Linear Transformations (线性变换)

Lecture 11 and 12

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Linear Transformations

Linear Transformations (线性变换)

- We know how a matrix moves subspaces around when we multiply by A. The nullspace goes into the zero vector. All vectors go into the column space, since Ax is always a combination of the columns.
- You will soon see something beautiful–that A takes its row space into its column space, and on those spaces of dimension r it is 100% invertible.
- That is the real action of A. It is partly hidden by nullspaces and left nullspaces, which lie at right angles and go their own way (toward zero).
- What matters now is what happens inside the space–which means inside n-dimensional space, if A is n by n. That demands a closer look.

Introduction

Suppose x is an n-dimensional vector. When A multiplies x, it transforms that vector into a new vector Ax. This happens at every point x of the n-dimensional space $\mathbb{R}^n$. The whole space is transformed, or "mapped into itself," by the matrix A. Here we consider four transformations that come from matrices:

1. A multiple of the identity matrix, $A = cI$, stretches every vector by the same number c.
2. A rotation matrix turns the whole space around the origin.
3. A reflection matrix transforms every vector into its image on the opposite side of a mirror.
4. A projection matrix takes the whole space onto a lower-dimensional subspace.

Four Typical Linear Transformations

See the figure:

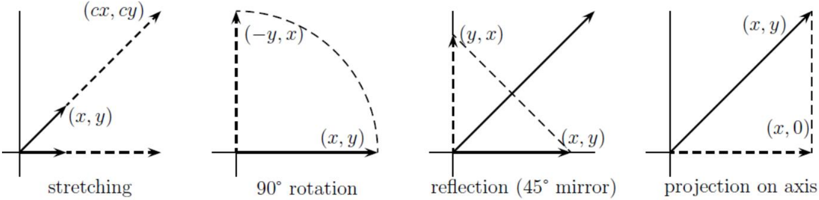

Figure 2.9: Transformations of the plane by four matrices.

The above figure illustrates four transformations that come from matrices:

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Linear Transformation

Those examples could be lifted into three dimensions. There are matrices to stretch the earth or spin it or reflect it across the plane of the equator. There is a matrix that projects everything onto that plane.

For all numbers c and d and all vectors x and y, matrix multiplication satisfies the rule of linearity:

$$A(cx+dy) = c(Ax) + d(Ay).$$

Every transformation $T(x)$ that meets this requirement is a linear transformation.

Definition

Definition

Let V, W be real vector spaces and T be a function from vector space V to W. If T satisfies the following two properties:

(a) Additivity: $T(v_1 + v_2) = T(v_1) + T(v_2)$;

(b) Homogeneity: $T(\lambda v) = \lambda T v$;

then T is called a linear map from V to W. Where $v_1, v_2, v \in V$ and $\lambda \in \mathbb{R}$.

$Tx = Ax$ is a linear transformation.

It is also important to recognize that matrices can not do everything, and some transformations Tx are not possible with Ax.

A few remarks

- Any matrix leads immediately to a linear transformation. The more interesting question is in the opposite direction: Does every linear transformation lead to a matrix? The object of this section is to find the answer: **yes**.
- A transformation need not go from $\mathbb{R}^n$ to the same space $\mathbb{R}^n$. It is absolutely permitted to transform vectors in $\mathbb{R}^n$ to vectors in a different space $\mathbb{R}^m$.
- The operations in the linearity condition are addition and scalar multiplication, but x and y need not be column vectors in $\mathbb{R}^n$, and they may actually be polynomials or matrices or functions. As long as the transformation satisfies the rule of linearity, it is linear.

Linearity: Examples

- The operation of differentiation, $A = d/dt$, is linear:

$$Tp(t) = \frac{d}{dt}(a_0 + a_1 t + \cdots + a_n t^n) = a_1 + \cdots + n a_n t^{n-1}.$$

  Nullspace? Column space? Nullity? Rank?

