



南方科技大学  
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数 A

开课单位: 数学系

考试时长: 120 分钟

命题教师: 线性代数教学团队

题号	1	2	3	4	5	6	7	8
分值	15 分	25 分	10 分	10 分	10 分	10 分	10 分	10 分

本试卷共 (8) 大题, 满分 (100) 分. 请将所有答案写在答题本上.

This exam includes 8 questions and the score is 100 in total. **Write all your answers on the examination book.**

本套试卷为 A 卷 Version A

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Which one of the following statements must be true? ( )

- (A) If  $A$  and  $B$  are  $m \times n$  matrices and  $Ax = 0$  has the same solution set as  $Bx = 0$ , then  $A$  and  $B$  have the same column space.
- (B) If  $A$  is an  $n \times n$  real symmetric positive definite matrix, then all the square submatrices of  $A$  have positive determinants.
- (C) If real symmetric matrices  $A$  and  $B$  are congruent, then they are similar.
- (D) If  $AB = 0$  and  $A$  and  $B$  are not zero matrices, then  $A$  has linearly dependent columns and  $B$  has linearly dependent rows.

下列陈述一定正确的是? ( )

- (A) 若  $A$  和  $B$  为  $m \times n$  矩阵, 且  $Ax = 0$  与  $Bx = 0$  同解, 则  $A$  和  $B$  具有相同的列空间.
- (B) 若实对称矩阵  $A$  正定, 则它的所有正方形子矩阵的行列式都为正.
- (C) 若实对称矩阵  $A$  和  $B$  相合 (也称合同), 则它们相似.
- (D) 设  $A, B$  为满足  $AB = 0$  的两个非零矩阵, 则  $A$  的列向量组线性相关,  $B$  的行向量组线性相关.

(2) The plane curve defined by the equation  $2x^2 - 8xy + 2y^2 = 1$  is ( )

- (A) an ellipse.
- (B) a hyperbola.
- (C) a parabola.
- (D) a union of two intersecting lines.

由方程  $2x^2 - 8xy + 2y^2 = 1$  定义的平面曲线是 ( )

- (A) 椭圆.
- (B) 双曲线.
- (C) 抛物线.
- (D) 一对相交直线.

(3) Let  $A$  be an  $n \times n$  real matrix. Suppose that for all column vectors  $x \in \mathbb{R}^n$  we have  $x^T Ax = 0$ . Then ( )

- (A) The determinant  $|A|$  of  $A$  is 0.
- (B)  $A = 0$ .
- (C) The trace,  $\text{trace}(A)$ , of  $A$  is 0.
- (D) The only eigenvalue of  $A$  in  $\mathbb{C}$  is 0.

设  $A$  为  $n \times n$  实矩阵. 假设对任意列向量  $x \in \mathbb{R}^n$  都有  $x^T Ax = 0$ . 则 ( )

- (A)  $A$  的行列式  $|A|$  为 0.
- (B)  $A = 0$ .
- (C)  $A$  的迹,  $\text{trace}(A)$ , 为 0.
- (D)  $A$  在  $\mathbb{C}$  中唯一的特征值是 0.

(4) Let  $n \geq 2$ . Let  $A$  be an  $n \times n$  real matrix of rank 1. Then ( )

- (A)  $A$  is necessarily diagonalizable.
- (B)  $A$  has only one nonzero column.
- (C) The trace,  $\text{trace}(A)$ , of  $A$  is nonzero.
- (D)  $A$  has at least  $n - 1$  linearly independent eigenvectors.

设  $n \geq 2$ ,  $A$  是秩为 1 的  $n \times n$  实矩阵. 则 ( )

- (A)  $A$  一定可以对角化.
- (B)  $A$  只有一列是非零列.
- (C)  $A$  的迹,  $\text{trace}(A)$ , 不为零.
- (D)  $A$  有至少  $n - 1$  个线性无关的特征向量.

(5) Let  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . The matrices  $A$  and  $B$  are ( )

- (A) congruent and similar.
- (B) congruent but not similar.
- (C) similar but not congruent.
- (D) neither similar nor congruent.

