Frederick H. Soon

Student's Guide to CALCULUS by J. Marsden and A. Weinstein

Volume II

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Dedicated to:

Henry, Ora, Dennis, and Debbie

FOREWORD

This Student Guide is exceptional, maybe even unique, among such guides in that its author, Fred Soon, was actually a student user of the textbook during one of the years we were writing and debugging the book. (He was one of the best students that year, by the way.) Because of his background, Fred has taken, in the Guide, the point of view of an experienced student tutor helping you to learn calculus. While we do not always think Fred's jokes are as funny as he does, we appreciate his enthusiasm and his desire to enter into communication with his readers; since we nearly always agree with the mathematical judgements he has made in explaining the material, we believe that this Guide can serve you as a valuable supplement to our text.

To get maximum benefit from this Guide, you should begin by spending a few moments to acquaint yourself with its structure. Once you get started in the course, take advantage of the many opportunities which the text and Student Guide together provide for learning calculus in the only way that any mathematical subject can truly be mastered — through attempting to solve problems on your own. As you read the text, try doing each example and exercise yourself before reading the solution; do the same with the quiz problems provided by Fred.

Fred Soon knows our textbook better than anyone with the (possible) exception of ourselves, having spent hundreds of hours over the past ten years assisting us with its creation and proofreading. We have enjoyed our association with him over this period, and we hope now that you, too, will benefit from his efforts.

Jerry Marsden

Alan Weinstein

HOW TO USE THIS BOOK

As the title implies, this book is intended to guide the student's study of calculus. Realizing that calculus is not the only class on the college student's curriculum, my objective in writing this book is to maximize understanding with a minimum of time and effort.

For each new section of the text, this student guide contains sections entitled Prerequisites, Prerequisite Quiz, Goals, Study Hints, Solutions to Every Other Odd Exercise, Section Quiz, Answers to Prerequisite Quiz, and Answers to Section Quiz. For each review section, I have included the solutions to every other odd exercise and a chapter test with solutions.

A list of prerequisites, if any, is followed by a short quiz to help you decide if you're ready to continue. If some prerequisite seems vague to you, review material can be found in the section or chapter of the text listed after each prerequisite. If you have any difficulty with the simple prerequisite quizzes, you may wish to review.

As you study, keep the goals in mind. They may be used as guidelines and should help you to grasp the most important points.

The study hints are provided to help you use your time efficiently. Comments have been offered to topics in the order in which they appear in the text. I have tried to point out what is worth memorizing and what isn't. If time permits, it is advisable to learn the derivations of formulas rather than just memorizing them. You will find that the course will be more meaningful to you and that critical parts of a formula can be recalled even under the stress of an exam. Other aspects of the study hints include clarification of text material and "tricks" which will aid you in solving the exercises. Finally, please be aware that your instructor may choose to emphasize topics which I have considered less important.

Detailed solutions to every other odd exercise, i.e., 1,5,9, etc. are provided as a study aid. Some students may find it profitable to try the exercises first and then compare the method employed in this book. Since the authors of the text wrote most of the exercises in pairs, the answers in this book may also be used as a guide to solving the corresponding even exercises. In order to save space, fractions have been written on one line, so be careful about your interpretations. Thus, $1/x + y$ means y plus $1/x$, whereas $1/(x + y)$ means the reciprocal of $x + y$. Transcendental functions such as cos, sin, ln, etc. take precedence over division, so $\cos ax/b$ means take the cosine of ax and then divide by b, whereas $\cos (ax/b)$ has an unambiguous meaning. $\ln a/2$ means half of $\ln a$, not the natural logarithm of $a/2$. Also, everything in the term after the slash is in the denominator, so $1/2\int xdx + 1$ means add 1 to the reciprocal of $2\int xdx$. It does not mean add 1 to half of the integral. The latter would be denoted $(1/2)\int xdx + 1$.

Section quizzes are included for you to evaluate your mastery of the material. Some of the questions are intended to be tricky, so do not be discouraged if you miss a few of them. The answers to these "hard" questions should add to your knowledge and prepare you for your exams. Since most students seem to fear word problems, each quiz contains at least one word problem to help you gain familiarity with this type of question.

Finally, answers have been provided to both the prerequisite and section quizzes. If you don't understand how to arrive at any of the answers, be sure to ask your instructor.

In the review sections, I have written more questions and answers which may appear on a typical test. These may be used along with the section quizzes to help you study for your tests.

Since Calculus was intended for a three semester course, I have also included three-hour comprehensive exams at the end of Chapters 3, 6, 9, 12, 15, and 18. These should help you prepare for your midterms and final examinations. Best of luck with all of your studies.

ACKNOWLEDGEMENTS

Several individuals need to be thanked for helping to produce this book. I am most grateful to Jerrold Marsden and Alan Weinstein for providing the first edition of Calculus from which I, as a student, learned about derivatives and integrals. Also, I am deeply appreciative for their advice and expertise which they offered during the prepartion of this book. Invaluable aid and knowledgeable reviewing were provided by my primary assistants: Stephen Hook, Frederick Daniels, and Karen Pao. Teresa Ling should be recognized for laying the groundwork with the first edition of the student guide. Finally, my gratitude goes to my father, Henry, who did the artwork; to Charles Olver and Betty Hsi, my proofreaders; and to Ruth Edmonds, whose typing made this publication a reality.

Frederick H. Soon
Berkeley, California

CONTENTS

CHAPTER 12 -- INFINITE SERIES

CHAPTER 7

BASIC METHODS OF INTEGRATION

7.1 Calculating Integrals

PREREQUISITES

1. Recall how to integrate polynomials (Sections 2.5, 4.4, and 4.5).
2. Recall integration formulas involving exponentials and logarithms (Section 6.3).
3. Recall integration formulas involving trigonometric functions and their inverses (Section 5.2 and 5.4).
4. Recall the relationship between the integral and area (Section 4.6).

PREREQUISITE QUIZ

1. Perform the following integrations:

   (a) $\int_0^4 (x^3 + x)dx$

   (b) $\int (e^x - 1/\sqrt{1 - x^2})dx$

   (c) $\int (\cos x - 1/x + 2/x^2)dx$

2. Suppose $g(x) \geqslant f(x)$ on $[0,2]$ and $f(x) \geqslant g(x)$ on $[2,3]$. Write an expression for the area between the graphs of $g(x)$ and $f(x)$ on $[0,3]$.

GOALS

1. Be able to evaluate integrals involving sums of polynomials, trigonometric functions, exponentials, and inverse trigonometric functions.

2. Be able to use integration for solving area and total change problems.

STUDY HINTS

1. Definite integrals. In the box preceding Example 2, restrictions are placed on a and b. Can you explain why? If $n = -2, -3, -4, \ldots$ a and b must have the same sign to avoid the discontinuity in the integrand at $x = 0$. If n is not an integer, one must impose condition that avoid roots of negative numbers. Finally, recall that $\ln x$ is undefined for $x \leqslant 0$.

2. Checking answers. Remember that integration is the inverse of differentiation. Thus, you should always check your answer by differentiating it to get the integrand. Differentiation can often detect a wrong sign or a wrong factor.

3. Word problems. Be sure all quantities are expressed in compatible units. (See Example 9.)

4. Review of integration methods. The material in this section is review. If any of the examples didn't make sense, go back to the appropriate sections in Chapters 1-6 and review until you understand the examples.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By combining the sum rule, the constant multiple rule, and the power rule for antidifferentiation, we get $\int(3x^2 + 2x + x^{-3})dx = 3x^3/3 + 2x^2/2 + x^{-2}/(-2) + C = x^3 + x^2 - 1/2x^2 + C$.

5. We guess that the antiderivative of $\sin 2x$ is $a \cos 2x$. Differentiation gives $-2a \sin 2x$, so a should be $-1/2$. Thus, $\int(\sin 2x + 3x)dx = -\cos 2x/2 + 3x^2/2 + C$. *

9. Using the sum rule, the constant multiple rule, and the power rule for antidifferentiation, we get $F(x) = \int(x^8 + 2x^2 - 1)dx = x^9/9 + 2x^3/3 - x + C$. Note that $F(-a) = -F(a)$, so $F(a) - F(-a) = 2F(a)$. Therefore, $\int_{-2}^{2}(x^8 + 2x^2 - 1)dx = 2F(2) = 2(512/9 + 16/3 - 2) = 1084/9$.

13. Using the power rule for antidifferentiation, $\int_{16}^{81} \sqrt[4]{s}\, ds = \int_{16}^{81} s^{1/4} ds = [s^{5/4}/(5/4)]\Big|_{16}^{81} = (4/5)[(81)(3) - (16)(2)] = 844/5$.

17. $\int_{-\pi}^{\pi} \cos x\, dx = \sin x \Big|_{-\pi}^{\pi} = 0$.

21. By the constant multiple rule, $\int_0^1 [3/(x^2 + 1)]\, dx = 3\int_0^1 [1/(x^2 + 1)]\, dx = 3 \tan^{-1} x\Big|_0^1 = 3\pi/4$.

25. From the basic trigonometric antidifferentiation formulas, $\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = 1$.

29. By the logarithm differentiation formula, $\int_1^5 (dt/t) = \ln t\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$.

33. By the sum rule, the integral is $\int_{-200}^{200}(90x^{21} - 80x^{33} + 5580x^{97})dx + \int_{-200}^{200} 1\, dx$. Since the first integrand is an odd function, the antiderivative will be an even function, $F(x)$; therefore $F(200) = F(-200)$ and the integral is 0. Thus, we are left with $\int_{-200}^{200} 1\, dx = x\Big|_{-200}^{200} = 400$.

37. (a) According to the fundamental theorem of calculus, the derivative of the integral is the integrand. Using the chain rule, we have $(d/dx)\left[e^{(x^2)}/2 + C\right] = 2xe^{(x^2)}/2 = xe^{(x^2)}$. Therefore, the formula is correct.

*Throughout the student guide, we take $\cos 2x/2$ to mean $(1/2)\cos 2x$, not $\cos(2x/2)$.

37. (b) According to Example 7, $\int \ln x\,dx = x\ln x - x + C$. Thus, $\int_1^e \left(2xe^{(x^2)} + 3\ln x\right)dx = \left(e^{(x^2)} + 3x\ln x - 3x\right)\Big|_1^e = e^{(e^2)} + 3e -$

$3e - e - 0 + 3 = e^{(e^2)} - e + 3$.

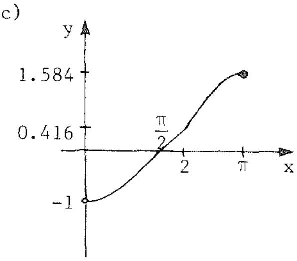
41. We may write $g(t) = \int_{t^2}^{4} \sqrt{e^x + \sin 5x^2}\,dx = -\int_4^{t^2} \sqrt{e^x + \sin 5x^2}\,dx$ as $f(t^2)$, where $f(u) = \int_4^u -\sqrt{e^x + \sin 5x^2}\,dx$. By the fundamental theorem of calculus (alternative version), $f'(u) = -\sqrt{e^u + \sin 5u^2}$. By the chain rule, $g'(t) = f'(t^2)\cdot(d/dt)t^2 = -2t\sqrt{e^{(t^2)} + \sin 5t^4}$.

45. Property 4 of the definite integral, the endpoint additivity rule, is used in this exercise.

(a) $\int_{-\pi/2}^{\pi/2} f(x)g(x)dx = \int_{-\pi/2}^{\pi/2} \sin x = -\cos x\Big|_{-\pi/2}^{\pi/2} = 0$.

(b) $\int_1^3 g(x)h(x)dx = \int_1^2 dx/x^2 + \int_2^3 2\,dx/x^2 = (-1/x)\Big|_1^2 + (-2/x)\Big|_2^3 =$ $-1/2 + 1 - 2/3 + 1 = 5/6$.

(c)



If $0 < x \leqslant 2$, then $\int_{\pi/2}^{x} f(t)g(t)dt = \int_{\pi/2}^{x} \sin t\,dt = -\cos t\Big|_{\pi/2}^{x} = -\cos x$. If $2 \leqslant x \leqslant \pi$, then $\int_{\pi/2}^{x} f(t)g(t)dt = \int_{\pi/2}^{2} \sin t\,dt + \int_2^x (2\sin t)dt = -\cos t\Big|_{\pi/2}^{2} + (-2\cos t)\Big|_2^x = -\cos(2) - 2\cos x + 2\cos(2) = \cos(2) - 2\cos x$. Therefore,

$$\int_{\pi/2}^{x} f(t)g(t)dt = \begin{cases} -\cos x & \text{if } 0 < x \leqslant 2 \\ \cos(2) - 2\cos x & \text{if } 2 \leqslant x \leqslant \pi \end{cases}.$$

49. Since the function is positive on $[1,4]$, the area is $\int_a^b f(x)dx$. $\int_1^4 [(x^2 + 2)/\sqrt{x}]dx = \int_1^4 (x^{3/2} + 2x^{-1/2})dx = [x^{5/2}/(5/2) + 2x^{1/2}/(1/2)]\Big|_1^4 =$ 16.4.

53. The y-intercept of $y = 1 - x^2/4$ is 1, so the white area in the center is a circle of radius 1. We will find the area in the first quadrant and then multiply by 4 to find the total area. The x-intercept

53. (continued)

of $y = 1 - x^2/4$ is 2. The area of the white quarter circle is $\pi/4$, so the area of the entire shaded region is $4\left[\int_0^2 (1 - x^2/4)dx - \pi/4\right] =$ $4[(x - x^3/12)|_0^2 - \pi/4] = 4(4/3 - \pi/4) = 16/3 - \pi$.

57. Integrating the identity $\sin^2 x + \cos^2 x = 1$ from 0 to $\pi/2$ and using the sum rule gives $\int_0^{\pi/2} \sin^2 x dx + \int_0^{\pi/2} \cos^2 dx = \int_0^{\pi/2} 1\, dx$. The two integrals on the left-hand side are equal by assumption and the right-hand side is $\pi/2$, so $\int_0^{\pi/2} \sin^2 x\, dx = \pi/4$. (Note that we did this problem without ever finding $\int \sin^2 x\, dx$.)

61. The area is given by the integral $\int_a^b [dx/(1 + x^2)] = \tan^{-1} x|_a^b = \tan^{-1} b -$ $\tan^{-1} a$. By the definition of the inverse tangent function, the value of $\tan^{-1} x$ always lies in the interval $(-\pi/2, \pi/2)$, so the difference between two values must be less than π, regardless of the length of the interval $[a,b]$.

65. (a) Guess that the antiderivative has the form ae^{-kt}. We differentiate to get $-ake^{-kt}$, so $a = -R/k$. $A = \int_0^T Re^{-kt}dt = (-R/k)e^{-kt}|_0^T =$ $(-R/k)(e^{-kT} - 1) = (R/k)(1 - e^{-kt})$.

(b) For this problem, $k = 0.0825$; $R = 4(12)(230) = 11040$; $T = 5$. Therefore, $A = (11040/0.0825)(1 - e^{-(0.0825)(5)}) = \$45{,}231.46$.

69. Using the identity, $\int_1^e [dt/t(t + 1)] = \int_1^e (dt/t) - \int_1^e [dt/(t + 1)] =$ $\ln t\,|_1^e - \ln(t + 1)|_1^e = 1 - 0 - \ln(e + 1) + \ln(2) \approx 0.380$. (The second integral was found by guessing that $\int [dt/(t + 1)] = a \ln|t + 1| + C$. Then differentiation showed that $a = 1$.)

SECTION QUIZ

1. Evaluate the following integrals:

   (a) $\int[(x^4 + 3x^3 - 1)/x]\,dx$

   (b) $\int_{-93}^{93}(x^5 + 8x^3 - 27x + 1)\,dx$

   (c) $\int(x^{3/2} + 1/\sqrt{x} + e^x)\,dx$

   (d) $\int_{\pi}^{\pi/2}(2\sin\theta + \cos\theta)\,d\theta$

   (e) $\int_{1/3}^{1/3}\left[dt/\sqrt{t + t^2}\right]$

   (f) $\int(3/u\sqrt{u^2 - 1})\,du$, $u > 0$

   (g) $\int[(t^4 + 2t^2 + 2)/(t^2 + 1) - e^t + \csc t \cot t]\,dt$

2. $\int[dt/(1 + t^2)] = \tan^{-1}t + C$ and $\int[dt/(1 + t^2)] = -\cot^{-1}t + C$. Since the integrals are equal, we have $\tan^{-1}t + C = -\cot^{-1}t + C$. The C's cancel, leaving $\tan^{-1}t = -\cot^{-1}t$. Obviously, $\tan^{-1}t \neq -\cot^{-1}t$. What is the fallacy in this argument?

3. Show that $\int e^x \sin x\,dx \neq (1/2)e^x(\sin x + \cos x) + C$.

4. Once again, Guilty Gary had borrowed his neighbor's tools without asking. This time, he was spray painting his house when a friendly policeman stopped to compliment Guilty Gary on the fine job he was doing. Startled, Guilty Gary forgot to turn off the spray before turning around. When he realized he was spraying the policeman, he accidentally increased the spray rate. For 15 seconds, paint was coming out at $(0.25 + t)$ liters/min., where t is in minutes. How much paint did Guilty Gary spray on the policeman?*

*Dear Reader: I realize that many of you hate math but are forced to complete this course for graduation. Thus, I have attempted to maintain interest with "entertaining" word problems. They are not meant to be insulting to your intelligence. Obviously, most of the situations will never happen; however, calculus has several practical uses and such examples are found throughout Marsden and Weinstein's text. I would appreciate your comments on whether my "unusual" word problems should be kept for the next edition.

ANSWERS TO PREREQUISITE QUIZ

1. (a) 72

   (b) $e^x - \sin^{-1}x + C$

   (c) $\sin x - \ln|x| - 2/x + C$

2. $\int_0^2 [g(x) - f(x)]\,dx + \int_2^3 [f(x) - g(x)]\,dx$

ANSWERS TO SECTION QUIZ

1. (a) $x^4/4 + x^3 - \ln|x| + C$

   (b) 186

   (c) $2x^{5/2}/5 + 2\sqrt{x} + e^x + C$

   (d) -1

   (e) 0

   (f) $3 \sec^{-1}u + C$

   (g) $t^3/3 + t + \tan^{-1}t - e^t - \csc t + C$

2. The arbitrary constants have different values.

3. Differentiate the right-hand side.

4. 3/32 liters

7.2 Integration by Substitution

PREREQUISITES

1. Recall how to differentiate rational, trigonometric, logarithmic, and exponential functions (Sections 1.5, 5.2, and 6.3).
2. Recall how to differentiate by the chain rule (Section 2.2).
3. Recall how to integrate basic functions (Section 7.1).

PREREQUISITE QUIZ

1. Differentiate the following expressions:

   (a) $\cos x + \sec 2x + 1/2x$

   (b) $e^{\sin x} + 3^{x/2} + x^{-3}$

   (c) $x^6 + 3x^2 - \ln(x^3 + 4x)$
2. Evaluate $\int (x^6 + \cos x + 1/x)dx$.
3. Evaluate $\int [1/(1 + x^2) + e^x - 3]\,dx$.

GOALS

1. Be able to recognize the types of integrals which may be evaluated by substitution.
2. Be able to evaluate integrals using the technique of substitution.

STUDY HINTS

1. Integration by substitution. Be sure you understand this technique well. It is one of the most important techniques of integration.
2. Choosing a substitution. Practice and study the examples. Note that in the first three examples, u is an expression which is raised to a power.

3. <u>More common substitutions</u>. If x^n appears, see if x^{n-1} also appears in the integrand, where n again does not have to be an integer. If so, try substituting $u = x^n$. Two other good choices for u are used to derive the shifting and scaling rules.

4. <u>Shifting rule</u>. Do not memorize the rule. It is simply a substitution of $u = x + a$.

5. <u>Scaling rule</u>. Do not memorize the rule. It is simply a substitution of $u = bx$.

6. <u>Differential notation</u>. As an alternative, you may differentiate both sides of $u = g(x)$ to get $du = g'(x)dx$, and then solving to get $du/g'(x) = dx$. All of the methods presented are the same. Use the one which you feel most comfortable with.

7. <u>Giving an answer</u>. Remember to express your answer in terms of the original variable. And don't forget the arbitrary constant.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x^2 + 4$, so $du = 2x\,dx$. Then $\int 2x(x^2 + 4)^{3/2}dx = \int u^{3/2}du = u^{5/2}/(5/2) + C = 2u^{5/2}/5 + C = 2(x^2 + 4)^{5/2}/5 + C$.

   Differentiating, we get $(d/dx)[2(x^2 + 4)^{5/2}/5 + C] = (2/5)(5/2)(x^2 + 4)^{3/2}(2x) = 2x(x^2 + 4)^{3/2}$. Differentiation yields the integrand, so the answer is verified.

5. Let $u = \tan\theta$, so $du = \sec^2\theta\, d\theta$. Then $\int(\sec^2\theta/\tan^3\theta)d\theta = \int(du/u^3) = u^{-2}/(-2) + C = -1/2\tan^2\theta + C$.

   Upon differentiating, we get $(d/d\theta)[-1/2\tan^2\theta + C] = 2(2)\tan\theta\sec^2\theta/(2\tan^2\theta)^2 = \sec^2\theta/\tan^3\theta$. Differentiation yields the integrand, so the answer is verified.

9. Let $u = x^4 + 2$, so $du = 4x^3 dx$. Then $\int (x^3/\sqrt{x^4+2})dx = \int (x^3/u^{1/2})(du/4x^3) = (1/4)\int u^{-1/2}du = (1/4)[u^{1/2}/(1/2)] + C = \sqrt{x^4+2}/2 + C$.

Differentiating the answer gives $(d/dx)[(x^4+2)^{1/2}/2 + C]$ $(1/2)(1/2)(x^4+2)^{-1/2}(4x^3) = x^3/\sqrt{x^4+2}$, which is the integrand. Thus, the answer is verified.

13. Let $u = \cos(r^2)$, so $du = -\sin(r^2)\cdot 2r\,dr$. Then $\int 2r\sin(r^2)\times \cos^3(r^2)dr = \int -u^3 du = -u^4/4 + C = -\cos^4(r^2)/4 + C$.

Differentiating the answer gives $(d/dr)[-\cos^4(r^2)/4 + C] = (-1/4)(4)(\cos^3(r^2))(-\sin(r^2))(2r) = 2r\sin(r^2)\cos^3(r^2)$, which is the integrand. Thus, the answer is verified.

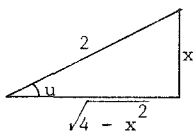
17. Let $u = \theta + 4$, so $du = d\theta$. Then $\int \sin(\theta+4)d\theta = \int \sin u\,du = -\cos u + C = -\cos(\theta+4) + C$. (The shifting rule may have been applied to this integral.)

Differentiating the answer yields $(d/d\theta)[-\cos(\theta+4) + C] = \sin(\theta+4)$, so the answer is verified.

21. Let $u = t^2 + 2t + 3$, so $du = (2t+2)dt = 2(t+1)dt$. Then $\int[(t+1)/\sqrt{t^2+2t+3}]dt = \int u^{-1/2}(du/2) = (1/2)(u^{1/2}/(1/2)) + C = u^{1/2} + C = \sqrt{t^2+2t+3} + C$.

Differentiating the answer gives $(d/dt)(\sqrt{t^2+2t+3} + C) = (1/2)(t^2+2t+3)^{-1/2}(2t+2) = (t+1)/\sqrt{t^2+2t+3}$, which is the integrand. Thus, the answer is verified.

25. Use the hint to get $\int\cos^3\theta d\theta = \int(\cos\theta)(\cos^2\theta)d\theta = \int(\cos\theta)(1-\sin^2\theta)d\theta$ Now let $u = \sin\theta$, so $du = \cos\theta\,d\theta$. Then $\int\cos^3\theta\,d\theta = \int(1-u^2)du = u - u^3/3 + C = \sin\theta - \sin^3\theta/3 + C$.

29. 

Let $x = 2 \sin u$, so $dx = 2 \cos u\, du$; therefore, $\int \sqrt{4 - x^2}\, dx = \int \sqrt{4 - 4 \sin^2 u} \times (2 \cos u\, du) = 4\int \cos^2 u\, du = 2\int (1 + \cos 2u) du$ (using the half-angle formula) $= 2(u + \sin 2u/2) + C = 2u + 2 \sin u \times \cos u + C$. Since $u = \sin^{-1}(x/2)$ (See the figure), the integral is $2 \sin^{-1}(x/2) + x\sqrt{4 - x^2}/2 + C$.

33. Let $u = \ln t$, so $du = dt/t$; therefore, $\int [\sin(\ln t)/t] dt = \int \sin u\, du = -\cos u + C = -\cos(\ln t) + C$.

37. (a) Let $u = \sin x$, so $\cos x\, dx = du$. Then $\int \sin x \cos x\, dx = \int u\, du = u^2/2 + C = \sin^2 x/2 + C$.

(b) Let $u = \cos x$, so $\sin x\, dx = -du$. Then $\int \sin x \cos x\, dx = \int u(-du) = -u^2/2 + C = -\cos^2 x/2 + C$.

(c) From $\sin 2x = 2 \sin x \cos x$, we get $\sin x \cos x = \sin 2x/2$. Hence $\int \sin x \cos x\, dx = (1/2)\int \sin 2x\, dx = (1/2)[-\cos 2x/2] + C = -\cos 2x/4 + C$ by the scaling rule.

To show that the three answers we got are really the same, we need to show that they differ from one another by constants. $\sin^2 x/2 - (-\cos^2 x/2) = 1/2$, which is a constant. Also, $-\cos^2 x/2 - (-\cos 2x/4) = -\cos^2 x/2 + (2 \cos^2 x - 1)/4 = -1/4$, which is a constant. Thus, we have shown that all three answers are equivalent.

SECTION QUIZ

1. Evaluate the following integrals:

   (a) $\int (t + 4)(t + 5)^{3/2} dt$ [Hint: Let $u = t + 5$.]

   (b) $\int \left[2e^{3x}/\left(1 + e^{6x}\right)\right] dx$

   (c) $\int \left(5/\sqrt{4u - u^2}\right) du$ [*Hint: Complete the square.*]

   (d) $\int \sec^5 t \tan t \, dt$

   (e) $\int x\left(e^{4x}\right)^x dx$

2. What is wrong with the following statements?

   (a) $\int (3 - x)^{3/2} dx = 2(3 - x)^{5/2}/5 + C$.

   (b) $\int (x - 3)^{-2} dx = (x - 3)^{-3}/(-3) + C$.

   (c) $\int \cos 5x \, dx = 5 \sin 5x + C$.

   (d) $\int [(4x)^2 + (4x) + 1] dx = (1/4)[(4x)^3/3 + (4x)^2/2 + (4x)]$.

3. The rate at which Schizophrenic Sam spends listening to voices t months after his psychiatric visit is given by $t^2/(t + 4)$, $0 \leqslant t \leqslant 2.5$. Find a formula describing the amount of time spent in auditory hallucination since his last psychiatric appointment. (Hint: Let $u = t + 4$

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-\sin x + 2 \sec 2x \tan 2x - 1/2x^2$

   (b) $(\cos x)e^{\sin x} + (1/2)(\ln 3)3^{x/2} - 3x^{-4}$

   (c) $6x^5 + 6x - (3x^2 + 4)/(x^3 + 4x)$

2. $x^7/7 + \sin x + \ln |x| + C$

3. $\tan^{-1} x + e^x - 3x + C$

ANSWERS TO SECTION QUIZ

1. (a) $2(t+5)^{7/2}/7 - 2(t+5)^{5/2}/5 + C$; let $u = t + 5$.

   (b) $(2/3)\tan^{-1}\left(e^{3x}\right) + C$; let $u = e^{3x}$.

   (c) $5\sin^{-1}[(u-2)/2] + C$; let $v = u - 2$.

   (d) $\sec^5 t/5 + C$; let $u = \sec x$.

   (e) $\exp(4x^2)/8 + C$; let $u = 4x^2$.

2. (a) Minus sign is missing.

   (b) Exponent and denominator should be -1 rather than -3 .

   (c) Answer should be $\sin 5x/5 + C$.

   (d) Arbitrary constant is missing.

3. $(t+4)^2/2 - 8(t+4) + 16\ln|t+4|$.

7.3 Changing Variables in the Definite Integral

PREREQUISITES

1. Recall how to integrate by the method of substitution (Section 7.2).
2. Recall the fundamental theorem of calculus (Section 4.4).

PREREQUISITE QUIZ

1. Express the integral $\int_a^b F'(x)dx$ in terms of F.
2. Perform the following integrations:

   (a) $\int \cos x \sin x\, dx$

   (b) $\int [3x/(1 + x^4)]\, dx$

   (c) $\int (2x - 3)^4 dx$

GOALS

1. Be able to change the limits of integration while integrating by substitution.

STUDY HINTS

1. Changing limits. Once you have learned integration by substitution for indefinite integrals, definite integrals can easily be computed by changing the limits every time you change variables. Simply choose $u = g(x)$ and then change the limits from $x = a$ and b to $u = g(a)$ and $g(b)$.
2. Substitution may not work. If the integral becomes harder, try a different substitution or maybe integration by substitution is not the correct method to be used for solving the integral. When you are proficient, going through a mental checklist shouldn't take long.

3. Use of tables. Don't rely on tables. By the time you finish this course, you should be able to derive the integration formulas found in the tables. They are provided for your convenience . Check with your instructor regarding access to a table during an exam.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x + 2$, so $du = dx$; therefore, $\int_{-1}^{1}\sqrt{x+2}\,dx = \int_{1}^{3}\sqrt{u}\,du = [(u^{3/2})/(3/2)]\big|_1^3 = 2(\sqrt{27} - 1)/3 = 2\sqrt{3} - 2/3$.

5. Let $u = x^2 + 2x + 1$, so $du = (2x + 2)dx = 2(x + 1)dx$; therefore, $\int_2^4 (x + 1)(x^2 + 2x + 1)^{5/4}dx = (1/2)\int_9^{25} u^{5/4}du = [(1/2)u^{9/4}/(9/4)]\big|_9^{25} = 2[(25)^{9/4} - (9)^{9/4}]/9$.

9. Let $u = x^2$, so $du = 2xdx$; therefore, $\int_0^1 x\exp(x^2)dx = (1/2)\int_0^1 e^u du = (e^u/2)\big|_0^1 = (e - 1)/2$.

13. Let $u = \cos x$, so $du = -\sin x\,dx$; therefore, $\int_{-\pi/2}^{\pi/2} 5\cos^2 x\sin x\,dx = -\int_0^0 5u^2 du = 0$.

17. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$; therefore, $\int_{\pi/8}^{\pi/4}\tan\theta\,d\theta = \int_{\pi/8}^{\pi/4}(\sin\theta/\cos\theta)d\theta = -\int_{\cos(\pi/8)}^{\sqrt{2}/2}(du/u) = -\ln|u|\Big|_{\cos(\pi/8)}^{\sqrt{2}/2} = \ln(\cos(\pi/8)) - \ln(\sqrt{2}/2) = \ln(\sqrt{2}\cos(\pi/8))$. We used the identity $\tan\theta = \sin\theta/\cos\theta$ to solve this problem.

21. Division yields $\int_1^3[(x^3 + x - 1)/(x^2 + 1)]dx = \int_1^3[x - 1/(x^2 + 1)]dx = (x^2/2 - \tan^{-1}x)\big|_1^3 = 4 - \tan^{-1}(3) + \pi/4$.

25. If we make the substitution $u = x^3 + 3x^2 + 1$, we have $du = (3x^2 + 6x)dx$, so $\int_0^1\left[(x^2 + 3x)/\sqrt[3]{x^3 + 3x^2 + 1}\right]dx = \int_{u=1}^{5}[(x^2 + 3x)/\sqrt[3]{u}(3x^2 + 6x)]\,du = \int_{u=1}^{5}[(x + 3)/\sqrt[3]{u}(3x + 6)]\,du$. There is no simple way to express the quantity $(x + 3)/(3x + 6)$ in terms of u . (We

25. (continued)

would have to solve the equation $u = x^3 + 3x^2 + 1$ for x in terms of u .) We conclude that the substitution was not effective in this case.

29. We use the first half of formula 65 because $a = 3 > 0$ and $b = 2$, $c = 1$. Therefore, $\int_0^1 \left(dx/\sqrt{3x^2 + 2x + 1}\right) = (1/\sqrt{3}) \ln \left|2(3)x + 2 + 2\sqrt{3}\sqrt{3x^2 + 2x + 1}\right| \Big|_0^1 = (1/\sqrt{3})[\ln |8 + 6\sqrt{2}| - \ln |2 + 2\sqrt{3}|] = (1/\sqrt{3}) \times \ln[(4 + 3\sqrt{2})/(1 + \sqrt{3})]$.

33. Let $u = x^2 + 2x + 2$, so $du = (2x + 2)dx = 2(x + 1)dx$. Thus, the area is $\int_0^1 [(x + 1)/(x^2 + 2x + 2)^{3/2}]dx = \int_2^5 (du/2u^{3/2}) = [(1/2)u^{1/2}/(-1/2)]\big|_2^5 = (-1/\sqrt{u})\big|_2^5 = -1/\sqrt{5} + 1/\sqrt{2} = -\sqrt{5}/5 + \sqrt{2}/2 = (5\sqrt{2} - 2\sqrt{5})/10$.

37. (a) By substituting $u = \cos x$, we have $du = -\sin x\, dx$, $u(\pi/2) = 0$, and $u(0) = 1$. Thus, $\int_0^{\pi/2} \cos^2 x \sin x\, dx = -\int_1^0 u^2 du = (-u^3/3)\big|_1^0 = 1/3$.

