Orthogonal Vectors and Subspaces (正交向量和正交子空间)

Lecture 13

Dept. of Math., SUSTech

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Orthogonal Vectors and Subspaces

Orthogonality

- A basis is a set of independent vectors that span a space.
- Geometrically, it is a set of coordinate axes. In choosing a basis, we tend to choose an orthogonal basis.
- The idea of an orthogonal basis is one of the foundations of linear algebra.
- We need a basis to convert geometric constructions into algebraic calculations, and we need an orthogonal basis to make those calculations simple.
- A further specialization makes the basis just about optimal: The vectors should have length 1.

Orthogonality

For an orthonormal basis(orthogonal unit vectors), we will find

(1) the length $||x||$ of a vector.

(2) the test $x^T y = 0$ for perpendicular vectors; and

(3) how to create perpendicular vectors from linearly independent vectors.

More than just vectors, subspaces can also be perpendicular.

(a) We will discover, so beautifully and simply that it will be a delight to see, that the fundamental subspaces meet at right angles.

(b) Those four subspaces are perpendicular in pairs, two in $\mathbb{R}^m$ and two in $\mathbb{R}^n$.

(c) That will complete the fundamental theorem of linear algebra.

Length of a vector

The first step is to find the length of a vector.

Definition

The length $\|x\|$ in $\mathbb{R}^n$ is the positive square root of $x^T x$.

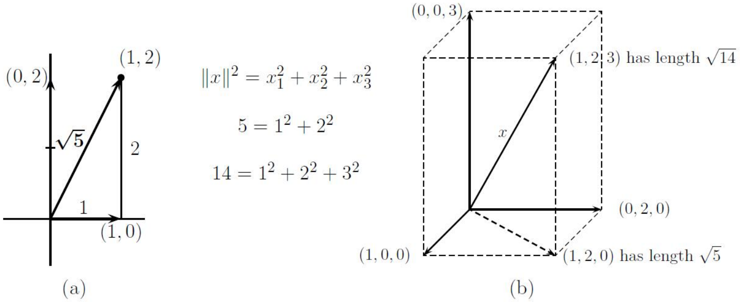

Figure 3.1: The length of vectors (x_1, x_2) and (x_1, x_2, x_3).

Orthogonal Vectors

How can we decide whether two vectors x and y are perpendicular?

What is the test for orthogonality in Figure 3.2?

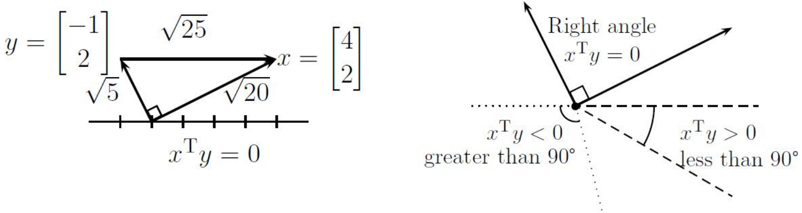

Figure 3.2: A right triangle with $5+20=25$. Dotted angle 100°, dashed angle 30°.

Inner Product

In the plane spanned by x and y, those vectors are orthogonal provided they form a right triangle. We go back to $a^2+b^2=c^2$:

$$||x||^2+||y||^2=||x-y||^2.$$

Sides of a right triangle

$$||x||^2+||y||^2=||x-y||^2.$$

Inner Product

Applying the length formula, this test for orthogonality in $\mathbb{R}^n$ becomes

$$(x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2.$$

The right-hand side has an extra $-2x_iy_i$ from each $(x_i - y_i)^2$:

$$\text{Right-hand side} = (x_1^2 + \cdots + x_n^2) - 2(x_1y_1 + \cdots + 2x_ny_n) + (y_1^2 + \cdots + y_n^2)$$

We have a right triangle when that sum of cross-product terms x_iy_i is zero:

$$\text{Orthogonal vectors} \quad x^Ty = x_1y_1 + \cdots + x_ny_n = 0$$

Orthogonal Vectors

This sum is $x^T y = \sum x_i y_i = y^T x$, the row vector x^T times the column vector y:

$$x^T y = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

This number is sometimes called the scalar product or dot product, and denoted by (x, y) or $x \cdot y$. We will use the name inner product and keep the notation $x^T y$.

Definition

Definition

The inner product $x^T y$ is zero if and only if x and y are orthogonal vectors. If $x^T y > 0$, their angle is less than 90°. If $x^T y < 0$, their angle is greater than 90°.

Useful Fact

Proposition

If nonzero vectors $v_1, v_2, \cdots, v_k$ are mutually orthogonal (every vector is perpendicular to every other), then those vectors are linearly independent.

Proof.

Suppose $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}$. Taking the inner product of both sides with v_1 to obtain

$$v_1^T (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = c_1 v_1^T v_1 = v_1^T \cdot \mathbf{0} = 0.$$

Since v_1 is nonzero, therefore $c_1 = 0$. A similar argument shows all c_i's are zero. Thus, $v_1, v_2, \cdots, v_k$ are linearly independent by definition. $\square$

Examples

1. The coordinate vectors $e_1, e_2, \cdots, e_n$ in $\mathbb{R}^n$ are the most important orthogonal vectors. Those are the columns of the identity matrix. They form the simplest basis for $\mathbb{R}^n$, and they are unit vectors–each has length $||e|| = 1$. They point along the coordinate axes.
2. Orthogonal vectors in $\mathbb{R}^2$. If these axes are rotated, the result is a new **orthonormal basis**: a new system of mutually orthogonal unit vectors. In $\mathbb{R}^2$, we have $\cos^2\theta + \sin^2\theta = 1$. Orthonormal vectors in $\mathbb{R}^2$:

$$v_1 = (\cos\theta, \sin\theta) \quad \text{and} \quad v_2 = (-\sin\theta, \cos\theta)$$

Orthogonal Subspaces

- A line can be orthogonal to another line, or it can be orthogonal to a plane, but a plane cannot be orthogonal to a plane.
- We come to the orthogonality of two subspaces.
- Every vector in one subspace must be orthogonal to every vector in the other subspace.

