1 State-Dependent Riccati Equation Method

1.1 Results on Control Design for LTI Systems

The Linear Quadratic Regulator (LQR) Consider the linear time invariant (LTI) system

$$\dot{x} = Ax + Bu \tag{1}$$

and the performance criteria

$$J[x_0, u(\cdot)][= \int_0^\infty [x^T Q x + u^T R u] dt, \dots Q \geq 0, R > 0 \tag{2}$$

Problem: Calculate function $u : [0, \infty] \mapsto \Re^p$ such that $J[u]$ is minimized.

The LQR controller has the following form

$$u(t) = -R^{-1} B^T P x(t) \tag{3}$$

where $P \in \Re^{n \times n}$ is given by the positive (symmetric) semi definite solution of

$$0 = PA + A^T P + Q - PBR^{-1} B^T P. \tag{4}$$

This equation is called Riccati equation. It is solvable if and only if the pair (A, B) is controllable and (Q, A) is detectable.

Note that

1. LQR assumes full knowledge of the state x.
2. (A, B) is given by "design" and can not be modified at this stage
3. (Q, R) are the controller design parameters. Large Q penalizes transients of x, large R penalizes usage of control action u.

The Linear Quadratic Gaussian Regulator (LQG) In LQR we assumed that the whole state is available for control at all times (see formula for control action above). This is unrealistic as the very least there is always measurement noise.

One possible generalization is to look at

$$\begin{aligned} \dot{x} &= Ax + Bu + w & (5) \\ y(t) &= Cx + v & (6) \end{aligned}$$

where v, w are stochastic processes called measurement and process noise respectively. For simplicity one assumes these processes to be white noise (ie zero mean, uncorrelated, Gaussian distribution).

Crucially, now only $y(t)$ is available for control. It turns out that for linear systems a separation principle holds

1. First, calculate $x_e(t)$ estimate the full state $x(t)$ using the available information

2. Secondly, apply the LQR controller, using the estimation $x_e(t)$ replacing the true (but unknown!) state $x(t)$.

Observer Design (Kalman Filter) The estimation $x_e(t)$ is calculated by integrating in real time the following ODE

$$\dot{x_e} = Ax_e + Bu + L(y - Cx_e) \tag{7}$$

With the following matrices calculated offline:

$$\begin{aligned}
L &= PC^T R^{-1} && (8)\\
0 &= AP + PA^T - PC^T R^{-1} CP + Q^Y, P \geq 0 && (9)\\
Q &= E(ww^T) && (10)\\
R &= E(vv^T) && (11)
\end{aligned}$$

The Riccati equation above has its origin in the minimization of the cost functional

$$J[\hat{x}(\cdot)] = \int_{-\infty}^{0} [(\hat{x} - x)(\hat{x} - x)^T] dt$$

1.2 State Dependent Riccati Equation Approach

The LQR/LQG method is extremely powerful and widely used in applications, where linearisations of the nonlinear process representations are valid over large operating areas. How can this framework be extended beyond that obvious case?

The State-Dependent Riccati Equation (SDRE) strategy provides an effective algorithm for synthesizing nonlinear feedback controls by allowing for nonlinearities in the system states, while offering design flexibility through state-dependent weighting matrices.

The method entails factorization of the nonlinear dynamics into the state vector and its product with a matrix-valued function that depends on the state itself. In doing so, the SDRE algorithm brings the nonlinear system to a non-unique linear structure having matrices with state-dependent coefficients. The method includes minimizing a nonlinear performance index having a quadratic-like structure. An algebraic Riccati equation (ARE), as given by the SDC matrices, is then solved on-line to give the suboptimum control law.

The coefficients of this Riccati equation vary with the given point in state space. The algorithm thus involves solving, at a given point in the state space, an algebraic state-dependent Riccati equation. The non-uniqueness of the factorization creates extra degrees of freedom, which can be used to enhance controller performance.

