考试时长: 120 分钟　　命题教师: 线性代数教学团队

题 号	1	2	3	4	5	6
分 值	15 分	25 分	20 分	10 分	15 分	15 分

本试卷共 (6) 大题, 满分 (100) 分. 请将所有答案写在答题本上.

This exam includes **6** questions and the score is 100 in total. **Write all your answers on the examination book.**

<u>本套试卷为 B 卷　Version B</u>

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.

(共 15 分, 每小题 3 分) 选择题. 每题只有一个选项是正确的.

(1) Let

$$\alpha_1=\begin{bmatrix}0\\0\\c_1\end{bmatrix},\alpha_2=\begin{bmatrix}0\\1\\c_2\end{bmatrix},\alpha_3=\begin{bmatrix}2\\-1\\c_3\end{bmatrix},\text{ and }\alpha_4=\begin{bmatrix}-2\\1\\c_4\end{bmatrix},$$

where c_1, c_2, c_3, c_4 are arbitrary constants. Which of the following vector systems must be linearly dependent? ( )

(A) $\alpha_1,\ \alpha_2,\ \alpha_3$.

(B) $\alpha_1,\ \alpha_3,\ \alpha_4$.

(C) $\alpha_1,\ \alpha_2,\ \alpha_4$.

(D) $\alpha_2,\ \alpha_3,\ \alpha_4$.

设

$$\alpha_1=\begin{bmatrix}0\\0\\c_1\end{bmatrix},\alpha_2=\begin{bmatrix}0\\1\\c_2\end{bmatrix},\alpha_3=\begin{bmatrix}2\\-1\\c_3\end{bmatrix},\ \alpha_4=\begin{bmatrix}-2\\1\\c_4\end{bmatrix},$$

其中 c_1, c_2, c_3, c_4 为任意常数. 以下向量组一定线性相关的是 ( )

(A) $\alpha_1,\ \alpha_2,\ \alpha_3$.

(B) $\alpha_1,\ \alpha_3,\ \alpha_4$.

(C) $\alpha_1,\ \alpha_2,\ \alpha_4$.

(D) $\alpha_2,\ \alpha_3,\ \alpha_4$.

(2) Let A and B be $n \times n$ complex matrices. Which of the following statements is correct? ( )

(A) If A and B are diagonalizable, so is $A + B$.

(B) If A and B are diagonalizable, so is AB.

(C) If A is invertible and A^2 is diagonalizable, then A is diagonalizable.

(D) If the distinct eigenvalues of A are 1 and 0, then A is diagonalizable.

设 A 和 B 均为 $n \times n$ 复矩阵. 下列陈述正确的是 ( )

(A) 若 A 和 B 均可对角化, 则 $A + B$ 也如此.

(B) 若 A 和 B 均可对角化, 则 AB 也如此.

(C) 若 A 可逆并且 A^2 可对角化, 则 A 也可对角化.

(D) 若 A 的所有互异特征值为 1 和 0, 则 A 可以对角化.

(3) Let Q be a 3×3 real orthogonal matrix. Which of the following is **false**? ( )

(A) For every 3×3 real symmetric matrix A, $Q^{-1}AQ$ is symmetric.

(B) For every column vector $v \in \mathbb{R}^3$, the vectors Qv and v have the same length.

(C) There is a nonzero column vector $v \in \mathbb{R}^3$ such that $Qv = v$ or $Qv = -v$.

(D) There is an invertible 3×3 real matrix P of order 3 such that $P^{-1}QP$ is diagonal.

设 Q 为 3×3 实正交矩阵. 下列陈述错误的是 ( )

(A) 对任意 3×3 实对称阵 A, $Q^{-1}AQ$ 是对称阵.

(B) 对任意列向量 $v \in \mathbb{R}^3$, 向量 Qv 和 v 的长度相同.

(C) 存在非零列向量 $v \in \mathbb{R}^3$ 使得 $Qv = v$ 或 $Qv = -v$.

(D) 存在可逆的 3×3 实矩阵 P 使得 $P^{-1}QP$ 为对角阵.

(4) Let A and B be $n \times n$ real symmetric matrices. Which of the following must be true? ( )

(A) If A and B are both negative definite, then they are congruent.

(B) The product AB is also symmetric.

(C) If A and B are congruent, then they have the same column space.

(D) The complex matrix $A + iB$ is a Hermitian matrix.