(b) Let F be the antiderivative of f . Then, $\int_a^b f(g(x))g'(x)dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a))$. This is the same as $\int_{g(a)}^{g(b)} f(u)du = F(u)\big|_{g(a)}^{g(b)}$. Thus, the formula is valid independent of the relationship between $g(a)$ and $g(b)$.

SECTION QUIZ

1. Evaluate the following integrals:

(a) $\int_{-1}^1 (x + 1)^5 x\, dx$ [Hint: Let $u = x + 1$.]

(b) $\int_{-3}^3 x \exp(-x^2)dx$

(c) $\int_0^1 [t^5/(1 + t^3)^3]dt$ [Hint: Let $u = 1 + t^3$.]

(d) $\displaystyle\int_0^{-\sqrt{1/2}} \left[2y/(1 - y^4)^{1/2}\right] dy$ [Hint: Let $u = y^2$.]

2. Consider the integral $\int_{-\pi}^{\pi} 3x^2 \sin(x^3)[\exp(\cos(x^3))]^2 dx$.

   (a) Rewrite the integral by substituting $u = x^3$.

   (b) Rewrite the integral in (a) by substituting $v = \cos u$.

   (c) Rewrite the integral in (b) by substituting $w = 2v$.

   (d) Evaluate the integral.

   (e) What substitution can be used to evaluate the integral in one step?

3. Lost in a magic cave, you read, scribbled on the wall, "RIGHT $\int_0^{\pi/4} 72 \sin 4x\, dx$, LEFT $\int_{-2}^{0} 6(t + 2)^3 dt$, RIGHT $\int_{-1/2}^{1/2} (70/\pi) \times [2dx/(1 + 4x^2)]$" . From that sign, an arrow points to a combination lock marked "EXIT." What combination will probably bring you back to the outside world?

ANSWERS TO PREREQUISITE QUIZ

1. $F(b) - F(a)$

2. (a) $\sin^2 x/2 + C$ or $-\cos^2 x/2 + C$

   (b) $(3/2) \tan^{-1}(x^2) + C$

   (c) $(2x - 3)^5/10 + C$

ANSWERS TO SECTION QUIZ

1. (a) $128/7 - 32/3$

   (b) 0

   (c) $1/24$

   (d) $\pi/6$

2. (a) $\int_{-\pi^3}^{\pi^3} \sin u\ [\exp(\cos u)]^2 du$

(b) $-\int_{-\cos(\pi^3)}^{\cos(\pi^3)} [\exp v]^2\ dv$

(c) $-\int_{-2(\cos(\pi^3))}^{2(\cos(\pi^3))} (e^w\ dw/2)$

(d) $\{\exp[-2\cos(\pi^3)] - \exp[2\cos(\pi^3)]\}/2$

(e) Let $u = 2\cos(x^3)$.

3. 36 RIGHT, 24 LEFT, 35 RIGHT (Apologies to the female readers!)

7.4 Integration by Parts

PREREQUISITES

1. Recall the product rule for differentiation (Section 1.5).

2. Recall the definition of an inverse function (Section 5.3).

PREREQUISITE QUIZ

1. Differentiate the following expressions:

   (a) $x \ln x$

   (b) $(5x - 3)(x^2 - 4x + 1)$

   (c) $x^2 \tan x$

2. If $f(x) = 2x + 3$, find a formula for $f^{-1}(x)$.

GOALS

1. Be able to use the technique of integration by parts.

2. Be able to integrate inverse functions.

STUDY HINTS

1. Integration by parts. Memorize the formula $\int u\,dv = uv - \int v\,du$ and don't forget the minus sign. The formula will be used quite often during your studies of mathematics. Learn the formula well enough so that it becomes second nature to you.

2. Choice of u. When integrating by parts, the factor x^n , where n is a positive integer, is often chosen to be u . After n repetitions of integration by parts, this factor will be eliminated. Another common situation is when one of the factors is $\ln x$, in which case we let $u = \ln x$. Note the choice of u in Examples 1 , 2 , and 3 .

3. I method. Example 4 demonstrates a technique which is useful for integrals involving exponentials and trigonometric functions. The method works because repeated differentiation of the factors yields the original factors.

4. "Other" I methods. Try Example 4 again by letting $u = e^x$ both times. Now, try again with $u = \sin x$ in the first step and then letting $u = e^x$ in the second step. The last method demonstrates that you must let u be the exponential or the trigonometric functions throughout the integration. Don't change in the middle.

5. Integrating inverse functions. Once you learn the formula $\int u\,dv = uv - \int v\,du$, simply rename the variables to get $\int y\,dx = xy - \int x\,dy$. All we did was replace u with y and v with x. Finally, remember to express your answer to $\int y\,dx$ in terms of x, not y.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $u = x + 1$ and $dv = \cos x\,dx$, so $du = dx$ and $v = \sin x$; therefore, $\int(x + 1)\cos x\,dx = (x + 1)\sin x - \int\sin x\,dx = (x + 1)\sin x + \cos x + C$.

5. Let $u = x^2$ and $dv = \cos x\,dx$, so $du = 2x\,dx$ and $v = \sin x$; therefore, $\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx$. For the integral $\int 2x \sin x\,dx$, apply integration by parts again with $u = 2x$, so $dv = \sin x\,dx$, $du = 2\,dx$ and $v = -\cos x$; therefore $\int 2x \sin x\,dx = -2x \cos x - \int -\cos x\,(2\,dx) = -2x\cos x + 2\sin x + C$. Hence $\int x^2 \cos x\,dx = x^2 \sin x - [-2x\cos x + 2\sin x] + C = (x^2 - 2)\sin x + 2x\cos x + C$.

9. Let $u = \ln(10x)$, so $dv = dx$, $du = (10/10x)dx = dx/x$, and $v = x$; therefore, $\int \ln(10x)dx = x \ln(10x) - \int x(dx/x) = x \ln(10x) - x + C$.

13. Let $u = s^2$, so $dv = e^{3s}ds$, $du = 2s\ ds$, and $v = e^{3s}/3$; therefore, $\int s^2e^{3s}ds = s^2e^{3s}/3 - \int 2se^{3s}ds/3$. Apply parts again: Let $u = 2s$, so $dv = e^{3s}ds/3$, $du = 2\ ds$, and $v = e^{3s}/9$; therefore, $\int s^2e^{3s}ds = s^2e^{3s}/3 - 2se^{3s}/9 + \int 2e^{3s}/9 = s^2e^{3s}/3 - 2se^{3s}/9 + 2e^{3s}/27 + C = e^{3s}(9s^2 - 6s + 2)/27 + C$.

17. Let $u = t^2$, so $dv = 2t \cos(t^2)dt$, $du = 2t\ dt$, and $v = \sin t^2$ (v is found by substituting $w = t^2$). Thus, $\int 2t^3 \cos t^2\ dt = t^2 \sin t^2 - \int 2t \sin t^2 dt$. Again, we substitute $w = t^2$ to get the answer $t^2 \sin t^2 + \cos t^2 + C$.

21. Let $u = \ln(\cos x)$, so $du = (-\sin x/\cos x)dx = -\tan x\ dx$; therefore, $\int \tan x \ln(\cos x)\ dx = -\int u\ du = -u^2/2 + C = -[\ln(\cos x)]^2/2 + C$.

25. We will use formula (8) with the role of x and y reversed. Let $x = \sqrt{1/y - 1}$, $y = 1/(x^2 + 1)$; therefore, $\int \sqrt{1/y - 1}\ dy = \int x\ dy = xy - \int y\ dx = \sqrt{1/y - 1}/(x^2 + 1) - \int [dx/(x^2 + 1)] = y\sqrt{1/y - 1} - \tan^{-1}x + C = y\sqrt{1/y - 1} - \tan^{-1}\sqrt{1/y - 1} + C$.

29. If we choose $f(x) = x$ and $G(x) = \sin x$, then $F(x) = x^2/2$ and $g(x) = \cos x$. We obtain $\int x \sin x\ dx = (x^2/2)\sin x - (1/2)\int x^2 \cos x\ dx$. The new integral on the right is more complicated than the one we started with, so this choice of f and G is not suitable.

33. Note that $\ln x^3 = 3 \ln x$, so let $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$. We get $\int_1^3 \ln x^3\ dx = 3\int_1^3 \ln x\ dx = 3[x \ln x|_1^3 - \int_1^3 1\ dx] = 3(3 \ln 3 - 2)$.

37. Use the formula for integrating inverse functions, so $y = \cos^{-1}(4x)$ and $dy = (-4/\sqrt{1 - 16x^2})dx$. Therefore, $\int_{1/8}^{1/4} \cos^{-1}(4x)dx =$ $x \cos^{-1}(4x)|_{1/8}^{1/4} + 4\int_{1/8}^{1/4} (x/\sqrt{1 - 16x^2})dx$. Let $u = 1 - 16x^2$, so $du = -32x\,dx$ or $-du/32 = x\,dx$. Hence $\int_{1/8}^{1/4} \cos^{-1}(4x)dx = (1/4)(0) -$ $(1/8)(\pi/3) - 4\int_{3/4}^{0} u^{-1/2}du/32 = -\pi/24 - \sqrt{u}/4|_{3/4}^{0} = \sqrt{3}/8 - \pi/24 \approx 0.09$.

41. First, substitute $\theta = 2x$, so $\int_{-\pi}^{\pi}\exp(2x)\sin(2x)\,dx$ becomes $(1/2)\int_{-2\pi}^{2\pi} e^{\theta}\sin\theta\,d\theta$. From Example 4, $\int e^x \sin x\,dx = e^x(\sin x - \cos x)/2 + C$. Thus, we get $(1/2)[(1/2)e^{\theta}(\sin\theta - \cos\theta)]|_{-2\pi}^{2\pi} = [-e^{2\pi} + e^{-2\pi}]/4$.

45. Use the formula for inverse functions with $y = \sin^{-1}2x$ and $dy = \left(2/\sqrt{1 - 4x^2}\right)dx$. Thus, $\int_0^{1/2\sqrt{2}}\sin^{-1}2x\,dx = x\sin^{-1}2x|_0^{1/2\sqrt{2}} - \int_0^{1/2\sqrt{2}}\left(2x/\sqrt{1 - 4x^2}\right)dx$. Substitute $u = 1 - 4x^2$ to get $(1/2\sqrt{2})(\pi/4) + (1/4) \times$ $\int_1^{1/2}(du/\sqrt{u}) = \pi/8\sqrt{2} + (1/4)2u^{1/2}|_1^{1/2} = \pi/8\sqrt{2} + 1/2\sqrt{2} - 1/2 = (\pi - 4)/8\sqrt{2} -$ $1/2$.

49. Let $u = x$, so $dv = \sin ax\,dx$, $du = dx$, and $v = -\cos ax/a$; therefore, $\int_0^{2\pi} x\sin ax\,dx = -x\cos ax/a|_0^{2\pi} + \int \cos ax\,dx/a = -x\cos ax/a|_0^{2\pi} + \sin ax/a^2|_0^{2\pi} = (-2\pi\cos 2\pi a)/a + (\sin 2\pi a)/a^2$. Since the terms in the numerator are finite as a tends to infinity, $\int_0^{2\pi} x\sin ax\,dx$ approaches 0 as a approaches ∞ .

Geometrically, $\sin ax$ oscillates more as a approaches ∞ . Note that the area of each hump gets smaller as a gets larger. Finally, the area of one hump above the x-axis almost cancels the area of the next hump below the x-axis; therefore, the total area approaches 0 .

53. (a) $\int \cos^n x\,dx = \int(\cos^{n-1}x)(\cos x)dx$. We apply integration by parts with $u = \cos^{n-1}x$, and $dv = \cos x\,dx$, so $du = (n - 1)\cos^{n-2}x \times (-\sin x)dx$ and $v = \sin x$. Therefore, $\int\cos^n x\,dx = \cos^{n-1}x\sin x -$ $[-(n - 1)\int \cos^{n-2}x(\sin x\,dx)(\sin x)] = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x \times$ $(1\ \cos^2 x)dx = \cos^{n-1}x\sin x + (n - 1)\int\cos^{n-2}x - (n - 1)\int\cos^n x\,dx$.

53. (a) (continued)

Rearrangement yields $n\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1) \times \int \cos^{n-2}x\,dx$. Division by n yields the desired result.

(b) Letting $n = 2$, we have $\int \cos^2 x\,dx = \cos^{2-1}x \sin x/2 + [(2-1)/2]\int \cos^{2-2}x\,dx = \cos x \sin x/2 + (1/2)\int dx = (1/2) \times (\cos x \sin x + x) + C$. Letting $n = 4$, we have $\int \cos^4 x\,dx = (\cos^3 x \sin x)/4 + (3/4)\int \cos^2 x\,dx = (\cos^3 x \sin x)/4 + (3/4) \times (\cos x \sin x/2 + x/2) + C = (1/4)[\cos^3 x \sin x + (3/2)\cos x \sin x + 3x/2] + C$.

57. (a) By the fundamental theorem of calculus, $Q = \int (dQ/dt)dt = \int i\,dt = \int EC(\alpha^2/\omega + \omega)e^{-\alpha t}\sin(\omega t)dt$.

(b) Let $A = EC(\alpha^2/\omega + \omega)$ and let $u = e^{-\alpha t}$, then $dv = \sin(\omega t)\,dt$, $du = -\alpha e^{-\alpha t}dt$, and $v = -\cos(\omega t)/\omega$; therefore, $Q = A\int e^{-\alpha t} \times \sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \int \alpha e^{-\alpha t}\cos(\omega t)\,dt/\omega]$. Now, let $u = \alpha e^{-\alpha t}/\omega$, so $dv = \cos(\omega t)\,dt$, $du = -\alpha^2 e^{-\alpha t}/\omega$, and $v = \sin(\omega t)/\omega$; therefore, $Q = A\int e^{-\alpha t}\sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2 - \int \alpha^2 e^{-\alpha t}\sin(\omega t)\,dt/\omega^2]$. Rearrangement yields $A(1 + \alpha^2/\omega^2)\int e^{-\alpha t}\sin(\omega t)\,dt = A[-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2] + C$. Division by $1 + \alpha^2/\omega^2 = (\omega^2 + \alpha^2)/\omega^2$ gives us $Q = EC[(\alpha^2 + \omega^2)/\omega][\omega^2/(\omega^2 + \alpha^2)][-e^{-\alpha t}\cos(\omega t)/\omega - \alpha e^{-\alpha t}\sin(\omega t)/\omega^2] + C = -ECe^{-\alpha t}[\cos(\omega t) + \alpha\sin(\omega t)/\omega] + C$. $Q(0) = -EC + C = 0$, so $C = EC$. Therefore, $Q(t) = EC\{1 - e^{-\alpha t}[\cos(\omega t) + \alpha\sin(\omega t)/\omega]\}$.

61. In each case, we must consider the case $n = 0$ separately because an n will appear in the denominator. It is always the case that $b_0 = (1/\pi)\int_0^{2\pi} f(x)\cdot 0\,dx = 0$.

(a) $a_0 = (1/\pi)\int_0^{2\pi} \cos 0\,dx = (1/\pi)\int_0^{2\pi} dx = (1/\pi)x\big|_0^{2\pi} = 2$; $a_n = (1/\pi) \times \int_0^{2\pi} \cos nx\,dx = (1/\pi)(\sin nx)/n\big|_0^{2\pi} = 0$; $b_n = (1/\pi)\int_0^{2\pi} \sin nx\,dx =$

61. (a) (continued)

$(1/\pi)(-\cos nx)/n\,\Big|_0^{2\pi} = 0$. Thus, $a_0 = 2$ and all other Fourier coefficients are 0 .

(b) $a_0 = (1/\pi)\int_0^{2\pi} x\,dx = (1/\pi)(x^2/2)\Big|_0^{2\pi} = 2\pi$. a_n is determined by letting $u = x$, $dv = \cos nx\,dx$, $du = dx$, and $v = \sin nx/n$, so $a_n = (1/\pi)\int_0^{2\pi} x\cos nx\,dx = (1/\pi)(x\sin nx/n\Big|_0^{2\pi} - \int_0^{2\pi}\sin nx\,dx/n) = (1/\pi)(0 + \cos nx/n^2\Big|_0^{2\pi}) = 0$. b_n is determined by letting $u = x$, $dv = \sin nx\,dx$, $du = dx$, and $v = -\cos nx/n$, so $b_n = (1/\pi)\int_0^{2\pi} x\sin nx\,dx = (1/\pi)(-x\cos nx/n\Big|_0^{2\pi} - \int_0^{2\pi}\cos nx\,dx/n) = (1/\pi)(-2\pi/n - \sin nx/n^2\Big|_0^{2}) = (1/\pi)(-2\pi/n - 0) = -2/n$. Thus, $a_0 = 2\pi$, $b_n = -2/n$ if $n \neq 0$, and all other Fourier coefficients are 0 .

(c) $a_0 = (1/\pi)\int_0^{2\pi} x^2\,dx = (1/\pi)(x^3/3)\Big|_0^{2\pi} = 8\pi^2/3$. a_n is determined by letting $u = x^2$, $dv = \cos nx\,dx$, $du = 2x\,dx$, and $v = \sin nx/n$, so $a_n = (1/\pi)\int_0^{2\pi} x^2\cos nx\,dx = (1/\pi)(x^2\sin nx/n\Big|_0^{2\pi} - \int_0^{2\pi} 2x\sin nx\,dx/n)$. Using the result from part (b) , $a_n = (1/\pi)\times(0) - (1/\pi)(2/n)\int_0^{2\pi} x\sin nx\,dx = -(2/n)(-2/n) = 4/n^2$. b_n is determined by letting $u = x^2$, $dv = \sin nx\,dx$, $du = 2x\,dx$, and $v = -\cos nx/n$, so $b_n = (1/\pi)\int_0^{2\pi} x^2\sin nx\,dx = (1/\pi)\times(-x^2\cos nx/n\Big|_0^{2\pi} + \int_0^{2\pi} 2x\cos nx\,dx/n)$. Using the result from part (b), $b_n = (1/\pi)(-4\pi^2/n) + (1/\pi)(2/n)\int_0^{2\pi} x\cos nx\,dx = -4\pi/n$. Thus, $a_0 = 8\pi^2/3$, $a_n = 4/n^2$ if $n \neq 0$, $b_0 = 0$, and $b_n = -4\pi/n$ if $n \neq 0$.

(d) This problem requires using $\int_0^{2\pi}\sin mx\sin nx\,dx$, $\int_0^{2\pi}\cos mx\times\cos nx\,dx$, and $\int_0^{2\pi}\sin mx\cos nx\,dx$. By Exercise 50(a), $\int_0^{2\pi}\sin mx\cos nx\,dx = (n\sin mx\sin nx + m\cos mx\cos nx)/(n^2 - m^2)\Big|_0^{2\pi} = 0$ if $n^2 \neq m^2$. If $n^2 = m^2$, $\int_0^{2\pi}\sin mx\cos nx\,dx =$

61. (d) (continued)

$(-\cos 2mx)/4m\big|_0^{2\pi} = 0$. Using the product formula, $\int_0^{2\pi} \sin mx \times \sin nx\, dx = (1/2)\int_0^{2\pi} [\cos(mx - nx) - \cos(mx + nx)]\, dx = (1/2) \times [\sin(mx - nx)/(m - n) - \sin(mx + nx)/(m + n)]\big|_0^{2\pi} = 0$ provided $m \neq n$. If $m = n$, then we apply the half-angle formula: $\int_0^{2\pi} \sin^2 mx\, dx = \int_0^{2\pi} [(1 - \cos 2mx)/2]\, dx = (x - \sin 2mx/2m)/2\big|_0^{2\pi} = \pi$. Applying the product formula, we get $\int_0^{2\pi} [\cos(mx + nx) + \cos(mx - nx)]\, dx = [\sin(mx + nx)/(m + n) + \sin(mx - nx)/(m - n)]\big|_0^{2\pi} = 0$ provided $m \neq n$. If $m = n$, then the half-angle formula implies $\int_0^{2\pi} \cos^2 mx\, dx = \int_0^{2\pi} [(1 + \cos 2mx)/2]\, dx = [(x + \sin 2mx/2m)/2]\big|_0^{2\pi} = \pi$.

$a_0 = (1/\pi)\int_0^{2\pi} (\sin 2x + \sin 3x + \cos 4x)dx = (1/\pi)(-\cos 2x/2 - \cos 3x/3 + \sin 4x/4)\big|_0^{2\pi} = 0$. $a_n = (1/\pi)[\int_0^{2\pi}(\sin 2x + \sin 3x + \cos 4x)(\cos nx\, dx)] = (1/\pi)(\pi) = 1$ if $n = 4$, and $a_n = 0$ if $n \neq 4$. $b_n = (1/\pi)[\int_0^{2\pi}(\sin 2x + \sin 3x + \cos 4x)(\sin nx\, dx)] = 1$ if $n = 2$ or 3 , and $b_n = 0$ otherwise. Therefore, $a_4 = b_2 = b_3 = 1$ and all other Fourier coefficients are 0 .

SECTION QUIZ

1. Which statement is the formula for integration by parts?

(a) $\int uv\, dx = uv - v\int u\, dx$

(b) $\int y\, dx = xy - \int x\, dy$

(c) $\int u\, dv = uv - v\int du$

2. Evaluate the following integrals:

   (a) $\int x^5 \ln x \, dx$

   (b) $\int x^5 \cos(x^2) dx$

   (c) $\int x^4 e^x dx$

   (d) $\int e^{4x} \sin 4x \, dx$

   (e) $\int \sin^{-1}(3x) dx$

3. Comment on the following: Differentiation of both sides shows that $\int \ln x \, dx = x \ln x - x + C$. Thus, $\int x \ln x \, dx = x(x \ln x - x) - \int (x \ln x - x) dx$.

   (a) Is integration by parts used correctly? Explain.

   (b) Evaluate $\int x \ln x \, dx$.

4. Jumping Janet's new inheritance is a porcupine farm with an odd-shaped plot of land. A new fence is needed, so Janet decides to make the holes for the posts. In preparation to hop away from flying quills, Janet jumps along the boundary on a pogo stick. The land is bounded by $y = x^2 \cos x$ and $y = -(x - 3\pi/4)^2 + 9\pi^2/16$. If one unit equals 10 meters, how much land area is allotted to each of Jumping Janet's 25 porcupines? (Hint: Draw a graph to determine the limits of integration.)

ANSWERS TO PREREQUISITE QUIZ

1. (a) $1 + \ln x$

   (b) $15x^2 - 46x + 17$

   (c) $2x \tan x + x^2 \sec^2 x$

2. $(x - 3)/2$

ANSWERS TO SECTION QUIZ

1. b

2. (a) $x^6 \ln x/6 - x^6/36 + C$

(b) $(x^4/2)\sin(x^2) + x^2\cos(x^2) - \sin(x^2) + C$

(c) $x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C$

(d) $(e^{4x}\sin 4x - e^{4x}\cos 4x)/8 + C$

(e) $x \sin^{-1}(3x) - \sqrt{1 - 9x^2}/3 + C$

3. (a) The integration is being performed correctly; however, it is more useful to let $u = \ln x$ and $dv = x\,dx$.

(b) $x^2 \ln x/2 - x^2/4 + C$

4. $(9\pi^3/4 + 9\pi^2 - 8)$ m^2/porcupine

7.R Review Exercises for Chapter 7

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the sum rule, power rule, and the basic trigonometric rules for antidifferentiation, we have $\int(x + \sin x)dx = x^2/2 - \cos x + C$.

5. Using integration formulas for sums, exponentials, logarithms, rational powers, and trigonometric functions, we get $\int(e^x - x^2 - 1/x + \cos x)dx = e^x - x^3/3 - \ln|x| + \sin x + C$.

9. Integrate by substitution. Let $u = x^3$, then $du = 3x^2dx$ or $du/3 = x^2dx$; therefore, $\int x^2 \sin x^3\, dx = \int \sin u\, du/3 = -\cos u/3 + C = -\cos x^3/3 + C$.

13. Integrate by substitution. Let $u = x + 2$, so $du = dx$; therefore, $\int(x + 2)^5dx = \int u^5du = u^6/6 + C = (x + 2)^6/6 + C$.

17. Substitute $u = \cos 2x$, so $du = -2 \sin 2x\, dx$; therefore, $\int 2 \cos^2 2x \sin 2x\, dx = -\int u^2du = -u^3/3 + C = -\cos^3 2x + C$.

21. Factor out $1/\sqrt{4} = 1/2$ and substitute $u = t/2$, so $du = dt/2$. Thus, $\int(1/\sqrt{4 - t^2} + t^2)dt = (1/2)\int\left[dt/\sqrt{1 - (t/2)^2}\right] + \int t^2dt = (1/2)\int\left[2\, du/\sqrt{1 - u^2}\right] + t^3/3 = \sin^{-1}u + t^3/3 + C = \sin^{-1}(t/2) + t^3/3 + C$.

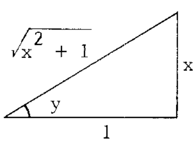
25. Integrate by parts. Let $u = x^2$, then $dv = \cos x\, dx$, $du = 2x\, dx$, and $v = \sin x$; therefore, $\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx$. Now, repeat the integration by parts with $u = 2x$, $dv = \sin x\, dx$, $du = 2dx$, and $v = -\cos x$; therefore, $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - \int 2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$.

29. Integrate by parts. Let $u = \ln 3x$, then $dv = x^2dx$, $du = 3dx/3x = dx/x$, and $v = x^3/3$; therefore, $\int x^2 \ln 3x\, dx = x^3 \ln 3x/3 - \int x^2dx/3 = x^3 \ln 3x/3 - x^3/9 + C$.

33. Integrate by parts with $u = x$, $dv = \cos 3x\,dx$, $du = dx$, and $v = \sin 3x/3$; therefore, $\int x \cos 3x\,dx = x \sin 3x/3 - \int \sin 3x\,dx/3 = x \sin 3x/3 + \cos 3x/9 + C$.

37. Substitute $w = x^2$, then $dw = 2x\,dx$; therefore, $\int x^3 e^{(x^2)} dx = \int x^2 \left(e^{(x^2)}\right)(x\,dx) = \int we^w dw/2$. Now, integrate by parts with $u = w$, so $dv = e^w dw/2$, $du = dw$, and $v = e^w/2$; therefore, $\int x^3 e^{(x^2)} dx = we^w/2 - \int e^w dw/2 = we^w/2 - e^w/2 + C = x^2 e^{(x^2)}/2 - e^{(x^2)}/2 + C$.

41. Substitute $w = \sqrt{x}$, then $dw = (1/2\sqrt{x})dx = dx/2w$; therefore, $\int e^{\sqrt{x}} dx = \int e^w(2w\,dw)$. Now, integrate by parts with $u = 2w$, so $dv = e^w dw$, $du = 2\,dw$, and $v = e^w$. Thus, $\int e^{\sqrt{x}} dx = 2we^w - \int 2e^w dw = 2we^w - 2e^w + C = e^w(2w - 2) + C = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$.

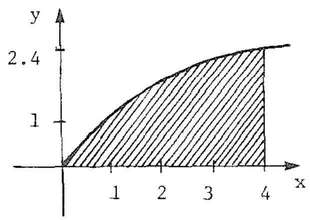
45. 

Using the formula for integrating inverse functions, we have $y = \tan^{-1}x$, and $x = \tan y$. Thus, $\int \tan^{-1}x\,dx = \tan y(\tan^{-1}x) - \int \tan y\,dy = \tan(\tan^{-1}x)(\tan^{-1}x) - \int(\sin y/\cos y)dy$. Let $u = \cos y$, so $du = -\sin y\,dy$; therefore, $\int \tan^{-1}x\,dx = x \tan^{-1}x + \int(du/u) = x \tan^{-1}x + \ln|u| + C = x \tan^{-1}x + \ln|\cos(\tan^{-1}x)| + C = x \tan^{-1}x + \ln\left(1/\sqrt{x^2+1}\right) + C = x \tan^{-1}x - \ln(1 + x^2)/2 + C$.

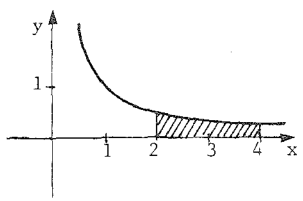
49. Let $u = x$, so $dv = \sin 5x\,dx$, $du = dx$, and $v = -\cos 5x/5$; therefore, $\int_0^{\pi/5} x \sin 5x\,dx = -x \cos 5x/5\Big|_0^{\pi/5} + \int_0^{\pi/5} \cos 5x\,dx/5 = -(\pi/5)(-1/5) + \sin 5x/25\Big|_0^{\pi/5} = \pi/25$.

53. Let $u = \tan^{-1}x$, then $dv = x\,dx$, $du = 1/(1 + x^2)$, and $v = x^2/2$. Integrating by parts and using long division, we have $\int_0^{\pi/4} x \tan^{-1}x\,dx = x^2 \tan^{-1}x/2\Big|_0^{\pi/4} - \int_0^{\pi/4}[x^2 dx/2(1 + x^2)] = (\pi^2/32)\tan^{-1}(\pi/4) - \int_0^{\pi/4}(dx/2) + \int_0^{\pi/4}[dx/2(1 + x^2)] = (\pi^2/32)\tan^{-1}(\pi/4) + (-x/2 + \tan^{-1}x/2)\Big|_0^{\pi/4} = ((\pi^2 + 16)/32)\tan^{-1}(\pi/4) - \pi/8$.

57. Integrate by parts with $u = x$, $dv = \sqrt{2x + 3}$, $du = dx$, and $v = (2x + 3)^{3/2}/3$ (using integration by substitution); therefore, $\int_0^1 x\sqrt{2x + 3}\, dx = x(2x + 3)^{3/2}/3\big|_0^1 - \int_0^1 (2x + 3)^{3/2} dx/3 = 5^{3/2}/3 - (2x + 3)^{5/2}/15\big|_0^1 = 5\sqrt{5}/3 - 25\sqrt{5}/15 + 9\sqrt{3}/15 = 3\sqrt{3}/5$.

61. 

The area under the curve of $f(x)$ on $[a,b]$ is $\int_a^b f(x)dx$. Let $u = x^2 + 9$, so $du = 2x\, dx$; therefore, $\int_0^4 \left(3x\, dx/\sqrt{x^2 + 9}\right) = \int_9^{25} (3\, du/2\sqrt{u}) = (3/2)\sqrt{u}(2)\big|_9^{25} = 3(5 - 3) = 6$.

65. 

The area under the curve of $f(x)$ on $[a,b]$ is $\int_a^b f(x)dx$. Thus, the area is $\int_2^4 (dx/x) = \ln x\big|_2^4 = \ln 4 - \ln 2 = \ln(4/2) = \ln 2$.

69. In $[0,\pi/2]$, $e^x + \cos x > -x^3 - 2x - 6$, so the area is $\int_0^{\pi/2} [(e^x + \cos x) - (-x^3 - 2x - 6)]\, dx = (e^x + \sin x + x^4/4 + x^2 + 6x)\big|_0^{\pi/2} = e^{\pi/2} - 1 + 1 + \pi^4/64 + \pi^2/4 + 3\pi = e^{\pi/2} + (\pi^4 + 16\pi^2 + 192\pi)/64 \approx 18.225$.

73. In the first method, let $u = \sin(\pi x/2)$, then $dv = \cos(\pi x)dx$, $du = (\pi/2)\cos(\pi x/2)dx$, and $v = \sin(\pi x)/\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = \sin(\pi x/2)\sin(\pi x)/\pi - \int \sin(\pi x)\cos(\pi x/2)dx/2$. Now, let $u = \cos(\pi x/2)$, so $dv = \sin(\pi x)dx/2$, $du = -(\pi/2)\sin(\pi x/2)dx$, $v = -\cos(\pi x)/2\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = \sin(\pi x/2)\sin(\pi x)/\pi + \cos(\pi x/2)\cos(\pi x)/2\pi + (1/4)\int \sin(\pi x/2)\cos(\pi x)dx$. Rearrangement yields $\int \sin(\pi x/2)\cos(\pi x)dx = 4[\sin(\pi x/2)\sin(\pi x)/\pi + \cos(\pi x/2)\cos(\pi x)/2\pi]/3 + C$.

By the second method, let $u = \cos(\pi x)$, then $dv = \sin(\pi x/2)dx$, $du = -\pi \sin(\pi x)dx$, and $v = -2\cos(\pi x/2)/\pi$; therefore, $\int \sin(\pi x/2) \times \cos(\pi x)dx = 2\cos(\pi x)\cos(\pi x/2)/\pi - \int 2\cos(\pi x/2)\sin(\pi x)dx$. Now, let

73. (continued)

$u = \sin(\pi x)$, then $dv = 2\cos(\pi x/2)dx$, $du = \pi\cos(\pi x)dx$, and $v = 4\sin(\pi x/2)/\pi$; therefore, $\int \sin(\pi x/2)\cos(\pi x)dx = -2\cos(\pi x)\cos(\pi x/2)/\pi - 4\sin(\pi x)\sin(\pi x/2)/\pi + 4\int \sin(\pi x/2)\cos(\pi x)dx$. Rearrangement yields $\int \sin(\pi x/2)\cos(\pi x)dx = [4\sin(\pi x)\sin(\pi x/2)/\pi + 2\cos(\pi x)\cos(\pi x/2)/\pi]/3 + C$.

77. (a) Let $u = \ln x$, then $du = dx/x$; therefore, $\int (\ln x/x)dx = \int u\,du = u^2/2 + C = (\ln x)^2/2 + C$.

