

考试科目: 线性代数 A

开课单位: 数学系

考试时长: 120 分钟

题 号	1	2	3	4	5	6	7	8
分 值	15 分	25 分	10 分	16 分	10 分	6 分	16 分	12 分

本试卷共 ( 8 ) 大题, 满分 ( 110 ) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 110 in total. Write all your answers on the examination book.

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

- (1) Let  $A$  be an  $m \times n$  real matrix and  $b$  be a column vector in  $\mathbb{R}^m$ . Which of the following statements is correct? ( )
- (A) If  $Ax = b$  has infinite many solutions, then  $Ax = 0$  has a nonzero solution.
- (B) If the system  $Ax = 0$  has only zero solution, then  $Ax = b$  has one and only one solution.
- (C) If the rank of  $A$  is  $n$ , then the system  $Ax = b$  must have a solution.
- (D) If  $A$  is a square matrix (i.e.,  $m = n$ ), then the system  $Ax = b$  is consistent if and only if  $A$  is invertible.

设  $A$  为  $m \times n$  实矩阵,  $b$  是  $\mathbb{R}^m$  中的列向量. 下列陈述中哪个是正确的? ( )

- (A) 如果  $Ax = b$  有无穷多个解, 则  $Ax = 0$  有非零解.
- (B) 如果方程组  $Ax = 0$  只有零解, 则  $Ax = b$  有且仅有一个解.
- (C) 如果  $A$  的秩为  $n$ , 则方程组  $Ax = b$  必有解.
- (D) 如果  $A$  是方阵 (即  $m = n$ ), 则方程组  $Ax = b$  是相容的当且仅当  $A$  可逆.
- (2) Suppose  $A$  is an  $m \times n$  matrix,  $B$  is an  $n \times m$  matrix, and  $I$  is the  $m \times m$  identity matrix. If  $AB = I$ , then ( )
- (A) the column vectors of  $A$  are linearly independent, and the row vectors of  $B$  are linearly independent.
- (B) the column vectors of  $A$  are linearly independent, and the column vectors of  $B$  are linearly independent.
- (C) the row vectors of  $A$  are linearly independent, and the column vectors of  $B$  are linearly independent.
- (D) the row vectors of  $A$  are linearly independent, and the row vectors of  $B$  are linearly independent.

设  $A$  为  $m \times n$  型矩阵,  $B$  为  $n \times m$  型矩阵,  $I$  为  $m$  阶单位矩阵. 若  $AB = I$ , 则 ( )

- (A)  $A$  的列向量组线性无关,  $B$  的行向量组线性无关.

(B)  $A$  的列向量组线性无关,  $B$  的列向量组线性无关.

(C)  $A$  的行向量组线性无关,  $B$  的列向量组线性无关.

(D)  $A$  的行向量组线性无关,  $B$  的行向量组线性无关.

- (3) Let  $A$  be a  $3 \times 3$  matrix, and let  $B$  be the matrix formed by adding the second column of  $A$  to its first column. Suppose that after exchanging the second and third rows of  $B$ , the resulting matrix is the  $3 \times 3$  identity matrix. Let  $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ .

Then  $A = ( \quad )$

(A)  $P_1 P_2$ .

(B)  $P_1^{-1} P_2$ .

(C)  $P_2 P_1$ .

(D)  $P_2 P_1^{-1}$ .

设  $A$  为 3 阶方阵, 将  $A$  的第二列加到第一列得矩阵  $B$ . 假设交换  $B$  的第二行与第三行可以得到 3 阶单位矩阵. 记  $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ . 则  $A = ( \quad )$

(A)  $P_1 P_2$ .

(B)  $P_1^{-1} P_2$ .

(C)  $P_2 P_1$ .

(D)  $P_2 P_1^{-1}$ .

- (4) Let  $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$ , where  $t \in \mathbb{R}$ . Suppose  $\text{rank}(A) = 2$ . Then ( )

(A)  $t = -6$ .

(B)  $t = 6$ .