Extended Linearization of a Nonlinear System Consider the deterministic, infinite-horizon nonlinear optimal regulation (stabilization) problem, where the system is full- state observable, autonomous, nonlinear in the state, and affine in the input, represented in the form

$$\begin{aligned} \dot{x} &= f(x) + B(x)u(t) \qquad (12) \\ x(0) &= x_0 \qquad (13) \end{aligned}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the input vector, with smooth functions f and B of appropriate domains such that

- $B(x) \neq 0$ for all x
- $f(0) = 0$.

Extended Linearization is the process of factorizing a nonlinear system into a linear-like structure which contains SDC matrices. Under the fairly generic assumptions

$$f(0) = 0, \quad f \in C^1(\Re^n)$$

a continuous nonlinear matrix-valued function $A : \Re^n \mapsto \Re^{n \times n}$ always exists such that

$$f(x) = A(x)x \qquad (14)$$

where $A : \Re^n \mapsto \Re n \times n$ is found by algebraic factorization and is clearly non-unique when $n > 1$.

After extended linearization of the input-affine nonlinear system 12 becomes

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u(t) \\ x(0) &= x_0. \end{aligned}$$

which has a linear structure with state dependent matrices $A(x)$ and $B(x)$.

- Note that these parameterizations are not unique for $n > 1$. For instance, if $A(x)x = f(x)$, then $(A(x)+E(x))x = f(x)$ for any $E(x)$ such that $E(x)x = 0$.
- We also note that given $A_1(x)x = f(x)$ and $A_2(x)x = f(x)$, then for any $\alpha \in \Re$

  $$A(x, \alpha) = \alpha A_1(x) + (1-\alpha)A_2(x) = \alpha f(x) + (1-\alpha)f(x) = f(x)$$

  is also a valid paramerization.

In general, one needs to answer the question about the optimal choice of α for the given application at hand.

Pointwise Hurwitz A Matrix does not imply stability We could naively think that if we find a controller $u = K(x)$ such that the closed loop matrix

$$A_{cl}(x) = A(x) - B(x)K(x)$$

is point-wise Hurtwitz then we would have a good design for stabilization of the nonlinear system given by Equation 15. However, this condition is not sufficient for closed loop stability as the following example shows.

Example 1 *Let us consider the system in form of Equation 15*

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{1+\epsilon\sqrt{x_1^2+x_2^2}} & 1+\sqrt{x_1^2+x_2^2}) \\ 0 & -\frac{1}{1+\epsilon\sqrt{x_1^2+x_2^2})} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We note that the eigenvalues of the matrix, given simply by the matrix components a_{11} and a_{22} are negative for any value of x. Moreover, we also notice that for $\epsilon = 0$ the eigenvalues are both equal to -1, with the effect that

