Quiz B (Week 7) Linear Algebra

Part I

1. Find a basis for the following subspace $\mathbb{R}^4$:

   (a) The vectors for which $x_1 = 2x_4$.

   (b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

   (c) The subspace spanned by $(1,1,1,1), (1,2,3,4)$, and $(2,3,4,5)$.

2. Find bases for the four fundamental subspaces associated with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

3. (a) Find the rank of A, and give a basis for its nullspace

$$A = LU = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 1 & 2 & \\ 3 & 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (b) The first three rows of U are a basis for the row space of A; true or false?

   Column $1, 3, 6$ of U are a basis for the column space of A; true or false?

   The four rows of A are a basis for the row space of A; true or false?

   (c) Find as many linearly independent vectors b as possible for which $Ax = b$ has a solution.

4. If e_1, e_2, e_3 are in the column space of a 3 by 5 matrix, does it have a right-inverse?

Part II

1. Let $\boldsymbol{\alpha}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{\alpha}_2 = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$, $\boldsymbol{\alpha}_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\boldsymbol{\alpha}_4 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$, $\boldsymbol{\alpha}_5 = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$.

   Please choose the maximal independent vectors from $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$ and $\boldsymbol{\alpha}_5$. In addition, express the other vectors as the linear combination of the independent vectors.

2. The augmented matrix of the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\left[\begin{array}{cccccc:c} 1 & -2 & 0 & 3 & 0 & 2 & 5 \\ 2 & -4 & 1 & 4 & 0 & 9 & 16 \\ 1 & -2 & 1 & 1 & 1 & 10 & 18 \end{array}\right]$$

   Please give a particular solution and the complete solutions of the systems.

3. Assume that the vectors $\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ are linearly independent and $\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r, \boldsymbol{\beta}$ are linearly dependent

   - Please show that the vector $\boldsymbol{\beta}$ can be represented as a linear combination of $\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ such that

$$\boldsymbol{\beta} = c_1\boldsymbol{\alpha}_1 + c_2\boldsymbol{\alpha}_2 + \cdots c_r\boldsymbol{\alpha}_r.$$

   - The coefficients $c_1, c_2, \cdots, c_r$ are unique.

4. Let $\boldsymbol{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\boldsymbol{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be non-zero n-vectors.

$$M = \begin{bmatrix} 0 & b_1 & b_2 & \cdots & b_n \\ a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

- $\text{Rank}(M) = ?$ Please give you argumentation.
- Give a decomposition of $M = AB$, where A is a full column rank matrix and B is a full row rank matrix.

5. We take the space $\mathbf{P}_2$ in which the vectors are polynomials $p(x)$ of degree 2. They look like

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

Suppose that

$$\boldsymbol{\varepsilon}_1 = 1, \quad \boldsymbol{\varepsilon}_2 = x, \quad \boldsymbol{\varepsilon}_3 = x^2,$$

and

$$\boldsymbol{\eta}_1 = 1, \quad \boldsymbol{\eta}_2 = 1 + x, \quad \boldsymbol{\eta}_3 = 1 + x + x^2,$$

are two different bases of $\mathbf{P}_2$. $\mathscr{A}$ is the differential transformation.

$$(\mathscr{A}\boldsymbol{\varepsilon}_1, \mathscr{A}\boldsymbol{\varepsilon}_2, \mathscr{A}\boldsymbol{\varepsilon}_3) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)A,$$

$$(\mathscr{A}\boldsymbol{\eta}_1, \mathscr{A}\boldsymbol{\eta}_2, \mathscr{A}\boldsymbol{\eta}_3) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)B.$$

Please write down the matrix A and B.

Part III

1. Let $A \in \Re^{m \times n}$. Please show that

$$\text{Rank}(A) = \text{Rank}(A^T A).$$