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Mathematical Modelling and Applications of Particle Swarm Optimization

by

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ABSTRACT

Optimization is a mathematical technique that concerns the finding of maxima or minima of functions in some feasible region. There is no business or industry which is not involved in solving optimization problems. A variety of optimization techniques compete for the best solution. Particle Swarm Optimization (PSO) is a relatively new, modern, and powerful method of optimization that has been empirically shown to perform well on many of these optimization problems. It is widely used to find the global optimum solution in a complex search space. This thesis aims at providing a review and discussion of the most established results on PSO algorithm as well as exposing the most active research topics that can give initiative for future work and help the practitioner improve better result with little effort. This paper introduces a theoretical idea and detailed explanation of the PSO algorithm, the advantages and disadvantages, the effects and judicious selection of the various parameters. Moreover, this thesis discusses a study of boundary conditions with the invisible wall technique, controlling the convergence behaviors of PSO, discrete-valued problems, multi-objective PSO, and applications of PSO. Finally, this paper presents some kinds of improved versions as well as recent progress in the development of the PSO, and the future research issues are also given.

Keywords: Optimization, swarm intelligence, particle swarm, social network, convergence, stagnation, multi-objective.

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CHAPTER 1

Introduction

Scientists, engineers, economists, and managers always have to take many technological and managerial decisions at several times for construction and maintenance of any system. Day by day the world becomes more and more complex and competitive so the decision making must be taken in an optimal way. Therefore optimization is the main act of obtaining the best result under given situations. Optimization originated in the 1940s, when the British military faced the problem of allocating limited resources (for example fighter airplanes, submarines and so on) to several activities [6]. Over the decades, several researchers have generated different solutions to linear and non-liner optimization problems. Mathematically an optimization problem has a fitness function, describing the problem under a set of constraints which represents the solution space for the problem. However, most of the traditional optimization techniques have calculated the first derivatives to locate the optima on a given constrained surface. Due to the difficulties in evaluation the first derivative for many rough and discontinuous optimization spaces, several derivatives free optimization methods have been constructed in recent time [15].

There is no known single optimization method available for solving all optimization problems. A lot of optimization methods have been developed for solving different types of optimization problems in recent years. The modern optimization methods (sometimes called nontraditional optimization methods) are very powerful and popular methods for solving complex engineering problems. These methods are particle swarm optimization algorithm, neural networks, genetic algorithms, ant colony optimization, artificial immune systems, and fuzzy optimization [6] [7].

The Particle Swarm Optimization algorithm (abbreviated as PSO) is a novel population-based stochastic search algorithm and an alternative solution to the complex non-linear optimization problem. The PSO algorithm was first introduced by Dr. Kennedy and Dr. Eberhart in 1995 and its basic idea was originally inspired by simulation of the social behavior of animals such as bird flocking, fish schooling and so on. It is based on the natural process of group communication to share individual knowledge when a group of birds or insects search food or migrate and so forth in a searching space, although all birds or insects do not know where the best position is. But from the nature of the social behavior, if any member can find out a desirable path to go, the rest of the members will follow quickly.

The PSO algorithm basically learned from animal's activity or behavior to solve optimization problems. In PSO, each member of the population is called a particle and the population is called a swarm. Starting with a randomly initialized population and moving in randomly chosen directions, each particle goes through the searching space and remembers the best previous positions of itself and its neighbors. Particles of a swarm communicate good positions to each other as well as dynamically adjust their own position and velocity derived from the best position of all particles. The next step begins when all particles have been moved. Finally, all particles tend to fly towards better and better positions over the searching process until the swarm move to close to an optimum of the fitness function $f: R^n \rightarrow R$.

The PSO method is becoming very popular because of its simplicity of implementation as well as ability to swiftly converge to a good solution. It does not require any gradient information of the function to be optimized and uses only primitive mathematical operators.

As compared with other optimization methods, it is faster, cheaper and more efficient. In addition, there are few parameters to adjust in PSO. That's why PSO is an ideal optimization problem solver in optimization problems. PSO is well suited to solve the non-linear, non-convex, continuous, discrete, integer variable type problems.

1.1 PSO is a Member of Swarm Intelligence

Swarm intelligence (SI) is based on the collective behavior of decentralized, self-organized systems. It may be natural or artificial. Natural examples of SI are ant colonies, fish schooling, bird flocking, bee swarming and so on. Besides multi-robot systems, some computer program for tackling optimization and data analysis problems are examples for some human artifacts of SI. The most successful swarm intelligence techniques are Particle Swarm Optimization (PSO) and Ant Colony Optimization (ACO). In PSO, each particle flies through the multidimensional space and adjusts its position in every step with its own experience and that of peers toward an optimum solution by the entire swarm. Therefore, the PSO algorithm is a member of Swarm Intelligence [3].

1.2 Motivation

PSO method was first introduced in 1995. Since then, it has been used as a robust method to solve optimization problems in a wide variety of applications. On the other hand, the PSO method does not always work well and still has room for improvement.

This thesis discusses a conceptual overview of the PSO algorithm and a number of modifications of the basic PSO. Besides, it describes different types of PSO algorithms and flowcharts, recent works, advanced topics, and application areas of PSO.

1.3 Research Questions

This thesis aims to answer the following questions:

Q.1: How can problems of premature convergence and stagnation in the PSO algorithm be prevented?

Q.2: When and how are particles reinitialized?

Q.3: For the PSO algorithm, what will be the consequence if
a) the maximum velocity V_{max} is too large or small?
b) the acceleration coefficients c_1 and c_2 are equal or not?
c) the acceleration coefficients c_1 and c_2 are very large or small?

Q.4: How can the boundary problem in the PSO method be solved?

Q.5: How can the discrete-valued problems be solved by the PSO method?

Q.1 is illustrated in Section 4.1 and 4.3; Q.2 in Section 5.1; Q.3 (a) in Section 4.1.1; Q.3 (b) and (c) in Section 3.3.5; Q.4 and Q.5 in Section 4.2 and 5.5 respectively.

CHAPTER 2

Background

This chapter reviews some of the basic definitions related to this thesis.

2.1 Optimization

Optimization determines the best-suited solution to a problem under given circumstances. For example, a manager needs to take many technological and managerial plans at several times. The final goal of the plans is either to minimize the effort required or to maximize the desired benefit. Optimization refers to both minimization and maximization tasks. Since the maximization of any function f is mathematically equivalent to the minimization of its additive inverse $-f$, the term minimization and optimization are used interchangeably [6]. For this reason, now-a-days, it is very important in many professions.

Optimization problems may be linear (called *linear optimization* problems) or non-linear (called *non-linear optimization* problems). *Non-linear optimization* problems are generally very difficult to solve.

Based on the problem characteristics, optimization problems are classified in the following:

2.1.1 Constrained Optimization

Many optimization problems require that some of the decision variables satisfy certain limitations, for instance, all the variables must be non-negative. Such types of problems are said to be *constrained optimization problems* [4] [8] [11] and defined as

$$\begin{aligned}
\text{minimize} \quad & f(\mathrm{x}), \qquad \mathrm{x} = (x_1, x_2, x_3, \dots, x_\mathrm{n}) \\
\text{subject to} \quad & g_m(\mathrm{x}) \leq 0, \;\; m = 1, 2, \dots, n_g \\
& h_m(\mathrm{x}) = 0, \;\; m = n_g + 1, \dots, n_g + n_h \\
& \forall \mathrm{x} \in R^n
\end{aligned} \tag{2.1}$$

where n_g and n_h are the number of inequality and equality constraints respectively.

Example: Minimize the function

$$\begin{aligned}
& f(\mathrm{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\
\text{subject to} \quad & x_1 + x_2^2 \geq 0 \\
& x_1^2 + x_2 \geq 0 \\
& \text{with } x_1 \in [-0.5, 0.5] \text{ and } x_2 \leq 1.0.
\end{aligned}$$

Then, the global optimum is $x^* = (0.5, 0.25)$, with $f(x^*) = 0.25$.

2.1.2 Unconstrained Optimization

Many optimization problems place no restrictions on the values of that can be assigned to variables of the problem. The feasible space is simply the whole search space. Such types of problems are said to be *unconstrained optimization problems* [4] and defined as

$$\text{minimize} \quad f(\mathrm{x}),\ \mathrm{x} \in \mathrm{R}^{\mathrm{n}} \tag{2.2}$$

where n is the dimension of x.

2.1.3 Dynamic Optimization

Many optimization problems have objective functions that change over time and such changes in objective function cause changes in the position of optima. These types of problems are said to be *dynamic optimization problems* [4] and defined as

$$\begin{aligned} &\text{minimize} \quad f(\mathrm{x}, \varpi(\mathrm{t})), \qquad \mathrm{x} = (x_1, x_2, \dots, x_{\mathrm{n}}), \varpi(\mathrm{t}) = (\varpi_1(\mathrm{t}), \varpi_2(\mathrm{t}), \dots, \varpi_{\mathrm{n}_\varpi}(\mathrm{t})) \\ &\text{subject to} \quad g_m(\mathrm{x}) \le 0,\ m = 1, 2, \dots, n_g \\ &\qquad\qquad\quad h_m(\mathrm{x}) = 0,\ m = n_g + 1, \dots, n_g + n_h \\ &\qquad\qquad\quad \forall \mathrm{x} \in R^n \end{aligned} \tag{2.3}$$

where $\varpi(\mathrm{t})$ is a vector of time-dependent objective function control parameters, and $x^*(t)$ is the optimum found at time step t.

There are two techniques to solve optimization problems: Global and Local optimization techniques.

2.2 Global Optimization

A global minimizer is defined as x^*such that

$$f(x^*) \le f(x),\ \forall x \in S \tag{2.4}$$

where S is the search space and $S = R^n$ for unconstrained problems.

Here, the term global minimum refers to the value $f(x^*)$, and x^* is called the global minimizer. Some global optimization methods require a starting point $z_0 \in S$ and it will be able to find the global minimizer x^* if $z_0 \in S$.

2.3 Local Optimization

A local minimizer x_L^* of the region L, is defined as

$$f(\ x_L^*\ ) \le f(x),\ \forall x \in L \tag{2.5}$$

where $L \subseteq R^n$.

Here, a local optimization method should guarantee that a local minimizer of the set L is found.

Finally, local optimization techniques try to find a local minimum and its corresponding local minimizer, whereas global optimization techniques seek to find a global minimum or lowest function value and its corresponding global minimizer.

Example: Consider a function $y = f(x) = x^4 - 12\,x^3 + 47\,x^2 - 75x + 10$, and then the following figure 2.1.1 illustrates the difference between the global minimizer x^* and the local minimizer x_L^*.

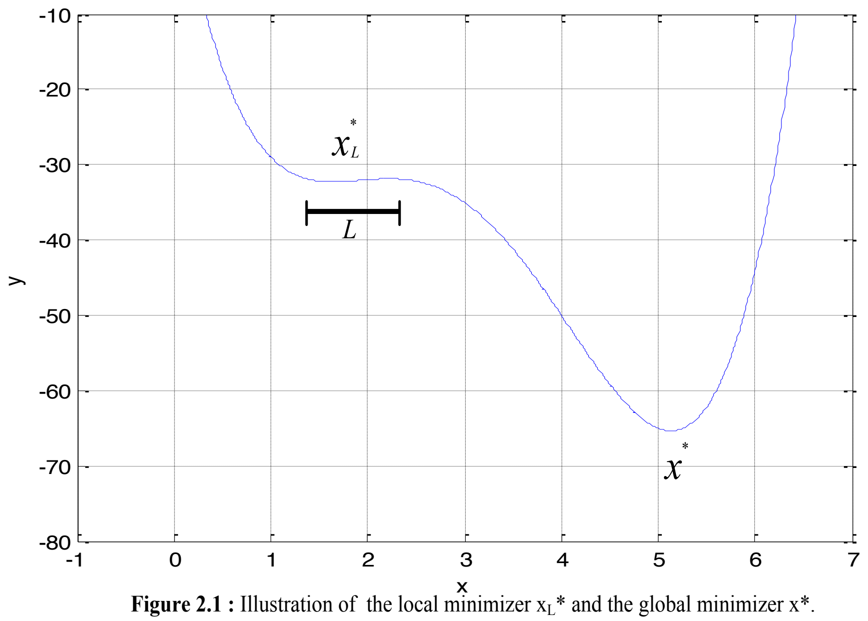
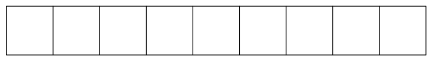


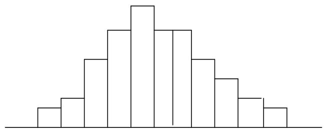
Figure 2.1 : Illustration of the local minimizer x_L* and the global minimizer x*.

2.4 Uniform Distribution

A uniform distribution, sometimes called a rectangular distribution, is a distribution where the probability of occurrence is the same for all values of x , i.e. it has constant probability. For instance, if a die is thrown, then the probability of obtaining any one of the six possible outcomes is 1/6. Now, since all outcomes are equally probable, the distribution is uniform.

Therefore, if a uniform distribution is divided into equally spaced intervals, there will be an equal number of members of the population in each interval. The distribution is defined by $U(a, b)$, where a and b are its minimum and maximum values respectively.

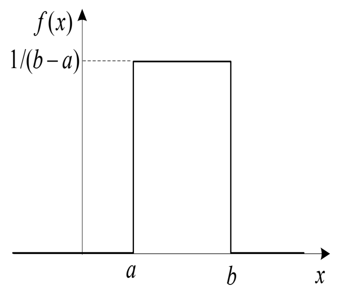

A uniform distribution


A nonuniform distribution

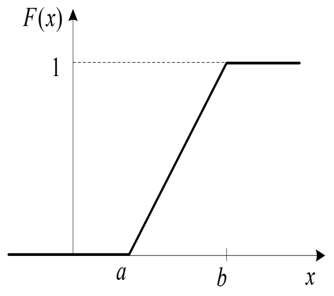
The probability density function (PDF) and cumulative distribution function (CDF) for a continuous uniform distribution on the interval $[a, b]$ are respectively

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for } x > b \end{cases} \tag{2.6}$$

and
$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases} \tag{2.7}$$



Uniform PDF



Uniform CDF

$U(0,1)$ is called a **standard uniform distribution**.

2.5 Sigmoid function

Sigmoid function, sometimes called a logistic function, is an ’S’ shape curve and defined by the formula

$$s(t) = \frac{1}{1+e^{-t}} \tag{2.8}$$

It is a monotonically increasing function with

$$s(t) = \begin{cases} \to 1 & \text{if } t \to \infty \\ \frac{1}{2} & \text{if } t = 0 \\ \to 0 & \text{if } t \to -\infty \end{cases} \tag{2.9}$$

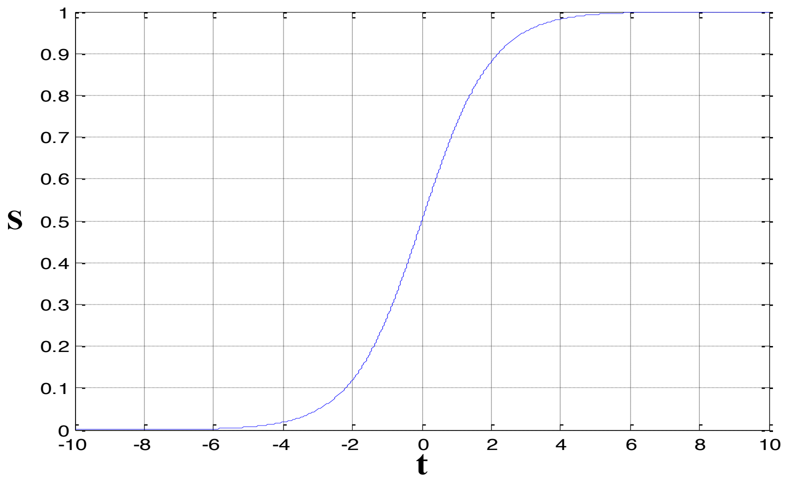


Figure 2.2: Sigmoid function.

Since, sigmoid function is monotonically increasing, we can write

$$s(t) < s(t+\epsilon), \epsilon > 0. \tag{2.10}$$

CHAPTER 3

Basic Particle Swarm Optimization

This chapter discusses a conceptual overview of the PSO algorithm and its parameters selection strategies, geometrical illustration and neighborhood topology, advantages and disadvantages of PSO, and mathematical explanation.

