The Poisson-GP to NHPP transformation and its derivatives

Yves Deville deville.yves@alpestat.com

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1 Poisson-GP to PP

1.1 Expression

Let $\boldsymbol{\theta} = [\lambda,\, \sigma,\, \xi]$ and $\boldsymbol{\theta}_0 = [\mu_0,\, \sigma_0,\, \xi_0]$ denote the Poisson-GP and PP parameter vectors. The threshold u being considered as fixed, the relation between the PP parameters $\boldsymbol{\theta}_0$ and the Poisson-GP parameters $\boldsymbol{\theta}$ is given by

$$\begin{aligned}
\mu_0 &= u + \frac{(\lambda w)^\xi - 1}{\xi}\,\sigma,\\
\sigma_0 &= (\lambda w)^\xi \sigma,\\
\xi_0 &= \xi.
\end{aligned}$$

1.2 Taylor approximation for $\xi \approx 0$

Raw expressions

$$
\begin{aligned}
\mu_0 &= \log\left(w\,\lambda\right)\,\sigma + u + \frac{\left(\log\left(w\,\lambda\right)\right)^2\,\sigma\,\xi}{2} + \frac{\left(\log\left(w\,\lambda\right)\right)^3\,\sigma\,\xi^2}{6} + \cdots \\
\sigma_0 &= \sigma + \log\left(w\,\lambda\right)\,\sigma\,\xi + \frac{\left(\log\left(w\,\lambda\right)\right)^2\,\sigma\,\xi^2}{2} + \cdots \\
\xi_0 &= +\xi + \cdots
\end{aligned}
$$

Simplified expressions

With $L := \log(\lambda w)$, we have

$$
\begin{aligned}
\mu_0 &= u + \sigma\,\left\{L + \frac{L^2}{2}\,\xi + \frac{L^3}{3}\xi^2\right\} + o(\xi^2) \\
\sigma_0 &= \sigma\,\left\{1 + L\,\xi + \frac{L^2}{2}\xi^2\right\} + o(\xi^2) \\
\xi_0 &= \xi + o(\xi^2)
\end{aligned}
$$

2 First-order derivatives for μ_0: expressions

Raw expressions

$$
\begin{aligned}
\frac{\partial\mu_0}{\partial\lambda} &= \frac{\sigma\,\left(w\,\lambda\right)^{\xi}}{\lambda} \\
\frac{\partial\mu_0}{\partial\sigma} &= \frac{\left(w\,\lambda\right)^{\xi} - 1}{\xi} \\
\frac{\partial\mu_0}{\partial\xi} &= \frac{\sigma\,\left(w\,\lambda\right)^{\xi}\,\log\left(w\,\lambda\right)}{\xi} - \frac{\sigma\,\left(\left(w\,\lambda\right)^{\xi} - 1\right)}{\xi^2}
\end{aligned}
$$

Simplified expressions

With $\nu := \lambda w$, we have

$$
\begin{aligned}
\frac{\partial\mu_0}{\partial\lambda} &= \sigma\,\frac{\nu^{\xi}}{\lambda} \\
\frac{\partial\mu_0}{\partial\sigma} &= \frac{\nu^{\xi} - 1}{\xi} \\
\frac{\partial\mu_0}{\partial\xi} &= \frac{\sigma}{\xi}\,\left\{\nu^{\xi}L - \frac{\nu^{\xi} - 1}{\xi}\right\} = \frac{\sigma}{\xi}\,\left\{\nu^{\xi}L - \frac{\partial\mu_0}{\partial\sigma}\right\}
\end{aligned}
$$

3 First-order derivatives for μ_0: Taylor approximation

Raw expressions

$$\frac{\partial \mu_0}{\partial \lambda} = \frac{\sigma}{\lambda} + \frac{\log(w\,\lambda)\,\sigma\,\xi}{\lambda} + \cdots$$
$$\frac{\partial \mu_0}{\partial \sigma} = \log(w\,\lambda) + \frac{(\log(w\,\lambda))^2\,\xi}{2} + \cdots$$
$$\frac{\partial \mu_0}{\partial \xi} = \frac{\sigma\,(\log(w\,\lambda))^2}{2} + \frac{\sigma\,(\log(w\,\lambda))^3\,\xi}{3} + \cdots$$

Simplified expressions

$$\frac{\partial \mu_0}{\partial \lambda} = \frac{\sigma}{\lambda}\,\{1 + L\,\xi\} + o(\xi)$$
$$\frac{\partial \mu_0}{\partial \sigma} = \frac{L}{2}\,\{2 + L\,\xi\} + o(\xi)$$
$$\frac{\partial \mu_0}{\partial \xi} = \sigma\,\frac{L^2}{6}\,\{3 + 2L\,\xi\} + o(\xi)$$

4 First-order derivatives for σ_0: expressions

Raw expressions

$$\frac{\partial \sigma_0}{\partial \lambda} = \frac{\sigma\,\xi\,(w\,\lambda)^{\xi}}{\lambda}$$
$$\frac{\partial \sigma_0}{\partial \sigma} = (w\,\lambda)^{\xi}$$
$$\frac{\partial \sigma_0}{\partial \xi} = \sigma\,(w\,\lambda)^{\xi}\,\log(w\,\lambda)$$

Simplified expressions

$$\frac{\partial \sigma_0}{\partial \lambda} = \sigma\,\xi\,\frac{\nu^{\xi}}{\lambda} = \xi\frac{\partial \mu_0}{\partial \lambda}$$
$$\frac{\partial \sigma_0}{\partial \sigma} = \nu^{\xi}$$
$$\frac{\partial \sigma_0}{\partial \xi} = \sigma\nu^{\xi}L$$

5 First-order derivatives for ξ_0: expressions

$$\frac{\partial \xi_0}{\partial \lambda} = 0$$
$$\frac{\partial \xi_0}{\partial \sigma} = 0$$
$$\frac{\partial \xi_0}{\partial \xi} = 1$$

6 First-order derivatives for σ_0: Taylor approximation

Raw expressions

$$\frac{\partial \sigma_0}{\partial \lambda} = +\frac{\sigma\,\xi}{\lambda} + \cdots$$
$$\frac{\partial \sigma_0}{\partial \sigma} = 1 + \log\left(w\,\lambda\right)\,\xi + \cdots$$
$$\frac{\partial \sigma_0}{\partial \xi} = \sigma\,\log\left(w\,\lambda\right) + \sigma\,\left(\log\left(w\,\lambda\right)\right)^2\,\xi + \cdots$$

Simplified expressions

$$\frac{\partial \sigma_0}{\partial \lambda} = \frac{\sigma}{\lambda}\xi + o(\xi)$$
$$\frac{\partial \sigma_0}{\partial \sigma} = 1 + L\xi + o(\xi)$$
$$\frac{\partial \sigma_0}{\partial \xi} = \sigma L\,\left\{1 + L\xi\right\} + o(\xi)$$