Similarity Transformations (相似变换)

Lecture 25 and 26

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Similarity Transformations

Similar Matrices

Now we look at all combinations $M^{-1}AM$– formed with any invertible M on the right and its inverse on the left.

A whole family of matrices $M^{-1}AM$ is similar to A, there are two questions:

1. What do these similar matrices $M^{-1}AM$ have in common?
2. With a special choice of M, what special form can be achieved by $M^{-1}AM$?

Theorem

Suppose that $B = M^{-1}AM$. Then A and B have the same eigenvalues. Every eigenvector x of A corresponds to an eigenvector $M^{-1}x$ of B.

Example 1

Example 1 $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 1 and 0. Each B is $M^{-1}AM$:

- If $M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$: triangular with $\lambda = 1$ and 0.
- If $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: projection with $\lambda = 1$ and 0.
- If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (invertible), then $B =$ an arbitrary matrix with $\lambda = 1$ and 0.

Change of Basis = Similarity Transformation

Similar matrices represent the same transformation T with respect to different bases.

Theorem

The matrices A and B that represent the same linear transformation T with respect to two different bases (the v's and the V's) are similar:

$$[T]_{V\to V} = [I]_{v\to V}[T]_{v\to v}[I]_{V\to v}$$
$$B = M^{-1}AM.$$

Proof: Sketch

If

$$T(V_1, V_2, \cdots, V_n) = (V_1, V_2, \cdots, V_n)B$$
$$T(v_1, v_2, \cdots, v_n) = (v_1, v_2, \cdots, v_n)A$$
$$(V_1, V_2, \cdots, V_n) = (v_1, v_2, \cdots, v_n)M$$
$$(v_1, v_2, \cdots, v_n) = (V_1, V_2, \cdots, V_n)M^{-1},$$

then

$$\begin{aligned}
T(V_1, V_2, \cdots, V_n) &= T((v_1, v_2, \cdots, v_n)M) \\
&= (T(v_1, v_2, \cdots, v_n))M \\
&= (v_1, v_2, \cdots, v_n)AM \\
&= (V_1, V_2, \cdots, V_n)M^{-1}AM.
\end{aligned}$$

Therefore, $B = M^{-1}AM$.

Figure 5.5

Example Suppose T is projection onto the line L at angle $\theta (= 135^\circ)$. This linear transformation is completely determined without the help of a basis. But to represent T by a matrix, we do need a basis. Figure 5.5 offers two choices, the standard basis $v_1 = (1,0), v_2 = (0,1)$ and a basis V_1, V_2 chosen especially for T.

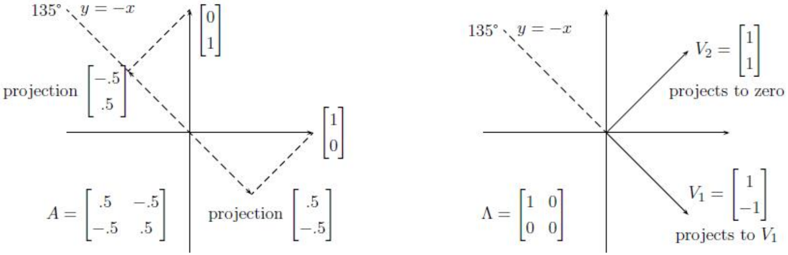

Figure 5.5: Change of basis to make the projection matrix diagonal.

Summary

- The way to simplify that matrix A–in fact to diagonalize it–is to find its eigenvectors. They go into the columns of M (or S) and $M^{-1}AM$ is diagonal. The algebraist says the same thing in the language of linear transformations: **Choose a basis consisting of eigenvectors.** The standard basis led to A, which was not simple. The right basis led to B, which was diagonal.
- $M^{-1}AM$ does not arise in solving $Ax = b$. There the basic operation was to multiply A (on the left side only!) by a matrix that subtracts a multiple of one row from another. Such a transformation preserved the nullspace and row space of A; it normally changes the eigenvalues.

Triangular Forms with a Unitary M

Theorem

(Schur's lemma) *There is a unitary matrix $M = U$ such that $U^{-1}AU = T$ is triangular. The eigenvalues of A appear along the diagonal of this similar matrix T.*

Can you prove this theorem?

Remark:

- This lemma applies to all matrices, with no assumption that A is diagonalizable.
- We could use it to prove that the powers A^k approach zero when all $|\lambda_i| < 1$, and the exponentials e^{At} approach zero when all Re $\lambda_i < 0$–even without the full set of eigenvectors which was assumed in sections 5.3 and 5.4.

Example

Example 2. $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ has the eigenvalues $\lambda = 1$ (twice).

