1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ and $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$. Then $f(A) =$

(A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

设 $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, 且 $f(t) = 1 + 2t + t^2 + t^4 - 5t^8$, 则 $f(A) =$

(A) $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 8 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

(2) Let A and B be invertible matrices. If A is similar to B, which of the following statements is **NOT** correct?

(A) A^T is similar to B^T.

(B) A^{-1} is similar to B^{-1}.

(C) $A + A^T$ is similar to $B + B^T$.

(D) $A + A^{-1}$ is similar to $B + B^{-1}$.

假定 A 和 B 都是可逆矩阵，且 A 和 B 相似, 下列陈述中哪个是**不正确**的？

(A) A^T 和 B^T 相似.

(B) A^{-1} 和 B^{-1} 相似.

(C) $A + A^T$ 和 $B + B^T$ 相似.

(D) $A + A^{-1}$ 和 $B + B^{-1}$ 相似.

(3) Let $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$. Then the number of positive eigenvalues of A is

(A) 0.

(B) 1.

(C) 2.

(D) 3.

设 $A = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 6 & 8 \\ 3 & 8 & 5 \end{bmatrix}$, 则矩阵 A 的正的特征值的个数为

(A) 0.

(B) 1.

(C) 2.

(D) 3.

(4) The equation $2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ represents a graph of

(A) An ellipsoid.

(B) Hyperboloid of one sheet.

(C) Hyperboloid of two sheets.

(D) Hyperbolic paraboloid.

$2x_1x_2 - 2x_1x_3 + 2x_2x_3 = 1$ 表示的曲面是

(A) 椭球面.

(B) 单叶双曲面.

(C) 双叶双曲面.

(D) 双曲抛物面.

(5) Which of the following statements is correct？

(A) If A is a Hermitian matrix, and $x^H Ax = 0$ for all complex vectors x, then $A = O$, where O denotes the zero matrix.

(B) An $n \times n$ matrix with real eigenvalues and n linearly independent real eigenvectors is symmetric.

(C) If A is a complex matrix, and $A^T = A$, then A is diagonalizable.

(D) Let A, B be $n \times n$ real matrices, then $\det(A + B) = \det A + \det B$.

下面的哪个陈述是正确的？

(A) 如果 A 是厄密特矩阵, 而且对所有的复向量 x 都有 $x^H Ax = 0$, 那么 $A = O$, 这里 O 表示零矩阵.

(B) 一个 n 阶的方阵的所有特征值和 n 个线性无关的特征向量都是实的, 则这个矩阵是对称的.

(C) 如果 A 是一个复矩阵, 且满足 $A^T = A$, 则 A 是可对角化的.

(D) 设 A, B 都是 n 阶实方阵, 则 $\det(A+B) = \det A + \det B$.

ANS: (1) A (2) C (3) C (4) B (5) A

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Suppose A is a 5×4 real matrix with 3 linearly independent columns. The dimension of the row space of A is __________, and the dimension of the left nullspace of A is __________.

设一个 5×4 的实矩阵 A 有三个线性无关的列向量, 则 A 的行空间的维数为 __________, A 的左零空间的维数为 __________.

(2) If the real quadratic form $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ is changed to standard form $f = 6y_1^2$ by orthogonal transformation $x = Qy$, then $a =$ __________.

如果实二次型 $f(x_1, x_2, x_3) = a(x_1^2 + x_2^2 + x_3^2) + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ 经过正交变换 $x = Qy$ 化为标准形 $f = 6y_1^2$, 则 $a =$ __________.

(3) The eigenvalues of $I_3 - uv^T$ are __________. Where I_3 is the 3×3 identity matrix, and u and v are nonzero vectors in $\mathbb{R}^3$.

矩阵 $I_3 - uv^T$ 的特征值为 __________. 这里 I_3 表示 3 阶单位阵, u 和 v 是 $\mathbb{R}^3$ 中的非零向量.

(4) If $A^2 = A$ and rank $(A) = r$, then trace $(A) =$__________.

如果 $A^2 = A$ 且 rank $(A) = r$, 则 trace $(A) =$__________.

(5) Let $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.

If the dimension of the vector space generated by $\alpha_1, \alpha_2, \alpha_3$ is 2, then $a =$__________.

设 $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ a \end{bmatrix}$.

如果由 $\alpha_1, \alpha_2, \alpha_3$ 生成的子空间的维数为 2, 则 $a =$__________.

ANS: (1) 3,2 (2) 2 (3) $1,1 - u^Tv$ (4) r (5) 6

3. (12 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find an orthonormal basis for the column space of A;

(b) Write A as QR, where Q has orthonormal columns and R is upper triangular.