3.1 The Basic Model of PSO algorithm

Kennedy and Eberhart first established a solution to the complex non-linear optimization problem by imitating the behavior of bird flocks. They generated the concept of function-optimization by means of a particle swarm [15]. Consider the global optimum of an *n*-dimensional function defined by

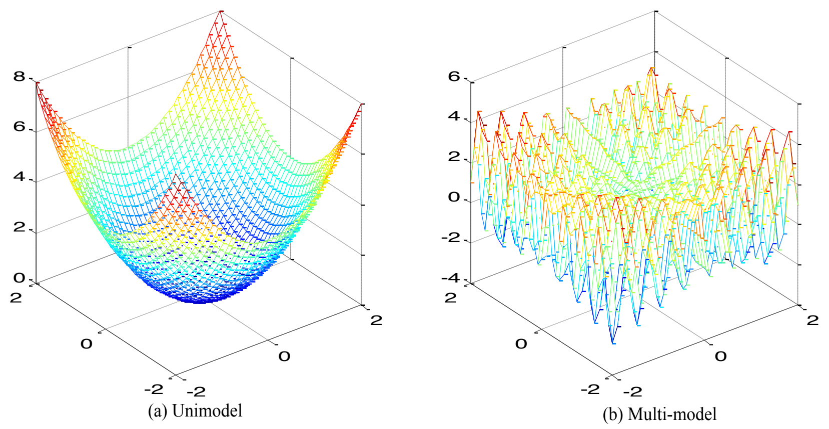
$$f(x_1, x_2, x_3, \dots, x_n) = f(X) \tag{3.1}$$

where x_i is the search variable, which represents the set of free variables of the given function. The aim is to find a value x^* such that the function $f(x^*)$ is either a maximum or a minimum in the search space.

Consider the functions given by

$$f_1 = x_1^2 + x_2^2 \tag{3.2}$$

$$\text{and} \quad f_2 = x_1 \sin(4\pi x_2) - x_2 \sin(4\pi x_1 + \pi) + 1 \tag{3.3}$$



(a) Unimodel (b) Multi-model

Figure 3.1: Plot of the functions f_1 and f_2.

From the figure 3.1 (a), it is clear that the global minimum of the function f_1 is at $(x_1, x_2) = (0,0)$, i.e. at the origin of function f_1 in the search space. That means it is a *unimodel* function, which has only one minimum. However, to find the global optimum is not so easy for *multi-model* functions, which have multiple local minima. Figure 3.1 (b) shows the function f_2 which has a rough search space with multiple peaks, so many agents have to start from different initial locations and continue exploring the search space until at least one agent reach the global optimal position. During this process all agents can communicate and share their information among themselves [15]. This thesis discusses how to solve the *multi-model* function problems.

The Particle Swarm Optimization (PSO) algorithm is a multi-agent parallel search technique which maintains a swarm of particles and each particle represents a potential solution in the swarm. All particles fly through a multidimensional search space where each particle is adjusting its position according to its own experience and that of neighbors. Suppose x_i^t denote the position vector of particle i in the multidimensional search space (i.e. R^n) at time step t, then the position of each particle is updated in the search space by

$$x_i^{t+1} = x_i^t + v_i^{t+1} \text{ with } x_i^0 \sim U(x_{min}, x_{max}) \tag{3.4}$$

where,

v_i^t is the velocity vector of particle i that drives the optimization process and reflects both the own experience knowledge and the social experience knowledge from the all particles;

$U(x_{min}, x_{max})$ is the uniform distribution where x_{min} and x_{max} are its minimum and maximum values respectively.

Therefore, in a PSO method, all particles are initiated randomly and evaluated to compute fitness together with finding the personal best (best value of each particle) and global best (best value of particle in the entire swarm). After that a loop starts to find an optimum solution. In the loop, first the particles' velocity is updated by the personal and global bests, and then each particle's position is updated by the current velocity. The loop is ended with a stopping criterion predetermined in advance [22].

Basically, two PSO algorithms, namely the Global Best (*gbest*) and Local Best (*lbest*) PSO, have been developed which differ in the size of their neighborhoods. These algorithms are discussed in Sections 3.1.1 and 3.1.2 respectively.

3.1.1 Global Best PSO

The global best PSO (or *gbest* PSO) is a method where the position of each particle is influenced by the best-fit particle in the entire swarm. It uses a star social network topology (Section 3.5) where the social information obtained from all particles in the entire swarm [2] [4]. In this method each individual particle, $i \in [1, \dots, n]$ where $n > 1$, has a current position in search space x_i, a current velocity, v_i, and a personal best position in search space, $P_{best,i}$. The personal best position $P_{best,i}$ corresponds to the position in search space where particle i had the smallest value as determined by the objective function f, considering a minimization problem. In addition, the position yielding the lowest value amongst all the personal best $P_{best,i}$ is called the global best position which is denoted by G_{best} [20]. The following equations (3.5) and (3.6) define how the personal and global best values are updated, respectively.

Considering minimization problems, then the personal best position $P_{best,i}$ at the next time step, $t+1$, where $t \in [0, \dots, N]$, is calculated as

$$P_{best,i}^{t+1} = \begin{cases} P_{best,i}^{t} & \text{if } f(x_i^{t+1}) > P_{best,i}^{t} \\ x_i^{t+1} & \text{if } f(x_i^{t+1}) \leq P_{best,i}^{t} \end{cases} \qquad (3.5)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the fitness function. The global best position G_{best} at time step t is calculated as

$$G_{best} = \min\{P_{best,i}^{t}\}, \text{where } i \in [1, \dots, n] \text{ and } n > 1 \qquad (3.6)$$

Therefore it is important to note that the personal best $P_{best,i}$ is the best position that the individual particle i has visited since the first time step. On the other hand, the global best position G_{best} is the best position discovered by any of the particles in the entire swarm [4].

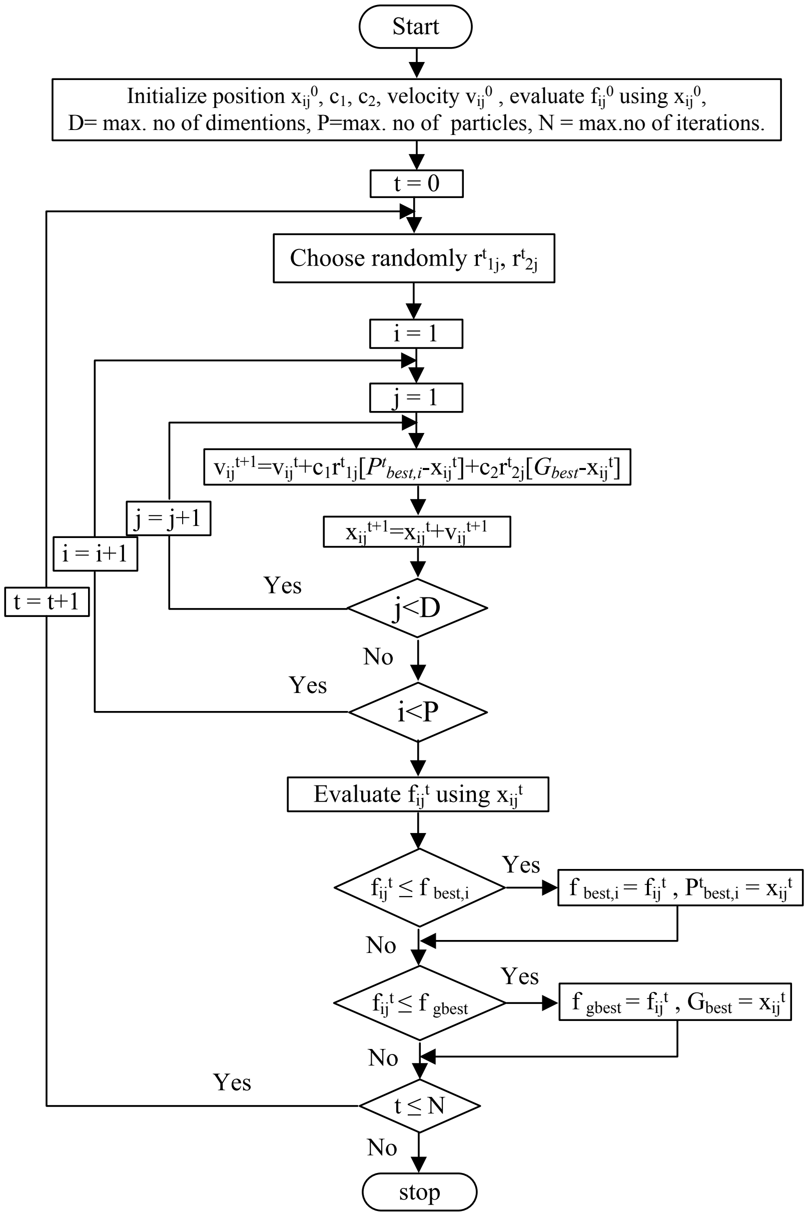
For gbest PSO method, the velocity of particle i is calculated by

$$v_{ij}^{t+1} = v_{ij}^{t} + c_1 r_{1j}^{t}[P_{best,i}^{t} - x_{ij}^{t}] + c_2 r_{2j}^{t}[G_{best} - x_{ij}^{t}] \qquad (3.7)$$

where

v_{ij}^{t}	is the velocity vector of particle i in dimension j at time t ;
x_{ij}^{t}	is the position vector of particle i in dimension j at time t ;
$P_{best,i}^{t}$	is the personal best position of particle i in dimension j found from initialization through time t;
G_{best}	is the global best position of particle i in dimension j found from initialization through time t;
c_1 and c_2	are positive acceleration constants which are used to level the contribution of the cognitive and social components respectively;
r_{1j}^{t} and r_{2j}^{t}	are random numbers from uniform distribution $U(0,1)$ at time t.

The following Flowchart 1 shows the *gbest* PSO algorithm.


Start
Initialize position x_{ij}^0, c_1, c_2, velocity v_{ij}^0 , evaluate f_{ij}^0 using x_{ij}^0,
D= max. no of dimentions, P=max. no of particles, N = max.no of iterations.
t = 0
Choose randomly r^t_{1j}, r^t_{2j}
i = 1
j = 1
$v_{ij}^{t+1}=v_{ij}^t+c_1r^t_{1j}[P^t_{best,i}-x_{ij}^t]+c_2r^t_{2j}[G_{best}-x_{ij}^t]$
$x_{ij}^{t+1}=x_{ij}^t+v_{ij}^{t+1}$
j = j+1
i = i+1
t = t+1
Yes
j<D
No
Yes
i<P
Evaluate f_{ij}^t using x_{ij}^t
$f_{ij}^t \le f_{best,i}$
Yes
$f_{best,i} = f_{ij}^t$, $P^t_{best,i} = x_{ij}^t$
No
$f_{ij}^t \le f_{gbest}$
Yes
$f_{gbest} = f_{ij}^t$, $G_{best} = x_{ij}^t$
No
Yes
$t \le N$
No
stop

Flowchart 1: gbest PSO

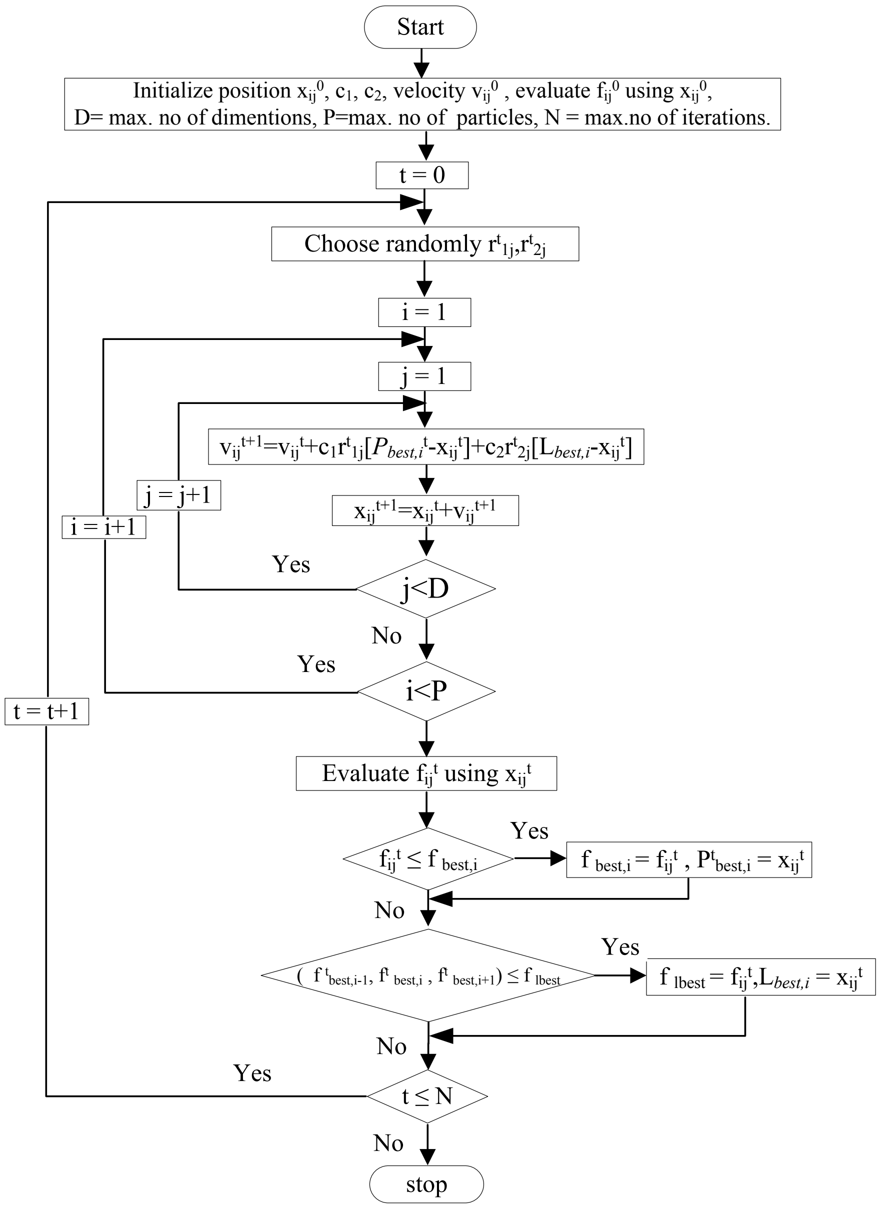
3.1.2 Local Best PSO

The local best PSO (or *lbest* PSO) method only allows each particle to be influenced by the best-fit particle chosen from its neighborhood, and it reflects a ring social topology (Section 3.5). Here this social information exchanged within the neighborhood of the particle, denoting local knowledge of the environment [2] [4]. In this case, the velocity of particle i is calculated by

$$v_{ij}^{t+1} = v_{ij}^t + c_1 r_{1j}^t [P_{best,i}^t - x_{ij}^t] + c_2 r_{2j}^t [L_{best,i} - x_{ij}^t] \tag{3.8}$$

where, $L_{best,i}$ is the best position that any particle has had in the neighborhood of particle i found from initialization through time t.

The following Flowchart 2 summarizes the *lbest* PSO algorithm:



Start
Initialize position x_{ij}^0, c_1, c_2, velocity v_{ij}^0 , evaluate f_{ij}^0 using x_{ij}^0, D= max. no of dimentions, P=max. no of particles, N = max.no of iterations.
t = 0
Choose randomly r^t_{1j},r^t_{2j}
i = 1
j = 1
$v_{ij}^{t+1}=v_{ij}^t+c_1r^t_{1j}[P_{best,i}^t-x_{ij}^t]+c_2r^t_{2j}[L_{best,i}-x_{ij}^t]$
$x_{ij}^{t+1}=x_{ij}^t+v_{ij}^{t+1}$
j = j+1
i = i+1
Yes
j<D
No
Yes
i<P
t = t+1
Evaluate f_{ij}^t using x_{ij}^t
Yes
$f_{ij}^t \le f_{best,i}$
$f_{best,i} = f_{ij}^t$, $P^t_{best,i} = x_{ij}^t$
No
Yes
$(f^t_{best,i-1}, f^t_{best,i}, f^t_{best,i+1}) \le f_{lbest}$
$f_{lbest} = f_{ij}^t$,$L_{best,i} = x_{ij}^t$
No
Yes
t ≤ N
No
stop

Flowchart 2: lbest PSO

Finally, we can say from the Section 3.1.1 and 3.1.2 respectively, in the *gbest* PSO algorithm every particle obtains the information from the best particle in the entire swarm, whereas in the *lbest* PSO algorithm each particle obtains the information from only its immediate neighbors in the swarm [1].

3.2 Comparison of '*gbest*' to '*lbest*'

Originally, there are two differences between the '*gbest*' PSO and the '*lbest*' PSO: One is that because of the larger particle interconnectivity of the *gbest* PSO, sometimes it converges faster than the *lbest* PSO. Another is due to the larger diversity of the *lbest* PSO, it is less susceptible to being trapped in local minima [4].

3.3 PSO Algorithm Parameters

There are some parameters in PSO algorithm that may affect its performance. For any given optimization problem, some of these parameter's values and choices have large impact on the efficiency of the PSO method, and other parameters have small or no effect [9]. The basic PSO parameters are swarm size or number of particles, number of iterations, velocity components, and acceleration coefficients illustrated bellow. In addition, PSO is also influenced by inertia weight, velocity clamping, and velocity constriction and these parameters are described in Chapter IV.