- Integration from 0 to t is also linear(it takes P_n to P_{n+1}):

$$Tp(t) = \int_0^t (a_0 + a_1 t + \cdots + a_n t^n) dt = a_0 t + \cdots + \frac{a_n}{n+1} t^{n+1}.$$

  No nullspace?

- Multiplication by a fixed polynomial like $2 + 3t$ is linear:

$$Tp(t) = (2 + 3t)(a_0 + a_1 t + \cdots + a_n t^n) = 2a_0 + \cdots + 3a_n t^{n+1}.$$

  No nullspace?

Transformations Represented by Matrices

Linearity has a crucial consequence:

Theorem

If we know Ax for each vector in a basis, then we know Ax for each vector in the entire space.

Example 4 What linear transformation takes x_1 and x_2 to Ax_1 and Ax_2?

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ goes to } Ax_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}; x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ goes to } Ax_2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}.$$

Solution. $T(x) = Ax$, where $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$.

Transformation from one space to another

In short, the matrix carries all the essential information. If the basis is known, and the matrix is known, then the transformation of every vector is known.

The coding of the information is simple. To transform a space to itself, one basis is enough. A transformation from one space to another requires a basis for each.

Matrix representation of Linear Transformation

Definition

Suppose the vectors

$$v_1, v_2, \cdots, v_n$$

are a basis for the space V, and vectors

$$w_1, w_2, \cdots, w_m$$

are a basis for W.
Each linear transformation T from V to W is represented by a matrix A. The j th column is found by applying T to the j th basis vector v_j, and writing $T(v_j)$ as a combination of the y's:

$$\text{Column } j \text{ of } A: \; T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m.$$

Example

Example

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\left[\begin{array}{cc} x_1 & x_2 \end{array}\right]^T\right) = \left[\begin{array}{ccc} x_2 & x_1 + x_2 & x_1 - x_2 \end{array}\right]^T$$

Find the matrix representation of T with respect to the ordered bases $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$, where

$$u_1 = \left[\begin{array}{cc} 1 & 2 \end{array}\right]^T, u_2 = \left[\begin{array}{cc} 3 & 1 \end{array}\right]^T$$

and

$$v_1 = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]^T,\ v_2 = \left[\begin{array}{ccc} 1 & 1 & 0 \end{array}\right]^T,\ v_3 = \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right]^T.$$

Solution.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix}.$$

Then the matrix representation is given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = A^{-1}B = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}.$$

Examples

Example 5 Differentiation matrix.

$$A_{diff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix can differentiate $p(t)$, because matrices built in linearity!

$$\frac{dp}{dt} = Tp \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \end{bmatrix} \to 1 - 2t - 3t^2.$$

Example

Example 6 Integration matrix.

$$\int_0^1 1dt = t \text{ or } Ax_1 = y_2, \cdots, \int_0^t t^3 dt = \frac{1}{4}t^4 \text{ or } Ax_4 = \frac{1}{4}y_5.$$

Integration matrix

$$A_{int} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

It can be verified that differentiation is a left-inverse of integration.

Inverse Operations

Differentiation and integration are inverse operations.

$$
\begin{aligned}
A_{diff}A_{int} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Rotations Q

This section began with 90° rotations, projections onto the x-axis, and reflections through the 45° line. But rotations through other angles, projections onto other lines, and reflections in other mirrors are almost as easy to visualize. Figure 2.10:

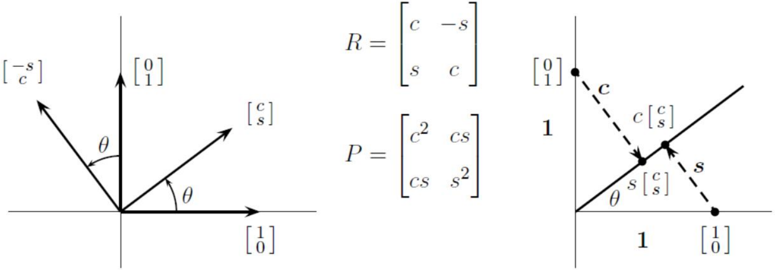

Figure 2.10: Rotation through θ (left). Projection onto the θ-line (right).

