Triangular Factors and Row Exchanges (三角分解和行交换)

Lecture 4

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LU Factorization (LU 分解)

Elementary Matrices

An elementary matrix of Type I is a matrix obtained by interchanging two rows of I.

Example

Let $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$E_1A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad AE_1 = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}.$$

Elementary Matrices

An elementary matrix of Type II is a matrix obtained by multiplying a row of I by a nonzero constant.

Example

Let $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$E_2A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

and

$$AE_2 = \begin{bmatrix} a_{11} & a_{12} & 3a_{13} \\ a_{21} & a_{22} & 3a_{23} \\ a_{31} & a_{32} & 3a_{33} \end{bmatrix}.$$

Elementary Matrices

An elementary matrix of Type III is a matrix obtained from I by adding a multiple of one row to another row.

Example

Let $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$E_3A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{11}+a_{31} & 3a_{12}+a_{32} & 3a_{13}+a_{33} \end{bmatrix}$$

and

$$AE_3 = \begin{bmatrix} a_{11} & a_{12} & 3a_{11}+a_{13} \\ a_{21} & a_{22} & 3a_{21}+a_{23} \\ a_{31} & a_{32} & 3a_{31}+a_{33} \end{bmatrix}.$$

Upper triangular matrix

Now we look again at the following linear system $Ax = b$:

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

After the following three steps, the coefficient matrix will be in its strictly triangular form:

1 : Subtract 2 times the first equation from the second;

2 : Subtract -1 times the first equation from the third;

3 : Subtract -1 times the second equation from the third.

The result was an equivalent system $Ux = c$, with a new coefficient matrix U.

Upper-Triangular Matrix

See

$$Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c.$$

This U is **upper triangular**–all entries below the diagonal are zero.

The new right side c was derived from the original vector b by the same steps that took A to U.

Forward elimination amounted to three row operations:

(a) Start with A and b;

(b) Apply steps 1, 2, 3 in that order;

(c) End with U and c. $Ux = c$ is solved by back-substitution.

From A to U

Here we concentrate on connecting A to U.

The elementary matrices for step 1, step 2 and step 3, E, F, G, were introduced in the previous section.

The result of all three steps is $GFEA = U$.

The question is how can we undo the steps of Gaussian Elimination?

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

From U back to A: $E^{-1}F^{-1}G^{-1}U = A$. If we denote $L = E^{-1}F^{-1}G^{-1}$, then $LU = A$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Triangular factorization $A = LU$

If the computer stores each multiplier l_{ij}–the number that multiplies the pivot row j when it is subtracted from row i, and produces a zero in the i,j position–then these multipliers give a complete record of elimination. The numbers l_{ij} fit right into the matrix L that takes U back to A.

Theorem

Triangular factorization $A = LU$ with no exchange of rows:

(a) L *is lower triangular, with 1s on the diagonal.*

(b) *The multipliers l_{ij} (taken from elimination) are below the diagonal.*

(c) U *is the upper triangular matrix which appears after forward elimination.*

(d) *The diagonal entries of U are the pivots.*

$A = LU$: The 4 by 4 case

Example

($A = LU$, with zeros in the empty spaces)

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= LU.
\end{aligned}
$$

One more example

Example

Find the triangular factorization for

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}.$$

In other words, find lower triangular matrix L and upper triangular matrix U, such that $A = LU$.

Solution

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}.$$

One Linear System = Two Triangular Systems

- Splitting of $Ax = b$. First $Lc = b$ and then $Ux = c$.
- This process of solving a linear system involves two steps:
  1. Factor (from A find its factors L and U).
  2. Solve (from L and U and b find the solution x).
- The triangular factorization can be written $A = LDU$, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.
- The separation into Factor and Solve means that a series of b's can be processed. The Solve subroutine obeys equation (8)(see page 35): Two triangular systems in $n^2/2$ steps each. The solution for any two right-hand side b can be found in only n^2 operations. That is far below the $n^3/3$ steps needed to factor A on the left-hand side.

Example 6

This is the previous matrix A with a right-hand side $b = (1, 1, 1, 1)$.

$$Ax = b : \begin{cases} x_1 - x_2 & & & = 1 \\ -x_1 + 2x_2 - x_3 & & & = 1 \\ - x_2 + 2x_3 - x_4 & & & = 1 \\ - x_3 + 2x_4 & & & = 1 \end{cases} \text{ splits into } Lc = b \text{ and } Ux = c.$$

$$Lc = b : \begin{cases} c_1 & = 1 \\ -c_1 + c_2 & = 1 \\ - c_2 + c_3 & = 1 \\ - c_3 + c_4 & = 1 \end{cases} \text{ gives } c = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Example 6

$$Ux = c : \begin{cases} x_1 - x_2 & = 1 \\ \quad x_2 - x_3 & = 2 \\ \quad\quad x_3 - x_4 & = 3 \\ \quad\quad\quad x_4 & = 4 \end{cases} \text{ gives } x = \begin{bmatrix} 10 \\ 9 \\ 7 \\ 4 \end{bmatrix}.$$

Triangular Factorization

The triangular factorization can be written as $A = LDU$, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.

There is no freedom in the final L, D, and U. That is our main point.

Theorem

If $A = L_1 D_1 U_1$ and also $A = L_2 D_2 U_2$, where the L's are lower triangular with unit diagonal, the U's are upper triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then

$$L_1 = L_2,\ D_1 = D_2,\ U_1 = U_2.$$

The LDU factorization and the LU factorization are uniquely determined by A.

The proof is a good exercise with inverse matrices in the next section.

Row Exchanges and Permutation Matrices

A permutation matrix has the same rows as the identity (in some order). There is a single "1" in every row and column. The most common permutation matrix is $P = I$ (It exchanges nothing). The product of two permutation matrices is another permutation–the rows of I get reordered twice.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition

Let P be a permutation matrix. P^{-1} is always the same as P^T.

Can you figure out how to prove this proposition?

Permutation Matrices

All 3 by 3 permutations(there are $3! = (3)(2)(1) = 6$ matrices):

$$
I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad
P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \quad
P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}
$$

$$
P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \quad
P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \quad
P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.
$$

- Repeat: A permutation matrix P has the same rows as the identity (in some order).
- There are $n! = n(n-1)\cdots(1)$ permutations of size n.
- A zero in the pivot location raises two possibilities: The trouble may be easy to fix, or it may be serious.

Elimination in a Nutshell: $PA = LU$

The main point is this: If elimination can be completed with the help of row exchanges, then we can imagine that those exchanges are done first. The matrix will not need row exchanges. In other words, PA allows the standard factorization into L and U. The theory of Gaussian Elimination can be summarized in a few lines:

Theorem

In the nonsingular case, there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. Then $Ax = b$ has a unique solution:

With the rows reordered in advance, PA can be factored into LU.

In the singular case, no P can produce a full set of pivots: elimination fails.

Example 7

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U.$$

That row exchange recovers LU–but now $l_{31} = 1$ and $l_{21} = 2$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } PA = LU.$$

A good elimination code saves L and U and P. Those matrices carry the information that originally came in–and then carry it in a more usable form. $Ax = b$ reduces to two triangular systems. This is the practical equivalent of the calculation we do next–to find the inverse matrix A^{-1} and the solution $x = A^{-1}b$.

Homework Assignment

1.5: 2, 3, 5, 8, 17, 21, 29, 43, 45.