(b) Letting $x = 3\tan u$, we have $dx = 3\sec^2 u\,du$ and $u = \tan^{-1}(x/3)$; therefore, $\int_{\sqrt{3}}^{3\sqrt{3}} \left[dx/x^2\sqrt{x^2+9}\right] = \int_{\pi/6}^{\pi/3}\left[(3\sec^2 u\,du)/9\tan^2 u\sqrt{9\tan^2 u + 9}\right] = \int_{\pi/6}^{\pi/3}[3\sec^2 u\,du/(9\tan^2 u)(3\sec u)] = (1/9)\int_{\pi/6}^{\pi/3}(\sec u\,du/\tan^2 u) = (1/9)\int_{\pi/6}^{\pi/3}(\cos u\,du/\sin^2 u)$. Let $v = \sin u$, so $dv = \cos u\,du$; therefore, $\int_{\sqrt{3}}^{3\sqrt{3}}\left(dx/x^2\sqrt{x^2+9}\right) = (1/9)\int_{1/2}^{\sqrt{3}/2}(dv/v^2) = (1/9)(-1/v)\Big|_{1/2}^{\sqrt{3}/2} = (1/9)(-2/\sqrt{3} + 2) = (2/9)(-\sqrt{3}/3 + 1)$.

81. (a) Let $u = e^{25x}$, so $dv = \cos 5x\,dx$, $du = 25e^{25x}$, and $v = \sin 5x/5$; therefore, $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/5 - 5\int(\sin 5x)e^{25x}dx$. Now, let $u = e^{25x}$, so $dv = 5\sin 5x$, $du = 25e^{25x}dx$, and $v = -\cos 5x$. Thus, $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/5 + e^{25x}\cos 5x - 25\int e^{25x}\cos 5x\,dx$. Rearrangement gives us $\int e^{25x}\cos 5x\,dx = e^{25x}\sin 5x/130 + e^{25x}\cos 5x/26$. Evaluating at the limits, 0 and t , gives us $e^{25t}\sin 5t/130 + e^{25t}\cos 5t/26 - 1/26$. Multiplying this by $100e^{-25t}$ gives us $(100/26)(\sin 5t/5 + \cos 5t - e^{-25t})$.

(b) $Q(1.01) = (100/26)[(\sin 5.05)/5 + \cos 5.05 - e^{-25.25}] \approx (3.8462) \times (-0.18871 + 0.33123 - 1.0816 \times 10^{-11}) \approx (3.8462)(0.14252) \approx 0.548$ coulomb.

85. Let $u = x$, then $dv = e^{ax}\cos(bx)\,dx$, $du = dx$, and $v = \int e^{ax}\cos(bx)\,dx$. Now, let $u = e^{ax}$, so $dv = \cos(bx)\,dx$, $du = ae^{ax}dx$, and $v = \sin(bx)/b$; therefore, $\int e^{ax}\cos(bx)\,dx = e^{ax}\sin(bx)/b - \int ae^{ax}\sin(bx)dx/b$. Repeat integration by parts with $u = ae^{ax}$, $dv = \sin(bx)\,dx/b$, $du = a^2e^{ax}dx$, and $v = -\cos(bx)/b^2$; therefore, $\int e^{ax}\cos(bx)dx = e^{ax}\sin(bx)/b + ae^{ax}\cos(bx)/b^2 - \int a^2e^{ax} \times \cos(bx)\,dx/b^2$. Rearrangement yields $\int e^{ax}\cos(bx)dx = [e^{ax}\sin(bx)/b + ae^{ax}\cos(bx)/b^2][b^2/(a^2 + b^2)] = [be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2)$.

Thus, $\int xe^{ax}\cos(bx)\,dx = x[be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) - \int [be^{ax}\sin(bx) + ae^{ax}\cos(bx)]\,dx/(a^2 + b^2)$. For $\int be^{ax}\sin(bx)dx/(a^2 + b^2)$, let $u = be^{ax}/(a^2 + b^2)$, so $dv = \sin(bx)\,dx$, $du = abe^{ax}dx/(a^2 + b^2)$, and $v = -\cos(bx)/b$. Thus, $\int be^{ax}\sin(bx)dx/(a^2 + b^2) = -e^{ax}\cos(bx)/(a^2 + b^2) + \int ae^{ax}\cos(bx)dx/(a^2 + b^2)$.

Using the result for $\int e^{ax}\cos(bx)\,dx$ in the first part of this problem, we have $\int xe^{ax}\cos(bx)\,dx = x[be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) + [e^{ax}\cos(bx)/(a^2 + b^2)]\int ae^{ax}\cos(bx)\,dx/(a^2 + b^2) - \int ae^{ax}\cos(bx)\,dx/(a^2 + b^2) = [bxe^{ax}\sin(bx) + axe^{ax}\cos(bx)]/(a^2 + b^2) + e^{ax}\cos(bx)/(a^2 + b^2) - [2a/(a^2 + b^2)][be^{ax}\sin(bx) + ae^{ax}\cos(bx)]/(a^2 + b^2) + C = xe^{ax}[b\sin(bx) + a\cos(bx)]/(a^2 + b^2) + e^{ax}[(b^2 - a^2) \times \cos(bx) - 2ab\sin(bx)]/(a^2 + b^2)^2 + C$.

TEST FOR CHAPTER 7

1. True or false:

(a) Substituting $u = -x^2$ into $\int_0^1 \exp(-x^2)dx$ yields $\int_0^{-1} e^u du$.

(b) If f and g are integrable functions and a , b , c are real numbers such that $a < b < c$, then $\int_a^b [f(x) + g(x)]dx = \int_a^c f(x)dx + \int_c^b g(x)dx + \int_c^b f(x)dx + \int_a^c g(x)dx$.

(c) For any constant $a > 0$, $\int_0^1 a^x dx = a - 1$.

(d) $\int_0^1 [dx/(1 + x^2)] = \tan^{-1}(1) - \tan^{-1}(0) = 45 - 0 = 45$.

(e) The area of the region bounded by $x = 2 - y^2$ and the y-axis is $\int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)dy$.

2. Show that $\int [dx/(x^3 - x^2)] = 1/x - \ln|x| + \ln|x - 1| + C$.

3. Substitute $x = \sin\theta$ into the integral $\int_{-1}^1 \sqrt{1 - x^2}\, dx$ and write the integral as a definite integral in terms of θ .

4. Evaluate the following by making a substitution:

(a) $\int [(x + 2)/(x^2 + 4x - 3)]dx$

(b) $\int [e^t/(1 + e^{2t})]dt$

(c) $\int [\sin x/(1 + \cos^2 x)]dx$

5. Evaluate the following integrals using integration by parts:

(a) $\int t^2 \sin t\, dt$

(b) $\int \ln(x^4)dx$

(c) $\int x^2 e^{-x} dx$

6. Let $u = 1/x$ and $dv = dx$, so integration by parts gives $\int (dx/x) = (1/x)x - \int(-1/x^2)x\, dx = 1 + \int(dx/x)$. Subtracting $\int(dx/x)$ from both sides yields $0 = 1$. Explain what went wrong.

7. Find the area under the graph of $f(x)$ on $[0,1]$ if $f(x)$ is:

(a) $x\exp(-x^2)$

(b) $x^3\exp(-x^2)$

8. Evaluate $\int\tan^{-1}(3x)dx$.

9. A particle's acceleration at time t is $\sqrt{(1-t)/(t+1)}$. If its velocity at $t = 0$ is 0 , what is the velocity as a function of t? (Hint: $\sqrt{(1-x)/(1-x)} = 1$.)

10. An oddly dressed gentleman, nicknamed Odd Ollie, came into the Odd Furniture Store. He bought a table top whose edges, according to the salesman, are given by $f(x) = x/(x+1)$, and the lines $x = 0$, $x = 1$, and $y = 0$. Each unit represents 1 meter. Odd Ollie also has a collection of "DO NOT REMOVE UNDER PENALTY OF LAW" tags, which he wants to make into a tablecloth. The average tag measures 3 cm. by 5 cm. Odd Ollie is willing to cut the tags to match the shape of his table. How many tags does Odd Ollie minimally need for his tablecloth?

ANSWERS TO CHAPTER TEST

1. (a) False; substitution yields $\int_0^{-1}(-e^u/2\sqrt{-u})du$.

(b) True

(c) False; it is $(\ln a)(a-1)$.

(d) False; $\tan^{-1}(1) - \tan^{-1}(0) = \pi/4$.

(e) True

2. Differentiate the right-hand side and simplify.

3. $\int_{-\pi/2}^{\pi/2}\cos^2\theta\, d\theta$.

4. (a) $\ln\sqrt{x^2 + 4x - 3} + C$

   (b) $\tan^{-1}(e^t) + C$

   (c) $-\tan^{-1}(\cos x) + C$

5. (a) $-t^2\cos t + 2t \sin t + 2 \cos t + C$

   (b) $4x \ln x - 4x + C$

   (c) $-(x^2 + 2x + 2)e^{-x} + C$

6. The integral $\int(dx/x)$ may have different additive constants on each side.

7. (a) $1/2 - 1/2e$

   (b) $1/2 - 1/e$

8. $x \tan^{-1}(3x) + (1/6)\ln(1 + 9x^2) + C$

9. $\sin^{-1}t + \sqrt{1 - t^2} - 1$

10. 205 tags

CHAPTER 8

DIFFERENTIAL EQUATIONS

8.1 Oscillations

PREREQUISITES

1. Recall how to use the chain rule for differentiation (Section 2.2).

2. Recall how to differentiate trigonometric functions (Section 5.2).

3. Recall how to convert between cartesian and polar coordinates (Section 5.1).

4. Recall how to graph trigonometric functions (Section 5.5).

PREREQUISITE QUIZ

1. Differentiate the following expressions:

   (a) $\sin 3x^2$

   (b) $\cos(x^3 + 2)$

2. Convert the polar coordinates $(2,\pi/3)$ to cartesian coordinates.

3. Convert the cartesian coordinates $(-5/2,0)$ to polar coordinates.

4. Sketch the graph of $y = 2\cos(x/2)$.

GOALS

1. Be able to solve differential equations of the form $x'' = -\omega^2 x$.

2. Be able to convert from $A \sin \omega t + B \cos \omega t$ to $\alpha\cos(\omega t - \theta)$ and sketch the graph.

STUDY HINTS

1. Notation and definitions. In the force law $F = -kx$, k is called the spring constant. The frequency of oscillations, measured in radians per second, is $\omega = \sqrt{k/m}$.

2. Spring equation. $d^2x/dt^2 = -\omega^2 x$ describes simple harmonic motion, and it is known as the spring equation. If $x = x_0$ and $dx/dt = v_0$ at $t = 0$, the solution is $x = x_0 \cos \omega t + (v_0/\omega)\sin \omega t$. You should memorize this solution.

3. Uniqueness. Just be aware that the spring equation has a unique solution if the initial values for position and velocity are given. The proof of uniqueness is usually not needed for solving problems.

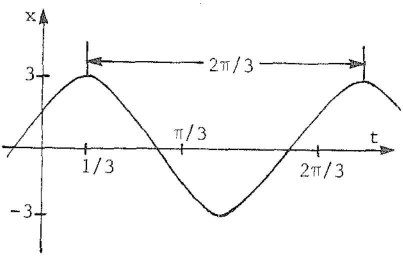
4. Graphing $x = \alpha \cos(\omega t - \theta)$. In order to graph $x = A \cos \omega t + B \sin \omega t$, convert it to $x = \alpha \cos(\omega t - \theta)$ by using the relationships $\alpha = \sqrt{A^2 + B^2}$ and $\theta = \tan^{-1}(B/A)$. Knowing this, we also know that the maximum occurs and equals α when $\omega t - \theta = 0$, i.e., $t = \theta/\omega$, which is the phase shift. The graph repeats itself when $\omega t - \theta = 2\pi$; therefore, the period is $2\pi/\omega$. Finally, the amplitude is α.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We want to show that $f(t + 2\pi/3) = f(t)$. $f(t + 2\pi/3) = \cos 3(t + 2\pi/3) = \cos(3t + 2\pi)$. Since the cosine function has period 2π, $\cos(3t + 2\pi) = \cos(3t) = f(t)$.

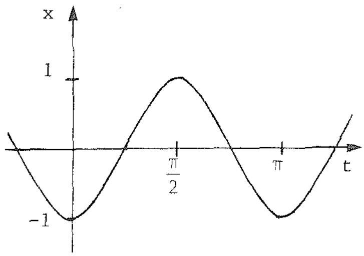
5. The solution of $d^2x/dt^2 + \omega^2 x = 0$ is $x = x_0 \cos \omega t + (v_0/\omega)\sin \omega t$ where $x_0 = x(0)$ and $v_0 = dx/dt$ at $t = 0$. Here, $\omega = 3$, so the solution is $x = 1 \cos 3t + (-2/3)\sin 3t = \cos 3t - 2 \sin 3t/3$.

9.



A function of the form $x = \alpha\cos(\omega t - \theta)$ has period $2\pi/\omega$, amplitude α, and phase shift θ/ω. In this case, the period is $2\pi/3$; the amplitude is 3; the phase shift is $1/3$.

13.



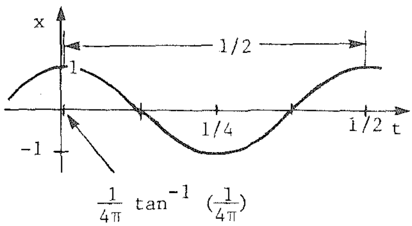
The solution of $d^2x/dt^2 + \omega^2 x = 0$ is $x = x_0\cos\omega t + (v_0/\omega)\sin\omega t$ where $x_0 = x(0)$ and $v_0 = dx/dt$ at $t = 0$. Here, $\omega = 2$, so the solution is $x = -\cos 2t$.

17. Here, $\omega = 2$, $y_0 = 1$, and $v_0 = 3$, so the solution is $y = y_0\cos\omega t + (v_0/\omega)\sin\omega t = \cos 2t + (3/2)\sin 2t$.

21. The frequency $\omega/2\pi = 2$ is given, so $\omega = 4\pi$.

(a) By definition, $\omega = \sqrt{k/m}$, so $4\pi = \sqrt{k/1}$; therefore, $k = 16\pi^2$ is the spring constant.

(b)



Here $x_0 = v_0 = 1$, so the solution of the spring equation becomes $x = \cos 4\pi t + (1/4\pi)\sin 4\pi t$. To graph it, rewrite $(1, 1/4\pi)$ in polar coordinates: $\left(\sqrt{1 + 1/16\pi^2}, \tan^{-1}(1/4\pi)\right)$, or approximately $(1, 0.08)$, so $x \approx \cos(4\pi t - 0.08)$. This is used to sketch the graph.

25. (a) The equation of motion is $m(d^2x/dt^2) = f(x)$, so this specific equation is $27(d^2x/dt^2) = -3x + 2x^3$.

(b) The linearized equation of motion is $m(d^2x/dt^2) = f'(x_0)(x - x_0)$ where $f(x_0) = 0$. We are told that $f(0) = 0$. $f'(x) = -3 + 6x^2$ implies $f'(0) = -3$, so the linearized equation is $27(d^2x/dt^2) = -3x$.

(c) The period of linearized oscillations is $2\pi/\sqrt{-f'(x_0)/m}$, which is $2\pi/\sqrt{3/27} = 6\pi$.

29. Since $f(t)$ satisfies the spring equation, $f(t)$ has the form $A\cos\omega t + B\sin\omega t$. Therefore, $f\circ g = f(g(t)) = A\cos\omega(at + b) + B\sin\omega(at + b)$. Now if $f\circ g$ satisfies the spring equation, $d^2(f\circ g)/dt^2 + \omega^2(f\circ g) = 0$. Now, $d(f\circ g)/dt = -a\omega A\sin\omega(at + b) + a\omega B\cos\omega(at + b)$, so $d^2(f\circ g)/dt^2 = -a^2\omega^2A\cos\omega(at + b) - a^2\omega^2B \times \sin\omega(at + b)$. Hence $d^2(f\circ g)/dt^2 + \omega^2(f\circ g) = 0 = -a^2\omega^2[A\cos\omega(at + b) + B\sin\omega(at + b)] + \omega^2[A\cos\omega(at + b) + B\sin\omega(at + b)] = \omega^2[A\cos\omega(at + b) + B\sin\omega(at + b)](1 - a^2)$. Since this is zero, one of the above factors must be zero. Since $\omega \neq 0$ and $(f\circ g)$ can be nonzero, $1 - a^2$ must be 0 . Therefore $a = \pm 1$.

Notice that the choice of b does not affect the differentiation of $f\circ g$. Thus, there is no restriction on b .

33. (a) By the definition of antiderivatives $V'(x_0) = -f(x_0) = 0$ and $V''(x_0) = -f'(x_0) > 0$. By the second derivative test, x_0 is a local minimum of V .

(b) By the chain rule, we have $dE/dt = (1/2)m(2)(dx/dt)(d^2x/dt^2) + V'(x)(dx/dt) = m(dx/dt)(dx^2/dt^2) + (-f(x))(dx/dt) = (dx/dt)[m(dx^2/dt^2) - f(x)] = (dx/dt)(0) = 0$.

33. (c) By part (b), we know that E is constant. If dx/dt and $x - x_0$ are sufficiently small at $t = 0$, then $E = (1/2)m(dx/dt)^2 + V(x)$ is also small. Since E must remain a small constant, the two terms which comprise E must both be small also. Therefore, dx/dt and $x - x_0$ will both remain small.

SECTION QUIZ

1. Solve the differential equation $d^2y/dx^2 + \pi^2 y = 0$, assuming that $dy/dx = 3$ when $x = -2$ and $y = 2$ when $x = 1$.

2. Let $x = 8\sin(t/4) + 3\cos(t/4)$.

   (a) What differential equation of the form $x''(t) = -kx(t)$ does x solve? Remember to specify the initial position and velocity.

   (b) Convert the given equation into the form $x = \alpha\cos(\omega t - \theta)$.

   (c) Sketch the graph of x.

   (d) What is the spring constant if the mass is 2 ?

3. As a money-saving concept, the latest lines of economical cars are not equipped with shock absorbers. After going over a pothole, it has been determined that these 800,000 gram cars have a spring constant of 400,000. Initially, after going over a pothole, the car is 10 cm. from equilibrium and bouncing with a velocity of 5 cm/sec.

   (a) Write an equation of the form $x = A\cos\omega t + B\sin\omega t$ describing the car's vertical motion.

   (b) Sketch the graph of the solution.

ANSWERS TO PREREQUISITE QUIZ

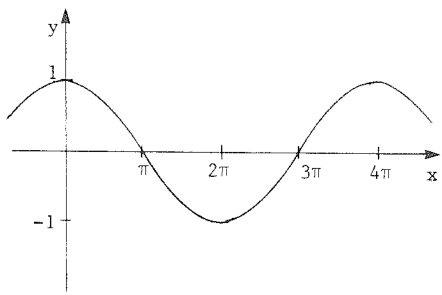
1. (a) $6x \cos 3x^2$

   (b) $-3x^2 \sin(x^3 + 2)$

2. $(1,\sqrt{3})$

3. $(-5/2,0)$ or $(5/2,\pi)$

4.



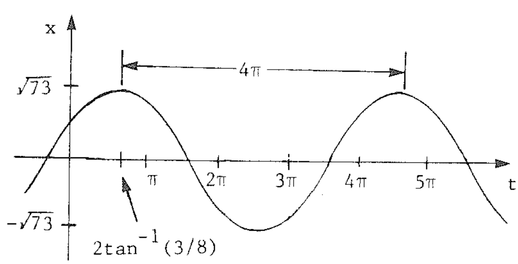
ANSWERS TO SECTION QUIZ

1. $y = -2 \cos \pi x + (3/\pi)\sin \pi x$

2. (a) $x'' = -x/16$; $x_0 = 3$, $v_0 = 2$

   (b) $x = \sqrt{73} \cos(t/2 - \tan^{-1}(8/3))$

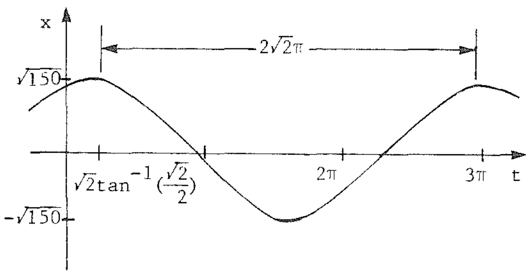
   (c)



   (d) 1/2

3. (a) $x = 10 \cos(\sqrt{2}t/2) + 5\sqrt{2} \sin(\sqrt{2}t/2)$

3. (b)



8.2 Growth and Decay

PREREQUISITES

1. Recall how to differentiate exponential functions (Section 6.3).

PREREQUISITE QUIZ

1. Differentiate the following:

   (a) $\exp(3t)$

   (b) $\exp(x^2)$

   (c) $\exp(-2t + 4)$

GOALS

1. Be able to solve differential equations of the form $f'(t) = \gamma f(t)$.
2. Be able to understand the concept of half-life and compute it.

STUDY HINTS

1. Decay and growth. $f'(t) = \gamma f(t)$ is known as the decay or growth equation depending on whether γ is negative or positive. Regardless of sign, the solution should be memorized; it is $f(t) = f(0)\exp(\gamma t)$.
2. Half-life. Rather than memorizing the formula for half-life, it is easiest to apply the definition. By definition, $f(t_{1/2}) = (1/2)f(0)$; from the solution, we also have $f(t_{1/2}) = f(0)\exp(-\kappa t_{1/2})$. Therefore $1/2 = \exp(-\kappa t_{1/2})$. See how this is used in Example 5, Method 2.
3. Doubling time. This is similar to the half-life concept except that the rate constant is positive rather than negative.

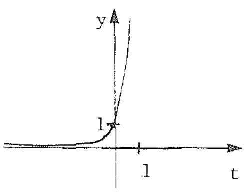
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This exercise is similar to Example 1. We have $\gamma = -0.11$, so a differential equation for the iron's temperature is $dT/dt = -0.11(T - 20)$.

5. The solution of $f'(t) = \gamma f(t)$ is $f(t) = f(0)\exp(\gamma t)$. Here, $f(0) = 2$ and $\gamma = -3$, so $f(t) = 2\exp(-3t)$.

9. The solution of $f'(t) = \gamma f(t)$ is $f(t) = f(0)\exp(\gamma t)$. Here $\gamma = 8$, so $y(t) = y(0)\exp(8t)$. Substituting $y = 2$ and $t = 1$ gives $2 = y(0)e^8$, so $y(0) = 2e^{-8}$. Therefore, $y(t) = 2e^{-8}(e^{8t}) = 2\exp(8t - 8)$.

13. As in Example 3, we let $f(t) = T - 20$, so $f'(t) = -(0.11)f(t)$ and $f(0) = 210 - 20 = 190$. Therefore, $f(t) = 190\exp(-0.11t)$. We want to find out when $T = 100$ or $f(t) = 80$. Thus $80 = 190\exp(-0.11t)$ or $-0.11t = \ln(8/19)$, i.e., $t = [\ln(8/19)]/(-0.11) \approx 7.86$ minutes.

17. 

Rearrangement yields $f' = 3f$, so $\gamma = 3$ and $f(0) = 1$; therefore, the solution is $f(t) = e^{3t}$.

21. Since $\gamma = 3$ is positive, x must be an increasing function. See Fig. 8.2.1.

25. If the decay law is $f'(t) = -\kappa f(t)$, then the half-life equation is $t_{1/2} = (1/\kappa)\ln 2$. Here, $\kappa = 0.000021$, so $t_{1/2} = (1/0.000021)\ln 2 \approx$ 33,000 years.

29. Rearrangement yields $\kappa = \ln 2/t_{1/2} = \ln 2/450000000$. After 90% decays, 0.10 gram is left, so $0.10 = e^{-(\ln 2/450000000)t}$, i.e., $\ln(0.10) = -(\ln 2/450000000)t$, i.e., $t = -\ln(0.10)450000000/\ln 2 \approx 1.5 \times 10^9$ years.

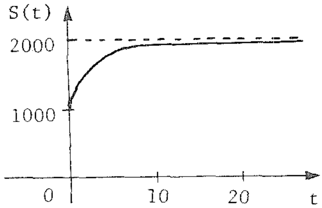
33. $f(t)/f(0) = 2 = e^{\gamma(10)}$ implies $\gamma = \ln 2/10$. Then, $3000 = 100e^{(\ln 2/10)t}$ implies $\ln 30 = (\ln 2/10)t$, i.e., $t = 10\ln 30/\ln 2 \approx 49$ minutes.

37. As in Example 8, if $P(t) = 4P_0$, then $4 = e^{0.075t}$, so $\ln 4 = 0.075t$, i.e., $t = \ln 4/(0.075) \approx 18.5$ years.

41. (a) Differentiation gives $S'(t) = 300e^{-0.3t}$.

(b) $\lim_{t\to\infty} S(t) = 2000 - 1000 \cdot 0 = 2000$, so 2000 books will be sold eventually. This is the difference between a constant and a natural decay.

(c)



45. (a) Upon differentiation of the solution, we get $da/dt = \int_1^t [h(s)/s^2]ds + th(t)/t^2 + C$; therefore, $t(da/dt) = t\int_1^t [h(s)/s^2]ds + tC + t^2h(t)/t^2 = a + h$.

(b) Here, $h(s) = e^{-1/s}$, so the solution is $t\int_1^t (e^{-1/s}/s^2)ds + tC$. Substituting $u = -1/s$ yields $du = ds/s^2$, so $\int_1^t (e^{-1/s}/s^2)ds = \int_{-1}^{-1/t} e^u du = e^u\Big|_{-1}^{-1/t} = \exp(-1/t) - 1/e$. Also, $a(1) = 1\int_1^1 [h(s)/s^2]ds + 1(C) = 0 + C = 1$, so $C = 1$. Thus, the solution is $a(t) = t/e^{-1/t} - t/e + t$.

SECTION QUIZ

1. Element Z decays exponentially. 80% remains after one month. What is the half-life of element Z ?

2. A population obeys exponential growth. In 25 years, the population increases from 500 to 750 . How long would it take for the same population to increase from 3 million to 4 million?
3. Solve the differential equation $y' = y/2$, assuming $y(3) = 1$.
4. Solve and sketch the solution of $5y' = -y$ if $y(0) = 3$.
5. Suppose an object shrinks exponentially. Initially, it weighs 15 grams. Exactly one hour later, it weighs 14 grams. When will it weigh 4 grams?
6. A stranger is trying to decide what to eat at a Mexican restaurant. He asks the waiter, "What's this - Jalepeño peppers?" The waiter tells him, "Try it. You'll like it." After one bite, the stranger's tongue feels like it's at 60° C . If ice water requires 90 seconds to bring his tongue temperature back down to 38° C (normal tongue temperature is 37° C), what is the decay constant? Assume the tongue obeys Newton's law of cooling.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $3\exp(3t)$

   (b) $2x \exp(x^2)$

   (c) $-2\exp(-2t + 4)$

ANSWERS TO SECTION QUIZ

1. 3.11 months
2. 17.74 years
3. $y = e^{(x-3)/2}$
4. $y = 3e^{-x/5}$
5. 19.16 hours
6. $-0.0348 \text{ sec}^{-1} = -2.090 \text{ min}^{-1}$

8.3 The Hyperbolic Functions

PREREQUISITES

1. Recall how to differentiate exponential functions (Section 6.3).

2. Recall how to apply the chain rule for differentiating (Section 2.2).

PREREQUISITE QUIZ

1. Differentiate the following with respect to t :

   (a) $e^t + e^{-t}$

   (b) $(e^t + e^{-t})/(e^t - e^{-t})$

   (c) $\exp(t^2 + t)$

GOALS

1. Be able to define the hyperbolic trigonometric functions as a function of exponentials.

2. Be able to differentiate and integrate expressions involving hyperbolic functions.

3. Be able to solve differential equations of the form $x'' = \omega^2 x$.

STUDY HINTS

1. Definitions. You should memorize $\sinh t = (e^t - e^{-t})/2$ and $\cosh t = (e^t + e^{-t})/2$. They are the same except that $\sinh t$ has a minus sign. Remembering that $\sinh 0 = 0$ and $\cosh 0 = 1$ may help. As with $\sin$ and $\cos$, $\sinh$ is odd and $\cosh$ is even. One usually pronounces $\sinh$ as "cinch", $\cosh$ as it is written, and $\tanh$ as "tanch."

2. Other hyperbolic functions. Notice the similarities of formulas (4) with their trigonometric counterparts.

3. Derivatives of hyperbolic functions. Except for the sign of the derivative and the fact that they are hyperbolic functions, the formulas are the same as their trigonometric counterparts. Note that the "commonly" used functions sinh , cosh , and tanh have a positive sign in front of the derivative, whereas the others have a negative sign.

4. Half-angle formulas. Formulas (8) are useful for integration. They are analogous to the trigonometric half-angle formulas; note that the negative sign is associated with $\sin^2 x$ and $\sinh^2 x$.

5. Antiderivatives of hyperbolic functions. As usual, the simplest antiderivatives are determined by reversing the differentiation formulas.

6. The equation $d^2x/dt^2 = \omega^2 x$. Memorize the fact that the solution is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$, where $x = x_0$ and $dx/dt = v_0$ at $t = 0$. Alternatively, one can memorize $x = A \cosh \omega t + B \sinh \omega t$ and derive the solution by determining A and B .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrangement of formula (3) yields $\cosh^2 x = 1 + \sinh^2 x$. By definition, $\tanh^2 x + \operatorname{sech}^2 x = \sinh^2 x/\cosh^2 x + 1/\cosh^2 x = (\sinh^2 x + 1)/\cosh^2 x = \cosh^2 x/\cosh^2 x = 1$.

5. Prove the identity by the method of Example 3. $(d/dx)\cosh x = (d/dx)(e^x + e^{-x})/2 = (e^x - e^{-x})/2 = \sinh x$.

9. Using the fact that $(d/dx)\sinh x = \cosh x$ and the chain rule, we have $(d/dx)\sinh(x^3 + x^2 + 2) = (3x^2 + 2x)\cosh(x^3 + x^2 + 2)$.

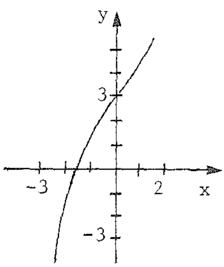
13. Using the fact that $(d/dx)\sinh x = \cosh x$ and the chain rule, we have $(d/dx)\sinh(\cos(8x)) = \cosh(\cos(8x))\cdot(d/dx)\cos(8x) = -8 \sin 8x \cosh(\cos 8x)$.

17. Since $(d/dx)\coth x = \operatorname{csch}^2 x$, the chain rule gives $(d/dx)\coth 3x = -3 \operatorname{csch}^2 3x$.

21. Since $(d/dx)\cosh x = \sinh x$ and $(d/dx)\tanh x = \operatorname{sech}^2 x$, the quotient rule gives $(d/dx)[\cosh x/(1 + \tanh x)] = [\sinh x(1 + \tanh x) - \cosh x(\operatorname{sech}^2 x)]/(1 + \tanh x)^2 = [\sinh x(1 + \tanh x) - \operatorname{sech} x]/(1 + \tanh x)^2$.

25. The solution of $d^2x/dt^2 - \omega^2 x = 0$ is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$ where $x = x_0$ and $dx/dt = v_0$ when $t = 0$. Here, $\omega = 3$, so the solution is $y = 0 \cosh 3t + (1/3)\sinh 3t = \sinh 3t/3$.

29. The solution of $d^2x/dt^2 - \omega^2 x = 0$ is $x = x_0 \cosh \omega t + (v_0/\omega)\sinh \omega t$ where $x = x_0$ and $dx/dt = v_0$ when $t = 0$. Here, $\omega = 3$, so the solution is $x = \cosh 3t + \sinh 3t/3$.

33. 

Begin with the graph of $y = \sinh x$ as shown in Fig. 8.3.3. Shift it up 3 units to obtain the graph of $y = 3 + \sinh x$.

37. Substitute $u = 3x$ to get $\int \cosh 3x\, dx = \sinh 3x/3 + C$.

41. Use the identity $\sinh^2 x = (\cosh 2x - 1)/2$ to get $\int \sinh^2 x\, dx = \int [(\cosh 2x - 1)/2]dx = \sinh 2x/4 - x/2 + C$.

45. Let $u = \cosh x$, so $du = \sinh x\, dx$, and $\int \cosh^2 x \sinh x\, dx = \int u^2 du = u^3/3 + C = \cosh^3 x/3 + C$.

49. Using the technique of implicit differentiation, we have $3(y + x\, dy/dx)\operatorname{sech}^2(3xy) + (\cosh y)dy/dx = 0$. Thus, $dy/dx = -3y \operatorname{sech}^2 3xy/(\cosh y + 3x \operatorname{sech}^2 3xy)$.

53. By the definition of $\cosh x$ and $\sinh x$, $(\cosh x + \sinh x)^n =$ $[(e^x + e^{-x}) + (e^x - e^{-x})]^n/2^n = (2e^x)^n/2^n = e^{nx}$. Also, $\cosh nx +$ $\sinh nx = (e^{nx} + e^{-nx})/2 + (e^{nx} - e^{-nx})/2 = 2e^{nx}/2 = e^{nx}$. Therefore, $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx = e^{nx}$.