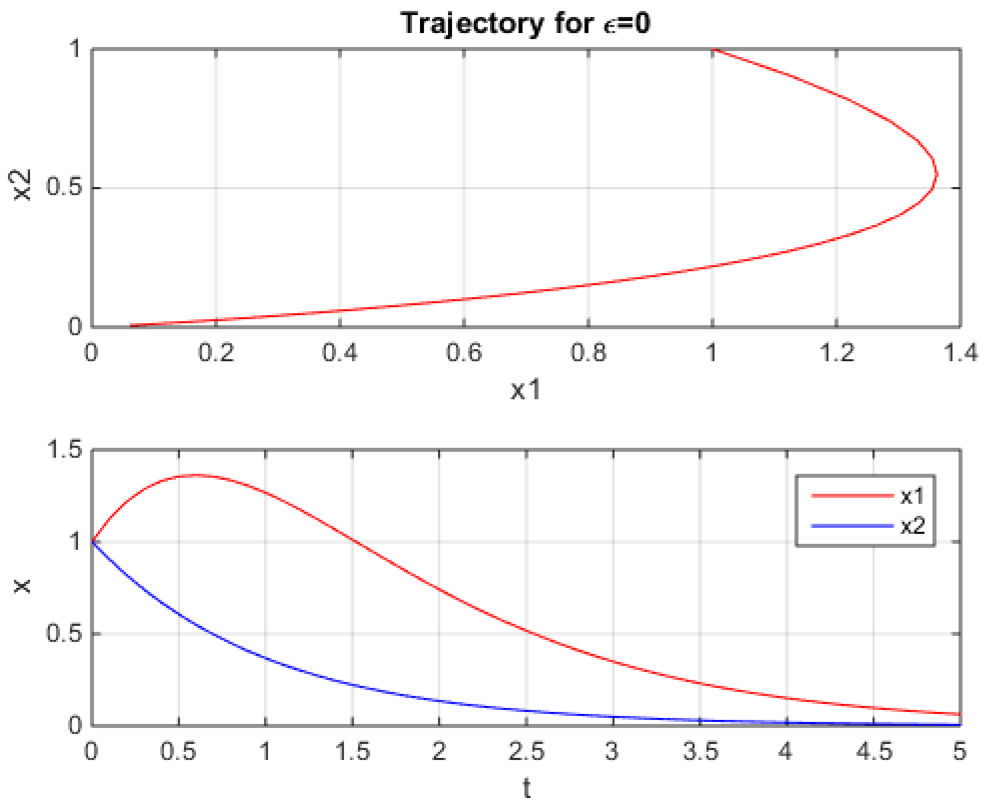

Figure 1: System behavior with $\epsilon = 0$

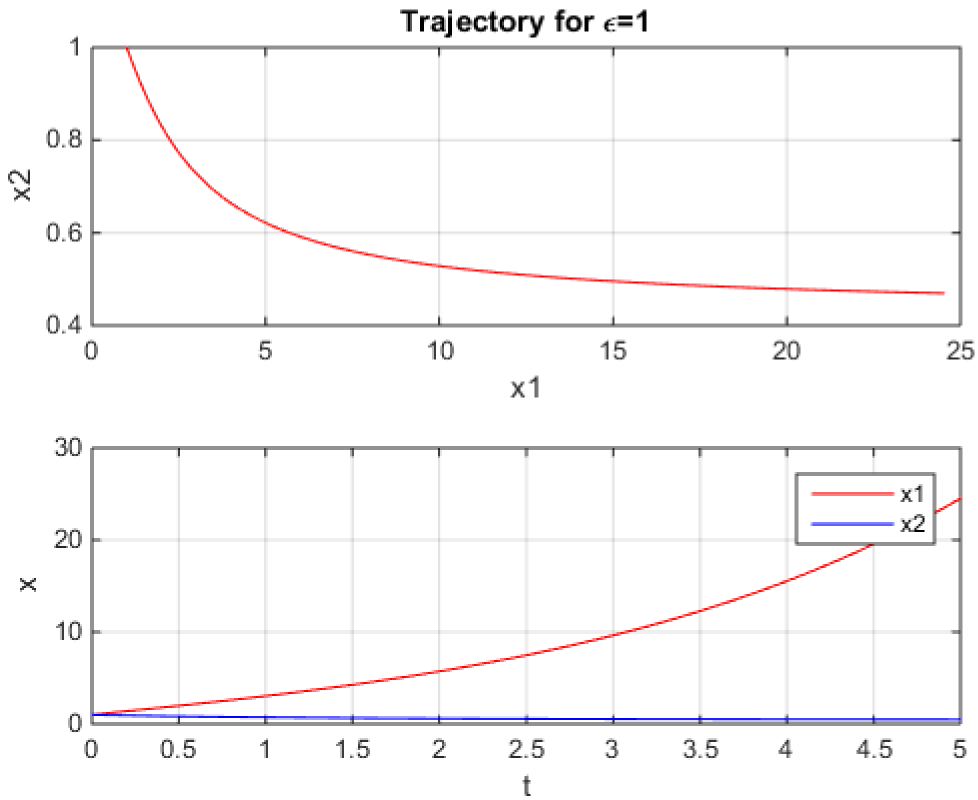

Figure 2: System behavior with $\epsilon = 1$

the state x_2 will converge to zero rapidly. This will make the term a_{12} also small, and eventually lead to x_1 to also converge to zero.

