

Practice Problems Set 1: Answers.

Question 1: (1) False (2) True (3) True (4) True (5) False.

Question 2: (1) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, (2) $T(x, y) = (x+y, -\frac{x}{2} + \frac{y}{2})$, (3) I , (4) 2, (5) $2\sqrt{2}, 1$.Question 3: (a) $a_3 = 4, a_4 = 5$.

(b) $a_n = 2a_{n-1} - a_{n-2}$.

(c) $a_n = n+1$.

Question 4: $y = \frac{1}{2}x + 2$.Question 5: (a) $\begin{bmatrix} -1 & -2 & -3 \\ -2 & -5 & -4 \\ -3 & -4 & -9 \end{bmatrix}$ (b) No (c) Yes.Question 6: (a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (b) 2, 2 (c) No. $M^T M v = 0 \Leftrightarrow M v = 0$.

$$\begin{aligned} \text{Question 7: } 0 &= (c_1 \alpha_1 + \dots + c_n \alpha_n)^T A (c_1 \alpha_1 + \dots + c_n \alpha_n) \\ &= c_1^2 + c_2^2 + \dots + c_n^2 \\ \Rightarrow c_1 &= \dots = c_n = 0. \end{aligned}$$

Question 8:

(a) $Av = \lambda v \quad (v \neq 0)$

$$\Rightarrow \lambda v = Av = A^2 v = A(\lambda v) = \lambda^2 v \Rightarrow \lambda = \lambda^2 \Rightarrow \lambda = 0 \text{ or } \lambda = 1$$

(b) Must have an eigenvalue of 0.

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Question 9: (a) $Av = \lambda v \Rightarrow Bv = f(\lambda)v (= f(A)v)$.(b) B is symmetric.Question 10: (a) $v^T v, 0$.(b) $n-1$.

Over

Practice Problems

Set 2.

Answers

Page 1.

Question 1: (a) True (b) True (c) False (d) True (e) False (f) False

Question 2: (a) 2. (b) 0, 0, 0. (c) 20.

Question 3: (i). eigenvalues: $1, i, i^{-2}, i^{-3}$.

eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}, \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix}, \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}$.

(ii). Diagonalizable. The four eigenvalues are distinct.

Question 4: (i). $A = A^H$.

(ii). $U = \begin{bmatrix} \frac{3-i}{\sqrt{14}} & \frac{3-i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}, U^H A U = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$

Question 5: (i) $\sigma_1 = \sqrt{3}, \sigma_2 = 1$.

(ii) $A = U \Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Question 6: (i) $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, Q^T A Q = \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(ii) $A^k = \begin{bmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 0 & 2^k & 0 \\ -2^{k-1} & 0 & 2^k \end{bmatrix}$

Question 7: (i) $A = \begin{bmatrix} t & 1 & 1 & 0 \\ 1 & t & -1 & 0 \\ 1 & -1 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(ii) $t > 2$.

Question 8: (i) $\|N_x\|^2 = (N_x)^H N_x = x^H N^H N_x = x^H N N^H x = \|N^H x\|^2$.

(ii) Let x be $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i^{\text{th}} \text{ component.}$

(iii)
$$N = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & t_{nn} \end{bmatrix}$$

$$N^H = \begin{bmatrix} \bar{t}_{11} & 0 & \dots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \bar{t}_{1n} & \bar{t}_{2n} & \dots & \bar{t}_{nn} \end{bmatrix}$$

Since $N^H N = N N^H$, comparing the diagonal entries of both sides, we obtain $t_{ij} = 0$ for $i \neq j$.

Question 9: (i) $A = Q \Lambda Q^T$, diagonal entries of Λ are positive (A is positive definite)

$$|A + I_n| = |Q \Lambda Q^T + Q I Q^T| = |Q| |\Lambda + I| |Q^T| = |\Lambda + I| > 1.$$

(ii) $A = U \Sigma V^T$, A is $n \times n$.

