Answer to Midterm Exam

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Problem 1. Multiple Choice.

Solution. 1.A 2.C 3.D 4.D 5.C

Problem 2.

1. $\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$ 2. 1 3. $\begin{bmatrix} 1 & & \\ -a & 1 & \\ ac-b & -c & 1 \end{bmatrix}$ 4. 12 5. $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$

Problem 3. Find the LU decomposition of the matrix $A=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

Solution. We use elementary matrice to do Gaussian elimiantion to A.

- $A \to E_{12}A = \begin{bmatrix} 1 & & \\ -1/3 & 1 & \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{bmatrix} = A_2$
- $A_2 \to E_{13}A_2 = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -1/3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} = A_3$
- $A_3 \to E_{23}A_3 = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1/4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix} = U$

So we have $E_{23}E_{13}E_{12}A = U \Rightarrow A = E_{12}^{-1}E_{13}^{-1}E_{23}^{-1}U = LU = \begin{bmatrix} 1 & & \\ 1/3 & 1 & \\ 1/3 & 1/4 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix}$.

Remark: 1. *Do not directly write the answer! You should write the process like the answer above.*

2. *If you didn't know the definition or computation of LU decomposition in midterm, you're near the end. I hope you to take a series of self-help steps to save yourselves.*

Problem 4. Let $A=\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. Find the bases of four fundamental subspaces.

Solution. Notice that all basis vectors are **column** vectors, not **row** vectors.

- Column space $C(A)$ has basis: $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}\right\}$.
- Row space $C(A^T)$ has basis: $\left\{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}\right\}$.
- Nullspace $N(A)$ has basis: $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right\}$.
- Left nullspace $N(A^T)$ has basis: $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

Remark: *Some of you write $N(A^T)$ in $\mathbb{R}^5$ not in $\mathbb{R}^3$. Incredible ! I hope you to read the textbook from the begining of chapter 2 to save yourselves.*

Problem 5. $E = \{u_1 = (1, 0, -1)^T, u_2 = (1, 2, 1)^T, u_3 = (-1, 1, 1)^T\}$; $F = \{v_1 = (1, -1)^T, v_2 = (2, -1)^T\}$

Solution. Suppose

$$\begin{cases} T(u_1) = a_{11}v_1 + a_{21}v_2 \\ T(u_2) = a_{12}v_1 + a_{22}v_2 \\ T(u_3) = a_{13}v_1 + a_{23}v_2 \end{cases} \tag{1}$$

By definition, the matrix representation of T under bases $E = \{u_1, u_2, u_3\}$, $F = \{v_1, v_2\}$ is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$.

So we only need to find the combination coefficients of (1). We can write the three equations of (1) as matrix form:

$$[\ T(u_1)\ \ T(u_2)\ \ T(u_3)\ ] = [\ v_1\ \ v_2\ ]A \Leftrightarrow \begin{bmatrix} 0 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} A$$

We can do Gaussian elimination to find A:

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 0 & 1 & -1 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -2 & -4 \\ 0 & 1 & -1 & 3 & 3 \end{bmatrix}$$

So $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$.

Problem 6. Let A, B be n by n matrices. Suppose A and B are both symmetric. Is AB necessarily symmetric?

Solution. No, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ is not symmetric. This is a counter-example.

Remark: *When you want to give an example, you should firstly consider the 2 by 2 matrices, not higher orders. Because the computation is simplest for 2 by 2 matrices.*

Problem 7.

a) A is the 2 by 2 rotation matrix of ratating $\frac{\pi}{3}$. Find A and A^{2020}.

b) Three planes $\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3$ in the space $\mathbb{R}^3$ are given by the equations:

$$\begin{cases} \mathbb{I}_1: x + y + z = 0 \\ \mathbb{I}_2: 2x - y + 4z = 0 \\ \mathbb{I}_3: -x + 2y - z = 0 \end{cases}$$

Determine a matrix representation(in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to $\mathbb{I}_1$, the yz plane to $\mathbb{I}_2$, and the zx plane to $\mathbb{I}_3$.