However, if $\epsilon = 1$ then at least for some initial conditions, x_1 may grow fast and make the term a_{22} converge to zero, which will lead to x_2 becoming a constant. Since also $a_{11} = a_{22}$ will be small, the term a_{12} will be dominant in the dynamics of x_1 and will lead x_1 to grow to infinite.

SDRE Method Formulation To be effective in this context we should rather look at the minimization of the infinite-time performance criterion

$$J[x_0, u(\cdot)] = \int_0^\infty [x^T Q(x) x + u^T R(x) u] dt, \ldots Q(x) \geq 0, R(x) > 0. \qquad (15)$$

The state and input weighting matrices are assumed *state dependent*.

Under the specified conditions, we apply point-wise the LQR method for $(A(x), B(x))$,$(Q(x), R(x))$ generating a control law

$$u(t) = -K(x)x = R(x)^{-1}B(x(^T P(x)x(t), \quad K : \Re^n \mapsto \Re^{p\times n}$$

where $P : \Re^n \mapsto \Re^{n\times n}$ satisfies

$$P(x)A(x) + A(x)^T P(x) - P(x)B(x)R(x)^{-1}B(x)^T P(x) + Q(x) = 0.$$

By applying this method one expects to obtain the good properties of the LQR control design, namely, that

- the control law minimizes the cost given by Equation 15
- regulates the system to the origin $\lim_{t\to\infty} x(t) = 0$.

Main Stability Results Below we establish how close we are to that "wish". Indeed, we have the following theorem.

Theorem 1 *(Mracek & Cloutier, 1998) Let us assume the following conditons hold.*

1. *The matrix valued functions $A(x), B(x), Q(x), R(x) \in C^1(\Re^n)$.*
2. *The pairs $(A(x), B(x))$ and $(A(x), Q^{\frac{1}{2}}(x))$ are pointwise stabilizable, respectively, detectable, state dependent parameterizations of the nonlinear system for all $x\Re^n$.*

Then, the nonlinear multivariable system given by 15 is rendered locally asymptotically stable by the control law

$$u(t) = -K(x)x = R(x)^{-1}B(x(^T P(x)x(t), \quad K : \Re^n \mapsto \Re^{p\times n} \tag{16}$$

where P is the unique, symmetric, positive-definite solution of the algebraic State-Dependent Riccati Equation

$$P(x)A(x) + A(x)P(x) - P(x)B(x)R^{-1}B(x)^T(x)P(x) + Q(x) = 0. \tag{17}$$

Proof: The dynamics is given by the pointwise Hurwitz matrix

$$\dot{x} = (A(x) - B(x)R^{-1}B^T P(x)]x = A_{cl}(x)x.$$

Under the assumptions of the theorem one can show that

$$A_{cl}(x) = A_{cl}(0) + \phi(x), \quad \lim_{x\to 0} \phi(x) = 0.$$

which means that the linear term is dominant near the origin. □

Remark 1 *Note that global stability has not been established, this is a local result. In general, as we saw in Example 1, even when*

$$A_{cl}(x) = A(x) - B(x)K(x)$$

is Hurtwitz for all x, global stability can not be guaranteed.

Remark 2 *One can prove though that if $A_{cl}(x)$ Hurtwitz and* symmetric *for all x, then global stability holds. The proof is simply obtained by showing that under these conditions $V(x) = x^T x$ is a Lyapunov function for system (15).*

