Quiz D (Week 15) Linear Algebra

(A) Problems for Chapter 5

1. Find the matrix $P = \dfrac{aa^T}{a^T a}$ that projects any vector onto the line through $a = (2, 1, 2)^T$.

   (a) What is the only nonzero eigenvalue of P,

   (b) Every eigenvalue of a Skew-Hermitian matrix is pure imaginary.

   (c) Two eigenvectors of a Skew-Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

2. Let $\alpha, \beta \in R^n$ be nonzero vectors, $\alpha = (1, a_2, \dots, a_n)^T$.

   Write everything you know about the matrix $A = (I + \beta\alpha^T)$:

   (a) All eigenvalues of A and the corresponding eigenvectors,

   (b) Is A diagonalizable ? If yes, give you the matrix S such that $AS = S\Lambda$.

3. Prove that $A^H A$ is always a Hermitian matrix.

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$$

   - Compute $B = AA^H$ and $C = A^H A$.
   - Compute the eigenvalues of B.
   - Give a unitary matrix U such that $B = U^H \Lambda U$.

4. For the matrices of B and C in the last problem,

   (a) Show that the both matrices have the same nonzero eigenvalues.

   (b) If x is an eigenvector of B, show that $A^H x$ is an eigenvector of C.

   (c) Show that 0 an eigenvalue of C.

   (d) Compute the eigenvector of C corresponding to the eigenvalue 0 of C.

5. If $A^H = A$, the matrix is a Hermitian matrix. Please show that

   (a) If $A = A^H$, then for all complex vectors x, the $x^H Ax$ is real.

   (b) Every eigenvalue of a Hermitian matrix is real.

   (c) Two eigenvectors of a Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

6. If $K^H = -K$, the matrix is Skew-Hermitian. Please show that
   (a) If $K^H = -K$, then for all complex vectors x, the $x^H K x$ is pure imaginary.
   (b) Every eigenvalue of a Skew-Hermitian matrix is pure imaginary or zero.
   (c) Two eigenvectors of a Skew-Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.
   (d) If $A^T = -A$ is a n by n matrix and n is an odd number, then $\det(A) = 0$.

7. If $A = R + iS$ is a Hermitian matrix, are the real matrices R and S symmetric. Please give you argument.

8. If $A^2 = -I$, what are the eigenvalues of A ? If A is a real n by n matrix show that n must be even, and give an example.

9. If K is a skew-symmetric matrix, show that $Q = (I - K)(I + K)^{-1}$ is an orthogonal matrix. Find Q if $K = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

(B) Problems for Chapter 6

1. Let A be a n by n symmetric matrix, please show that there is a constant c such that
$$|x^T A x| \leq c \cdot (x^T x), \quad \forall x \in \Re^n.$$

2. If A is positive definite and a_{11} is increased prove that the determinant is increased.

3. Let A be a positive semi-definite matrix, please show the generalized Cauchy-Schwarz inequality $|x^T A y| \leq (x^T A x)(y^T A y)$.

4. Let $\boldsymbol{I}$ be n times n Identity matrix and $\boldsymbol{e} \in \boldsymbol{R}^n$ whose every entry is 1. Please show that :
   (a) The matrix $(n\boldsymbol{I} - \boldsymbol{e}\boldsymbol{e}^T)$ is positive semi-definite.
   (b) For any $\boldsymbol{x} \in \boldsymbol{R}^n$,
$$\sum_{j=1}^{n} x_j^2 \geq \frac{1}{n}(\sum_{j=1}^{n} x_j)^2.$$

5. Please show that for every symmetric matrix A, there is a constants s_0 and t_0, such that

   (a) $s_0 I + A$ is positive semi-definite, $sI + A$ is positive definite for any $s > s_0$.

   (b) $t_0 I + A$ is negative semi-definite, $tI + A$ is positive definite for any $t < t_0$.

6. Let $\quad f(x) = \sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} x_i x_j$

   (a) Write this function in the quadratic form $\quad x^T A x$.

   (b) What is the eigenvalues and the corresponding eigenvectors of A.

7. Let A be an $n \times n$ positive definite matrix, $\alpha, \beta \in \Re^n$.

$$H = \begin{bmatrix} A & \alpha \\ \alpha^T & a \end{bmatrix}, \qquad L = \begin{bmatrix} I_n & 0_{n\times 1} \\ \beta^T & 1 \end{bmatrix},$$

   (a) If $LH = \begin{bmatrix} A & \alpha \\ 0_{1\times n} & x \end{bmatrix}$, $\beta =?$

   (b) If H is positive definite, $a >?$

8. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be a symmetric matrix, $\det(A_{11}) \neq 0$. Please give a matrix $C = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ such that $C^T A C = \begin{bmatrix} A_{11} & 0 \\ 0 & Z \end{bmatrix}$. What are the matrices X and Z ?