Complex Matrices(复矩阵)

Lecture 23 and 24

Dept. of Math.

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Complex Matrices

Complex Numbers

- It is no longer possible to work only with real vectors and real matrices. A real matrix has real coefficients in $\det(A-\lambda I)$, but the eigenvalues may be complex.
- We now introduce the space $\mathbb{C}^n$ of vectors with n complex components. Addition and matrix multiplication follow the same rules as before. Length is computed differently.
- We want to find out about symmetric matrices and Hermitian matrices: Where are their eigenvalues, and what is special about their eigenvectors?

Proposition

Every symmetric matrix(and Hermitian matrix) has real eigenvalues. Its eigenvectors can be chosen to be orthonormal.

The Arithmetic of Complex Numbers

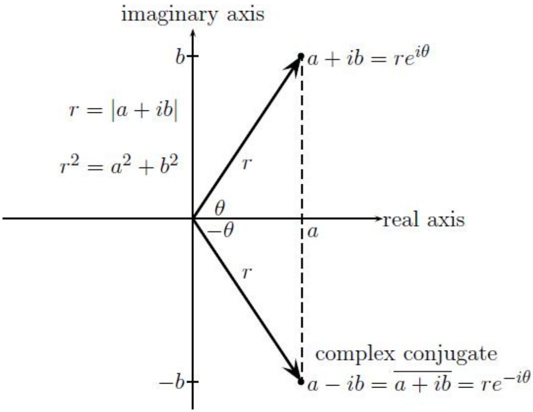

Figure 5.4: The complex plane, with $a+ib=re^{i\theta}$ and its conjugate $a-ib=re^{-i\theta}$.

The Arithmetic of Complex Numbers

Addition:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

Multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

Complex Conjugate:

$$\overline{a+ib}=a-ib$$

The sign of the imaginary part is reversed.

Complex Conjugate and Polar Form

The complex conjugate has the following three important properties:

(a) The conjugate of a product equals the product of the conjugates.

(b) The conjugate of a sum equals the sum of the conjugates.

(c) Multiplying any $a+bi$ by its conjugate $a-bi$ produces a real number a^2+b^2.

Polar form:

$$a+bi = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Lengths and Transposes in the Complex Case

- Complex vector. The complex vector space $\mathbb{C}^n$ contains all vectors x with n complex components:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  with components $x_j = a_j + ib_j$.

- Length Squared

$$||x||^2 = |x_1|^2 + \cdots + |x_n|^2.$$

- Inner Product

$$\bar{x}^T y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n.$$

Lengths and Transposes in the Complex Case

- A Hermitian

$$A^H = \overline{A}^T$$

- Conjugate Transpose.

$$\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}$$

Properties

Definition

The inner product of x *and* y *is* $x^H y$*. The squared length of* x *is* $||x||^2 = x^H x = |x_1|^2 + \cdots + |x_n|^2$*.*

Orthogonal vectors have $x^H y = 0$.

Proposition

Conjugating $(AB)^T = B^T A^T$ *produces* $(AB)^H = B^H A^H$*.*

Hermitian matrix

Definition

Matrices that equal their conjugate transpose are called Hermitian Matrices.

Our main goal is to establish three basic properties of Hermitian matrices. These properties apply equally well to symmetric matrices. A real symmetric matrix is certainly Hermitian.

1. If $A = A^H$, then for all complex vectors x, the number $x^H A x$ is real.
2. If $A = A^H$, every eigenvalue is real.
3. Two eigenvectors of a real symmetric matrix or a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

Real Symmetric Matrix

Now we state one of the great theorems of linear algebra:

Theorem

A real symmetric matrix can be factored into $A = Q\Lambda Q^T$. Its orthonormal eigenvectors are in the orthogonal matrix Q and its eigenvalues are in Λ.

Example 3 A is a combination of two projections.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

All symmetric matrices are combinations of one-dimensional projections–which are symmetric matrices of rank 1.

Unitary matrices

Definition

A complex matrix with orthonormal columns is called a unitary matrix.

A Hermitian matrix can be compared to a real number. A unitary matrix can be compared to a number on the unit circle.

Three properties of U:

1' : $(Ux)^H(Uy) = x^H U^H Uy = x^H y$ and lengths are preserved by U.

2' : Every eigenvalue of U has absolute value $|\lambda| = 1$.

3' : Eigenvectors corresponding to different eigenvalues are orthogonal.

Examples

Example 4 $U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ has eigenvalues e^{it} and e^{-it}.

Example 5

$$U = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdot & 1 \\ 1 & \omega & \cdot & \omega^{n-1} \\ . & . & . & . \\ 1 & \omega^{n-1} & \cdot & \omega^{(n-1)^2} \end{bmatrix} = \frac{\text{Fourier matrix}}{\sqrt{n}}.$$

Example 6 $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Final Note

Definition

Skew-Hermitian matrices satisfy $K^H = -K$.

Just as skew-symmetric matrices satisfy $K^T = -K$. Their properties follow immediately from their close link to Hermitian matrices:

Proposition

If A is Hermitian then $K = iA$ is skew-Hermitian.

- The eigenvalues of K are purely imaginary instead of purely real.
- The eigenvectors are still orthogonal, and we still have $K = U\Lambda U^H$–with a unitary U instead of a real orthogonal Q.

Real versus Complex

Real versus Complex

$\mathbf{R}^n$ (n real components)	$\leftrightarrow$	$\mathbf{C}^n$ (n complex components)
length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\|x\|^2 = \lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2$
transpose: $A_{ij}^{\mathrm{T}} = A_{ji}$	$\leftrightarrow$	Hermitian transpose: $A_{ij}^{\mathrm{H}} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
inner product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $x^{\mathrm{H}}y = \overline{x}_1y_1 + \cdots + \overline{x}_ny_n$
$(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	$\leftrightarrow$	$(Ax)^{\mathrm{H}}y = x^{\mathrm{H}}(A^{\mathrm{H}}y)$
orthogonality: $x^{\mathrm{T}}y = 0$	$\leftrightarrow$	orthogonality: $x^{\mathrm{H}}y = 0$
symmetric matrices: $A^{\mathrm{T}} = A$	$\leftrightarrow$	Hermitian matrices: $A^{\mathrm{H}} = A$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian $K^{\mathrm{H}} = -K$
orthogonal $Q^{\mathrm{T}}Q = I$ or $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary $U^{\mathrm{H}}U = I$ or $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$	$\leftrightarrow$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Ux\| = \|x\|$

The columns, rows, and eigenvectors of Q and U are orthonormal, and every $\lvert\lambda\rvert = 1$

Homework Assignment 23 and 24

5.5: 6, 7, 10, 11, 13, 16, 17, 30, 49.