2022-2023 Fall B final

1.(1) D (2) C (3) D (4) A (5) C

2.(1) $\lambda=-1$ (2) 2 (3) 3 (4) $-3, 1, -3$ (5) $Q=\begin{bmatrix} -\frac{\sqrt2}{2} & \frac{\sqrt6}{6} & \frac{\sqrt3}{3} \\ -\frac{\sqrt2}{2} & \frac{\sqrt6}{6} & \frac{\sqrt3}{3} \\ 0 & -\frac{\sqrt6}{3} & \frac{\sqrt3}{3} \end{bmatrix}$ $R=\begin{bmatrix} -\sqrt2 & \sqrt2 & 3\sqrt2 \\ & \sqrt6 & -\sqrt6 \\ & & \sqrt3 \end{bmatrix}$

3.(a) $A^2-A=O$ Suppose $x\in\mathbb{R}^n$

$(A-I_n)Ax=0$

$Ax\in C(A)$

$Ax\in N(A-I_n)$

$\dim C(A)=\dim(A-I_n)$

$\operatorname{rank}(A)=n-\operatorname{rank}(A-I_n)$

(b) Suppose

$V_{\lambda_1}=\operatorname{span}\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_r\}$ $(r\le n, r\in N)$

$V_{\lambda_2}=\operatorname{span}\{\varepsilon_{r+1},\varepsilon_{r+2}\cdots,\varepsilon_n\}$

$\dim V_{\lambda_1}=r=n-\operatorname{rank}(A-I_n)$

$\dim V_{\lambda_2}=n-r=n-\operatorname{rank}(A)$

$A\varepsilon_1=\lambda_1\varepsilon_1,\ A\varepsilon_2=\lambda_1\varepsilon_1\cdots A\varepsilon_r=\lambda_1\varepsilon_r$

$A\varepsilon_{r+1}=\lambda_2\varepsilon_{r+1},\cdots A\varepsilon_n=\lambda_2\varepsilon_n$

$\because V_{\lambda_1}\oplus V_{\lambda_2}=\mathbb{R}^n$

$\angle\ \beta=\{\varepsilon_1,\varepsilon_2,\cdots\varepsilon_r,\varepsilon_{r+1}\cdots\varepsilon_n\}$ is a basis for $\mathbb{R}^n$.

$\angle A_{(\beta)}=\begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & \\ & & & & \ddots \ \lambda_2 \end{bmatrix}$

$\det(\lambda I_n-A)=(\lambda-\lambda_1)^r(\lambda-\lambda_2)^{n-r}$

(c) $\because A^2=A$

$\therefore \Lambda^2=\Lambda$

$\therefore \lambda_i\in\{0,1\}$

$\operatorname{rank}(A)=3$

$\therefore p(\lambda)=(\lambda-1)^3\lambda^{n-3}$

$\operatorname{trace}(A)=3$

$\Lambda(e_{ii})\quad e_{ii}=\begin{cases}1, & 3\\ 0, & n-3\end{cases}$

$\operatorname{trace}(A)=3+2n$

$\det(A+2I_n)=\det(\Lambda+2I_n)=2^{n-3}\cdot3^3=27\cdot2^{n-3}$

(d) $A^k=A$

$f(A)=(1+C_{d-1}+\cdots+C_1)A+C_0I$

suppose $E_{A,\lambda=1}=\operatorname{span}\{e_1,e_2,e_3\}$

$E_{A,\lambda=0}=\operatorname{span}\{e_4,\cdots,e_n\}$

$x^Tf(A)x=x^T[(1+C_{d-1}+\cdots+C_1)A+C_0I]x$

$=(1+C_{d-1}+\cdots+C_1)x^TAx+C_0x^Tx$

$C_0x^Tx\le x^Tf(A)x\le(1+C_{d-1}+\cdots+C_1+C_0)x^Tx$

$\therefore$

4.(a) $\frac{x_1}{\|x_1\|}=\begin{bmatrix}-\frac{\sqrt3}{3}\\ \frac{\sqrt3}{3}\\ -\frac{\sqrt3}{3}\end{bmatrix}$ $\frac{x_2}{\|x_2\|}=\begin{bmatrix}-\frac{\sqrt6}{6}\\ -\frac{\sqrt6}{3}\\ -\frac{\sqrt6}{6}\end{bmatrix}\Rightarrow x_3=\begin{bmatrix}1\\0\\1\end{bmatrix}$

(b) $A=Q\Lambda Q^T=\begin{bmatrix}-\frac{\sqrt3}{3} & \frac{\sqrt6}{6} & \frac{\sqrt2}{2}\\ \frac{\sqrt3}{3} & -\frac{\sqrt6}{3} & 0\\ -\frac{\sqrt3}{3} & -\frac{\sqrt6}{6} & \frac{\sqrt2}{2}\end{bmatrix}\begin{bmatrix}1 & & \\ & 2 & \\ & & 3\end{bmatrix}\begin{bmatrix}-\frac{\sqrt3}{3} & \frac{\sqrt3}{3} & -\frac{\sqrt3}{3}\\ \frac{\sqrt6}{6} & -\frac{\sqrt6}{3} & -\frac{\sqrt6}{6}\\ \frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2}\end{bmatrix}=\begin{bmatrix}\frac{13}{6} & -\frac13 & \frac56\\ -\frac13 & \frac53 & \frac13\\ \frac56 & \frac13 & \frac{13}{6}\end{bmatrix}$

