Brief Solutions
to Linear Algebra 2018 Fall Midterm

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(a). True. List the basis as columns of matrix A, its left nullspace denotes the wanted matrix.
(b). True. $AA^T = I \Rightarrow A^T A = I$.
(c). False. Counterexample:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

.
(d). False. Counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(e). False. Counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(f). True. The independent rows in A and B keeps independent in left matrix.

2

(a). $rank(A) = m$ because $C(A) \supseteq \mathbb{R}^m$.
(b). $t = 3$ and $rank(B) = 1$ because A must be singular, and $C(B) \subseteq N(A)$.
(c). $(\frac{9}{7}, \frac{6}{7}, \frac{3}{7})^T$. Use $p = \frac{aa^T}{a^T a} b$.

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(i) The detailed process is skipped.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) The detailed process is skipped as well.

$$A^{-1} = \begin{bmatrix} 3 & 4 & -2 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 3 & -2b_1 + b_2 \\ 0 & 0 & 0 & 0 & 5b_1 - b_2 + b_3 \end{bmatrix}$$

(i). Basis for $C(A^T)$: $\{(1,3,3,2),(0,0,3,3)\}$.
Basis for $N(A)$: $\{(-3,1,0,0)^T,(1,0,-1,1)^T\}$.
Basis for $C(A)$: $\{(1,2,-1)^T,(3,9,3)^T\}$.
Basis for $N(A^T)$: $\{(5,-2,1)\}$. (As long as it is orthogonal to $C(A)$.)
(ii). $5b_1 - b_2 + b_3 = 0$.
(iii). By asking free variables to be 0, we can find a particular solution $x_p = (-3,0,1,0)^T$. Thus

$$x = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, u \in \mathbb{R}, v \in \mathbb{R}.$$

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We just live so long, so we skip the detailed process. Here we just follow Gram-Schmidt method to find out the first two columns, while the last column is found by calculating $N(A)$.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{66}} & \frac{-3}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{7}{\sqrt{66}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{6}} & \frac{-4}{\sqrt{66}} & \frac{1}{\sqrt{11}} \end{bmatrix}$$

You should be reminded that the possible solution is not unique.

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(i). Whatever it happens, we always need basic observations. Here we may observe that the two columns are already orthogonal to each other, thus we only need to let there length to be 1.
Orthonormal basis: $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})^T, (\frac{-6}{\sqrt{90}},\frac{-2}{\sqrt{90}},\frac{1}{\sqrt{90}},\frac{7}{\sqrt{90}})^T$.
(ii). From what we did just now, this question is easy to answer.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-6}{\sqrt{90}} \\ \frac{1}{2} & \frac{-2}{\sqrt{90}} \\ \frac{1}{2} & \frac{1}{\sqrt{90}} \\ \frac{1}{2} & \frac{7}{\sqrt{90}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{90} \end{bmatrix}$$

(iii). To find the least squared solution, we multiply A^T on both sides.

$$
\begin{aligned}
&A^T A\hat{x} = A^T b \\
&\begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \hat{x} = \begin{bmatrix} 8 \\ 45 \end{bmatrix} \\
&\hat{x} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

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(i). Since α has two vectors, with $2 = dim(\mathbb{R}^2)$, we only need to show the two vectors are independent.

$$
\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}
$$

Thus the matrix has rank 2, which meets the requirement.
Similarly, we need to show the three vectors in γ are independent to each other.

$$
\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}
$$

Thus this matrix has rank 3, which meets the requirement as well.
(ii). By requiring $A[a_1, a_2]^T = T([a_1, a_2]^T)$, we can calculate the matrix A.

$$
A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}
$$

When considering the basis, we can find matrix T, which is what we want.

$$
T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}
$$

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$$
M = \begin{bmatrix} 2 & -6 & 3 & 2 & -1 \\ -3 & 9 & -2 & -8 & 1 \\ 4 & -12 & 7 & 2 & 2 \\ -3 & 9 & -8 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 & 0 \\ 2 & \frac{2}{5} & 1 & 0 \\ \frac{2}{3} & -\frac{5}{7} & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 3 & 2 & -1 \\ 0 & 0 & \frac{5}{2} & -5 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{21}{5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

(i). Form matrix $A = [1, 1, 1, 1]$, then $W = N(A)$, thus $dim(W) = dim(N(A)) = 4 - rank(A) = 3$.
(ii). It is easy to check every $u \in W$, thus $span(u_1, u_2 ... u_5) \subseteq W$. Since the matrix M has rank 3, we deduce $dim(span(u_1, u_2 ... u_5)) = 3 = dim(W)$, which forces these two spaces to be equal.
(iii). Since the pivots sit on column 1, column 3 and column 5. We know the set of $\{u_1, u_3, u_5\}$ is a basis of W.

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(i). $\exists x, Ax = b \Leftrightarrow b \in C(A) \Leftrightarrow C(A) = C(A|b)$. When $C(A) = C(A|b)$, then their dimensions must be the same, thus $rank(A) = rank(A|b)$. When $rank(A) = rank(A|b)$, there is $dim(C(A)) = dim(C(A|b))$, and $C(A) \subseteq C(A|b)$ forces the two spaces to be equal. Hence $\exists x, Ax = b \Leftrightarrow C(A) = C(A|b) \Leftrightarrow rank(A) = rank(A|b)$.

(ii). $AB = 0$ shows that every column of B is in $N(A)$, in other words, $C(B) \subseteq N(A)$, thus $dim(C(B)) \leq dim(N(A)) \Rightarrow rank(B) \leq n - rank(A) \Rightarrow rank(A) + rank(B) \leq n$.

(iii). $A^T A$ is an n by n matrix, it suffices that $rank(A) = rank(A^T A)$, i.e. $n - rank(A) = n - rank(A^T A)$. We need only to show $N(A^T A) = N(A)$, and its dimension will meet the requirement.
$\forall x \in N(A), Ax = 0 \Rightarrow A^T Ax = 0 \Rightarrow x \in N(A^T A)$. Thus $N(A) \subseteq N(A^T A)$.
$\forall x \in N(A^T A), A^T Ax = 0 \Rightarrow x^T A^T Ax = 0 \Rightarrow \|Ax\| = 0 \Rightarrow x \in N(A)$. Thus $N(A) \supseteq N(A^T A)$.
As a result, $N(A^T A) = N(A)$, which ends the proof.

As for P, we claim that P is invertible if and only if $m = n$. The reasons are shown below.
If $m = n$, then A is an invertible matrix because it has full rank. Thus $P = A(A^T A)^{-1} A^T = A(A^{-1}(A^T)^{-1})A^T = I$ is obviously invertible.
If $m \neq n$, since $rank(A) = n$, we deduce $m > n$. Obviously P is an m by m matrix, and since $rank(P) = rank(A[(A^T A)^{-1} A^T]) \leq rank(A) = n < m$, we obtain that P does not have full rank. Hence P is not invertible.
From the two statements above, we derive that the claim is true.