(12 分) 设

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) 求 A 的列空间的一组标准正交基;

(b) 将 A 分解成 QR, 其中 Q 的列是标准正交的向量, R 是上三角矩阵.

Solution.

(a) An orthonormal basis for the column space of A is:

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

(b) The QR factorization of A is as follows:

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} & \frac{5\sqrt{2}}{2} \\ 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

4. (10 points) Compute the determinant of an $n \times n$ matrix A:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

(10 分) 计算 n 阶矩阵 A 的行列式:

$$|A| = \begin{vmatrix} a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & a & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & a & 0 \\ 1 & 0 & \cdots & \cdots & 0 & a \end{vmatrix}, \quad n \geq 2.$$

Solution. $a^n - a^{n-2}$.

5. (10 points) Suppose $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, and A is similar to B.

   (a) Find a and b;

   (b) Find an invertible matrix S, such that $S^{-1}AS = B$.

(10 分) 假定 $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$, 并且 A 和 B 相似.

   (a) 求 a 和 b 的值;

   (b) 求一个可逆矩阵 S, 使得 $S^{-1}AS = B$.

Solution.

   (a) $a = 5$ and $b = 6$;

   (b) S can be chosen to be

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}.$$

6. (12 points) Let

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) Find all the singular values of A;

(b) Find a singular value decomposition of A.

(12 分) 设

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) 求 A 的所有奇异值;

(b) 求矩阵 A 的一个奇异值分解.

Solution.

(a) The singular values of A are $3\sqrt{2}, \sqrt{2}$;

(b) A singular value decomposition of A is as follows:

$$A = U\Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

7. (8 points) Let A be a real symmetric $n \times n$ positive definite matrix and B be an $m \times n$ real matrix.

   (a) Show that the matrix $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ is congruent to the matrix $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$.

   (b) Suppose $\operatorname{rank}(B) = r$. Find the number of positive eigenvalues, the number of negative eigenvalues, and the number of zero eigenvalues of M (counted with multiplicities).

   (8 分) 假定 A 是一个 $n \times n$ 实对称正定矩阵， B 为一个 $m \times n$ 实矩阵.

   (a) 证明：矩阵 $M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ 和矩阵 $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$ 相合.

   (b) 假定 $\operatorname{rank}(B) = r$. 求矩阵 M 的正的特征值的个数, 负的特征值的个数, 以及零特征值的个数 (重根按重复的次数计).

Solution.

(a) Since

$$\begin{bmatrix} I_n & 0 \\ -BA^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} I_n & -A^{-1}B^T \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$$

Therefore, M is congruent to the matrix $\begin{bmatrix} A & 0 \\ 0 & -BA^{-1}B^T \end{bmatrix}$.

(b) Since $\operatorname{rank}(B) = r$ and $-BA^{-1}B^T$ is negative definite, the number of positive eigenvalues is n, the number of negative eigenvalues r, and the number of zero eigenvalues of M is $m - r$. (counted with multiplicities).

8. (8 points) Let A be an $n \times n$ real symmetric positive definite matrix, and B be an $n \times n$ real symmetric positive semidefinite matrix.

   (a) Prove that the eigenvalues of AB are all nonnegative real numbers.

   (b) Prove that AB is diagonalizable.

(8 分) 设 A 为 n 阶正定实对称矩阵, B 为 n 阶半正定实对称矩阵.

   (a) 证明: AB 的所有特征值都是非负实数.

   (b) 证明: AB 可对角化.

Solution.

(a) Since A is positive definite, and B is positive semidefinite, then we can find P and Q such that

$$A = P^T P, B = Q^T Q,$$

where P is an invertible matrix. It follows that

$$AB = P^T P Q^T Q = P^T P Q^T Q P^T (P^T)^{-1}.$$

This means that AB is similar to $PQ^T QP^T$. This together with the fact that $PQ^T QP^T$ is a positive semidefinite matrix complete the proof.

(b) Since A is positive definite, then there is an invertible matrix C such that

$$C^T AC = I_n,\ C^T AB(C^T)^{-1} = C^T ACC^{-1} B(C^T)^{-1}.$$

$M = C^{-1} B(C^T)^{-1}$ is a positive semidefinite matrix, therefore we can find an orthogonal matrix Q such that

$$Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$Q^T C^T AB(C^T)^{-1} Q = Q^T C^T ACC^{-1} B(C^T)^{-1} Q = Q^T MQ = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that AB is diagonalizable.