SECTION QUIZ

1. Differentiate the following functions of x:
   (a) $\sinh 3x$
   (b) $\cosh x \sinh 2x$
   (c) $\tan x \sinh 2x$
   (d) $\coth x/\operatorname{csch} 2x$
   (e) $\tanh(x/2) - \operatorname{sech} x$
2. Write $\cosh(x/2)$ in terms of exponentials.
3. Solve the following differential equations:
   (a) $f''(x) = 16f(x)$; $f'(0) = 5$; $f(0) = 2$.
   (b) $f''(x) = -25f(x)$; $f'(0) = 3$; $f(0) = 1$.
   (c) $d^2y/dx^2 = 9y$; $(dy/dx)|_0 = 6$; $y(0) = 0$.
4. Perform the following integrations:
   (a) $\int x \cosh 2x\, dx$
   (b) $\int e^x \cosh x\, dx$
   (c) $\int \sinh^5 x \cosh x\, dx$
5. One day, two teen-agers decided to equip their grandfather's electric wheelchair with rocket jets. When the elderly man went for his afternoon ride down the street, the faulty rockets did not work immediately. When Grandpa had ridden 100 m. down the street, the rocket jets began firing. At that time, he was at an equilibrium position and v_0 was 1 . As he accelerated down the street, the ride became bumpier and bumpier,

5. (continued)

and his height off the seat can be described by $d^2x/dt^2 = x$, where x is the position of the chair.

(a) Solve the differential equation.

(b) Sketch the graph of the solution.

(c) How fast was Grandpa moving, i.e., find dx/dt.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $e^t - e^{-t}$

(b) $4/(e^t - e^{-t})^2$

(c) $(2t + 1)\exp(t^2 + t)$

ANSWERS TO SECTION QUIZ

1. (a) $3 \cosh 3x$

(b) $\sinh x \sinh 2x + 2 \cosh x \cosh 2x$

(c) $\sec^2 x \sinh 2x + 2 \tan x \cosh 2x$

(d) $(-\operatorname{csch}^2 x \operatorname{csch} 2x + 2 \coth x \operatorname{csch} 2x \coth 2x)/\operatorname{csch}^2 2x$

(e) $(1/2)\operatorname{sech}^2(x/2) - \operatorname{sech} x \tanh x$

2. $(e^{x/2} + e^{-x/2})/2$

3. (a) $f(x) = 2 \cosh 4x + (5/4)\sinh 4x$

(b) $f(x) = \cos 5x + (3/5)\sin 5x$

(c) $y = 2 \sinh 3x$

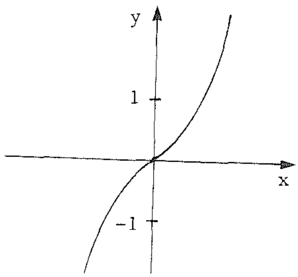
4. (a) $x \sinh 2x/2 - \cosh 2x/4 + C$

(b) $e^{2x}/4 + x/2 + C$

(c) $\sinh^6 x/6 + C$

5. (a) $x = \sinh t$

5. (b)



(c) $dx/dt = \cosh t$

8.4 The Inverse Hyperbolic Functions

PREREQUISITES

1. Recall the definition of an inverse function and how to differentiate them (Section 5.3).

2. Recall how to differentiate the hyperbolic trigonometric functions (Section 8.3).

PREREQUISITE QUIZ

1. (a) On what intervals is $y = -x^2 + 4$ invertible?

   (b) Find a decreasing function which is an inverse of $y = -x^2 + 4$.

2. Differentiate the following with respect to x :

   (a) $\cosh(x^2 + 1)$

   (b) $\sinh x$

3. Let $f(x) = x^5 + x^3 + x$. Find $(f^{-1})'(2)$.

GOALS

1. Be able to differentiate and integrate expressions involving the inverse trigonometric hyperbolic functions.

STUDY HINTS

1. Inverse hyperbolic derivatives. Study the method of deriving the derivative of $\sinh^{-1}x$ and note its similarity to that for $\sin^{-1}x$ (Chapter 5). All of the others are derived analogously. The only difference between $(d/dx)\sinh^{-1}x$ and $(d/dx)\cosh^{-1}x$ *is that the first has a plus sign in the* denominator and the second has a minus sign. A similar statement may be said for the denominator of the derivatives of $\text{sech}^{-1}x$ and $\text{csch}^{-1}x$. The derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ look the same, but $\tanh^{-1}x$ is defined for $|x| < 1$; $\coth^{-1}$ for $|x| > 1$. Think about the graph to determine if the sign is correct.

2. Inverse hyperbolic logarithmic expressions. Again, study how to derive the formula $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$. The others are derived by a similar method. The formulas are normally not worth memorizing. Learn to derive them (for exams), or in many cases (for homework), one can simply look them up. Consult your instructor to see what is expected on exams.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since $(d/dx)\cosh^{-1}x = 1/\sqrt{x^2 - 1}$, the chain rule gives $(d/dx)\cosh^{-1}(x^2 + 2) = [1/\sqrt{(x^2 + 2)^2 - 1}]2x = 2x/\sqrt{x^4 + 4x^2 + 3}$.

5. Since $(d/dx)\tan^{-1}x = 1/(1 - x^2)$, the chain rule and product rule give $(d/dx)x\tan^{-1}(x^2 - 1) = \tan^{-1}(x^2 - 1) + 2x^2/[1 - (x^2 - 1)^2] =$ $\tan^{-1}(x^2 - 1) - 2x^2/(x^4 - 2x^2) = \tan^{-1}(x^2 - 1) + 2/(2 - x^2)$.

9. Since $(d/dx)\sinh^{-1}x = 1/\sqrt{x^2 + 1}$, the chain rule gives $(d/dx)\exp(1 + \sinh^{-1}x) = [\exp(1 + \sinh^{-1}x)]/\sqrt{x^2 + 1}$.

13. $\tanh^{-1}x = (1/2)\ln[(1 + x)/(1 - x)]$, so $\tanh^{-1}(0.5) = (1/2)\ln(1.5/0.5) =$ $\ln 3/2 \approx 0.55$.

17. Let $y = \cosh x = (e^x + e^{-x})/2$. Multiply through by $2e^x$ and rearrange to get $(e^x)^2 - 2ye^x + 1 = 0$. By the quadratic formula, $e^x = (2y \pm \sqrt{4y^2 - 4})/2$. Since $e^x > 0$, we take the positive square root to get $e^x = y + \sqrt{y^2 - 1}$. Thus, $x = \cosh^{-1}y = \ln(y + \sqrt{y^2 - 1})$. Change the variables to get the desired result.

21. Differentiate $\tanh^{-1}x = (1/2)\ln[(1 + x)/(1 - x)]$. By the chain rule, we get $(1/2)[(1 - x)/(1 + x)]\cdot[(1 - x) + (1 + x)]/(1 - x)^2 =$ $1/(1 + x)(1 - x) = 1/(1 - x^2)$.

25. By the chain rule, we get $\left[1/(x + \sqrt{x^2 - 1})\right] \cdot [1 + (1/2)(x^2 - 1)^{-1/2} \times (2x)] = \left[1/(x + \sqrt{x^2 - 1})\right]\left[1 + x/\sqrt{x^2 - 1}\right] = \left[1/(x + \sqrt{x^2 - 1})\right]\left[(\sqrt{x^2 - 1} + x)/\sqrt{x^2 - 1}\right] = 1/\sqrt{x^2 - 1}$. Differentiation yields the integrand, so the formula is verified.

29. Substitute $u = 2x$, so $du/2 = dx$. Therefore, $\int [dx/(1 - 4x^2)] = (1/2)\int [du/(1 - u^2)] = (1/4)\ln|(1 + u)/(1 - u)| + C = (1/4)\ln|(1 + 2x)/(1 - 2x)| + C$.

33. Substitute $u = \sin x$, so $du = \cos x\, dx$; therefore, $\int (\cos x/\sqrt{\sin^2 x + 1})dx = \int (du/\sqrt{u^2 + 1}) = \sinh^{-1} u + C = \sinh^{-1}(\sin x) + C = \ln(\sin x + \sqrt{\sin^2 x + 1}) + C$.

37. By definition, $\cosh^{-1}(\sqrt{x^2 + 1}) = \ln(\sqrt{x^2 + 1} + \sqrt{x^2}) = \ln(\sqrt{x^2 + 1} + x)$ if $x \geqslant 0$ or $\ln(\sqrt{x^2 + 1} - x)$ if $x < 0$. Now, if $x > 0$, differentiation yields $f'(x) = 1/\sqrt{x^2 + 1}$. If $x < 0$, we get $f'(x) = -1/\sqrt{x^2 + 1}$. $f'(x)$ is not continuous at $x = 0$ because $\lim_{x\to 0-} f'(x) = -1$ and $\lim_{x\to 0+} f'(x) = 1$. Thus, $\cosh^{-1}(\sqrt{x^2 + 1})$ is not even once differentiable for all x .

SECTION QUIZ

1. Differentiate the following functions:

   (a) $\sinh^{-1}(x/2)$

   (b) $\cosh^{-1}x \operatorname{sech}^{-1}(2x)$, $x < -1$

   (c) $\coth^{-1}x \tan^{-1}(x^2)$, $|x| > 1$

2. Perform the following integrations:

   (a) $\int (x/\sqrt{x^4 + 1})dx$

   (b) $\int [(1 + x)/(1 - x^2)]\,dx$

   (c) $\int [dx/(x^2 - 1)]$

   (d) $\int \sinh^{-1}x\, dx$

3. A highway patrolwoman had just stopped a driver with alcoholic breath. The driver explained that he was an astrophysicist and had spotted a flying pink elephant travelling along the path $y = \cosh^{-1}x$.

(a) Use logarithms to determine the elephant's position at $x = 2$.

(b) Sketch the elephant's flight path.

(c) If the y-axis points north and the x-axis points east, estimate the elephant's flight direction at $x = 2$. (Choose from N , S , E , W , NW , SW , NE , and SE).

ANSWERS TO PREREQUISITE QUIZ

1. (a) $(-\infty,0)$ and $(0,\infty)$

(b) $f^{-1}(x) = -\sqrt{-x + 4}$

2. (a) $2x \sinh(x^2 + 1)$

(b) $\cosh x$

3. 1/93

ANSWERS TO SECTION QUIZ

1. (a) $1/\sqrt{x^2 + 4}$

(b) $\operatorname{sech}^{-1}2x/\sqrt{x^2 - 1} + \cosh^{-1}x/x\sqrt{1 + x^2}$

(c) $\tan^{-1}(x^2)/(1 - x^2) + 2x \coth^{-1}x/(1 + x^2)$

2. (a) $\sinh^{-1}(x^2)/2 + C = \ln(x^2 + \sqrt{x^4 + 1})/2 + C$

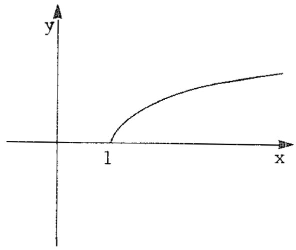
(b) $-\ln|1 - x| + C$ [Factor the denominator and simplify the integrand first].

(c) $-\tanh^{-1}x + C$ if $|x| < 1$; $-\coth^{-1}x + C$ if $|x| > 1$; or $(1/2)\ln|(1 - x)/(1 + x)| + C$ if $x \neq 1$.

(d) $x \sinh^{-1}x - \sqrt{x^2 + 1} + C = x \ln(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + C$

3. (a) $\ln(2 + \sqrt{3}) \approx 1.32$

(b)



(c) NE (The actual angle is $\pi/6$.)

8.5 Separable Differential Equations

PREREQUISITES

1. Recall the geometric interpretation of the linear approximation (Section 1.6).
2. Recall the rules of integration (Chapter 7).

PREREQUISITE QUIZ

1. Evaluate the following:
   (a) $\int [dx/(1 - x)]$
   (b) $\int (e^y + y^4 - \sin y)dy$
2. (a) Write a formula which approximates $f(x_0 + \Delta x)$.
   (b) How is the linear approximation related to the tangent line of a graph?
   (c) Use the linear approximation to estimate $(0.97)^2$.

GOALS

1. Be able to recognize separable differential equations and solve them.
2. Be able to use direction fields or the Euler method for estimating solutions of differential equations.

STUDY HINTS

1. Separable equations. These must be first-order differential equations. They are separable in the sense that everything involving y can be placed on one side of the equals sign, and everything involving x can be placed on the other side.

2. Method of solution. Begin by thinking of dx and dy as separate entities. Multiply and divide to put all of the terms involving y and all of the terms involving x on separate sides of the equation. Then integrate both sides. See Examples 1, 2, and 3.

3. Electric circuits. The result of Example 4 should not be memorized; however, the result is useful for doing the exercises.

4. Transforming an equation. Example 6 demonstrates an interesting method of solution. By letting a new variable represent a derivative, the original equation became a separable first-order equation.

5. Direction fields. These fields are represented by tiny lines which are the tangent lines to the solution curves. If $y(x)$ is a solution of $dy/dx = f(x,y)$, then the slope of the graph at the point (x,y) is $dy/dx = f(x,y)$. Line segments through (x,y) with slope $f(x,y)$ are used to depict the direction field. By drawing a curve which follows the direction of these lines, one can sketch a solution without knowing an explicit formula. See Figures 8.5.7 and 8.5.8.

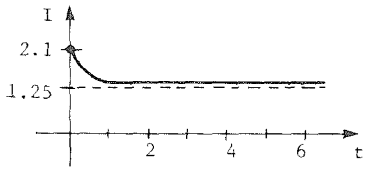
6. Euler method. This method is based upon the concept of the linear approximation (Section 1.6). By starting at a point and moving along the tangent line for a short distance, one should remain near the curve. You may find it more efficient to understand the concept and derive the formula, rather than memorizing it.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrange $dy/dx = \cos x$ to get $dy = \cos x\, dx$. Integrating both sides yields $\int dy = \int \cos x\, dx$, so $y = \sin x + C$. The initial condition $y(0) = 1$ implies $C = 1$, so $y = \sin x + 1$.

5. Rearrange the equation to get $dy/y = dx/x$. Integration yields $\ln|y| = \ln|x| + C$. Exponentiation gives $y = e^C x$. Substituting $y(1) = 2$ gives $e^C = -2$, so $y = -2x$ is the solution.

9. Multiply through by $dx/(1 + y)$ to get $dy/(1 + y) = dx/(1 + x)$. Integrating both sides yields $\ln(1 + y) = \ln(1 + x) + C$. Substituting $y = 1$ and $x = 0$ gives $\ln 2 = \ln 1 + C = C$. Thus, $\ln(1 + y) = \ln(1 + x) + \ln 2$, or upon exponentiating, $1 + y = 2(1 + x)$, i.e., $y = 2x + 1$.

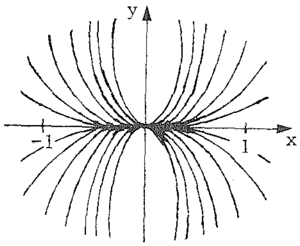
13. 

This is Example 4 with $L = 3$, $R = 8$, $E = 10$, and $I_0 = 2.1$. Thus, the solution is $I = 10/8 + (2.1 - 10/8)\exp(-8t/3) = 1.25 + 0.85\exp(-8t/3)$. At $t = 0$, $I = 2.1$; then the graph drops exponentially toward $E/R = 1.25$.

17. $dy/dt = by - rxy = b(s/c) - r(b/r)(s/c) = bs/c - bs/c = 0$ and $dx/dt = -sx + cxy = -s(b/r) + c(b/r)(s/c) = -sb/r + sb/r = 0$. Since both x and y are constant, this is the equilibrium point which also solves the predator-prey equations.

21. Substituting into $y = (T_0/mg)[\cosh(mgx/T_0) - 1] + h$ yields $y = (T_0/9.8)[\cosh (9.8/T_0)x - 1]$. Now, $y'(x) = (T_0/9.8)(9.8/T_0) \times [\sinh (9.8/T_0)x] = \sinh(9.8/T_0)x$. As T_0 gets large, $9.8/T_0$ approaches 0 and $\sinh x = 0$. Thus, the solution is not only straight, but also constant as T_0 gets large.

25. (a)



(b) Differentiation gives the differential equation, $dy/dx = 3cx^2$.

(c) The slopes must be negative reciprocals for the curves to be orthogonal, so $dy/dx = -1/3cx^2$. Separating variables gives $dy = -dx/3cx^2$ and integration yields $y = 1/3cx + C$.

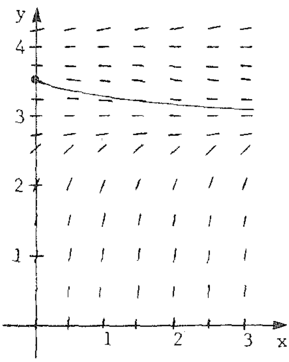
29. Starting with $(x_0,y_0) = (0,1)$, we want to find $y_{10} = y(1)$. We use the recursive formula $y_n = (y_{n-1} - x_{n-1}^2)(0.1) + y_{n-1}$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	1.1	1.209	1.3259	1.44949	1.578439	1.711283

x	0.7	0.8	0.9	1.0
y	1.846411	1.982052	2.116258	2.246883

The ten-step Euler procedure gives us $y(1) \approx 2.2469$.

33.



This exercise is analogous to Example 10. We have $dy/dx = (y - 4)(y - 3)$. As shown by the direction field, $\lim_{x\to\infty} y(x)$ for $y(0) = 3.5$ is 3 .

37. Differentiate to get $f''(x) = 2y(dy/dx)e^x + e^x y^2 + 4y^5 + 4x \cdot 5y^4(dy/dx)$. Differentiate again, so $f'''(x) = e^x[2y(dy/dx) + y^2 + 2(dy/dx)^2 + 2y(d^2y/dx^2) + 2y(dy/dx)] + 20y^4(dy/dx) + 20y^4(dy/dx) + 20x[4y^3(dy/dx) + y^4(d^2y/dx^2)]$. Substitute $y(0) = 1$ into the given equation giving $dy/dx = 1$. Substitute these into the equation for $f''(x)$ to get $f''(0) = 2 + 1 + 4 = 7$. Substitute these into the equation for $f'''(x)$ to get $f'''(0) = 2 + 1 + 2 + 14 + 2 + 20 + 20 = 61$.

SECTION QUIZ

1. (a) Solve the differential equation $dy/dx = (x + 4)(y - 2)/(x + 7)$, given $y(0) = 1$.

   (b) Find the interval in x for which the solution is valid.

2. (a) Solve $dy/dx = y/2$, assuming $y(0) = 1$.

   (b) Plot the solution to part (a) for $x = x_i/10$, where $x_i = 0$, ... , 10 . Then, draw a smooth curve through the plotted points.

   (c) Use Euler's method to approximate $y(1)$ for $dy/dx = y/2$, $y(0) = 1$. Use ten steps.

   (d) Plot the eleven points obtained from part (c) onto the same graph as in (b) and connect the points with straight line segments.

3. Solve the differential equation $y'' = 1 + (y')^2$, assuming $(dy/dx)|_0 = 0$ and $y_0 = 2$.

4. Which of the following equations are separable?

   (a) $dy/dx = x^3y - y \ln x + y$

   (b) $dy/dx = x^2 - xy^2$

   (c) $(y/(x - 1))dy/dx = x(y^2 - y)$

   (d) $(x + y)dy/dx = xy$

5. At the annual witches' convention, new advances in brewology (the science of brewing magic potions) were discussed. One of the newest potions prevents broom thefts. Depending on the location (x,y), broom thieves will be swept off their feet and taken for a ride. The ride is designed to take the path given by $dy/dx = (x^2 + x + 1)/y$. Determine the path $y(x)$ if $y(1) = 1$.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-\ln|1 - x| + C$

   (b) $e^y + y^5/5 + \cos y + C$

2. (a) $f(x_0) + f'(x_0)\Delta x$

   (b) The linear approximation and the equation of a tangent line have the same formula.

   (c) 0.94

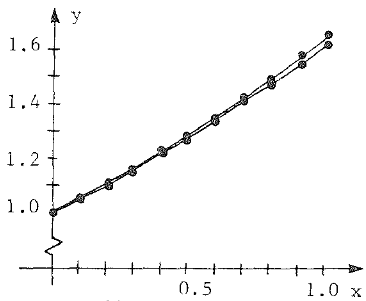
ANSWERS TO SECTION QUIZ

1. (a) $y = -343e^x/(x + 7)^3 + 2$

   (b) $x \geqslant -7$

2. (a) $y = e^{x/2}$

   (b),(d)



   (c) $y(1) \approx 1.6289$

3. $y = -\ln|\cos x| + 2$

4. a and c

5. $y^2/2 = x^3/3 + x^2/2 + x - 4/3$

8.6 Linear First-Order Equations

PREREQUISITES

1. Recall basic rules of differentiation, especially the product and chain rules (Chapters 1 and 2).
2. Recall basic methods of integration, especially exponential functions and substitution (Chapter 7).

PREREQUISITE QUIZ

1. Differentiate the following:
   (a) $\exp(-\sin x)$
   (b) $t\exp(t^2)$
2. Evaluate the following integrals:
   (a) $\int t\exp(t^2)dt$
   (b) $\int (x + 2)\exp(2x^2 + 8x)dx$
   (c) $\int te^t dt$

GOALS

1. Be able to solve linear first-order differential equations.

STUDY HINTS

1. Linearity defined. Differential equations are linear in the sense that, in the usual usage, dy/dx and y appear only to the first power.
2. Method of solution. Equation (6) gives the solution of $dy/dx = P(x)y + Q(x)$. You should not memorize the solution. Instead, learn the method of solution, which is summarized in the box in the middle of p. 409. As always, practice is the best way to learn.

3. Choice of method. Always look for the simplest solution. For instance, Example 1 is separable. You get another (simpler) solution by noting that the right-hand side is $x(y + 1)$. (However, note that the equations in Example 2 are not separable.)

4. Integrating factor defined. These are multiplicative factors which transform an expression into one which can be integrated. $\exp(-\int P(x)dx)$ is the integrating factor which is discussed in this section.

5. Applications. You should understand how the results in Examples 3-6 were derived. There is no need to memorize the results; however, the results may be useful for solving the exercises.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Here, $P(x) = 1/(1 - x)$ and $Q(x) = 2/(1 - x) + 3$. $-\int P(x)dx = -\ln|1 - x|$, dropping the integration constant. Thus we have $[1/(1 - x)][dy/dx - y/(1 - x)] = [1/(1 - x)][2/(1 - x) + 3] = (d/dx)[y/(1 - x)] = 2(1 - x)^{-2} + 3/(1 - x)$. Integration gives $y/(1 - x) = 2/(1 - x) - 3\ln|1 - x| + C$ or $y = 2 + (-3\ln|1 - x| + C)(1 - x)$.

5. $P(x) = \cos x$ and $Q(x) = 2\cos x$. $-\int P(x)dx = -\sin x$, so $\exp(-\sin x)(dy/dx - y\cos x) = 2\exp(-\sin x)\cos x$. Letting $u = -\sin x$, we get $y\exp(-\sin x) = -2\exp(-\sin x) + C$ or $y = -2 + C\exp(\sin x)$. When $y = 0$, $x = 0$, so $0 = -2 + C$ or $C = 2$. Thus, the solution is $y = -2 + 2\exp(\sin x)$.

9. The equation becomes $dI/dt = -RI/L + (E_0\sin\omega t + E_1)/L$. $P(t)$ is still the same, so $\exp(tR/L)(dI/dt + RI/L) = \exp(tR/L)(E_0\sin\omega t + E_1)/L = (d/dt)\exp(tR/L)\cdot I$. Using formula 82 of the integration table on the inside back cover of the text, we integrate to get $I\exp(tR/L) = (E_0/L)[\exp(tR/L)/((R/L)^2 + \omega^2)](R\sin\omega t/L - \omega\cos\omega t) + (E_1/L)(L/R)\exp(tR/L) +$

9. (continued)

C . Multiplying by $\exp(-tR/L)$ gives $I = (E_0/L)(R \sin \omega t/L - \omega\cos \omega t)/((R/L)^2 + \omega^2) + E_1/R + C \exp(-tR/L)$. (See Example 3 for more details.)

13. From the solution to Example 4, $y = (1 - \exp(-2.67 \times 10^{-7}t))(2.51 \times 10^6)$. We want to find t so that $y(t) = 0.9(2.51 \times 10^6)$, i.e., $0.9 = 1 - \exp(-2.67 \times 10^{-7}t)$ or $0.1 = \exp(-2.67 \times 10^{-7}t)$. Therefore, $-2.67 \times 10^{-7}t = \ln(0.1)$ and $t \approx 8.63 \times 10^6$ seconds $\approx$ 100 days.

17. Use the result of Example 5: $v = (mg/\gamma)[1 - e^{-\gamma t/m}]$. The distance travelled is $\int v\, dt = mgt/\gamma + ge^{-\gamma t/m} + C$. Since no distance has been travelled at $t = 0$, we have $c = -g$. As $t \to \infty$, v approaches $mg/\gamma = 64$, so $m/\gamma = 64/g$. Now, we want to know when $v = 0.9(64) = 57.6 = 64(1 - e^{-gt/64})$, or $\ln(1 - 57.6/64) = -gt/64$, i.e., $t = (-64/g)\ln(1 - 57.6/64) \approx 15.0$ seconds. At this time, the person has fallen 951 meters.

21. The acceleration is $dv/dt = [F(M_0 - rt) - (-r)Ft]/(M_0 - rt)^2 - [(gM_0 - grt)(M_0 - rt) - (-r)(gM_0t - grt^2/2)]/(M_0 - rt)^2 = FM_0/(M_0 - rt)^2 - [g(M_0 - rt)^2 + grt(M_0 - rt/2)]/(M_0 - rt)^2$. Substituting $M_1 = M_0 - rt$, we get $FM_0/M_1^2 - [gM_1^2 + grt(2M_0 - rt)/2]/M_1^2 = FM_0/M_1^2 - [2gM_1^2 + g(M_0 - M_1)(M_0 + M_0 - rt)]/2M_1^2 = FM_0/M_1^2 - [2gM_1^2 + g(M_0^2 - M_1^2)]/2M_1^2 = FM_0/M_1^2 - g(M_0^2 - M_1^2)/2M_1^2$.

25. Any solution y of $y' = P(x)y + Q(x)$ must be of the form $y = \exp(\int P(x)dx)\{\int [Q(x)\exp(-\int P(x)dx)dx] + C\}$. Since $y(0) = y_0$, we can show that C is unique: $y_0 = \exp(\int P(x)dx)|_{x=0}\{\int [Q(x) \times \exp(-\int P(x)dx)dx]|_{x=0} + C\}$. Therefore, $C = y_0\exp(-\int P(x)dx)|_{x=0} - \int [Q(x)\exp(-\int P(x)dx)dx]|_{x=0}$. Now, the right side of this equation is

25. (continued)

a constant, determined by operations on y_0, $P(x)$, and $Q(x)$, and evaluated for $x = 0$. Hence C is a constant, not a function, and there is exactly one y. [Alternatively, if y_1 and y_2 are two solutions, look at the equation for $w = y_1 - y_2$.]

29. (a) The equation is $(d/dt)(Mv) = F - Mg - \gamma v$, or $(d/dt)[(M_0 - rt)v] = F - (M_0 - rt)g - \gamma v = (M_0 - rt)dv/dt - rv$. Rearrangement yields $dv/dt = rv/(M_0 - rt) - \gamma v/(M_0 - rt) + F/(M_0 - rt) - g$. $P(x) = (r - \gamma)/(M_0 - rt)$, so $-\int P(x)dx = \ln(M_0 - rt) - (\gamma/r)\ln(M_0 - rt)$ and $\exp(-\int P(x)dx) = (M_0 - rt)^{1-\gamma/r}$. Therefore, $(M_0 - rt)^{1-\gamma/r}[dv/dt - (r - \gamma)v/(M_0 - rt)] = F(M_0 - rt)^{-\gamma/r} - g(M_0 - rt)^{1-\gamma/r} = (d/dt)[v(M_0 - rt)^{1-\gamma/r}]$. Integrate to get $v(M_0 - rt)^{1-\gamma/r} = -F(M_0 - rt)^{1-\gamma/r}/(r - \gamma) + g(M_0 - rt)^{2-\gamma/r}/(2r - \gamma) + C$ or $v = F/(\gamma - r) - g(M_0 - rt)/(\gamma - 2r) + C/(M_0 - rt)^{1-\gamma/r}$. $v = 0$ when $t = 0$, so $0 = F/(\gamma - r) - gM_0/(\gamma - 2r) + C/M_0^{1-\gamma/r}$ or $C = [gM_0/(\gamma - 2r) - F/(\gamma - r)]M_0^{1-\gamma/r}$. Thus, the solution is $v = F/(\gamma - r) - g(M_0 - rt)/(\gamma - 2r) + \{[gM_0/(\gamma - 2r) - F/(\gamma - r)]M_0^{1-\gamma/r}\}(M_0 - rt)^{\gamma/r-1}$.

(b) At burnout, $M_0 - rt = M_1$, so $v = F/(\gamma - r) - gM_1/(\gamma - 2r) + [gM_0/(\gamma - 2r) - F/(\gamma - r)](M_0/M_1)^{1-\gamma/r}$.

SECTION QUIZ

1. Solve $x(dy/dx) + 2y = x^2 - x + 1$ if $y(1) = 1$.

2. Solve $y'(x) - y = e^{3x}$ if $y(0) = 1$.

3. Find $x(y)$ if $dy/dx = 1/(x + y)$. [Hint: What is the relationship between dy/dx and dx/dy?].

4. A stuntwoman is going down Niagara Falls inside a barrel. Due to resistance from the water, her velocity can be described by $m(dv/dt) = mg - 0.7v$, where m is her mass (50 kg) and g is $9.8 m/s^2$, the acceleration due to gravity.

   (a) If $v = 0$ when $t = 0$, find the velocity function.

   (b) When does her speed become 25 m/s (freeway driving speed)?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $-(\cos x)\exp(-\sin x)$

   (b) $(1 + 2t^2)\exp(t^2)$

2. (a) $\exp(t^2)/2 + C$

   (b) $\exp(2x^2 + 8x)/4 + C$

   (c) $te^t - e^t + C$

ANSWERS TO SECTION QUIZ

1. $y = x^2/4 - x/3 + 1/2 + 7/12x^2$

2. $y = (e^{3x} + e^x)/2$

3. $x = -y - 1 + Ce^y$

4. (a) $v = (mg/0.7)[1 - \exp(-0.7t/m)]$

   (b) 2.60 seconds

8.R Review Exercises for Chapter 8

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Rearrangement gives us $dy/y = 3\,dt$. Then, integrating gives $\ln y = 3t + C$, so for $k = e^C$, $y = ke^{3t}$. Substitute $y(0) = 1$, giving $1 = ke^0 = k$. So the final solution is $y = e^{3t}$.

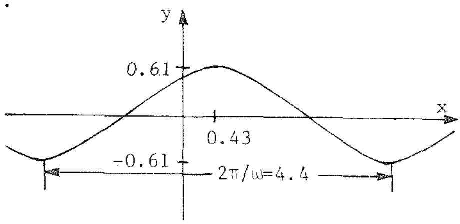
5. With $P(t) = 3$ and $Q(t) = 1$, note that $\int P(t)dt = 3t$. Subtract $3t$ and multiply by e^{-3t}, giving $e^{-3t}(dy/dt - 3y) = e^{-3t} = (d/dt)(ye^{-3t})$. Integrate, so $ye^{-3t} = (-1/3)e^{-3t} + C$, and $y = (-1/3) + Ce^{3t}$. Substitute $y(0) = 1$ to get $1 = (-1/3) + C$, so $C = 4/3$ and the solution is $y = (4e^{3t} - 1)/3$.

9. This is a case of natural growth with $\gamma = 4$, so $f(x) = Ce^{4x}$. Substitute $f(0) = 1$ to get $C = 1$. Therefore, $f(x) = e^{4x}$.

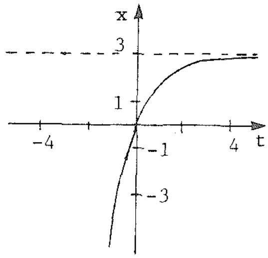
13. This is a case of simple harmonic motion with $\omega = 1$, $x_0 = 1$, and v_0 is unknown. The solution is $x(t) = \cos t + v_0 \sin t$. Substituting $x(\pi/4) = 0$ gives $0 = \sqrt{2}/2 + v_0\sqrt{2}/2$, so $v_0 = -1$. Therefore, the solution is $x(t) = \cos t - \sin t$.

17. $dy/dx = e^{x+y} = e^x e^y$, so rearrangement gives $dy/e^y = e^x dx$. Integration yields $-e^{-y} = e^x + C$. Substituting $y(0) = 1$ gives $-1/e = 1 + C$, so $C = 1 - 1/e$. Thus, $e^{-y} = 1/e - 1 - e^x$ or $y = -\ln|1/e - 1 - e^x|$.

21. Separate variables to get $dy/(y + 1) = dt/(1 - t)$ and integrate to get $\ln(y + 1) = \ln(1 - t) + C$. $y(0) = 0$ implies $C = 0$, so $y + 1 = 1 - t$ or $y = -t$.

25. 

This is simple harmonic motion with $\omega = \sqrt{2}$, $y_0 = 1/2$, and $v_0 = 1/2$, so the solution is $y(x) = (1/2)\cos(\sqrt{2}x) + (1/2\sqrt{2}) \times \sin(\sqrt{2}x)$. Change $(1/2, 1/2\sqrt{2})$ to polar coordinates: $(\sqrt{3/8}, \tan^{-1}(1/\sqrt{2}))$ so $y(x) = \sqrt{3/8}\cos(\sqrt{2}x - \tan^{-1}(1/\sqrt{2}))$. To plot the graph, $y(x) \approx 0.612\cos(1.41x - 0.196\pi)$.

29. 

