南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线性代数　　　　开课单位: 数学系
考试时长: 120 分钟　　　　命题教师: 线性代数教学团队

题号	1	2	3	4	5	6	7
分值	15 分	20 分	10 分	24 分	20 分	5 分	6 分

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师)

This exam paper contains 7 questions and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends.)

$\frac{2c-7}{3} = \frac{3c-3}{7}$

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

$\begin{bmatrix} 2 & 1 & 7 \\ 3 & -1 & 3 \\ 1 & 2 & c \end{bmatrix}$

(1) Let

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

$\begin{bmatrix} 0 & 3 & 2c-7 \\ 0 & 7 & 3c-3 \\ 2 & 1 & 7 \end{bmatrix}$

If α_1, α_2, α_3 are linearly dependent, then c equals

(A) 5.
(B) 6.
(C) 7.
(D) 8.

$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 7 & 3 & c \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & -3 \\ 0 & 10 & c-14 \end{bmatrix}$

假定

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 7 \\ 3 \\ c \end{bmatrix}.$$

$\to \begin{bmatrix} 2 & 3 & 1 \\ & 5 & -3 \\ & 0 & c-14+6 \end{bmatrix}$ -8

D) 若 α_1, α_2, α_3 线性相关, 则 c 的取值为

(A) 5.
(B) 6.
(C) 7.
(D) 8.

(2) Let A be an $m \times n$ real matrix and $\mathbf{b}$ be an $m \times 1$ real column vector. Which of the following statements is correct?

(A) If $A\mathbf{x} = \mathbf{b}$ does not have any solution, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

(B) If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(C) If $m < n$, both $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ have infinitely many solutions.

(D) If the rank of A is n, then $A\mathbf{x} = \mathbf{0}$ has only the zero solution.

$R(A)<m$
$R(b)=m$

D 设 A 为一个 $m \times n$ 实矩阵, $\mathbf{b}$ 为一个 m 维实列向量. 以下说法一定是正确的是?

(A) 若 $A\mathbf{x} = \mathbf{b}$ 无解, 则 $A\mathbf{x} = \mathbf{0}$ 只有零解.

(B) 若 $A\mathbf{x} = \mathbf{0}$ 有无穷多解, 则 $A\mathbf{x} = \mathbf{b}$ 有无穷多解.

(C) 若 $m < n$, 则 $A\mathbf{x} = \mathbf{b}$ 和 $A\mathbf{x} = \mathbf{0}$ 都有无穷多解.

(D) 若 A 的秩为 n, 则 $A\mathbf{x} = \mathbf{0}$ 只有零解.

$A=0$ AB B 非满秩

(3) For which value of k does the system

$$
\begin{cases}
x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\
x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\
x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0,
\end{cases}
$$

have exactly two free variables?

(A) 5.

(B) 4.

(C) 3.

(D) 2.

C 如果以下线性方程组有两个自由变量

$$
\begin{cases}
x_1 + 2x_2 - 4x_3 + 3x_4 = 0, \\
x_1 + 3x_2 - 2x_3 - 2x_4 = 0, \\
x_1 + 5x_2 + (5-k)x_3 - 12x_4 = 0,
\end{cases}
$$

$$
\begin{bmatrix} 1 & 2 & -4 & 3 \\ 1 & 3 & -2 & -2 \\ 1 & 5 & 5-k & -12 \end{bmatrix}
\to \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 1 & 2 & -5 \\ 0 & 3 & 7-k & -15 \end{bmatrix}
\to \begin{bmatrix} 1 & 0 & -8 & 13 \\ & 1 & 2 & -5 \\ & 0 & 3-k & 0 \end{bmatrix}
$$

$3-k=0 \quad \exists\, k=3.$

k 的取值为

(A) 5.

(B) 4.

(C) 3.

(D) 2.

(4) Let $u, v \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Which of the following statements is **false**?

(A) If u and v are nonzero vectors satisfying $u^T v = 0$, then u and v are linearly independent.

(B) If $u + v$ is orthogonal to $u - v$, then $||u|| = ||v||$.

(C) $u^T v = 0$ if and only if $u = 0$ or $v = 0$.

