南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目:	线 性 代 数	开课单位:	数 学 系
考试时长:	120 分钟	命题教师:	线性代数教学团队

题 号	1	2	3	4	5	6
分 值	15 分	25 分	24 分	15 分	15 分	6 分

本试卷共 （6）大题, 满分 （100）分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This exam paper contains 6 questions and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

A

(1) Suppose that α_1, α_2, α_3 are a basis for the nullspace of a matrix A, $N(A)$. Which of the following lists of vectors is also a basis for $N(A)$?

(A) $\alpha_1+\alpha_2-\alpha_3$, $\alpha_1+\alpha_2+5\alpha_3$, $4\alpha_1+\alpha_2-2\alpha_3$.

(B) $\alpha_1+2\alpha_2+\alpha_3$, $2\alpha_1+\alpha_2+2\alpha_3$, $\alpha_3+\alpha_1+\alpha_2$.

(C) $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$.

(D) $\alpha_1-\alpha_2$, $\alpha_2-\alpha_3$, $\alpha_3-\alpha_1$.

设 α_1, α_2, α_3 为矩阵 A 的零空间 $N(A)$ 的一组基. 下列哪一组向量也是矩阵 A 的零空间的一组基?

(A) $\alpha_1+\alpha_2-\alpha_3$, $\alpha_1+\alpha_2+5\alpha_3$, $4\alpha_1+\alpha_2-2\alpha_3$.

(B) $\alpha_1+2\alpha_2+\alpha_3$, $2\alpha_1+\alpha_2+2\alpha_3$, $\alpha_3+\alpha_1+\alpha_2$.

(C) $\alpha_1+\alpha_2$, $\alpha_1+\alpha_2+\alpha_3$.

(D) $\alpha_1-\alpha_2$, $\alpha_2-\alpha_3$, $\alpha_3-\alpha_1$.

D

(2) Which of the following statements is correct?

(A) If the columns of A are linearly independent, then $A\mathbf{x}=\mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

(B) Any 5×7 matrix has linearly dependent columns.

(C) If the columns of a matrix A are linearly dependent, so are the rows.

(D) The column space and row space of a 10×12 matrix may have different dimensions.

以下说法一定是正确的是?

(A) 如果矩阵 A 的列向量线性无关, 那么对任意的 $\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$ 有唯一的解. 无解

(B) 任意 5×7 矩阵的列向量一定是线性相关的.

(C) 如果矩阵 A 的列向量线性相关, 该矩阵的行向量也线性相关.

1 2 1
2 1 2
1 1 1

(D) 一个 10×12 矩阵的行空间和列空间可能具有不同的维数.

$\frac{12}{7}\alpha_1 + \frac{1}{7}\alpha_2 = \beta$

D

(3) Let

$$\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \ \beta = \begin{bmatrix} 2 \\ t9 \\ 3 \end{bmatrix}.$$

β can be written as a linear combination of $\alpha_1, \alpha_2, \alpha_3$ if $t = (\quad)$

(A) 1.

(B) 3.

(C) 6.

(D) 9.

设

$$\alpha_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \ \alpha_3 = \begin{bmatrix} 6 \\ 2 \\ -16 \end{bmatrix}, \ \beta = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix}.$$

当 $t = (\quad)$ 时, β 可用 $\alpha_1, \alpha_2, \alpha_3$ 线性表示.

(A) 1.

(B) 3.

(C) 6.

(D) 9.

B

(4) Which of the following statements is correct?

(A) Suppose that $EA = B$ and E is an invertible matrix, then the column space of A and the column space of B are the same.

(B) Let A be a $n \times n$ matrix with rank 1, then $A^n = cA$, where n is a positive integer and c is a real number.

(C) Let A, B be symmetric matrices, then AB is symmetric.

(D) If A is of full row rank, then $Ax = 0$ has only the zero solution.

以下说法一定是正确的是?

(A) 设 E 为一个可逆矩阵. 如果 A, B 矩阵满足 $EA = B$, 则 A 和 B 的列空间相同.

(B) 设 A 为秩为 1 的 n 阶的方阵, 则 $A^n = cA$, 其中 n 为正整数, c 为实数.

(C) 如果 A, B 为对称矩阵, 则 AB 为对称矩阵.

(D) 如果矩阵 A 为一个行满秩矩阵, 那么 $Ax = 0$ 只有零解.

C

D

(5) Let A and B be two $n \times n$ matrices. If A is a non-zero matrix and $AB = 0$, then

(A) $BA = 0$.

(B) $B = 0$.

(C) $(A + B)(A - B) = A^2 - B^2$.

(D) rank $B < n$.

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & c \\ a & c \end{bmatrix}$

$A^2 + \underbrace{BA}_{\neq 0} - \cancel{AB} - B^2$

设 A 和 B 都为 n 阶矩阵. A 为非零矩阵, 且 $AB = 0$, 则

(A) $BA = 0$.

(B) $B = 0$.

(C) $(A+B)(A-B) = A^2 - B^2$.

(D) rank $B < n$.

2. (25 points, 5 points each) Fill in the blanks.

(共 25 分, 每小题 5 分) 填空题.

(1) Denote the vector space of 7×7 real matrices by $M_{7\times 7}(\mathbb{R})$, and let W be the subspace of $M_{7\times 7}(\mathbb{R})$ consisting of skew-symmetric real matrices, then $\dim W =$__________.

A matrix A is called skew symmetric if $A^T = -A$.