This is a first-order equation with $P(t) = -1$ and $Q(t) = 3$. The integrating factor is e^t . Thus, $e^t(dx/dt + x) = 3e^t = (d/dt)xe^t$. Integration gives $xe^t = 3e^t + C$. $x(0) = 0$ implies $0 = 3 + C$ or $C = -3$, so $x = 3 - 3e^{-t}$. Since $\lim_{t\to\infty} e^{-t} = 0$, we have $\lim_{t\to\infty} x(t) = 3$.

33. Using the hint, $dw/dx + w = x$. With $P(x) = -1$ and $Q(x) = x$, note that $\int P(x)dx = -x$. Multiply by e^x , giving $e^x(dw/dx + w) = xe^x = (d/dx)(we^x)$. Integrate to get $we^x = xe^x - e^x + C$. Divide by e^x , yielding $w = x - 1 + Ce^{-x}$. Substitute $y'(0) = 1 = w(0)$ to get $1 = -1 + C$, so $C = 2$. Therefore $w = x - 1 + 2e^{-x} = dy/dx$. Integrate again to get $y = x^2/2 - x - 2e^{-x} + D$. Substitute $y(0) = 0$ to get $0 = -2 + D$, meaning $D = 2$. Then the solution is $y = x^2/2 - x - 2e^{-x} + 2$.

37. Since $(d/dx)\sinh x = \cosh x$, the chain rule gives $(d/dx)\sinh(3x^2) = 6x\cosh(3x^2)$.

41. Using the product rule with $(d/dx)\sinh^{-1}x = 1/\sqrt{x^2+1}$ and $(d/dx)\cosh x = \sinh x$, we get $(d/dx)[(\sinh^{-1}x)(\cosh 3x)] = \cosh 3x/\sqrt{x^2+1} + 3\sinh 3x \sinh^{-1}x$.

45. Substitute $u = \sinh x$, so $du = \cosh x\,dx$; therefore, the integral becomes $\int[du/(1+u^2)] = \tan^{-1}u + C = \tan^{-1}(\sinh x) + C$.

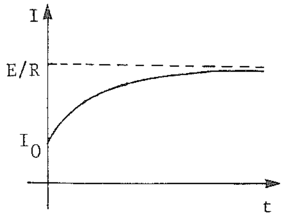
49. Integrate by parts with $u = x$ and $dv = \sinh x\,dx$, so $du = dx$ and $v = \cosh x$. Therefore, $\int x \sinh x\,dx = x\cosh x - \int\cosh x\,dx = x\cosh x - \sinh x + C$.

53. (a) Here, $m = 10$ and $\sqrt{k/m} = 8$, so $k/10 = 64$, i.e., the spring constant is $k = 640$.

(b) The force is $m(d^2x/dt^2)$. $dx/dt = 80\cos(8t)$ and $d^2x/dt^2 = -640\sin(8t)$, so the force is $-6400\sin(8t)$. At $t = \pi/16$, the force is $-6400\sin(\pi/2) = -6400$ newtons.

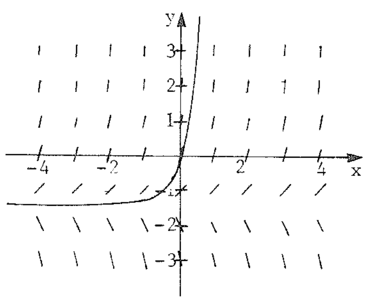
57. This is natural growth and it obeys $f(t) = f(0)e^{kt}$. In this case, $f(0) = 100{,}000$ and $f(10) = 200{,}000$, so $2 = e^{k(10)}$ or $k = \ln 2/10$. We want to determine t for $f(t) = 10$ million $= 100{,}000e^{(\ln 2/10)t}$, i.e., $\ln 100 = (\ln 2/10)t$ or $t = 10\ln 100/\ln 2 \approx 66.4$ years.

61. Let $x(t)$ be the temperature above $18\,^\circ C$ in $^\circ C$. Then $dx/dt = kx$ for some constant k. Separate variables to get $dx/x = k\,dt$. Integration yields $\ln x = kt + C$. Exponentiate to get $x = e^{kt}e^C$. Let $D = e^C$, so $x = De^{kt}$. Now $x(0) = 82$ and $x(8) = 62$, so substitute each of these, giving $82 = D$ and $62 = 82\,e^{8k}$. Therefore $31/41 = e^{8k}$, so $\ln(31/41) = 8k$, and thus $(1/8)\ln(31/41) = k$. Therefore, $x = 82\exp[(1/8)\ln(31/41)t]$. At $50\,^\circ C$, $x = 32$, so $32/82 = \exp[(1/8)\ln(31/41)t]$. Take logs to get $\ln(16/41) = (1/8)\ln(31/41)t$, or $t = 8\ln(16/41)/\ln(31/41) \approx 27$ minutes.

65. 

By Example 4 of Section 8.5, the solution is $I = E/R + (I_0 - E/R)e^{-Rt/L}$. If $I_0 < E/R$, then the graph begins below E/R and increases, rather than decreases, toward E/R .

69. Let $x(t)$ be the number of gallons of antifreeze in the radiator at t minutes. Note that $x(0) = 4/3$. Now, $dx/dt = -(\text{flow out}) = -(1/2)x/4 = -x/8$. This is exponential decay, so $x(t) = (4/3)e^{-t/8}$. When the mixture is 95% fresh water, $x(t) = (0.05)(4) = 1/5$, so solve $1/5 = (4/3)e^{-t/8}$. Multiply by $3/4$ and take logs, so $\ln(3/20) = -t/8$, or $t = 8\ln(20/3) \approx 15.2$ minutes. If you drain the radiator and then add fresh water, you will spend $[4/(1/2)]2 = 16$ minutes. Therefore, draining the radiator first is no faster.

73. 

$P(x) = 3$, so the integrating factor is e^{-3x} ; therefore, $e^{-3x}(y' - 3y) = 4e^{-3x} = (d/dx)ye^{-3x}$. Integration gives $ye^{-3x} = -4e^{-3x}/3 + C$ or $y = -4/3 + Ce^{3x}$.

77. Starting with $(x_0, y_0) = (0,1)$, we want $y_{10} = y(1)$ and $y_{20} = y(1)$. For y_{10} , we use the formula $y_n = y_{n-1}(0.1) + y_{n-1} = (1.1)y_{n-1}$. $y_0 = 1$; $y_1 = 1(1.1)$; $y_2 = 1(1.1)(1.1)$; $y_3 = 1(1.1)(1.1)(1.1)$; ... ; $y_{10} = 1(1.1)^{10} = 2.5937$. For the twenty-step method, $y_n = y_{n-1}(0.05) + y_{n-1} = (1.05)y_{n-1}$, which implies $y_{20} = 1(1.05)^{20} = 2.6533$. For the exact solution, $dy/dx = y$ implies $\int(dy/y) = \int dx$, so $\ln|y| = x + C$. $y(0) = 1$ implies $C = 0$; therefore $y = e^x$, and $y(1) = e \approx 2.718282$

77. (continued)

The ten-step method gives us an error of 4.58%, while the twenty-step method has a 2.39% error.

81. (a) Differentiate w to get $w' = y'$. Multiply w by a to get $aw = ay + b = y'$. Therefore $w' = aw$, a case of natural growth, so $w(t) = Ce^{at} = y(t) + b/a$. Thus, $y(t) = Ce^{at} - b/a$.

(b) From part (a), $dw/dx = aw$. Separate variables to get $dw/w = a\,dt$, and integrate to get $\ln w = at + k$. Exponentiation gives $w = e^k e^{at} = Ce^{at}$ for $C = e^k$. Substitute $w = y + b/a$ and subtract b/a to get $y = Ce^{at} - b/a$.

(c) With $P(t) = a$ and $Q(t) = b$, note that $\int P(t)\,dt = at$. Subtract ay and multiply by e^{-at} to get $e^{-at}(y' - ay) = e^{-at} = (d/dt)(ye^{-at})$. Integrate, giving $ye^{-at} = (-b/a)e^{-at} + C$. Multiply by e^{at} to get $y = -b/a + Ce^{-at}$. All three answers are the same.

85. (a) In other words, find $y = f(x)$ such that $f(x) = \sqrt{1 + [f'(x)]^2}$, i.e., $y^2 = 1 + (y')^2$. Now, if y is constant, y' becomes zero and we are left with $y^2 = 1$ or $y = \pm 1$. The solution $y = -1$ is not valid since the integrand on the right is positive. Thus, $y = 1$ is one solution.

If y is not constant, then rearrangement yields $(y')^2 = y^2 - 1$ or the differential equation $y' = \sqrt{y^2 - 1} = dy/dx$. Separating variables yields $dy/\sqrt{y^2 - 1} = dx$. Integration gives us $\cosh^{-1} y = x + C$ or $y = \cosh(x + C)$.

(b) We recognize $\int_a^b f(x)dx$ as the area under $f(x)$ on $[a,b]$. In Chapter 10, we will derive $\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ for the length of $f(x)$ on $[a,b]$. Thus the formula equates area and arc length.

TEST FOR CHAPTER 8

1. True or false:

   (a) If y is a function of x, then $y' + y = 0$ can be solved by integrating to get $y + y^2/2 = C$.

   (b) The domain of $\cosh^{-1}(x^2 + 1)$ is all real x.

   (c) The most general solution to the differential equation $y' = -ay$, where a is constant and $y(0) = 2$, is $y = 2e^{-ax}$.

   (d) As long as $y(x_0)$ is specified for some constant x_0, $y'' - y = 0$ has a unique solution for $y(x)$.

   (e) For all x, $\cosh^2 x = 1 + \sinh^2 x$.

2. Solve the following differential equations with the given conditions:

   (a) $d^2x/dt^2 + 9x = 0$, $x(0) = 1$, $x'(0) = 1$

   (b) $d^2x/dt^2 - 9x = 0$, $x(0) = 1$, $x'(0) = 1$

   (c) $d^2x/dt^2 + 9t = 0$, $x(0) = 2$, $x'(1) = 2$

3. (a) Find a solution of the form $y = A\cos(\omega t - \theta)$ for $4y'' = -y$, assuming $y(0) = 1$ and $y'(0) = 4$.

   (b) Sketch the graph of y.

4. Find the solution of $dx/dt + x = \sin t + 2e^{-t}$ if $x(0) = 1$.

5. Solve the differential equation $dy/dx + \sin^2 y = 1$, assuming $y(0) = 0$.

6. Solve the following differential equations with the given initial conditions:

   (a) $dy/dx = -2x$; $y(0) = 1$

   (b) $f'(x) + x^2 f(x) = 0$; $f(0) = A$, a constant

7. Evaluate the following:

   (a) $(d/dt)\sqrt{\cosh 5t}$

   (b) $\int \tanh(x/3)\, dx$

   (c) $\int \left[(2x + 5)/\sqrt{1 + x^2}\right] dx$

   (d) $(d/dy) \tanh^{-1}(e^{-y+2})$

8. (a) Find an approximate solution for $y(2)$ if $y' = x^2 + 2y$ and $y(0) = 0$ by using a 10-step Euler method.

   (b) Compare the answer in (a) with the exact solution.

9. An electric circuit is governed by the equation $C(dV/dt) + V/R = I_0\cos(\omega t)$ where C, R, I_0, and ω are constants. Find $V(t)$ satisfying $V(0) = 0$.

10. Scientific investigators have recently concluded that stupid question asking obeys the law of exponential decay. At the age of five, the average person's stupid questioning peaks, and then decays exponentially.

    (a) Suppose a young boy asked an average of 1 stupid question daily when he was five. He is now thirteen and asks an average of 0.65 stupid questions daily. Write a formula for $q(t)$, the average number of stupid questions asked daily, in terms of t, the person's age.

    (b) How many years does it take for this person's stupid question asking to decrease by 50% ?

ANSWERS TO CHAPTER TEST

1. (a) False; integration needs to be done with respect to x, not y.

   (b) True

   (c) True; it is the <u>only</u> solution.

   (d) False; $y'(x_1)$ must also be specified for some x_1.

   (e) True

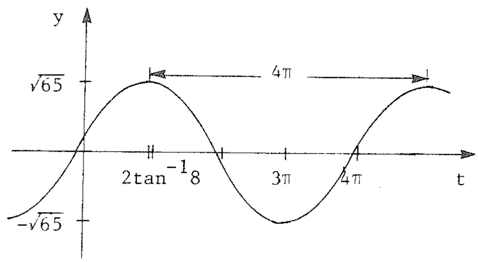
2. (a) $x = \cos 3t + (1/3) \sin 3t$

(b) $x = \cosh 3t + (1/3) \sinh 3t$

(c) $x = -3t^3/2 + 13t/2 + 2$

3. (a) $y = \sqrt{65}\cos(t/2 - \tan^{-1}8)$

(b)



4. $x = e^{-t}/2 + 2te^{-t} + (\sin t - \cos t)/2$

5. $y = \tan^{-1}x$

6. (a) $y = e^{-2x}$

(b) $f(x) = A \exp(-x^3/3)$

7. (a) $5 \sinh 5t/2\sqrt{\cosh 5t}$

(b) $3 \ln [\cosh(x/3)] + C$

(c) $2\sqrt{1 + x^2} + 5 \sinh^{-1}x + C$

(d) $e^{-y+2}/(e^{-2y+4} - 1)$

8. (a) $y(2) \approx 5.378$

(b) $y(x) = -x^2/2 - x/2 - 1/4 + e^{2x}/4$, so $y(2) = 10.3995$

9. $V = [I_0R / (R^2C^2\omega^2 + 1)] [\cos \omega t + (RC\omega)\sin \omega t - \exp(-t/RC)]$

10. (a) $q(t) = \exp[-0.0538(t - 5)]$

(b) 12.9 years

CHAPTER 9

APPLICATIONS OF INTEGRATION

9.1 Volumes by the Slice Method

PREREQUISITES

1. Recall how to derive the integration formula for area by using the infinitesimal argument (Section 4.6).
2. Recall the various methods of integration (Chapter 7).

PREREQUISITE QUIZ

1. The area under the graph of a positive function $f(x)$ is $\int_a^b f(x)\,dx$. Sketch a typical graph of $f(x)$ and use it to explain the geometric meaning of $f(x)\,dx$.
2. Evaluate $\int (x + 1)^2\,dx$.
3. Evaluate $\int e^{4y}\,dy$.

GOALS

1. Be able to compute volumes by using the slice method.
2. Be able to compute volumes by using the disk method.

STUDY HINTS

1. Basic formula. All of the volume formulas are based upon $V = \int_a^b A(x)\,dx$, where $A(x)$ is a solid's cross-sectional area. Remembering this and deriving the formulas in this section will prove to be more beneficial than just memorizing.

2. Slice method. Study Examples 1, 2, and 3 to see how elementary geometry is used to compute the cross-sectional area. Note that $A(x)$ in the volume formula corresponds to $\ell(x)$ in the area formula.

3. Radius dependent upon height. Example 2 shows how similar triangles are commonly used as an aid in computing a cross-sectional area.

4. Disk method. By this method, each cross-sectional area is simply a circle whose radius is $f(x)$; therefore, $A(x) = \pi[f(x)]^2$. Learn this derivation rather than memorizing the formula. Note that the formula works even if $f(x) < 0$.

5. Washer method. The area is the difference between that of two circular regions; therefore, if $g(x) \geqslant f(x)$, then $A(x) = \pi[g(x)]^2 - \pi[f(x)]^2$. Note that this reduces to the disk method if $f(x) = 0$. Again, it is best to learn the derivation. WARNING: The integrand is not $[f(x) - g(x)]^2$.

6. Step function argument (p. 425). This is just for the theoretically inclined. Except in honors courses, most instructors will not expect their students to reproduce the argument.

7. Cavalieri's principle. Basically, it states that two volumes with equal cross-sectional areas have equal volumes. Thus, the "tilted" solids in Exercises 1-4 have the same volume as those which stand "straight up." Note that $\Sigma_{i=1}^{n} \pi r_i^2 \Delta x_i$ in the discussion is simply the disk method formula

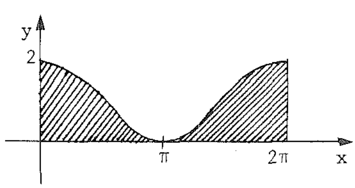
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let the x-axis be vertical as in Example 2. Then apply the slice method. Each infinitesimal slice has a circular base with area $\pi r^2 = \pi(1)^2$ and a thickness dx . Thus, $V = \int_0^3 \pi dx = 3\pi$.

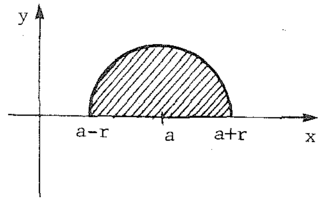
5. Since the cross-section is a square, its area is $\{[(x - 6)^2 - 1]/6\}^2$. Therefore, the volume is $V = (1/36)\int_0^5 [(x - 6)^4 - 2(x - 6)^2 + 1]\, dx = (1/36)[(x - 6)^5/5 - 2(x - 6)^3/3 + x]\big|_0^5 = 2125/54$.

9. As shown in Example 2, the volume from height x_1 to height x_2 is $(\pi r^2/h^2)(h^2x - hx^2 + x^3/3)\big|_{x_1}^{x_2}$. The volume of a short cone with height from x to h is $\pi r^2(h - x)^3/3h^2$. When we bisect the cone, we want $\pi r^2(h - x)^3/3h^2 = V/2$, where $V = \pi r^2 h/3$, the volume of the entire cone. Therefore, $(h - x)^3 = h^3/2$, i.e., $x = (1 - 1/\sqrt[3]{2})h$. Similarly, we equate $\pi r^2(h - x)^3/3h^2$ to $V/4$ and $(3/4)V$. Thus, the cuts are at $x_1 = (1 - \sqrt[3]{1/4})h$, $x_2 = (1 - \sqrt[3]{1/2})h$, and $x_3 = (1 - \sqrt[3]{3/4})h$, respectively.

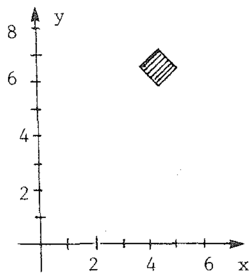
13. The volume of the entire cylinder is $\pi(5)^2(20) = 500\pi\ cm^3$ since the radius is 5 cm. The volume of the wedge is determined by $r = 5$ and $\tan\theta = 5/5 = 1$, so the removed volume is $2(5)^3/3 = 250/3\ cm^3$. Therefore, the entire solid's volume is $(500\pi - 250/3) \approx 1487.5\ cm^3$.

17. 

Here, $f(x) = \cos x + 1$, so the disk method gives the volume as $V = \pi\int_0^{2\pi}(\cos x + 1)^2 dx = \pi\int_0^{2\pi}(\cos^2 x + 2\cos x + 1)dx = \pi\int_0^{2\pi}((1 + \cos 2x)/2 + 2\cos x + 1)dx = \pi(3x/2 + \sin 2x/4 + 2\sin x)\big|_0^{2\pi} = 3\pi^2$.

21. 

The equation of the entire circle is $(x - a)^2 + y^2 = r^2$, so the equation of the semicircle is $y = \sqrt{r^2 - (x - a)^2}\, dx$. By the disk method, $V = \pi\int_{a-r}^{a+r}\left(\sqrt{r^2 - (x - a)^2}\right)^2 dx = \pi\int_{a-r}^{a+r}[r^2 - (x - a)^2]\, dx = \pi[xr^2 - (x - a)^3/3]\big|_{a-r}^{a+r} = \pi(2r^3 - 2r^3/3) = 4\pi r^3/3$

25. 

The square in Exercise 23 is centered at $(9/2,13/2)$. Since each side of the square has length 1, each vertex is $\sqrt{2}/2$ from the center; therefore, the vertices are $(9/2,13/2\pm\sqrt{2}/2)$ and $(9/2\pm\sqrt{2}/2,13/2)$. The square can be divided in two regions. One is the region between $y = x + (4 + \sqrt{2})/2$ and $y = -x + (22 - \sqrt{2})/2$ on $[(9 - \sqrt{2}/2,9/2]$ and the other lies between $y = -x + (22 + \sqrt{2})/2$ and $y = x + (4 - \sqrt{2})/2$ on $[9/2,(9 + \sqrt{2})/2]$. For revolution around the x-axis, the disk method gives the volume as $V = \pi\int_{(9-\sqrt{2})/2}^{9/2}\{[x + (4 + \sqrt{2})/2]^2 - [-x + (22 - \sqrt{2})/2]^2\}dx + \pi\int_{9/2}^{(9+\sqrt{2})/2}\{[-x + (22 + \sqrt{2})/2]^2 - [x + (4 - \sqrt{2})/2]^2\}dx = \pi\int_{(9-\sqrt{2})/2}^{9/2}(26x - 117 + 13\sqrt{2})\, dx + \pi\int_{9/2}^{(9+\sqrt{2})/2}(-26x + 117 + 13\sqrt{2})dx = \pi[(13x^2 - 117x + 13\sqrt{2}x)\big|_{(9-\sqrt{2})/2}^{9/2} + (-13x^2 + 117x + 13\sqrt{2}x)\big|_{9/2}^{(9+\sqrt{2})/2}] = \pi[13(-1 + 9\sqrt{2})/2 - 117(\sqrt{2}/2) + 13\sqrt{2}(\sqrt{2}/2) - 13(1 + 9\sqrt{2})/2 + 117(\sqrt{2}/2) + 13\sqrt{2}(\sqrt{2}/2) = 13\pi$.

29. A doughnut can be made by revolving a circle around the x-axis. To form the desired doughnut, revolve the circle centered at $(0,(R + r)/2)$ with radius $(R - r)/2$. The equation of the circle is $x^2 + (y - (R + r)/2)^2 = (R - r)^2/4$. Solving for y, we get $\pm\sqrt{(R - r)^2/4 - x^2} + (R + r)/2$. Therefore, the volume is $\pi\int_{-(R-r)/2}^{(R-r)/2}\Big[\sqrt{(R - r)^2/4 - x^2} +$

29. (continued)

$$(R + r)/2\Big]^2 dx - \pi\int_{-(R-r)/2}^{(R-r)/2}\Big[-\sqrt{(R - r)^2/4 - x^2} + (R + r)/2\Big]^2 dx =$$

$$\pi\int_{-(R-r)/2}^{(R-r)/2} 2(R + r)\sqrt{(R - r)^2/4 - x^2}\,dx = 2\pi(R + r)\int_{-(R-r)/2}^{(R-r)/2}\sqrt{(R - r)^2/4 - x^2}\,dx .$$

Note that the integral is the area of a semicircle centered at the origin with radius $(R - r)/2$, which is $\pi(R - r)^2/8$. Thus, the volume of the doughnut is $\pi^2(R + r)(R - r)^2/4$.

SECTION QUIZ

1. From elementary geometry, we know that the volume of a cone is $(1/3)\pi r^2 h$, where r is the radius of the base and h is the height of the cone. Suppose we revolve $y = x/2$ on $[0,2]$ around the x-axis, then the base radius is 1 and $h = 2$. Thus, the volume is $2\pi/3$. On the other hand, $V = \pi\int_a^b [f(x)]^2 dx = \pi[f(x)]^3/3|_a^b$. For the cone, $f(x) = x/2$, $a = 0$, and $b = 2$, so the volume is $\pi(x/2)^3/3|_0^2 = (\pi x^3/24)|_0^2 = \pi/3$. What's wrong?

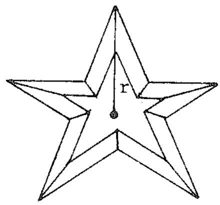
2. The line $y = x + 1$ is revolved about the x-axis to form a solid of revolution.

   (a) A vertical cut is made at $x = 9$. What is the volume of the resulting solid between $x = 0$ and $x = 9$?

   (b) Where should another cut be made parallel to the y-axis to get two equal volumes from the solid in (a)?

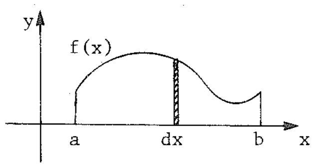
3. The curve $y = x^5$ on $[0,1]$ is revolved around the y-axis. Use the disk method to find the volume of the resulting solid.

4. The cross-sectional area of a solid at height h is given by $h|\cos h|$. The solid extends from $h = 0$ to $h = 3\pi/2$. Is enough information given to compute the volume? If not, what is missing? If yes, compute it.

5. 

Bruce, the boozing butcher has just returned from his afternoon vodka break. As he begins to trim the fat off a piece of rib roast, the alcohol begins to take effect and all he can see before his eyes are stars. Consequently, the beef is cut into the shape shown. Define the "radius" r of the cross-sectional stars to be the segment from the center to one of the outer vertices. When our boozing friend was sober, he determined the area of the star to be $5r^2/4$. If the bottom base has area $125\ \text{cm}^2$, the top base has area $20\ \text{cm}^2$, and the height is 6 cm, what is the volume of the meat that Boozer Bruce cut?

ANSWERS TO PREREQUISITE QUIZ

1. 

The shaded region is a very thin "rectangle". Its width is dx and since it is so thin, its height, $f(x)$, is "constant". Therefore, $f(x)dx$ is the area of the region.

2. $(x + 1)^3/3 + C$
3. $e^{4y}/4 + C$

ANSWERS TO SECTION QUIZ

1. In performing the integration, a factor of 2 was forgotten when the substitution $u = x/2$ was made.
2. (a) 333π

   (b) $\sqrt[3]{1001/2} - 1 \approx 6.94$
3. $5\pi/7$
4. Yes; $5\pi/2 - 1$
5. $390\ \text{cm}^3$

9.2 Volumes by the Shell Method

PREREQUISITES

1. Recall how to compute volumes by the disk method (Section 9.1).

PREREQUISITE QUIZ

1. Find the volume of the solid obtained by revolving the graph of $y = x^2 - 1$ on $[2,4]$ around the x-axis.
2. Repeat Question 1 for $y = \sec x$ on $[0,\pi/4]$.

GOALS

1. Be able to compute volumes by using the shell method.

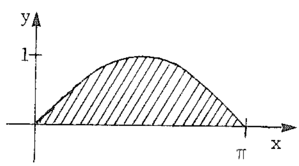
STUDY HINTS

1. Shell method. Learn the derivation. It is simply a summation of infinitesimal cylindrical volumes, dV. Each shell has radius x, so its circumference is $2\pi x$. Multiplying by the thickness dx gives the area of the base as $2\pi x\, dx$. Finally, for a region between the curves $f(x)$ and $g(x)$, the height is $f(x) - g(x)$. Thus, we sum $dV = 2\pi x\, dx(f(x) - g(x))$ to get $V = 2\pi\int_a^b x[f(x) - g(x)]dx$. If $g(x) = 0$, this reduces to $V = 2\pi\int_a^b xf(x)dx$. Note that if $f(x) < 0$, we need to use $|f(x)|$.
2. Useful trick. At this point in your studies, you do not know how to integrate $\sqrt{1 - x^2}$; however, many times it is possible to make a substitution so that even though you cannot compute an integral directly, you can determine it by computing the area under the curve using elementary geometry. See how this method is used in Example 6.

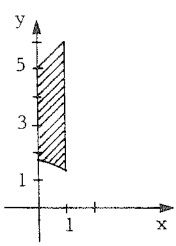
3. Step function argument. Again, as with the disk method, you will probably not need to regurgitate the step function argument unless you are enrolled in an honors course. Ask your instructor.

4. Choosing a method. In most cases, if $y = f(x)$ is revolved around the y-axis to generate a solid, the volume is best found by using the shell method. Similarly, revolution around the x-axis implies the use of the disk method. If you have a thorough understanding of these two methods, it is possible to do the same problem using either method. However, the simplest method should be used to promote efficiency.

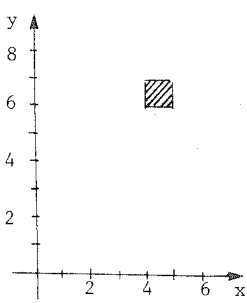
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

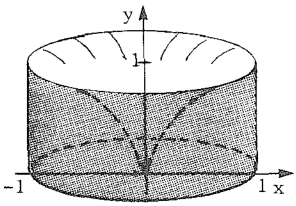
By the shell method, the volume is $2\pi\int_0^{\pi} x \sin x \, dx$. Integration by parts with $u = x$ and $v = -\cos x$ yields $2\pi(-x \cos x + \sin x)\big|_0^{\pi} = 2\pi(\pi) = 2\pi^2$.

5. 

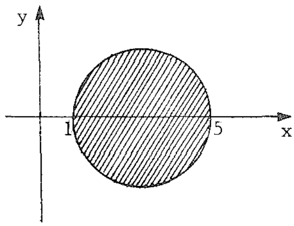
For revolution around the y-axis, the shell method gives $V = 2\pi\int_0^1 x(5 + x)dx - 2\pi\int_0^1 x\sqrt{3 - x^2}\, dx$. Let $u = 3 - x^2$, so $du/2 = -x\, dx$; therefore, $V = 2\pi(5x^2/2 + x^3/3)\big|_0^1 + 2\pi\int_3^2 \sqrt{u}\, du/2 = 17\pi/3 + 2\pi u^{3/2}/3\big|_3^2 = \pi(17 + 4\sqrt{2} - 6\sqrt{3})/3$.

9. 

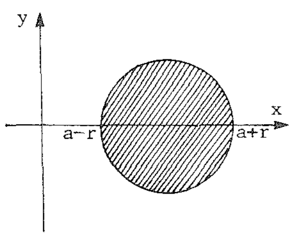
The description of the square is the region between $y = 6$ and $y = 7$ on $[4,5]$. For revolution around the y-axis, the shell method gives $V = 2\pi\int_4^5 7x\, dx - 2\pi\int_4^5 6x\, dx = 2\pi(7x^2/2 - 6x^2/2)\big|_4^5 = 2\pi(1/2)x^2\big|_4^5 = \pi(25 - 16) = 9\pi$.

13. 

The volume is $2\pi\int_0^1 x\sqrt{x}\,dx = 2\pi(2x^{5/2}/5)\big|_0^1 = 4\pi/5$. If we put the resulting volume of Example 5 of the previous section on top of the solid generated here, it will produce a cylinder of radius 1 and height 1 .

17. 

Use the method of Example 6. The volume of the top half is $2\pi\int_1^5 x[4 - (x - 3)^2]^{1/2}dx = 2\pi\int_{-2}^{2}(u + 3)\sqrt{4 - u^2}\,du = 2\pi\int_{-2}^{2}u\sqrt{4 - u}\,du + 6\pi\int_{-2}^{2}\sqrt{4 - u^2}\,du = 0 + 6\pi(2\pi) = 12\pi^2$. Thus, the total volume is $24\pi^2$.

21. (a) 

By symmetry, we only need to revolve the upper semicircle. Thus, $f(x) = [a^2-(x-b)^2]^{1/2}$, and the volume is $V = 2\pi\int_{b-a}^{b+a} x[a^2 - (x - b)^2]^{1/2}dx$. Using the method of Example 6, let $u = x - b$ to get $2\pi\int_{-a}^{a}(u + b) \times \sqrt{a^2 - u^2}\,du = 2\pi\int_{-a}^{a} u\sqrt{a^2 - u^2}\,du + 2\pi\int_{-a}^{a} b\sqrt{a^2 - u^2}\,du = 0 + 2\pi b(\pi a^2)$. Thus, the total volume is $2\pi^2a^2b$.

(b) The volume of $T_{a,b}$ is $2\pi^2a^2b$ by part (a). Substituting $a + h$ for a , we compute the volume of $T_{a+h,b}$ to be $2\pi^2(a + h)^2b = 2\pi^2b(a^2 + 2ah + h^2)$. Therefore, the difference in the two volumes is $2\pi^2b(2ah + h^2)$.

(c) As in Exercise 19, we expect the surface area to be the derivative $(d/dh)[\text{volume }(T_{a+h,b}) - \text{volume}(T_{a,b})]$, evaluated at $h = 0$. Thus, the surface area is $2\pi^2b(2a + 2h)\big|_{h=0} = 4\pi^2ab$.

SECTION QUIZ

1. The curve $y = x^4$ on $[0,1]$ is revolved around the x-axis to generate a solid of revolution. Use the shell method to compute the volume. Compute the same volume using the disk method.

2. The region between the curves $y = x$ and $y = x^2$ on $[0,2]$ is revolved around the y-axis. What is the volume of the resulting solid?

3. (a) The line $y = x + 5$ on $[2,3]$ is revolved around the line $x = 1$. What is the volume of the solid generated by revolving the region between $y = x + 5$ and the x-axis?

   (b) Find a general formula for the volume generated by revolving the region under $y = f(x)$ on $[a,b]$ about the line $x = A$. Assume $A < a < b$, and $f(x) \geqslant 0$ for x in $[a,b]$.

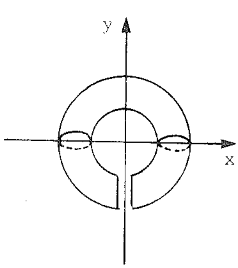
4. Suppose $f(x) = \cos x$ on $[0,\pi]$. The region between $f(x)$ and the x-axis is revolved around the y-axis. Which of the following is true?

   (a) The volume can not be computed because $f(x) < 0$ for some x in $[0,\pi]$.

   (b) The volume can be computed; it is negative.

   (c) The volume can be computed; it is positive.

   (d) The volume is $2\pi^2 - 4\pi$ because the volume on $[0,\pi]$ is $\pi^2 - 2\pi$ and symmetry can be applied.

5. 

As a practical joke, you give your hungry little cousin a stale doughnut. She breaks her front teeth trying to bite into it. Angrily, she throws the doughnut at you. Seeing her front teeth still stuck in it, you realize its value as a conversation piece. In order to stand the doughnut up, you drill a 1-centimeter diameter hole along the y-axis as shown.