3.3.1 Swarm size

Swarm size or population size is the number of particles n in the swarm. A big swarm generates larger parts of the search space to be covered per iteration. A large number of particles may reduce the number of iterations need to obtain a good optimization result. In contrast, huge amounts of particles increase the computational complexity per iteration, and more time consuming. From a number of empirical studies, it has been shown that most of the PSO implementations use an interval of $n \in [20,60]$ for the swarm size.

3.3.2 Iteration numbers

The number of iterations to obtain a good result is also problem-dependent. A too low number of iterations may stop the search process prematurely, while too large iterations has the consequence of unnecessary added computational complexity and more time needed [4].

3.3.3 Velocity Components

The velocity components are very important for updating particle's velocity. There are three terms of the particle's velocity in equations (3.7) and (3.8):

1. The term v_{ij}^{t} is called inertia component that provides a memory of the previous flight direction that means movement in the immediate past. This component represents as a momentum which prevents to drastically change the direction of the particles and to bias towards the current direction.

2. The term $c_1 r_{1j}^t [P_{best,i}^t - x_{ij}^t]$ is called cognitive component which measures the performance of the particles i relative to past performances. This component looks like an individual memory of the position that was the best for the particle. The effect of the cognitive component represents the tendency of individuals to return to positions that satisfied them most in the past. The cognitive component referred to as the nostalgia of the particle.

3. The term $c_2 r_{2j}^t [G_{best} - x_{ij}^t]$ for *gbest* PSO or $c_2 r_{2j}^t [L_{best,i} - x_{ij}^t]$ for *lbest* PSO is called social component which measures the performance of the particles i relative to a group of particles or neighbors. The social component's effect is that each particle flies towards the best position found by the particle's neighborhood.

3.3.4 Acceleration coefficients

The acceleration coefficients c_1 and c_2, together with the random values r_1 and r_2, maintain the stochastic influence of the cognitive and social components of the particle's velocity respectively. The constant c_1 expresses how much confidence a particle has in itself, while c_2 expresses how much confidence a particle has in its neighbors [4]. There are some properties of c_1 and c_2:

●When $c_1 = c_2 = 0$, then all particles continue flying at their current speed until they hit the search space's boundary. Therefore, from the equations (3.7) and (3.8), the velocity update equation is calculated as

$$v_{ij}^{t+1} = v_{ij}^t \tag{3.9}$$

●When $c_1 > 0$ and $c_2 = 0$, all particles are independent. The velocity update equation will be

$$v_{ij}^{t+1} = v_{ij}^t + c_1 r_{1j}^t [P_{best,i}^t - x_{ij}^t] \tag{3.10}$$

On the contrary, when $c_2 > 0$ and $c_1 = 0$, all particles are attracted to a single point ($i.e.\ G_{best}$) in the entire swarm and the update velocity will become

$$v_{ij}^{t+1} = v_{ij}^t + c_2 r_{2j}^t [G_{best} - x_{ij}^t] \text{ for gbest PSO,} \tag{3.11}$$

$$\text{or, } v_{ij}^{t+1} = v_{ij}^t + c_2 r_{2j}^t [L_{best,i} - x_{ij}^t] \text{ for lbest PSO.} \tag{3.12}$$

●When $c_1 = c_2$, all particles are attracted towards the average of $P_{best,i}^t$ and G_{best}.

●When $c_1 \gg c_2$, each particle is more strongly influenced by its personal best position, resulting in excessive wandering. In contrast, when $c_2 \gg c_1$ then all particles are much more influenced by the global best position, which causes all particles to run prematurely to the optima [4] [11].

Normally, c_1 and c_2 are static, with their optimized values being found empirically. Wrong initialization of c_1 and c_2 may result in divergent or cyclic behavior [4]. From the different empirical researches, it has been proposed that the two acceleration constants should be $c_1 = c_2 = 2$.

3.4 Geometrical illustration of PSO

The update velocity for particles consist of three components in equations (3.7) and (3.8) respectively. Consider a movement of a single particle in a two dimensional search space.

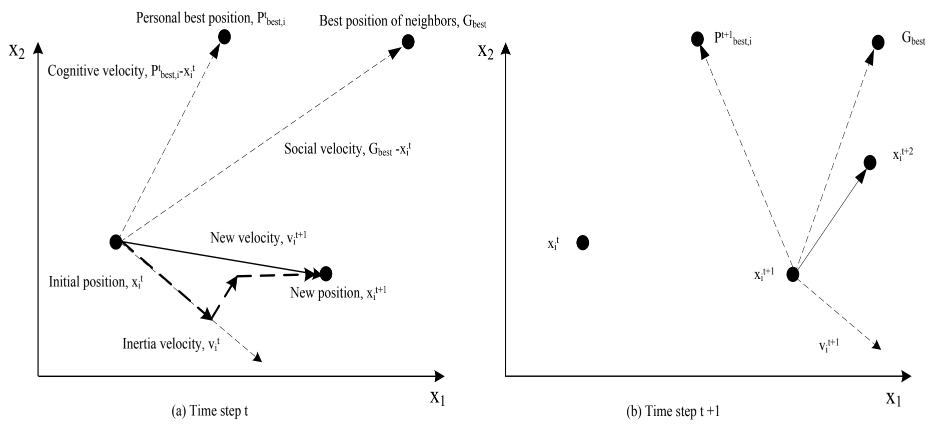

Personal best position, $P^t_{best,i}$
Best position of neighbors, G_{best}
x_2
Cognitive velocity, $P^t_{best,i}-x_i^t$
Social velocity, $G_{best}-x_i^t$
New velocity, v_i^{t+1}
Initial position, x_i^t
New position, x_i^{t+1}
Inertia velocity, v_i^t
x_1
(a) Time step t
$P^{t+1}_{best,i}$
G_{best}
x_i^{t+2}
x_i^t
x_i^{t+1}
v_i^{t+1}
(b) Time step t +1

Figure 3.2: velocity and position update for a particle in a two-dimensional search space.

Figure 3.2 illustrates how the three velocity components contribute to move the particle towards the global best position at time steps t and $t + 1$ respectively.

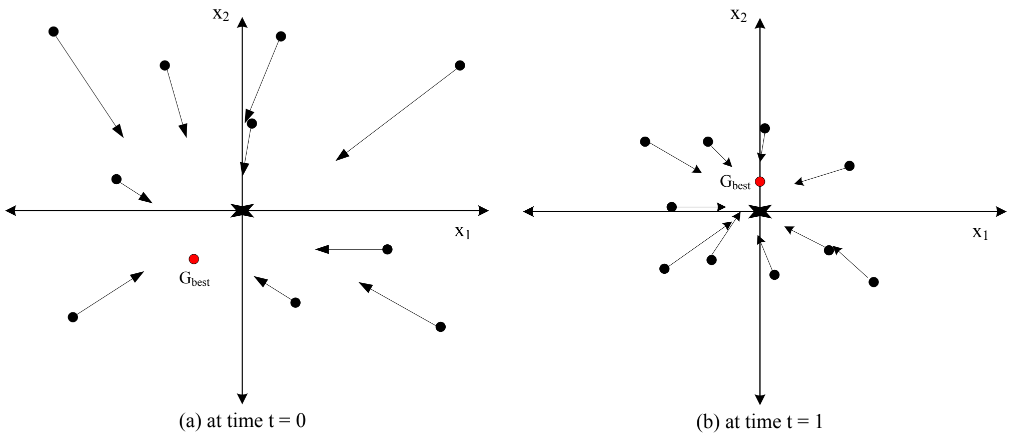


Figure 3.3: Velocity and Position update for Multi-particle in gbest PSO.

Figure 3.3 shows the position updates for more than one particle in a two dimensional search space and this figure illustrates the gbest PSO. The optimum position is denoted by the symbol ' ⨯ '. Figure 3.3 (a) shows the initial position of all particles with the global best position. The cognitive component is zero at $t = 0$ and all particles are only attracted toward the best position by the social component. Here the global best position does not change. Figure 3.3 (b) shows the new positions of all particles and a new global best position after the first iteration i.e. at $t = 1$.

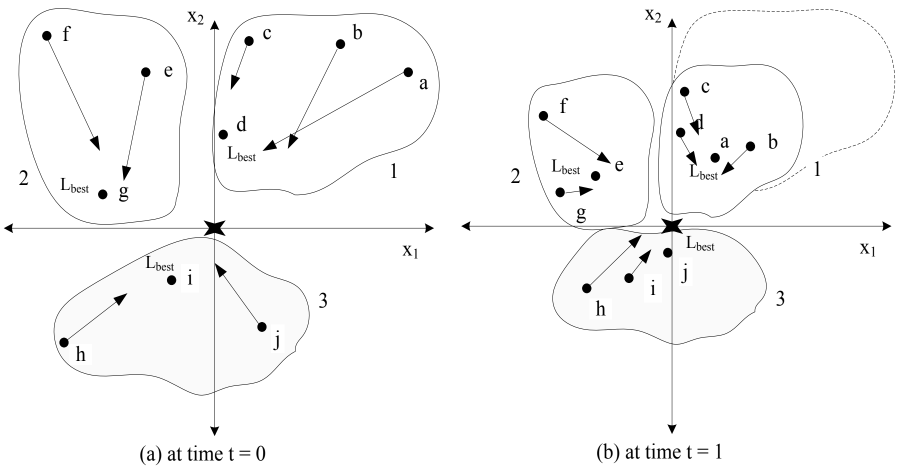


Figure 3.4: Velocity and Position update for Multi-particle in lbest PSO.

Figure 3.4 illustrates how all particles are attracted by their immediate neighbors in the search space using lbest PSO and there are some subsets of particles where one subset of particles is defined for each particle from which the local best particle is then selected. Figure 3.4 (a) shows particles *a*, *b* and *c* move towards particle *d*, which is the best position in subset 1. In subset 2, particles *e* and *f* move towards particle *g*. Similarly, particle *h* moves towards particle *i*, so does *j* in subset 3 at time step $t = 0$. Figure 3.4 (b) for time step $t = 1$, the particle *d* is the best position for subset 1 so the particles *a*, *b* and *c* move towards *d*.

3.5 Neighborhood Topologies

A neighborhood must be defined for each particle [7]. This neighborhood determines the extent of social interaction within the swarm and influences a particular particle's movement. Less interaction occurs when the neighborhoods in the swarm are small [4]. For small neighborhood, the convergence will be slower but it may improve the quality of solutions. For larger neighborhood, the convergence will be faster but the risk that sometimes convergence occurs earlier [7]. To solve this problem, the search process starts with small neighborhoods size and then the small neighborhoods size is increased over time. This technique ensures an initially high diversity with faster convergence as the particles move towards a promising search region [4].

The PSO algorithm is social interaction among the particles in the entire swarm. Particles communicate with one another by exchanging information about the success of each particle in the swarm. When a particle in the whole swarm finds a better position, all particles move towards this particle. This performance of the particles is determined by the particles' neighborhood [4]. Researchers have worked on developing this performance by designing different types of neighborhood structures [15]. Some neighborhood structures or topologies are discussed below:

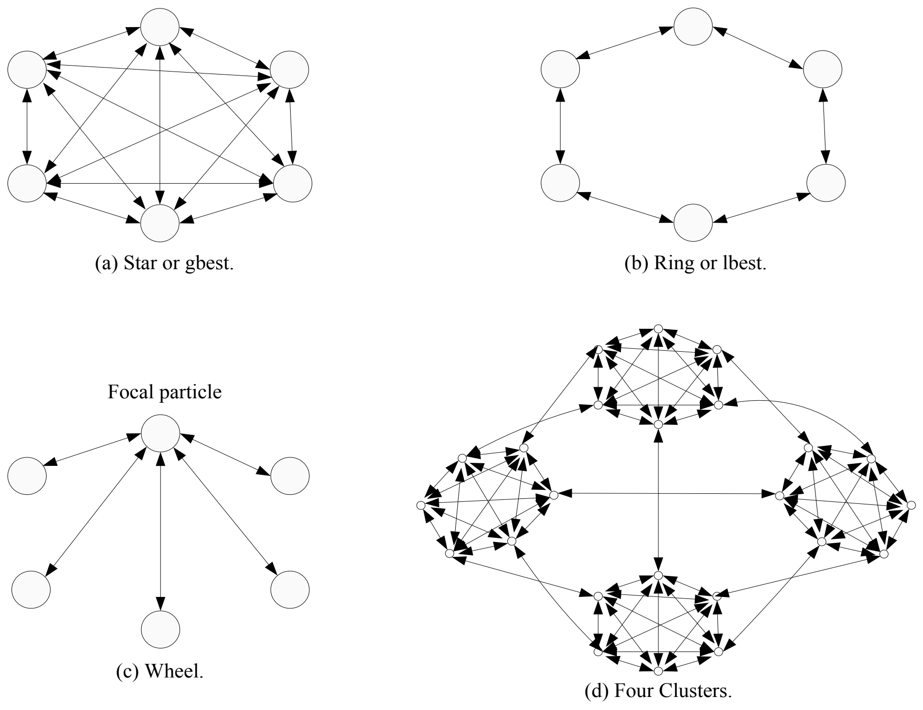


Figure 3.5: Neighborhood topologies.

Figure 3.5 (a) illustrates the star topology, where each particle connects with every other particle. This topology leads to faster convergence than other topologies, but there is a susceptibility to be trapped in local minima. Because all particles know each other, this topology is referred to as the gbest PSO.

Figure 3.5 (b) illustrates the ring topology, where each particle is connected only with its immediate neighbors. In this process, when one particle finds a better result, this particle passes it to its immediate neighbors, and these two immediate neighbors pass it to their immediate neighbors, until it reaches the last particle. Thus the best result found is spread very slowly around the ring by all particles. Convergence is slower, but larger parts of the search space are covered than with the star topology. It is referred as the lbest PSO.

Figure 3.5 (c) illustrates the wheel topology, in which only one particle (a *focal particle*) connects to the others, and all information is communicated through this particle. This focal particle compares the best performance of all particles in the swarm, and adjusts its position towards the best performance particle. Then the new position of the focal particle is informed to all the particles.

Figure 3.5 (d) illustrates a four clusters topology, where four clusters (or cliques) are connected with two edges between neighboring clusters and one edge between opposite clusters.

There are more different neighborhood structures or topologies (for instance, pyramid topology, the Von Neumann topology and so on), but there is no the best topology known to find the optimum for all kinds of optimization problems.

3.6 Problem Formulation of PSO algorithm

Problem: Find the maximum of the function

$$f(x) = -x^2 + 5x + 20 \text{ with } -10 \leq x \leq 10$$

using the PSO algorithm. Use 9 particles with the initial positions $x_1 = -9.6$, $x_2 = -6$, $x_3 = -2.6$, $x_4 = -1.1$, $x_5 = 0.6$, $x_6 = 2.3$, $x_7 = 2.8$, $x_8 = 8.3$, and $x_9 = 10$. Show the detailed computations for iterations 1, 2 and 3.

Solution:

Step1: Choose the number of particles: $x_1 = -9.6$, $x_2 = -6$, $x_3 = -2.6$, $x_4 = -1.1$, $x_5 = 0.6$, $x_6 = 2.3$, $x_7 = 2.8$, $x_8 = 8.3$, and $x_9 = 10$.

The initial population (i.e. the iteration number $t = 0$) can be represented as $x_i^0, i = 1,2,3,4,5,6,7,8,9$:

$$x_1^0 = -9.6\,,\ x_2^0 = -6,\ x_3^0 = -2.6,$$
$$x_4^0 = -1.1,\ x_5^0 = 0.6,\ x_6^0 = 2.3,$$
$$x_7^0 = 2.8,\quad x_8^0 = 8.3,\ x_9^0 = 10.$$

Evaluate the objective function values as

$$f_1^0 = -120.16,\ f_2^0 = -46,\ f_3^0 = 0.24,$$
$$f_4^0 = 13.29,\ f_5^0 = 22.64\,,\ f_6^0 = 26.21,$$
$$f_7^0 = 26.16,\ f_8^0 = -7.39,\ f_9^0 = -30.$$

Let $c_1 = c_2 = 1$.

Set the initial velocities of each particle to zero:
$$v_i^0 = 0, i.e.\, v_1^0 = v_2^0 = v_3^0 = v_4^0 = v_5^0 = v_6^0 = v_7^0 = v_8^0 = v_9^0 = 0.$$

Step2: Set the iteration number as $t = 0 + 1 = 1$ and go to step 3.

