**Problem Set 4 —— Linear Algebra A (Fall 2021)**

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Please hand in your assignment at the beginning of your FIFTH tutorial session!

1. 如果方阵 $A$ 适合 $A^2 = A$, 则 $A$ 称为幂等的. 求出所有 2 阶幂等方阵.

2. 设 $A$ 为 $m \times n$ 矩阵, $B$ 为 $n \times p$ 矩阵, $C$ 为 $p \times q$ 矩阵. 证明：

$$\operatorname{rank} AB + \operatorname{rank} BC - \operatorname{rank} B \le \operatorname{rank} ABC.$$

同时探讨一下在什么时候上面的等号成立.

3. Suppose:

$$W_1 := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + 2y + 3z = 0 \right\} \text{ and } W_2 := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x + 2y + z = 0 \right\}$$

   (a) Is $W_1 \cup W_2 := \left\{ x \in \mathbb{R}^3 : x \in W_1 \text{ and } x \in W_2 \right\}$ a subspace of $\mathbb{R}^3$. Explain your answer.

   (b) Let $W_3$ be another subspace of $\mathbb{R}^3$. Show that $(W_1 + W_2) \cap W_3$ is a subspace of $\mathbb{R}^3$, where

$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1,\ w_2 \in W_2\}.$$

4. Let $A$ be an $m \times n$ matrix. Suppose that $P$ is an $m \times m$ invertible matrix, and $Q$ is an $n \times n$ invertible matrix. Show that

$$\operatorname{rank} PAQ = \operatorname{rank} A.$$

5. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ v_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix},\ v_4 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix},\ v_5 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Find a subset of $v_1, v_2, v_3, v_4, v_5$, which forms a basis of $\operatorname{span}(v_1, v_2, v_3, v_4, v_5)$.