Definition

Two subspaces V and W of the same space $\mathbb{R}^n$ are orthogonal if **every** vector v in V is orthogonal to **every** vector w in W: $v^T w = 0$ for all v and w.

Fundamental Theorem of Orthogonality

- The important orthogonal subspaces don't come by accident, and they come two at a time. In fact orthogonal subspaces are unavoidable: They are the fundamental subspaces!
- The first pair is the nullspace and row space. Those are subspaces of $\mathbb{R}^n$–the rows have n components and so does the vector x in $Ax = 0$. We have to show, using $Ax = 0$, that the rows of A are orthogonal to the nullspace vector x.

Theorem

The row space is orthogonal to the nullspace (in $\mathbb{R}^n$). The column space is orthogonal to the left nullspace(in $\mathbb{R}^m$).

Let us prove this theorem in two different ways!

Orthogonal complement

It is certainly true that the nullspace is perpendicular to the row space–but it is not the whole truth. $N(A)$ contains **every** vector orthogonal to the row space. The nullspace was formed from all solutions to $Ax = 0$.

Definition

Given a subspace V of $\mathbb{R}^n$, the space of **all** vectors orthogonal to V is called the **orthogonal complement** of V. It is denoted by $V^{\perp}$="V perp."

Using this terminology, the nullspace is the orthogonal complement of the row space:

$$N(A) = (C(A^T))^{\perp}, C(A^T) = (N(A))^{\perp}.$$

The same reasoning applied to A^T produces the dual result.

Fundamental Theorem of Linear Algebra, Part II

Theorem

The nullspace is the orthogonal complement of the row space in $\mathbb{R}^n$.
The left nullspace is the orthogonal complement of the column space in $\mathbb{R}^m$.

Theorem

$Ax = b$ is solvable if and only if $y^T b = 0$ whenever $y^T A = 0$.

Remarks:

- b must be a combination of the columns.
- b must be orthogonal to every vector that is orthogonal to the columns.

The Matrix and the Subspaces

We emphasize that V and W can be orthogonal without being complements.

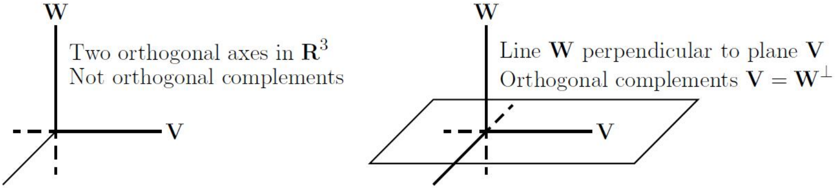

Figure 3.3: Orthogonal complements in $\mathbf{R}^3$: a plane and a line (not two lines).

Remarks:

- Splitting $\mathbb{R}^n$ into orthogonal parts will split every vector into $x = v + w$.
- The vector v is the projection onto the subspace V. The orthogonal component w is the projection of x onto W.

Theorem

Theorem

From the row space to the column space, A is actually invertible. Every vector b in the column space comes from exactly one vector x_r in the row space.

Proof.

Every b in the column space is a combination Ax of the columns. In fact, b is Ax_r, with x_r is in the row space, since the nullspace component gives $Ax_n = 0$. If another vector x_r' in the row space gives $Ax_r' = b$, then $A(x_r - x_r') = b - b = 0$. This puts $x_r - x_r'$ in the nullspace and the row space, which makes it orthogonal to itself. Therefore it is zero, and $x_r = x_r'$. Exactly one vector in the row space is carried to b. □

Figure 3.4

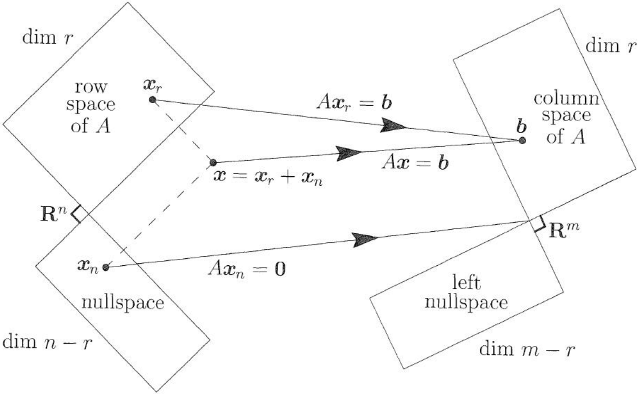

Figure 3.4: The true action $Ax = A(x_{\text{row}} + x_{\text{null}})$ of any m by n matrix.

Proposition

Proposition

Every matrix transforms its row space onto its column space.

Remarks:

- On those r-dimensional spaces A is invertible.
- On its nullspace A is zero.
- A^T goes in the opposite direction, from $\mathbb{R}^m$ to $\mathbb{R}^n$ and from $C(A)$ back to $C(A^T)$.
- When A^{-1} fails to exist, the best substitute is the pseudoinverse A^+.
- One formula for A^+ depends on the singular value decomposition–for which we first need to know about eigenvalues.

Homework Assignment 13

3.1: 5, 6, 11, 12, 19, 21, 25, 35, 44.