南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数 A　　开课单位: 数学系
考试时长: 120 分钟　　命题教师: 线性代数教学团队

题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	16 分	10 分	6 分	16 分	12 分

本试卷共 (8) 大题, 满分 (110) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let A and B be $n \times n$ complex matrices. Which of the following statements is correct? (　　)

(A) If A and B are diagonalizable, so is $A+B$.

(B) If A and B are diagonalizable, so is AB.

(C) If A is invertible and A^2 is diagonalizable, then A is diagonalizable.

(D) If the distinct eigenvalues of A are 1 and 0, then A is diagonalizable.

设 A 和 B 为 n 阶复方阵. 下列陈述中哪个是正确的? (　　)

(A) 如果 A 和 B 都可对角化, 则 $A+B$ 也可对角化.

(B) 如果 A 和 B 都可对角化, 则 AB 也可对角化.

(C) 如果 A 为可逆且 A^2 可对角化, 则 A 也可对角化.

(D) 如果 A 的互不相同的特征值为 0 或 1, 则 A 可对角化.

(2) The equation $2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ represents a graph of (　　)

(A) An ellipsoid.

(B) Hyperboloid of one sheet.

(C) Hyperboloid of two sheets.

(D) Hyperbolic paraboloid.

$2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ 表示的曲面是 (　　)

(A) 椭球面.

(B) 单叶双曲面.

(C) 双叶双曲面.

(D) 双曲抛物面.

(3) Let A be a 3×3 matrix, and let B be the matrix formed by adding the second column of A to its first column. Suppose that after exchanging the second and third rows of B, the resulting matrix is the 3×3 identity matrix. Let $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then $A =$ ( )

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

设 A 为 3 阶方阵, 将 A 的第二列加到第一列得矩阵 B. 假设交换 B 的第二行与第三行可以得到 3 阶单位矩阵. 记 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. 则 $A =$ ( )

(A) P_1P_2.

(B) $P_1^{-1}P_2$.

(C) P_2P_1.

(D) $P_2P_1^{-1}$.

(4) Let $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, where $t \in \mathbb{R}$. Suppose $\operatorname{rank}(A) = 2$. Then ( )

(A) $t = -6$.

(B) $t = 6$.

(C) $t \neq 0$.

(D) t can be any real number.

设 $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$, 其中 $t \in \mathbb{R}$. 假设 $\operatorname{rank}(A) = 2$. 则 ( )

(A) $t = -6$.

(B) $t = 6$.

(C) $t \neq 0$.

(D) t 可以是任意实数.

(5) Which of the following statements is incorrect? ( )

(A) For any matrix A, $\operatorname{rank}(A) = \dim C(A)$.

(B) If $v_1, \cdots, v_m$ are pairwise orthogonal nonzero vectors, then the vectors $v_1, \cdots, v_m$ are linear independent.

(C) If A is an upper triangular $n \times n$ matrix such that $A^2 = 0$, then $A = 0$.

(D) Let A, B be $n \times n$ matrices such that AB is invertible. Then both A and B are invertible.

下列哪个论断是错误的？(　　)

(A) 对于任意矩阵 A, $\operatorname{rank}(A) = \dim C(A)$.

(B) 如果 $v_1, \cdots, v_m$ 是一组两两正交的非零向量, 则向量组 $v_1, \cdots, v_m$ 线性无关.

(C) 如果 A 是 $n \times n$ 上三角矩阵且 $A^2 = 0$, 则 $A = 0$.

(D) 设 A, B 为 $n \times n$ 矩阵且 AB 可逆. 则 A 和 B 都可逆.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A, B be invertible $n \times n$ matrices. Then the inverse of the block matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is __________

设 A, B 均为 $n \times n$ 可逆矩阵. 则分块矩阵 $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ 的逆为 __________

(2) Suppose A is a 3×4 matrix and $\dim N(A) = 2$. Then $\dim N(A^T) =$ __________

设 A 为 3×4 矩阵且 $\dim N(A) = 2$. 则 $\dim N(A^T) =$ __________

(3) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Then $A^{-1} =$__________

设 $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. 则 $A^{-1} =$__________

(4) Let u, v be vectors in $\mathbb{R}^n$ such that $\|u\| = 3$, $\|v\| = 4$ and $u^T v = -3$.

Then $\|2u + 3v\| =$__________

设 u, v 为 $\mathbb{R}^n$ 中的向量, 满足 $\|u\| = 3$, $\|v\| = 4$ 以及 $u^T v = -3$. 则 $\|2u+3v\| =$__________

(5) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

Then the least squares solution to $Ax = b$ is $\hat{x} =$__________

设 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$.

则 $Ax = b$ 的最小二乘解是 $\hat{x} =$__________

3. (10 points) Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Find an orthonormal basis, q_1, q_2, q_3, for the subspace V of $\mathbb{R}^4$ spanned by the column vectors v_1, v_2, v_3.

(b) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 2020 \\ 2021 \\ 2022 \\ 2023 \end{bmatrix}.$$

Find the least-squares solution of $Ax = b$.

假定

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) 设 V 为由 v_1, v_2, v_3 生成的 $\mathbb{R}^4$ 的子空间，求 V 的一组标准正交基.

(b) 记

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 2020 \\ 2021 \\ 2022 \\ 2023 \end{bmatrix}.$$

求 $Ax = b$ 在最小二乘意义下的解.

4. (16 points) Compute the nth order determinant:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, n \geq 2.$$

(16 分) 计算如下的 n 阶行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, n \geq 2.$$

5. (10 points) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, and A is similar to B. Find an orthogonal matrix Q, such that $Q^{-1}AQ = B$.

(10 分) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, and A is similar to B. Find an orthogonal matrix Q, such that $Q^{-1}AQ = B$.

6. (6 points) Let A, B be $n \times n$ matrices. Suppose A and B are both symmetric. Is AB necessarily symmetric? If yes, please give a proof. Otherwise please give a counterexample.

(6 分) 设 A, B 均为 $n \times n$ 矩阵. 假设 A 和 B 都是对称矩阵. AB 是否一定是对称矩阵? 若是, 请给出证明. 否则请给出一个反例.

7. (16 points)

(16 分)

8. (12 points) Let A be a 3×3 matrix such that $\operatorname{rank}(A) = 2$ and $A^3 = 0$.

(a) Prove that $\operatorname{rank}(A^2) = 1$.

(b) Let $\alpha_1 \in \mathbb{R}^3$ be a nonzero vector such that $A\alpha_1 = 0$. Prove that there exist vectors α_2, α_3 such that $A\alpha_2 = \alpha_1$, $A^2\alpha_3 = \alpha_1$.

(c) For any vectors α_2, α_3 described as above, show that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(12 分) 设 A 是 3×3 矩阵, 它满足 $\operatorname{rank}(A)=2$ 及 $A^3=0$.

(a) 证明 $\operatorname{rank}(A^2)=1$.

(b) 设 $\alpha_1\in\mathbb{R}^3$ 是满足 $A\alpha_1=0$ 的非零向量. 证明: 存在向量 α_2, α_3 使得 $A\alpha_2=\alpha_1$, $A^2\alpha_3=\alpha_1$.

(c) 证明: 对于任意满足上述条件的向量 α_2,α_3, 向量组 $\alpha_1,\alpha_2,\alpha_3$ 线性无关.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)