1. The only line of eigenvectors goes through $(1,1)$.
2. After dividing by $\sqrt{2}$, this is the first column of U, and the triangle $U^{-1}AU = T$ has the eigenvalues on its diagonal.
3. The triangular T is given as follows:

$$U^{-1}AU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T.$$

This triangular form will show that any symmetric or Hermitian matrix–whether its eigenvalues are distinct or not–has a **complete set of orthonormal eigenvectors.**

Spectral Theorem

We need a unitary matrix such that $U^{-1}AU$ is diagonal. **Schur's lemma** has just found it. This triangular T must be diagonal, because it is also Hermitian when $A = A^H$:

$$T = T^H$$

$$(U^{-1}AU)^H = U^H A^H (U^{-1})^H = U^{-1}AU.$$

Spectral Theorem

The diagonal matrix $U^{-1}AU$ represents a key theorem in linear algebra:

Theorem

Every real symmetric A can be diagonalized by an orthogonal matrix Q. Every Hermitian matrix can be diagonalized by a unitary U:

(Real)

$$Q^{-1}AQ = \Lambda \text{ or } A = Q\Lambda Q^T.$$

(Complex)

$$U^{-1}AU = \Lambda \text{ or } A = U\Lambda U^H.$$

The columns of Q (or U) contain orthonormal eigenvectors of A.

Remarks

- In the real symmetric case, the eigenvalues and eigenvectors are real at every step. That produces a real unitary U–an orthogonal matrix.
- A is the limit of symmetric matrices with distinct eigenvalues. As the limit approaches, the eigenvectors stay perpendicular. This can fail if $A \neq A^T$:

$$A(\theta) = \begin{bmatrix} 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix}$$

has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. As $\theta \to 0$, the only eigenvector of the nondiagonalizable matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Example

Example 3 The spectral theorem says that this $A = A^T$ can be diagonalized:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) A has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.

(b) Every Hermitian matrix with k different eigenvalues has a spectral decomposition into $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$, where P_i is the projection onto the eigenspace for λ_i.

(c) Since there is a full set of eigenvectors, the projections add up to the identity. And since the eigenspaces are orthogonal, two projections produce zero: $P_j P_i = 0$.

Normal Matrices

We are very close to answering an important question, so we keep going: For which matrices is $T = \Lambda$?

Theorem

The matrix N is normal if it commutes with N^H: $NN^H = N^H N$. For such matrices, and no others, the triangular $T = U^{-1}NU$ is the diagonal Λ. Normal matrices are exactly those that have a complete set of orthonormal eigenvectors.

Remarks:

- Symmetric, skew-Symmetric, and Orthogonal are normal.
- Hermitian, skew-Hermitian, and Unitary are normal.

Proof: Sketch

Step 1: If N is normal, then so is the triangular $T = U^{-1}NU$:

$$TT^H = U^{-1}NUU^HN^HU = U^{-1}NN^HU$$
$$= U^{-1}N^HNU = U^HN^HUU^{-1}NU = T^HT.$$

Step 2: A triangular matrix T that is normal must be diagonal. (See Problems 19-20 at the end of this section).

Thus, if N is normal, the triangular $T = U^{-1}NU$ must be diagonal. Since T has the same eigenvalues as N, it must be Λ. The eigenvectors of N are the columns of U, and they are orthonormal. That is the good case. We turn now from the best possible matrices (normal) to the worst possible (defective). See:

$$\textbf{Normal} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } \textbf{Defective} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The Jordan Form

- Our next goal is to make $M^{-1}AM$ as nearly diagonal as possible.
- The result of this supreme effort at diagonalization is the **`Jordan form`** J.
- If A has a full set of eigenvectors, we take $M = S$ and we arrive at $J = S^{-1}AS = \Lambda$. Then the Jordan form coincides with the diagonal Λ.
- This is impossible for a nondiagonalizable matrix. For every missing eigenvector, the Jordan form will have a 1 just above its main diagonal.

The Jordan Block

Theorem

If A has s independent eigenvectors, it is similar to a matrix with s blocks:

$$J = M^{-1}AM = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}.$$

Each Jordan block J_i is a triangular matrix that has only a single eigenvalue λ_i and only one eigenvector.

Jordan Block

The Jordan Block:

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

- The same λ_i will appear in several blocks, if it has several independent eigenvectors.

Jordan Block

The Jordan Block:

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

- The same λ_i will appear in several blocks, if it has several independent eigenvectors.
- Two matrices are similar if and only if they share the same Jordan form J.

Example

Example 4 $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ all lead to $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(T)

$$M^{-1}TM = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(B)

$$P^{-1}BP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

(A)

$$U^{-1}AU = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = T \quad \text{and then} \quad M^{-1}TM = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = J.$$

Example

Example 5 $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

- Zero is a triple eigenvalue for A and B, so it will appear in all their Jordan blocks.
- There can be a single 3 by 3 block, or a 2 by 2 and a 1by 1 block, or three 1 by 1 blocks.
- A count of the eigenvectors will determine J when there is nothing more complicated than a triple eigenvalue.

