南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数 A　　开课单位: 数学系
考试时长: 120 分钟　　命题教师: 线性代数教学团队

题号	1	2	3	4	5	6	7	8
分值	15 分	25 分	12 分	10 分	12 分	12 分	8 分	6 分

本试卷共 (8) 大题, 满分 (100) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 100 in total. Write all your answers on the examination book.

本套试卷为 B 卷 Version B

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ and let $\alpha_1, \alpha_2, \alpha_3$ be linearly independent column vectors in $\mathbb{R}^3$. Then the rank of the vector system $A\alpha_1, A\alpha_2, A\alpha_3$ (　　)

(A) must be 1.

(B) must be 2.

(C) must be 3.

(D) can be 1 or 2.

设 $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 中线性无关的向量组. 则向量组 $A\alpha_1, A\alpha_2, A\alpha_3$ 的秩 (　　)

(A) 一定是 1.

(B) 一定是 2.

(C) 一定是 3.

(D) 可能是 1 也可能是 2.

(2) Let I_n be the identity matrix of order n and let α be a column vector of length 1 in $\mathbb{R}^n$. Then (　　)

(A) $I_n - \alpha\alpha^T$ is not invertible.

(B) $I_n + \alpha\alpha^T$ is not invertible.

(C) $I_n + 2\alpha\alpha^T$ is not invertible.

(D) $I_n - 2\alpha\alpha^T$ is not invertible.

设 I_n 为 n 阶单位矩阵, α 是 $\mathbb{R}^n$ 中长度为 1 的列向量. 则 (　　)

(A) $I_n - \alpha\alpha^T$ 不可逆.

(B) $I_n + \alpha\alpha^T$ 不可逆.

(C) $I_n + 2\alpha\alpha^T$ 不可逆.

(D) $I_n - 2\alpha\alpha^T$ 不可逆.

(3) Let $M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. Which of the following matrices is congruent to M? (　　)

(A) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(B) $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(C) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(D) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

设 $M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. 下列哪个矩阵与 M 相合 (也称合同)? (　　)

(A) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(B) $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

(C) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(D) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

(4) Let $\alpha_1, \alpha_2, \alpha_3$ be column vectors in $\mathbb{R}^3$ such that the matrix $P = (\alpha_1, \alpha_2, \alpha_3)$ is invertible. Suppose A is a 3×3 matrix such that $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then $A(\alpha_1 + \alpha_2 + \alpha_3) =$ (　　)

(A) $\alpha_1 + \alpha_2$.

(B) $\alpha_2 + 3\alpha_3$.

(C) $\alpha_2 + \alpha_3$.

(D) $\alpha_1 + 2\alpha_3$.

设 $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 中的列向量, 使得矩阵 $P=(\alpha_1, \alpha_2, \alpha_3)$ 可逆. 假设 A 为 3×3 矩阵, 满足 $P^{-1}AP=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. 则 $A(\alpha_1+\alpha_2+\alpha_3)=(\quad)$

(A) $\alpha_1+\alpha_2$.

(B) $\alpha_2+2\alpha_3$.

(C) $\alpha_2+\alpha_3$.

(D) $\alpha_1+2\alpha_3$.

(5) Which of the following statements is true? (　　)

(A) Let A be an $m\times n$ complex matrix and $B=A^HA$. (Here $A^H=\overline{A}^T$ denotes the conjugate transpose of A.) Then the matrix $B+iI_n$ is invertible.

(B) If A is a real square matrix of order n and $x^TAx=0$ for all $x\in\mathbb{R}^n$, then A must be the zero matrix.

(C) Suppose A and B are invertible square matrices. If A and B are similar to each other, then $A+A^T$ is similar to $B+B^T$.

(D) If A is an upper triangular (square) matrix and A is similar to a diagonal matrix, then A must be a diagonal matrix.

下列哪个论断是正确的? (　　)

(A) 设 A 为 $m\times n$ 复矩阵, $B=A^HA$. (这里 $A^H=\overline{A}^T$ 表示 A 的共轭转置.) 则矩阵 $B+iI_n$ 可逆.

(B) 如果 A 是 n 阶实方阵, 且对任意 $x\in\mathbb{R}^n$ 均有 $x^TAx=0$, 则 A 一定是零矩阵.

(C) 假设 A 和 B 都是可逆方阵. 如果 A 和 B 相似, 则 $A+A^T$ 和 $B+B^T$ 相似.

(D) 若 A 是上三角(方)阵并且 A 相似于某个对角阵, 则 A 一定是对角阵.