5.(a) $AQ_1=Q_1\begin{bmatrix}\lambda_1 & \cdots & \\ 0 & 0 & \\ 0 & \cdots & \end{bmatrix}$

$Q_m^T\cdots Q_2^TQ_1^TAQ_1Q_2\cdots Q_m=T=\begin{bmatrix}\lambda_1 & * & \cdots & * \\ & \ddots & & \vdots \\ & & & \lambda_m\end{bmatrix}$

$Q^TAQ=T$ T for upper triangular matrix

$T^T=(Q^TAQ)^T=Q^TAQ=T$

$\therefore T$ is symmetric $\therefore T$ is diagonal.

$\angle\ T=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_m\end{bmatrix}$

(b) $x^TAx=x^TQ\Lambda Q^Tx=(Q^Tx)^T\Lambda Q^Tx=y^T\Lambda y=\sum_{i=1}^m\lambda_iy_i^2\ge\sum_{i=1}^m\lambda_{min}y_i^2=\lambda_{min}\sum_{i=1}^my_i^2=\lambda_{min}\|x\|=\lambda_{min}$

$\min\{x^TAx\}=\lambda_{min}\le\lambda$.

(c) $\min_{\|x\|=1}\{x^TAx\}=\lambda_{min}$

$\overset{\omega^T\omega=1}{=}\lambda_{min}\omega^T\omega$

$=\omega^T(\lambda_{min}\omega)$

if ω is the eigenvector to $\lambda=\lambda_{min}$

$\therefore \min\{x^TAx\}=\omega^TA\omega$

$f(x,y)=[x\ y]\begin{bmatrix}1 & -2\sqrt3\\ -2\sqrt3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+2025=v^TAv+b^Tb$

$\mu=\min_{\|v\|=1}\{v^TAv+b^Tb\}=\lambda_{min}+b^Tb=-2+2025=2023$

(e) $\lambda_{min}^{n} x^T x \le x^T A^n x \le \lambda_{max}^{n} x^T x$

$$\lim_{n\to\infty} \frac{n\ln\lambda_{min} + \ln x^T x}{n^2} = \lim_{n\to\infty} \frac{\ln a_n}{n^2} \le \lim_{n\to\infty} \frac{\ln(|\lambda|_{max}^{n} x^T x)}{n^2} = \lim_{n\to\infty} \frac{\ln(x^T x) + n\ln(\lambda_{max})}{n^2} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\ln a_n}{n^2} = 0$$

$$\therefore \lim_{n\to\infty} \sqrt[n^2]{a_n} = 1$$

1.(a) $A^T(I_n + A) = A^T + A^T A = A^T + I_n = (A + I_n)^T$

$\det(A^T)\det(I_n + A) = \det(I_n + A)^T$

$\therefore (\det A)\det(I_n + A) = \det(I_n + A)$

(b) $\because A^T A = I \quad B^T B = I$

$\therefore |A|^2 = 1 \quad |B|^2 = 1$

$\because |A| + |B| = 0.$

$\therefore |B| = -|A|$

$\Leftarrow A^T(A+B) = I_n + A^T B$

$\Rightarrow |A^T||A+B| = |I_n + A^T B|$

$\Leftarrow |A||A+B| = |I_n + A^T B|$

$\because (A^T B)^T A^T B = I$

$\therefore A^T B$ is orthogonal

$\therefore |A||A+B| = |A^T B||I_n + A^T B|$

$\therefore |B||A+B| = |I_n + A^T B|$

$\therefore |A||A+B| = -|A||A+B|$

$\because |A| \ne 0. \quad \therefore |A+B| = 0$

(c)

$|I_n + A| = |A||I_n + A|$

$|I_n + A| = -|B||I_n + A|$

$A^T(A+B) = I_n + A^T B$

$|A||A+B| = |I_n + A^T B| \ne 0$

$\therefore A^T B$ has no $\lambda = -1$.

$|A^T B| = |A||B| = -|A|^2 = -1$

$$\therefore \Lambda_{A^T B} = \begin{bmatrix} 3 & & \\ & 1 & \\ & & -\frac{1}{3} \end{bmatrix}$$

$$A^T B = S\Lambda_{A^T B}S^{-1} = \begin{bmatrix} 1 & & 3 \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 3 & & \\ & 1 & \\ & & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & & -3 \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & & -10 \\ & 1 & \\ & & -\frac{1}{3} \end{bmatrix}$$

$$B = AA^T B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & & -10 \\ & 1 & \\ & & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} & 1 & \\ 3 & & -10 \\ & & -\frac{1}{3} \end{bmatrix}$$