Step3: Find the personal best for each particle by

$$P_{best,i}^{t+1} = \begin{cases} P_{best,i}^t & \text{if } f_i^{t+1} > P_{best,i}^t \\ x_i^{t+1} & \text{if } f_i^{t+1} \leq P_{best,i}^t \end{cases}$$

So,

$P_{best,1}^{1} = -9.6$, $P_{best,2}^{1} = -6$, $P_{best,3}^{1} = -2.6$,

$P_{best,4}^{1} = -1.1$, $P_{best,5}^{1} = 0.6$, $P_{best,6}^{1} = 2.3$,

$P_{best,7}^{1} = 2.8$, $P_{best,8}^{1} = 8.3$, $P_{best,9}^{1} = 10$.

Step4: Find the global best by

$$G_{best} = \min\{P_{best,i}^{t}\} \text{ where } i = 1,2,3,4,5,6,7,8,9.$$

Since, the maximum personal best is $P_{best,6}^{1} = 2.3$, thus $G_{best} = 2.3$.

Step5: Considering the random numbers in the range (0, 1) as $r_1^1 = 0.213$ and $r_2^1 = 0.876$, and find the velocities of the particles by

$$v_i^{t+1} = v_i^t + c_1 r_1^t\left[P_{best,i}^t - x_i^t\right] + c_2 r_2^t[G_{best}^t - x_i^t];\ i = 1, \dots ,9.$$

so

$v_1^1 = 0 + 0.213(-9.6 + 9.6) + 0.876(2.3 + 9.6) = 10.4244$

$v_2^1 = 7.2708$, $v_3^1 = 4.2924$, $v_4^1 = 2.9784$, $v_5^1 = 1.4892$,

$v_6^1 = 0$, $v_7^1 = -0.4380$, $v_8^1 = 5.256$, $v_9^1 = -6.7452$.

Step6: Find the new values of $x_i^1, i = 1, \dots ,9$ by

$x_i^{t+1} = x_i^t + v_i^{t+1}$

So

$x_1^1 = 0.8244$, $x_2^1 = 1.2708$, $x_3^1 = 1.6924$,

$x_4^1 = 1.8784$, $x_5^1 = 2.0892$, $x_6^1 = 2.3$,

$x_7^1 = 2.362$, $x_8^1 = 3.044$, $x_9^1 = 3.2548$.

Step7: Find the objective function values of x_i^1 , $i = 1, \dots ,9$:

$f_1^1 = 23.4424$, $f_2^1 = 24.7391$, $f_3^1 = 25.5978$,

$f_4^1 = 25.8636$, $f_5^1 = 26.0812$, $f_6^1 = 26.21$,

$f_7^1 = 26.231$, $f_8^1 = 25.9541$, $f_9^1 = 25.6803$.

Step 8: Stopping criterion:

If the terminal rule is satisfied, go to step 2,
Otherwise stop the iteration and output the results.

Step2: Set the iteration number as $t = 1 + 1 = 2$, and go to step 3.

Step3: Find the personal best for each particle.

$$P_{best,1}^{2} = 0.8244, P_{best,2}^{2} = 1.2708, P_{best,3}^{2} = 1.6924,$$
$$P_{best,4}^{2} = 1.8784, P_{best,5}^{2} = 2.0892, P_{best,6}^{2} = 2.3,$$
$$P_{best,7}^{2} = 2.362, P_{best,8}^{2} = 3.044, P_{best,9}^{2} = 3.2548.$$

Step4: Find the global best.

$$G_{best} = 2.362\ .$$

Step5: By considering the random numbers in the range (0, 1) as
$r_1^2 = 0.113$ and $r_2^2 = 0.706$, find the velocities of the particles by

$$v_i^{t+1} = v_i^t + c_1 r_1^t \left[P_{best,i}^t - x_i^t\right] + c_2 r_2^t [\, G_{best} - x_i^t];\ i = 1, \dots, 9.$$

so

$$v_1^2 = 11.5099,\ v_2^2 = 8.0412,\ v_3^2 = 4.7651,$$
$$v_4^2 = 3.3198,\ v_5^2 = 1.6818,\ v_6^2 = 0.0438,$$
$$v_7^2 = -0.4380,\ v_8^2 = -5.7375,\ v_9^2 = -7.3755.$$

Step6: Find the new values of x_i^2 , $i = 1, \dots, 9$ by

$$x_i^{t+1} = x_i^t + v_i^{t+1}$$

so

$$x_1^2 = 12.3343,\ x_2^2 = 9.312,\ x_3^2 = 6.4575,$$
$$x_4^2 = 5.1982,\ x_5^2 = 3.7710,\ x_6^2 = 2.3438,$$
$$x_7^2 = 1.9240,\ x_8^2 = -2.6935,\ x_9^2 = -4.1207.$$

Step7: Find the objective function values of $f_i^2, i = 1, \dots, 9$:

$$f_1^2 = -70.4644, f_2^2 = -20.1532, f_3^2 = 10.5882,$$
$$f_4^2 = 18.9696,\ \ f_5^2 = 24.6346,\ \ \ f_6^2 = 26.2256,$$
$$f_7^2 = 25.9182, f_8^2 = -0.7224,\ f_9^2 = -17.5839.$$

Step 8: Stopping criterion:

If the terminal rule is satisfied, go to step 2,
Otherwise stop the iteration and output the results.

Step2: Set the iteration number as $t = 1 + 1 = 3$, and go to step 3.

Step3: Find the personal best for each particle.

$$P_{best,1}^{3} = 0.8244, P_{best,2}^{3} = 1.2708, P_{best,3}^{3} = 1.6924,$$
$$P_{best,4}^{3} = 1.8784,\ P_{best,5}^{3} = 2.0892, P_{best,6}^{3} = 2.3438,$$
$$P_{best,7}^{3} = \ 2.362,\ P_{best,8}^{3} = 3.044, \quad P_{best,9}^{3} = 3.2548.$$

Step4: Find the global best.

$$G_{best} = 2.362$$

Step5: By considering the random numbers in the range (0, 1) as
$r_1^3 = 0.178$ and $r_2^3 = 0.507$, find the velocities of the particles by

$$v_i^{t+1} = v_i^t + c_1 r_1^t\left[P_{best,i}^t - \ x_i^t\right] + c_2 r_2^t[\ G_{best} - x_i^t];\ i = 1, \dots, 9.$$

so

$$v_1^3 = 4.4052,\ v_2^3 = 3.0862,\ v_3^3 = 1.8405,$$
$$v_4^3 = 1.2909,\ v_5^3 = 0.6681,\ v_6^3 = 0.053,$$
$$v_7^3 = -0.1380,\ v_8^3 = -2.1531,\ v_9^3 = -2.7759.$$

Step6: Find the new values of x_i^3 , $i = 1, \dots, 9$ by

$$x_i^{t+1} = \ x_i^t + \ v_i^{t+1}$$

so

$$x_1^3 = 16.7395,\ x_2^3 = 12.3982,\ x_3^3 = 8.298,$$
$$x_4^3 = 6.4892,\ x_5^3 = 4.4391,\ x_6^3 = 2.3968,$$
$$x_7^3 = 1.786,\ x_8^3 = -4.8466,\ x_9^3 = -6.8967.$$

Step7: Find the objective function values of $f_i^3, i = 1, \dots, 9$:

$$f_1^3 = -176.5145, f_2^3 = -71.7244, f_3^3 = -7.3673,$$
$$f_4^3 = 10.3367, \qquad f_5^3 = 22.49, \quad f_6^3 = 26.2393,$$
$$f_7^3 = 25.7402, \ \ f_8^3 = -27.7222,\ f_9^3 = -62.0471.$$

Step 8: Stopping criterion:

If the terminal rule is satisfied, go to step 2,
Otherwise stop the iteration and output the results.

Finally, the values of $x_i^2, i = 1,2,3,4,5,6,7,8,9$ did not converge, so we increment the iteration number as $t = 4$ and go to step 2. When the positions of all particles converge to similar values, then the method has converged and the corresponding value of x_i^t is the optimum solution. Therefore the iterative process is continued until all particles meet a single value.

3.7 Advantages and Disadvantages of PSO

It is said that PSO algorithm is the one of the most powerful methods for solving the non-smooth global optimization problems while there are some disadvantages of the PSO algorithm. The advantages and disadvantages of PSO are discussed below:

Advantages of the PSO algorithm [14] [15]:

1) PSO algorithm is a derivative-free algorithm.
2) It is easy to implementation, so it can be applied both in scientific research and engineering problems.
3) It has a limited number of parameters and the impact of parameters to the solutions is small compared to other optimization techniques.
4) The calculation in PSO algorithm is very simple.
5) There are some techniques which ensure convergence and the optimum value of the problem calculates easily within a short time.
6) PSO is less dependent of a set of initial points than other optimization techniques.
7) It is conceptually very simple.

Disadvantages of the PSO algorithm [13]:

1) PSO algorithm suffers from the partial optimism, which degrades the regulation of its speed and direction.
2) Problems with non-coordinate system (for instance, in the energy field) exit.

CHAPTER 4

Empirical Analysis of PSO Characteristics

This chapter discusses a number of modifications of the basic PSO, how to improve speed of convergence, to control the exploration-exploitation trade-off, to overcome the stagnation problem or the premature convergence, the velocity-clamping technique, the boundary value problems technique, the initial and stopping conditions, which are very important in the PSO algorithm.

4.1 Rate of Convergence Improvements

Usually, the particle velocities build up too fast and the maximum of the objective function is passed over. In PSO, particle velocity is very important, since it is the step size of the swarm. At each step, all particles proceed by adjusting the velocity that each particle moves in every dimension of the search space [9]. There are two characteristics: exploration and exploitation. Exploration is the ability to explore different area of the search space for locating a good optimum, while exploitation is the ability to concentrate the search around a searching area for refining a hopeful solution. Therefore these two characteristics have to balance in a good optimization algorithm. When the velocity increases to large values, then particle's positions update quickly. As a result, particles leave the boundaries of the search space and diverge. Therefore, to control this divergence, particles' velocities are reduced in order to stay within boundary constraints [4]. The following techniques have been developed to improve speed of convergence, to balance the exploration-exploitation trade-off, and to find a quality of solutions for the PSO:

4.1.1 Velocity clamping

Eberhart and Kennedy first introduced velocity clamping; it helps particles to stay within the boundary and to take reasonably step size in order to comb through the search space. Without this velocity clamping in the searching space the process will be prone to explode and particles' positions change rapidly [1]. Maximum velocity V_{max} controls the granularity of the search space by clamping velocities and creates a better balance between global exploration and local exploitation.

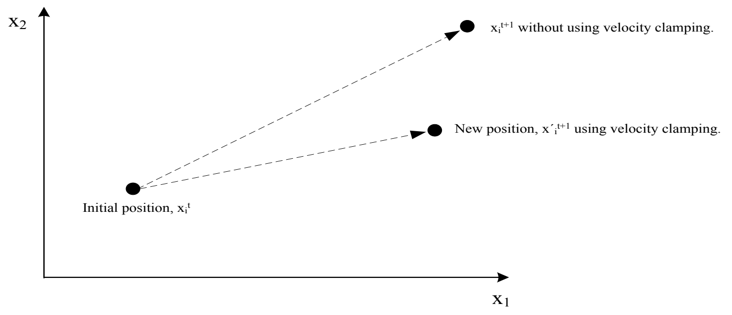

x_2
x_i^{t+1} without using velocity clamping.
New position, x'^{t+1}_i using velocity clamping.
Initial position, x_i^t
x_1

Figure 4.1: Illustration of effects of Velocity Clampnig for a particle in a two-dimensinal search space.

Figure 4.1 illustrates how velocity clamping changes the step size as well as the search direction when a particle moves in the process. In this figure, x_i^{t+1} and x'^{t+1}_i denote respectively the position of particle i without using velocity clamping and the result of velocity clamping [4].

Now if a particle's velocity goes beyond its specified maximum velocity V_{max} , this velocity is set to the value V_{max} and then adjusted before the position update by,

$$v_i^{t+1} = \min\ (v'^{t+1}_i ,\ V_{max}) \tag{4.1}$$

where, v'^{t+1}_i is calculated using equation (3.7) or (3.8).

If the maximum velocity V_{max} is too large, then the particles may move erratically and jump over the optimal solution. On the other hand, if V_{max} is too small, the particle's movement is limited and the swarm may not explore sufficiently or the swarm may become trapped in a local optimum.

This problem can be solved when the maximum velocity V_{max} is calculated by a fraction of the domain of the search space on each dimension by subtracting the lower bound from the upper bound, and is defined as

$$V_{max} = \varepsilon\ (x_{max} - x_{min}) \tag{4.2}$$

where, $x_{max}\ and\ x_{min}$ are respectively the maximum and minimum values of x, and $\varepsilon \in (0,1]$. For example, if $\varepsilon = 0.5$ and $x = [-150,150]$ on each dimension of the search space, then the range of the search space is 300 per dimension and velocities are then clamped to a percentage of that range according to equation (4.2), then the maximum velocity is $V_{max} = 150$ [8].

There is another problem when all velocities are equal to the maximum velocity V_{max} . To solve this problem V_{max} can be reduced over time. The initial step starts with large values of V_{max}, and then it is decreased it over time. The advantage of velocity clamping is that it controls the explosion of velocity in the searching space. On the other hand, the disadvantage is that the best value of V_{max} should be chosen for each different optimization problem using empirical techniques [4] and finding the accurate value for V_{max} for the problem being solved is very critical and not simple, as a poorly chosen V_{max} can lead to extremely poor performance [1].

Finally, V_{max} was first introduced to prevent explosion and divergence. However, it has become unnecessary for convergence because of the use of inertia-weight ω (Section 4.1.2) and constriction factor χ (Section 4.1.3) [15].

4.1.2 Inertia weight

The inertia weight, denoted by ω, is considered to replace V_{max} by adjusting the influence of the previous velocities in the process, i.e. it controls the momentum of the particle by weighing the contribution of the previous velocity. The inertia weight 'ω' will at every step be multiplied by the velocity at the previous time step, i.e. v_{ij}^{t}. Therefore, in the gbest PSO, the velocity equation of the particle i with the inertia weight changes from equation (3.7) to

$$v_{ij}^{t+1} = \omega v_{ij}^{t} + c_1 r_{1j}^{t}[P_{best,i}^{t} - x_{ij}^{t}] + c_2 r_{2j}^{t}[G_{best} - x_{ij}^{t}] \tag{4.3}$$

In the lbest PSO, the velocity equation changes in a similar way as the above velocity equation do.

The inertia weight was first introduced by Shi and Eberhart in 1999 to reduce the velocities over time (or iterations), to control the exploration and exploitation abilities of the swarm, and to converge the swarm more accurately and efficiently compared to the equation (3.7) with (4.3). If $\omega \geq 1$, then the velocities increase over time and particles can hardly change their direction to move back towards optimum, and the swarm diverges. If $\omega \ll 1$, then little momentum is only saved from the previous step and quick changes of direction are to set in the process. If $\omega = 0$, particles velocity vanishes and all particles move without knowledge of the previous velocity in each step [15].

The inertia weight can be implemented either as a fixed value or dynamically changing values. Initial implementations of ω used a fixed value for the whole process for all particles, but now dynamically changing inertia values is used because this parameter controls the exploration and exploitation of the search space. Usually the large inertia value is high at first, which allows all particles to move freely in the search space at the initial steps and decreases over time. Therefore, the process is shifting from the exploratory mode to the exploitative mode. This decreasing inertia weight ω has produced good results in many optimization problems [16]. To control the balance between global and local exploration, to obtain quick convergence, and to reach an optimum, the inertia weight whose value decreases linearly with the iteration number is set according to the following equation [6] [14]:

$$\omega^{t+1} = \omega_{max} - \left(\frac{\omega_{max} - \omega_{min}}{t_{max}}\right)t \quad , \quad \omega_{max} > \omega_{min} \tag{4.4}$$

where,

ω_{max} *and* ω_{min} are the initial and final values of the inertia weight respectively,

t_{max} is the maximum iteration number,

and t is the current iteration number.

Commonly, the inertia weight ω decreases linearly from 0.9 to 0.4 over the entire run.

Van den Bergh and Engelbrecht, Trelea have defined a condition that

$$\omega > \frac{1}{2}(c_1 + c_2) - 1 \tag{4.5}$$

guarantees convergence [4]. Divergent or cyclic behavior can occur in the process if this condition is not satisfied.

Shi and Eberhart defined a technique for adapting the inertia weight dynamically using a fuzzy system [11]. The fuzzy system is a process that can be used to convert a linguistic description of a problem into a model in order to predict a numeric variable, given two inputs (one is the fitness of the global best position and the other is the current value of the inertia weight). The authors chose to use three fuzzy membership functions, corresponding to three fuzzy sets, namely low, medium, and high that the input variables can belong to. The output of the fuzzy system represents the suggested change in the value of the inertia weight [4] [11]. The fuzzy inertia weight method has a greater advantage on the unimodal function. In this method, an optimal inertia weight can be determined at each time step. When a function has multiple local minima, it is more difficult to find an optimal inertia weight [11].

