = \frac{1}{\xi^2}\,\left[A - \xi(\xi + 1)B\right]$$

1.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

For $\xi \approx 0$ we have

$$
\begin{aligned}
\frac{\partial \log f}{\partial \sigma} &= \frac{y-\sigma}{\sigma^2} - \frac{\left(y^2 - \sigma\, y\right)\, \xi}{\sigma^3} + \cdots \\
\frac{\partial \log f}{\partial \xi} &= \frac{y^2 - 2\,\sigma\, y}{2\,\sigma^2} - \frac{\left(2\, y^3 - 3\,\sigma\, y^2\right)\, \xi}{3\,\sigma^3} + \cdots
\end{aligned}
$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial \log f}{\partial \sigma} &= \frac{z-1}{\sigma} - \frac{z(z-1)}{\sigma}\,\xi + o(\xi) \\
\frac{\partial \log f}{\partial \xi} &= \frac{z(z-2)}{2} - \frac{z^2(2z-3)}{3}\,\xi + o(\xi)
\end{aligned}
$$

1.5 Second-order derivatives: expressions

Raw expressions

$$
\begin{aligned}
\frac{\partial^2}{\partial \sigma^2}\,\log f &= -\left(\frac{2\,\left(\frac{1}{\xi}+1\right)\,\xi\, y}{\sigma^3\,\left(\frac{\xi\, y}{\sigma}+1\right)}\right) + \frac{\left(\frac{1}{\xi}+1\right)\,\xi^2\, y^2}{\sigma^4\,\left(\frac{\xi\, y}{\sigma}+1\right)^2} + \frac{1}{\sigma^2} \\
\frac{\partial^2}{\partial \sigma \partial \xi}\,\log f &= \frac{\left(\frac{1}{\xi}+1\right)\, y}{\sigma^2\,\left(\frac{\xi\, y}{\sigma}+1\right)} - \frac{y}{\sigma^2\,\xi\,\left(\frac{\xi\, y}{\sigma}+1\right)} - \frac{\left(\frac{1}{\xi}+1\right)\,\xi\, y^2}{\sigma^3\,\left(\frac{\xi\, y}{\sigma}+1\right)^2} \\
\frac{\partial^2}{\partial \xi^2}\,\log f &= -\left(\frac{2\,\log\left(\frac{\xi\, y}{\sigma}+1\right)}{\xi^3}\right) + \frac{2\, y}{\sigma\,\xi^2\,\left(\frac{\xi\, y}{\sigma}+1\right)} + \frac{\left(\frac{1}{\xi}+1\right)\, y^2}{\sigma^2\,\left(\frac{\xi\, y}{\sigma}+1\right)^2}
\end{aligned}
$$

Simplified expressions

$$
\begin{aligned}
\frac{\partial^2}{\partial \sigma^2}\,\log f &= \frac{1}{\sigma^2}\,\left[1 - 2(\xi+1)B + \xi(\xi+1)B^2\right] \\
\frac{\partial^2}{\partial \sigma \partial \xi}\,\log f &= \frac{1}{\sigma}\,\left[B - (\xi+1)B^2\right] \\
\frac{\partial^2}{\partial \xi^2}\,\log f &= \frac{1}{\xi^3}\,\left[-2A + 2\xi B + \xi^2(\xi+1)B^2\right]
\end{aligned}
$$

1.6 Second-order derivatives: limits for $\xi \to 0$

Raw expressions

Here are the limits for $\xi \to 0$

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \sigma^2} &= -\left(\frac{2\,y - \sigma}{\sigma^3}\right) \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= -\left(\frac{y^2 - \sigma\,y}{\sigma^3}\right) \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \xi^2} &= -\left(\frac{2\,y^3 - 3\,\sigma\,y^2}{3\,\sigma^3}\right)
\end{aligned}
$$