5. (a) The inner radius of the doughnut is 2 centimeters. Its outer radius is 4 centimeters. What was the original volume of the doughnut? (*Hint:* See Example 6.)

   (b) Suppose you revolve the line $x = 1$ between $y = -2$ and $y = -4$ around the y-axis and you subtract the resulting volume from the doughnut's original volume. Neglecting the volume of the teeth, is your answer an approximation or is it exact for the volume of the doughnut with the drilled hole? Explain. (*Hint:* Take a coffee break and buy yourself a doughnut.)

   (c) What answer did you get for the volume by using the method described in part (b)?

ANSWERS TO PREREQUISITE QUIZ

1. $(163.07)\pi$
2. π

ANSWERS TO SECTION QUIZ

1. $2\pi\int_0^1 y(1 - y^{1/4})dy = \pi/9$; $\pi\int_0^1 x^8 dx = \pi/9$
2. 3π
3. (a) $68\pi/3$

   (b) $2\pi\int_a^b (x - A)f(x)dx$
4. c
5. (a) $3\pi^2$

   (b) Approximate; the "curvature" at $y = -2$ is greater than that at $y = -4$.

   (c) $3\pi^2 - 2\pi$

9.3 Average Values and the Mean Value Theorem for Integrals

PREREQUISITES

1. Recall how to use the summation notation (Section 4.1).
2. Recall the intermediate value theorem (Section 3.1).
3. Recall the meaning of the mean value theorem for differentiation (Section 3.6).

PREREQUISITE QUIZ

1. State the mean value theorem for derivatives.
2. What conditions are necessary to apply the mean value theorem?
3. Explain the intermediate value theorem and state any necessary conditions.
4. Suppose $a_1 = 1$, $a_2 = 4$, $a_3 = -2$ and $b_j = j$. Compute the following:

   (a) $\Sigma_{i=1}^{3} a_i b_i$

   (b) $\Sigma_{i=2}^{5} b_i$

GOALS

1. Be able to compute the average of a function on a given interval.
2. Be able to state the mean value theorem for integrals and understand its meaning.

STUDY HINTS

1. Notation. Remember that the bar over the function indicates that the average value is desired. The interval is indicated as a subscript in $\overline{f(x)}_{[a,b]}$.

2. Average value. If you remember that the average is weighted, you should have no problem deriving the formula. It is simply the area under the curve divided by the length of the interval, i.e., the integral divided by the length. Fig. 9.3.1 may help you remember this.
3. Clarification. The numbers m and M as used on p. 435 are often chosen to be the minimum and the maximum values of $f(x)$ on $[a,b]$.
4. Integral mean value theorem. As with several other previous theorems, this is an existence theorem. It states that the average is attained somewhere in the interval, possibly many times; however, it doesn't specify exactly where it occurs, nor does the theorem help you find it. Note that continuity on a closed interval is required.
5. Example 5. Notice that the solution uses the fact that $\int_a^a f(x)dx = 0$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The average value is $\overline{f(x)}_{[a,b]} = \int_a^b f(x)dx/(b - a)$. In this case, it is $[1/(1 - 0)]\int_0^1 x^3dx = (x^4/4)\big|_0^1 = 1/4$.
5. This is just like Exercise 1 except that the limits of integration range from 0 to 2 . Thus, the average is $[1/(2 - 0)]\int_0^2 x^3dx = (1/2)(x^4/4)\big|_0^2 = 2$.
9. The average is $[1/(1 - 0)]\int_0^1 \sin^{-1}x\ dx$. Using the formula for integrating inverse functions from Section 7.4, we get $\int_0^1 \sin^{-1}x\ dx = x\sin^{-1}x\big|_0^1 - \int_0^{\pi/2}\sin y\ dy = x\sin^{-1}x\big|_0^1 + \cos y\big|_0^{\pi/2} = \pi/2 - 1$.
13. The average is $[1/(3 - 1)]\int_1^3 (x^3 + \sqrt{1/x})dx = (1/2)\int_1^3 (x^3 + x^{-1/2})dx = (1/2)(x^4/4 + 2\sqrt{x})\big|_1^3 = (1/2)(20 + 2\sqrt{3} - 2) = 9 + \sqrt{3}$.

17. Let t be the number of hours after midnight, then

$$f(t) = \begin{cases} 50 & 0 \leqslant t \leqslant 3 \\ 145/3 + 5t/9 & 3 \leqslant t \leqslant 12 \\ 15 + 10t/3 & 12 \leqslant t \leqslant 15 \\ 90 - 5t/3 & 15 \leqslant t \leqslant 24 \end{cases}.$$

The formula for average values gives us $[1/(24 - 0)]\,[\int_0^3 50\,dt + \int_3^{12}(145/3 + 5t/9)dt + \int_{12}^{15}(15 + 10t/3)dt + \int_{15}^{24}(90 - 5t/3)dt] = (1/24)\,[(50t)|_0^3 + (145t/3 + 5t^2/18)|_3^{12} + (15t + 5t^2/3)|_{12}^{15} + (90t - 5t^2/6)|_{15}^{24}] = (1/24)[150 + (435 + 37.5) + (45 + 135) + (810 - 292.5)] = (1/24)(1320) = 55^\circ F$.

21. We apply the fundamental theorem of calculus: $\overline{f'(x)}_{[a,b]} = [1/(b - a)] \times \int_a^b f'(x)dx = f(x)|_a^b/(b - a) = [f(b) - f(a)]/(b - a)$. This can only be 0 if $f(b) = f(a)$.

25. According to the mean value theorem for integrals, there is some t_0 in $[a,b]$ such that $\overline{f'(t)}_{[a,b]} = [1/(b - a)]\int_a^b f'(t)dt$. By the fundamental theorem of calculus, the right-hand side is $[f(b) - f(a)]/(b - a)$. We have shown that the average derivative is attained at some t_0 in $[a,b]$, so $f'(t_0) = [f(b) - f(a)]/(b - a)$, which is the mean value theorem for derivatives.

29. We need to show that $\lim_{x\to x_0} F(x) = F(x_0)$. By the definition used in Section 11.1, we must show that $|F(x) - F(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. By definition, $F(x) - F(x_0) = \int_a^x f(s)ds - \int_a^{x_0} f(s)ds = \int_{x_0}^x f(s)ds$. Now, by the extreme value theorem, $|f(s)|$ has a maximum $|M|$ on $[a,b]$, so $|f(s)| \leqslant |M|$ on $[a,b]$ and $|\int_{x_0}^x f(s)ds| \leqslant |M|\,|x - x_0|$. Thus, if we let $\delta = |x - x_0| = \varepsilon/|M|$, we have $|F(x) - F(x_0)| \leqslant |M|\varepsilon/|M| = \varepsilon$.

SECTION QUIZ

1. Find the average of the following functions on the given intervals:

   (a) $g(t) = t\sqrt{t-1}$ on $[2,3]$

   (b) $f(x) = 2^x$ on $[1,3]$

   (c) $y = \sin(x/2)$ for $0 \leqslant x \leqslant \pi$

2. Suppose $\overline{f(x)}_{[0,3]} = 8$, $\overline{f(x)}_{[3,5]} = 2$, and $\overline{f(x)}_{[5,10]} = 5$.

   (a) What is $\overline{f(x)}_{[0,10]}$?

   (b) Find $\overline{f(x)}_{[0,5]}$, if possible.

   (c) Find $\overline{f(x)}_{[3,8]}$, if possible .

3. Consider the function in Question 2.

   (a) Can we conclude that $f(x_0) = 2$ for some x_0 in $[3,5]$? Explain.

   (b) If f is not differentiable somewhere in $[0,10]$, but it is continuous, can we conclude that $f(x_0) = 5$ for some x_0 in $[0,10]$? Explain.

   (c) If f is continuous, $f(7.5) = 5$, and $f(10) = 7$, then can we conclude that $f(5) = 3$? Explain.

4. True or false: A continuously differentiable function can not attain its average at a critical point.

5. On a typical day, the Smith family, consisting of three talkative teenagers and their parents, uses the telephone for $3 + t^2$ hours, where t is the number of days before or after Wednesday, 12 noon. Thus, $t = 1$ on Tuesday or Thursday, etc.

   (a) What is the average number of hours the Smith family talks on the phone daily? (Hint: Integrate on the interval $-3.5 \leqslant t \leqslant 3.5$.)

   (b) Show that the mean value theorem for integrals is valid for this situation.

ANSWERS TO PREREQUISITE QUIZ

1. If f is continuous on $[a,b]$ and differentiable in (a,b), then $f'(x_0) = [f(b) - f(a)]/(b - a)$ for some x_0 in (a,b).

2. Continuity and differentiability.

3. Continuity is a necessity. If f is continuous, and $f(x_1)$, $f(x_2)$ lie on opposite sides of the line $y = C$, then one must cross the line somewhere in (x_1,x_2).

4. (a) 3

   (b) 14

ANSWERS TO SECTION QUIZ

1. (a) $(44\sqrt{2} - 16)/15$

   (b) $3/\ln 2$

   (c) $2/\pi$

2. (a) 5.3

   (b) 5.6

   (c) Not possible

3. (a) No, f may not be continuous.

   (b) Yes, particularly in $(5,10)$ by the mean value theorem for integrals

   (c) No, there are numerous functions which satisfy $f(7.5) = 5$, $f(10) =$ and $\int_5^{10} f(x)dx = 25$. The function doesn't have to be symmetric.

4. False; consider $f(x) = x^3$ on $[-1,1]$.

5. (a) 28/3 hours/day

   (b) $x_0 = (-3\pm\sqrt{139/3})/2 \approx -4.90$ and 1.90; 1.90 is in the interval $(-3.5,3.5)$.

9.4 Center of Mass

PREREQUISITES

1. Recall how integration formulas for areas and volumes were derived by using the infinitesimal argument (Sections 4.6, 9.1, and 9.2).
2. Recall how to use the summation notation (Section 4.1).

PREREQUISITE QUIZ

1. Use an infinitesimal argument to derive $V = 2\pi\int_a^b xf(x)dx$.
2. Use an infinitesimal argument to derive $A = \int_a^b [f(x) - g(x)]dx$, the area between two graphs such that $f(x) \geqslant g(x)$ on $[a,b]$.
3. (a) In general, does $\Sigma_{i=1}^n m_i x_i = (\Sigma_{i=1}^n m_i)(\Sigma_{i=1}^n x_i)$?

   (b) In general, does $\Sigma_{i=1}^n m_i x_i / \Sigma_{i=1}^n m_i = \Sigma_{i=1}^n x_i$?

GOALS

1. Be able to state and understand the consolidation principle.
2. Be able to find the center of mass on a line.
3. Be able to find the center of mass for a plane region.

STUDY HINTS

1. Consolidation principle. This allows you to concentrate the entire mass of an object at its center of mass. This principle is important in the derivation of formulas. "Negative" masses may also be used if a mass needs to be subtracted. The usefulness of this principle is illustrated quite well in Example 7.
2. Center of mass on the line. By remembering that the center of mass is simply a weighted average of position, you should be able to recall the formula $\bar{x} = \Sigma_{i=1}^n m_i x_i / \Sigma_{i=1}^n m_i$.

3. Warning. If you think the m_i's cancel in the center of mass formula (3), you need to review Section 4.1.

4. Center of mass in the plane. The only difference between $\bar{x}$ and $\bar{y}$ is that one formula uses x_i and the other uses y_i.

5. Symmetry principle. Properly used, this saves a lot of needless work. Remember that uniform density is a requirement. Two axes of symmetry is all that is needed to determine the center of mass of a region in the plane.

6. Center of mass of a region. It is recommended that you learn to derive the formula with the aid of Fig. 9.4.10. Again, this is a weighted average where the mass is density ρ times the area of the region. The center of mass of the "infinitesimal rectangle" is simply $(x,f(x)/2)$. Now, use the consolidation principle and the fact that integration is a continuous summation. The same formula holds even if $f(x)$ is negative. Exercise 28 gives a more general formula.

7. Density cancels. In all of the formulas, mass is used. For region in the plane, mass is proportional to density × area. Density cancels when it is uniform. Be careful when density is not uniform.

8. Step functions. As with the other step function arguments presented in this chapter, you will probably not be expected to recall the material in the supplement.

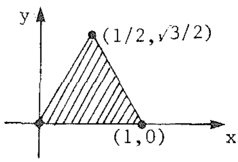
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. In this case, $M_1 = m_1$, $M_2 = m_2 + m_3$, $X_1 = x_1$, and $X_2 = (m_2x_2 + m_3x_3)/(m_2 + m_3)$. The center of mass is $(M_1X_1 + M_2X_2)/(M_1 + M_2) = [m_1x_1 + (m_2 + m_3)(m_2x_2 + m_3x_3)/(m_2 + m_3)]/[m_1 + (m_2 + m_3)] = (m_1x_1 + m_2x_2 + m_3x_3)/(m_1 + m_2 + m_3)$, which is the same formula derived in Example 1.

5. Using the formula $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$, we get $\bar{x} = (1\cdot7 + 3\cdot3 + 5\cdot5 + 7\cdot1)/(1 + 3 + 5 + 7) = 48/16 = 3$.

9. Using the formulas $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$ and $\bar{y} = \Sigma_{i=1}^{n} m_i y_i / \Sigma_{i=1}^{n} m_i$, we get $\Sigma_{i=1}^{2} m_i x_i = 10(1) + 20(1) = 30$; $\Sigma_{i=1}^{2} m_i y_i = 10(0) + 20(2) = 40$; $\Sigma_{i=1}^{2} m_i = 10 + 20 = 30$. Thus, $\bar{x} = 30/30 = 1$ and $\bar{y} = 40/30 = 4/3$.

13. (a)

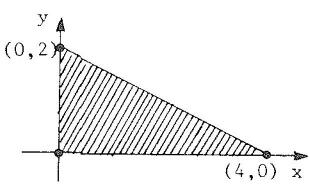


All of the sides of the triangle must have unit length. Thus, the third vertex may be determined by solving $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{(x - 1)^2 + (y - 0)^2} = 1$ or by using plane geometry. Thus, the thrid vertex is $(1/2,\sqrt{3}/2)$. $\bar{x} = m(0 + 1 + 1/2)/3m = 1/2$, where m is the mass of the object. $\bar{y} = m(\sqrt{3}/2 + 0 + 0)/3m = \sqrt{3}/6$. Thus, *the center of mass is* $(1/2,\sqrt{3}/6)$.

(b) If the mass at $(0,0)$ is doubled, then $\bar{x} = [2m(0) + m(1/2) + m(1)]/(2m + m + m) = 3/8$ and $\bar{y} = [2m(0) + m(\sqrt{3}/2) + m(0)]/(2m + m + m) = \sqrt{3}/8$. Thus, the center of mass is $(3/8,\sqrt{3}/8)$.

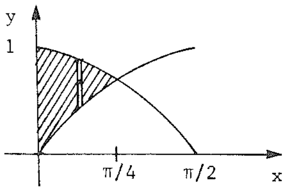
17. Using the formulas $\bar{x} = \int_a^b xf(x)dx / \int_a^b f(x)dx$ and $\bar{y} = (1/2)\int_a^b [f(x)]^2 dx / \int_a^b f(x)dx$, we get $\bar{x} = \int_1^3 x(4/x^2)dx / \int_1^3 (4/x^2)dx = \int_1^3 (4/x)dx/(-4/x)\big|_1^3 = 4 \ln x\big|_1^3/(8/3) = 4 \ln 3/(8/3) = 3 \ln 3/2$. And $\bar{y} = (1/2)\int_1^3 (4/x^2)^2 dx / \int_1^3 (4/x^2)dx = (1/2)\int_1^3 16x^{-4}dx/(8/3) = 3(x^{-3}/-3)\big|_1^3 = 26/27$. Thus, the center of mass is $((3/2)\ln 3, 26/27) \approx (1.65,0.96)$.

21.



As shown in the figure, we want to find the center of mass of the region under $f(x) = -x/2 + 2$. Therefore, $\bar{x} = \int_0^4 x(-x/2 + 2)dx / \int_0^4 (-x/2 + 2)dx = (-x^3/6 + x^2)\big|_0^4/(-x^2/4 + 2x)\big|_0^4 = 4/3$. $\bar{y} = (1/2)\int_0^4 (-x/2 + 2)^2 dx / \int_0^4 (-x/2 + 2)dx = -(-x/2 + 2)^3/3\big|_0^4/4 = (8/3)/4 = 2/3$. Therefore, the center of mass is $(4/3,2/3)$.

25. First, recall that $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$. Since the masses are independent of time, we differentiate to get the velocity at the center of mass, $v = d\bar{x}/dt = \Sigma_{i=1}^{n} m_i (dx_i/dt)/\Sigma_{i=1}^{n} m_i$. By definition, $P = \Sigma_{i=1}^{n} m_i v_i = \Sigma_{i=1}^{n} m_i (dx_i/dt)$, and $M = \Sigma_{i=1}^{n} m_i$. Rearrange $P = Mv$ to get $v = P/M = \Sigma_{i=1}^{n} m_i (dx_i/dt)/\Sigma_{i=1}^{n} m_i = d\bar{x}/dt = v$.

29. 

Note that $\cos x \geqslant \sin x$ on $[0,\pi/4]$. On an infinitesimal strip, the center of mass is $(x,(\cos x + \sin x)/2)$ (see figure). We take a weighted average by weighing the center of mass against its mass, i.e., area. The area of the rectangular infinitesimal strip is $(\cos x - \sin x)dx$. Now the continuous sum of $\bar{x} = \Sigma_{i=1}^{n} m_i x_i / \Sigma_{i=1}^{n} m_i$ becomes $\int_0^{\pi/4} x(\cos x - \sin x)dx / \int_0^{\pi/4}(\cos x - \sin x)dx$. Integration by parts with $u = x$ and $v = \sin x + \cos x$ yields $[x(\sin x + \cos x)|_0^{\pi/4} - \int_0^{\pi/4}(\sin x + \cos x)dx]/ \sin x + \cos x)|_0^{\pi/4} = [(\pi/4)\sqrt{2} - (-\cos x + \sin x)|_0^{\pi/4}]/(\sqrt{2} - 1) = (\sqrt{2}\pi/4 - 1)/(\sqrt{2} - 1) \approx 0.27 = \bar{x}$. Similarly, the continuous sum of $\bar{y} = \Sigma_{i=1}^{n} m_i y_i / \Sigma_{i=1}^{n} m_i$ is $\int_0^{\pi/4}(1/2)(\cos x + \sin x)(\cos x - \sin x)dx / \int_0^{\pi/4}(\cos x - \sin x)dx = (1/2)\int_0^{\pi/4}(\cos^2 x - \sin^2 x)dx/(\sqrt{2} - 1)$. By a trigonometric substitution, we get $(1/2)\int_0^{\pi/4}\cos 2x\, dx/(\sqrt{2} - 1) = (1/2)(\sin 2x/2)|_0^{\pi/4}/(\sqrt{2} - 1) = 1/4(\sqrt{2} - 1)$. Therefore, the center of mass is $((\sqrt{2}\pi/4 - 1)/(\sqrt{2} - 1), 1/4(\sqrt{2} - 1)) \approx (0.27, 0.60)$.

SECTION QUIZ

1. (a) A square region with vertices at $(-1,-1)$, $(-1,1)$, $(1,-1)$, and $(1,1)$ has density x^2 at (x,y) . Explain why the symmetry principle applies for a vertical axis.

1. (b) A square region with vertices at $(3,-1)$, $(3,1)$, $(5,1)$, and $(5,-1)$ has density x^2 at (x,y). Explain why the symmetry principle does not apply for a vertical axis.

2. Find the centers of mass of each individual square region described in Question 1.

3. Find the center of mass of the combined regions described in Question 1.

4. The earth has a radius of 6.4×10^3 km and a mass of 6.0×10^{24} kg. The moon has a radius of 1.6×10^3 km and a mass of 7.4×10^{22} kg. The moon is 3.9×10^5 km from earth.

   (a) Assuming the earth and moon are perfect spheres, where is the center of mass of the earth-moon system?

   (b) What information was not necessary and why?

5. A famous artist likes to focus on the center of mass. For her latest painting, the lovely artist has selected three objects. All of the individual centers of mass are located in the same plane. One object is a circular plate with center at $(-3,4)$ and radius $\sqrt{5/\pi}$. A square plate has vertices at $(2,2)$, $(4,2)$, $(2,4)$, and $(4,4)$. Each plate has density $1\ \mathrm{kg/m^2}$ and each unit on the xy-plane is 1 m. The third object is a frog whose mass is 2 kg and whose center of mass is at the origin.

   (a) Where is the center of mass for the three objects?

   (b) At the end of the day, the artist kisses the frog and he turns into a handsome prince whose mass increases to 70 kg and whose center of mass remains at $(0,0)$. Where is the new center of mass for the three objects?

ANSWERS TO PREREQUISITE QUIZ

1. This is the shell method. Revolving a thin rectangle at x around the y-axis gives us a cylinder with base circumference or "length" $2\pi x$. Its height is $f(x)$ and its width is dx, so its volume is $2\pi x f(x)dx$. Integrate to get the entire volume.

2. Consider a thin rectangle at x. Its height is $f(x) - g(x)$ and its width is dx. Therefore, its area is $[f(x) - g(x)]dx$. Integrate to get the entire area.

3. (a) No

   (b) No

ANSWERS TO SECTION QUIZ

1. (a) The mass at $(-x,y)$ equals the mass at (x,y).

   (b) The mass at $(4 - x_0,y)$ does not equal the mass at $(4 + x_0,y)$.

2. $(0,0)$ and $(204/49,0) \approx (4.16,0)$

3. $\approx (4.14,0)$

4. (a) 4.8×10^3 km from the earth on the line between the earth and the moon.

   (b) The radii are not needed because the consolidation principle can be used.

5. (a) $(-3/11,32/11)$

   (b) $(-3/79,32/79)$

9.5 Energy, Power, and Work

PREREQUISITES

1. Recall the physical interpretation of integrating rates of change (Section 4.6).

PREREQUISITE QUIZ

1. Given the following quantities for $f(x)$, what is $\int f(x)dx$?
   (a) Velocity (meters per second)
   (b) Water flow rate (gallons per minute)
   (c) Melting rate of a candle (grams per minute)

GOALS

1. Be able to state the relationship between power and energy and apply it for problem solving.
2. Be able to state the relationship between work and force and apply it for problem solving.

STUDY HINTS

1. Psychology. Many students fear this section because it is "Physics." Remember that you are enrolled in a math course and being a physics major is not a requirement. You will find the text self contained in what you need to know.
2. Units. The unit of work is the joule which is $1\ kg\cdot m^2/s^2$. This can be remembered by using $dW = Fdx = ma\ dx$ and using the units of m , a , and dx . Another unit to know is the watt which is 1 joule/second.

3. Energy vs. power. Energy is the total sum of power, which is rate. They are related just as distance (integration) is to velocity (differentiation).

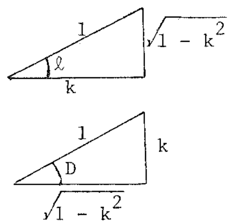
4. Important formulas. Memorize the fact that work is the integral of force with respect to position, not time. It is not essential to know that $dK/dx = F$. However, note that work is a form of kinetic energy; this is a special form of the equation $dW/dx = F$.

5. Good example. Example 4 should be studied thoroughly. It uses the important relationships $dW = Fdx$, $F = mg$, and $m = \rho V$. At this point, we still need to determine the volume term, which is $\pi r^2 dx$, by using similar triangles. Remember that gravity acts in a downward direction.

6. Sunshine formula. This is an interesting application of the calculus you have learned so far. Consult your instructor to see if you need to study it. None of the equations should be memorized. The energy equation (3) is the important one for doing the exercises.

SOLUTIONS TO EVERY OTHER ODD EXERCISE (SUPPLEMENT)

1. We use the formula, $E = \sqrt{\cos^2 \ell - \sin^2 D} + \sin \ell \sin D \cos^{-1}(-\tan \ell \tan D)$ where $\sin D = \sin \alpha \cos(2\pi T/365)$. On June 21, we have $T = 0$, so $\sin D = \sin \alpha$, or $D = \alpha$. Also, we have $\cos \ell = \sin \alpha$ and $\sin \ell = \cos \alpha$, $\tan \ell = \cot \alpha$, so $E = \sqrt{\sin^2 \alpha - \sin^2 \alpha} + \cos \alpha \sin \alpha \times \cos^{-1}(-\tan \ell \cot \ell) = \cos \alpha \sin \alpha \cos^{-1}(-1) = (\pi/2)\sin(2\alpha)$. Evaluating at $\alpha = 23.5^\circ$, we find $E \approx 1.15$, which is about 1.25 times the energy received at the equator on June 21. This excess is due, of course, to the long day at the Arctic Circle.

5. 

To simplify the calculations, let $k = \sin\alpha \cos(2\pi T/365)$. On a day on which the sun does not set, $S = 12 = (24/2\pi)\cos^{-1}\left[-\tan \ell \left(k/\sqrt{1 - k^2}\right)\right]$. The equation requires $\tan \ell \left(k/\sqrt{1 - k^2}\right) = 1$ or $\tan^2\ell(k^2) = 1 - k^2$. Equivalently, $k^2(\tan^2\ell + 1) = 1$, which implies $k^2\sec^2\ell = 1$. Thus, $k = \cos \ell$ or $\ell = \cos^{-1}k$. Using the notation of the text, $\sin D = k$. Now substituting into the energy equation, $E = \sqrt{\cos^2\ell - \sin^2 D} + \sin \ell \sin D \cos^{-1}(-\tan \ell \tan D) = \sqrt{k^2 - k^2} + \sqrt{1 - k^2}(k) \cos^{-1} \times \left[-\left(\sqrt{1 - k^2}/k\right)\left(k/\sqrt{1 - k^2}\right)\right] = \sqrt{1 - k^2}(k) \cos^{-1}(-1) = \pi \sin \ell \sin \alpha \cos(2\pi T/365) = \pi \sin \ell \sin D$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We use $E = \int_a^b P\,dt$. During one hour, $0 \leqslant t \leqslant 3600$ seconds, so the energy output is $E = \int_0^{3600} 1050 \sin^2(120\pi t)dt$. By the half-angle formula, $E = (1/2)(1050)\int_0^{3600}(1 - \cos 240\pi t)dt = 525(t - \sin 240\pi t/240\pi)\Big|_0^{3600} = 525(3600) = 1{,}890{,}000$ joules.

5. Work is the integral of force; $W = \int_a^b F\,dx$. In this case, $W = \int_0^1 3x\,dx = (3x^2/2)\Big|_0^1 = 3/2$.

9. Power is dE/dt, so the units are joules/second. As in Example 2, the work done in one second $= (1\text{ kg})(9.8\text{ m/sec}^2)(10\text{ m}) = 98$ joules. Therefore, the required power is 98 joules/1 second = 98 watts.

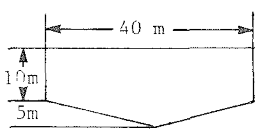
13. Recall that $\Delta K = \int_a^b F\,dx$. In this case, $\Delta K = \int_0^1(-3x)dx = (-3x^2/2)\Big|_0^1 = -3/2$, so kinetic energy decreases by 1.5 joules.

17. If the depth of the water is between 0 and 2, then the volume, with a thickness dx, is $(15)(30)\,dx$. If the water is below the 2m level we use similar triangles to find that the volume is $(15)[15 + 5(5 - x)]\,dx$ Since a cubic meter of water has mass 10^3 kilograms, we multiply the volume by 1000 to find the total mass. Each layer of water with thickness dx is lifted x meters against gravity with $g = 9.8\ \text{m/sec}^2$, so $W = 1000g[\int_0^2 450\,x\,dx + \int_2^5 (600 - 75x)x\,dx] = 9800[225x^2|_0^2 + (300x^2 - 25x^3)|_2^5] = 9800(900 + 6300 - 2925) = 9800(4275) = 41{,}895{,}000$ joules.

21. Since -3 newtons is needed to stretch the spring 5 cm, $+3$ newtons is needed to compress the spring 5 cm. Work is the integral of force, so $W = F(\Delta x) = (3)(0.05) = 0.15$ joule since the force is constant.

SECTION QUIZ

1. While doing pull-ups at your local athletic club, you realize that you must lift your 70 kg body off the ground by 0.5 meter. How much work is done by your arm muscles?

2. 

Algae needs to be removed from a drydock which is 300 m long. The cross-sectional area is shown at the left. How much energy is needed to pump the entire dock dry?

3. Late last night, a vampire was seen breaking into the blood bank. He had sucked the blood out of 50 cylindrical test tubes. Each test tube was 10 cm long and had a diameter of 1 cm. If blood has a density of $1.03\ \text{g/cm}^3$, how much work was done by the vampire while sucking up his dinner?

ANSWERS TO PREREQUISITE QUIZ

1. (a) Net distance travelled

   (b) Volume of water flowing through over time

   (c) Amount of candle which melted

ANSWERS TO SECTION QUIZ

1. 343 joules

2. 9.3×10^9 joules

3. 0.2 joules

9.R Review Exercises for Chapter 9

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. (a) Use the disk method and the half-angle formula, so $V = \pi\int_0^\pi \sin^2 x\,dx = \pi\int_0^\pi [(1 - \cos 2x)/2]\,dx = \pi(x/2 - \sin 2x/4)\big|_0^\pi = \pi(\pi/2) = \pi^2/2$.

   (b) Use the shell method and integrate by parts, so $V = 2\pi\int_0^\pi x \sin x\,dx = 2\pi(-x\cos x\big|_0^\pi + \int_0^\pi \cos x\,dx) = 2\pi(-x\cos x + \sin x)\big|_0^\pi = 2\pi(\pi) = 2\pi^2$.

5. Use the result of Example 5, Section 9.2 with $R = 1$ and $r = 1/3$. The volume of the original ball is $(4/3)\pi$. The volume of the removed material is $(4/3)\pi[1 - (1 - 1/9)^{3/2}]$. Thus, the volume of the resulting solid is $(4/3)\pi(8/9)^{3/2} = 64\sqrt{2}\pi/81$.

9. The average value is $\overline{f(t)}_{[a,b]} = [1/(b - a)]\int_a^b f(t)dt$. In this case, it is $[1/(1 - 0)]\int_0^1 (1 + t^3)dt = 1(t + t^4/4)\big|_0^1 = 5/4$.

13. The average value of $g(x)$ is $\int_0^2 g(x)dx/(2 - 0) = \int_0^2 3f(x)dx/2 = (3/2)\int_0^2 f(x)dx = (3/2)(4) = 6$.

17. The average of f on $[a,b]$ is $\mu = [1/(b - a)]\int_a^b f(t)dt = [1/(1 - 0)]\int_0^1 x^2 dx = (x^3/3)\big|_0^1 = 1/3$. The variance is $[1/(b - a)] \times \int_a^b [f(t) - \mu]dt = [1/(1 - 0)]\int_0^1 [x^2 - 1/3]^2 dx = \int_0^1 (x^4 - 2x^2/3 + 1/9)dx = (x^5/5 - 2x^3/9 + x/9)\big|_0^1 = 1/5 - 2/9 + 1/9 = 4/45$. The standard deviation is the square root of the variance, which is $\sqrt{4/45} = 2/3\sqrt{5} = 2\sqrt{5}/15$.

21. The average of $f(x)$ on $[a,b]$ is $\mu = [1/(b - a)]\int_a^b f(x)dx$, so $\mu = [1/(2 - 0)]\int_0^2 f(x)dx = (1/2)[\int_0^1 dx + \int_1^2 2\,dx] = (1/2)[x\big|_0^1 + 2x\big|_1^2] = 3/2$. The variance is $[1/(b - a)]\int_a^b [f(x) - \mu]^2 dx$, so variance $= (1/2)\int_0^2 [f(x) - \mu]^2 dx = (1/2)[\int_0^1 (1 - 3/2)^2 dx + \int_1^2 (2 - 3/2)^2 dx] = (1/2)((x/4)\big|_0^1 + (x/4)\big|_1^2) = 1/4$. The square root of the variance is the standard deviation, which is $1/2$.

25. Use the formulas $\bar{x} = \int_a^b xf(x)dx/\int_a^b f(x)dx$ and $\bar{y} = (1/2)\int_a^b [f(x)]^2 dx/\int_a^b f(x)dx$. Thus, the center of mass is $\bar{x} = \int_0^2 x(x^4)dx/\int_0^2 x^4 dx = \left(x^6/6\Big|_0^2\right)/\left(x^5/5\Big|_0^2\right) = (32/3)/(32/5) = 5/3$ and $\bar{y} = \int_0^2 (x^4)^2 dx/2\int_0^2 x^4 dx = \left(x^9/9\Big|_0^2\right)/2(32/5) = (512/9)/2(32/5) = 40/9$. Therefore, the center of mass is $(5/3, 40/9)$.

29. Use the formulas from Exercise 28, Section 9.4 with $g(x) = x^3$ and $f(x) = -x^2$. Therefore, $\bar{x} = \int_0^1 x(x^3 + x^2)dx/\int_0^1 (x^3 + x^2)dx = (x^5/5 + x^4/4)\Big|_0^1/(x^4/4 + x^3/3)\Big|_0^1 = (9/20)/(7/12) = 27/35$; $\bar{y} = (1/2)\int_0^1 (x^3 - x^2) \times (x^3 + x^2)dx/\int_0^1 (x^3 + x^2)dx = (1/2)\int_0^1 (x^6 - x^4)dx/(7/12) = (x^7/7 - x^5/5)\Big|_0^1/(7/6) = (-2/35)/(7/6) = -12/245$. Thus, the center of mass is $(27/35, -12/245)$.

