

Exercises Fall 2019

(1) True or false. No need to justify.

- (a) The diagonal entries of a positive definite matrix are positive. ()
- (b) If A is similar to B , then A^2 is similar to B^2 . ()
- (c) If A and B are diagonalizable, so is AB . ()
- (d) If A is a 3×3 skew-symmetric ($A^T = -A$), then $|A| = 0$. ()
- (e) If A is negative definite, then all the upper left submatrices A_k of A have negative determinants. ()
- (f) Let A be an $n \times n$ matrix, then the number of nonzero eigenvalues of A (counting the multiplicities) is equal to the rank of A . ()

(2) Fill in the blanks.

- (a) Let A be a 3×3 real matrix whose column vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. If $A\alpha_1 = \alpha_1 + \alpha_2, A\alpha_2 = \alpha_2 + \alpha_3, A\alpha_3 = \alpha_3 + \alpha_1$, then $|A| = \underline{\hspace{2cm}}$.
- (b) If $A \in \mathbb{R}^{3 \times 3}$ has eigenvalues $0, 1, 2$, then the eigenvalues of $A(A - I)(A - 2I)$ are $\underline{\hspace{2cm}}$.
- (c) A box has edges from $(0, 0, 0)$ to $(3, 1, 1), (1, 3, 1), (1, 1, 3)$, then its volume is $\underline{\hspace{2cm}}$.

(3) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

- (i) Find all the eigenvalues of A and their associated eigenvectors.
- (ii) Is A diagonalizable? Explain why.

(4) Let

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}.$$

- (i) Verify that A is Hermitian.
- (ii) Find a unitary matrix U that diagonalizes A .

(5) Let

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- (i) Find all the singular values of A .
- (ii) Find the singular value decomposition of A , in other words, find orthogonal matrices U and V , such that $A = U\Sigma V^T$.

(6) Let A be

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

- (i) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.
- (ii) Find A^k , where k is positive integer.

(7) Consider the quadratic form

$$f(x_1, x_2, x_3, x_4) = t(x_1^2 + x_2^2 + x_3^2) + x_4^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$$

- (i) Find A , such that $f(x_1, x_2, x_3, x_4) = x^T Ax$.
- (ii) For which t does $f(x_1, x_2, x_3, x_4)$ positive definite?

(8) Let N be a normal matrix ($N^H N = N N^H$).

- (i) Show that $\|Nx\| = \|N^H x\|$ for every vector x .
- (ii) Deduce that the i th row of N has the same length as the i th column.
- (iii) If N is upper triangular, then N must be diagonal.

(9) Prove the following statements:

- (i) Suppose A is an $n \times n$ real symmetric positive definite matrix, show that $|A + I_n| > 1$.
- (ii) Let A be an $n \times n$ matrix, show that $A^T A$ is similar to AA^T .

(10) If $A^k = O$ for some positive integer k , then A is called a “nilpotent” matrix. O is the $n \times n$ zero matrix.

- (i) Show that all the eigenvalues of a nilpotent matrix must be zero.
- (ii) Prove that a nonzero nilpotent matrix can not be symmetric.

(11) Let A be an $n \times n$ real symmetric positive definite matrix, and $\alpha \in \mathbb{R}^n$ be a nonzero vector. Consider

$$M = \begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix}$$

- (i) Find a condition on b to guarantee that M is positive definite.
- (ii) In the case that M is positive semi-definite, find a basis for the nullspace of M , $N(M)$.