1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ and $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$. Then $f(A) =$

(A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

设 $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, 且 $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$, 则 $f(A) =$

(A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

(2) Let $A$ and $B$ be invertible matrices. If $A$ is similar to $B$, which of the following statements is **NOT** correct?

(A) $A^T$ is similar to $B^T$.

(B) $A^{-1}$ is similar to $B^{-1}$.

(C) $A + A^T$ is similar to $B + B^T$.

(D) $A + A^{-1}$ is similar to $B + B^{-1}$.

假定 $A$ 和 $B$ 都是可逆矩阵, 且 $A$ 和 $B$ 相似, 下列陈述中哪个是不正确的？

(A) $A^T$ 和 $B^T$ 相似.

(B) $A^{-1}$ 和 $B^{-1}$ 相似.

(C) $A + A^T$ 和 $B + B^T$ 相似.

(D) $A + A^{-1}$ 和 $B + B^{-1}$ 相似.

(3) Let $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$. Then the number of positive eigenvalues of $A$ is

(A) 0.

(B) 1.

(C) 2.

(D) 3.

设 $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$, 则矩阵 $A$ 的正的特征值的个数为

(A) 0.

(B) 1.

(C) 2.

(D) 3.

(4) The equation $2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ represents a graph of

(A) An ellipsoid.

(B) Hyperboloid of one sheet.

(C) Hyperboloid of two sheets.

(D) Hyperbolic paraboloid.

$2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ 表示的曲面是

(A) 椭球面.

(B) 单叶双曲面.

(C) 双叶双曲面.

(D) 双曲抛物面.

(5) Which of the following statements is correct?

(A) If $A$ is a Hermitian matrix, and $x^HAx = 0$ for all complex vectors $x$, then $A = O$, where $O$ denotes the zero matrix.

(B) An $n \times n$ matrix with real eigenvalues and $n$ linearly independent real eigenvectors is symmetric.

(C) If $A$ is a complex matrix, and $A^T = A$, then $A$ is diagonalizable.

(D) Let $A$, $B$ be $n \times n$ real matrices, then $\det(A + B) = \det A + \det B$.

下面的哪个陈述是正确的?

(A) 如果 $A$ 是厄密特矩阵, 而且对所有的复向量 $x$ 都有 $x^HAx = 0$, 那么 $A = O$, 这里 $O$ 表示零矩阵.

(B) 一个 $n$ 阶的方阵的所有特征值和 $n$ 个线性无关的特征向量都是实的, 则这个矩阵是对称的.

(C) 如果 $A$ 是一个复矩阵, 且满足 $A^T = A$, 则 $A$ 是可对角化的.

(D) 设 $A, B$ 都是 $n$ 阶实方阵, 则 $\det(A+B) = \det A + \det B$.

**ANS:** (1) A (2) C (3) C (4) B (5) A

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Suppose $A$ is a $5 \times 4$ real matrix with 3 linearly independent columns. The dimension of the row space of $A$ is ________, and the dimension of the left nullspace of $A$ is ________.

设一个 $5 \times 4$ 的实矩阵 $A$ 有三个线性无关的列向量, 则 $A$ 的行空间的维数为 ________, $A$ 的左零空间的维数为 ________.

(2) If the real quadratic form $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ is changed to standard form $f = 6y_1^2$ by orthogonal transformation $x = Qy$, then $a =$ ________.

如果实二次型 $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ 经过正交变换 $x = Qy$ 化为标准形 $f = 6y_1^2$, 则 $a =$ ________.

(3) The eigenvalues of $I_3 - uv^T$ are ________. Where $I_3$ is the $3 \times 3$ identity matrix, and $u$ and $v$ are nonzero vectors in $\mathbb{R}^3$.

矩阵 $I_3 - uv^T$ 的特征值为 ________. 这里 $I_3$ 表示 3 阶单位阵, $u$ 和 $v$ 是 $\mathbb{R}^3$ 中的非零向量.

(4) If $A^2 = A$ and rank $(A) = r$, then trace $(A) =$ ________.

如果 $A^2 = A$ 且 rank $(A) = r$, 则 trace $(A) =$ ________.

(5) Let $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.

If the dimension of the vector space generated by $\alpha_1, \alpha_2, \alpha_3$ is 2, then $a =$ ________.

设 $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.

如果由 $\alpha_1, \alpha_2, \alpha_3$ 生成的子空间的维数为 2, 则 $a =$ ________.

**ANS:** (1) 3,2 (2) 2 (3) $1, 1 - u^Tv$ (4) $r$ (5) 6

3. (12 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find an orthonormal basis for the column space of $A$;

(b) Write $A$ as $QR$, where $Q$ has orthonormal columns and $R$ is upper triangular.

(12 分) 设

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) 求 $A$ 的列空间的一组标准正交基;

(b) 将 $A$ 分解成 $QR$, 其中 $Q$ 的列是标准正交的向量, $R$ 是上三角矩阵.

**Solution.**

(a) An orthonormal basis for the column space of $A$ is:

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

(b) The $QR$ factorization of $A$ is as follows:

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

4. (10 points) Compute the determinant of an $n \times n$ matrix $A$:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

(10 分) 计算 $n$ 阶矩阵 $A$ 的行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

**Solution.** $a^n - a^{n-2}$.

4. (10 points) Compute the determinant of an $n \times n$ matrix $A$:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

(10 分) 计算 $n$ 阶矩阵 $A$ 的行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

**Solution.** $a^n - a^{n-2}$.

6. (12 points) Let

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) Find all the singular values of $A$;

(b) Find a singular value decomposition of $A$.

(12 分) 设

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) 求 $A$ 的所有奇异值;

(b) 求矩阵 $A$ 的一个奇异值分解.

**Solution.**

(a) The singular values of $A$ are $3\sqrt{2}, \sqrt{2}$;

(b) A singular value decomposition of $A$ is as follows:

$$A = U\Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

8. (8 points) Let $A$ be an $n \times n$ real symmetric positive definite matrix, and $B$ be an $n \times n$ real symmetric positive semidefinite matrix.

   (a) Prove that the eigenvalues of $AB$ are all nonnegative real numbers.

   (b) Prove that $AB$ is diagonalizable.

   (8 分) 设 $A$ 为 $n$ 阶正定实对称矩阵, $B$ 为 $n$ 阶半正定实对称矩阵.

   (a) 证明: $AB$ 的所有特征值都是非负实数.

   (b) 证明: $AB$ 可对角化.

**Solution.**

(a) Since $A$ is positive definite, and B is positive semidefinite, then we can find $P$ and $Q$ such that

$$A = P^T P, B = Q^T Q,$$

where $P$ is an invertible matrix. It follows that

$$AB = P^T P Q^T Q = P^T P Q^T Q P^T (P^T)^{-1}.$$

This means that $AB$ is similar to $PQ^T QP^T$. This together with the fact that $PQ^T QP^T$ is a positive semidefinite matrix complete the proof.

(b) Since $A$ is positive definite, then there is an invertible matrix $C$ such that

$$C^T AC = I_n,\ C^T AB(C^T)^{-1} = C^T ACC^{-1} B(C^T)^{-1}.$$

$M = C^{-1}B(C^T)^{-1}$ is a positive semidefinite matrix, therefore we can find an orthogonal matrix $Q$ such that

$$Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$Q^T C^T AB(C^T)^{-1}Q = Q^T C^T ACC^{-1}B(C^T)^{-1}Q = Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $AB$ is diagonalizable.