The inertia weight technique is very useful to ensure convergence. However there is a disadvantage of this method is that once the inertia weight is decreased, it cannot increase if the swarm needs to search new areas. This method is not able to recover its exploration mode [16].

4.1.3 Constriction Coefficient

This technique introduced a new parameter 'χ', known as the constriction factor. The constriction coefficient was developed by Clerc. This coefficient is extremely important to control the exploration and exploitation tradeoff, to ensure convergence behavior, and also to exclude the inertia weight ω and the maximum velocity V_{max} [19]. Clerc's proposed velocity update equation of the particle i for the j dimension is calculated as follows:

$$v_{ij}^{t+1} = \chi[v_{ij}^{t} + \phi_1(P_{best,i}^{t} - x_{ij}^{t}) + \phi_2(G_{best} - x_{ij}^{t})] \tag{4.6}$$

where

$$\chi = \frac{2}{\left|2-\phi-\sqrt{\phi^2-4\phi}\right|}, \phi = \phi_1 + \phi_2, \phi_1 = c_1 r_1 \text{ and } \phi_2 = c_2 r_2.$$

If $\phi < 4$, then all particles would slowly spiral toward and around the best solution in the searching space without convergence guarantee. If $\phi > 4$, then all particles converge quickly and guaranteed [1].

The amplitude of the particle's oscillation will be decreased by using the constriction coefficient and it focuses on the local and neighborhood previous best points [7] [15]. If the particle's previous best position and the neighborhood best position are near each other, then the particles will perform a local search. On the other hand, if their positions are far from each other then the particles will perform a global search. The constriction coefficient guarantees convergence of the particles over time and also prevents collapse [15]. Eberhart and Shi empirically illustrated that if constriction coefficient and velocity clamping are used together, then faster convergence rate will be obtained [4].

The disadvantage of the constriction coefficient is that if a particle's personal best position and the neighborhood best position are far apart from each other, the particles may follow wider cycles and not converge [16].

Finally, a PSO algorithm with constriction coefficient is algebraically equivalent to a PSO algorithm with inertia weight. Equation (4.3) and (4.6) can be transformed into one another by the mapping $\omega \leftrightarrow \chi$ and $\phi_i \leftrightarrow \chi\phi_i,\ i = 1{,}2$ [19].

4.2 Boundary Conditions

Sometimes, the search space must be limited in order to prevent the swarm from exploding. In other words, the particles may occasionally fly to a position beyond the defined search space and generate an invalid solution. Traditionally, the velocity clamping technique is used to control the particle's velocities to the maximum value V_{max}. The maximum velocity V_{max} , the inertia weight ω , and the constriction coefficient value χ do not always confine the particles to the solution space. In addition, these parameters cannot provide information about the space within which the particles stay. Besides, some particles still run away from the solution space even with good choices for the parameter V_{max}.

There are two main difficulties connected with the previous velocity techniques: first, the choice of suitable value for V_{max} can be nontrivial and also very important for the overall performance of the method, and second, the previous velocity techniques cannot provide information about how the particles are enforced to stay within the selected search space all the time [18]. Therefore, the method must be generated with clear instructions on how to overcome this situation and such instructions are called the *boundary condition* (BC) of the PSO algorithm which will be parameter-free, efficient, and also reliable.

To solve this problem, different types of boundary conditions have been introduced and the unique features that distinguish each boundary condition are showed in Figure 4.2 [17] [18]. These boundary conditions form two groups: restricted boundary conditions (namely, absorbing, reflecting, and damping) and unrestricted boundary conditions (namely, invisible, invisible/reflecting, invisible/damping) [17].

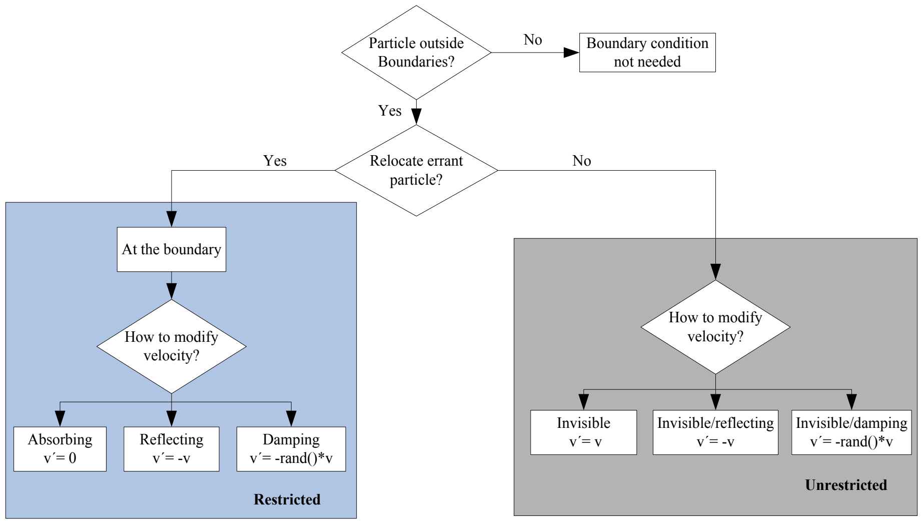


Figure 4.2: Various boundary conditions in PSO.

The following Figure 4.3 shows how the position and velocity of errant particle is treated by boundary conditions.

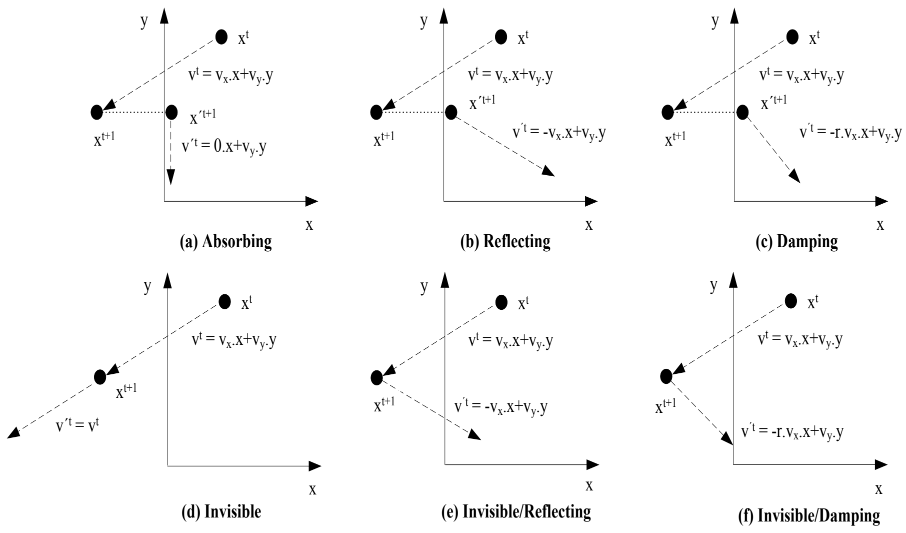


Figure 4.3: Six different boundary conditions for a two-dimensional search space. x′ and v′ represent the modified position and velocity repectively, and r is a random factor [0,1].

The six boundary conditions are discussed below [17]:

● **Absorbing boundary condition (ABC):** When a particle goes outside the solution space in one of the dimensions, the particle is relocated at the wall of the solution space and the velocity of the particle is set to zero in that dimension as illustrated in Figure 4.3(a). This means that, in this condition, such kinetic energy of the particle is absorbed by a soft wall so that the particle will return to the solution space to find the optimum solution.

● **Reflecting boundary condition (RBC):** When a particle goes outside the solution space in one of the dimensions, then the particle is relocated at the wall of the solution space and the sign of the velocity of the particle is changed in the opposite direction in that dimension as illustrated in Figure 4.3(b). This means that, the particle is reflected by a hard wall and then it will move back toward the solution space to find the optimum solution.

● **Damping boundary condition (DBC):** When a particle goes outside the solution space in one of the dimensions, then the particle is relocated at the wall of the solution space and the sign of the velocity of the particle is changed in the opposite direction in that dimension with a random coefficient between 0 and 1 as illustrated in Figure 4.3(c). Thus the damping boundary condition acts very similar as the reflecting boundary condition except randomly determined part of energy is lost because of the imperfect reflection.

● **Invisible boundary condition (IBC):** In this condition, a particle is considered to stay outside the solution space, while the fitness evaluation of that position is skipped and a bad fitness value is assigned to it as illustrated in Figure 4.3(d). Thus the attraction of personal and global best positions will counteract the particle's momentum, and ultimately pull it back inside the solution space.

● **Invisible/Reflecting boundary condition (I/RBC):** In this condition, a particle is considered to stay outside the solution space, while the fitness evaluation of that position is skipped and a bad fitness value is assigned to it as illustrated in Figure 4.3(e). Also, the sign of the velocity of the particle is changed in the opposite direction in that dimension so that the momentum of the particle is reversed to accelerate it back toward in the solution space.

● **Invisible/Damping boundary condition (I/DBC):** In this condition, a particle is considered to stay outside the solution space, while the fitness evaluation of that position is skipped and a bad fitness value is assigned to it as illustrated in Figure 4.3(f). Also, the velocity of the particle is changed in the opposite direction with a random coefficient between 0 and 1 in that dimension so that the reversed momentum of the particle which accelerates it back toward in the solution space is damped.

4.3 Guaranteed Convergence PSO (GCPSO)

When the current position of a particle coincides with the global best position, then the particle moves away from this point if its previous velocity is non-zero. In other words, when $x_{ij}^{t} = P_{best,i}^{t} = G_{best}^{t}$, then the velocity update depends only on the value of ωv_{ij}^{t}. Now if the previous velocities of particles are close to zero, all particles stop moving once and they catch up with the global best position, which can lead to premature convergence of the process. This does not even guarantee that the process has converged to a local minimum, it only means that all particles have converged to the best position in the entire swarm. This leads to stagnation of the search process which the PSO algorithm can overcome by forcing the global best position to change when $x_{ij}^{t} = P_{best,i}^{t} = G_{best}^{t}$ [11].

To solve this problem a new parameter is introduced to the PSO. Let τ be the index of the global best particle, so that

$$y_\tau = G_{best} \tag{4.7}$$

A new velocity update equation for the globally best positioned particle, y_τ , has been suggested in order to keep y_τ moving until it has reached a local minimum. The suggested equation is

$$v_{\tau j}^{t+1} = -x_{\tau j}^{t} + G_{best}^{t} + \omega v_{\tau j}^{t} + \rho^t(1 - 2r_{2j}^{t}) \tag{4.8}$$

where

'ρ^t' is a scaling factor and causes the PSO to perform a random search in an area surrounding the global best position G_{best}. It is defined in equation (4.10) below,

'$-x_{\tau j}^{t}$' resets the particle's position to the position G_{best}^{t},

'$\omega v_{\tau j}^{t}$' represents the current search direction,

'$\rho^t(1 - 2r_{2j}^{t})$' generates a random sample from a sample space with side lengths $2\rho^t$.

Combining the position update equation (3.4) and the new velocity update equation (4.8) for the global best particle τ yields the new position update equation

$$x_{\tau j}^{t+1} = G_{best}^{t} + \omega v_{\tau j}^{t} + \rho^t(1 - 2r_{2}^{t}) \tag{4.9}$$

while all other particles in the swarm continue using the usual velocity update equation (4.3) and the position update equation (3.4) respectively.

The parameter ρ^t controls the diameter of the search space and the value of ρ^t is adapted after each time step, using

$$\rho^0 \quad = 1.0$$

$$\rho^{t+1} \quad = \begin{cases} 2\rho^t & \text{if } \#successes(t) > \epsilon_s \\ 0.5\rho^t & \text{if } \#failures\ (t) > \epsilon_f \\ \rho^t & \text{otherwise} \end{cases} \tag{4.10}$$

where $\#successes$ and $\#failures$ respectively denote the number of consecutive successes and failures, and a failure is defined as $f(G_{best}^{t+1}) = f(G_{best}^{t})$. The following conditions must also be implemented to ensure that equation (4.10) is well defined:

$$\#successes(t+1) > \#successes(t) \Rightarrow \#failures\ (t+1) = 0$$

and

$$\#failures\ (t+1) > \#failures(t) \Rightarrow \ \#successes(t+1) = 0 \tag{4.11}$$

Therefore, when a success occurs, the failure count is set to zero and similarly when a failure occurs, then the success count is reset.

The optimal choice of values for ϵ_s and ϵ_f depend on the objective function. It is difficult to get better results using a random search in only a few iterations for high- dimensional search spaces, and it is recommended to use $\epsilon_s = 15$ and $\epsilon_f = 5$. On the other hand, the optimal values for ϵ_s and ϵ_f can be found dynamically. For instance, ϵ_s may be increased every time that $\#failures > \epsilon_f$ i.e. it becomes more difficult to get the success if failures occur frequently which prevents the value of ρ from fluctuating rapidly. Such strategy can be used also for ϵ_f [11].

GCPSO uses an adaptive ρ to obtain the optimal of the sampling volume given the current state of the algorithm. If a specific value of ρ repeatedly results in a success, then a large sampling volume is selected to increase the maximum distance traveled in one step. On the other hand, when ρ produces ϵ_f consecutive failures, then the sampling volume is too large and must be consequently reduced. Finally, stagnation is totally prevented if $\rho^t > 0$ for all steps [4].

4.4 Initialization, Stopping Criteria, Iteration Terms and Function Evaluation

A PSO algorithm includes particle initialization, parameters selection, iteration terms, function evaluation, and stopping condition. The first step of the PSO is to initialize the swarm and control the parameters, the second step is to calculate the fitness function and define the iteration numbers, and the last step is to satisfy stopping condition. The influence and control of the PSO parameters have been discussed in Sections 3.3 and 4.1 respectively. The rest of the conditions are discussed below:

4.4.1 Initialization

In PSO algorithm, initialization of the swarm is very important because proper initialization may control the exploration and exploitation tradeoff in the search space more efficiently and find the better result. Usually, a uniform distribution over the search space is used for initialization of the swarm. The initial diversity of the swarm is important for the PSO's performance, it denotes that how much of the search space is covered and how well particles are distributed. Moreover, when the initial swarm does not cover the entire search space, the PSO algorithm will have difficultly to find the optimum if the optimum is located outside the covered area. Then, the PSO will only discover the optimum if a particle's momentum carries the particle into the uncovered area. Therefore, the optimal initial distribution is to located within the domain defined by x_{min} and x_{max} which represent the minimum and maximum ranges of x for all particles i in dimension j respectively [4]. Then the initialization method for the position of each particle is given by

$$x(0) = x_{min,j} + r_j(x_{max,j} - x_{min,j}) \tag{4.12}$$

where $r_j \sim U\ (0,1)$.

The velocities of the particles can be initialized to zero, i.e. $v_i(0) = 0,$ since randomly initialized particle's positions already ensure random positions and moving directions. In addition, particles may be initialized with nonzero velocities, but it must be done with care and such velocities should not be too large. In general, large velocity has large momentum and consequently large position update. Therefore, such large initial position updates can cause particles to move away from boundaries in the feasible region, and the algorithm needs to take more iterations before settling the best solution [4].

4.4.2 Iteration Terms and Function Evaluation

The PSO algorithm is an iterative optimization process and repeated iterations will continue until a stopping condition is satisfied. Within one iteration, a particle determines the personal best position, the local or global best position, adjusts the velocity, and a number of function evaluations are performed. Function evaluation means one calculation of the fitness or objective function which computes the optimality of a solution. If n is the total number of particles in the swarm, then n function evaluations are performed at each iteration [4].

4.4.3 Stopping Criteria

Stopping criteria is used to terminate the iterative search process. Some stopping criteria are discussed below:

1) The algorithm is terminated when a maximum number of iterations or function evaluations (FEs) has been reached. If this maximum number of iterations (or FEs) is too small, the search process may stop before a good result has been found [4].

2) The algorithm is terminated when there is no significant improvement over a number of iterations. This improvement can be measured in different ways. For instance, the process may be considered to have terminated if the average change of the particles' positions are very small or the average velocity of the particles is approximately zero over a number of iterations [4].

3) The algorithm is terminated when the normalized swarm radius is approximately zero. The normal swarm radius is defined as

$$R_{norm} = \frac{R_{max}}{\text{diametr(S)}} \tag{4.13}$$

where diameter(S) is the initial swarm's diameter and R_{max} is the maximum radius,

$$R_{max} = \|x_m - G_{best}\|$$

with

$$\|x_m - G_{best}\| \geq \|x_i - G_{best}\|, m, \forall i = 1, \dots, n,$$

and $\|\blacksquare\|$ is a suitable distance norm.

The process will terminate when $R_{norm} < \epsilon$. If ϵ is too large, the process can be terminated prematurely before a good solution has been reached while if ϵ is too small, the process may need more iterations [4].