设 A 和 B 为 $n \times n$ 实对称阵. 下列陈述一定正确的是 ( )

(A) 若 A 和 B 都是负定的, 则它们是合同的.

(B) 乘积矩阵 AB 也是对称阵.

(C) 若 A 和 B 合同, 则它们的列空间相同.

(D) 复矩阵 $A + iB$ 是一个 Hermite 矩阵.

(5) If a square matrix A is only similar to itself, then ( )

(A) A is the identity matrix or the zero matrix.

(B) A can be any real symmetric matrix.

(C) A commutes with all matrices of the same size.

(D) A can be any diagonal matrix.

若方阵 A 只和自己相似, 则 ( )

(A) A 必为单位矩阵或零矩阵.

(B) A 可以是任意实对称阵.

(C) A 与任意相同大小的矩阵可交换.

(D) A 可以是任意对角阵.

2. (25 points, 5 points each) Fill in the blanks. (共 25 分, 每小题 5 分) 填空题.

(1) Let $\lambda \in \mathbb{R}$. Determine for which values of λ the plane curve $-\lambda x^2+4xy+(3-\lambda)y^2=2023$ is an ellipse. (A circle is also considered as an ellipse.) Answer: __________

设 $\lambda \in \mathbb{R}$. 求 λ 的取值范围使平面曲线 $-\lambda x^2+4xy+(3-\lambda)y^2=2023$ 是一个椭圆. (圆也认为是椭圆.) 答: __________

(2) Let A be a 3×3 matrix with determinant $|A|=4$. Then $|2A^{-1}|=$ __________.

设 A 为 3×3 矩阵, 其行列式为 $|A|=4$. 则 $|2A^{-1}|=$ __________.

(3) Suppose $A=\begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & a \\ 4 & 0 & 5 \end{bmatrix}$ is diagonalizable, then $a=$ __________.

如果 $A=\begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & a \\ 4 & 0 & 5 \end{bmatrix}$ 为可对角化的, 则 $a=$ __________.

(4) Let A be a 3×3 matrix. Suppose that the trace of A is -5, and $A^2+2A-3I=0$. Then the three eigenvalues of A are __________.

设 A 为 3×3 矩阵. 假设 A 的迹是 -5, 且 $A^2+2A-3I=0$. 则 A 的三个特征值为 __________.

(5) A QR factorization of the matrix $A=\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$ is $Q=$ __________, $R=$ __________.

矩阵 $A=\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix}$ 的一个 QR 分解为 $Q=$ __________, $R=$ __________.

3. (20 points) Let A be an $n\times n$ real matrix satisfying $A^2=A$. Let I_n denote the $n\times n$ identity matrix.

(a) Prove that $\operatorname{rank}(A)+\operatorname{rank}(A-I_n)=n$.

(b) Show that A is diagonalizable.

(c) Suppose that the rank of A is 3. Find the trace and the determinant of the matrix $2I_n+A$.

(d) Assume further that A is symmetric. Show that for any real monic polynomial

$$f(x)=x^d+c_{d-1}x^{d-1}+\cdots+c_1x+c_0,$$

the matrix $f(A)$ is positive definite if and only if $c_0>0$ and $1+c_{d-1}+\cdots+c_1+c_0>0$.

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(20 分) 设 A 为 $n \times n$ 实矩阵, 它满足 $A^2 = A$. 用 I_n 表示 $n \times n$ 单位矩阵.

(a) 证明 $\operatorname{rank}(A) + \operatorname{rank}(A - I_n) = n$.

(b) 证明 A 可以对角化.

(c) 假设 A 的秩为 3. 求矩阵 $2I_n + A$ 的迹和行列式.

(d) 进一步假设 A 是对称阵. 证明: 对任意首项系数为一的实多项式

$$f(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0,$$

矩阵 $f(A)$ 正定当且仅当 $c_0 > 0$ 且 $1 + c_{d-1} + \cdots + c_1 + c_0 > 0$.

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)

4. (10 points) Suppose A is a 3×3 real symmetric matrix with eigenvalues 1, 2, 3, and suppose

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

are eigenvectors of A corresponding to the eigenvalues 1 and 2 respectively.

(a) Find an eigenvector corresponding to the eigenvalue 3.

(b) Find the matrix A.