(C)  $t \neq 0$ .

(D)  $t$  can be any real number.

设  $A = \begin{bmatrix} -1 & 2 & 3 \\ -3 & 6 & 8 \\ 2 & -4 & t \end{bmatrix}$ , 其中  $t \in \mathbb{R}$ . 假设  $\text{rank}(A) = 2$ . 则 ( )

(A)  $t = -6$ .

(B)  $t = 6$ .

(C)  $t \neq 0$ .

(D)  $t$  可以是任意实数.

- (5) Which of the following statements is incorrect? ( )

(A) For any matrix  $A$ ,  $\text{rank}(A) = \dim C(A)$ .

- (B) If  $v_1, \dots, v_m$  are pairwise orthogonal nonzero vectors, then the vectors  $v_1, \dots, v_m$  are linear independent.
- (C) If  $A$  is an upper triangular  $n \times n$  matrix such that  $A^2 = 0$ , then  $A = 0$ .
- (D) Let  $A, B$  be  $n \times n$  matrices such that  $AB$  is invertible. Then both  $A$  and  $B$  are invertible.

下列哪个论断是错误的? ( )

- (A) 对于任意矩阵  $A$ ,  $\text{rank}(A) = \dim C(A)$ .
- (B) 如果  $v_1, \dots, v_m$  是一组两两正交的非零向量, 则向量组  $v_1, \dots, v_m$  线性无关.
- (C) 如果  $A$  是  $n \times n$  上三角矩阵且  $A^2 = 0$ , 则  $A = 0$ .
- (D) 设  $A, B$  为  $n \times n$  矩阵且  $AB$  可逆. 则  $A$  和  $B$  都可逆.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

- (1) Let  $A, B$  be invertible  $n \times n$  matrices. Then the inverse of the block matrix  $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$  is \_\_\_\_\_

设  $A, B$  均为  $n \times n$  可逆矩阵. 则分块矩阵  $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$  的逆为 \_\_\_\_\_

- (2) Suppose  $A$  is a  $3 \times 4$  matrix and  $\dim N(A) = 2$ . Then  $\dim N(A^T) =$  \_\_\_\_\_

设  $A$  为  $3 \times 4$  矩阵且  $\dim N(A) = 2$ . 则  $\dim N(A^T) =$  \_\_\_\_\_

- (3) Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ . Then  $A^{-1} =$  \_\_\_\_\_

设  $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ . 则  $A^{-1} =$  \_\_\_\_\_

- (4) Let  $u, v$  be vectors in  $\mathbb{R}^n$  such that  $\|u\| = 3$ ,  $\|v\| = 4$  and  $u^T v = -3$ .

Then  $\|2u + 3v\| =$  \_\_\_\_\_

设  $u, v$  为  $\mathbb{R}^n$  中的向量, 满足  $\|u\| = 3$ ,  $\|v\| = 4$  以及  $u^T v = -3$ . 则  $\|2u + 3v\| =$  \_\_\_\_\_

- (5) Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ .

Then the least squares solution to  $Ax = b$  is  $\hat{x} =$  \_\_\_\_\_

设  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ .

则  $Ax = b$  的最小二乘解是  $\hat{x} =$  \_\_\_\_\_

3. (10 points) Find the LU factorization of the matrix  $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ .

求矩阵  $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$  的 LU 分解.

4. (16 points) Let  $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ .

Please give a basis for each of the four fundamental subspaces  $C(A)$ ,  $N(A)$ ,  $C(A^T)$  and  $N(A^T)$ , respectively.

(16 分) 设  $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ .

请对四个基本子空间  $C(A)$ ,  $N(A)$ ,  $C(A^T)$  和  $N(A^T)$  分别给出各自的一组基.

5. (10 points) Let  $E = \{u_1, u_2, u_3\}$  and  $F = \{v_1, v_2\}$ , where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Define the linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

Find the matrix  $A$  representing  $T$  with respect to the ordered bases  $E$  and  $F$ .