CHAPTER 5

Recent Works and Advanced Topics of PSO

This chapter describes different types of PSO methods which help to solve different types of optimization problems such as Multi-start (or restart) PSO for when and how to reinitialize particles, binary PSO (BPSO) method for solving discrete-valued problems, Multi-phase PSO (MPPSO) method for partition the main swarm of particles into sub-swarms or subgroups, Multi-objective PSO for solving multiple objective problems.

5.1 Multi-Start PSO (MSPSO)

In the basic PSO, one of the major problems is lack of diversity when particles start to converge to the same point. To prevent this problem of the basic PSO, several methods have been developed to continually inject randomness, or chaos, into the swarm. These types of methods are called the Multi-start (or restart) Particle Swarm Optimizer (MSPSO). The Multi-start method is a global search algorithm and has as the main objective to increase diversity, so that larger parts of the search space are explored [4] [11]. It is important to remember that continual injection of random positions will cause the swarm never to reach an equilibrium state that is why, in this algorithm, the amount of chaos reduces over time. Kennedy and Eberhart first introduced the advantages of randomly reinitializing particles and referred to as craziness. Now the important questions are when to reinitialize, and how are particles reinitialized? These aspects are discussed below [4]:

Randomly initializing position vectors or velocity vectors of particles can increase the diversity of the swarm. Particles are physically relocated to a different random position in the solution space by randomly initializing positions. When position vectors are kept constant and velocity vectors are randomized, particles preserve their memory of current and previous best solutions, but are forced to search in different random directions. When randomly initialized particle’s velocity cannot found a better solution, then the particle will again be attracted towards its personal best position. When the positions of particles are reinitialized, then the particles’ velocities are typically set to zero and to have a zero momentum at the first iteration after reinitialization. On the other hands, particle velocities can be initialized to small values. To ensure a momentum back towards the personal best position, G. Venter and J. Sobieszczanski-Sobieski initialize particle velocities to the cognitive component before reinitialization. Therefore the question is when to consider the reinitialization of particles. Because, when reinitialization occurs too soon, then the affected particles may have too short time to explore their current regions before being relocated. If the reinitialization time is too long, however it may happen that all particles have already converged [4].

A probabilistic technique has been discussed to decide when to reinitialize particles. X. Xiao, W. Zhang, and Z. Yang reinitialize velocities and positions of particles based on chaos factors which act as probabilities of introducing chaos in the system. Let c_v and c_l denote the chaos factors for velocity and location. If $r_{ij} \sim U(0,1) < c_v$, then the particle velocity component is reinitialized to $v_{ij}^{t+1} = U(0,1)V_{max,j}$, where r_{ij} is random number for each particle i and each dimension j. Again, if $r_{ij} \sim U(0,1) < c_l$, then the particle position component is initialized to $x_{ij}^{t+1} \sim U\left(x_{min,j}, x_{max,j}\right)$. In this technique, start with large chaos factors that decrease over time to ensure that an equilibrium stat can be reached. Therefore the initial large chaos factors increase diversity in the first stages of the solution space, and allow particles to converge in the final steps [4].

A convergence criterion is another technique to decide when to reinitialize particles, where particles are allowed to first exploit their local regions before being reinitialized [4]. All particles are to initiate reinitialization when particles do not improve over time. In this technique, a variation is to evaluate in particle fitness of the current swarm, and if the variation is small, then particles are close to the global best position. Otherwise, particles that are at least two standard deviations away from the swarm center are reinitialized.

M. Løvberg and T. Krink have developed reinitialization of particles by using self-organized criticality (SOC) which can help control the PSO and add diversity [21]. In SOC, each particle maintains an additional variable, C_i, where C_i is the *criticality* of the particle i. If two particles are closer than a threshold distance ϵ, from one another, then both particles have their criticality increased by one. The particles have no neighborhood restrictions and this neighborhood is full connected network (i.e. star type) so that each particle can affect all other particles [21].

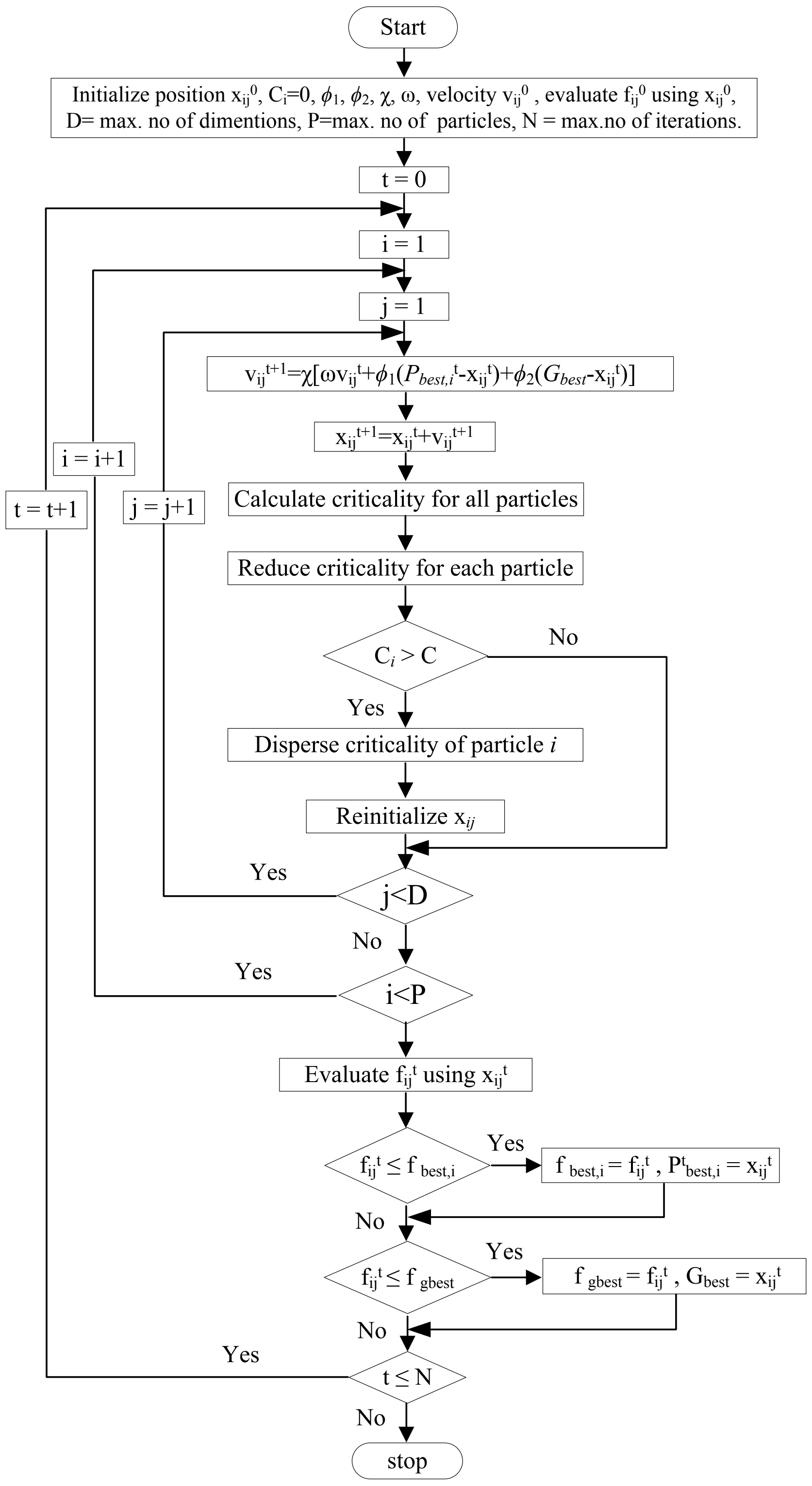
In SOCPSO model the velocity of each particle is updated by

$$v_{ij}^{t+1} = \chi[\omega v_{ij}^{t} + \phi_{1j}(P_{best,i}^{t} - x_{ij}^{t}) + \phi_{2j}(G_{best} - x_{ij}^{t})] \tag{5.1}$$

where χ is known as the constriction factor, ω is the inertia-weight, ϕ_{1j} and ϕ_{2j} are random values different for each particle and for each dimension [21].

In each iteration, each C_i is decreased by a fraction to prevent criticality from building up [21]. When $C_i > C$, C is the global criticality limit, then the criticality of the particle i is distributed to its immediate neighbors and is reinitialized. The authors also consider the inertia weight value of each particle i to $w_i = 0.2 + 0.1C_i$, this forces the particle to explore more when it is too similar to other particles [4].

Flowchart 3 shows criticality measures for SOCPSO.


Start
Initialize position x_{ij}^0, C_i=0, ϕ_1, ϕ_2, χ, ω, velocity v_{ij}^0 , evaluate f_{ij}^0 using x_{ij}^0, D= max. no of dimentions, P=max. no of particles, N = max.no of iterations.
t = 0
i = 1
j = 1
$v_{ij}^{t+1}=\chi[\omega v_{ij}^t+\phi_1(P_{best,i}^t-x_{ij}^t)+\phi_2(G_{best}-x_{ij}^t)]$
$x_{ij}^{t+1}=x_{ij}^t+v_{ij}^{t+1}$
Calculate criticality for all particles
Reduce criticality for each particle
$C_i > C$
No
Yes
Disperse criticality of particle *i*
Reinitialize x_{ij}
Yes
j<D
No
Yes
i<P
Evaluate f_{ij}^t using x_{ij}^t
$f_{ij}^t \leq f_{best,i}$
Yes
$f_{best,i} = f_{ij}^t$, $P^t_{best,i} = x_{ij}^t$
No
$f_{ij}^t \leq f_{gbest}$
Yes
$f_{gbest} = f_{ij}^t$, $G_{best} = x_{ij}^t$
No
Yes
$t \leq N$
No
stop
i = i+1
t = t+1
j = j+1

Flowchart 3: Self-Organized Criticality PSO

5.2 Multi-phase PSO (MPPSO)

Multi-phase PSO (MPPSO) method partitions the main swarm of particles into sub-swarms or subgroups, where each sub-swarm performs a different task, exhibits a different behavior and so on. This task or behavior performed by a sub-swarm usually changes over time and information are passed among sub-swarms in this process.

In 2002, B. Al-Kazemi and C. Mohan described the MPPSO method and they divided the main swarm into two sub-swarms with the equal size. In this algorithm, particles are randomly assigned, and each sub-swarm may be in one of two phases [4]. These phases are discussed below:

● **Attraction phase:** in this phase, the particles of the corresponding sub-swarm are influenced to move towards the global best position.

● **Repulsion phase**: in this phase, the particles of the corresponding sub-swarm go away from the global best position [4].

In MPPSO algorithm, the particle velocity updating equation is presented as follows:

$$v_{ij}^{t+1} = \omega v_{ij}^{t} + c_1 x_{ij}^{t} + c_2 G_{best} \qquad (5.2)$$

where ω is the inertia-weight, $c_1$and c_2 are acceleration coefficients respectively.

In this method, the personal best position is eliminated from the main velocity equation (4.3), since a particle's position is only updated when the new position improves the performance in the solution space [4] [23].

Another MPPSO algorithm is based on the groups PSO and multi-start PSO algorithm, and it was introduced by H. Qi *et al* [23]. The advantage of the MPPSO algorithm is that when the fitness of a particle doesn't changed any more, then the particle's flying speed and direction in the searching space are changed by the adaptive velocity strategy. Therefore, MPPSO differ from basic PSO in three ways:

1. Particles divide into multiple groups to increase the diversity of the swarm and extensiveness of the exploration space.
2. Different phases introduce in the algorithm which have different searching ways and flying directions.
3. Searching direction will increase particle's fitness [23].

5.3 Perturbed PSO (PPSO)

The basic particle swarm optimization (PSO) has some disadvantages, for example, high speed of convergence frequently generates a quick loss of diversity during the process of optimization. Then, the process leads to undesirable premature convergence. To overcome this disadvantage, Zhao Xinchao described a perturbed particle swarm algorithm which is based upon a new particle updating strategy and the concept of perturbed global best (p-gbest) within the swarm. The perturbed global best (p-gbest) updating strategy is based on the concept of possibility measure to model the lack of information about the true optimality of the gbest [24]. In PPSO, the particle velocity is rewritten by

$$v_{ij}^{t+1} = \omega v_{ij}^{t} + c_1 r_{1j}^{t}\left[P_{best,i}^{t} - x_{ij}^{t}\right] + c_2 r_{2j}^{t}[G'_{best} - x_{ij}^{t}] \tag{5.3}$$

where

$$G'_{best} = N(G_{best}, \sigma) \tag{5.4}$$

is the j-th dimension of p-gbest in iteration t.

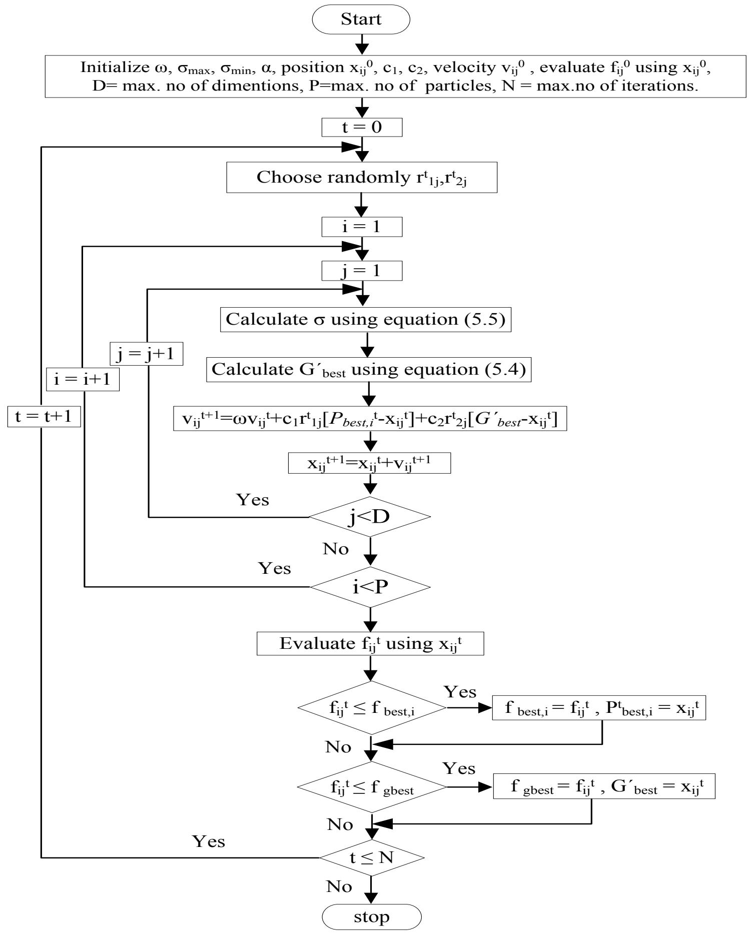
Here, $N(G_{best}, \sigma)$ is the normal distribution, and σ represents the degree of uncertainty about the optimality of the gbest and is modeled as some non-increasing function of the number of iterations, defined as

$$\sigma = \begin{cases} \sigma_{max} & \text{iterations} < \alpha \times \max\text{\_iterations} \\ \sigma_{min} & \text{otherwise} \end{cases} \tag{5.5}$$

where

$\sigma_{max}, \sigma_{min}$, and α are manually set parameters.

The Flowchart 4 shows the perturbed PSO algorithm.



Start
Initialize ω, σ_{max}, σ_{min}, α, position x_{ij}^0, c_1, c_2, velocity v_{ij}^0 , evaluate f_{ij}^0 using x_{ij}^0, D= max. no of dimentions, P=max. no of particles, N = max.no of iterations.
t = 0
Choose randomly r^t_{1j},r^t_{2j}
i = 1
j = 1
Calculate σ using equation (5.5)
Calculate G'_{best} using equation (5.4)
$v_{ij}^{t+1}=\omega v_{ij}^t+c_1 r^t_{1j}[P_{best,i}^t-x_{ij}^t]+c_2 r^t_{2j}[G'_{best}-x_{ij}^t]$
$x_{ij}^{t+1}=x_{ij}^t+v_{ij}^{t+1}$
j = j+1
i = i+1
t = t+1
j<D
Yes
No
i<P
Yes
Evaluate f_{ij}^t using x_{ij}^t
$f_{ij}^t \le f_{best,i}$
Yes
$f_{best,i} = f_{ij}^t$, $P^t_{best,i} = x_{ij}^t$
No
$f_{ij}^t \le f_{gbest}$
Yes
$f_{gbest} = f_{ij}^t$, $G'_{best} = x_{ij}^t$
No
t ≤ N
Yes
No
stop

Flowchart 4: Perturbed PSO

The p-gbest function encourages the particles to explore a solution space beyond that defined by the search trajectory. If σ is large, then the p-gbest algorithm generates a simple and efficient exploration at the initial stage, while it encourages a local fine-tuning at the latter stage when σ is small. Moreover, the p-gbest function reduces the likelihood of premature convergence and also helps to direct the search toward the most promising search area [24].

