(C) kA is orthogonal.

(D) $P^{-1}AP$ is orthogonal.

若 A, B 是正交矩阵, k 是非零实数, P 是可逆矩阵, 则 ( )

(A) AB 也是正交矩阵.

(B) $A+B$ 也是正交矩阵.

(C) kA 也是正交矩阵.

(D) $P^{-1}AP$ 也是正交矩阵.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let u be a n-dimensional nonzero real column vector and $A = I + uu^T$, where I is the $n \times n$ identity matrix. Then all the distinct eigenvalues of A are ________ with algebraic multiplicities ________ .

设 u 为一个 n 维非零实列向量, 且 $A = I + uu^T$, 其中 I 为 n 阶单位阵, 则 A 的所有互不相同的特征值为 ________, 相应的代数重数为 ________ .

(2) The singular values of the matrix $A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ are ________.

矩阵 $A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ 的奇异值是 ________.

(3) Let $A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Then $A^{2024} =$ ________

设 $A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, 则 $A^{2024} =$ ________

(4) Let x_n be a sequence defined by $x_0 = 0, x_1 = 1$ and $x_n = 2x_{n-1} + x_{n-2}$ for all $n \geq 2$. Then $x_{100} =$ ________.

设 x_n 为按照以下方式定义的一个数列: $x_0 = 0, x_1 = 1$, $x_n = 2x_{n-1} + x_{n-2}$ 对所有 $n \geq 2$, 则 $x_{100} =$ ________.

(5) Consider the following system of linear equations:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + ax_3 = 0 \\ 3x_1 + x_2 - x_3 = 0 \end{cases}$$

If the columns of a nonzero 3×3 matrix B are solutions to the above system, then
$a =$ __________, $|B| =$ __________.

考虑以下线性方程组:

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + ax_3 = 0 \\ 3x_1 + x_2 - x_3 = 0 \end{cases}$$

如果非零 3 阶矩阵 B 的列向量都是以上线性方程组的解, 则
$a =$ __________, $|B| =$ __________.

3. (12 points) Consider the 5×5 matrix

$$A = \begin{bmatrix} 3 & 5 & 8 & 1 & -6 \\ -3 & -4 & -7 & 1 & 9 \\ 3 & 4 & 7 & -1 & -9 \\ -3 & -4 & -7 & 1 & 9 \\ 2 & 3 & 5 & 0 & -5 \end{bmatrix}.$$

The characteristic polynomial of A is $p(x) = -x^3(x-1)^2$. And the reduced row echelon form of $A - I$ is given by

$$A - I \to \begin{bmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find an invertible matrix S such that $S^{-1}AS$ is diagonal.

(b) Compute A^k for any positive integer k.

(12 分) 考虑以下 5×5 的矩阵

$$A = \begin{bmatrix} 3 & 5 & 8 & 1 & -6 \\ -3 & -4 & -7 & 1 & 9 \\ 3 & 4 & 7 & -1 & -9 \\ -3 & -4 & -7 & 1 & 9 \\ 2 & 3 & 5 & 0 & -5 \end{bmatrix}.$$

已知矩阵 A 的特征多项式为 $p(x) = -x^3(x-1)^2$. 矩阵 $A - I$ 的简化阶梯型矩阵如下:

$$A - I \to \begin{bmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) 求一个可逆矩阵 S 使得 $S^{-1}AS$ 为对角矩阵.

(b) 求 A^k, 其中 k 为任意的正整数.

4. (8 points) Find the matrix Q in the QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(8 points) 求以下矩阵 A 的 QR 分解中的 Q 矩阵:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

5. (20 points) Let

$$A = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}$$

(a) Show that A is Hermitian.

(b) Find the eigenvalues and eigenvectors of A.

(c) Can we find a unitary matrix U such that $U^H AU$ is diagonal? If yes, find one such matrix. Otherwise, explain.

(20 分) 设

$$A = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}$$

(a) 证明: A 是埃尔米特矩阵.

(b) 求矩阵 A 的特征值和特征向量.

(c) 是否可以找到酉矩阵 U 使得 $U^H AU$ 为对角矩阵? 如果可以, 求出一个满足要求的酉矩阵, U. 如若不然, 请说明理由.

6. (10 points) *Let B be an $n \times n$ real symmetric matrix. A quadratic form $g(y) = y^T By$ is called negative definite if $y^T By < 0$ for all nonzero real vectors $y \in \mathbb{R}^n$.* Consider the following quadratic form

$$f(x_1, x_2, x_3) = (\lambda - 3)x_1^2 + 4x_1x_2 + \lambda x_2^2 + (\lambda - 1)x_3^2.$$

(a) Find a real symmetric matrix A, such that $f(x_1, x_2, x_3) = x^T Ax$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(b) Determine the range of λ such that $f(x_1, x_2, x_3)$ is negative definite.

(10 分) 设 B 为一个 n 阶实对称矩阵. 如果对所有非零的 $y \in \mathbb{R}^n$ 都有 $y^T B y < 0$, 则称二次型 $g(y) = y^T B y$ 为负定的. 考虑以下二次型

$$f(x_1, x_2, x_3) = (\lambda - 3)x_1^2 + 4x_1x_2 + \lambda x_2^2 + (\lambda - 1)x_3^2.$$

(a) 找一个实对称矩阵 A, 使得 $f(x_1, x_2, x_3) = x^T A x$, 其中 $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(b) 求出 λ 的取值范围使得 $f(x_1, x_2, x_3)$ 为负定的.

7. (10 points) Let A, B be two $n \times n$ real matrices. Suppose A has n distinct eigenvalues and $AB = BA$.

(a) Show that B is diagonalizable.

(b) Suppose $n = 3$. Show that there exists a polynomial of degree at most 2, $g(x)$, such that $B = g(A)$.

(c) Suppose $n > 3$. Show that there exists a polynomial of degree at most $n - 1$, $f(x)$, such that $B = f(A)$.

(10 分) 设 A, B 为两个 $n \times n$ 实矩阵. 假定 A 有 n 个互不相同的特征值, 且 $AB = BA$.

(a) 证明: B 为可对角化矩阵.

(b) 证明: 若 $n = 3$, 存在次数不超过 2 的多项式, $g(x)$, 使得 $B = g(A)$.

(c) 证明: 若 $n > 3$, 存在次数不超过 $n - 1$ 的多项式, $f(x)$, 使得 $B = f(A)$.