Theorem 2 *Under conditions of Theorem 1 , the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality of the nonlinear optimal regulator problem*

$$u(x) = \arg\min H(x, \lambda, u), \quad \lambda = P(x)x$$

where

$$H(x, \lambda, u) = 0.5x^T Q(x)x + 0.5u^T R(x)u + \lambda^T [A(x)x + B(x)u]$$

Proof: The Hamiltonian of this problem is given by

$$H(x, \lambda, u) = 0.5x^T Q(x)x + 0.5u^T R(x)u + \lambda^T [A(x)x + B(x)u].$$

Thus, it is clear that

$$H_u = R(x)u + B^T(x)\lambda$$

This implies that for any choice of λ

$$u(x) = -R^{-1}(x)B^T(x)\lambda \implies H_u = 0.$$

In particular, the choice $\lambda = P(x)x$ renders a controller given by Equation 16. $\square$

When using the SDRE method one observes that as the state converges to zero, the solution also converges to the optimal solution given by the Pontriaguin Maximum Principle. This observation is supported by the following result.

Theorem 3 *Assume that all state dependent matrices are bounded functions along the trajectories. Then, under the conditions of Theorem 1 the Pontriaguin optimality condition*

$$\dot{\lambda} = -H_x(x, \lambda, u)$$

is satisfied approximately by

$$\lambda = P(x)x$$

at a quadratic rate along the trajectory. Here $P(x)$ denotes the solution of Equation 17.

Proof: The proof of this statement regarding $\lambda(\cdot)$ is quite technical and is obtained as follows. First, we observe

$$\lambda = P(x)x \implies \dot{\lambda} = \dot{P}(x)x + P(x)\dot{x}.$$

Thus, we need it to hold:

$$-H_x(x, \lambda, u) \approx \dot{P}(x)x + P(x)\dot{x}.$$

Equating these expressions, grouping linear and quadratic terms and using Equation 17, one obtains that the residuals are all quadratic functions of x. Theorem 1 proves that these quadratic terms in x decay quadratically as x converges to the origin. $\square$

Example 2 . *Steer to $x = (d, 0)$ the following system.*

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -a\sin(x_1) - bx_2 + cu(t). \end{aligned}$$

Indeed,

$$A(x) = \begin{pmatrix} 0 & 1 \\ -a\sin(x_1 - d)/(x_1 - d) & -b \end{pmatrix} \quad B(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We choose

$$Q(x) = \begin{pmatrix} 1 + (x_1 - d)^2 & 0 \\ 0 & 1 + x_2^2 \end{pmatrix}, \quad R = 1.$$

The choice of $Q(x)$ ensures larger control actions for large deviations from the equilibrium.