5.4 Multi-Objective PSO (MOPSO)

Multi-objective optimization problems have several objective functions that need to be optimized simultaneously. In multiple-objectives cases, due to lack of common measure and confliction among objective functions, there does not necessarily exist a solution that is best with respect to all objectives. There exist a set of solutions for the multi-objective problem which cannot normally be compared with each other. Such solutions are called non-dominated solutions (or Pareto optimal solutions) only when no improvement is possible in any objective function without sacrificing at least one of the other objective functions [25].

Concepts of Multi-Objective Optimization: Consider $S \subset \mathbb{R}^n$ is an n-dimensional search space, and $f_i(x),\ i = 1, \dots, k$ is k-objective functions defined over S. Then, a general multi-objective minimization optimization problem can be expressed as:

$$\text{minimize } f(x) = [f_1(x), f_2(x), \dots, f_k(x)] \tag{5.6}$$

subject to:

$$g_i(x) \leq 0 \qquad i = 1,2, \dots, m \tag{5.7}$$

$$h_j(x) = 0 \qquad j = 1,2, \dots, p \tag{5.8}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the decision-making vector on the search space; $f(x)$ is the goal vector ; and $g_i(x)$ and $h_j(x)$ are the constraint functions(or bound conditions) of the problem. The objective functions $f_i(x)$ can be conflicting with each other so that the detection of a single global minimum cannot possibly be at the same point in S. To solve this problem, optimality of a solution in multi-objective problems needs to be redefined properly.

Let $u = (u_1, u_2, \dots, u_k)$ and $v = (v_1, v_2, \dots, v_k)$ be two vectors of the search space S. Therefore, $u\ dominates\ v$ (denoted by $u \prec v$) if and only if u is better than v (i.e. $u_i \leq v_i$) for all $i = 1, 2, \dots, k$, and $u_i < v_i$ for at least one component. This property is called *Pareto dominance*. Now a solution, x, of the multi-objective problem $f(x)$ is said to be *Pareto optimal*, if and only if there is no other solution, y, in S such that $f(y)$ dominates $f(x)$, that means x is not dominated such that $y \prec x$. The set of non-dominated (or all Pareto optimal) solutions of a problem $f(x)$ in the solution space is called the *Pareto optimal set*, denoted by P^*, and the set

$$\mathcal{PF}^* = \{ f(x) \colon x \in P^* \} \tag{5.9}$$

is called the *Pareto front* ($\mathcal{PF}^*$) [26].

In multi-objective optimization algorithms, these cases are considered the most difficult. From the definition of Pareto optimality, it is true that the main goal in multi-objective optimization problems is the detection of all Pareto optimal solutions. Since the Pareto optimal set may be infinite and all the computational problems are time and space limited, we are compelled to set more realistic goals [26].

A number of approaches have been proposed to extend the PSO for multiple objective problems. Some approaches will be discussed in this section.

5.4.1 Dynamic Neighborhood PSO (DNPSO)

This method for solving multi-objective optimization problems was first proposed by Hu and Eberhart. In this algorithm, the multiple objectives are divided into two groups: f_1 and f_2, where f_1 is defined as the neighborhood objective, and f_2 is defined as the optimization objective. The choices of $f_1$and f_2 are arbitrary. In each iteration, each particle dynamically determines a new neighborhood by calculating the distance to all other particles and choosing the m nearest neighbors. The distance is discribed as the difference between fitness values for the first group of objective functions f_1. When the neighborhood has been determined, the best local value is selected among the neighbors in terms of the fitness value of the second objective functions f_2. Finally, the global best updating system considers only the solution that dominates the current personal best value [16]. The main drawback is that it is useful only for two objectives [27].

5.4.2 Multi-Objective PSO (MOPSO)

The MOPSO algorithm was developed by Coello-Coello *et. al*. There are two main fundamental approaches in this algorithm [26]. The first approach is that each particle is evaluated only by one objective function at a time, and the finding of the best positions is performed similarly to the single-objective optimization case. In such cases, the main challenge is the proper management of the information coming from each objective function so that the particles go toward Pareto optimal solutions. For maintaining the identified Pareto optimal solutions, the most trivial solution would be to store non-dominated solutions as the particles' best positions. The second approach is that each particle is evaluated by all objective functions and based on the concept of Pareto optimality, and they produce non-dominated best positions (called leaders). Nevertheless, there can be many non-dominated solutions in the neighborhood of a particle, and the determination of leaders is not easy. On the other hand, only the one that will be used as the best position of a particle is usually selected to participate in the velocity update. Therefore, the selection of the best position is an important task in making particles move to the Pareto optimal front. To solve this problem, consider an additional set that is called external archive for storing the non-dominated solutions discovered during search [26].

5.4.3 Vector Evaluated PSO (VEPSO)

The vector evaluated particle swarm optimization (VEPSO) algorithm was first proposed by Parsopoulos and Vrahatis [26]. In VEPSO algorithm, one swarm is evaluated only for one objective function and the information is exchanged among other swarms. As a result in the swarm update, the best position of one swarm is used for the velocity update of another swarm that corresponds to a different objective function. Therefore, in this algorithm k swarms are used for k objective functions and the velocity update equation for an k-objective function problem can be defined as

$$v_{ij}^{[s]}(t+1) = \omega v_{ij}^{[s]}(t) + c_1 r_1 \left[P_{best,i}^{[s]}(t) - x_{ij}^{[s]}(t)\right] + c_2 r_2 [G_{best}^{[q]} - x_{ij}^{[s]}(t)] \qquad (5.10)$$

where

s	defines the swarm number($s = 1,2,\ldots,k$);
i	corresponds to the particle number($i = 1,2,\ldots,n$);
$v_{ij}^{[s]}$	is the velocity of the i-th particle in the s-th swarm;
$G_{best}^{[q]}$	is the best position found for any particle in the q-th swarm which is evaluated with the q-th objective function.

The VEPSO algorithm is called parallel VEPSO because this algorithm also enables the swarms to be implemented in parallel computers that are connected in an Ethernet network [16].

In 2005, Raquel and Naval first introduced Multi-Objective PSO with Crowing Distance (MOPSO-CD). This algorithm is based on a crowding distance mechanism for the selection of the global best particle and also for the deletion of non-dominated solutions from the external archive. The MOPSOCD method has a constraint handling technique for solving constrained optimization problems.

5.5 Binary PSO (BPSO)

In the beginning, PSO algorithm was developed for continuous-valued search spaces and most of its modified versions worked in the continuous space, which could not be used to optimize for a discrete-valued search spaces. In 1997, Kennedy and Eberhart firstly extended the basic PSO algorithm to the discrete space to solve this problem [4]. They developed the PSO to operate on the binary search spaces, because real-valued domains can be transformed into the binary-valued domains. The proposed algorithm is called binary PSO (BPSO) algorithm where the particles represent position in binary space and particle's position vectors can take on the binary value 0 or 1 i.e. $x_{ij} \in \{0,1\}$. In this case, it maps from the n-dimensional binary space B^n (i.e. bit strings of the length n) to the real numbers $f : B^n \to R$ (where f and R is a fitness function and a real number set respectively). That means a particle's positions must belong to B^n in order to be calculated by f [15].

In BPSO, a particle's velocity v_{ij}^t is connected to the possibility that the particle's position x_{ij}^t takes a value of 0 or 1. The update equation for the velocity does not change from that used in the original PSO and the equation (3.7),

$$v_{ij}^{t+1} = v_{ij}^t + c_1 r_{1j}^t [P_{best,i}^t - x_{ij}^t] + c_2 r_{2j}^t [G_{best} - x_{ij}^t]$$

Now, the jth bit of the ith particle,x_{ij}^t , is updated by

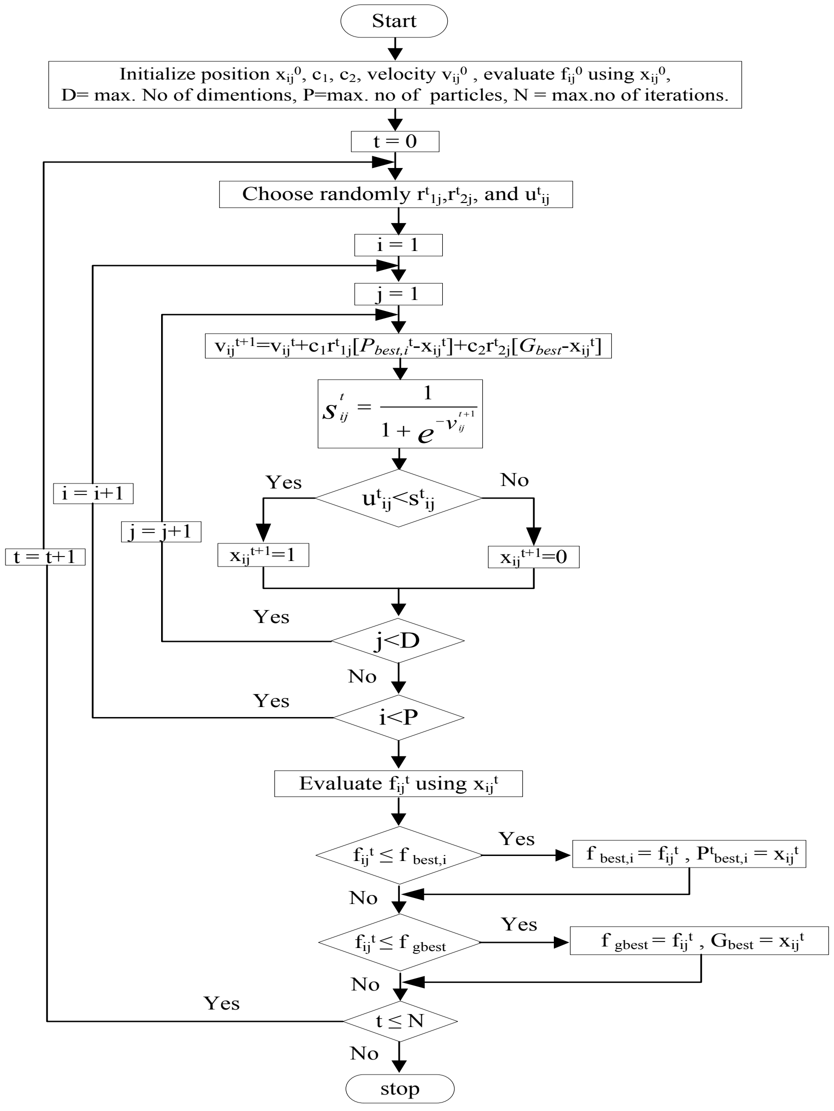
$$x_{ij}^t = \begin{cases} 1 & \text{if } u_{ij}^t < s_{ij}^t \\ 0 & \text{if } u_{ij}^t \geq s_{ij}^t \end{cases} \tag{5.11}$$

where, u_{ij}^t is a random number selected from a uniform distribution in (0, 1), and

s_{ij}^t is the sigmoid function , denoted by,

$$s_{ij}^t = \frac{1}{1+e^{-v_{ij}^{t+1}}} \tag{5.12}$$

The Flowchart 5 shows the Binary PSO algorithm.


Start
Initialize position x_{ij}^0, c_1, c_2, velocity v_{ij}^0 , evaluate f_{ij}^0 using x_{ij}^0, D= max. No of dimentions, P=max. no of particles, N = max.no of iterations.
t = 0
Choose randomly r^t_{1j},r^t_{2j}, and u^t_{ij}
i = 1
j = 1
$v_{ij}^{t+1}=v_{ij}^t+c_1r^t_{1j}[P_{best,i}^t-x_{ij}^t]+c_2r^t_{2j}[G_{best}-x_{ij}^t]$
$s_{ij}^t = \frac{1}{1+e^{-v_{ij}^{t+1}}}$
Yes
$u^t_{ij}<s^t_{ij}$
No
$x_{ij}^{t+1}=1$
$x_{ij}^{t+1}=0$
i = i+1
j = j+1
t = t+1
Yes
j<D
No
Yes
i<P
Evaluate f_{ij}^t using x_{ij}^t
$f_{ij}^t \leq f_{best,i}$
Yes
$f_{best,i} = f_{ij}^t$, $P^t_{best,i} = x_{ij}^t$
No
$f_{ij}^t \leq f_{gbest}$
Yes
$f_{gbest} = f_{ij}^t$, $G_{best} = x_{ij}^t$
No
Yes
$t \leq N$
No
stop

Flowchart 5: Binary PSO

Now if the bit is not flipped into a 1 at the tth iteration, then the velocity increases again at the $t+1$th iteration, along with a greater probability of s_{ij}^{t} to flip the bit. In continuous valued problems of the PSO, the maximum velocity V_{max} can be large number for the particles exploration. On the other hand, in binary PSO the maximum velocity V_{max} will be small numbers for exploration, even if a good solution is found [12]. It is suggested that the maximum velocity $V_{max} = 6$, which corresponds to a maximum probability of 0.997 that a bit is flipped into 1, whereas the minimum velocity $V_{min} = -6$, corresponds to a minimum probability of 0.002 that the bit remains 0. Since each bit of x_{ij}^{t} is always binary-valued in the solution space, so no boundary conditions need to be specified in BPSO [10].

The velocity v_{ij}^{t} is a probability for the particle position x_{ij}^{t} to be 0 or 1. For instance, if $v_{ij}^{t} = 0$, then $P(x_{ij}^{t+1} = 1) = 0.5$ (or 50%). On the other hand, if $v_{ij}^{t} < 0$, then $P(x_{ij}^{t+1} = 1) < 0.5$, and if $v_{ij}^{t} > 0$, then $P(x_{ij}^{t+1} = 1) > 0.5$. Besides, it is true that $P(x_{ij}^{t} = 0) = 1 - P(x_{ij}^{t} = 1)$. Due to the random number r in the above equation (5.11), x_{ij} can change even if the value of v_{ij} does not change [4].

The binary PSO algorithm is very important to practical and commercial use in discrete problems solving, therefore this algorithm completely needs a lot more attention in the future. Finally, in the following section the Lot sizing problem is used to illustrate the details of computations for iterations 1 and 2 using the BPSO method.

5.5.1 Problem Formulation (Lot Sizing Problem)

The problem is to find order quantities that will minimize total ordering and holding costs of an ordering budget. I wish to find the solution of the following problem by the Binary PSO method:

$$\text{Minimize} \quad f(x) = \sum_{i=1}^{n} (px_i + cI_i) \qquad (1)$$

$$\text{Subject to} \quad I_0 = 0 \qquad (2)$$

$$I_{i-1} + xQ_i - I_i = R_i \qquad (3)$$

$$I_i \geq 0 \qquad (4)$$

$$Q_i \geq 0 \qquad (5)$$

$$x_i \in \{0,1\} \quad \forall i \qquad (6)$$

where,

i = number of periods,

p = ordering cost per period,

c = holding cost per unit per period,

R_i = net requirement for period i,

Q_i = order quantity for period i,

I_i = projected inventory balance for period i,

$$x_i = \begin{cases} 1 & \text{if an order is placed in period i} \\ 0 & \text{otherwise} \end{cases}$$

The objective function equation (1) is to minimize the total cost with constraints that include some limits, no initial inventory is available equation (2),equation (3) represents the inventory balance equation in which the order quantity covers all the requirements until the next order, equation (4) shows that projected inventory is always positive, equation (5) satisfies the condition that no shortages are allowed, and finally equation (6) denotes the decision variable that x_i is either 1 (place an order) or 0 (not place an order) [22].

Solution:

The various steps of the procedure are illustrated using the binary particle swarm optimization:

Step1: Consider the number of particles $i = 1, 2, 3$; the number of dimensions or periods $j = 1, 2, 3, 4, 5$; the ordering cost per period $p = \$\ 200$; the holding cost per unit per period $c = \$1$; also the net requirements R_j, the lot size of the particles Q_{ij}^0, the initial particles positions x_{ij}^0 and corresponding velocities v_{ij}^0, the inventory balance of particles I_{ij}^0 are given below and finally evaluate each particle in the swarm using the objective function $f_i^0(x)$ in period j at iteration $t = 0$.

j	1	2	3	4	5
R_j	80	60	40	30	70
Q_{1j}	140	100	70	100	80
x_{1j}^0	1	0	1	0	0
v_{1j}^0	3.50	1.50	3.80	-2.20	-1.00
I_{1j}^0	60		30		
$f_1^0(x) =$	260	+	230		= 490
Q_{2j}	120	80	50	40	90
x_{2j}^0	1	1	0	0	0
v_{2j}^0	2.80	3.00	-1.20	-3.00	2.5
I_{2j}^0	40	20			
$f_2^0(x) =$	240 +	220			= 460
Q_{3j}	100	70	30	25	95
x_{3j}^0	1	0	0	1	1
v_{3j}^0	3.40	-2.50	-1.00	1.50	3.00
I_{3j}^0	20			15	25
$f_3^0(x) =$	220			+ 215 +	225 =660

Find the personal best for each particle in the swarm:

j	1	2	3	4	5	$f^0_{best,i}(X)$
$P^0_{1j} = X^0_{1j}$	1	0	1	0	0	490
$P^0_{2j} = X^0_{2j}$	1	1	0	0	0	460
$P^0_{3j} = X^0_{3j}$	1	0	0	1	1	660

Find the global best in the entire swarm:

j	1	2	3	4	5	G^0_{best}
$G^0_{best} = X^0_{2j}$	1	1	0	0	0	460

Step 2: Set the iteration number as $t = t + 1 = 1$ and go to step 3.