Solution.

a) We know that the 2 by 2 rotation matrix is $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. So $A = R(\frac{\pi}{3}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. As for A^{2020}, we notice that $R(\theta) = R(\theta + 2k\pi), k \in \mathbb{Z}^+(\text{Positive integer})$. So

$$A^{2020} = R\left(\frac{\pi}{3} \times 2020\right) = R\left(\frac{4}{3}\pi\right) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

b) **Method 1:** We want to find the matrix representation A under standard basis of $\mathbb{R}^3$. So we only need to compute Ae_1, Ae_2, Ae_3, where $\{e_1, e_2, e_3\}$ is the standard basis. The question is we don't know A, how to compute Ae_i ?

Consider the plane transformation. We notice that e_1, e_2 are in the xy plane, and A transform xy plane to $\mathbb{I}_1$ plane. This means that each vector in xy plane should in $\mathbb{I}_1$ plane after transformation. So Ae_1, Ae_2 should in $\mathbb{I}_1$ plane. Similarly, we know that Ae_2, Ae_3 are in $\mathbb{I}_2$ plane, and Ae_1, Ae_3 are in $\mathbb{I}_3$ plane. So we have

$$Ae_1 \in \mathbb{I}_1 \cap \mathbb{I}_3, Ae_2 \in \mathbb{I}_1 \cap \mathbb{I}_2, Ae_3 \in \mathbb{I}_2 \cap \mathbb{I}_3$$

Thus, we know that Ae_i is in the intersection of two planes. Find the 3 intersection lines of the 3 planes:

$$\mathbb{I}_1 \cap \mathbb{I}_3 \colon \begin{cases} x+y+z=0 \\ -x+2y-z=0 \end{cases}, \mathbb{I}_1 \cap \mathbb{I}_2 \colon \begin{cases} x+y+z=0 \\ 2x-y+4z=0 \end{cases}, \mathbb{I}_2 \cap \mathbb{I}_3 \colon \begin{cases} 2x-y+4z=0 \\ -x+2y-z=0 \end{cases}$$

Notice that we only determine the directions of Ae_i, can not determine the norm(length) of Ae_i. So A is not unique. We choose

$$Ae_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, Ae_1 = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}, Ae_1 = \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix}$$

This implies that $A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & -2 & 2 \\ -1 & -3 & -3 \end{bmatrix}$.

Remark: *1. Some of students do this question by finding normal vector of each plane. The normal vector of xy plane is $n_1 = (0,0,1)^T$, normal vector of $\mathbb{I}_1$ plane is $n_2 = (1,1,1)^T$, then they have $An_1 = n_2$. They find 3 equations between 6 normal vectors to find A. But this is incorrect, since A is not a rotation matrix in $\mathbb{R}^3$, so the angle will be changed after transformation. Right angle will be not right after transformation. You can image a simple case to figure out this: T is a transformation in $\mathbb{R}^2$.*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \end{bmatrix}$$

T is a transformation which transforms $x-\text{axis}$ to the line $x-y=0$ and leaves $y-\text{axis}$ unchanged. If you want to find the matrix representation A of T under standard basis by finding the normal vectors of each lines. We have

$$\begin{cases} \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{is the normal vector of } x-\text{axis}, v = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{is the normal vector of the line: } x-y=0 \\ \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{is the normal vector of } y-\text{axis, normal vector is unchanged since } y-\text{axis is unchanged} \end{cases}$$

But we notice that $T(\hat{y}) \neq v, T(\hat{x}) \neq \hat{x}$. This means that transformation T don't transform the normal vector of $x-\text{axis}$ to normal vector of the line: $x-y=0$, although T transform $x-\text{axis}$ to the line: $x-y=0$.

2. Actually, only when T is a comformal transformation(保角变换), the angle leaves unchanged after transforamtion. In $\mathbb{R}^n$, there are two special conformal transformation: rotation and reflection. These two kinds of matrices both are orthogonal matrices. You can verify that $(x,y) = (Ox, Oy)$ for any otrhogonal matrix O and vectors x, y. Since the inner product of two vectors is unchanged after transforamtion, the angle between them is unchanged.

Method 2: We can directly compute A, not compute from Ae_1, Ae_2, Ae_3.

