

题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	12 分	10 分	10 分	12 分	8 分	8 分

本试卷共 ( 8 ) 大题, 满分 ( 100 ) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 100 in total. **Write all your answers on the examination book.**

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let  $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$  and  $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$ . Then  $f(A) =$

(A)  $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B)  $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C)  $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D)  $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

设  $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ , 且  $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$ , 则  $f(A) =$

(A)  $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B)  $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C)  $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D)  $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

(2) Let  $A$  and  $B$  be invertible matrices. If  $A$  is similar to  $B$ , which of the following statements is **NOT** correct?

(A)  $A^T$  is similar to  $B^T$ .

(B)  $A^{-1}$  is similar to  $B^{-1}$ .

(C)  $A + A^T$  is similar to  $B + B^T$ .

(D)  $A + A^{-1}$  is similar to  $B + B^{-1}$ .

假定  $A$  和  $B$  都是可逆矩阵, 且  $A$  和  $B$  相似, 下列陈述中哪个是**不正确**的?

- (A)  $A^T$  和  $B^T$  相似.
- (B)  $A^{-1}$  和  $B^{-1}$  相似.
- (C)  $A + A^T$  和  $B + B^T$  相似.
- (D)  $A + A^{-1}$  和  $B + B^{-1}$  相似.

(3) Let  $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$ . Then the number of positive eigenvalues of  $A$  is

- (A) 0.
- (B) 1.
- (C) 2.
- (D) 3.

设  $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$ , 则矩阵  $A$  的正的特征值的个数为

- (A) 0.
- (B) 1.
- (C) 2.
- (D) 3.

(4) The equation  $2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$  represents a graph of

- (A) An ellipsoid.
- (B) Hyperboloid of one sheet.
- (C) Hyperboloid of two sheets.
- (D) Hyperbolic paraboloid.

$2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$  表示的曲面是

- (A) 椭球面.
- (B) 单叶双曲面.
- (C) 双叶双曲面.
- (D) 双曲抛物面.

(5) Which of the following statements is correct?

- (A) If  $A$  is a Hermitian matrix, and  $x^H Ax = 0$  for all complex vectors  $x$ , then  $A = O$ , where  $O$  denotes the zero matrix.
- (B) An  $n \times n$  matrix with real eigenvalues and  $n$  linearly independent real eigenvectors is symmetric.
- (C) If  $A$  is a complex matrix, and  $A^T = A$ , then  $A$  is diagonalizable.
- (D) Let  $A, B$  be  $n \times n$  real matrices, then  $\det(A + B) = \det A + \det B$ .

下面的哪个陈述是正确的?

- (A) 如果  $A$  是厄密特矩阵, 而且对所有的复向量  $x$  都有  $x^H Ax = 0$ , 那么  $A = O$ , 这里  $O$  表示零矩阵.
- (B) 一个  $n$  阶的方阵的所有特征值和  $n$  个线性无关的特征向量都是实的, 则这个矩阵是对称的.
- (C) 如果  $A$  是一个复矩阵, 且满足  $A^T = A$ , 则  $A$  是可对角化的.
- (D) 设  $A, B$  都是  $n$  阶实方阵, 则  $\det(A + B) = \det A + \det B$ .

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

- (1) Suppose  $A$  is a  $5 \times 4$  real matrix with 3 linearly independent columns. The dimension of the row space of  $A$  is \_\_\_\_\_, and the dimension of the left nullspace of  $A$  is \_\_\_\_\_.

设一个  $5 \times 4$  的实矩阵  $A$  有三个线性无关的列向量, 则  $A$  的行空间的维数为 \_\_\_\_\_,  $A$  的左零空间的维数为 \_\_\_\_\_.

- (2) If the real quadratic form  $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$  is changed to standard form  $f = 6y_1^2$  by orthogonal transformation  $x = Qy$ , then  $a =$  \_\_\_\_\_.

如果实二次型  $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$  经过正交变换  $x = Qy$  化为标准形  $f = 6y_1^2$ , 则  $a =$  \_\_\_\_\_.