Step 3: By considering $c_1 = c_2 = 0.5$ and the random numbers in the range (0, 1) as $r^0_{1j} = r^0_{2j} = 0.5$, find the update velocities of the particles by

$$v^{t+1}_{ij} = v^t_{ij} + c_1 r^t_{1j}\left[P^t_{best,i} - x^t_{ij}\right] + c_2 r^t_{2j}[G_{best} - x^t_{ij}]$$

So

$$v^1_{11} = v^0_{11} + 0.5 * 0.5[1 - 1] + 0.5 * 0.5[0 - 1] = 3.50 - 0.25 = 3.25$$

$$v^1_{12} = v^0_{12} + 0.5 * 0.5\ [0 - 0] + 0.5 * 0.5\ [0 - 0] = 1.50$$

$$v^1_{13} = 3.80 - 0.25 = 3.55,\ v^1_{14} = -2.20,\ v^1_{15} = -1.00$$

$$v^1_{21} = v^0_{21} + 0.5 * 0.5\ [1 - 1] + 0.5 * 0.5\ [1 - 1] = 2.80 - 0 = 2.80$$

$$v^1_{22} = 3.00,\ v^1_{23} = -1.20,\ v^1_{24} = -3.00,\ v^1_{25} = 2.50$$

$$v^1_{31} = 3.40 - 0.25 = 3.15,\ v^1_{32} = -2.50,\ v^1_{33} = -1.00,$$

$$v^1_{34} = 1.50 - 0.25 = 1.25,\ v^1_{35} = 3.00 - 0.25 = 2.75$$

Step 4: Find the values of s^0_{ij} by the following equation

$$s^t_{ij} = \frac{1}{1 + e^{-v^{t+1}_{ij}}}$$

So

$$s^1_{11} = \frac{1}{1+e^{-v^1_{11}}} = 0.96,\ s^1_{12} = 0.82,\ s^1_{13} = 0.98,\ s^1_{14} = 0.10,\ s^1_{15} = 0.27.$$

$$s^1_{21} = 0.94,\ s^1_{22} = 0.95,\ s^1_{23} = 0.24,\ s^1_{24} = 0.04,\ s^1_{25} = 0.92.$$

$$s^1_{31} = 0.96,\ s^1_{32} = 0.08,\ s^1_{33} = 0.27,\ s^1_{34} = 0.78,\ s^1_{35} = 0.94.$$

Step 5: Update the particle's position x_{1j}^{1} by using the sigmoid function,

$$x_{1j}^{1} = \begin{cases} 1 & \text{if } u_{1j}^{1} < s_{1j}^{1} \\ 0 & \text{if } u_{1j}^{1} \geq s_{1j}^{1} \end{cases}$$

Where, u_{1j}^{1} is a random number selected from a uniform distribution in (0, 1).

The following table is given the particles update position after completing the first iteration:

j	1	2	3	4	5
x_{1j}^{1}	1	1	0	0	0
v_{1j}^{1}	3.50	1.50	3.55	-2.20	-1.00
s_{1j}^{1}	0.96	0.82	0.98	0.10	0.27
u_{1j}^{1}	0.60	0.10	0.99	0.35	0.53
I_{1j}^{1}	120	40			
x_{2j}^{1}	0	1	0	0	1
v_{2j}^{1}	2.80	3.00	-1.20	-3.00	2.5
s_{2j}^{1}	0.94	0.95	0.24	0.04	0.92
u_{2j}^{1}	0.96	0.70	0.99	0.05	0.30
I_{2j}^{1}		40			20
x_{3j}^{1}	1	0	0	0	1
v_{3j}^{1}	3.15	-2.50	-1.00	1.25	2.75
s_{3j}^{1}	0.96	0.08	0.27	0.78	0.94
u_{3j}^{1}	0.56	0.70	0.80	0.85	0.35
I_{3j}^{1}	40				50

Step 6: Evaluate $f_i^1(x)$ using x_{ij}^{1} in the swarm:

$$f_1^1(x) = 320 + 240 = 560$$

$$f_2^1(x) = 240 + 220 = 460$$

$$f_3^1(x) = 240 + 250 = 490$$

Step 7: Update the personal best for each particle in the swarm:

Since

$f_1^1(x) = 560 > f_{best,1}^0(X) = 490$, then $f_{best,1}^1(X) = 490$ with X_{1j}^0.

$f_2^1(x) = f_{best,2}^0(X) = 460$, then $f_{best,2}^1(X) = 460$ with X_{2j}^1.

$f_3^1(x) = 490 < f_{best,3}^0(X) = 660$, then $f_{best,3}^1(X) = 490$ with X_{3j}^1.

The new positions of particles are

j	1	2	3	4	5	$f_{best,i}^{1}(X)$
$P_{1j}^{1} = X_{1j}^{1}$	1	0	1	0	0	490
$P_{2j}^{1} = X_{2j}^{1}$	0	1	0	0	1	460
$P_{3j}^{1} = X_{3j}^{1}$	1	0	0	0	1	490

Step 8: Update the global best in the entire swarm:

$$G_{best} = \min\{f_{best,i}^{1}(X)\} = 460$$

Since $G_{best}^{1} = G_{best}^{0} = 460$, then $G_{best}^{1} = 460$ with X_{2j}^{1}.

j	1	2	3	4	5	G_{best}^{1}
$G_{best}^{1} = X_{2j}^{1}$	0	1	0	0	1	460

Step 9: Stopping criterion:

If the terminal rule is satisfied, go to step 2,
Otherwise stop the iteration and output the results.

CHAPTER 6

Applications of PSO

This chapter discusses the various application areas of PSO method.

Kennedy and Eberhart established the first practical application of Particle Swarm Optimization in 1995. It was in the field of neural network training and was reported together with the algorithm itself. PSO have been successfully used across a wide range of applications, for instance, telecommunications, system control, data mining, power systems, design, combinatorial optimization, signal processing, network training, and many other areas. Nowadays, PSO algorithms have also been developed to solve constrained problems, multi-objective optimization problems, problems with dynamically changing landscapes, and to find multiple solutions, while the original PSO algorithm was used mainly to solve unconstrained, single-objective optimization problems [29].Various areas where PSO is applied are listed in Table1 [5]:

Table 1. Application areas of Particle Swarm Optimization

Antennas Design	The optimal control and design of phased arrays, broadband antenna design and modeling, reflector antennas, design of Yagi-Uda arrays, array failure correction, optimization of a reflect array antenna, far-field radiation pattern reconstruction, antenna modeling, design of planar antennas, conformal antenna array design, design of patch antennas, design of a periodic antenna arrays, near-field antenna measurements, optimization of profiled corrugated horn antennas, synthesis of antenna arrays, adaptive array antennas, design of implantable antennas.
Signal Processing	Pattern recognition of flatness signal, design of IIR filters, 2D IIR filters, speech coding, analogue filter tuning, particle filter optimization, nonlinear adaptive filters, Costas arrays, wavelets, blind detection, blind source separation, localization of acoustic sources, distributed odour source localization, and so on.
Networking	Radar networks, bluetooth networks, auto tuning for universal mobile telecommunication system networks, optimal equipment placement in mobile communication, TCP network control, routing, wavelength division-multiplexed network, peer-to-peer networks, bandwidth and channel allocation, WDM telecommunication networks, wireless networks, grouped and delayed broadcasting, bandwidth reservation, transmission network planning, voltage regulation, network reconfiguration and expansion, economic dispatch problem, distributed generation,

	microgrids, congestion management, cellular neural networks, design of radial basis function networks, feed forward neural network training, product unit networks, neural gas networks, design of recurrent neural networks, wavelet neural networks, neuron controllers, wireless sensor network design, estimation of target position in wireless sensor networks, wireless video sensor networks optimization.
Biomedical	Human tremor analysis for the diagnosis of Parkinson's disease, inference of gene regulatory networks, human movement biomechanics optimization, RNA secondary structure determination, phylogenetic tree reconstruction, cancer classification, and survival prediction, DNA motif detection, biomarker selection, protein structure prediction and docking, drug design, radiotherapy planning, analysis of brain magneto encephalography data, electroencephalogram analysis, biometrics and so on.
Electronics and electromagnetic	On-chip inductors, configuration of FPGAs and parallel processor arrays, fuel cells, circuit synthesis, FPGA-based temperature control, AC transmission system control, electromagnetic shape design, microwave filters, generic electromagnetic design and optimization applications, CMOS RF wideband amplifier design, linear array antenna synthesis, conductors, RF IC design and optimization, semiconductor optimization, high-speed CMOS, frequency selective surface and absorber design, voltage flicker measurement, shielding, digital circuit design.
Robotics	Control of robotic manipulators and arms, motion planning and control, odour source localization, soccer playing, robot running, robot vision, collective robotic search, transport robots, unsupervised robotic learning, path planning, obstacle avoidance, swarm robotics, unmanned vehicle navigation, environment mapping, voice control of robots, and so forth.
Design and Modelling	Conceptual design, electromagnetics case, induction heating cooker design, VLSI design, power systems, RF circuit synthesis, worst case electronic design, motor design, filter design, antenna design, CMOS wideband amplifier design, logic circuits design, transmission lines, mechanical design, library search, inversion of underwater acoustic models, modeling MIDI music, customer satisfaction models, thermal process system identification, friction models, model selection, ultrawideband channel modeling, identifying ARMAX models, power plants and systems, chaotic time series modeling, model order reduction.
	Image segmentation, autocropping for digital photographs, synthetic aperture radar imaging, locating treatment

Image and Graphics	planning landmarks in orthodontic x-ray images, image classification, inversion of ocean color reflectance measurements, image fusion, photo time-stamp recognition, traffic stop-sign detection, defect detection, image registration, microwave imaging, pixel classification, detection of objects, pedestrian detection and tracking, texture synthesis, scene matching, contrast enhancement, 3D recovery with structured beam matrix, character recognition, image noise cancellation.
Power generation and Controlling	Automatic generation control, power transformer protection, power loss minimization, load forecasting, STATCOM power system, fault-tolerant control of compensators, hybrid power generation systems, optimal power dispatch, power system performance optimization, secondary voltage control, power control and optimization, design of power system stabilizers, operational planning for cogeneration systems, control of photovoltaic systems, large-scale power plant control, analysis of power quality signals, generation planning and restructuring, optimal strategies for electricity production, production costing, operations planning.
Fuzzy systems, Clustering, data mining	Design of neurofuzzy networks, fuzzy rule extraction, fuzzy control, membership functions optimization, fuzzy modeling, fuzzy classification, design of hierarchical fuzzy systems, fuzzy queue management, clustering, clustering in large spatial databases, document and information clustering, dynamic clustering, cascading classifiers, classification of hierarchical biological data, dimensionality reduction, genetic-programming-based classification, fuzzy clustering, classification threshold optimization, electrical wader sort classification, data mining, feature selection.
Optimization	Electrical motors optimization, optimization of internal combustion engines, optimization of nuclear electric propulsion systems, floor planning, travelling-sales man problems, n-queens problem, packing and knapsack, minimum spanning trees, satisfiability, knights cover problem, layout optimization, path optimization, urban planning, FPGA placement and routing.
Prediction and forecasting	Water quality prediction and classification, prediction of chaotic systems, streamflow forecast, ecological models, meteorological predictions, prediction of the floe stress in steel, time series prediction, electric load forecasting, battery pack state of charge estimation, predictions of elephant migrations, prediction of surface roughness in end milling, urban traffic flow forecasting, and so on.

CHAPTER 7

Conclusion

This thesis discussed the basic Particle Swarm Optimization algorithm, geometrical and mathematical explanation of PSO, particles' movement and the velocity update in the search space, the acceleration coefficients and particles' neighborhood topologies in Chapter 3.

In Chapter 4, a set of convergence techniques, *i.e.* velocity clamping, inertia weight and constriction coefficient techniques which can be used to improve speed of convergences and control the exploration and exploitation abilities of the entire swarm, was illustrated. The Guaranteed Convergence PSO (GCPSO) algorithm was analyzed. This algorithm is very important to solve a problem when all particles face premature convergence or stagnation in the search process. Boundary conditions were presented which are very useful in the PSO algorithm.

Chapter 5 presented five different types of PSO algorithms which solve different types of optimization problems. The Multi-Start PSO (MSPSO) algorithm attempts to detect when the PSO has found lack of diversity. Once lack of diversity is found, the algorithm re-starts the algorithm with new randomly chosen initial positions for the particles. The Multi-phase PSO (MPPSO) algorithm partitions the main swarm into sub-swarms or subgroups, where each sub-swarm performs a different task, exhibits a different behavior and so on. Then the swarms cooperate to solve the problem by sharing the best solutions they have discovered in their respective sub-swarms. During the optimization process, high speed of convergence sometimes generates a quick loss of diversity which lead to undesirable premature convergence. To solve this problem, the perturbed particle swarm algorithm (PPSO) illustrated in this chapter. The Multi-Objective PSO (MOPSO) algorithm is very important when an optimization problem has several objective functions. One discrete optimization problem was solved by the Binary PSO (BPSO) algorithm.

The PSO algorithm has some problems that ought to be resolved. Therefore, the future works on the PSO algorithm will probably concentrate on the following:

1. Find a particular PSO algorithm which can be expected to provide good performance.

2. Combine the PSO algorithm with other optimization methods to improve the accuracy.

3. Use this algorithm to solve the non-convex optimization problems.

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Appendix A

Glossary terms of this thesis:

ABC: Absorbing boundary condition.

BPSO: Binary Particle Swarm Optimization.

CSS-MOPSO: Cross-Searching Strategy Multi-Objective

DBC: Damping boundary condition.

DNPSO: Dynamic Neighborhood Particle Swarm Optimization.

gbest PSO: Global Best Particle Swarm Optimization.

GCPSO: Guaranteed Convergence Particle Swarm Optimization.

IBC: Invisible boundary condition.

I/DBC: Invisible/Damping boundary condition.

I/RBC: Invisible/Reflecting boundary condition.

lbest PSO: Local Best Particle Swarm Optimization.

$\mathcal{PF}^*$: Pareto front.

PPSO: Perturbed Particle Swarm Optimization.

PSO: Particle Swarm Optimization.

RBC: Reflecting boundary condition.

MOPSO: Multi-Objective Particle Swarm Optimization.

MPPSO: Multi-phase Particle Swarm Optimization.

MSPSO: Multi-Start Particle Swarm Optimization.

VEPSO: Vector Evaluated Particle Swarm Optimization.

Appendix B

Commonly-used symbols of this thesis:

f: The function being minimized or maximized. It takes a vector input and returns a scalar value.

v_{ij}^{t}: The velocity vector of particle i in dimension j at time t.

x_{ij}^{t}: The position vector of particle i in dimension j at time t.

$P_{best,i}^{t}$: The personal best position of particle i in dimension j found from initialization through time t.

G_{best}: The global best position of particle i in dimension j found from initialization through time t.

c_1, c_2 : Positive acceleration constants which are used to level the contribution of the cognitive and social components respectively.

r_{1j}^{t}, r_{2j}^{t}: Random numbers from uniform distribution $U(0,1)$ at time t.

$\mathrm{gbest}_1^t, \mathrm{gbest}_2^t$: Two chosen gbests (local guides) for each particle in iteration t.

n: The swarm size or number of particles.

t: Denotes time or time steps.

D: The maximum no. of dimensions.

P: The maximum no. of particles.

N: Total number of iterations.

V_{max}: The maximum velocity.

ω: The inertia weight.

χ: The constriction coefficient.

ρ^t: The parameter controls the diameter of the search space.

C_i: The *criticality* of the particle i.

C: The global criticality limit.

G'_{best}: The j-th dimension of p-gbest in iteration t.

$N(G_{best}, \sigma)$: The normal distribution.

σ: The degree of uncertainty about the optimality of the gbest and is modeled as some non-increasing function of the number of iterations.

$v_{ij}^{[s]}$: The velocity of the i-th particle in the s-th swarm.

$G_{best}^{[q]}$: The best position found for any particle in the q-th swarm which is evaluated with the q-th objective function.

s_{ij}^{t}: The sigmoid function.