Let v is a vector in xy plane, we know that Av must be a vector in plane $\mathbb{I}_1$. This means that

$$\forall v \text{ with} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0 \quad s.t. \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$$

The first equation $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0$ means that v is in xy plane, the second equation $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$ means that Av is in the plane $\mathbb{I}_1$. Then we have the following result

Claim: $k\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A, k \in \mathbb{R}$

Proof. Any vector v in xy plane satisfies $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v = 0$, this means that $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is orthogonal to xy plane. And we know that Any vector v in xy plane also satisfies $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} Av = 0$, this means that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$ is orthogonal to xy plane. In other words, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$ are both in orthogonal complement of xy plane, we know that the orthogonal complement has dimension 1. So they two vectors are linearly dependent. □

Let $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A$(we choose the simplest case $k=1$, they are same)

Simiarly, we have $\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}2 & -1 & 4\end{bmatrix} A$ and $\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 2 & -1\end{bmatrix} A$. Write the three equations together, we have

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix} A$$

We can compute A by Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 2 & -1 & 4 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -3 & 2 & 1 & 0 & -2 \\ 0 & 3 & 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -3 & 2 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{5}{6} & \frac{7}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \text{So } A = \begin{bmatrix} -\frac{1}{2} & -\frac{5}{6} & \frac{7}{6} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Since A is not unique, the columns of A can multiple any constants. We multiple -2 to column 1, -6 to column 2, and 6 to column 3. We have

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & -2 & 2 \\ -1 & -3 & -3 \end{bmatrix}$$

This is the same result as we get in method 1.

Problem 8. Let $A \in \mathbb{R}^{3\times 3}$ with $r(A) = 2$, and $A^3 = 0$.

a) Prove that $r(A^2) = 1$.

b) Let $\alpha_1 \in \mathbb{R}^3$ is a vector s.t. $A\alpha_1 = 0$. Prove that there exist vectors α_2 and α_3 s.t. $A\alpha_2 = \alpha_1, A^2\alpha_3 = \alpha_2$.

c) Prove that $\alpha_1, \alpha_2, \alpha_3$ are linearly independent.

Proof.

a) From rank theorem: $AB = 0 \Rightarrow r(A) + r(B) \leqslant n$, we know that

$$r(A) + r(A^2) \leqslant 3 \,\text{since}\, A(A^2) = 0$$

So we have $r(A^2) \leqslant 1$. Next, we need to prove $r(A^2) \neq 0$.

Argue by contradiction, suppose $r(A^2) = 0$, we have $A^2 = 0$. Using rank theorem again, then $r(A) + r(A) \leqslant 3$. But we already know that $r(A) = 2$. Contradiction!

So $r(A^2) = 1$.

b) We only need to prove the two equations $A\alpha_2 = \alpha_1$ and $A^2\alpha_3 = \alpha_2$ has solution, namely, $\alpha_1 \in C(A)$ and $\alpha_1 \in C(A^2)$

- $A^3 = A^2(A) = 0 \Rightarrow C(A) \subset N(A^2)$. And we know that

  $$\dim C(A) = r(A) = 2, \dim N(A^2) = 3 - \dim C(A^2) = 3 - r(A^2) = 2$$

  So $C(A) = N(A^2)$, then $\alpha_1 \in N(A) \subset N(A^2) = C(A)$.

- $A^3 = A(A^2) = 0 \Rightarrow C(A^2) \subset N(A)$. And we know that

  $$\dim C(A^2) = r(A^2) = 1, \dim N(A) = 3 - \dim C(A) = 3 - r(A) = 1$$

  So $C(A^2) = N(A)$, then $\alpha_1 \in N(A) = C(A^2)$.

c) Suppose $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$, then

$$A^2(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3) = c_1A^2\alpha_1 + c_2A^2\alpha_2 + c_3A^2\alpha_3 = c_3\alpha_1 = 0 \Rightarrow c_3 = 0 \,\text{since}\, \alpha_1 \neq 0$$

Now we have $c_1\alpha_1 + c_2\alpha_2 = 0$, then

$$A(c_1\alpha_1 + c_2\alpha_2) = c_1A\alpha_1 + c_2A\alpha_2 = c_2\alpha_1 = 0 \Rightarrow c_2 = 0 \,\text{since}\, \alpha_1 \neq 0$$

Now we have $c_1\alpha_1 = 0 \Rightarrow c_1 = 0 \,\text{since}\, \alpha_1 \neq 0$. All $c_i's$ are zero, so $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. □