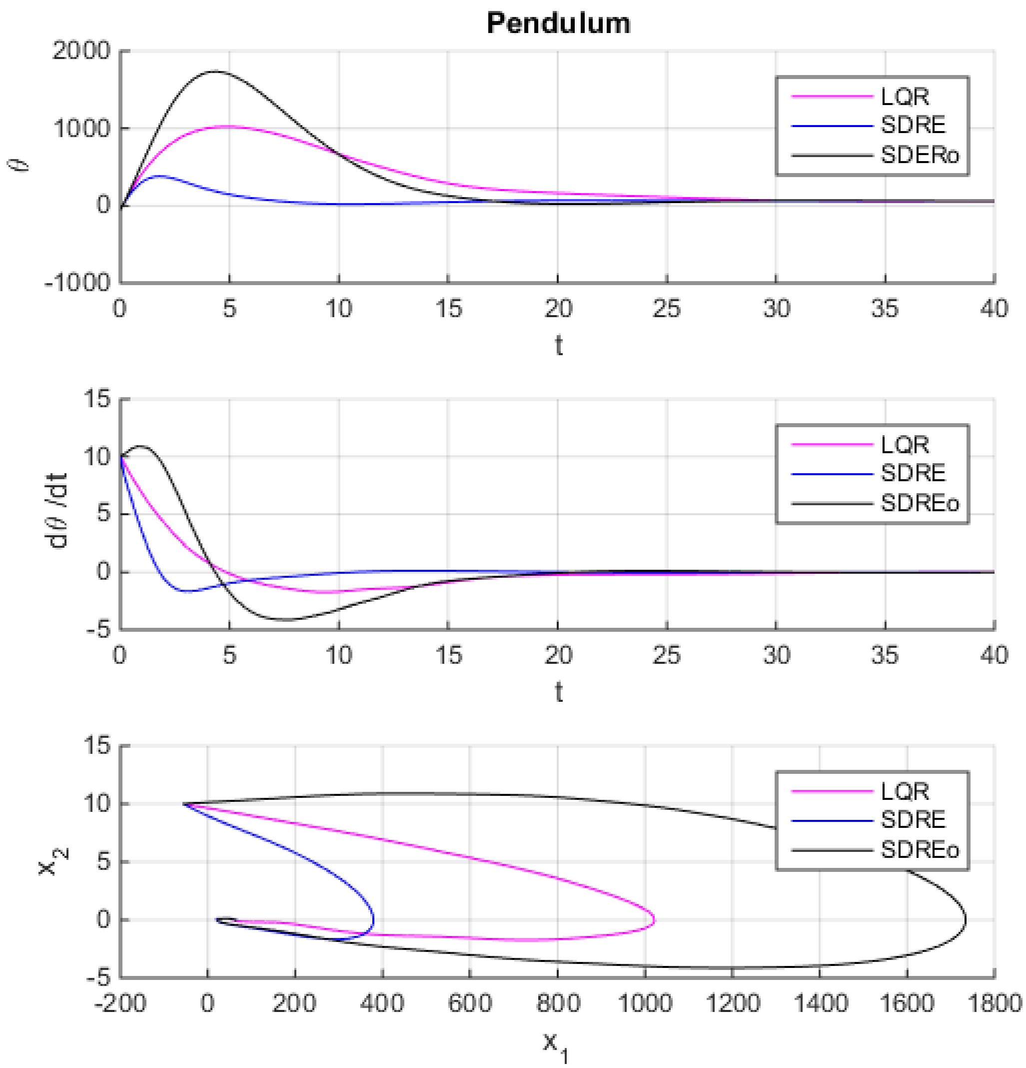

Figure 3: SDRE versus LQR

The magenta trajectory in Figure 3 is obtained using LQR on the standard linearization of the original system with

$$Q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1.$$

Note how the state is brought faster to the equilibrium in the SDRE case.

Output Feedback and SDRE Approach One can extend these same ideas to the case where the state is not available for control. As expected the method reduces to use the matrix functions $A(\cdot), B(\cdot), C(\cdot)$ to calculate pointwise the corresponding Kalman gain L described in Equation 8. Clearly, since x is not available, one must evaluate the matrix functions at the current estimate x_e, which is calculated following Equation 7.

In Figure 3 the line in black depicts the result of this strategy for the previous example, but using only the position x_1 for calculation the control. One observes that the controller is able to bring the system to the desired position.

We also observe though that the transient is now much longer. The length of this transient being dependent, among other issues, on the quality of the initial estimate and on the measurement noise. Figure 4 shows the transients of x and x_e. Note that the controller has only x_e to plan its actions.

1.3 H_∞ control for nonlinear systems

H_∞ control is a systematic method for generating robust controllers for linear systems that implies designing a controller that minimizes the effect on a performance related variable $z = C_z x + D_u u$ of the worst possible disturbance w, see Figure 5. In complete analogy to LQR and LQG controllers, the SDRE method is similarly deployed to implement H_∞ controllers for nonlinear systems.

H_∞ control creates a beautiful mathematical apparat related to game theory that unfortunately we will not touch further due to time and space constraints. Interested readers are referred to the work by Helton and James listed in the references.

