Tips for Proof

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1 Tips for Proof

1.1 problem 2

Find two matrix with no full rank and the same column space in distinct bases, which give a different linear combination of the column vectors to get the 0 vector.

e.g

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.2 problem 4

B must be a matrix 3×4 with rank2 from the linear independence of the column vectors.

$$n = 4, n - r = 4 - 2 = 2 \neq 1$$

Uncomputable.

1.3 problem 5

No. Check by definition. In fact, suppose that there are two matricesA, B with full rank with $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ in their column space, which implies the only linear combination giving $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$. For the sum of A, B

$$Ax = \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{1}$$

$$By = \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{2}$$

$$(A+B)z = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} = Az + Bz = mAx + nBz \tag{3}$$

$$z \text{ is linearly dependent to x,y} \tag{4}$$

Just find such thing.

Another method is to get a matrix and times it with 0, and amazing things happen.

1.4 problem 13

From the property of orthogonality. It's easy to prove that it is wrong.

1.5 problem 17

The sum of the columns and rows of A is $\begin{bmatrix}0\\0\\0\end{bmatrix}$

1.6 problem 18

Consider the relationship of four fundamental spaces.

1.7 problem 19

Suppose

$$a = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{5}$$

Then the trace is (6)

$$\frac{x_1^2}{x_1^2 + x_2^2 + \cdots + x_n^2} + \frac{x_2^2}{x_1^2 + x_2^2 + \cdots + x_n^2} + \cdots + \frac{x_1^2}{x_n^2 + x_2^2 + \cdots + x_n^2} = 1 \tag{7}$$

Using the method of multiplication!

1.8 problem 21

The column space is the same as the row space.

1.9 problem 24

Doing the multiplication columns by columns, applying the property of orthogonality in the same time to 'eliminate' ('remove', or so to speak) the off-diagonal entries.

1.10 problem 26

Gram-Schmidt.

1.11 problem 35

1.11.1 a

$$|Q^T||Q| = |Q|^2 = 1 \Rightarrow |Q| = \pm 1 \tag{8}$$

p.s under the condition of unitary, please remember that $|Q^H| = |\bar{Q}|$.

1.12 problem 36

1.12.1 b

$$P^2 = P \Rightarrow |P|^2 = |P| \tag{9}$$

1.13 problem 39

Do elimination!

1.14 problem 40

THE RIGHT SIDE SHOULD BE $detB$. Do the computation in the reverse order.

1.15 problem 43

Like problem 39, doing it in the columns.

1.16 problem 45

1.16.1 c

Big formula! Do elimination in the columns/rows!

1.17 problem 51

Diagonalization.

1.18 problem 60

non-zero case

$$ABx = \lambda x \Rightarrow BA(Bx) = \lambda Bx \tag{10}$$

same eigenvalue.

zero case

$$ABx = 0 \Rightarrow BA(Bx) = B * 0 = 0 \tag{11}$$

In all, the same.

1.19 problem 61

1.19.1 a

Distinct eigenvalue.

1.20 problem 62

Like problem 60, there is contradiction of eigenvalues.

1.21 problem 63

As problem 60.

1.22 problem 64

1.22.1 a

Do the multiplication rows by rows $((C - \lambda_i I) * x)$. Substitute the polynomial when necessary.

1.22.2 b

Expand by big formula with PATIENCE! The distinct roots result in distinct eigenvalue, which fulfils the characteristic polynomial and ensure the existence of all the eigenvalues.

1.23 problem 65

$$(A - \delta I)\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\\vdots\\0\end{bmatrix} \tag{12}$$

1.24 problem 66

$$||Qv|| = ||\lambda v|| \Rightarrow ||Qv||^2 = ||\lambda||^2||v||^2 \Rightarrow ||v^TQ^TQv|| = ||\lambda||^2||v||^2 \tag{13}$$
$$\Rightarrow ||\lambda||^2 = 1 \tag{14}$$

Remember to use determinant!

1.25 problem 68

1.25.1 a

Elimination work. Think in the way of linear combination.

1.25.2 b

$$tr(A) = 0 + 0 + \cdots + \lambda_n \tag{15}$$

prove by multiplication and the special use of transpose.

1.25.3 c

n linearly independent eigenvectors.(distinct eigenvalue also gives 1)

1.26 problem 69

$$A^2 = SIS^{-1} = I \tag{16}$$

1.27 problem 72

Multiplication!

1.28 problem 77

Take an example in the low dimension.

1.29 problem 82

$$(A+I)x = (\lambda+1)x \tag{17}$$

different eigenvalue.

1.30 problem 84

1.30.1 c

If symmetric, then diagonalizable $\Rightarrow S0S^{-1} = 0$, which is contradict to the non-zero condition.

1.31 problem 85

$$Ax = \lambda x \tag{18}$$

$$\Rightarrow A^2 x = \lambda A x \Rightarrow -x = \lambda A x \Rightarrow Ax = -\frac{1}{\lambda} x \tag{19}$$

$$\Rightarrow \lambda^2 = -1 \tag{20}$$

For n being even,

$$\text{characteristic polynomial is } \sum_i^n \lambda^2 + a_i \lambda + b_i \tag{21}$$

or (odd condition) there must be $(\lambda - ki)$ in the polynomial, which is not 'real'!

1.32 problem 89

Consider eigenvalue.

1.33 problem 92

Find F s.t.

$$F^T F = C \tag{22}$$

1.34 problem 93

Take the transpose after diagonalization.

1.35 problem 94

1.35.1 a

As problem 89

1.35.2 b

Find R s.t.

$$A = RR^T \tag{23}$$

1.35.3 c

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} R & R \\ R & R \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} R^T & R^T \\ R^T & R^T \end{bmatrix} \tag{24}$$

not invertible, then positive semi-definite.

1.36 problem 95

$$Ax = y \tag{25}$$

y is non-zero

1.37 problem 96

See the file, 行列式的计算1、2、3 in the QQ group.