(D) $\lambda v = 0$ if and only if $v = 0$ or $\lambda = 0$.

C 设 $u, v \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$. 以下说法错误的是?

(A) 如果 u 和 v 为满足 $u^T v = 0$ 的非零向量, 则 u 和 v 线性无关.

(B) 如果 $u + v$ 和 $u - v$ 正交, 则 $||u|| = ||v||$.

(C) $u^Tv = 0$ 当且仅当 $u = 0$ or $v = 0$. ✗

(D) $\lambda v = 0$ 当且仅当 $v = 0$ or $\lambda = 0$. ✓

(5) Let A and B be two $n \times n$ matrices. Which of the following assertions is **false**?

(A) If A, B are symmetric matrices, then AB is a symmetric matrix.

(B) If A, B are invertible matrices, then AB is an invertible matrix.

(C) If A, B are permutation matrices, then AB is a permutation matrix.

(D) If A, B are upper triangular matrices, then AB is an upper triangular matrix.

A　设 A 和 B 都为 n 阶矩阵. 以下说法错误的是?

(A) 如果 A, B 为对称矩阵, 则 AB 也为一个对称矩阵. ✗

(B) 如果 A, B 为可逆矩阵, 则 AB 也为一个可逆矩阵. ✓

(C) 如果 A, B 为置换矩阵, 则 AB 也为一个置换矩阵.

(D) 如果 A, B 为上三角矩阵, 则 AB 也为上三角矩阵.

$$\underset{A}{\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

2. (20 points, 5 points each) Fill in the blanks.

(共 20 分, 每小题 5 分) 填空题.

(1) Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix},\ a, b \in \mathbb{R}.$$

Then $A^{-1} =$ __________.

设

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 3 & 2 \end{bmatrix},\ a, b \in \mathbb{R},$$

则 $A^{-1} = \begin{bmatrix} 1 & & \\ -a & 1 & \\ \frac{3a-b}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$.

$[A \quad I]$

$$= \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ a & 1 & & & 1 & \\ b & 3 & 2 & & & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ 0 & 1 & & -a & 1 & \\ \cancel{b} & 3 & 2 & -b & \frac{-3}{2} & \frac{1}{2} \end{array}\right]$$

$0 \quad 0 \quad 1 \quad \cancel{3a/2}$

$$\begin{bmatrix} 1 & & \\ -a & 1 & \\ \frac{3a-b}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & & \\ a & 1 & \\ b & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{3a-b-3a+b}{}$　$-3+3$

(2) Let A be a 4×3 real matrix with rank 2 and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. Then the rank AB is __________.

设 A 为一个 4×3 的实矩阵, B 为 $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. 如果矩阵 A 的秩为 2, 则 AB 的秩为 2.

$\to \begin{bmatrix} 1 & 0 & 2 \\ & 2 & \\ 0 & 0 & 5 \end{bmatrix}$　$R(B) = 3$

$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 3 \\ & & \\ & & \end{bmatrix}$

$R(AB) \geq R(A) + R(B) - n$

(3) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$. Then $A^{2024} =$ __________.

设 $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$, 则 $A^{2024} = 4^{2023}A$.

$$\begin{bmatrix} 1 & & \\ & 1 & \\ 0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ & 2 & \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R(A^2) \geq 2R(A) - n = 2 - 3 = -1$

$R(A + A) \leq 2R(A) = 2$

$\times 2+2$

$4^0 \quad A^2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$

$A^3.$　$16 \cdot 4$　4^1

A^1	A^2	A^3	A^4
1	4	16	64
4^0	4^1	4^2	

$4^2 \quad A^3 = \begin{bmatrix} 4 & -4 & 4 \\ -4 & 4 & -4 \\ 8 & -8 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -16 & 16 \\ -16 & +16 & -16 \\ 32 & -32 & 32 \end{bmatrix}$

$\times 4$　$A^4 = [\ 32+32 = 64$

(4) Consider the system of linear equations:

$$A\mathbf{x}=\mathbf{b}:\begin{cases} x &= 2\\ y &= 3\\ x+y &= 6\end{cases}$$

The least-squares solution for the system is ________.

