

# Diagonalization of a Matrix (矩阵的对角化)

Lecture 22

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# Diagonalization of a Matrix

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# Introduction

We start right off with the one essential computation. It is perfectly simple and will be used in every section of this chapter. The eigenvectors diagonalize a matrix:

## Theorem

*Suppose the  $n$  by  $n$  matrix  $A$  has  $n$  linearly independent eigenvectors. If these eigenvectors are the columns of a matrix  $S$ , then  $S^{-1}AS$  is a diagonal matrix  $\Lambda$ . The eigenvalues of  $A$  are on the diagonal of  $\Lambda$ :*

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

# Diagonalization of a Matrix

We call  $S$  the “eigenvector matrix” and  $\Lambda$  the “eigenvalue matrix”.

There are four remarks before giving any examples or applications.

- 1: Any matrix with distinct eigenvalues can be diagonalized. If the matrix  $A$  has no repeated eigenvalues—the numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct—then its  $n$  eigenvectors are automatically independent. Therefore any matrix with distinct eigenvalues can be diagonalized.
- 2: The diagonalizing matrix is not unique. An eigenvector  $x$  can be multiplied by a constant, and remains an eigenvector.

## Remarks

- 3: Other matrices  $S$  will not produce a diagonal  $\Lambda$ .
- 4: Not all matrices possess  $n$  linearly independent eigenvectors, so **not all matrices are diagonalizable**. The standard example of a “defective matrix” is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$\lambda = 0$  is a double eigenvalue—its algebraic multiplicity is 2. But the geometric multiplicity is 1—there is only independent eigenvector. We can not construct  $S$ .

# Diagonalizability and Invertibility

There is **no** connection between diagonalizability and invertibility. Diagonalization can fail **only if** there are **repeated** eigenvalues. The problem is the shortage of eigenvectors—which are needed for  $S$ . That needs to be emphasized:

## Proposition

*Diagonalizability of  $A$  depends on enough eigenvectors. Invertibility of  $A$  depends on nonzero eigenvalues.*

Algebraic multiplicity and geometric multiplicity.

# Theorem

## Theorem

*If eigenvectors  $x_1, x_2, \dots, x_k$  correspond to different eigenvalues  $\lambda_1, \dots, \lambda_k$ , then those eigenvectors are linearly independent.*

## Proof.

Suppose first that  $k = 2$ , and that some combination of  $x_1$  and  $x_2$  produces zero:  $c_1x_1 + c_2x_2 = 0$ . Multiplying by  $A$ , we find  $c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0$ .

Subtracting  $\lambda_2$  times the previous equation, the vector  $x_2$  disappears:

$c_1(\lambda_1 - \lambda_2)x_1 = 0$ . Since  $\lambda_1 \neq \lambda_2$  and  $x_1 \neq 0$ , we are forced into  $c_1 = 0$ .

Similarly  $c_2 = 0$ , and the two vectors are independent; only the trivial combination gives zero. The same argument extends to any number of vectors. □

# Examples of Diagonalization

The main point of this section is  $S^{-1}AS = \Lambda$ . The eigenvector matrix  $S$  converts  $A$  into its eigenvalue matrix  $\Lambda$  (diagonal). We see this for projections and rotations.

**Example 1.** The projection  $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$  has eigenvalue matrix

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$



## Example 2

**Example 2.** The eigenvalues and eigenvectors of a rotation matrix.  $90^\circ$  rotation:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

How can a vector be rotated and still have its direction unchanged?

Imaginary numbers? Complex numbers are needed even for real matrices! See section 5.5 for more!

## Powers and Products: $A^k$ and $AB$

The eigenvalues of  $A^2$  are exactly  $\lambda_1^2, \dots, \lambda_n^2$ , and every eigenvector of  $A$  is also an eigenvector of  $A^2$ .

