

15'

1. (1) D (2) C (3) B (4) B (5) A

25'

2. (1) $u^T u + 1, 1$; $1, n-1$.

(2) $\sqrt{5}, \sqrt{7}$

(3)
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2024 & 10/2 \times 2023 \\ 0 & 0 & 0 & 1 & 2024 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(4)
$$\frac{1}{2\sqrt{2}} \left[(1+\sqrt{2})^{100} - (1-\sqrt{2})^{100} \right]$$

(5) $1, 0$.

2'

Question 3: 12'

(a) There are two eigenvalues of A : $\lambda_1 = 0$ & $\lambda_2 = 1$ A basis for the eigenspace $N(A)$ corresponding to $\lambda_1 = 0$ is:

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigenspace, $N(A-I)$, corresponding to $\lambda_2 = 1$ is:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$S = \begin{bmatrix} -1 & 3 & 7 & 1 & 3 \\ -1 & -2 & -3 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$(b) \text{ Since } S^{-1}AS = \Lambda = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},$$

$$A = S\Lambda S^{-1} \Rightarrow A^k = S\Lambda^k S^{-1} \\ = S\Lambda S^{-1} = A.$$

8'
Question 4: 8'

Orthonormalize the columns of A by Gram-Schmidt to obtain

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

20'
Question 5: 20'

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$$(a) \quad A^H = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}^H = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix} = A \Rightarrow A \text{ is Hermitian.}$$

$$(b) \quad \begin{vmatrix} A - \lambda I \end{vmatrix} = \begin{vmatrix} 0-\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & 0-\lambda \end{vmatrix} = (-\lambda)(-1)^{1+1} \left[(1-\lambda)(-\lambda) + i^2 \right]$$
$$+ (-i)(-1)^{1+2} [(-\lambda)i]$$

$$= -\lambda(\lambda^2 - \lambda - 1) + \lambda = -\lambda(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow \lambda = 0, -1, 2.$$

$$\lambda = 0: (A - \lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = -1: (A - \lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

$$\lambda = 2: (A - \lambda I)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix}$$

(c)

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{i}{\sqrt{3}} & \frac{2i}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

10'

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Question 6: 10'

$$(a) \quad A = \begin{bmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{bmatrix}.$$

$$(\lambda-4)(\lambda+1) > 0$$

$$(\lambda-1)(\lambda-4)(\lambda+1) < 0$$

(b) ^{negative} A is definite if and only if

$$1) \det A_1 = \lambda - 3 < 0$$

$$2) \det A_2 = \lambda^2 - 3\lambda - 4 > 0 \rightarrow$$

$$3) \det A_3 = (\lambda-1)(\lambda^2 - 3\lambda - 4) < 0$$



$$\lambda - 4 > 0 \text{ or } \lambda < -1$$

$$\lambda - 3 < 0$$

$$\lambda - 1 < 0$$

$$\left. \begin{array}{l} \lambda - 4 > 0 \text{ or } \lambda < -1 \\ \lambda - 3 < 0 \\ \lambda - 1 < 0 \end{array} \right\} \Rightarrow \lambda < -1.$$

Question 7. (a) Since A has n distinct eigenvalues, A is diagonalizable, i.e., there exists an invertible P , such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$AB = BA \Rightarrow P^{-1}APP^{-1}BP = P^{-1}ABP = P^{-1}BAP = P^{-1}BP P^{-1}AP.$$

Therefore, we can assume A is a diagonal matrix, that is,

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$$AB = \begin{bmatrix} \lambda_1 b_{11} & \lambda_1 b_{12} & \cdots & \lambda_1 b_{1n} \\ \lambda_2 b_{21} & \lambda_2 b_{22} & \cdots & \lambda_2 b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n b_{n1} & \lambda_n b_{n2} & \cdots & \lambda_n b_{nn} \end{bmatrix} = \begin{bmatrix} \lambda_1 b_{11} & \lambda_2 b_{12} & \cdots & \lambda_n b_{1n} \\ \lambda_1 b_{21} & \lambda_2 b_{22} & \cdots & \lambda_n b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 b_{n1} & \lambda_2 b_{n2} & \cdots & \lambda_n b_{nn} \end{bmatrix}$$

Comparing the entries, we see that $\lambda_i b_{ij} = \lambda_j b_{ij}$, $\lambda_i \neq \lambda_j$, ($i \neq j$) hence $b_{ij} = 0$ ($i \neq j$). It follows that B is a diagonal matrix.

(b) (c) A and B can be diagonalized by some invertible matrix P , i.e.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}, \quad P^{-1}BP = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_n \end{bmatrix}$$

Where λ_i, μ_i are the eigenvalues of A and B correspondingly.

Choose n polynomials as follows,

$$f_i(x) = \frac{(x-\lambda_1) \cdots (x-\lambda_{i-1})(x-\lambda_{i+1}) \cdots (x-\lambda_n)}{(\lambda_i-\lambda_1) \cdots (\lambda_i-\lambda_{i-1})(\lambda_i-\lambda_{i+1}) \cdots (\lambda_i-\lambda_n)}, \quad i=1, 2, \dots, n$$

Let $f(x) = \mu_1 f_1(x) + \mu_2 f_2(x) + \cdots + \mu_n f_n(x)$. Then it can be verified that $f(\lambda_i) = \mu_i$, $i=1, 2, \dots, n$.

$$P^{-1}BP = \begin{bmatrix} f(\lambda_1) & & \\ & f(\lambda_2) & \\ & & \ddots \\ & & & f(\lambda_n) \end{bmatrix} = f(P^{-1}AP) = P^{-1}f(A)P,$$

thus $B = f(A)$.