MID-SMESTER TEST

Linear Algebra I A

This three-hour long test has 11 problems in total. Write ***all your answers*** on the examination book.

(1) (10 points, 2 points each) True or false. No need to justify.

(a) If $\alpha_1, \alpha_2, \cdots, \alpha_r$ are linearly independent, and let $\beta_i = \sum_{j=1}^{i} \alpha_j, i = 1, 2, \cdots, r$ then $\beta_1, \beta_2, \cdots, \beta_r$ are linearly independent.

(b) If the rows of a matrix are linearly independent, then its columns are also linearly independent.

(c) If $Ax = b$ is not solvable, then $Ax = 0$ has only 0 as its solution.

(d) Let A be an m by n matrix, then $\text{rank}(A^T A) = \text{rank}(A)$.

(e) If the nullspace of AB is contained in the nullspace of B, namely, $N(AB) \subset N(B)$, then $\text{rank}(AB) = \text{rank}(B)$.

Solution. (a) True, (b) False, (c) False, (d) True, (e) True.

(2) (12 points, 3 points each) Fill in the blanks.

(a) Suppose $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$, then $X = \underline{\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}}$.

(b) Let A be an $n \times n$ square matrix and $A^2 + A - 5I = 0$, then the inverse of $A + 2I$ is $\underline{\frac{1}{3}(A - I)}$.

(c) The projection of a vector $b = (1, 2, 3)^T$ onto the line through $a = (1, 1, 1)^T$ is $\underline{(2, 2, 2)^T}$.

(d) If $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, then $A^6 = \underline{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$.

(3) (12 points) This question is about the matrix

$$A = \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 5 & -1 & 5 \\ 4 & -2 & 9 & 1 & 4 \end{bmatrix}$$

(a) Find a LU factorization of A.
(b) What is the rank of A?
(c) Give a basis for the row space of A.
(d) Give a basis of the column space of A.
(e) What is the dimension of the left nullspace of A?
(f) What is the general solution to $Ax = 0$?

Solution:

(a)

$$A = \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 5 & -1 & 5 \\ 4 & -2 & 9 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) The rank of A is 3.
(c) A basis for the row space of A:

$$\begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

(d) The first, third, and fifth column of matrix is a basis of the column space of A:

$$\begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}.$$

(e) The dimension of the left nullspace is $m - r = 4 - 3 = 1$.
(f) The general solution to $Ax = 0$ is:

$$c_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, c_1, c_2 \in \mathbb{R}.$$

(4) (9 points)For which values of a does the following system of linear equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 3x_2 + (a+2)x_3 = 3 \\ x_1 + ax_2 - 2x_3 = 0 \end{cases}$$

have no solution, one solution, or infinitely many solutions? When the system has infinitely many solutions, find all its solutions.

Solution:

$$[A \mid b] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & a+2 & 3 \\ 1 & a & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -a & -1 \\ 0 & 0 & (a-3)(a+1) & a-3 \end{bmatrix}$$

Case 1: $a = -1$, it has no solution.
Case 2: $a \neq -1$ and $a \neq 3$, it has a unique solution.
Case 3: $a = 3$, it has infinitely many solutions. The solutions are

$$(x_1, x_2, x_3)^T = (3, -1, 0)^T + k(-7, 3, 1)^T, \forall k \in \mathbb{R}.$$

(5) (9 points) Which of the following subsets are actually subspaces? If the subset is a subspace, find its basis and dimension. If not, explain why.
(i) All vectors in $\mathbb{R}^2$ whose components are positive or zero.
(ii) The plane of vectors $(x, y, z, t) \in \mathbb{R}^4$ that satisfy $x + y - 2z - t = 0$.
(iii) All skew-symmetric 3 by 3 matrices ($A^T = -A$).

Solution:

(1) This subset is not a subspace, since it is not closed under scalar multiplication.
(2) This subset is a subspace, it is the nullspace of the 1 by 4 matrix $[1\ 1\ -2\ -1]$. It has dimension 3, and its basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(3) The subset is closed under addition and scalar multiplication, hence it is a subspace. It has dimension 3, and its basis is

$$\left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

(6) (10 points) Let

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

(i) Use the Gauss-Jordan method to find its inverse A^{-1}.

(ii) Notice that A could be written as the addition of an identity matrix and a rank one matrix as follows:

$$A = I_3 + \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Now we choose two general vectors $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $1+\mathbf{u}^T\mathbf{v} \neq 0$,

$$A = I_3 + \mathbf{u}\mathbf{v}^T$$

is invertible and its inverse takes the form

$$A^{-1} = I_3 + k\mathbf{u}\mathbf{v}^T.$$

Find k.

(iii) Use the result from (ii) to find the inverse of

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Solution:

(i) We use the Gauss Jordan Method to find the inverse of A:

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(ii) Assume $B = I_3 + k\mathbf{u}\mathbf{v}^T$. From $AB = (I_3 + \mathbf{u}\mathbf{v}^T)(I_3 + k\mathbf{u}\mathbf{v}^T) = I_3$, we get $k = -\frac{1}{1+\mathbf{u}^T\mathbf{v}}$. Hence, the inverse of A is $I_3 - \frac{1}{1+\mathbf{u}^T\mathbf{v}}\mathbf{u}\mathbf{v}^T$.

(iii) The inverse of B is

$$B^{-1} = \left(I_4 + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{bmatrix}.$$

(7) (i) (4 points) If A is an n-by-n matrix such that $A^2 = A$ and $\text{rank}(A) = n$, prove that $A = I$.

(ii) (4 points) Consider the following rank 1 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Find A^{2017}.