### Theorem

*The eigenvalues of  $A^k$  are  $\lambda_1^k, \dots, \lambda_n^k$ , and each eigenvector of  $A$  is still an eigenvector of  $A^k$ . When  $S$  diagonalizes  $A$ , it also diagonalizes  $A^k$ :*

$$A^k = S^{-1} A^k S$$

*Each  $S^{-1}$  cancels an  $S$ , except for the first  $S^{-1}$  and the last  $S$ .*

If  $A$  is invertible, this rule also applies to its inverse (the power  $k = -1$ ). The eigenvalues of  $A^{-1}$  are  $1/\lambda_i$ . That can be seen without diagonalizing:

$$\text{if } Ax = \lambda x \text{ then } x = \lambda A^{-1}x \text{ and } \frac{1}{\lambda}x = A^{-1}x.$$

## Example 3

**Example 3.** If  $K$  is rotation through  $90^\circ$ , then  $K^2$  is rotation through  $180^\circ$  (which means  $-I$ ) and  $K^{-1}$  is rotation through  $-90^\circ$ :

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of  $K$  are  $i$  and  $-i$ ; their squares are  $-1$  and  $-1$ ; their reciprocals are  $1/i = -i$  and  $1/(-i) = i$ . Then  $K^4$  is a complete rotation through  $360^\circ$ .

$$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and also } \Lambda^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Notice also that complex numbers are needed even for real matrices.

# Product

- For a product of two matrices, we can ask about the eigenvalues of  $AB$ —but we won't get a good answer.
- In general, if  $A$  has an eigenvalue  $\lambda$  and  $B$  has an eigenvalue  $\mu$ ,  $AB$  does not have  $\lambda\mu$  as its eigenvalue. For instance, we have two matrices with zero eigenvalues, while  $AB$  has  $\lambda = 1$ :

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors of this  $A$  and  $B$  are completely different, which is typical.

- Similarly, the eigenvalues of  $A + B$  generally have nothing to do with  $\lambda + \mu$ .

## Theorem

If the eigenvector is the same for  $A$  and  $B$ , then the eigenvalues multiply and  $AB$  has the eigenvalue  $\mu\lambda$ . But there is something more important.

There is an easy way to recognize when  $A$  and  $B$  share a full set of eigenvectors, and that is a key question in quantum mechanics:

### Theorem

*Diagonalizable matrices share the same eigenvector matrix  $S$  if and only if  $AB = BA$ .*

## Proof

**Proof.** If the same  $S$  diagonalizes both  $A = S\Lambda_1 S^{-1}$  and  $B = S\Lambda_2 S^{-1}$ , we can multiply in either order:

$$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1 \Lambda_2 S^{-1} \text{ and } BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2 \Lambda_1 S^{-1}.$$

Since  $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$  (diagonal matrices always commute) we have  $AB = BA$ .

# Proof

In the opposite direction, suppose  $AB = BA$ . Starting from  $Ax = \lambda x$ , we have

$$ABx = BAx = B\lambda x = B(\lambda x) = \lambda Bx.$$

Thus  $x$  and  $Bx$  are both eigenvectors of  $A$ , sharing the same  $\lambda$ . If we assume for convenience that the eigenvalues of  $A$  are distinct—the eigenspaces are all one-dimensional—then  $Bx$  must be a multiple of  $x$ . In other words,  $x$  is an eigenvector of  $B$  as well as  $A$ . The proof with repeated eigenvalues is a little longer (left as an exercise).

# Heisenberg's uncertainty principle

Heisenberg's uncertainty principle comes from non-commuting matrices, like position  $P$  and momentum  $Q$ . Position is symmetric, momentum is skew-symmetric, and together they satisfy:

$$QP - PQ = I.$$

The uncertainty principle follows directly from the Schwarz inequality  $(Qx)^T(Px) \leq \|Qx\| \|Px\|$  of Section 3.2:

$$\|x\|^2 = x^T x = x^T (QP - PQ)x \leq 2\|Qx\| \|Px\|$$

The product of  $\|Qx\|/\|x\|$  and  $\|Px\|/\|x\|$ —momentum and position errors, when the wave function is  $x$ —is at least  $\frac{1}{2}$ . It is impossible to get both errors small, because when you try to measure the position of a particle you change its momentum.



## Final Note

At the end we come back to  $A = S\Lambda S^{-1}$ .

- That factorization is particularly suited to take powers of  $A$ , and the simplest case  $A^2$  makes the point.
- The  $LU$  factorization is hopeless when squared, but  $S\Lambda S^{-1}$  is perfect. The square is  $S\Lambda^2 S^{-1}$ , and the eigenvectors are unchanged.
- By following those eigenvectors we will solve difference equations and differential equations.

# Homework Assignment 22

5.2: 1, 6, 7, 8, 11, 13, 19, 38, 43.