考试科目: 线 性 代 数 A **开课单位:** 数 学 系

题 号	1	2	3	4	5	6	7
分 值	12 分	15 分	24 分	14 分	15 分	10 分	10分

本试卷共 (7) 大题, 满分 (100) 分.

This 2-hour long test includes 7 questions. Write ***all your answers*** on the examination book.

1. (12 points, 2 points each) Label the following statements as **True** or **False**. No need to justify. (12 分, 2 分一道) 判断正误, 不需要说明理由.

   (a) If A and B are invertible, then BA is invertible.

   如果 A 和 B是可逆矩阵，则 BA 也是可逆矩阵.

   (b) Let A be an $m \times n$ matrix with rank n, then $Ax = b$ is solvable for all $b \in \mathbb{R}^m$.

   设 A 为 $m \times n$ 矩阵且秩为 n, 则对于任意的 $b \in \mathbb{R}^m$, $Ax = b$ 都是可解的.

   (c) If x_p is a particular solution to $Ax = b$, then x_p is always in the row space of A.

   如果 x_p 是 $Ax = b$ 的一个特解, 那么 x_p 一定在矩阵 A 的行空间里.

   (d) Let the vectors v_1, v_2, v_3 be linearly independent. If $w_1 = v_1, w_2 = v_1 + v_2, w_3 = v_1 + v_2 + v_3$, then w_1, w_2, w_3 are linearly independent.

   假定向量 v_1, v_2, v_3 线性无关. 如果 $w_1 = v_1, w_2 = v_1 + v_2, w_3 = v_1 + v_2 + v_3$, 则 w_1, w_2, w_3 线性无关.

   (e) The transformation that takes x to $2x + 1$ is linear (from $\mathbb{R}^1$ to $\mathbb{R}^1$).

   把 x 变为 $2x + 1$ 的变换是线性的 (从 $\mathbb{R}^1$ 到 $\mathbb{R}^1$).

   (f) If the row space of A is the same as the column space of A, then the nullspace of A and the left nullspace of A must be the same.

   如果矩阵 A 的行空间和列空间相同, 则 A 的零空间和左零空间必定相同.

2. (15 points, 5 points each) Fill in the blanks. (15 分, 5 分一道) 填空题.

(a) If $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & a+2 \\ 1 & a & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ has no solution, then $a =$ __________.

如果 $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & a+2 \\ 1 & a & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ 无解, 那么 $a =$ __________.

(b) Suppose A is a 4×3 matrix, and rank $A = 2$, and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, then rank $(AB) =$ __________.

如果 A 是一个 4×3 矩阵, 且 rank $A = 2$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, 则 rank $(AB) =$ __________.

(c) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & -6 & 7 \end{bmatrix}$, and $B = (I+A)^{-1}(I-A)$, then $(I+B)^{-1} =$ __________ (Here I is the 4×4 identity matrix).

设 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & -6 & 7 \end{bmatrix}$, $B = (I + A)^{-1}(I - A)$, 那么 $(I + B)^{-1} =$ __________ (这里 I 是 4×4 单位矩阵).

3. (24 points) Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}.$$

(a) Find the complete solution to $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Find the complete solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(c) Find the rank of A and dimensions of the four fundamental subspaces of A.

(d) Find bases of the four fundamental subspaces of A.

(24 分) 设

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}.$$

(a) 求 $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ 的所有解.

(b) 求 $Ax = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ 的所有解.

(c) 求 A 的秩和矩阵 A 的四个基本子空间的维数.

(d) 求矩阵 A 的四个基本子空间的基.

4. (14 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}.$$

(a) Find the symmetric factorization of $A = LDL^T$.

(b) Use the Gauss-Jordan method to find A^{-1}.

(14 分) 假设

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}.$$

(a) 求 A 的一个 LDL^T 分解.

(b) 用高斯约旦方法求 A 的逆矩阵, A^{-1}.

5. (15 points) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}.$$

(a) Explain why $Ax = b$ is inconsistent.

(b) Find the lease squares solution to $Ax = b$.

(c) Split b into a column space component x_c and a left nullspace component x_l, i.e., $b = x_c + x_l$.

(15 分) 设

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}.$$

(a) 说明为什么线性方程组 $Ax = b$ 没有解.

(b) 求 $Ax = b$ 的最小二乘解.

(c) 把 b 分解成一个列空间分量 x_c 和一个左零空间分量 x_l, 换言之, $b = x_c + x_l$.

6. (10 points) The space of all 2×2 real matrices, denoted $\mathbb{R}^{2\times 2}$, has the four basis "vectors"

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define the transformation of transposing from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ as follows:

$$T(X) = X^T.$$

(a) Show that T is a linear transformation.

(b) Find the matrix A representing T with respect to the above basis for $\mathbb{R}^{2\times 2}$.

(c) Explain why $A^2 = I$.

(10 points) 包含所有 2×2 实矩阵的向量空间 $\mathbb{R}^{2\times 2}$ 有以下四个基向量

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

从 $\mathbb{R}^{2\times 2}$ 到 $\mathbb{R}^{2\times 2}$ 的转置变换定义如下:

$$T(X) = X^T.$$

(a) 证明 T 是一个线性变换.

(b) 找出线性变换 T 在上述基向量组下的矩阵表示, A.

(c) 为什么有 $A^2 = I$? 说明理由.

7. (10 points)

(a) Let $v_1, v_2, \cdots, v_m$ be linearly independent vectors in $\mathbb{R}^n$ ($n > m$), and

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}.$$

If follows that A is an $m \times n$ matrix with rank m. Let

$$w_1, w_2, \cdots, w_{n-m}$$

be a sequence of linearly independent vectors in $\mathbb{R}^n$ satisfying

$$Aw_j = 0,\ j = 1, 2, \cdots, n - m.$$

Show that

$$v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_{n-m}$$

are linearly independent.

(b) Let A be an $n \times n$ real matrix and A^T be its transpose. Show that the column spaces of $A^T A$ and A^T are the same, i.e., $C(A^T A) = C(A^T)$.

(10 分)

(a) 如果 $v_1, v_2, \cdots, v_m$ 是 $\mathbb{R}^n$ 中的线性无关向量($n > m$). 假定

$$A = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}.$$

由此可见, A 是一个 $m \times n$ 行满秩矩阵. 如果 $\mathbb{R}^n$ 中线性无关向量组

$$w_1, w_2, \cdots, w_{n-m}$$

满足 $Aw_j = 0,\ j = 1, 2, \cdots, n - m$. 证明:

$$v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_{n-m}$$

线性无关.

(b) 设 A 为一个 $n \times n$ 实矩阵, A^T 为它的转置. 证明: $A^T A$ 和 A^T 的列空间相同, 换言之, $C(A^T A) = C(A^T)$.