Solving $Ax = 0$ and $Ax = b$

Lecture 7

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Complete Solution

Questions

(a) Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$, and it was $x = A^{-1}b$. That solution was found by elimination. A rectangular matrix brings new possibilities–U may not have a full set of pivots. This section goes onward to a reduced form R–the simplest matrix that elimination can give. R reveals all solutions immediately.

(b) For an invertible matrix, the nullspace contains only $x = 0$. The column space is the whole space ($Ax = b$ has a solution for every b).

(c) The new questions appear when:
   (i) The nullspace contains more than zero, how can you get the complete solution to $Ax = 0$?
   (ii) The column space contains less than all vectors, how can you solve $Ax = b$?

Answers

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

   $$\text{Complete solution } Ax_p = b \text{ and } Ax_n = 0 \text{ produce } A(x_p + x_n) = b.$$

2. When the column space doesn't contain every b in $\mathbb{R}^m$, we need the conditions on b that make $Ax = b$ solvable.
3. A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable).

A few examples

(a) Let's begin to look at several examples:

The 1×1 system $0x = b$, one equation and one unknown, shows two possibilities:

1. $0x = b$ has no solution unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.
2. $0x = 0$ has infinitely many solutions.

Simple!

(b) If you move up to 2 by 2, it's more interesting. The matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

is not invertible: $y + z = b_1$ and $2y + 2z = b_2$ usually have no solution. In other words, we consider the following 2×2 system:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Infinitely many solutions

There is no solution unless $b_2 = 2b_1$. The column space of A contains only those b's, the multiples of $(1,2)$.

When $b_2 = 2b_1$ there are infinitely many solutions. A particular solution to $y+z=2$ and $2y+2z=4$ is $x_p = (1,1)$. The nullspace of A contains $(-1,1)$ and all its multiples $x_n = (-c,c)$:

$$x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 1+c \end{bmatrix}$$

Figure 2.2

The particular solution will be one point on the line. Adding the nullspace vectors x_n will move us along the line in Figure 2.2.

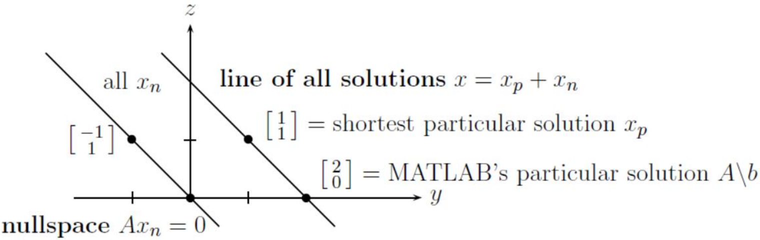

Figure 2.2: The parallel lines of solutions to $Ax_n = 0$ and $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Echelon Form U and Row Reduced Form R

What about a 3×4 system?

Basic Example

We start by simplifying this 3 by 4 matrix, first to U and then further to R:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The candidate for the second pivot has become zero: unacceptable.

(b) Next column.

(c) U is upper triangular, but its pivots are not on the main diagonal.

Remarks

(d) The nonzero entries of U have a “staircase pattern,” or echelon form.

(e) We can always reach this echelon form U, with zeros below the pivots.

Recall:

Definition

A matrix is said to be in row echelon form if

- The pivots are the first nonzero entries in their rows.
- Below each pivot is a column of zeros, obtained by elimination.
- Each pivot lies to the right of the pivot in the row above. This produces the staircase pattern, and zero rows come last.

Echelon Form U and Row Reduced Form R

Example

The entries of a 5 by 8 echelon matrix U and its reduced form R.

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & * & 0 & * & * & * & 0 \\ 0 & 1 & * & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The LU factorization for an m by n matrix A

(a) Do we have $A = LU$ as before? There is no reason why not, since the elimination steps have not changed. Each step still subtracts a multiple of one row from a row beneath it.

(b) The inverse of each step adds back the multiple that was subtracted. These inverses come in the right order to put the multipliers directly into L.

(c) Note that L is square. It has the same number of rows as A and U.

Theorem

For any m by n matrix A there is a permutation P, a lower triangular L with unit diagonal, and an m by n echelon matrix U, such that $PA = LU$.

Reduced Row Echelon Form

Now comes R. We can go further than U, to make the matrix even simpler. Recall:

Definition

A matrix is said to be in Reduced Row Echelon Form if

1. The matrix is in row echelon form.
2. The first nonzero entry in each row is the only nonzero entry in its column.

What is the row reduced form of a square invertible matrix?

The identity matrix.

Example

Here is one example of converting a matrix in row echelon form to reduced row echelon form:

Example

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) This matrix R is the final result of elimination on A

(b) MATLAB would use the command $R = rref(A)$.

(c) For a 5×8 matrix with four pivots, figure 2.4 shows the reduced form R. It still contains an identity matrix, in the four pivot rows and four pivot columns.

Pivot Variables and Free Variables

Our next goal is to read off all the solutions to $Rx = 0$. The pivots are crucial:

$$Rx = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) The unknowns go in to two groups. One group contains the **pivot variables**, those that correspond to columns with pivots. The other group is made up of the **free variables**, corresponding to columns without pivots.

(b) To find the most general solution to $Rx = 0$, we may assign arbitrary values to the free variables. Suppose we call these values simply v and y.

Free variables and pivot variables

The pivot variables are completely determined in terms of v and y:

$$Rx = 0$$

$\Leftrightarrow$

$$\begin{aligned} u + 3v - y &= 0 \\ w + y &= 0 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} u &= -3v + y \\ w &= -y \end{aligned}$$

There is a “double infinity” of solutions, with v and y free and independent.

Summary

The complete solution is a combination of two **special solutions**:

$$x = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(a) After reaching $Rx = 0$, identify free variables and pivot variables.

(b) Give one free variable the value 1, set other free variables to 0, and solve $Rx = 0$ for the pivot variables. This x is a special solution.

(c) Every free variable produces its own “special solution” by step 2. The combinations of special solutions form the nullspace–all solutions to $Ax = 0$.

Theorem

This is the place to recognize one extremely important theorem. Suppose a matrix has more columns than rows, $n > m$. Since m rows can hold at most m pivots, there must be at least $n - m$ free variables.

Theorem

If $Ax = 0$ has more unknowns than equations $(n > m)$, it has at least one special solution: There are more solutions than trivial $x = 0$.

Remarks:

(a) The nullspace has the same “dimension” as the number of free variables and special solutions.

(b) The dimension of a subspace is made precise in the next section. We count the free variables for the nullspace. We count the pivot variables for the column space!

Homework Assignment 7

2.2: 1, 2, 6, 9, 15, 24.