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^T \Sigma V^T \Leftrightarrow V^T A^T A V = \Sigma^T \Sigma$$

$$A A^T = (U \Sigma V^T) (U \Sigma V^T)^T = U \Sigma \Sigma^T U^T \Leftrightarrow U^T A A^T U = \Sigma \Sigma^T$$

$\Rightarrow A^T A$ is similar to $A A^T$.

Question 10: (i) $Ax = \lambda x \Rightarrow A^k x = \lambda^k x$ $A^k = 0 \Rightarrow \lambda^k = 0$
 $x \neq 0 \Rightarrow \lambda = 0$

(ii) A symmetric $\Rightarrow A = Q \Lambda Q^T$, $A^k = Q \Lambda^k Q^T \Rightarrow A = \mathbf{0}$ zero matrix.

Question 11: (i) $b > \alpha^T A^{-1} \alpha$.

(ii) $\left\{ \begin{bmatrix} -A^{-1} \alpha \\ 1 \end{bmatrix} \right\}$.

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Practice Problems Set 3

Answers

Page 1.

Question 1: (1) C (2) B (3) B (4) A (5) D

Question 2: (1) CA^T , $D - CA^T B$.

(2) 2.

(3) 1, -3, -3.

(4) $[-\frac{1}{9} \frac{2}{9} -\frac{2}{9}]^T$

(5) 2.

Question 3: (a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}$ (b) No (c) No.Question 4: (a) Indefinite (b) $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ (c) C.Question 5: (a) $\sqrt{3}, \sqrt{2}$ (b) $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{4} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.Question 6: Suppose $d = c_1 \alpha_1 + \dots + c_n \alpha_n$, then

$$d^T A d = \sum_{i=1}^n c_i^2 \|\alpha_i\|^2 \lambda_i = 0 \Rightarrow c_1 = \dots = c_n = 0.$$

Therefore, $\alpha_1, \dots, \alpha_n$ are linearly independent.Question 7: (a) $Bv = \lambda v$, $v^H B v = \lambda \|v\|^2$. On the other hand,
 $v^H B v = (Bv)^H v = \overline{\lambda} v^H v = \overline{\lambda} \|v\|^2 \Rightarrow \lambda = \overline{\lambda}$.(b) $\text{rank}(B) \leq m < n$.(c) No. Consider the SVD of A , $A = U \Sigma V^T$, then $B = A^H A$
 $= V \Sigma^2 V^H$.~~Question 8:~~

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Practice Problems Set 4

Answers

Page 1.

Question 1: (1) A (2) C (3) C (4) B (5) A

Question 2: (1) 3, 2 (2) 2 (3) 1, $1 - u^T v$ (4) r (5) 6.

Question 3: (a) $q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$, $q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(b) $A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

Question 4: $|A| = a^n - a^{n-2}$.

Question 5: (a) $a = 5$, $b = 6$.

(b) $S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.

Question 6: (a) $3\sqrt{2}$, $\sqrt{2}$.

(b) $A = U\Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Question 7: (a) $\begin{bmatrix} I_n & 0 \\ -BA^T & I_m \end{bmatrix} \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} I_n & -A^T B^T \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -BA^T B^T \end{bmatrix}$.

(b) $\text{rank}(B) = r$, $-BA^T B^T$ is negative definite

(c) n , r , $m-r$.

Question 8: (a) $A = P^T P$, P is invertible

$B = Q^T Q$. $AB = P^T P Q^T Q = P^T P Q^T Q P^T (P^T)^{-1}$

AB is similar to $PQ^T Q P^T$, $PQ^T Q P^T$ is positive semidefinite $\Rightarrow AB$ is positive semidefinite.

(b) $C^T A C = I_n$, $C^T A B (C^T)^{-1} = C^T A C C^{-1} B (C^T)^{-1}$, $Q^T M Q = \begin{bmatrix} \wedge & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow Q^T \underbrace{C^T A C}_{I_n} \underbrace{C^{-1} B (C^T)^{-1}}_M Q = Q^T M Q = \begin{bmatrix} \wedge & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, AB is diagonalizable.