Example

Example 6 Application to difference and differential equations(powers and exponentials). If $A = MJM^{-1}$, we have

$$A^k = MJM^{-1}MJM^{-1}\cdots MJM^{-1} = MJ^kM^{-1}$$

J is block diagonal, and the powers of each block can be taken separately:

$$(J_i)^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

This block J_i will enter when λ is a triple eigenvalue with a single eigenvector.

Exponential

Its exponential is in the solution to the corresponding differential equation:

$$e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

Here

$$I + J_i t + \frac{(J_i t)^2}{2!} + \cdots$$

produces

$$1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots = e^{\lambda t}$$

on the diagonal.

Similarity Transformations

1. A is diagonalizable: The columns of S are eigenvectors and $S^{-1}AS = \Lambda$.
2. A is arbitrary: The columns of M include "generalized eigenvectors " of A, and the Jordan form $M^{-1}AM$ is block diagonal.
3. A is arbitrary: The unitary U can be chosen so that $U^{-1}AU = T$ is triangular.
4. A is normal, $AA^H = A^HA$: then U can be chosen so that $U^{-1}AU = \Lambda$.

Special Cases of Normal Matrices, all with orthonormal eigenvectors

(a) If $A = A^H$ is Hermitian, then all λ_i are real.

(b) If $A = A^T$ is real symmetric, then Λ is real and $U = Q$ is orthogonal.

(c) If $A = -A^H$ is skew-Hermitian, then all λ_i are purely imaginary.

(d) If $A = A^H$ is orthogonal or unitary, then all $|\lambda_i| = 1$ are on the unit circle.

Exercise

设 A 是三阶实对称矩阵， A 的秩为 2，即 $r(A)=2$, 且

$$A\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

(I) 求 A 的所有特征值和特征向量;

(II) 求矩阵 A.

Exercise

已知矩阵 $A=\begin{pmatrix} -2 & -2 & 1 \\ 2 & x & -2 \\ 0 & 0 & -2 \end{pmatrix}$ 与 $B=\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & y \end{pmatrix}$ 相似.

(I) 求 x, y;

(II) 求可逆矩阵 P, 使得 $P^{-1}AP=B$.

Properties of Eigenvalues and Eigenvectors

1. Symmetric Matrices: $A = A^T$;
   real λ's; orthogonal eigenvectors: $x_i^T x_j = 0$.
2. Orthogonal: $Q^T = Q^{-1}$;
   all $|\lambda| = 1$; orthogonal $\overline{x_i}^T x_j = 0$.
3. Skew-symmetric: $A^T = -A$
   imaginary λ's; orthogonal $\overline{x_i}^T x_j = 0$.
4. Complex Hermitian: $\overline{A}^T = A$
   real λ's; orthogonal eigenvectors: $\overline{x_i}^T x_j = 0$.
5. Positive definite: $x^T A x > 0$, A is symmetric
   all $\lambda > 0$; eigenvectors can be chosen to be orthogonal

Properties of Eigenvalues and Eigenvectors

6. Similar Matrices:$B = M^{-1}AM$;
   $\lambda(A) = \lambda(B)$; $\quad x(B) = M^{-1}x(A)$.
7. Projection: $P = P^2 = P^T$;
   $\lambda = 1;0$; $\quad$ column space; nullspace.
8. Reflection: $I - 2uu^T$
   $\lambda = -1; 1, 1, \cdots, 1$; $\quad u; u^{\perp}$.
9. Rank-1 matrix: uv^T
   $\lambda = v^T u; 0, \cdots, 0 \quad u; v^{\perp}$.
10. Inverse: A^{-1}
    $\frac{1}{\lambda(A)}$; $\quad$ eigenvectors of A.

Properties of Eigenvalues and Eigenvectors

11. Shift: $A+cI$;
    $\lambda(A)+c$; eigenvectors of A.
12. Cyclic permutation: $P^n=I$;
    $\lambda_k = e^{\frac{2\pi ik}{n}}$; $x_k=(1,\lambda_k,\cdots,\lambda_k^{n-1})$.
13. Diagonalizable: $S\Lambda S^{-1}$
    diagonal of Λ; columns of S are independent.
14. Symmetric: $Q\Lambda Q^T$
    diagonal of Λ(real); columns of Q are orthonormal.
15. Jordan: $J=M^{-1}AM$
    diagonal of J; each block gives 1 eigenvector
16. Every matrix: $A=U\Sigma V^T$
    rank(A)=rank(Σ); eigenvectors of A^TA, AA^T in V,U.

Homework Assignment 25 and 26

5.6: 2, 5, 6, 7, 15, 19, 20, 21, 30, 39.