设  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . 则  $A$  与  $B$  ( )

- (A) 合同且相似. (这里的“合同”也称“相合”.)

- (B) 合同但不相似.  
 (C) 相似但不合同.  
 (D) 既不相似也不合同.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Evaluate the determinant: 
$$\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = \underline{\hspace{2cm}}.$$

计算行列式: 
$$\begin{vmatrix} 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & n-1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 2 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = \underline{\hspace{2cm}}.$$

(2) Let  $\mathbb{R}^{2 \times 2}$  be the real vector space of  $2 \times 2$  real matrices. Let  $V$  be the subspace of  $\mathbb{R}^{2 \times 2}$  spanned by the 4 matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Then  $\dim V = \underline{\hspace{2cm}}.$

设  $\mathbb{R}^{2 \times 2}$  为所有  $2 \times 2$  实矩阵构成的实向量空间. 令  $V$  为以下 4 个矩阵在  $\mathbb{R}^{2 \times 2}$  中张成 (也称“生成”) 的子空间:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} 5 & 1 \\ 1 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

则  $\dim V = \underline{\hspace{2cm}}.$

(3) Let  $A$  be a  $2 \times 2$  matrix and suppose  $\alpha_1, \alpha_2$  are 2-dimensional linearly independent column vectors such that  $A\alpha_1 = 0, A\alpha_2 = 4\alpha_1 + 2\alpha_2$ . Then the eigenvalues of  $A$  are  $\underline{\hspace{2cm}}.$

设  $A$  是 2 阶方阵,  $\alpha_1, \alpha_2$  是线性无关的二维列向量, 满足  $A\alpha_1 = 0, A\alpha_2 = 4\alpha_1 + 2\alpha_2$ . 则  $A$  的所有特征值为  $\underline{\hspace{2cm}}.$

(4) Suppose that the quadratic form  $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$  can be transformed by an orthogonal transformation  $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$  to  $y_2^2 + 4y_3^2$ . Then  $a = \underline{\hspace{2cm}}, b = \underline{\hspace{2cm}}.$

假定二次型  $f(x_1, x_2, x_3) = x_1^2 + ax_2^2 + x_3^2 + 2bx_1x_2 + 2x_1x_3 + 2x_2x_3$  可由正交变换  $(x_1, x_2, x_3)^T = Q(y_1, y_2, y_3)^T$  化为  $y_2^2 + 4y_3^2$ . 则  $a = \underline{\hspace{2cm}}, b = \underline{\hspace{2cm}}.$

(5) Let  $A$  be a  $3 \times 3$  matrix with eigenvalues  $-1, 0, 1$ . Suppose that

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors belonging to the eigenvalues  $-1, 0, 1$  respectively. Then  $A^{2021} =$ \_\_\_\_\_.

假设  $A$  为  $3 \times 3$  矩阵, 其特征值为  $-1, 0, 1$ . 假设

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

分别为属于特征值  $-1, 0, 1$  的特征向量. 则  $A^{2021} =$ \_\_\_\_\_.

3. (10 points 本题共 10 分) Suppose  $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & -1 \\ 0 & 1 & a \end{bmatrix}$  and  $A^3 = 0$ .

- (a) Find  $|A|$ .
- (b) Find the value of  $a$ .
- (c) Show that  $I - A$  is invertible. (Here  $I$  denotes the  $3 \times 3$  identity matrix.)
- (d) Find an invertible matrix  $X$  of order 3 such that  $(I - A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$ .  
(Hint:  $X^{-1} - X^{-1}A = X^{-1}(I - A)$ .)

设  $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & -1 \\ 0 & 1 & a \end{bmatrix}$ , 且  $A^3 = 0$ .

- (a) 求  $A$  的行列式  $|A|$ .
- (b) 求  $a$  的值.
- (c) 证明  $I - A$  可逆. (这里  $I$  表示 3 阶单位矩阵.)
- (d) 求一个 3 阶可逆矩阵  $X$  使得  $(I - A)^{-1}X = (X^{-1} - X^{-1}A)^{-1}A^2 + I$ .  
(提示:  $X^{-1} - X^{-1}A = X^{-1}(I - A)$ .)

