MIT 18.06 Exam 1 Solutions, Fall 2017
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Problem 1:

You are given three vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$. Your goal is to find a *linear combination of these three vectors* (that is, multiply them by some numbers x_1, x_2, x_3 and add them) to give the vector $\vec{b} = \begin{pmatrix} 2 \\ -2 \\ 12 \end{pmatrix}$.

(a) Write the equation in matrix form.

(b) Solve it to find the correct linear combination (x_1, x_2, x_3) of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(c) Change *one number* in $\vec{v}_3$ to make the problem have *no* solution for *most* vectors $\vec{b}$, but give a new vector $\vec{b}'$ for which there *is* still a solution. This new $\vec{b}'$ is in the ________________ space of the matrix_________________.

(There are multiple correct answers for your new $\vec{v}_3$ and your new $\vec{b}'$.)

Solution:

(a) **Write the equation in matrix form.**

Recall that if $A = [c_1|c_2|c_3]$ is a matrix with three columns c_1, c_2, c_3 then

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1c_1 + x_2c_2 + x_3c_3.$$

So, finding numbers x_1, x_2, x_3 such that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{b}$ is the same as solving the equation $Ax = \vec{b}$ for the unknown vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, where $A = [\vec{v_1}|\vec{v_2}|\vec{v_3}] = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 2 \\ 3 & 5 & 4 \end{pmatrix}$.

(b) **Solve it to find the correct linear combination (x_1, x_2, x_3) of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.**

We can solve the matrix equation by performing elimination on the augmented matrix $(A|b)$ to make it upper triangular:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -2 & 0 & 2 & -2 \\ 3 & 5 & 4 & 12 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 4 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right).$$

Then backsubstitution yields $2x_3 = 4 \implies x_3 = 2$, $2x_2 + 2x_3 = 2 = 2x_2 + 4 \implies x_2 = -1$, $x_1 + x_2 = 2 = x_1 - 1 \implies x_1 = 3$, or $\boxed{x = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}}$.

(c) **Change *one number* in $\vec{v}_3$ to make the problem have *no* solution for *most* vectors $\vec{b}$, but give a new vector $\vec{b}'$ for which there *is* still a solution. This new $\vec{b}'$ is in the __________________ space of the matrix__________________.**

To not have solutions for most right-hand sides, the matrix needs to be singular. We just need to change $\vec{v}_3$ so that we get a 0 instead of a 2 for the last step of Gaussian elimination, to eliminate the third pivot, so we just need to subtract 2 from the third component, and hence $\boxed{\vec{v}_3' = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}}$.

(Equivalently, this makes $\vec{v}_3' = \vec{v}_2 - \vec{v}_1$, so the column space becomes two-dimensional.) Now we have a new matrix $A' = [\vec{v}_1|\vec{v_2}|\vec{v_3}']$, and to have a solution to $A'\vec{x} = \vec{b}'$ we just need $\vec{b}'$ in the **column space** of A'. We can just pick any $\vec{x}$ we want and let $\vec{b}' = A'\vec{x}$, or equivalently we can pick $\vec{b}'$ to be any linear combination of the columns of A'. For example, we can pick $\vec{b}'$ to be *one of the columns* of A', or the sum of two columns, or even $\vec{b}' = \vec{0}$.

Problem 2:

Suppose A is some 3×3 matrix. We will transform this into a *new* 3×3 matrix B by doing operations on the rows or columns of A as follows. For each part, (i) **explain how to express B as B=AE or B=EA (say which!) for some matrix E (write down E!)**. Also, (ii) say **whether E is invertible** (that is, whether the transformation is reversible). (You don't need to compute E^{-1}, just say whether the inverse exists!)

(a) **Swap the first and second rows of A.**

Remember that left-multiplications do row operations and that right-multiplications do column operations

(i) Swapping the first and second rows is an elementary row operation, given by left-multiplication by the matrix

$$\boxed{E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

In this case we therefore have $\boxed{B = EA}$ with E as above. (ii) $\boxed{E \text{ is invertible}}$ (in fact $E^2 = I$), since you can undo a row swap by swapping again.

(b) **Keep the first row the same, *then* add the second row to the third row, *then* replace the second row with the sum of the first and third rows.**

(i) We are again performing row operations, so we'll have $\boxed{B = EA}$. To find E, we can simply apply the operations to the identity matrix I. Keeping the first row the same doesn't change I. Adding the second row to the third row yields the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Replacing the second row with the sum of the first and third rows gives the final answer:

$$\boxed{E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}$$

(ii) $\boxed{E \text{ is not invertible}}$; its columns are linearly independent. In fact, the last two columns are equal. This means that the vector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is in the nullspace of E. But the nullspace of an invertible matrix must include only the zero vector. Alternatively, we could just do elimination:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

to see that we only have two pivots, hence E is singular.