(10 分) 设  $E = \{u_1, u_2, u_3\}$ ,  $F = \{v_1, v_2\}$ , 其中

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

定义线性变换  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  如下

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}.$$

求  $T$  在  $E$  和  $F$  这两组有序基下的矩阵表示  $A$ .

6. (6 points) Let  $A, B$  be  $n \times n$  matrices. Suppose  $A$  and  $B$  are both symmetric. Is  $AB$  necessarily symmetric? If yes, please give a proof. Otherwise please give a counterexample.

(6 分) 设  $A, B$  均为  $n \times n$  矩阵. 假设  $A$  和  $B$  都是对称矩阵.  $AB$  是否一定是对称矩阵? 若是, 请给出证明. 否则请给出一个反例.

7. (16 points) The following two questions are independent:

- (a) Let  $A$  be the  $2 \times 2$  matrix such that the linear transformation  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $v \mapsto Av$  rotates every vector in  $\mathbb{R}^2$  through  $60^\circ$  counter-clockwise (about the origin).

Find  $A$  and  $A^{2020}$ .

- (b) Three planes  $\Pi_1, \Pi_2, \Pi_3$  in the space  $\mathbb{R}^3$  are given by the equations

$$\Pi_1 : x + y + z = 0,$$

$$\Pi_2 : 2x - y + 4z = 0,$$

$$\Pi_3 : -x + 2y - z = 0.$$

Determine a matrix representative (in the standard basis of  $\mathbb{R}^3$ ) of a linear transformation taking the  $xy$  plane to  $\Pi_1$ , the  $yz$  plane to  $\Pi_2$  and the  $zx$  plane to  $\Pi_3$ .

(16 分) 以下两个小题是相互独立的:

- (a) 设  $A$  是  $2 \times 2$  矩阵使得线性变换  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $v \mapsto Av$  把  $\mathbb{R}^2$  中每个向量 (绕原点) 逆时针转动  $60^\circ$ .

求  $A$  和  $A^{2020}$ .

- (b) 在空间  $\mathbb{R}^3$  中由以下方程给出三个平面  $\Pi_1, \Pi_2, \Pi_3$ :

$$\Pi_1 : x + y + z = 0,$$

$$\Pi_2 : 2x - y + 4z = 0,$$

$$\Pi_3 : -x + 2y - z = 0.$$

求一个矩阵, 使它 (在  $\mathbb{R}^3$  的标准基下) 表示的线性变换将  $xy$  平面映射成  $\Pi_1$ , 将  $yz$  平面映射成  $\Pi_2$  并将  $zx$  平面映射成  $\Pi_3$ .

8. (12 points) Let  $A$  be a  $3 \times 3$  matrix such that  $\text{rank}(A) = 2$  and  $A^3 = 0$ .

- (a) Prove that  $\text{rank}(A^2) = 1$ .

- (b) Let  $\alpha_1 \in \mathbb{R}^3$  be a nonzero vector such that  $A\alpha_1 = 0$ . Prove that there exist vectors  $\alpha_2, \alpha_3$  such that  $A\alpha_2 = \alpha_1$ ,  $A^2\alpha_3 = \alpha_1$ .

- (c) For any vectors  $\alpha_2, \alpha_3$  described as above, show that  $\alpha_1, \alpha_2, \alpha_3$  are linearly independent.

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(12 分) 设  $A$  是  $3 \times 3$  矩阵, 它满足  $\text{rank}(A) = 2$  及  $A^3 = 0$ .

(a) 证明  $\text{rank}(A^2) = 1$ .

(b) 设  $\alpha_1 \in \mathbb{R}^3$  是满足  $A\alpha_1 = 0$  的非零向量. 证明: 存在向量  $\alpha_2, \alpha_3$  使得  $A\alpha_2 = \alpha_1, A^2\alpha_3 = \alpha_1$ .

(c) 证明: 对于任意满足上述条件的向量  $\alpha_2, \alpha_3$ , 向量组  $\alpha_1, \alpha_2, \alpha_3$  线性无关.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)