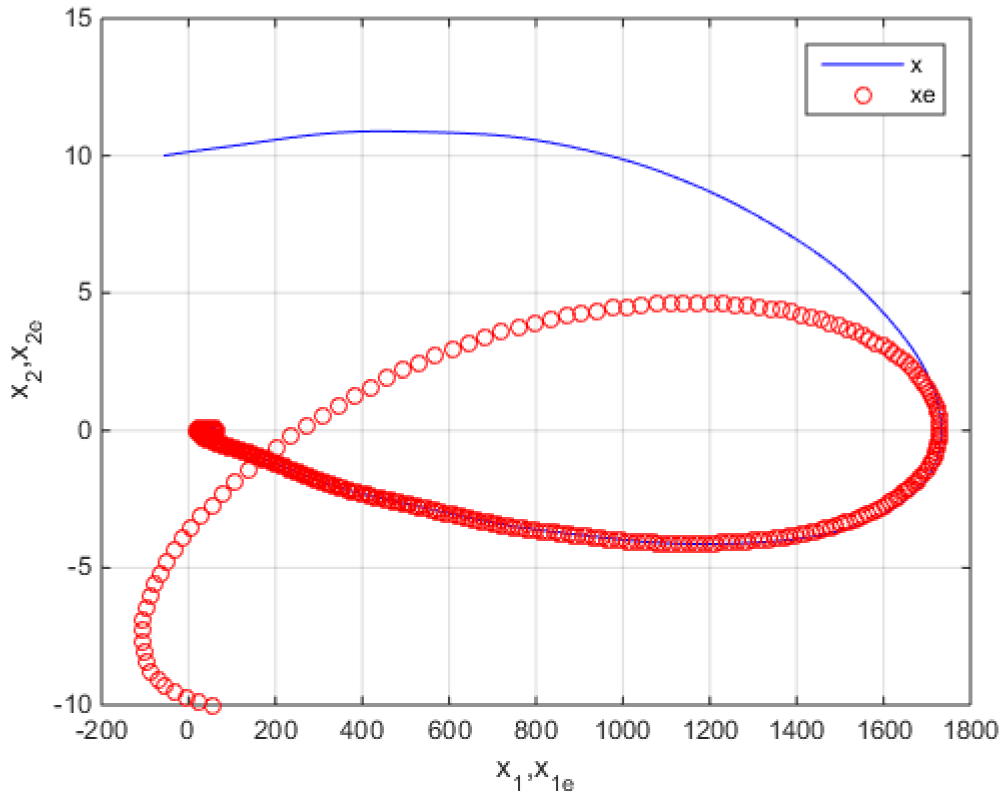

Figure 4: Estimate and true state using the SDRE approach

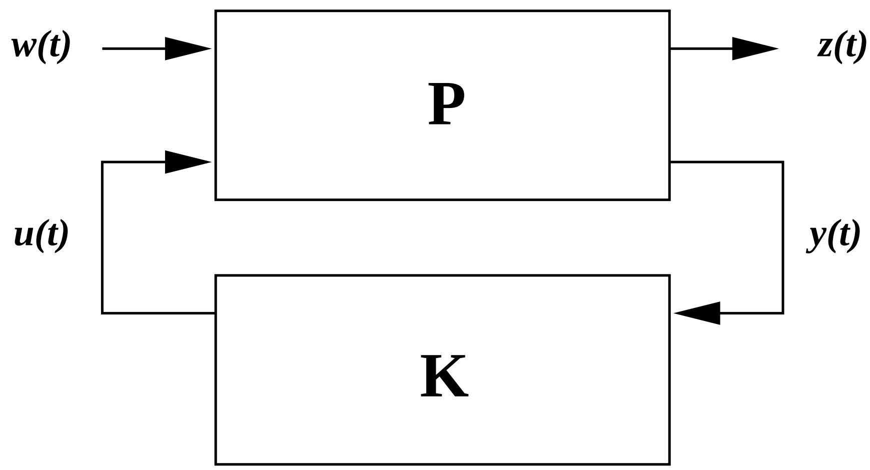

Figure 5: H_∞ control components

1.4 References

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2. Nonlinear Regulation and Nonlinear H_∞ Control via the State Dependent Riccati Equation Technique. James Cloutier, et. al. Proceedings of First International Conference on Nonlinear Problems in Aviation and Aerospace. Florida, May, 1996.

3. J.W. Helton & M.R. James, *A General Framework for Extending H-Infinity Control to Nonlinear Systems*, SIAM 1998.