Problem Set 10 —— Linear Algebra A (Fall 2021)

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Please hand in your assignment at the beginning of your 12th tutorial session!

1. 如果 A 是一个 n 阶复方阵, 且 λ_i 是 A 一个特征值. 证明：λ_i 的代数重数大于等于它的几何重数.

2. 设 A 是一个 n 阶复方阵, 如果矩阵 A 的所有特征值的几何重数和代数重数都相等. 证明：矩阵 A 是可对角化的.

3. 设 A, B 都为 n 阶复方阵. 考虑线性变换 $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$:

$$T(X) = AX - XB.$$

   证明：T 可逆的充要条件是 A, B 没有公共的特征值 (可逆映射等价于既是单射又是满射).

4. (方阵的同时对角化问题) Suppose A and B are diagonalizable and $AB = BA$. Show that there is an invertible matrix S, such that

$$S^{-1}AS = \Lambda_1,\ S^{-1}BS = \Lambda_2,$$

   where Λ_1 and Λ_2 are diagonal matrices.

5. Let A be an $n \times n$ matrix. If some positive power of T is the identity matrix, say $T^r = I$, show that T is diagonalizable.