Southern University of Science and Technology

Linear Algebra I Final Examination–Solutions
Fall 2018 A

Department: Math Class: __________

Student ID: __________ Name: __________

Answer all parts of Questions (1)-(11). Total is 100 points.

(1) (12 points, 2 points each) True or false. No need to justify.

(a) The diagonal entries of an $n \times n$ $(n > 1)$ real symmetric positive definite matrix are positive. (T)

(b) If A is similar to B, then A^2 is similar to B^2. (T)

(c) If A and B are diagonalizable, so is AB. (F)

(d) If A is a 3×3 skew-symmetric $(A^T = -A)$, then $|A| = 0$. (T)

(e) If A is negative definite, then all the upper left submatrices A_k of A have negative determinants. (F)

(f) Let A be an $n \times n$ matrix, then the number of nonzero eigenvalues of A (counting the multiplicities) is equal to the rank of A. (F)

(2) (9 points, 3 points each) Fill in the blanks.

(a) Let A be a 3×3 real matrix whose column vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly independent. If $A\alpha_1 = \alpha_1 + \alpha_2, A\alpha_2 = \alpha_2 + \alpha_3, A\alpha_3 = \alpha_3 + \alpha_1$, then $|A| =$ 2

(b) If $A \in \mathbb{R}^{3\times3}$ has eigenvalues $0, 1, 2$, then the eigenvalues of $A(A - I)(A - 2I)$ are 0,0,0 .

(c) A box has edges from $(0, 0, 0)$ to $(3, 1, 1), (1, 3, 1), (1, 1, 3)$, then its volume is 20 .

(3) (10 points) Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find all the eigenvalues of A and their associated eigenvectors.

(ii) Is A diagonalizable? Explain why.

Solution.

(i) The eigenvalues are $1, i, i^2, i^3$, their corresponding eigenvectors are:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}, \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix}, \begin{bmatrix} 1 \\ i^3 \\ i^6 \\ i^9 \end{bmatrix}.$$

(ii) A is diagonalizable, since it has four distinct eigenvalues.

(4) (9 points) Let

$$A = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}.$$

(i) Verify that A is Hermitian.

(ii) Find a unitary matrix U that diagonalizes A.

Solution.

(i) It can be easily verified that $A = A^H$, and therefore A is Hermitian.

(ii) The eigenvalues and their corresponding eigenvectors are determined as follows:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3-i \\ 3+i & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = 0, \lambda_1 = -1, \lambda_2 = 6.$$

For $\lambda = -1$:

$$(A - \lambda_1 I)x = 0, \text{the eigenvectors are}: \ \beta(3+i, -2)^T.$$

For $\lambda = 6$:

$$(A - \lambda_2 I)x = 0, \text{the eigenvectors are}: \ \gamma(3+i, 5)^T.$$

Therefore, we can choose

$$U = \begin{bmatrix} \frac{3+i}{\sqrt{14}} & \frac{3+i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}.$$

Then

$$U^{-1}AU = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}.$$

(5) (12 points) Let

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(i) Find all the singular values of A.

(ii) Find the singular value decomposition of A, in other words, find orthogonal matrices U and V, such that $A = U\Sigma V^T$.

Solution.

(i) First we compute $A^T A$:

$$A^T A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

We know that $A^T A$ has eigenvalues $\lambda_1 = 3, \lambda_2 = 1$, therefore the singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = 1$.

(ii) In order to construct V, we compute the eigenvectors of $A^T A$:

For $\lambda_1 = 3$:

$$(A^T A - \lambda_1 I)x = 0 : v_1 = (1/\sqrt{2}, 1/\sqrt{2})^T.$$

For $\lambda_2 = 1$:

$$(A^T A - \lambda_2 I)x = 0 : v_1 = (1/\sqrt{2}, -1/\sqrt{2})^T.$$

To construct U, we find the first two columns of U as follow:

$$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} Av_1 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$Av_2 = \sigma_1 u_2 \Rightarrow u_1 = \frac{1}{\sigma_2} Av_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{6} \end{bmatrix}$$

The third column of U is determined by finding a basis of the left nullspace of A:

$$A^T x = 0 \Rightarrow u_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

And therefore,

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(6) (8 points) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

(i) Find an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^T$.

(ii) Find A^k, where k is a positive integer.

Solution. We first compute the eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda(\lambda-2)^2 = 0.$$

There are two distinct eigenvalues: 2 and 0.

For $\lambda_1 = 2$:

$$(A - \lambda_1 I)x = 0 : v_1 = (1/\sqrt{2}, 0, -1/\sqrt{2})^T, v_2 = (0, 1, 0)^T.$$

For $\lambda_2 = 0$:

$$(A - \lambda_2 I)x = 0 : v_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})^T.$$

Therefore, $A = Q\Lambda Q^T$ and $A^k = Q\Lambda Q^T$:

$$A^k = \begin{bmatrix} 2^{k-1} & 0 & -2^{k-1} \\ 0 & 2^k & 0 \\ -2^{k-1} & 0 & 2^{k-1} \end{bmatrix}.$$

(7) (8 points) Consider the following quadratic form

$$f(x_1, x_2, x_3, x_4) = t(x_1^2 + x_2^2 + x_3^2) + x_4^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3.$$

(i) Find A, such that $f(x_1, x_2, x_3, x_4) = x^T Ax$.

(ii) For which t is $f(x_1, x_2, x_3, x_4)$ positive definite?

Solution.

(i) A is given as follows:

$$A = \begin{bmatrix} t & 1 & 1 & 0 \\ 1 & t & -1 & 0 \\ 1 & -1 & t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ii) To make $f(x_1, x_2, x_3, x_4)$ positive definite, the determinants of all upper triangular matrices of A should have positive determinants.