Solution. (1) B (2) A (3) C (4) B (5) D.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let A be a 5×4 matrix of rank 3. Then the dimension of its left null space $N(A^T)$ is __________

设 A 为秩等于 3 的 5×4 矩阵. 则它的左零空间 $N(A^T)$ 维数是 __________

(2) Let A be a square matrix of order 3 and let $\alpha_1, \alpha_2, \alpha_3$ be linearly independent vectors in $\mathbb{R}^3$. Suppose $A\alpha_1=2\alpha_1+\alpha_2+\alpha_3$, $A\alpha_2=\alpha_2+2\alpha_3$ and $A\alpha_3=-\alpha_2+\alpha_3$. Then the real eigenvalues of A are __________

设 A 为 3 阶方阵, $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 中的线性无关向量组. 假设 $A\alpha_1=2\alpha_1+\alpha_2+\alpha_3$, $A\alpha_2=\alpha_2+2\alpha_3$ 且 $A\alpha_3=-\alpha_2+\alpha_3$. 则 A 的实特征值为 __________

(3) Let A, B be square matrices of order n. Suppose $|A|=3$, $|B|=2$ and $|A^{-1}+B|=2$. Then $|A+B^{-1}|=$__________

设 A, B 为 n 阶方阵. 假设 $|A|=3$, $|B|=2$ 而 $|A^{-1}+B|=2$. 则 $|A+B^{-1}|=$__________

(4) Let ξ_1, ξ_2, ξ_3 be a basis of $\mathbb{R}^3$, and let $\eta_1 = \xi_1$, $\eta_2 = \xi_1 + \xi_2$, $\eta_3 = \xi_3 - \xi_1$. Let $\mathrm{Id} : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity map.

Then the matrix A representing Id in the basis η_1, η_2, η_3 is ____________

设 ξ_1, ξ_2, ξ_3 为 $\mathbb{R}^3$ 的一组基, $\eta_1 = \xi_1$, $\eta_2 = \xi_1 + \xi_2$, $\eta_3 = \xi_3 - \xi_1$. 记 $\mathrm{Id} : \mathbb{R}^3 \to \mathbb{R}^3$ 为恒等映射. 则 Id 在 η_1, η_2, η_3 这组基下的矩阵表示是 $A =$____________

(5) Let $a \in \mathbb{R}$ and let $C \subseteq \mathbb{R}^2$ be the zero locus of the equation $3x^2 - 2axy + 3y^2 - 1 = 0$. Determine for which values of a the curve C is an ellipse. (A circle is also considered as an ellipse.) ____________

设 $a \in \mathbb{R}$, 记 $C \subseteq \mathbb{R}^2$ 为方程 $3x^2 - 2axy + 3y^2 - 1 = 0$ 的零点集. 求 a 的取值范围使得曲线 C 是个椭圆. (注意圆也认为是椭圆.) ____________

Solution. (1) 2 (2) 1, 2, 2 (3) 3 (4) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (5) $-3 < a < 3$.

3. (12 points)

(a) Find an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}.$$

(b) Write A as QR, where Q has orthonormal columns and R is upper triangular.

(12 分)

(a) 找出下列矩阵的列空间的一组标准正交 (也称规范正交) 基:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}.$$

(b) 将 A 分解成乘积 QR 的形式, 其中 Q 的列是标准正交 (也称规范正交) 的向量, R 是上三角矩阵.

Solution.

(a)

$$\begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \\ \frac{2}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}.$$

(b)

$$Q = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix}, \; R = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

4. (10 points) Let x, $y \in \mathbb{R}$, $n \geq 2$ and consider the following determinant of order n:

$$D_n(x, y) = \begin{vmatrix} x+y & xy & 0 & \dots & 0 & 0 \\ 1 & x+y & xy & \dots & 0 & 0 \\ 0 & 1 & x+y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & x+y \end{vmatrix}$$

For $n = 2$, we consider $D_2(x, y)$ as $\begin{vmatrix} x+y & xy \\ 1 & x+y \end{vmatrix}$.

(a) Find a recurrence relation relating $D_n(x, y)$ to $D_{n-1}(x, y)$ and $D_{n-2}(x, y)$ for $n \geq 4$.

(b) Compute the determinant $D_n(x, y)$ for all $n \geq 2$.

