MIT 18.06 Final Exam, Fall 2017
Johnson

Your name: ______________________________

Recitation: ______________________________

problem	**score**
1	/15
2	/15
3	/15
4	/15
5	/15
6	/15
7	/10
total	/100

Problem 1 (15 points):

A matrix $A = LU$ has the LU factors

$$L = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 0 & -2 & 1 & \\ -1 & -1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & -2 & 0 \\ & 1 & 0 & -2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

(a) If $b = \begin{pmatrix} -1 \\ 2 \\ 2 \\ -4 \end{pmatrix}$, what is $x = A^{-1}b$?

(b) Assuming you solved the previous part efficiently, roughly how much more arithmetic operations would be required for the same approach if the matrices were 8×8 instead of 4×4? It should be about ______ times more.

(c) If you form a new 4×5 matrix $B = \begin{pmatrix} A & b \end{pmatrix}$ by appending the vector b (from above) as an extra column after A, and perform the *same* elimination steps as were used to get the LU factors above, what upper-triangular matrix would you obtain? (Hint: if you did part (a) properly, this part can be done with *no arithmetic*.)

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Problem 2 (15 points):

You are given the recurrence relation

$$(2I + B^T B)x_{n+1} = (2I - B^T B)x_n$$

where B is a real 5×3 matrix. We start with a vector x_0 and compute $x_1, x_2, \ldots$.

(a) $x_n = A^n x_0$ for some matrix A (independent of x_0). What is A?

(b) If λ is an eigenvalue of $B^T B$, give an eigenvalue of A.

(c) **Circle all possible** behaviors of x_n for large n, given the information above: **decaying** to zero, **oscillating** but not growing or decaying in length, going to a **nonzero constant** vector, or **growing** longer and longer. Explain your answers by giving some property (or properties) that must be true of the eigenvalues of A.

(d) If $x_1 = 0$ for a nonzero x_0, give one of the singular values (σ) of B.

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Problem 3 (15 points):

The distance between a point b and a plane in $\mathbb{R}^3$ is defined as the *minimum* distance $\|b - y\|$ between b and *any* point y in the plane.

(a) Suppose the points y in the plane are of the form $y = c + \alpha a_1 + \beta a_2$ for all real numbers α and β, given vectors $c, a_1, a_2 \in \mathbb{R}^3$ that define the plane (a_1 and a_2 are linearly independent). Under **what condition(s)** on c, a_1, a_2 is the plane a **subspace** of $\mathbb{R}^3$?

(b) Write down a 2×2 **system of equations**, in terms of the vectors a_1, a_2, c, b (or matrices defined from these vectors) whose solution gives the $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for the *closest* point y in the plane to b.

(c) For this closest point y, $b - y$ is in **what subspace** of the matrix $A = (a_1 a_2)$? What is the **dimension** of this subspace?

(d) For $a_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, **find a vector** d such that the distance between any point b and the plane is equal to $|d^T(b-c)|$. **What subspace** of A contains d?

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Problem 4 (15 points):

You are given the following matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 4 \\ 1 & -2 & 1 & -4 \end{pmatrix}$$

(a) Find the **complete solution** x (i.e. all solutions) to $Ax = b$ for $b = \begin{pmatrix} 3 \\ 9 \\ -4 \end{pmatrix}$.

(b) $A^T y = d$ is solvable if and only if $d^T z = 0$ for some z. **Give such a vector** z.

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Problem 5 (15 points):

QR factorization of the matrix A (e.g. via Gram–Schmidt) yields $A = QR$, where

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 0 \\ & 1 & 2 \\ & & 2 \end{pmatrix}.$$

(a) Which columns of A were **orthogonal** to begin with, if any?

(b) What is the orthogonal **projection** p of the vector $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto $C(A)$?

(c) If we are minimizing $\|Ax - b\|$ (i.e. solving the least-square problem) for $b = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, you should be able to *quickly* **get an upper-triangular system** of equations $U\hat{x} = c$ for the least-square solution $\hat{x}$. **What are the** upper-triangular matrix U and the right-hand-side vector c?

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Problem 6 (15 points):

You are given the nonsymmetric, diagonalizable matrix

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & -3 & -2 \\ -1 & 0 & -3 \end{pmatrix}.$$

and we want to understand the solutions of the ODE

$$\frac{dx}{dt} = Ax$$

for some initial condition $x(0)$.

(a) Show (by any test you want, e.g. the pivot test) that the matrix $A + A^H = \begin{pmatrix} -2 & 2 & 2 \\ 2 & -6 & -2 \\ 2 & -2 & -6 \end{pmatrix}$ is **negative definite**.

(b) If $Av = \lambda v$ is an eigensolution of A (v and λ may be complex), look at $v^H \left(A + A^H\right) v$ and use the fact that $A + A^H$ is negative definite to **show** that the **real part** of λ must be **negative**.

(c) What can you conclude from the previous parts about the solutions $x(t)$ as $t \to \infty$?

(d) If $A + A^H$ is negative definite (so that A's eigenvalues have negative real parts), but A is **defective**, does your answer to the previous part about $x(\infty)$ **change? Why** or why not?

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Problem 7 (10 points):

The following parts can be **answered independently** (and refer to **different matrices**). Little or no calculation should be needed.

(a) If $C(B)$ is a subspace of $N(A)$, then either (**circle one**) AB or BA must be simply ______.

(b) If A is a real-symmetric 3×3 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and corresponding real eigenvectors v_1, v_2, v_3, then an explicit equation for $A^{-1}b$ in terms of sums/products involving these eigenvectors and b, with *no matrix inverses,* is:

______________________________________.

(c) If A is a 3×3 non-singular real matrix with singular values $\sigma_1, \sigma_2, \sigma_3$, then give formulas in terms of $\sigma_1, \sigma_2, \sigma_3$ for $\det(A^T A) =$____________ and $|\det(A)| =$____________.

(d) If $N(A)$ is spanned by the vector $v \neq 0$, then projection matrices onto *two* of the fundamental subspaces of A are:

_________________________________ and _________________________________
(write down two matrices and indicate which subspaces they project onto).

(e) If A is similar to the matrix $\begin{pmatrix} 3 & 6 & 2 \\ & 17 & 3 \\ & & 4 \end{pmatrix}$, then the eigenvalues of A are: ____________________.

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