记所有 7×7 实矩阵构成的向量空间为 $M_{7\times 7}(\mathbb{R})$, W 为$M_{7\times 7}(\mathbb{R})$ 中所有斜对称矩阵构成的子空间, 则 $\dim W =$ ~~6~~ 21

如果 $A^T = -A$, A 就称之为斜对称的.

(2) Let A, B be two $n\times n$ invertible matrices. Suppose the inverse of $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, where O is the $n\times n$ zero matrix. Then $D =$__________.

设 A, B 为两个 $n\times n$ 可逆矩阵. 假设 $\begin{bmatrix} A & C \\ O & B \end{bmatrix}$ 的逆矩阵为 $\begin{bmatrix} A^{-1} & D \\ O & B^{-1} \end{bmatrix}$, 其中 O 为 $n\times n$ 的零矩阵, 则 $D =$ $\cancel{-A^{-1}C^{-1}B}$.

$-A^{-1}CB^{-1}$

(3) Let $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ with $\operatorname{rank}(A) < 4$. Then $a =$__________.

设 $A = \begin{bmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{bmatrix}$ 且 $\operatorname{rank}(A) < 4$, 则 $a =$ -2或1.

(4) Consider the system of linear equations:

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x + 2y = 1 \\ x - y = 2 \\ \quad\;\; y = -1. \end{cases}$$

The least-squares solution for the system is __________.

考虑以下线性方程组:

$$A\mathbf{x} = \mathbf{b} : \begin{cases} x + 2y = 1 \\ x - y = 2 \\ \quad\;\; y = -1. \end{cases}$$

该线性方程组的最小二乘解为 $\begin{bmatrix} \frac{17}{11} \\ -\frac{5}{11} \end{bmatrix}$.

(5) Let H be the subspace of $\mathbb{R}^3$ be defined as follows:

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + 2x_2 + x_3 = 0 \right\}.$$

A unit vector orthogonal to H is ____________.

设 H 为如下定义的一个 $\mathbb{R}^3$ 中的子空间

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + 2x_2 + x_3 = 0 \right\}.$$

一个和子空间 H 正交的单位向量为 $\begin{bmatrix} \frac{\sqrt6}{6} \\ \frac{\sqrt6}{3} \\ \frac{\sqrt6}{6} \end{bmatrix}$

3. (24 points) Consider the following 4×5 matrix A with its reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the third column of matrix A.

考虑以下这个 4×5 矩阵 A 以及它的简化阶梯形矩阵 R:

$$A = \begin{bmatrix} 1 & 2 & * & 1 & * \\ 0 & 1 & * & 1 & * \\ -1 & 1 & * & 3 & * \\ 2 & 0 & * & 1 & * \end{bmatrix}, \; R = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) 分别求矩阵 A 的四个基本子空间的一组基向量.

(b) 求出矩阵 A 的第三个列向量.

4. (15 points) Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{bmatrix}, \; B = \begin{bmatrix} 2 & 2 \\ 1 & a \\ -a-1 & -2 \end{bmatrix}.$$

For which value(s) of a, the matrix equation $AX = B$ has no solution, a unique solution, or infinitely many solutions? Where X is a 3×2 matrix. Solve $AX = B$ if it has at least one solution.

设

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & a & 1 \\ -1 & 1 & a \end{bmatrix}, \; B = \begin{bmatrix} 2 & 2 \\ 1 & a \\ -a-1 & -2 \end{bmatrix}.$$

当 a 为何值时, 矩阵方程 $AX = B$ 无解、有唯一解、有无穷多解? 在有解时, 求解此方程. 这里的 X 为一个 3×2 矩阵.

5. (15 points) Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 real matrices. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the map

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3, \ T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

for any 2×2 real matrix X, where $tr(D)$ denotes the trace of a matrix D.

The trace of an $n \times n$ matrix D is defined to be the sum of all the diagonal entries of D, in other words,

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) Show that T is a linear transformation.

(b) Find the matrix representation of T with respect to the ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for $M_{2\times 2}(\mathbb{R})$ and the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for $\mathbb{R}^3$.

(c) Can we find a matrix X such that $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? If yes, please find one such matrix.

Otherwise, give an explanation.

设 $M_{2\times 2}(\mathbb{R})$ 为所有 2×2 实矩阵构成的向量空间. 设

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

考虑以下映射

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3, \ T(X) = \begin{bmatrix} tr(A^T X) \\ tr(B^T X) \\ tr(C^T X) \end{bmatrix},$$

对任意的 2×2 实矩阵 X, 其中 $tr(D)$ 表示 n 阶矩阵 D 的迹.

方阵 D 的迹是指 D 的对角元之和, 也即

$$tr(D) = d_{11} + d_{22} + \cdots + d_{nn}.$$

(a) 证明 T 是一个线性变换.

(b) 求 T 在 $M_{2\times 2}(\mathbb{R})$ 的一组基

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ v_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

以及 $\mathbb{R}^3$ 的标准基

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

下的矩阵表示.

(c) 是否可以找到一个矩阵 X 使得 $T(X) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? 如果可以, 请求出一个符合要求的矩阵 X. 如果不存在, 请说明理由.

6. (6 points) Let A be an $m \times n$ matrix, B be an $m \times p$ matrix, and C be an $q \times p$ matrix. Show that

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

where O is the $q \times n$ zero matrix.

设 A 为 $m \times n$ 矩阵, B 为 $m \times p$ 矩阵, C 为 $q \times p$ 矩阵. 证明:

$$\operatorname{rank} \begin{bmatrix} A & B \\ O & C \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} C,$$

其中 O 为 $q \times n$ 的零矩阵.