Simplified expressions

The corresponding simplified expressions are

$$
\begin{aligned}
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \sigma^2} &= \frac{1 - 2z}{\sigma^2} \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \sigma \partial \xi} &= -\frac{z(z-1)}{\sigma} \\
\lim_{\xi \to 0} \frac{\partial^2 \log f}{\partial \xi^2} &= -\frac{z^2(2z-3)}{3}.
\end{aligned}
$$

2 Cumulated hazard (log-survival) $H = \log S$

2.1 Expression

The cumulated hazard $H(y) = \log S(y)$

$$H := \frac{\log\left(\frac{\xi\,y}{\sigma} + 1\right)}{\xi}$$

2.2 Taylor approximation for $\xi \approx 0$

Raw expression

$$H = \frac{y}{\sigma} - \frac{y^2\,\xi}{2\,\sigma^2} + \frac{y^3\,\xi^2}{3\,\sigma^3} + \cdots$$

Simplified expression

$$H = z - \frac{z^2}{2}\,\xi + \frac{z^3}{3}\,\xi^2 + o(\xi^2) \tag{3}$$

2.3 First-order derivatives: expressions

Raw expressions

$$\frac{\partial H}{\partial \sigma} = -\left(\frac{y}{\sigma^2\,\left(\frac{\xi\,y}{\sigma}+1\right)}\right)$$
$$\frac{\partial H}{\partial \xi} = \frac{y}{\sigma\,\xi\,\left(\frac{\xi\,y}{\sigma}+1\right)} - \frac{\log\left(\frac{\xi\,y}{\sigma}+1\right)}{\xi^2}$$

Simplified expressions

$$\frac{\partial H}{\partial \sigma} = -\frac{1}{\sigma}\,B$$
$$\frac{\partial H}{\partial \xi} = -\frac{1}{\xi^2}\,[A - \xi B]$$

2.4 First-order derivatives: Taylor expansions for $\xi \approx 0$

Raw expressions

Here are the limits for $\xi \approx 0$

$$\frac{\partial H}{\partial \sigma} = -\left(\frac{y}{\sigma^2}\right) + \frac{y^2\,\xi}{\sigma^3} + \cdots$$
$$\frac{\partial H}{\partial \xi} = -\left(\frac{y^2}{2\,\sigma^2}\right) + \frac{2\,y^3\,\xi}{3\,\sigma^3} + \cdots$$

Simplified expressions

$$\frac{\partial H}{\partial \sigma} = -\frac{z}{\sigma} + \frac{z^2}{\sigma}\,\xi + o(\xi)$$
$$\frac{\partial H}{\partial \xi} = -\frac{z^2}{2} + \frac{2z^3}{3}\,\xi + o(\xi)$$

2.5 Second-order derivatives: expressions

Raw expressions

$$\frac{\partial^2}{\partial\sigma^2} H = \frac{2\,y}{\sigma^3\,\left(\frac{\xi\,y}{\sigma}+1\right)} - \frac{\xi\,y^2}{\sigma^4\,\left(\frac{\xi\,y}{\sigma}+1\right)^2}$$
$$\frac{\partial^2}{\partial\sigma\partial\xi} H = \frac{y^2}{\sigma^3\,\left(\frac{\xi\,y}{\sigma}+1\right)^2}$$
$$\frac{\partial^2}{\partial\xi^2} H = \frac{2\,\log\left(\frac{\xi\,y}{\sigma}+1\right)}{\xi^3} - \frac{2\,y}{\sigma\,\xi^2\,\left(\frac{\xi\,y}{\sigma}+1\right)} - \frac{y^2}{\sigma^2\,\xi\,\left(\frac{\xi\,y}{\sigma}+1\right)^2}$$

Simplified expressions

$$\frac{\partial^2}{\partial\sigma^2} H = \frac{1}{\sigma^2}\,\left[2B - \xi B^2\right]$$
$$\frac{\partial^2}{\partial\sigma\partial\xi} H = \frac{1}{\sigma}\,B^2$$
$$\frac{\partial^2}{\partial\xi^2} H = \frac{1}{\xi^3}\,\left[2A - 2\xi B - \xi^2 B^2\right]$$

