Problem Set 6 —— Linear Algebra (Fall 2022)

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Please hand in your assignment at the beginning of your seventh tutorial session!

1. The points $P = (x, x, x)$ and $Q = (y, 3y, -1)$ are on two lines in space that don't meet. Choose x and y to minimize the squared distance

$$||P - Q||^2.$$

2. Three planes Π_1, Π_2, Π_3 in the space $\mathbb{R}^3$ are given by the equations

$$\begin{aligned} \Pi_1 &: \quad x + y + z = 0\,, \\ \Pi_2 &: \quad 2x - y + 4z = 0\,, \\ \Pi_3 &: \quad -x + 2y - z = 0\,. \end{aligned}$$

   Determine a matrix representative (in the standard basis of $\mathbb{R}^3$) of a linear transformation taking the xy plane to Π_1, the yz plane to Π_2 and the zx plane to Π_3.

3. Suppose P_1 and P_2 are projection matrices ($P_i^2 = P_i = P_i^T$). Prove: P_1P_2 is a projection if and only if $P_1P_2 = P_2P_1$.

4. The space M of 2 by 2 matrices has the basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Suppose T multiplies each matrix by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, i.e., $T(X) = AX$. Find the matrix representing this linear transformation T with respect to the above mentioned basis.

5. Consider two bases of $\mathbb{R}^3$

$$u_1 = (1, 0, 1),\ u_2 = (2, 1, 0),\ u_3 = (1, 1, 1)$$

   and

$$v_1 = (1, 2, -1),\ v_2 = (2, 2, -1),\ v_3 = (2, -1, -1).$$

   Define a linear transformation as follows：$T(u_i) = v_i, i = 1, 2, 3$.

   (a) Find the transition matrix from u_1, u_2, u_3 to v_1, v_2, v_3 ;

   (b) Find the matrix representation of T with respect to the basis u_1, u_2, u_3 ;

   (c) Find the matrix representation of T with respect to the basis v_1, v_2, v_3 .