Solution:

(i) $A(A-I) = 0$, since $\text{rank}(A) = n$, $Ax = 0$ has only zero as its solution, therefore all the columns of $A - I$ are zero vectors. Thus $A - I = 0$, which completes the proof.

(ii) A can be written as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

Therefore

$$A^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = 14 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

and

$$A^{2017} = 14^{2016} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

(8) (10 points)

(i) (4 points) Describe the Gram-Schmidt procedure in detail.

(ii) (6 points) Apply the Gram-Schmidt process to the columns of

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix},$$

and write the result in the form $A = QR$.

Solution:

(i) Gram-Schmidt Procedure:

(a) The Gram-Schmidt process starts with independent vectors $a_1, a_2, \cdots, a_n$ and ends with orthonormal vectors $q_1, q_2, \cdots, q_n$.

(b) At step j it subtracts from a_j its components in the directions $q_1, \cdots, q_{j-1}$ that are already settled:

$$A_j = a_j - (q_1^T a_j)q_1 - \cdots - (q_{j-1}^T a_j)q_{j-1}.$$

(c) Then q_j is the unit vector

$$q_j = \frac{A_j}{||A_j||}.$$

(ii) According to the Gram-Schmidt Procedure:

$$q_1 = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}, q_2 = \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \\ \frac{2}{5} \end{bmatrix}, q_3 = \begin{bmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}.$$

The QR factorization is as follows:

$$A = QR = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

(9) (10 points) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 4 \end{bmatrix}.$$

(i) Check the consistency of the system of linear equations: $Ax = b$.

(ii) If there is no solution, find the best estimate $\hat{x}$ by least squares.

Solution: (i) We convert the augmented matrix $[A \mid b]$ to its row echelon form:

$$[A \mid b] = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The system is inconsistent.

(ii) The normal equations are

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^T \begin{bmatrix} 3 \\ 2 \\ 4 \\ 4 \end{bmatrix}$$

These simplify to

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \hat{x} = \begin{bmatrix} 13 \\ 22 \\ 54 \end{bmatrix}$$

The solution of this system is $\hat{x} = \begin{bmatrix} 2.75 \\ -0.25 \\ 0.25 \end{bmatrix}$.

(10) (10 points)

(i) Let V and W be subspaces of $\mathbb{R}^n$ and $V \subset W$. Prove that $\mathsf{dim}V \leq \mathsf{dim}W$.

(ii) Let A and B be $m \times n$ and $n \times k$ matrices respectively. Prove that

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}.$$

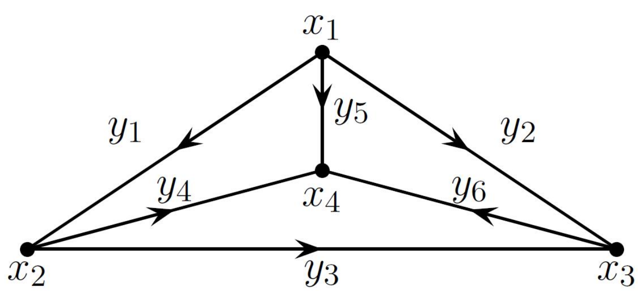
Solution: (a) Let $\mathcal{B} = \{v_1, v_2, \cdots, v_k\}$ be a basis of V (it could be empty if V is the trivial subspace). By definition of basis $\mathcal{B} \subset W$ is linearly independent and then, it can be completed to a basis. As a consequence $\mathsf{dim}\ W \geq k = \mathsf{dim}\ V$.

(b) On the one hand, the columns of AB are linear combinations of the columns of A. Then, $C(AB) \subset C(A)$. Item (a) implies that $\mathsf{dim}\ C(AB) \leq \mathsf{dim}\ C(A)$. Since, given any matrix, the dimension of its column space is equal to its rank, we have that $\text{rank}(AB) \leq \text{rank}(A)$.

One the other hand, the rows of AB are linear combinations of the rows of B. Then, $C((AB)^T) \subset C(B^T)$. Item (a) implies that $\mathsf{dim}\ C((AB)^T) \leq \mathsf{dim}\ C(B^T)$. Since, given any matrix, the dimension of its row space is also equal to its rank, we have that $\text{rank}(AB) \leq \text{rank}(B)$.

The two inequalities together give the the proof of part (b).

(11) (10 points) This question is about the directed graph



(i) (4 points) Write out the incidence matrix A for the graph. Verify that the vector $(1, 1, 1, 1)^T$ is in the nullspace of A.

(ii) (3 points) There will be three independent vectors that satisfy $A^T y = 0$, why? Find three vectors y and connect them to the loops in the graph.

(iii) (3 points) If the graph represents six games between four teams, and the score differences are $b_1, b_2, b_3, b_4, b_5, b_6$, when it is possible to assign potentials x_1, x_2, x_3, x_4 so that the potential differences agree with the b's? (Hint: You are finding the conditions that make $Ax = b$ solvable.)

Solution:

(i) The incidence matrix of the graph is:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

It can be easily verified that $(1, 1, 1, 1)^T$ is in the nullspace of A, since all the entries in each row add up to zero.

(ii) The left nullspace of A has the following $m - n + 1 = 3$ independent vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

They represent exactly the three small loops in the graph:$y_1y_4y_5, y_3y_6y_4, y_5y_6y_2$.
(iii) Consider $Ax = b$ as follows:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = b.$$

To make this system solvable, we can perform the row operations to discover the conditions that b must satisfy. The conditions on b are:

$$b_1 + b_4 - b_5 = 0, b_3 - b_4 + b_6 = 0, b_2 - b_5 + b_6 = 0.$$