- (3) The eigenvalues of  $I_3 - uv^T$  are \_\_\_\_\_. Where  $I_3$  is the  $3 \times 3$  identity matrix, and  $u$  and  $v$  are nonzero vectors in  $\mathbb{R}^3$ .

矩阵  $I_3 - uv^T$  的特征值为 \_\_\_\_\_. 这里  $I_3$  表示 3 阶单位阵,  $u$  和  $v$  是  $\mathbb{R}^3$  中的非零向量.

- (4) If  $A^2 = A$  and  $\text{rank}(A) = r$ , then  $\text{trace}(A) =$  \_\_\_\_\_.

如果  $A^2 = A$  且  $\text{rank}(A) = r$ , 则  $\text{trace}(A) =$  \_\_\_\_\_.

- (5) Let  $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ ,  $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ ,  $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$ .

If the dimension of the vector space generated by  $\alpha_1, \alpha_2, \alpha_3$  is 2, then  $a =$  \_\_\_\_\_.

设  $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ ,  $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ ,  $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$ .

如果由  $\alpha_1, \alpha_2, \alpha_3$  生成的子空间的维数为 2, 则  $a =$  \_\_\_\_\_.

3. (12 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Find an orthonormal basis for the column space of  $A$ ;  
 (b) Write  $A$  as  $QR$ , where  $Q$  has orthonormal columns and  $R$  is upper triangular.

(12 分) 设

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) 求  $A$  的列空间的一组标准正交基;  
 (b) 将  $A$  分解成  $QR$ , 其中  $Q$  的列是标准正交的向量,  $R$  是上三角矩阵.

4. (10 points) Compute the determinant of an  $n \times n$  matrix  $A$ :

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

(10 分) 计算  $n$  阶矩阵  $A$  的行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

5. (10 points) Suppose  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$ , and  $A$  is similar to  $B$ .

- (a) Find  $a$  and  $b$ ;
- (b) Find an invertible matrix  $S$ , such that  $S^{-1}AS = B$ .

(10 分) 假定  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$ , 并且  $A$  和  $B$  相似.

- (a) 求  $a$  和  $b$  的值;
- (b) 求一个可逆矩阵  $S$ , 使得  $S^{-1}AS = B$ .

6. (12 points) Let

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

- (a) Find all the singular values of  $A$ ;
- (b) Find a singular value decomposition of  $A$ .

(12 分) 设

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

- (a) 求  $A$  的所有奇异值;
- (b) 求矩阵  $A$  的一个奇异值分解.

7. (8 points) Let  $A$  be a real symmetric  $n \times n$  positive definite matrix and  $B$  be an  $m \times n$  real matrix.

- (a) Show that the matrix  $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$  is congruent to the matrix  $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$ .
- (b) Suppose  $\text{rank}(B) = r$ . Find the number of positive eigenvalues, the number of negative eigenvalues, and the number of zero eigenvalues of  $M$  (counted with multiplicities).

(8 分) 假定  $A$  是一个  $n \times n$  实对称正定矩阵,  $B$  为一个  $m \times n$  实矩阵.

- (a) 证明: 矩阵  $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$  和矩阵  $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$  相合.
- (b) 假定  $\text{rank}(B) = r$ . 求矩阵  $M$  的正的特征值的个数, 负的特征值的个数, 以及零特征值的个数 (重根按重复的次数计).

8. (8 points) Let  $A$  be an  $n \times n$  real symmetric positive definite matrix, and  $B$  be an  $n \times n$  real symmetric positive semidefinite matrix.

- (a) Prove that the eigenvalues of  $AB$  are all nonnegative real numbers.
- (b) Prove that  $AB$  is diagonalizable.

(8 分) 设  $A$  为  $n$  阶正定实对称矩阵,  $B$  为  $n$  阶半正定实对称矩阵.

- (a) 证明:  $AB$  的所有特征值都是非负实数.
- (b) 证明:  $AB$  可对角化.