

1. $T \supset C \supset B \supset B$

2. (1) $\begin{bmatrix} a & b \\ 2-a & 3-b \end{bmatrix}$, $a, b \in \mathbb{R}$

(2) $T \cong S'$

(3) $K = \mathbb{Q}$

(4) $\dim N(A^T A) = 1$

(5) $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. (1) let ξ_1, ξ_2, ξ_3 be three linearly independent solutions.

$\xi_1, \xi_2, \xi_1 - \xi_2$ linearly independent

Solutions to $Ax = 0$

$\Rightarrow 4 - \text{rank}(A) \geq 2 \Rightarrow \text{rank}(A) \leq 2$.

Also, $\text{rank}(A) \geq 2 \Rightarrow \text{rank}(A) = 2$.

(the first two columns of rows)

A are linearly independent)

(2) $\begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 4 & 3 & 5 & -1 & -1 \\ a & 1 & 3 & b & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -4 & 2 \\ 0 & 1 & -1 & 5 & -3 \\ 0 & 0 & 4-2a & 4a+b-5 & 4-2a \end{bmatrix}$

$\text{rank}(A) = 2 \Rightarrow 4-2a = 0 \quad 4a+b-5 = 0 \Rightarrow a = 2, b = -3$.

Complete solution:

$x = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \end{bmatrix}$, $k_1, k_2 \in \mathbb{R}$.

$$4. (1) A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & a^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2) A^{-1} = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & -a^2 & 0 & 1 \end{bmatrix}$$

$$(3) \kappa = \begin{bmatrix} 0 \\ \frac{1}{a} \\ \frac{1}{a} \\ 0 \end{bmatrix}$$

$$5. A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 0 & 5 \\ 1 & 1 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 9 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(a) N(A)'s \text{ basis: } \left\{ \begin{bmatrix} -9 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$(b) C(A^T)'s \text{ basis: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

$$(c) C(A)'s \text{ basis: } \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\}$$

$$(d) \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = 9 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$6. (a) v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \quad (b) L's \text{ basis: } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$(c) \text{Projection: } \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix}.$$

$$7. (a) \begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 0 & -4 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_n = \begin{bmatrix} +1/2 \\ -1/2 \\ 1 \end{bmatrix}, \quad x_p = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$$

$$A^1 = \begin{bmatrix} 2 & 2 & 0 \\ -2 & -2 & 0 \\ 4 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A^2 & b \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & -1 \\ -2 & -2 & 0 & 1 \\ 4 & 4 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \xi_3 = \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad k_1, k_2 \in \mathbb{R}.$$

$$(b) \quad c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_3 = 0$$

$$A \xi_1 = 0$$

$$A^2 (c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_3) = A^2 0$$

$$A \xi_2 = \xi_1$$

$$\Rightarrow c_3 = 0$$

$$A (c_1 \xi_1 + c_2 \xi_2) = 0$$

$$A \xi_3 = \xi_2$$

$$c_1 \xi_1 = 0$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

$$\Rightarrow c_2 = 0 \Rightarrow$$

ξ_1, ξ_2, ξ_3 are linearly independent.

$$8. (a) \quad A^{-1} = I_n + \frac{u u^T}{1 - u^T u}$$

$$(b) \quad (I_n - U U^T)^{-1} = I_n + U (I_m - V^T U)^{-1} V^T.$$

Assume $I_m - V^T U$ is invertible.

Suggested Solutions.

1. $ACDDC$

2. (1) $\begin{bmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$ (2) 1 (3) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$ (4) 12 (5) $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$

3. $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{8}{3} & \frac{2}{3} \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$

4. $C(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$N(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$

$C(A^T) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

$N(A^T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

5. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 3 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

7. (a) $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{bmatrix}$, $A^{2004} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} -1 & -5 & -7 \\ 0 & 2 & -2 \\ 1 & 3 & 3 \end{bmatrix}$ 每一行与零核为它的任意常数倍.

8. (a) $A^3 = 0 \Rightarrow \left. \begin{array}{l} C(A^2) \subseteq N(A) \\ \dim N(A) = 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \dim C(A^2) = \text{rank}(A^2) \leq 1 \\ \text{If } A^2 = 0 \text{ then } C(A) \subseteq N(A) \end{array} \right\} A^2 \neq 0 \Rightarrow \text{rank}(A) \leq 1$

$\Rightarrow \text{rank}(A^2) = 1$.

(b) $C(A^2) \subseteq N(A)$

$N(A) = \text{span}(\alpha_1)$

$A^2 \alpha_3 = A(A \alpha_3) = \alpha_1$, (c) $C_1 \alpha_1 + C_2 \alpha_2$

$A^2 \alpha_3 = \alpha_1$. existence.

$C(A^2) = \text{span}(\alpha_1)$

$A \alpha_2 = \alpha_1$. existence. $C_1 = C_2 = C_3 = 0$

Suggested solutions:

1. (a) True (b) True (c) ~~True~~ False (d) False (e) True

2. (a) m (b) 3, 1 (c) $\begin{bmatrix} 9/4 \\ 6/4 \\ 3/4 \end{bmatrix}$

3. (i) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A^{-1} = \begin{bmatrix} 3 & -4 & 2 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

4. (a) Row space: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

dimension: 2

Column space: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$

dimension: 2

Null space: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

dimension: 2

left nullspace: $\left\{ \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\}$

dimension: 1

(b) $b_3 - 2b_2 + 5b_1 = 0$

(c) $x = x_p + x_n = \begin{bmatrix} -1 \\ 0 \\ 1/3 \end{bmatrix} + u \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

5. $Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-3}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{6}} & \frac{-\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{11}} \end{bmatrix}$

6. (i) $g_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $g_2 = \begin{bmatrix} \frac{-6/\sqrt{90}}{7/\sqrt{90}} \\ \frac{-2/\sqrt{90}}{1/\sqrt{90}} \end{bmatrix}$

(ii) $A = \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} g_1^T a & g_1^T b \\ 0 & g_2^T b \end{bmatrix}$
 $= \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3\sqrt{10} \end{bmatrix}$

(iii) $\hat{x} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$

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7. (i) By definition.

$$(ii) \begin{bmatrix} -\frac{3}{2} & -\frac{11}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

8. (i) $\dim W = 3$.

(ii) Three independent vectors.

of independent vectors = $\dim W$.(iii) One possible choice: u_1, u_3, u_5 .

9. (i) Check the definitions: Columnspace, rank,

$$(ii) C(B) \subseteq N(A) \Rightarrow \dim C(B) \leq \dim N(A)$$

$$\text{rank}(B) \leq n - \text{rank } A \quad \checkmark$$

$$(iii) N(A^T A) = N(A) \quad (\text{prove this!})$$

$$\Rightarrow \text{rank}(A^T A) = \underline{\text{rank}(A)}$$

$$A_{m \times n}$$

If $\text{rank}(A) = n$, $A^T A$ is invertible \Rightarrow projection matrix is invertibleIf $m < n$, $\text{rank}(A) < m$, Projection matrix is NOT invertible.