期中考试题答案

1、 (1) D (2) B (3) A (4) C (5) C

2. (1) 0 (2) $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ (3) 1 (4) $\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix}$

(5) $\lambda_1+\lambda_2=1$.

3. Solution. $\begin{cases} h_{11}S_1+h_{21}S_2+h_{31}n=m_1 \\ h_{12}S_1+h_{22}S_2+h_{32}n=m_2 \\ h_{13}S_1+h_{23}S_2+h_{33}n=m_3 \end{cases}$

$H=[h_{ij}]=\begin{bmatrix} \frac{7}{8} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$, $H^T\begin{bmatrix} S_1 \\ S_2 \\ n \end{bmatrix}=\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$

$\leadsto S_2=20$ dB.

4. Solution. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}=U$

Since U is invertible, so is $[\alpha_1,\alpha_2,\alpha_3]$; that is, $\alpha_1,\alpha_2,\alpha_3$ are linearly independent.

$\{\alpha_1,\alpha_2,\alpha_3\}$ is a basis of $\mathbb{R}^3$.

(b) $T[e_1,e_2,e_3]=[e_1,e_2,e_3]A=I_3A=A$.

$[\alpha_1,\alpha_2,\alpha_3]Q=[e_1,e_2,e_3]=I_3$

$\therefore A=T[e_1,e_2,e_3]=T([\alpha_1,\alpha_2,\alpha_3]Q)\underset{\text{linear map}}{=}(T[\alpha_1,\alpha_2,\alpha_3])Q$

$A = [T\alpha_1, T\alpha_2, T\alpha_3]\, Q, \quad Q = [\alpha_1, \alpha_2, \alpha_3]^{-1}$

Using Gauss-Jordan Elimination, we get

$$Q = \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & -4 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 1 \\ -3 & -1 & 3 \end{bmatrix}$$

(c) Yes. Since Q is invertible,

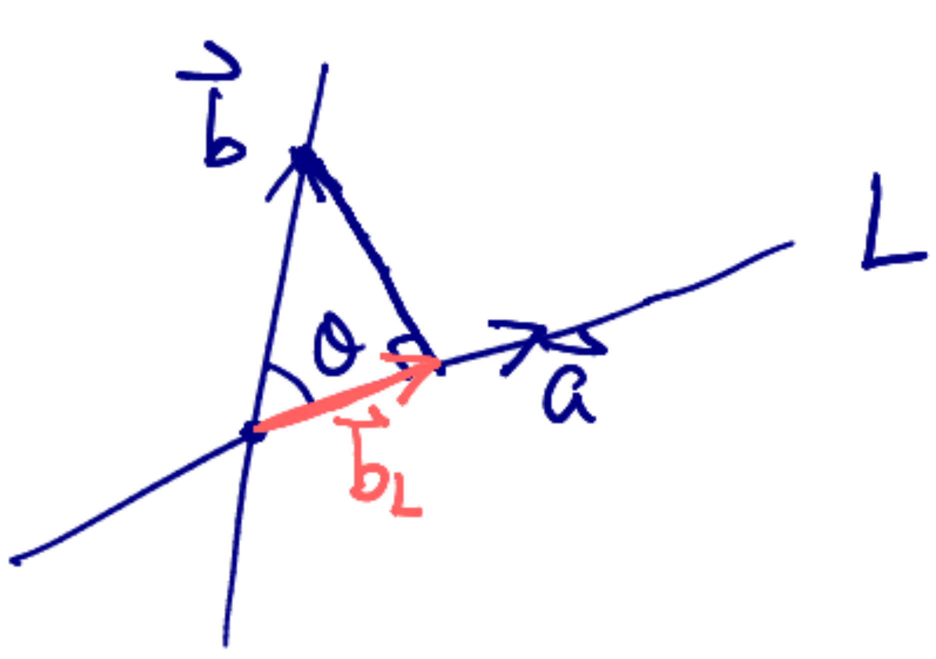
A is invertible $\Leftrightarrow P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ is invertible

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \therefore \operatorname{rank}(P) = 3, \ P \text{ is invertible}$$

$\square$

5. Solution.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_2 - 2x_3 = 0 \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} = t\vec{a}, \quad a = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$$



$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\vec{b}_L = |\vec{b}| \cos\theta \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}^T \vec{b}}{|\vec{a}|^2} \vec{a}$$

$$= \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a} = \frac{2}{13} \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix},$$

6. 略.

7. proof. (a) Since $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are linearly independent,

there exist numbers $x_1, x_2, \cdots, x_n$, in which at least one of them is non-zero, such that

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} = 0.$$

If $x_1 \neq 0$, we are done:

$$\alpha_1 = -\frac{x_2}{x_1}\alpha_2 - \cdots - \frac{x_{n-1}}{x_1}\alpha_{n-1} + 0\cdot\alpha_n$$

Assume $x_1 = 0$, then $x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + 0\cdot\alpha_n = 0$

Since $\alpha_2, \cdots, \alpha_n$ are linearly independent,

we have $x_2 = \cdots = x_{n-1} = 0$, which contradicts.

Thus $x_1 \neq 0$, and the proof of (a) is completed.

(b) $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n = [\alpha_1, \alpha_2, \cdots, \alpha_n]\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$\therefore$ $Ax = \beta$ has a solution $x_p = [1, 1, \cdots, 1]^T$.

Since $x = x_p + x_{null}$, $x_{null} \in N(A)$

it suffices to show $N(A) \neq 0$, or equivalently $\dim N(A) \geq 1$.

By (a) we get $\operatorname{rank}(A) = n-1$, hence

$$\dim N(A) = n - \operatorname{rank}(A) = 1 \geq 1.$$

Thus $N(A)$ is of dimension 1, $Ax = \beta$ has infinitely many solutions.

(c) "$n > 2 \Rightarrow A^2 \neq 0$" $\Leftrightarrow$ "$A^2 = 0 \Rightarrow n \leq 2$".

Suppose $A^2 = 0$, then $C(A) \subseteq N(A)$, $\underbrace{\operatorname{rank}(A)}_{\underset{n-1}{\parallel}} \leq \dim N(A) = 1$

Thus $n \leq 2$. $\square$