33. Use the equation $\Delta K = \int_a^b F\,dx = \int_2^4 30\sin(\pi x/4)dx = 30(4/\pi) \times [-\cos(\pi x/4)]\Big|_2^4 = (-120/\pi)(-1 - 0) = (120/\pi)$ joules.

37. (a) Consider a rectangular slab with width dx. Then, at a depth h, the force on the slab is $\rho gh\,dh\,dx$ because the pressure acts on an area $dh \cdot dx$. Integrate from 0 to $f(x)$ to find the total force on the rectangular slab. Then integrate the pressure of each slab from a to b to find the total force on the dam. $F = \int_a^b [\int_0^{f(x)} \rho gh\,dh]dx = \int_a^b [\rho g(h^2/2)\Big|_0^{f(x)}]dx = (1/2)\int_a^b \rho g[f(x)]^2 dx$.

(b) If we rotate $f(x)$ around the x-axis, then the volume V obtained is $\pi\int_a^b [f(x)]^2 dx$; therefore, $F = V\rho g/2\pi$.

(c) We use the formula in (b). Rotation of the dam face results in a volume consisting of two cones and a circular cylinder. The cones have radii of 100 m and heights of 125 m. The cylinder has a radius of 100 m and a height of 50 m. Therefore, $V = 2(\pi r^2 h/3) + \pi r^2 h = 2\pi(100)^2(125)/3 + \pi(100)^2(50) = 4{,}000{,}000\pi/3$, which means $F = (4{,}000{,}000\pi/3)(1000)(9.8)/2\pi = (19.6/3)(10^9) \approx (6.53 \times 10^9)$ newtons $= (2/3)(\rho g \times 10^6)$ newtons.

41. Suppose that $f(a) < 0$ and $f(b) > 0$. Let $I = \int_a^b f(x)dx$. If $I = 0$, the mean value theorem for integrals tells us that there is an x_0 in $[a,b]$ with $(b-a)f(x_0) = I = 0$, so that $f(x_0) = 0$, as required. If $I > 0$, then let $a_1 = a - I/f(a)$, and extend f to the interval $[a_1,b]$ by setting $f(x) = f(a)$ for x in $[a_1,b]$. This is still continuous, and now one may compute that $\int_{a_1}^b f(x)dx = 0$. By the mean value theorem for integrals, there is an x_0 in $[a_1,b]$ with $f(x_0) = 0$. Since $f(x) = f(a) < 0$ on $[a_1,a]$, this x_0 must be in $[a,b]$. The case $I < 0$ is handled in a similar way; extend the function past b instead. (See the hint on p.A.53 for another method).

TEST FOR CHAPTER 9

1. True or false:

   (a) The volume of the solid of revolution obtained by revolving the graph of $f(x) \geqslant 0$ around the x-axis is $V = 2\pi\int_a^b xf(x)dx$.

   (b) If $f(x) < 0$, it can be revolved around the x-axis to form a solid of revolution.

   (c) The center of mass of a plane region with uniform density always lies on the region.

   (d) The average of $f(x)$ on $[a,b]$ is attained at least once by f in (a,b).

   (e) The average of $f(x)$ on $[a,b]$ never occurs at a or b.

2. (a) Find the volume which results from revolving $y = x^2 - 1$, $0 \leqslant x \leqslant 2$, around the x-axis.

   (b) Does it matter that $f(x) < 0$ on part of the interval? Why?

3. A force $F(x) = 1/x^2$ is applied to a particle on the interval $1 \leqslant x \leqslant 10$. Find the work done by the force in moving the particle from $x = 1$ to $x = 10$.

4. Let $f(x) = \sqrt{x}$ on $[0,1]$. Find the volume of the solid of revolution generated by revolving the graph of $f(x)$ around the following lines:

   (a) $x = 3$

   (b) $y = -1$

5. Find the center of mass of the region between $g(x) = \sqrt{x+4}$ and the x-axis on the interval $0 \leqslant x \leqslant 4$.

6. Compute the average of $x\cos(x^2 + \pi)$ on $[0,\sqrt{\pi}]$.

7. What is the center of mass of the region bounded by $y = x^3$ and the x-axis on $[-1,2]$?

8. (a) Show that the mean value theorem for integrals applies to $f(x) = x^2 - 1$ on $[-1,0]$.

   (b) What is the average of $f(x)$ on $[-1,0]$?

9. Find the volume of the solid of revolution generated by revolving the region between the graphs of $2x$ and x^3 on $[0,1]$ around each axis.

10. One hot day in India, an elephant was down at the river cooling itself. The elephant filled its trunk with water to squirt on its back. Unfortunately, the last trunkful of water had a fly in it, which caused a sneeze attack. Suppose the elephant's cylindrical trunk has a diameter of 8 cm and a length of 1.5 m. How much work is minimally required to force the water out of its trunk if it is pointing straight up into the sky?

ANSWERS TO CHAPTER TEST

1. (a) False; $V = \pi\int_a^b [f(x)]^2 dx$.

   (b) True

   (c) False; the center of mass of a "ring" lies at the center, in the hole.

   (d) False; f should be continuous.

   (e) False; consider $f(x) = 1$ on $[-2,2]$.

2. (a) $(46/15)\pi$

   (b) No, because $[-f(x)]^2 = [f(x)]^2$. Thus, the formula is the same for any f .

3. $-9/10$

4. (a) $16\pi/5$

   (b) $17\pi/6$

5. $((40 + 24\sqrt{2})/35, (9 + 18\sqrt{2})/28)$

6. 0

7. $(511/85, 2047/238)$

8. (a) f is integrable and the average is attained at $x_0 = -\sqrt{1/3}$, which is in $[-1,0]$.

   (b) $-2/3$

9. $25\pi/21$ around x-axis; $14\pi/15$ around y-axis

10. 55 joules

COMPREHENSIVE TEST FOR CHAPTERS 7-9 (Time limit: 3 hours)

1. True or false. If false, explain why.

(a) The differential equation $dy/dx = y/(x + 1)(y + xy)$ is separable.

(b) $\int[dt/(1 + t^2)] = \tan^{-1}t + C = -\cot^{-1}t + C$ is valid for all t.

(c) $\int[dt/(1 - t^2)] = \tanh^{-1}t + C = \coth^{-1}t + C$ is valid for all $t \neq 1$.

(d) Revolving $y = x^2$ around the x-axis yields the same volume as revolving $y = x^2 + 2x + 1$ around the line $y = 1$ on any interval $[0,a]$, $a > 0$.

(e) Force is the derivative of work with respect to time.

(f) One solution of $dy/dx = -3x$ is $y = -3x^2/2$.

(g) The mean value theorem for integrals requires differentiability before it can apply to a function.

(h) $\int_0^5 f(t)g(t)dt = (\int_0^5 f(t)dt)g(0) + f(0)\int_0^5 g(t)dt$.

(i) The derivative of $\cosh x$ exists for all real x.

(j) $\int\tan^3\theta\, d\theta = \tan^4\theta/4 + C$

2. Multiple choice.

(a) Which is the solution of $y'' + 4y = 0$, where $y'(0) = y(0) = 1$?

(i) $\cosh 2t + (1/2)\sinh 2t$

(ii) $\cos 2t + (1/4)\sin 2t$

(iii) $\sin 2t + (1/2)\cos 2t$

(iv) $(1/2)\sin 2t + \cos 2t$

(b) The average value of $x \exp(x^2)$ on $[0,1]$ is:

(i) $e/2$

(ii) $(e - 1)/2$

(iii) $(x^2 + \ln x)\big|_0^1$

(iv) $(e + 1)/2$

2. (c) What is the most efficient way to evaluate $\int \sec^3\theta \tan\theta \, d\theta$?

(i) Integrate by parts with $u = \sec^3\theta$, $dv = \tan\theta \, d\theta$.

(ii) Integrate by parts with $u = \tan\theta$, $dv = \sec^3\theta \, d\theta$.

(iii) Substitute $u = \sec\theta$.

(iv) Substitute $u = \tan\theta$.

(d) Suppose f is symmetric with respect to the origin and $b > a$. Furthermore, suppose $f \geqslant 0$ on $[-b,-a]$, then $\overline{f}_{[-a,b]}$ is:

(i) Positive

(ii) Negative

(iii) Zero

(iv) There is not enough information.

(e) A rabbit population grows exponentially. It takes five years for the population to increase from 1000 to 2000 . How long does it take for the population to grow from 20,000 to 30,000 ?

(i) About 50 years

(ii) 2-4 years

(iii) 4-6 years

(iv) There is not enough information.

3. Perform the following integrations:

(a) $\int \cosh^{-1} x \, dx$

(b) $\int y^2 (\ln y)^2 dy$

(c) $\int_0^1 \left[3t^3/\sqrt{t^2 + 4}\right] dt$

(d) $\int [(x + 1)/(x^2 + 2x + 5)] dx$

(e) $\int_3^4 (\ln t/t) dt$

4. Short answers.

(a) Solve the differential equation $y' = -y/4$, assuming $y(0) = 3$.

(b) Differentiate $\tanh(x^2 + 3)$.

(c) Express $\sinh t$ as a sum or difference of exponentials.

(d) Sketch the graph of $y = \cosh x$.

(e) If $F' = f$, what is $\int f(3t)dt$?

5. Find the center of mass of the region between $y = x^2$ and $y = x^3$ on $[0,1]$.

6. The temperature of a swimming pool changes according to the formula $dT/dt + T = 80^\circ$. Initially, $T(0) = 50^\circ$. How long does it take for the temperature to change from 50° to 75° ?

7. Solve the following differential equations with the given conditions

(a) $dy/dx = xy$, $y(0) = 1$

(b) $dy/dx = \sin x + y$, $y(0) = 1$

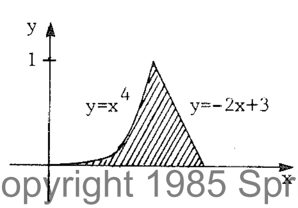
(c) $d^2y/dx^2 + 7x = 0$, $y(0) = 1$, $y'(0) = 1$

8. Let R be the region between x and x^3 on $[-1,2]$. What is the volume of the solid of revolution obtained by revolving R around each of the following?

(a) The x-axis.

(b) The y-axis.

9. Suppose the price of land in Manhattan increases at an instantaneous rate of 15% per year. How long does it take for the value of the land to triple?

10. 

A volcanic mountain top has a shape which can be described as the solid of revolution formed by revolving the shaded region around the y-axis. Each unit represents 1 kilometer and the mountain top has a uniform

10. density of $5000\ kg/m^3$. A violent eruption disintegrates the mountain top sending the particles to form a cloud cover at $y = 3$. How much energy was expended by the volcano in forming the cloud cover?

ANSWERS TO COMPREHENSIVE TEST

1. (a) True

   (b) True

   (c) False; it is $\tanh^{-1} + C$ if $|t| > 1$, $\coth^{-1} t + C$ if $|t| < 1$.

   (d) True

   (e) False; it is the derivative with respect to position.

   (f) True

   (g) False; it only requires integrability.

   (h) False; the right-hand side should be obtained from integration by parts.

   (i) True

   (j) False; differentiating the right-hand side yields $\tan^3\theta \sec^2\theta$.

2. (a) iv

   (b) ii

   (c) iii

   (d) i

   (e) ii

3. (a) $x \cosh^{-1}x - \sqrt{x^2 - 1} + C$

   (b) $(y^3/3)[(\ln y)^2 - 2(\ln y)/3 + 2/9] + C$

   (c) $16 - 7\sqrt{5}$

   (d) $(1/2)\ln|x^2 + 2x + 5| + C$

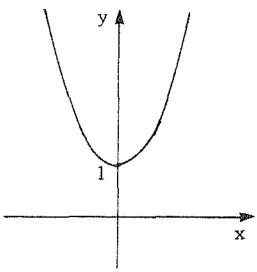
   (e) $[(\ln 4)^2 - (\ln 3)^2]/2$

4. (a) $y = 3 \exp(-y/4)$

(b) $2x \operatorname{sech}^2(x^2 + 3)$

(c) $(e^t - e^{-t})/2$

(d)



(e) $[F(3t)]/3 + C$

5. $(3/5, 12/35)$

6. $\ln 6$

7. (a) $y = \exp(x^2/2)$

(b) $y = (3e^x - \sin x - \cos x)/2$

(c) $\cos\sqrt{7}x + (1/\sqrt{7})\sin\sqrt{7}x$

8. (a) $340\pi/21$

(b) $124\pi/15$

9. $\ln 3/1.15$ year

10. 3.3×10^{14} joules

CHAPTER 10

FURTHER TECHNIQUES AND APPLICATIONS OF INTEGRATION

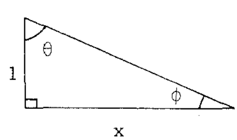
10.1 Trigonometric Integrals

PREREQUISITES

1. Recall how to differentiate and integrate trigonometric functions (Sections 5.2 and 7.1).
2. Recall how to integrate by substitution (Section 7.2).
3. Recall how to define trigonometric functions in terms of the sides of a right triangle (Section 5.1).
4. Recall how to complete a square (Section R.1).

PREREQUISITE QUIZ

1. Differentiate $y = \sin 4x - \cos^2 x + \tan x$.
2. (a) Evaluate $\int \cos 3x\, dx$.

   (b) Evaluate $\int \sin(2x - 3)dx$.
3. 

   Consider the figure at the left.

   (a) What is $\sin \phi$?

   (b) What is $\cos \theta$?
4. Complete the square in the following:

   (a) $x^2 + 2x + 4$

   (b) $x^2 - 3x - 1$

GOALS

1. Be able to use trigonometric identities for integrating expressions involving products of sines and cosines.

2. Be able to use trigonometric substitution for integrating expressions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{a^2 + x^2}$, or $a^2 + x^2$.

STUDY HINTS

1. Half-angle formulas. You should memorize or learn to derive $\sin^2 x = (1 - \cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$. If you forget which sign goes with which formula, substitute $x = 0$ as a check. These formulas can be derived by using $\cos 2x = \cos^2 x - \sin^2 x$ and $1 = \cos^2 x + \sin^2 x$. Adding and subtracting yields the desired results. The half-angle formulas are commonly used for integration.

2. Integrating $\sin^m x \cos^n x$. Basically, if one exponent is odd, use the identity $\cos^2 x + \sin^2 x = 1$ and substitute $u = \sin x$ or $u = \cos x$, whichever is the odd power . If both are even, use the half-angle formulas. The trigonometric integal box on p. 458 is a good one to know.

3. Substituting $u = \sec x$. When $\tan x$ and $\sec x$ appear, it is sometimes useful to substitute $u = \sec x$. Often, it is necessary to rewrite the integrand into a useful form. See Example 3(c).

4. Addition and product formulas. Knowing that $\sin 2x = 2 \sin x \cos x$ and that $\cos 2x = \cos^2 x - \sin^2 x$ may help you recall formulas (1a) and (1b) on p. 460. Simply substitute x for y . The addition formulas, along with $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, are useful for deriving the product formulas. If you choose to memorize the product formulas, which are useful for integration, note that the angle $x-y$ always appears first as it is written on p. 460. Also, it may be

4. (continued)

helpful to ask yourself, "What if $x = y$?" In addition, note that only $\sin x \cos y$ has sine terms on the right-hand side.

5. Trigonometric substitutions. This technique is often used when $(\pm a^2 \pm x^2)^{n/2}$ appears in the integrand, where n = integer and a are constants. Notice that this technique is based upon the Pythagorean identities $\cos^2 x + \sin^2 x = 1$ and $\sec^2 x = 1 + \tan^2 x$. Know what substitution to use for x in each of the three cases. Using the substitution equation, draw an appropriate triangle and label it like those in the box on p. 461. After the integration is completed, use the triangle to express your answer in the original variable.

6. Integrating $\sec x$ and $\csc x$. Just note the interesting trick used in the integration on p. 462 (lines 3 - 5).

7. Integrating $\sec^3 x$. Often, when $\sqrt{1 + x^2}$ appears in the integrand, trigonometric substitution will call for the integration of $\sec^3 x$. The technique is shown in the solution to Example 8(a), p. 496. Note that $\sec^3 x$ is integrated by parts.

8. Integrals involving $ax^2 + bx + c$. In many instances, the first step is to complete the square. Then, use a trigonometric substitution.

9. Example 5 comment. If $a = \pm b$, remember that $\cos(0) = 1$.

10. Practice. A lot of material has been covered in this section. Items placed in memory are easy to forget. Practice helps to reinforce what has been memorized.

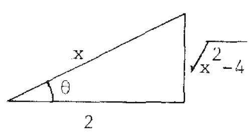
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using the identity, $\sin^2 x = 1 - \cos^2 x$, we get $\int \sin^3 x \cos^3 x\, dx = \int (1 - \cos^2 x)\sin x \cos^3 x\, dx$. Now, substitute $u = \cos x$, so $du = -\sin x$, and the integral becomes $-\int (1 - u^2)u^3 du = \int (u^5 - u^3) du = u^6/6 - u^4/4 + C = \cos^6 x/6 - \cos^4 x/4 + C$.

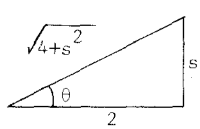
5. The half-angle formula $\cos^2 x = (1 + \cos 2x)/2$ gives us $\cos 2x - \cos^2 x = (\cos 2x - 1)/2$. Thus, $\int (\cos 2x - \cos^2 x)dx = \int (\cos 2x - 1)dx/2 = \sin 2x/4 - x/2 + C$.

9. Using the product formula $\sin x \sin y = (1/2)[\cos(x - y) - \cos(x + y)]$, we get $\int \sin 4x \sin 2x\, dx = \int (\cos 2x - \cos 6x)/2 = \sin 2x/4 - \sin 6x/12 + C$.

13. By using the hint, $\int \tan^3 x \sec^3 x\, dx$ becomes $\int (\sin^3 x/\cos^6 x)dx = \int [\sin x(1 - \cos^2 x)/\cos^6 x]dx$. Now, let $u = \cos x$, so $du = -\sin x\, dx$; therefore, the integral becomes $-\int [(1 - u^2)/u^6] du = \int (u^{-4} - u^{-6}) du = u^{-3}/(-3) - u^{-5}/(-5) + C = -1/3 \cos^3 x + 1/5 \cos^5 x + C = -\sec^3 x/3 + \sec^5 x/5 + C$.

17. 

Let $x = 2 \sec \theta$, so $dx = 2 \sec \theta \tan \theta\, d\theta$. Then $\int (\sqrt{x^2 - 4}/x)dx = \int (2\sqrt{\sec^2\theta - 1}/2 \sec \theta) \times 2 \sec \theta \tan \theta\, d\theta = 2\int \tan^2\theta\, d\theta = 2\int (\sin^2\theta/\cos^2\theta)d\theta = 2\int [(1 - \cos^2\theta)/\cos^2\theta] d\theta = 2\int (\sec^2\theta - 1)d\theta = 2(\tan \theta - \theta) + C$. We used the identities $\tan \theta = \sin \theta/\cos \theta$ and $\sin^2\theta + \cos^2\theta = 1$. Now, use the figure to get $2(\tan \theta - \theta) + C = 2(\sqrt{x^2 - 4}/2 - \cos^{-1}(2/x)) + C$.

21. 

Let $s = 2 \tan \theta$, so $ds = 2 \sec^2\theta\, d\theta$. Then $\int (s/\sqrt{4 + s^2})ds$ becomes $\int [2 \tan \theta/\sqrt{4(1 + \tan^2\theta)}] \times 2 \sec^2\theta\, d\theta = 2\int \tan \theta \sec \theta\, d\theta = 2 \sec \theta + C$.

From the figure, we get the final answer, $\sqrt{4 + s^2} + C$.

25. Completing the square, we have $4x^2 + x + 1 = 4(x + 1/8)^2 + (-1/16 + 1) = 4(x + 1/8)^2 + 15/16$. Let $u = x + 1/8$, so $du = dx$. Then, $\int(dx/\sqrt{4x^2 + x + 1}) = \int(du/\sqrt{4u^2 + 15/16})$. Factoring out $1/\sqrt{15/16}$ and letting $\theta = 8u/\sqrt{15}$, we have $\int(dx/\sqrt{4x^2 + x + 1}) = \int(2\ d\theta/\sqrt{\theta^2 + 1}) = 2 \sinh^{-1}\theta + C = (1/2) \sinh^{-1}(8u/\sqrt{15}) + C = (1/2) \times \sinh^{-1}((8x + 1)/\sqrt{15}) + C$.

29. Note that $\cos(\pi + x) = -\cos x$, so if n is odd, $\cos^n(\pi + x) = -\cos^n x$. Hence, $\int_0^{2\pi} \cos^n x\ dx = \int_0^{\pi} \cos^n x\ dx + \int_{\pi}^{2\pi} \cos^n x\ dx = \int_0^{\pi} \cos^n x\ dx + \int_0^{\pi} \cos^n(x + \pi)dx = \int_0^{\pi}(\cos^n x - \cos^n x)dx = \int_0^{2\pi} 0\ dx = 0$. Therefore, we need to consider only even values of n for four cases.

(a) For $n = 0$, $(1/2\pi)\int_0^{2\pi} \cos^0 x\ dx = (1/2\pi)\int_0^{2\pi} 1\cdot dx = 2\pi/2\pi = 1$.

(b) For $n = 2$, $(1/2\pi)\int_0^{2\pi} \cos^2 x\ dx = (1/4\pi)\int_0^{2\pi}(1 + \cos 2x)dx = (1/4\pi)(x + (1/2)\sin 2x)\Big|_0^{2\pi} = 1/2$.

(c) For $n = 4$, $(1/2\pi)\int_0^{2\pi} \cos^4 x\ dx = (1/2\pi)\int_0^{2\pi}(\cos^2 x)^2 dx = (1/8\pi)\int_0^{2\pi}(1 + \cos 2x)^2 dx = (1/8\pi)\int_0^{2\pi}(1 + 2 \cos 2x + \cos^2 2x)dx = (1/8\pi)\int_0^{2\pi}[1 + 2 \cos 2x + (1/2)(1 + \cos 4x)]dx = (1/8\pi)(x + \sin 2x + x/2 + (1/8)\sin 4x)\Big|_0^{2\pi} = 3/8$.

(d) For $n = 6$, $(1/2\pi)\int_0^{2\pi}\cos^6 x\ dx = (1/2\pi)\int_0^{2\pi}(\cos^2 x)^3 dx = \int_0^{2\pi}(1/(16\pi))(1 + \cos 2x)^3 dx = (1/(16\pi))\int_0^{2\pi}(1 + 3 \cos 2x + 3 \cos^2 2x + \cos^3 2x)dx = (1/16\pi)\int_0^{2\pi}(1 + 3 \cos 2x + (3/2)(1 + \cos 4x) + (1 - \sin^2 2x)\cos 2x)dx = (1/16\pi)(x + (3/2)\sin 2x + (3/2)x + (3/8)\sin 4x)\Big|_0^{2\pi} + (1/16\pi)\int_0^{2\pi}(\cos 2x - \cos 2x \sin^2 2x)dx = (1/16\pi)[(5\pi) + ((1/2)\sin 2x - (1/6)\sin^3 2x)\Big|_0^{2\pi}] = 5/16$.

33. Substitute $i = 10 \sin(377t)$ and a half-angle formula to get $P = (1/T)\int_0^T R\cdot 100 \sin^2(377t)dt = (50R/T)\int_0^T(1 - \cos(754t))dt = (50R/T)(t - (1/754)\sin(754t))\Big|_0^T = (50R/T)(T - (1/754)\sin(754T)) = 50R - (50R/754T) \times \sin(754T)$. Now, substitute $R = 2.5$ and $T = 2\pi/377$ to get $P = 125 - (125\cdot 377/(2\pi\cdot 754)) \sin 4\pi = 125$.

37. (a) Differentiation yields $(d/dt)[(S(t))^3] = 3(S(t))^2 S'(t) = 3(t \sin t + \sin^2 t \cos^2 t)$. Now, integrate to get $(S(t))^3 = 3\int_c^t (x \sin x + \sin^2 x \cos^2 x)dx$. Substitute $t = 0$ into the given equation to get $(S(0))^2 S'(0) = 0 = S(0)$. Thus, $S(0)^3 = 0 = 3\int_c^0 (x \sin x + \sin^2 x \cos^2 x)dx$, so $c = 0$. Therefore, $(S(t))^3 = 3\int_0^t (x \sin x + \sin^2 x \cos^2 x)dx$.

(b) Let $I(t) = \int_0^t (x \sin x + \sin^2 x \cos^2 x)dx$. Let $A(t) = \int_0^t x \sin x\, dx$ and let $B(t) = \int_0^t \sin^2 x \cos^2 x\, dx$. Then $I(t) = A(t) + B(t)$. In $A(t)$, let $u = x$ so $du = dx$, and let $dv = \sin x\, dx$, so $v = -\cos x$. Then $A(t) = -x \cos x\big|_0^t + \int_0^t \cos x\, dx = (-x \cos x + \sin x)\big|_0^t = -t \cos t + \sin t$. Since $\sin x \cos x = (1/2) \sin 2x$, $B(t) = (1/4)\int_0^t \sin^2 2x\, dx = (1/8) \times \int_0^t (1 - \cos 4x)dx = (1/8)(x - (1/4)\sin 4x)\big|_0^t = (1/8)/(t - (1/4)\sin 4t)$ Then $I(t) = -t \cos t + \sin t + t/8 - (1/32) \sin 4t$. Thus, $S(t) = \sqrt[3]{3I(t)} = [3(-t \cos t + \sin t + t/8 - (1/32)\sin 4t)]^{1/3}$.

(c) $S'(t) = 0$ whenever $t \sin t + \sin^2 t \cos^2 t = 0$; i.e., whenever $\sin t = 0$ or $t + \sin t \cos^2 t = 0$. Since $-1 \leqslant \sin t \leqslant 1$ and $-1 \leqslant \cos \leqslant 1$, $-1 < \sin t \cos^2 t < 1$. Thus, if $t > 1$, $t + \sin^2 t \cos^2 t > 1 - 1 = 0$. Therefore, the only zeros of $S'(t)$ for $t > 1$ are when $\sin t = 0$, i.e., $t = n\pi$. These are critical values of t , corresponding to relative maxima and minima of $S(t)$. When n is odd, $\sin t$, and correspondingly, $S'(t)$ changes from positive to negative, indicating a relative maximum excursion. There is no absolute maximum, since $S(t + 2\pi) > S(t)$.

SECTION QUIZ

1. Use the technique of trigonometric substitution to evaluate $\int \left[dx/\sqrt{1 - x^2} \right]$. Did you get the expected result?

2. Evaluate the following integrals:

   (a) $\int \cos^7\theta \, d\theta$

   (b) $\int \cos^6 t \, dt$

   (c) $\int \sec^{97} x \tan x \, dx$

   (d) $\int \left[\sin(3x/2)\cos 2x + 1/\sqrt{x^2 - 1} \right] dx$

   (e) $\int \sin 3x \sin 4x \, dx$

   (f) $\int [3/(1 + t^2)^{3/2}] \, dt$

3. Find the center of mass of the region bounded by $y = x \cos^2 x$, the x-axis, and the lines $x = 0$, $x = \pi/2$. [Hint: You may find it helpful to integrate $x^2 \cos \alpha x$ for a constant α.]

4. Evaluate the following integrals:

   (a) $\int \left[(3x + 5)/\sqrt{x^2 + 4x - 1} \right] dx$

   (b) $\int [(4x + 2)/(x^2 + 6x + 10)^{1/2}] \, dx$

   (c) $\int [(x - 1)/(x^2 - 2x - 5)] \, dx$

5. Underwater divers have recently discovered a sea monster at the ocean floor. Its shape has been described as follows: The region between $(4 - x^2)^{1/4}$ and $\sin x$ on $[0, \pi/4]$, revolved around the x-axis. If each unit represents 10 meters and the sea monster has a density of 1500 kg/m^3, how much does it weigh?

ANSWERS TO PREREQUISITE QUIZ

1. $4 \cos 4x + 2 \sin x \cos x + \sec^2 x$

2. (a) $\sin 3x/3 + C$

   (b) $-\cos(2x - 3)/2 + C$

3. (a) $1/\sqrt{1 + x^2}$

   (b) $1/\sqrt{1 + x^2}$

4. (a) $(x + 1)^2 + 3$

   (b) $(x - 3/2)^2 - 13/4$

ANSWERS TO SECTION QUIZ

1. $\sin^{-1}x + C$, as expected

2. (a) $\sin\theta - \sin^3\theta + 3\sin^5\theta/5 - \sin^7\theta/7 + C$

   (b) $5t/16 + \sin 2t/4 + 3\sin 4t/64 - \sin^3 2t/48 + C$

   (c) $\sec^{97}x/97 + C$

   (d) $\cos(x/2) - \cos(7x/2)/7 + \ln|x + \sqrt{x^2 - 1}| + C$

   (e) $\sin x/2 - \sin 7x/14 + C$

   (f) $3t/\sqrt{1 + t^2} + C$

3. $(\pi(\pi^2 - 6)/3(\pi^2 - 4), \pi(2\pi^2 - 15)/16(\pi^2 - 4))$

4. (a) $3\sqrt{x^2 + 4x - 1} - \ln|x + 2 + \sqrt{x^2 + 4x - 1}| + C$

   (b) $4\sqrt{x^2 + 6x + 10} - 10\ln|x + 3 + \sqrt{x^2 + 6x + 10}| + C$

   (c) $\ln\sqrt{x^2 - 2x - 5} + C$

5. $1500000\pi[(\pi/8)(\sqrt{4 - \pi^2/16} - 1) + 2\sin^{-1}(\pi/8) - 1/4] \approx 4.2 \times 10^6$ kg

10.2 Partial Fractions

PREREQUISITES

1. Recall how to factor a polynomial (Section R.1).
2. Recall how to integrate by the method of trigonometric substitution (Section 10.1).
3. Recall how to integrate by the method of substitution (Section 7.2).

PREREQUISITE QUIZ

1. Evaluate $\int\left(x^2/\sqrt{4 - x^2} + x + 2\right)dx$.
2. Evaluate $\int\left(1/\sqrt{x^2 + 2x + 2}\right)dx$.
3. Factor the following polynomials:
   (a) $x^3 - 27$
   (b) $x^3 - x^2$
   (c) $x^2 + 5x + 6$

GOALS

1. Be able to integrate rational expressions by the technique of partial fractions.

STUDY HINTS

1. Beginning partial fractions. Look before you leap! There may be an easier method. See Example 7. If partial fractions is the method of choice, be sure the degree of the denominator is larger than the degree of the numerator. If not, begin with long division.
2. Denominator factorization. All factors should be of degree one or two. If not, the denominator can be factored further. Check to be sure quadratic factors do not factor further (by using the quadratic formula,

2. (continued)

if necessary). Don't forget that $x = x - 0$, so x^2 is composed of the linear factors $(x - 0)\cdot(x - 0)$. See Example 4. Also, remember that a factor raised to the n^{th} power must be represented n times.

3. Determination of coefficients. This method is called comparing coefficients: Multiply so that both sides of the equation are over a common denominator. Rather than expanding, it is best to leave the expression as a sum of factored terms. Then, substitute values of x so that as many terms as possible become zero. The result should be a few, simple linear equations. If no such x's are left, choose any other constants.

4. Differentiating to determine coefficients. To solve Example 2, this author prefers Method 1. I find that there's a greater chance for error with the differentiation process; you may disagree.

5. Comparing integrands. After you have found the coefficients, use a calculator and an arbitrary number to compare the original integrand with your new one.

6. Rationalizing substitutions. If $[f(x)]^{p/q}$ appears in the integrand, you might be able to make a simplification by substituting $u = [f(x)]^{1/q}$, i.e., $u^q = f(x)$. See Example 8.

7. Integrals of rational trigonometric expressions. The technique used in Examples 10 and 11 is useful for rational functions in $\sin x$ and $\cos x$ Rather than memorizing equations (8), (9), and (10), it is suggested that you reproduce Fig. 10.2.2, and use the identities $\sin 2x = 2 \sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$ to complete the substitution. Often, partial fractions may be necessary to finish the integration.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $1/(x-2)^2(x^2+1)^2 = A/(x-2) + B/(x-2)^2 + (Cx+D)/(x^2+1) + (Ex+F)/(x^2+1)^2$. Then we need to solve $1 = A(x-2)(x^2+1)^2 + B(x^2+1)^2 + (Cx+D)(x-2)^2(x^2+1) + (Ex+F)(x-2)$. If $x = 2$, $1 = B(2^2+1)^2$, i.e., $B = 1/25$. Substituting in $B = 1/25$ and comparing coefficients gives $A + C = 0$; $-2A - 4C + D = -1/25$; $2A + 5C - 4D + E = 0$; $-4A - 4C + 5D - 4E + F = -2/25$; $A + 4C - 4D + 4E - 4F = 0$; and $-2A + 4D + 4F = 24/25$. Solving this set of equations gives: $A = -8/125$, $B = 5/125$, $C = 8/125$, $D = 11/125$, $E = 20/125$, and $F = 15/125$. Thus, $\int dx/[(x-2)^2(x^2+1)^2] = (1/125)\int[-8/(x-2) + 5/(x-2)^2 + (8x+11)/(x^2+1) + (20x+15)/(x^2+1)^2]dx = (1/125) \times \{-8\ \ln|x-2| - 5/(x-2) + 4\int[2x\,dx/(x^2+1)] + 11\tan^{-1}x + 10\int[2x/(x^2+1)^2]dx + 15\int[dx/(x^2+1)^2]\}$. The last integral can be evaluated by using trigonometric substitution with $\tan\theta = x$, i.e., $\int[dx/(x^2+1)^2] = \int(\sec^2\theta\, d\theta/\sec^4\theta) = \int\cos^2\theta\, d\theta = \int[(1+\cos 2\theta)/2]d\theta = \theta/2 + \sin 2\theta/4 + C = \theta/2 + \sin\theta\ \cos\ \theta/2 + C = (1/2)\tan^{-1}x + x/2(1+x^2) + C$. Hence, $\int[dx/(x-2)^2(x^2+1)^2] = (1/125)\{-8\ \ln|x-2| - 5/(x-2) + 4\ \ln(x^2+1) + 11\tan^{-1}x - 10/(x^2+1) + (15/2)\tan^{-1}x + 15x/2(1+x^2)\} + C = (1/125)\{4\ \ln[(x^2+1)/(x^2-4x+4)] + (37/2)\tan^{-1}x + (15x-20)/2(1+x^2) - 5/(x-2)\} + C$.