(10 分) 设 A 是 3×3 实对称矩阵, 它以 1, 2, 3 为特征值. 假设

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

分别是 A 的对应于特征值 1 和 2 的特征向量.

(a) 求一个对应于特征值 3 的特征向量.

(b) 求出矩阵 A.

5. (15 points) Let A be an $m \times m$ real symmetric matrix.

(a) Prove that there exists an $m \times m$ orthogonal matrix Q such that $Q^T A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$, where $\lambda_i \in \mathbb{R}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$.

(b) For each $a = (a_1, \cdots, a_m)^T \in \mathbb{R}^m$, put $\|a\| = \sqrt{a_1^2 + \cdots + a_m^2}$. Prove that every eigenvalue λ of A satisfies

$$\lambda \geq \min_{\|x\|=1} \{x^T A x\}.$$

(c) Show that there exists a vector $w \in \mathbb{R}^m$ with $\|w\| = 1$ such that $\min_{\|x\|=1} \{x^T A x\} = w^T A w$.

(d) Consider the binary function $f(x, y) = x^2 + 2y^2 - 4\sqrt{3}xy + 2025$. Find the minimum

$$\mu := \min\{f(x, y) \,|\, x^2 + y^2 = 1\}$$

and find a point (x_0, y_0) such that $f(x_0, y_0) = \mu$ and $x_0^2 + y_0^2 = 1$.

(e) Suppose A is positive definite and fix a nonzero column vector $x \in \mathbb{R}^m$. Define $a_n = x^T A^n x$ for each $n \geq 1$. Show that

$$\lim_{n \to +\infty} \sqrt[n^2]{a_n} = 1.$$

(*In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.*)

(15 分) 设 A 为 $m \times m$ 实对称阵.

(a) 证明: 存在 $m \times m$ 正交矩阵 Q 使得 $Q^T A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$, 其中 $\lambda_i \in \mathbb{R}$ 且 $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$.

(b) 对任意 $a = (a_1, \cdots, a_m)^T \in \mathbb{R}^m$, 令 $\|a\| = \sqrt{a_1^2 + \cdots + a_m^2}$. 证明: A 的每个特征值 λ 均满足

$$\lambda \geq \min_{\|x\|=1} \{x^T A x\}.$$

(c) 证明: 存在向量 $w \in \mathbb{R}^m$ 满足 $\|w\| = 1$ 以及 $\min_{\|x\|=1} \{x^T A x\} = w^T A w$.

(d) 考虑二元函数 $f(x, y) = x^2 + 2y^2 - 4\sqrt{3}xy + 2025$. 求最小值

$$\mu := \min\{f(x, y) \,|\, x^2 + y^2 = 1\}$$

并求一点 (x_0, y_0) 使得 $f(x_0, y_0) = \mu$ 且 $x_0^2 + y_0^2 = 1$.

(e) 假设 A 是正定的, 并取定一个非零列向量 $x \in \mathbb{R}^m$. 对每个 $n \geq 1$, 定义 $a_n = x^T A^n x$. 求证

$$\lim_{n \to +\infty} \sqrt[n^2]{a_n} = 1.$$

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)

6. (15 points) Let A be an $n \times n$ real orthogonal matrix. Let B be any $n \times n$ real matrix.

(a) Show that $\det(I_n + A) = \det(A)\det(I_n + A)$, where I_n denotes the $n \times n$ identity matrix.

(b) Show that if $\det(A) + \det(B) = 0$ and B is orthogonal, then $\det(A + B) = 0$.

(c) Give an example to show that if B is not orthogonal, it is possible to have $\det(A) + \det(B) = 0$ but $\det(A + B) \neq 0$. (You may choose a suitable value of n.)

(In this problem, you are allowed to assume the statements of some questions to answer subsequent questions.)

(15 分) 设 A 为 $n \times n$ 实正交矩阵, B 为任意 $n \times n$ 实矩阵.

(a) 证明: $\det(I_n + A) = \det(A)\det(I_n + A)$, 其中 I_n 表示 $n \times n$ 单位矩阵.

(b) 证明: 如果 $\det(A) + \det(B) = 0$, 则 $\det(A + B) = 0$.

(c) 举例说明: 如果 B 不是正交阵, 那么有可能 $\det(A) + \det(B) = 0$ 但 $\det(A + B) \neq 0$. (你可以将 n 取为合适的特殊值.)

(本题中, 允许承认前面小题的结果来用于后续问题的解答.)