2.6 Second-order derivatives: limits for $\xi \to 0$

Here are the limits for $\xi \to 0$

$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma^2} = \frac{2\,y}{\sigma^3}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma\partial\xi} = \frac{y^2}{\sigma^3}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\xi^2} = \frac{2\,y^3}{3\,\sigma^3}$$

The corresponding simplified expresions are

$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma^2} = \frac{2z}{\sigma^2}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\sigma\partial\xi} = \frac{z^2}{\sigma}$$
$$\lim_{\xi\to 0} \frac{\partial^2 H}{\partial\xi^2} = \frac{2z^3}{3}.$$

3 Quantile or return period

3.1 Expression

The quantile corresponding to an exceedance probability $q := 1 - p$ for $\xi \neq 0$

$$\rho = \frac{\left(\frac{1}{q^{\xi}} - 1\right)\,\sigma}{\xi}$$

3.2 Taylor approximation for $\xi \approx 0$

$$-\left(\log q\,\sigma\right) + \frac{\left(\log q\right)^2\,\sigma\,\xi}{2} - \frac{\left(\log q\right)^3\,\sigma\,\xi^2}{6} + \cdots \tag{4}$$

3.3 First-order derivatives: expressions

When $\xi \neq 0$ we have

$$\frac{\partial\rho}{\partial\sigma} = \frac{\frac{1}{q^{\xi}} - 1}{\xi}$$
$$\frac{\partial\rho}{\partial\xi} = -\left(\frac{\log q\,\sigma}{q^{\xi}\,\xi}\right) - \frac{\left(\frac{1}{q^{\xi}} - 1\right)\,\sigma}{\xi^2}$$

3.4 First-order derivatives: Taylor approximation

$$\frac{\partial\rho}{\partial\sigma} = -\log q + \frac{\left(\log q\right)^2\,\xi}{2} + \cdots$$
$$\frac{\partial\rho}{\partial\xi} = \frac{\left(\log q\right)^2\,\sigma}{2} - \frac{\left(\log q\right)^3\,\sigma\,\xi}{3} + \cdots$$

3.5 Second-order derivatives: expressions

Raw expressions

$$\frac{\partial^2\rho}{\partial\sigma^2} = 0$$
$$\frac{\partial^2\rho}{\partial\sigma\partial\xi} = -\left(\frac{\log q}{q^{\xi}\,\xi}\right) - \frac{\frac{1}{q^{\xi}} - 1}{\xi^2}$$
$$\frac{\partial^2\rho}{\partial\xi^2} = \frac{\left(\log q\right)^2\,\sigma}{q^{\xi}\,\xi} + \frac{2\,\log q\,\sigma}{q^{\xi}\,\xi^2} + \frac{2\,\left(\frac{1}{q^{\xi}} - 1\right)\,\sigma}{\xi^3}$$

Simplified expressions

Define $V := [q^{-\xi} - 1]/\xi$ so that $\rho = \sigma V$, and

$$W := \frac{\partial V}{\partial \xi} = -\frac{1}{\xi}\left\{V + q^{-\xi}\log q\right\}.$$

$$\begin{aligned}\frac{\partial^2 \rho}{\partial \sigma^2} &= 0\\ \frac{\partial^2 \rho}{\partial \sigma \partial \xi} &= W\\ \frac{\partial^2 \rho}{\partial \xi^2} &= \frac{\sigma}{\xi^2}\left\{2V + q^{-\xi}\log q\,[2 + \xi \log q]\right\}\end{aligned}$$

3.6 Second-order derivatives: limit for $\xi \to 0$

$$\begin{aligned}\lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \sigma^2} &= 0\\ \lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \sigma \partial \xi} &= \frac{(\log q)^2}{2}\\ \lim_{\xi \to 0} \frac{\partial^2 \rho}{\partial \xi^2} &= -\left(\frac{(\log q)^3\ \sigma}{3}\right)\end{aligned}$$