Rotations Q

Rotation through θ:

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Q_θ^{-1} and $Q_\theta Q_\varphi$:

Does the inverse of Q_θ equal $Q_{-\theta}$ (rotation backward through θ)? **Yes.**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Does the square of Q_θ equal $Q_{2\theta}$? **Yes.**

Does the product of Q_θ and Q_φ equal $Q_{\theta+\varphi}$ (rotation through θ and φ)? **Yes.**

The product of the transformations

Matrix multiplication is defined exactly so that **the product of the matrices corresponds to the product of the transformations.**

Theorem

Suppose S and T are linear transformations from V to W and from U to V.

(1) *Their product ST starts with a vector u in U, goes to Tu in V, and finishes with STu in W.*

(2) *This "composition" ST is again a linear transformation (from U to W). Its matrix is the product of the individual matrices representing S and T.*

Projections P

Projection onto the θ-line

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

Remark:

- **A projection matrix equals its own square:** $P^2 = P$.
- The matrix P has no inverse.
- Points on the perpendicular line are projected onto the origin; that line is the nullspace of P.

Reflections H

Figure 2.11:

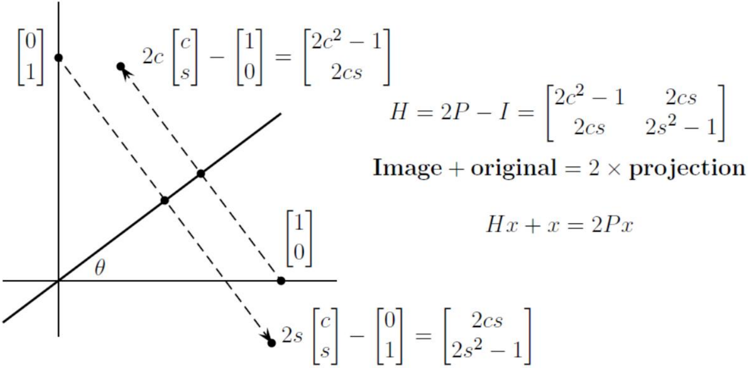

Figure 2.11: Reflection through the θ-line: the geometry and the matrix.

Remarks

Reflection Matrix:

$$H = \begin{bmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{bmatrix}$$

Remarks:

- Two reflections bring back the original, $H^2 = I$.
- $H = 2P - I$.

Remarks

- The matrix depends on the choice of basis.
- How to choose the best basis?
- Change of basis.
- A single transformation is represented by different matrices.
- The theory of eigenvalues will lead to this formula $S^{-1}AS$, and to the best basis.

练习题

在 $\mathbb{R}^3$ 中考虑以下两组向量:

$$u_1=(1,0,1),\ u_2=(2,1,0),\ u_3=(1,1,1)$$

和

$$v_1=(1,0,0),\ v_2=(1,1,0),\ v_3=(1,1,1).$$

可以验证这两组向量都是 $\mathbb{R}^3$ 的基. 假定线性变换 T 把基 u_1, u_2, u_3 映到基 v_1, v_2, v_3.

1. 求基 u_1, u_2, u_3 到基 v_1, v_2, v_3 的过渡矩阵;
2. 求 T 在基 u_1, u_2, u_3 下的矩阵;
3. 求 T 在基 v_1, v_2, v_3 下的矩阵;
4. 求 $T^2(u_1)=T(T(u_1))$;
5. 求 $(1,2,3)$ 在基 u_1, u_2, u_3 下的坐标.

练习题

设

$$v_1 = (1,-1,5,2),\ v_2 = (-2,3,1,0),\ v_3 = (4,-5,9,4),$$
$$v_4 = (0,4,2,-3),\ v_5 = (-7,18,2,-8).$$

求向量组 v_1, v_2, v_3, v_4, v_5 的一个极大线性无关组, 并用极大线性无关组线性表出向量组中的其它的向量.

Homework Assignment 12

2.6: 1, 3, 5, 6, 15, 28, 33, 46, 47, 50.