考虑以下线性方程组:

$$A\mathbf{x}=\mathbf{b}:\begin{cases} x &= 2\\ y &= 3\\ x+y &= 6\end{cases}$$

$A=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$ $b=\begin{bmatrix}2\\3\\6\end{bmatrix}$

该线性方程组的最小二乘解为 $\begin{bmatrix}\frac{7}{3}\\ \frac{10}{3}\end{bmatrix}$.

$\hat{x}=(A^TA)^{-1}A^Tb$

$=\left(\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}2\\3\\6\end{bmatrix}$

$=\begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}8\\9\end{bmatrix}$

$=\frac{1}{4-1}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}8\\9\end{bmatrix}$

$\begin{bmatrix}16-9\\-8+18\end{bmatrix}=\begin{bmatrix}7\\10\end{bmatrix}$

3. (10 points) Let

$$A=\begin{bmatrix}1&-2&3\\2&-5&1\\1&-4&-7\end{bmatrix}.$$

Find an LU factorization of A.

设

$$A=\begin{bmatrix}1&-2&3\\2&-5&1\\1&-4&-7\end{bmatrix}.$$

求矩阵 A 的一个 LU 分解.

4. (24 points) Consider the following 4×5 matrix A and 4-dimensional column vector $\mathbf{b}$:

$$A=\begin{bmatrix}0&2&4&1&6\\0&1&1&1&3\\0&4&10&1&2\\0&-1&-5&1&7\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}3\\2\\-5\\10\end{bmatrix}$$

(a) Find a basis for each of the four fundamental subspaces of A.

(b) Find the complete solution to $A\mathbf{x}=\mathbf{b}$.

考虑以下 4×5 矩阵 A 以及 4 维列向量 $\mathbf{b}$:

$$A=\begin{array}{c} \begin{matrix}&-5&1&3&1\end{matrix}\\ \begin{bmatrix}0&2&4&1&6\\0&1&1&1&3\\0&4&10&1&2\\0&-1&-5&1&7\end{bmatrix}\end{array},\quad \mathbf{b}=\begin{bmatrix}3\\2\\-5\\10\end{bmatrix}$$

(a) 分别求矩阵 A 的四个基本子空间的一组基向量.

(b) 求 $A\mathbf{x}=\mathbf{b}$ 的所有解.

5. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and T be the linear transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ defined by

$$T(X) = XA + AX, \ X \in \mathbb{R}^{2\times 2}.$$

Where $\mathbb{R}^{2\times 2}$ denotes the vector space consisting of all 2×2 real matrices.

(a) Find the matrix representation of T with respect to the following ordered basis

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Find a matrix B such that

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) Find a matrix C such that

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

设 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, T 为按照以下方式定义的从 $\mathbb{R}^{2\times 2}$ 到 $\mathbb{R}^{2\times 2}$ 线性变换:

$$T(X) = XA + AX, \ X \in \mathbb{R}^{2\times 2}.$$

其中 $\mathbb{R}^{2\times 2}$ 表示所有 2×2 实矩阵构成的向量空间.

(a) 求 T 在以下有序基

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

下的矩阵表示.

(b) 求一个矩阵 B 使得

$$T(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(c) 求一个矩阵 C 使得

$$T(C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

6. (5 points) Let A, B be two $n \times n$ real matrices satisfying $A^2 = A$ and $B^2 = B$. Show that if $(A+B)^2 = A+B$, then $AB = O$. Where O denotes the $n \times n$ zero matrix.

设 A, B 为满足 $A^2 = A$ 和 $B^2 = B$ 的 n 阶实矩阵. 证明: 如果 $(A+B)^2 = A+B$, 则 $AB = O$. 其中 O 表示 n 阶零矩阵.

7. (6 points) Let A be a 3×2 matrix, B be a 2×3 matrix such that

$$AB=\begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a) Compute $(AB)^2$.

(b) Find BA.

设 A 为 3×2 矩阵, B 为 2×3 矩阵, 并且

$$AB=\begin{bmatrix} 8 & 0 & -4 \\ -\frac{3}{2} & 9 & -6 \\ -2 & 0 & 1 \end{bmatrix}.$$

(a) 计算 $(AB)^2$.

(b) 求 BA.

b

$2\times3\times2$.