(c) **Subtract the first *column* from the second and third columns.**

(i) We are now operating on columns, so we'll have $\boxed{B = AE}$. To compute E, as usual we can just apply the operation in question to the identity matrix. Subtracting the first column from the second and third columns gives the matrix:

$$\boxed{E = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

(ii) $\boxed{E \text{ is invertible}}$, because the corresponding column operation is invertible: just add the first column back to the other two! In fact, from this we can see that the inverse of E is

$$E^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3:

Suppose you have a 3×3 matrix A satisfying $A = B^{-1}UL$ where

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ -2 & 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

(a) **The *second* column c of the matrix A^{-1} satisfies $Ac = b$ for what right-hand side b?**

Recall that if M is any 3×3 matrix and if $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then Me_2 is the second column of M. So $c = A^{-1}e_2$. We want to know what vector Ac is. Using our fomrula for c, we get

$$Ac = A(A^{-1}e_2) = (AA^{-1})e_2 = Ie_2 = e_2.$$

So $\boxed{b = e_2}$.

(b) **The *second* column c of the matrix A^{-1} also satisfies $ULc = d$ for what right-hand side d?**

We're given that $A = B^{-1}UL$, and from part (1) we have that $Ac = e_2$. Putting these together, we have $B^{-1}ULc = e_2$. Multipying both sides by B on the left, we then get $ULc = Be_2$. But Be_2 is just the second column of B, so we get:

$$ULc = Be_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

So $\boxed{d = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}$.

(c) **Compute the second column c of the matrix A^{-1}. (Important: you don't *have* to compute the inverse of any matrix!)**

By (2), to get $c$we can just solve the system $ULc = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ for c. First, we can solve for Lc by backsubstitution in the augmented matrix $(U|d)$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

so $Lc = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$. We can now solve this lower-triangular sysem for $c$by forward-substitution:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 3 & 1 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -10 \\ 0 & -2 & 1 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -23 \end{array}\right)$$

So $\boxed{c = \begin{pmatrix} 3 \\ -10 \\ -23 \end{pmatrix}}$.

Common mistake: Many students think that the inverse of a matrix like U or L can be found just by flipping the signs of the off-diagonal entries. It is true that sometimes inverses have a simple form for matrices of special types, and it *is* true that you can just flip the signs to invert the lower-triangular matrix describing a *single* elimination step, but it is *not* true that you can invert a general U or L matrix this way (even if they have 1's on the diagonal).

Problem 4 (20 points):

In class and homework, we showed that multiplying two arbitrary $m \times m$ matrices, doing Gaussian elimination, or inverting an $m \times m$ matrix requires $\sim m^3$ arithmetic operations (that is, roughly proportional to m^3 for large m). We found that adding matrices, multiplying an $m \times m$ matrix by a vector, or solving an $m \times m$ upper/lower triangular system of equations requires $\sim m^2$ operations.

Suppose that A is an $m \times m$ matrix, x is an m-component *column* vector (an $m \times 1$ matrix), and r is an m-component *row* vector (a $1 \times m$ matrix).

- You could compute the same result $xrAx$ by doing the multiplications in different orders, for example $x(r(Ax))$ (multiplying terms from *right to left*) or $((xr)A)x$ (multiplying from *left to right*). **Give the rough number of operations** (say whether proportional to $\sim m$, $\sim m^2$, $\sim m^3$, or $\sim m^4$) **for these two different orders (right to left and left to right).** Which one is the fastest for $m = 1000$?

Solution:

- Let's look at $x(r(Ax))$ first. A is a matrix and x is a column vector, so computing Ax takes $\sim m^2$ operations. Then r is a row vector and Ax is a column vector, so computing $r(Ax)$ is a dot product, just $\sim m$ operations. Finally $r(Ax)$ is a 1×1 matrix, so computing $x(r(Ax))$ takes $\sim m$ operations (multiplying each entry by a number). The largest power of m we saw was m^2, so the whole procedure takes $\boxed{\sim m^2 \text{ operations}}$.

- Now let's look at $((xr)A)x$. Note that x is a column vector and r is a row vector, so computing xr takes $\sim m^2$ operations — that is, the result is an $m \times m$ matrix, and each entry involves a different multiplication, hence exactly m^2 multiplications are needed. Then xr is a $m \times m$ matrix, and so is A, so computing $(xr)A$ is a matrix–matrix product that takes $\sim m^3$ operations (as given above). Finally result $(xr)A$ is then another $m \times m$ matrix, so computing the product $((xr)A)x$ is a matrix–vector multiply that takes another $\sim m^2$ operations (as given above). The largest power of m we saw was m^3, so the whole procedure takes $\boxed{\sim m^3 \text{ operations}}$.

When $m = 1000$, the first option (right to left) takes $\sim 10^6$ operations, while the second option (left to right) takes $\sim 10^9$ operations. So definitely right to left is better.

- **Remark:** More generally, if X is an $m \times n$ matrix and Y is an $n \times p$ matrix, then computing XY involves computing the mp entries of XY, and each entry involves computing a dot product of two length-m vectors ($\sim m$ operations, actually $\approx 2m$), so computing XY takes $\sim mnp$ operations (actually $\approx 2mnp$). This gives the square matrix–matrix and square matrix–vector results given in the problem, and also tells us that computing xr takes $\sim m^2$ operations, etc.