(10 分) 设 x, $y \in \mathbb{R}$, $n \geq 2$. 考虑以下 n 阶行列式:

$$D_n(x, y) = \begin{vmatrix} x+y & xy & 0 & \dots & 0 & 0 \\ 1 & x+y & xy & \dots & 0 & 0 \\ 0 & 1 & x+y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & x+y \end{vmatrix}$$

当 $n = 2$ 时, 我们认为 $D_2(x, y)$ 是 $\begin{vmatrix} x+y & xy \\ 1 & x+y \end{vmatrix}$.

(a) 在 $n \geq 4$ 时找出能将 $D_n(x, y)$ 和 $D_{n-1}(x, y), D_{n-2}(x, y)$ 建立联系的一个递推关系式.

(b) 对任意 $n \geq 2$ 计算行列式 $D_n(x, y)$.

Solution.

(a)

$$D_n = (x+y)D_{n-1} - xyD_{n-2}.$$

(b) $D_n(x, y) = x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + x^2y^{n-2} + xy^{n-1} + y^n$.

5. (12 points) Let a, $b \in \mathbb{R}$ and put

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{bmatrix}.$$

Suppose that $p = (1, 1, -1)^T$ is an eigenvector of A.

(a) Find the values of a, b and find the eigenvalue λ corresponding to the eigenvector p.

(b) Is the matrix A diagonalizable? Please explain your answer.

(12 分) 设 $a, b \in \mathbb{R}$, 令

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{bmatrix}.$$

假设 $p = (1, 1, -1)^T$ 是 A 的一个特征向量.

(a) 求 a, b 的值并找出特征向量 p 对应的特征值 λ.

(b) 矩阵 A 是否可对角化? 请解释你的答案.

Solution.

(a) $a = -3$, $b = 0$, $\lambda = -1$.

(b) The matrix is not diagonalizable, $\lambda = -1$ has multiplicity 3, but only has an eigenspace of dimension 1.

6. (12 points) Consider the ternary quadratic form

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3)^2 + (x_2 + x_3)^2 + (x_1 + ax_3)^2$$

where $a \in \mathbb{R}$ is a parameter.

(a) What are the possible values of a if the quadratic form f is positive definite?

(b) What are the possible values of a if the equation $f(x_1, x_2, x_3) = 0$ has infinitely many solutions?

(c) Let $y = (y_1, y_2, y_3)^T$ be a new system of variables. Suppose that $f(x_1, x_2, x_3) = 0$ has infinitely many solutions.
Find an invertible linear transformation $y = Px$, where $x = (x_1, x_2, x_3)^T$, such that in the variables y_1, y_2, y_3 the quadratic form f has a diagonal form.

(12 分) 考虑三元二次型

$$f(x_1, x_2, x_3) = (x_1 - x_2 + x_3)^2 + (x_2 + x_3)^2 + (x_1 + ax_3)^2$$

其中 $a \in \mathbb{R}$ 为参数.

(a) 如果 f 是正定的, a 的取值范围是什么?

(b) 如果方程 $f(x_1, x_2, x_3) = 0$ 有无穷多个解, a 的取值可能是什么?

(c) 设 $y = (y_1, y_2, y_3)^T$ 为一组新的变元. 假设 $f(x_1, x_2, x_3) = 0$ 有无穷多个解.
找出一个可逆线性变换 $y = Px$, 其中 $x = (x_1, x_2, x_3)^T$, 使得在 y_1, y_2, y_3 这组变元下二次型 f 可以表示为对角形式.

Solution.

(a) $a \neq -2$.

(b) $a = 2$.

(c)

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

7. (8 points) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(a) Find all the singular values of A.

(b) Find the singular value decomposition of A, in other words, find two orthogonal matrices U and V (of suitable size) such that $A = U\Sigma V^T$.

(8 分) 令 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(a) 求 A 的所有奇异值.

(b) 求 A 的奇异值分解. 即, 找出两个 (适当大小的) 正交矩阵 U 和 V 使得 $A = U\Sigma V^T$.

Solution.

(a) The singular values of A are 2, $\sqrt{2}$.

(b)

$$U = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix}, V = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

8. (6 points) Find a real 4×4 orthogonal matrix A such that A has no real eigenvalues but both A^2 and A^3 have real eigenvalues. Please explain why the matrix you give has the required properties.

(6 分) 给出一个 4×4 实正交矩阵 A 使得 A 没有实特征值, 而 A^2 和 A^3 都有实特征值. 请解释为什么你给出的矩阵满足要求.

Solution.

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has no real eigenvalues.

However,

$$A^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

have real eigenvalues.