Southern University of Science and Technology
Linear Algebra I Final Examination
Fall 2017   A

Department:____Math____Class:_____

Student ID:_____Name:_____

Answer all parts of Questions (1)-(10). Total is 100 points.

(1) (10 points, 2 points each) True or false. No need to justify.

(a) Let A be an $n \times n$ matrix ($n > 1$), then $\det(kA) = k\det(A)$.   (   )

(b) For any real matrix A and $\delta > 0$, then matrix $\delta I + A^T A$ is positive definite. (   )

(c) Let A be a 2×2 matrix whose eigenvalues are 2 and 3, then the matrix $A^2 - 3A + 6I$ is singular.   (   )

(d) Let A be an $n \times n$ matrix satisfying $A^2 = A$ and $A \neq I$, then $\det(A) = 0$.   (   )

(e) Let A be a real square matrix, then A and A^T have the same eigenvectors.   (   )

(2) (12 points, 3 points each) Fill in the blanks.

(a) Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$. $\mathbf{C_{ij} = (-1)^{i+j}\det M_{ij}}$, **Delete row i, column j**. According to the formula **Cofactors along row i** , $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}$, then $C_{11} + C_{12} + C_{13} =$ _____.

(b) Let A be a 3×2 real matrix whose column vectors a_1, a_2 are linearly independent. The eigenvalues of $P = A(A^T A)^{-1} A^T$ are _____.

(c) Let A be a 3×3 matrix and its eigenvalues are $-1, 2, 3$, then $\det(A^3 - 2A^2 + A + 2I) =$ _____.

(d) Which of following four assertions are true? They are _____ .

   1. If Q_1 and Q_2 are orthogonal matrices, then $Q_1 Q_2$ is orthogonal.
   2. If H_1 and H_2 are positive definite, then $H_1 H_2$ is positive definite.
   3. If A and B are similar, then they have the same eigenvalues.
   4. If A and B have same eigenvalues, then A and B are similar.

(3) (10 points) Given
$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(i) Find the determinant of A.

(ii) Decide whether A is positive definite, negative definite, semidefinite, or indefinite.

(iii) Find all the eigenvalues of A and their associated eigenvectors.

(iv) Is A diagonalizable? If so, diagonalize it. Otherwise, explain why.

(4) (10 points) Consider
$$\frac{du}{dt} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} u = Au.$$

(i) Find e^{At}.

(ii) If $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, solve for $u(t)$.

(5) (10 points) Let
$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

(i) Find AA^T and $A^T A$.

(ii) Find all the singular values of A.

(iii) Find all the eigenvectors of $A^T A$.

(iv) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

(v) Find the pseudoinverse of A, namely, $A^+ = V\Sigma^+ U^T$.

(6) (10 points) Consider

$$A = \begin{bmatrix} 1 & t \\ t & 4 \end{bmatrix}.$$

    (i) For which numbers t is matrix A positive definite?

    (ii) Factor $A = LDL^T$ when t is in the range for positive definiteness.

    (iii) Find the minimum value of $P(x) = \frac{1}{2}(x_1^2 + 2tx_1x_2 + 4x_2^2) - x_1 - x_2$ for t in the range found in (ii).

    (iv) What is the minimum if $t = 2$?

(7) (10 points) Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

Prove that $\det(A) = 0$ in the following ways:

    (i) Show that the columns of A are dependent.

    (ii) Explain why all 120 terms are zero in the "big formula" for $\det(A)$.

(8) (10 points) Consider a complex matrix $C = A + iB$ with A and B real and

$$D = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

    (i) If C is a Hermitian matrix, show that D is symmetric.

    (ii) If C is a unitary matrix, show that D is orthogonal.

(9) (10 points) Prove the following statements:

    (i) If eigenvectors $x_1, x_2, \cdots, x_k$ of matrix A correspond to different eigenvalues

$$\lambda_1, \lambda_2, \cdots, \lambda_k,$$

then those eigenvectors are linearly independent.

    (ii) Two eigenvectors of a real symmetric matrix B, if they come from different eigenvalues, are orthogonal to one another.