(i) $t > 0$.

(ii) $t^2 - 1 > 0$.

(iii)

$$\begin{vmatrix} t & 1 & 1 \\ 1 & t & -1 \\ 1 & -1 & t \end{vmatrix} = (t+1)^2(t-2) > 0.$$

Therefore, to make A positive definite, we need $t > 2$.

(8) (10 points) Let N be a normal matrix ($N^H N = NN^H$).

(i) Show that $||Nx|| = ||N^H x||$ for every vector x.

(ii) Deduce that the ith row of N has the same length as the ith column.

(iii) If N is upper triangular, then N must be diagonal.

Solution.

(i) We consider

$$||Nx||^2 = (Nx)^H(Nx) = x^H N^H Nx = X^H NN^H x = ||N^H x||^2.$$

Taking the positive square roots of both sides, we obtain that $||Nx|| = ||N^H x||$ for every vector x.

(ii) We take x to be e_i, then we immediately see that the ith row of N has the same length as the ith column.

(iii) If N is upper triangular, then N is of the following form:

$$N = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & & & t_{nn} \end{bmatrix}.$$

Therefore,

$$N^H = \begin{bmatrix} \overline{t_{11}} & 0 & \cdots & 0 \\ \overline{t_{12}} & \overline{t_{22}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \overline{t_{1n}} & \overline{t_{2n}} & & \overline{t_{nn}} \end{bmatrix}.$$

According to $NN^H = N^H N$, and comparing the diagonal entries of both sides, we obtain:

$$\begin{aligned} |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 &= |t_{11}|^2 \\ |t_{22}|^2 + \cdots + |t_{2n}|^2 &= |t_{12}|^2 + |t_{22}|^2 \\ &\cdots\cdots \\ |t_{nn}|^2 &= |t_{1n}|^2 + |t_{2n}|^2 + \cdots + |t_{nn}|^2 \end{aligned}$$

Therefore, $t_{ij} = 0$, for $i \neq j$. Thus N must be diagonal.

(9) (8 points) Prove the following two statements:

(i) Suppose A is an $n \times n$ real symmetric positive definite matrix, then $|A + I_n| > 1$.

(ii) Let A be an $n \times n$ matrix, then $A^T A$ is similar to AA^T.

Solution.

(i) Suppose A is an $n \times n$ real symmetric positive definite matrix, we can find an orthogonal matrix Q such that $A = Q\Lambda Q^T$.

$$|A + I_n| = |Q\Lambda Q^T + I_n| = |Q||\Lambda + I_n||Q^T| = |\Lambda + I_n| > 1.$$

(ii) Starting with a SVD of A, i.e., $A = U\Sigma V^T$, then

$$A^T A = U\Sigma\Sigma^T U^T,$$
$$AA^T = V\Sigma^T\Sigma V^T.$$

Note that if A is an $n \times n$ matrix, then $\Sigma\Sigma^T = \Sigma^T\Sigma$, and

$$AA^T = UV^T(V\Sigma\Sigma^T V^T)VU^T = UV^T(A^T A)VU^T.$$

Letting $Q = UV^T$, we obtain that $AA^T = QA^T AQ^T$. That is, $A^T A$ is similar to AA^T.

(10) (6 points) Let A be an $n \times n$ real matrix. If $A^k = O$ for some positive integer k, then A is called a "nilpotent" matrix. O is the $n \times n$ zero matrix.

(i) Show that all the eigenvalues of a nilpotent matrix must be zero.

(ii) Prove that a nonzero nilpotent matrix can not be symmetric.

Solution.

(i) Let λ be an eigenvalue of A and x be its corresponding eigenvector, then A^k will have an eigenvalue λ^k, and

$$A^k x = \lambda^k x = 0$$

It follows that $\lambda^k = 0$, and then $\lambda = 0$.

(ii) If A is nilpotent and symmetric, then it can be diagonalized by an orthogonal matrix Q:

$$A = Q\Lambda Q^T = O.$$

Since all the eigenvalues of A are zero, it follows that A can only be the zero matrix. Contradiction! That is A can not be symmetric.

(11) (8 points) Let A be an $n \times n$ real symmetric positive definite matrix, and $\alpha \in \mathbb{R}^n$ be a nonzero vector. Consider

$$M = \begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix}.$$

Here b is a real number.

(i) Under what condition on b is M positive definite?

(ii) In the case that M is positive semidefinite (not positive definite), find a basis for the nullspace of M, $N(M)$.

Solution.

(i) We first find the determinant of M:

$$\begin{bmatrix} I & 0 \\ \alpha^T A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix} = \begin{bmatrix} A & \alpha \\ 0 & b - \alpha^T A^{-1} \alpha \end{bmatrix}$$

It follows that $\det M = (\det A)(b - \alpha^T A^{-1} \alpha)$. If $\det(M) > 0$, then M is positive definite. Therefore, $b > \alpha^T A^{-1} \alpha$.

(ii) Now we are supposing that M is positive semidefinite (not positive definite), the dimension of $N(M)$ has to be 1, and $b = \alpha^T A^{-1} \alpha$. Therefore, we only need to find a nonzero vector in the nullspace of M. We can take a vector $\beta = (x, 1)^T$ such that

$$\begin{bmatrix} A & \alpha \\ \alpha^T & b \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = 0 \Rightarrow$$

$$\begin{aligned} Ax + \alpha &= 0 \\ \alpha^T x + b &= 0 \end{aligned}$$

Solve for x to obtain that $x = -A^{-1}\alpha$. Therefore, we find a basis for the nullspace of M:

$$\left\{ \begin{bmatrix} -A^{-1}\alpha \\ 1 \end{bmatrix} \right\}.$$