Solving $Ax = 0$ and $Ax = b$

Lecture 8

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Complete Solution

Solving $Ax = b$, $Ux = c$, and $Rx = d$

The case $b \neq 0$ is quite different from $b = 0$. The row operations on A must act also on the right-hand side(on b). We begin with letters (b_1, b_2, b_3) to find the solvability condition– for b lie in the column space.

For the original example $Ax = b = (b_1, b_2, b_3)$, apply to both sides the operations that led from A to U. The result is an upper triangular system $Ux = c$:

$$Ux = c \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}.$$

Remarks

(a) The equations are inconsistent unless $b_3 - 2b_2 + 5b_1 = 0$.

(b) The dependent columns, the second and the fourth, are exactly the ones without pivots.

Solving $Ax = b$, $Ux = c$, and $Rx = d$: Continue

For a specific example with $b_3 - 2b_2 + 5b_1 = 0$, choose $b = (1, 5, 5)$:

$$Ax = b \quad \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}.$$

The complete solution is as follows

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Remark

Remark:

Every solution to $Ax = b$ is the sum of one particular solution and a solution to $Ax = 0$:

$$x_{\textbf{complete}} = x_{\textbf{particular}} + x_{\textbf{nullspace}}$$

Rank

Elimination reveals the pivot variables and free variables. If there are r pivots, there are r pivot variables and $n-r$ free variables. That important number r will be given a name–it is the rank of the matrix.

Definition

The rank of A is the number of pivots. This number is r.

Theorem

Theorem

Suppose elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$, with r pivot rows and r pivot columns. ***The rank of those matrices is*** *r. The last $m-r$ rows of U and R are zero, so there is a solution only if the last $m-r$ entries of c and d are also zero.*

The complete solution is $x = x_p + x_n$. One particular solution x_p has all free variables zero. Its pivot variables are the first r entries of d, so $Rx_p = d$.

The nullspace solution x_n are combinations of $n-r$ ***special solutions****, with one free variable equal to 1. The pivot variables in that special solution can be found in the corresponding column of R (with sign reversed).*

Another Worked Example

There are several remarks regarding the previous theorem:

1. You see how the rank r is crucial. It counts the pivot rows in the "row space" and the pivot columns in the column space.
2. There are $n-r$ special solutions in the nullspace.
3. There are $m-r$ solvability conditions on b or c or d.

The full picture uses elimination and pivot columns to find the column space, nullspace, and rank. The 3 by 4 matrix A has rank 2:

$$Ax = b \ is \ \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Another Worked Example:Continue

$$Ax = b \text{ is } \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We follow the following steps to solve this system:

1. Reduce $[A \quad b]$ to $[U \quad c]$, to reach a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 to have a solution.
3. Describe the column space of A: Which plane in $\mathbb{R}^3$?
4. Describe the nullspace of A: Which special solutions in $\mathbb{R}^4$?

Remarks

5. Find a particular solution to $Ax = (0, 6, -6)$ and the complete $x_p + x_n$.
6. Reduce $[U \quad c]$ to $[R \quad d]$: Special solutions from R and x_p from d.

Now let us work out the details on blackboard together!

Two More Examples

Example

Describe the set of attainable right-hand sides b(in the column space) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

by finding the constraints on b that turn the third equation into $0 = 0$ (after elimination). What is the rank, and a particular solution?

Example

Suppose A and B are n by n matrices, and $AB = I$. Prove from rank$(AB) \leq$ rank(A) that the rank of A is n. So A is invertible and B must its two-sided inverse. Therefore, $BA = I$.

Homework Assignment 8

2.2: 33, 45, 48, 63.