4. (10 points 本题共 10 分) Let

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- (a) Find the determinant and the trace of  $H$ .
- (b) Find all the singular values of  $H$ .
- (c) Find a real number  $\alpha$  such that  $\text{rank}(\alpha I_4 - H)$  is the smallest possible.

令

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

- (a) 求  $H$  的行列式和迹.
- (b) 求  $H$  的所有奇异值.
- (c) 求一个实数  $\alpha$  使得  $\text{rank}(\alpha I_4 - H)$  达到最小可能的值.

5. (10 points 本题共 10 分) Suppose that the complex matrix  $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$  is a Hermitian matrix.

- (a) Find the value of  $\alpha$ .
- (b) Find all the complex eigenvalues of  $A$ .
- (c) Find a unitary matrix  $U$  such that  $U^{-1}AU$  is a diagonal matrix.
- (d) Let  $B = A + A^T$ , where  $A^T$  denotes the transpose of  $A$ . Show that  $B$  is a real symmetric matrix, and decide whether  $B$  is positive definite.

假设复矩阵  $A = \begin{bmatrix} 1 & 1+i \\ \alpha & 2 \end{bmatrix}$  是个埃尔米特矩阵.

- (a) 求  $\alpha$  的值.
- (b) 求  $A$  的所有复特征值.
- (c) 求一个酉矩阵  $U$  使得  $U^{-1}AU$  为对角阵.
- (d) 令  $B = A + A^T$ , 其中  $A^T$  表示  $A$  的转置. 证明  $B$  是实对称阵, 并确定  $B$  是否正定.

6. (10 points 本题共 10 分) Let  $A$  be a  $3 \times 3$  real matrix whose second and third columns are  $(1, 0, 0)^T$  and  $(2, 1, 0)^T$  respectively. Suppose that the QR factorization of  $A$  takes the form  $A = QR$  with

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) Find the values of  $x, y, z$  and  $a, b, c$ .
- (b) Find the determinant  $|A|$ .

设  $A$  为 3 阶实方阵, 其第二列和第三列分别为  $(1, 0, 0)^T$  和  $(2, 1, 0)^T$ . 假设  $A$  的 QR 分解  $A = QR$  满足

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & x \\ 0 & 0 & y \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & z \end{bmatrix} \quad \text{及} \quad R = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & \frac{1}{\sqrt{2}} & c \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a) 求  $x, y, z$  和  $a, b, c$  的值.  
 (b) 求行列式  $|A|$ .

7. (10 points 本题共 10 分) Let  $A$  be an  $n \times n$  real symmetric positive definite matrix.

- (a) Show that there exists an  $n \times n$  invertible matrix  $R$  such that  $A = R^T R$ .  
 (b) Show that for all column vectors  $x, y \in \mathbb{R}^n$ ,

$$(x^T A y)^2 \leq (x^T A x)(y^T A y).$$

设  $A$  为  $n$  阶正定实对称矩阵.

- (a) 证明: 存在  $n$  阶可逆阵  $R$  使得  $A = R^T R$ .  
 (b) 证明: 对任意列向量  $x, y \in \mathbb{R}^n$  都有

$$(x^T A y)^2 \leq (x^T A x)(y^T A y).$$

8. (10 points 本题共 10 分) Let  $A, B$  be two  $n \times n$  real symmetric matrices.

- (a) Suppose  $A$  is positive definite. Show that there exists an invertible  $n \times n$  matrix  $C$  such that  $C^T A C = I_n$  and  $C^T B C$  is diagonal. (Here  $I_n$  denotes the  $n \times n$  identity matrix).  
 (b) Suppose  $B - A$  and  $A$  are positive semidefinite matrices. Show that:  $\det B \geq \det A$ .

设  $A, B$  都为  $n$  阶实对称矩阵.

- (a) 设  $A$  为正定实对称阵. 证明: 存在  $n$  阶可逆实矩阵  $C$  使得  $C^T A C = I_n$  且  $C^T B C$  是对角阵. (这里  $I_n$  为  $n$  阶单位阵).  
 (b) 设  $B - A$  和  $A$  都是半正定矩阵. 证明:  $\det B \geq \det A$ .