5. Since the discriminant $(b^2 - 4ac)$ for $x^2 + 2x + 2$ is less than zero, we cannot factor it further. Thus, $x^2/(x-2)(x^2+2x+2) = A/(x-2) + (Bx+C)/(x^2+2x+2) = [(A+B)x^2 + (2A-2B+C)x + (2A-2C)]/(x-2)(x^2+2x+2)$. Comparing coefficients and solving simultaneous equations gives $A = 2/5$, $B = 3/5$, and $C = 2/5$. Thus, $\int[x^2/(x-2)(x^2+2x+2)]dx = (1/5)\int[2/(x-2) + (3x+2)/(x^2+2x+2)]dx$.

5. (continued)

The first term gives $\int[2/(x-2)]\,dx = \ln[(x-2)^2]$. For the second term, note that $(3x+2)/(x^2+2x+2) = (3/2)[(2x+2)/(x^2+2x+2)] - 1/(x^2+2x+1+1)$. The reason we break it up this way is because we want to have $d(x^2+2x+2)$ in the numerator and whatever is left constitutes another term. Thus, $\int[(3x+2)/(x^2+2x+2)]\,dx = (3/2)\times \int[(2x+2)/(x^2+2x+2)]\,dx - \int[1/((x+1)^2+1)]\,dx = (3/2)\ln(x^2+2x+2) - \tan^{-1}(x+1)$. All this information gives $\int[x^2/(x-2)(x^2+2x+2)]\,dx = (1/5)\{\ln(x-2)^2 + (3/2)\ln(x^2+2x+2) - \tan^{-1}(x+1)\} + C$.

9. Factoring x^4+2x^2-3 gives $(x^2+3)(x+1)(x-1)$. Using the technique of partial fractions, we have $x/(x^4+2x^2-3) = (Ax+B)/(x^2+3)$ $C/(x+1) + D/(x-1) = [(Ax+B)(x^2-1) + C(x^2+3)(x-1) + D(x^2+3)\times(x-1)]/(x^2+3)(x^2-1)$. Comparing coefficients gives the following results: x^3: $A+C+D=0$; x^2: $B-C+D=0$; x: $-A+3C+3D=1$ $x^0(1)$: $-B-3C+3D=0$. Solving these four equations simultaneously gives $A=-1/4$, $B=0$, $C=1/8$, and $D=1/8$. Thus, $\int[x/(x^4+2x^2-3)]\,dx = (1/8)\int[-2x/(x^2+3) + 1/(x+1) + 1/(x-1)]\,dx = (1/8)\times[-\ln(x^2+3) + \ln|x+1| + \ln|x-1|] + C = (1/8)\ln|(x^2-1)/(x^2+3)| + C$

13. We apply the method of Example 8 by letting $u=\sqrt{x}$, so $u^2=x$, and $2u\,du = dx$. Thus, $\int[\sqrt{x}/(1+x)]\,dx = \int[u/(1+u^2)]2u\,du = 2\int[u^2/(1+u^2)]\,du = 2\int[(u^2+1-1)/(u^2+1)]\,du = 2\int du - 2\int[1/(u^2+1)]\,du = 2u - 2\tan^{-1}u + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$.

17. We use the technique of Example 11. Substitute $u=\tan(x/2)$, so $\sin x = 2u/(1+u^2)$ and $dx = 2du/(1+u^2)$. Thus, $\int[dx/(1+\sin x)] = \int\{[2du/(1+u^2)]/[1+2u/(1+u^2)]\} = \int[2du/(1+u^2+2u)] = 2\int(u+1)^{-2}du = -2(u+1)^{-1} + C = -2/(1+\tan(x/2)) + C$.

21. By the shell method, $V = 2\pi\int_5^6 [x/((1-x)(1-2x))]\,dx$. Now, partial fractions yield $2\pi\int_5^6 [A/(1-x) + B/(1-2x)]\,dx$, with $A/(1-x) + B/(1-2x) = x/((1-x)(1-2x))$. Therefore, $A(1-2x) + B(1-x) = x$ for all x . Let $x = 1/2$, so $B/2 = 1/2$, i.e., $B = 1$. Let $x = 1$, so $-A = 1$, i.e., $A = -1$. Therefore, $V = 2\pi\int_5^6 [-1/(1-x) + 1/(1-2x)]\,dx = 2\pi[\ln(x-1) - (1/2)\ln(2x-1)]\Big|_5^6 = 2\pi(\ln 5 - \ln\sqrt{11} - \ln 4 + \ln 3) = \pi\ln(225/176)$.

25. (a) Integrating the right-hand side and using the method of partial fractions for the left-hand side, we get $kt + C = \int[A/(80-x) + B/(60-x)]\,dx$ for constants A , B , and C , where $A(60-x) + B(80-x) = 1$ for all x . Let $x = 60$, so $20B = 1$, or $B = 1/20$. Let $x = 80$, so $-20A = 1$, or $A = -1/20$. Thus, $kt + C = (1/20)\int[-1/(80-x) + 1/(60-x)]\,dx = (1/20)\int[1/(x-80) - 1/(x-60)]\,dx = (1/20)\ln|(x-80)/(x-60)|$. Since $x = 0$ when $t = 0$, we have $C = (1/20)\ln(4/3)$, and the formula becomes $kt + (1/20)\ln(4/3) = (1/20)\ln|(x-80)/(x-60)|$.

(b) Rearrangement of the equation in part (a) gives $20kt + \ln(4/3) = \ln|(x-80)/(x-60)|$ and exponentiation yields $(4/3)\exp(20kt) = |(x-80)/(x-60)|$. Since we assume $x < 60$, the formula becomes $(4/3)\exp(20kt) = (x-80)/(x-60)$ or $(4/3)(x-60) \times \exp(20kt) = x - 80$. Rearrange again to get $[(4/3)\exp(20kt) - 1]x = -80 + 80\exp(20kt)$. Thus, $x = 80[1 - \exp(20kt)]/[1 - (4/3)\exp(20kt)]$

(c) If $x = 20$ when $t = 10$, we substitute into the formula in part (a) to get $(1/20)\ln(3/2) = 10k + (1/20)\ln(4/3)$ or $(1/20)[\ln(3/2) - \ln(4/3)] = (1/20)\ln(9/8) = 10k$. Therefore, $k = \ln(9/8)/200$. Now, substitute $k = \ln(9/8)/200$ and $t = 15$ into the formula in part (b) to get $x = 80[1 - \exp(3\ln(9/8)/2)]/[1 - (4/3)\exp(3\ln(9/8)/2] = 80(1 - (9/8)^{3/2})/(1 - (4/3)(9/8)^{3/2}) \approx 26.2$ kg .

SECTION QUIZ

1. Evaluate $\int [dx/(2x^3 + 4x^2)]$.

2. Evaluate $\int_1^{\sqrt{2}} [dx/(x^2 + 2)^2 x]$.

3. Evaluate $\int [dx/(x^2 - 2)(x - 1)]$.

4. Find the average of $f(t) = t^8\sqrt{t^3 + 2}/(1 - t^6)$ on the interval $2 \leqslant t \leqslant 4$. (Hint: let $u = \sqrt{t^3 + 2}$.)

5. Evaluate $\int_0^{\pi/4} [\tan\theta/(1 + \sec\theta)]\, d\theta$.

6. When the minister asked for any objections, the bride's grandfather said, "Sonny-boy has to prove to me he's smart enough to be her husband. Look at that archway. $y = 10 - x/(x^2 - 2x - 3)$ on $[0,2]$ describes it. He can marry her if he knows the area under that archway." What answer would bring happiness to the young couple?

ANSWERS TO PREREQUISITE QUIZ

1. $2\sin^{-1}(x/2) - x\sqrt{4 - x^2}/2 + x^2/2 + 2x + C$

2. $x + 1 + \sqrt{x^2 + 2x + 2} + C$

3. (a) $(x - 3)(x^2 + 3x + 9)$

   (b) $x^2(x - 1)$

   (c) $(x + 3)(x + 2)$

ANSWERS TO SECTION QUIZ

1. $(1/8)\ln|(x + 2)/x| - 1/4x + C$

2. $(1/8)\ln(3/2) - 1/48$

3. $[(2 + \sqrt{2})/4]\ln|x + \sqrt{2}| + [(2 - \sqrt{2})/4]\ln|x - \sqrt{2}| - \ln|x - 1| + C$

4. $-(1/9)(66\sqrt{66} - 10\sqrt{10}) + (1/12)\ln[(\sqrt{66} - 1)(\sqrt{10} + 1)/(\sqrt{66} + 1)(\sqrt{10} - 1)] + (1/4\sqrt{3})\ln[(\sqrt{66} + \sqrt{3})(\sqrt{10} - \sqrt{3})/(\sqrt{66} - \sqrt{3})(\sqrt{10} + \sqrt{3})] \approx -56.14$

5. $\ln[1 + \tan^2(\pi/8)]$

6. $20 - \ln\sqrt{3}$

10.3 Arc Length and Surface Area

PREREQUISITES

1. Recall how to integrate by using trigonometric substitution (Section 10.1).
2. Recall how to compute the distance between two points in the plane (Section R.4).

PREREQUISITE QUIZ

1. What is the distance between the points (3,6) and (4,2) ?
2. Evaluate $\int (dx/\sqrt{x^2 - 1})$.

GOALS

1. Be able to express the length of a curve as an integral and perform the integration, if possible.
2. Be able to express the area of a surface of revolution as an integral and perform the integration, if possible.

STUDY HINTS

1. Definition. The notation ds is introduced in this section. It is an infinitesimal length defined by $\sqrt{(dx)^2 + (dy)^2}$. See Fig. 10.3.1.
2. Arc length. You should become familiar with $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. Either memorize this or learn to derive it. Starting with $ds = \sqrt{(dx)^2 + (dy)^2}$, substitute $dy = f'(x)dx$, factor out dx , and integrate. Since the integrand is positive, you made a mistake if your answer is negative or zero.
3. Constant with no effect. The length of $f + C$ is the same as that of f . Shifting a graph along the y-axis does not change its length. Compute the length of $(x - 1)^{3/2}$ on $[1,2]$ and compare with Example 2.

4. Tricks to find arc length. In general, many textbook and exam questions are chosen for their simplicity. Thus, the expression under the radical will often simplify; for example, it may be a perfect square. If it is not a perfect square, a trigonometric substitution will often be helpful.

5. Square roots in arc length problems. The appearance of square roots in arc length questions may sometimes present a special problem. When in doubt, take the absolute value of the expression inside the square root. For example, consider finding the arc length of $x^{2/3}$ on $[-8,-2]$. We get $\int_{-8}^{-2}\sqrt{(9x^{2/3}+4)/9x^{2/3}}\,dx = \int_{-8}^{-2}\left(\sqrt{9x^{2/3}+4}/3x^{1/3}\right)dx$ and substituting $u = 9x^{2/3}+4$ yields $\int_{\sqrt{40}}^{\sqrt{9(4)^{1/3}+4}}\sqrt{u}\,du/18 = (1/27)u^{3/2}\Big|_{\sqrt{40}}^{\sqrt{9(4)^{1/3}+4}}$, which is negative. The correct method is to use $\sqrt{9x^{2/3}} = 3|x^{1/3}|$.

6. Step function derivation (pp. 480-482). Except in honors classes, most instructors will not emphasize these theoretical aspects on their exams.

7. Area of revolution. Learn to derive formula (2) in the box on p. 483. It may be easier to think of the infinitesimal frustums as cylinders, so the area is circumference times width. The circumference is $2\pi r = 2\pi f(x)$ and the width is simply the arc length. Thus, $A = \int_a^b 2\pi f(x)L\,dx = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$.

8. Revolution around y-axis. Here, the circumference is $2\pi x$ rather than than $2\pi y$. The width is still the arc length. Thus, $A = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$. The only difference with the preceding formula is that $f(x)$ appears in one and x appears in the other.

9. Mathematical illusion? The bands in Fig. 10.3.8 are equal in area. Why? The smaller radius is compensated for by a larger ds.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Given that $f(x) = x^4/8 + 1/4x^2$, we have $f'(x) = x^3/2 - 1/2x^3$. Thus the length of the graph is $L = \int_1^3\sqrt{1 + (x^3/2 - 1/2x^3)^2}dx = \int_1^3\sqrt{x^6/4 + 1/2 + 1/4x^6}dx = \int_1^3(x^3/2 + 1/2x^3)dx = (x^4/8 - 1/4x^2)\big|_1^3 = 92/9$.

5. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$ and $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + n^2x^{2n-2}}$. Thus, the length is $L = \int_a^b\sqrt{1 + n^2x^{2n-2}}dx$.

9. This exercise is analogous to Example 5. By direct computation, we have $L = [(1 - 0)^2 + (2 - 0)^2]^{1/2} + [(2 - 1)^2 + (1 - 2)^2]^{1/2} + [(5 - 2)^2 + (0 - 1)^2]^{1/2} = \sqrt{5} + \sqrt{2} + \sqrt{10}$. On the other hand, $f'(x) = 2$ on $[0,1]$, -1 on $[1,2]$, and $-1/3$ on $[2,5]$. Thus, the length is $\int_0^1\sqrt{1 + (2)^2}dx + \int_1^2\sqrt{1 + (-1)^2}dx + \int_2^5\sqrt{1 + (-1/3)^2}dx = \sqrt{5}x\big|_0^1 + \sqrt{2}x\big|_1^2 + \sqrt{10/9}x\big|_2^5 = \sqrt{5} + \sqrt{2} + \sqrt{10}$.

13. Given that $f(x) = (x + 1)^{1/2}$, we have $f'(x) = (1/2)(x + 1)^{-1/2}$. Thus, the area of the surface obtained by revolving $f(x)$ on $[0,2]$ about the x-axis is $A = 2\pi\int_0^2(x + 1)^{1/2}\sqrt{1 + [(1/2)(x + 1)^{-1/2}]^2}dx = 2\pi\int_0^2[\sqrt{4x + 5}/2]dx = (\pi/6)(4x + 5)^{3/2}\big|_0^2 = (\pi/6)(13^{3/2} - 5^{3/2}) \approx 18.7$.

17. $y' = -\sin x$, so the area is $A = 2\pi\int_{-\pi/2}^{\pi/2}\cos x\sqrt{1 + \sin^2 x}\,dx$. Let $u = \sin x$ so $du = \cos x\,dx$. Then $A = 2\pi\int_{-1}^1\sqrt{1 + u^2}du$. Integrate as in Example 3 to get $A = 2\pi[(u/2)\sqrt{u^2 + 1} + (1/2)\ln|u + \sqrt{u^2 + 1}|]\Big|_{-1}^1 = (2\pi)[(1/2)\sqrt{2} + (1/2)\ln(1 + \sqrt{2}) + (1/2)\sqrt{2} - (1/2)\ln(\sqrt{2} - 1)] = [\sqrt{2} + \ln\sqrt{(1 + \sqrt{2})/(\sqrt{2} - 1)}]2\pi = 2\pi(\sqrt{2} + \ln(1 + \sqrt{2})) \approx 14.42$.

21. This exercise is analogous to Example 9. We have $f(x) = x$ on $[0,1]$ and $f(x) = -x + 2$ on $[1,2]$. Thus, the area is $A = 2\pi\left[\int_0^1 x\sqrt{1 + 1^2}dx + \int_1^2(-x + 2)\sqrt{1 + (-1)^2}dx\right] = 2\sqrt{2}\pi[\int_0^1 x\,dx + \int_1^2(-x + 2)dx] = 2\sqrt{2}\pi[(x^2/2)\big|_0^1 + (-x^2/2 + 2x)\big|_1^2] = 2\sqrt{2}\pi[(1/2) + (-3/2 + 2)] = 2\sqrt{2}\pi$.

25. $[f'(x)]^2 = (3a\sqrt{x+b}/2)^2 = 9a^2(x+b)/4$, so the length $= s = \int_0^1 \sqrt{1 + 9a^2(x+b)/4}\,dx$. Let $u = 1 + 9a^2(x+b)/4$, so $du = 9a^2dx/4$; i.e., $dx = 4\,du/9a^2$. Then $s = \int_{x=0}^{x=1} 4u^{1/2}du/9a^2 = (8/27a^2)(1 + 9a^2(x+b)/4)^{3/2}\big|_0^1 = (8/27a^2)[(1 + 9a^2(1+b)/4)^{3/2} - (1 + 9a^2b/4)^{3/2}]$. Note that $4^{3/2} = 8$, so $s = (1/27a^2)[(4 + 9a^2(1+b))^{3/2} - (4 + 9a^2b)^{3/2}]$. Changing c has no effect, since c dropped out when we found $f'(x)$.

29. $f'(x) = \sec^2 x + 2$, so the length is $\int_0^{\pi/2}\sqrt{1 + [f'(x)]^2}dx = \int_0^{\pi/2}\sqrt{5 + \sec^4 x + 4\sec^2 x}\,dx$. And the area is $2\pi\int_0^{\pi/2} f(x)\sqrt{1 + [f'(x)]^2}dx = 2\pi\int_0^{\pi/2}(\tan x + 2x)\sqrt{5 + \sec^4 x + 4\sec^2 x}\,dx$.

33. The area of a surface of revolution is $2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx = 2\pi\int_a^b f(x)ds$. For a small arc length where $f(x)$ is almost constant, this is approximately 2π multiplied by the function value multiplied by the small length. To find the area of the surface obtained by revolving the given curve, divide the curve into 1 mm segments. On each segment, measure the distance from the x-axis to each endpoint, and then take the average value as $f(x)$. Using this method, our answer is about $16\ \text{cm}^2$.

37. Let s_1 be the length of $\sin x$ and s_2 be the length of $1 + x^4$. Then $s_1 = \int_{0.1}^1 \sqrt{1 + \cos^2 x}\,dx$. Since $\cos x \leqslant 1$, $s_1 \leqslant \int_{0.1}^1 \sqrt{2}\,dx = \sqrt{2}(1 - 0.1) = 0.9\sqrt{2} \approx 1.28$. Now s_2 must be longer than the line from $(0.1, 1.0001)$ to $(1,2)$, so $s_2 > \sqrt{(0.9)^2 + (0.9999)^2} \approx 1.34$. Thus, $s_2 > s_1$.

41. (a) Cutting and unrolling the frustum as in Fig. 10.3.12, we found the area to be $\pi s(r_1 + r_2)$ if a linear function is revolved around the x-axis. Since the formula is independent of x and y, the formula for the area obtained by revolving a linear segment around

41. (a) (continued)

the y-axis is also $\pi s(r_1 + r_2)$, which works out to be $\pi(a + b)\sqrt{1 + m^2}(b - a)$, since $s = \sqrt{1 + m^2}(b - a)$.

(b) From part (a), the area is $\pi\sqrt{1 + m^2}(b^2 - a^2) = 2\pi\sqrt{1 + m^2}(b^2/2 - a^2/2) = 2\pi\sqrt{1 + m^2}(x^2/2)\big|_a^b = \int_a^b 2\pi x\sqrt{1 + m^2}dx$. Since $m = f'(x)$, we have $A = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}dx$. Let the entire surface be composed of a finite sequence of frustums of cones and use the additivity property of the integral to obtain formula (3).

SECTION QUIZ

1. Compute the arc length of the following functions:

   (a) $y = x^5/2 + 1/30x^3$ on $[1,t]$, $t > 1$

   (b) $y = \sqrt{x}$ on $[1,2]$ (Hint: Substitute $u = \sqrt{4x}$.)

   (c) $y = (2x + 5)^{3/2} + 3$ on $[0,3]$

2. Suppose the arc length of f on $[a,b]$ is L . What can you say about the arc length of $g = f + k$ on $[a,b]$, where k is a constant?

3. Find the area of the surfaces obtained by revolving the curves in Question 1 around the y-axis. (Hint: Use formulas 65 and 66 of the integral table for part (b).)

4. Find the area of the surfaces obtained by revolving the curves in Question 1, parts (a) and (b), around the x-axis.

5. Upon close inspection, you notice that your pet tarantula's legs can be described by $y = 1 - t^{3/2}$ on $[0,1]$.

   (a) How long are the tarantula's legs?

   One day, while you were walking your tarantula, a cyclone suddenly appeared and twirled the spider around so fast that it seemed like a

5. (continued)

solid surface was being formed by revolving one of the legs around the y-axis.

(b) What was the apparent surface area?

ANSWERS TO PREREQUISITE QUIZ

1. $\sqrt{17}$

2. $\ln|x + \sqrt{x^2 + 1}| + C$

ANSWERS TO SECTION QUIZ

1. (a) $t^5/2 + 1/30t^3 - 7/15$

   (b) $(1/4)[6\sqrt{2} - 2\sqrt{5} + \ln((\sqrt{8} + 3)/(\sqrt{5} + 2))]$

   (c) $(1000 - 46\sqrt{46})/27$

2. They are equal.

3. (a) $\pi(5t^6/6 - 1/10t^2 - 11/15)$

   (b) $(51\sqrt{2} - 9\sqrt{5})/16 - (1/64)\ln(17 + 12\sqrt{12})/(9 + 4\sqrt{5}))$

   (c) $2\pi(35000 - 2116\sqrt{46})/1215$

4. (a) $\pi(t^{10}/4 - t^2/30 + 1/900t^6 - 49/225)$

   (b) $(\pi/6)(27 - 5\sqrt{5})$

5. (a) $(13\sqrt{13} - 8)/27$

   (b) $16(884\sqrt{13} - 4)/1215$

10.4 Parametric Curves

PREREQUISITES

1. Recall how to sketch curves described by parametric equations (Section 2.4).
2. Recall how to compute arc lengths (Section 10.3).

PREREQUISITE QUIZ

1. Sketch the curve described by $x = 2t + 1$ and $y = t + 2$.
2. Sketch the curve described by $x = t^2$ and $y = t^4$.
3. Compute the length of the graph of $y = 2x + 3$ on $[0,1]$.
4. Write a formula for the length of the graph of $y = x^3 - x^2$ on $[1,2]$. (Do not evaluate.)

GOALS

1. Be able to find the tangent line of a parametric curve.
2. Be able to relate speed to arc length for parametric curves.
3. Be able to express the length of a parametric curve as an integral and perform the integration, if possible.

STUDY HINTS

1. Direction of motion. Parametric curves move in a specified direction. This is illustrated in Example 1.
2. Converting parametric equations. One of the simplest ways to convert a parametric equation into the form $y = f(x)$ is to solve one equation for t and then substitute into the other equation.
3. Tangent lines. You should definitely know how to compute a tangent line for parametric equations. Remember that for $y = f(x)$, the tangent

3. (continued)

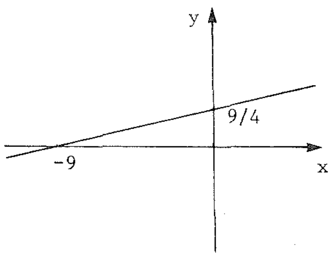
line is given by $y = y_0 + f'(x_0)(x - x_0)$. For the analogous parametric form, $f'(x_0)$ is replaced by $g'(t_0)/f'(t_0)$, y_0 by $g(t_0)$, and x_0 by $f(t_0)$.

4. Arc length. The infinitesimal argument is very easy to follow. Starting with $ds = \sqrt{dx^2 + dy^2}$, we multiply and divide by dt to get $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$. Integration yields the arc length formula.

5. Speed. It is probably best to memorize that speed is given by the formula $\sqrt{(dx/dt)^2 + (dy/dt)^2}$, which is the arc length integrand.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1.



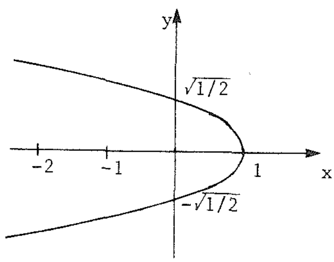
From $x = 4t - 1$, we get $t = (x + 1)/4$, and so $y = t + 2 = (x + 1)/4 + 2 = (1/4) \times (x + 9)$. The graph is a straight line with slope $1/4$ and y-intercept $9/4$.

5. One of the simplest parametric representations is to let $x = t$. $2x^2 + y^2 = 1$ becomes $y = \pm\sqrt{1 - 2x^2}$, so a parametric representation is $x = t$ and $y = \pm\sqrt{1 - 2t^2}$. Another one is $x = \sin t/\sqrt{2}$ and $y = \cos t$. Many other solutions are possible.

9. If we let $x = t$, then $y = t^3 + 1$. We can also let $x = \pm t^2$, so $y = \pm t^6 + 1$. Several other solutions are possible.

13. Given $x = f(t) = (1/2)t^2 + t$ and $y = g(t) = t^{2/3}$, we have $f'(t) = t + 1$ and $g'(t) = (2/3)t^{1/3}$. At $t_0 = 1$, we have $f(1) = 3/2$, $g(1) = 1$, $f'(1) = 2$, and $g'(1) = 2/3$. So the tangent line has the equation $y = [(2/3)/2](x - 3/2) + 1$, or $y = (1/3)(x + 3/2)$.

17. The parametric form of the tangent line equation is: $x = f'(t_0) \times (t - t_0) + f(t_0)$ and $y = g'(t_0)(t - t_0) + g(t_0)$. Here, $f'(t) = 2(2 - 3t)(-3)$ and $g'(t) = -3$. Thus, at $t_0 = 1$, we have $f(1) = 1$, $f'(1) = 6$, $g(1) = -1$ and $g'(1) = -3$; therefore, $x = 6(t - 1) + 1$ and $y = -3(t - 1) - 1$. At $t = 3$, the bead is at $(13,-7)$.

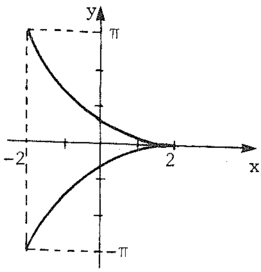
21. 

Since $\sin t = \pm\sqrt{(1 - \cos 2t)/2}$, $y = \pm\sqrt{(1 - x)/2}$, i.e., $y^2 = (1 - x)/2$. $x'(t) = -2 \sin 2t$ and $y'(t) = \cos t$. The tangent line should be horizontal when $y' = 0$, i.e., $t = (2n + 1)\pi/2$, where n is an integer. However, $x' = 0$ also at these points. From the graph, we see that the tangent is never horizontal. The graph should be vertical at $2t = n\pi$, i.e., $t = n\pi/2$. If n is even, $y' \neq 0$, so there is a vertical tangent. If n is odd, $y' = 0$, and the limit of y'/x' shows that the tangent is not vertical. So vertical tangents occur when $t = m\pi$ for all integers m .

25. Since $dx/dt = 2t$ and $dy/dt = 4t^3$, the length is $s = \int_0^1 \sqrt{4t^2 + 16t^6}\, dt = \int_0^1 2t\sqrt{1 + 4t^4}\, dt$. Let $u = 2t^2$ so $du = 4t\, dt$. Then $s = (1/2) \times \int_0^2 \sqrt{1 + u^2}\, du$. Integrate as in Example 3, Section 10.3, to get $s = (1/2) \times [(u/2)\sqrt{u^2 + 1} + (1/2) \ln(u + \sqrt{u^2 + 1})]\,\Big|_0^2 = (1/2)[\sqrt{5} + (1/2) \ln(2 + \sqrt{5})] \approx 1.48$.

29. (a) $dx/d\theta = -2 \sin \theta$ and $dy/d\theta = 1 - \cos \theta$. Then $dy/dx = (1 - \cos \theta)/(-2 \sin \theta)$. When $\theta = \pi/2$, $dy/dx = 1/(-2) = -1/2$, $x = 0$, and $y = \pi/2 - 1$. The tangent line is $[y - (\pi/2 - 1)]/x = -1/2$, so $y = -x/2 + \pi/2 - 1$.

29. (b)



$\theta = \cos^{-1}(x/2)$ and $\sin\theta = \sqrt{1 - x^2/4}$, so plot $y = \cos^{-1}(x/2) - \sqrt{1 - x^2/4}$.

(c) Let s denote the length. $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta$, so $s = \int_0^{\pi}\sqrt{4\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}\, d\theta =$ $\int_0^{\pi}\sqrt{5 - 3\cos^2\theta - 2\cos\theta}\, d\theta$.

33. (a) $\dot{x} = k(\cos\omega t - \omega t \sin\omega t)$; $\dot{y} = k(\sin\omega t + \omega t\cos\omega t)$.

(b) The speed is $\sqrt{\dot{x}^2 + \dot{y}^2} = k(\cos^2\omega t - 2\omega t\sin\omega t\cos\omega t +$ $\omega^2 t^2 \sin^2\omega t + \sin^2\omega t + 2\omega t\sin\omega t\cos\omega t + \omega^2 t^2\cos^2\omega t)^{1/2} =$ $k\sqrt{1 + \omega^2 t^2}$.

(c) $\ddot{x} = k(-\omega\sin\omega t - \omega\sin\omega t - \omega^2 t\cos\omega t) = -k\omega(2\sin\omega t +$ $\omega t\cos\omega t)$, and $\ddot{y} = k(\omega\cos\omega t + \omega\cos\omega t - \omega^2 t\sin\omega t) =$ $k\omega(2\cos\omega t - \omega t\sin\omega t)$. Hence, $\ddot{x}(0) = 0$ and $\ddot{y}(0) = 2k\omega$, so the Coriolis force is $m\sqrt{(2k\omega)^2} = 2mk\omega$.

37. (a) Using string on the map of The United States in the National Geographic Atlas of the World, p. 25, (scale 1:7,800,000), the coastline of Maine was estimated at 338 miles.

(b) Using string on the map of Maine in the State Farm Road Atlas, Rand McNally, 1974, p. 47, (scale 1 in. = 20 mi) gives 688 miles.

(c) It would probably be longer.

(d) The measurement will depend on the definition and on the scale (and therefore the available detail) of the maps used.

(e) From the World Almanac and Book of Facts - 1974, Newspaper Enterprise Assoc., New York, 1973, p. 744, we have coastline: 228 miles; shoreline: 3,478 miles.

SECTION QUIZ

1. (a) Sketch the curve $x = t^2$, $y = t^3$.

   (b) What is the tangent line when $t = 0$?

2. A curve is described by $x = t^5 + 3t^2 - t + 7$ and $y = 3t^4 - 2t^3/3 + t - 4$. What is the equation of the tangent line when $t = 1$?

3. What is the length of the curve in Question 1 for $-1 \leqslant t \leqslant 1$?

4. What is the speed of a particle moving according to the parametric equations given in Question 2 when $t = 0$?

5. The human cannonball's path can be parametrically described by $x = 5t$ and $y = 5\sqrt{3}t - 5t^2$. She is shot through a ring of fire at the highest point of her flight and ends her feat by diving into a narrow tube of water at $y = 0$. The cannon is located at $(0,0)$.

   (a) How fast is she going through the ring of fire?

   (b) At what speed and at what angle does she enter the water?

ANSWERS TO PREREQUISITE QUIZ

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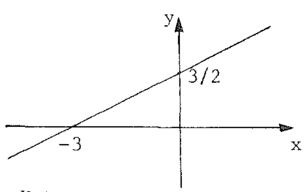

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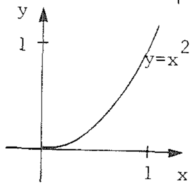
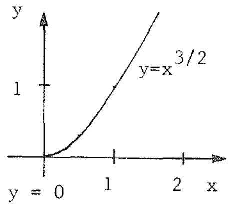


3. $\sqrt{5}$

4. $L = \int_1^2 \sqrt{1 + (3x^2 - 2x)^2}\,dx$

ANSWERS TO SECTION QUIZ

1. (a)



   (b) $y = 0$

2. $30y - 33x + 350 = 0$

3. $2(13^{3/2} - 8)/27$

4. $\sqrt{2}$

5. (a) 5

   (b) 10 ; $-\pi/3$ radians from ground

10.5 Length and Area in Polar Coordinates

PREREQUISITES

1. Recall how to compute arc lengths of parametric curves (Section 10.4).
2. Recall how to convert from cartesian to polar coordinates (Section 5.1).
3. Recall how to sketch graphs described in polar coordinates (Section 5.6).
4. Recall the trigonometric half-angle formulas for sine and cosine (Section 5.1 and inside front cover).

PREREQUISITE QUIZ

1. Which of the following is equivalent to $\sin^2\theta$?
   (a) $2 \sin\theta \cos\theta$
   (b) $(1 + \cos 2\theta)/2$
   (c) $(1 - \cos 2\theta)/2$
2. Sketch the graph of $r = \sin\theta$ in the xy-plane.
3. What is the length of the curve described by $x = 2\cos\theta$, $y = \sin\theta$ for $0 \leqslant \theta \leqslant \pi$? (Write a formula, but don't evaluate it.)
4. Convert the cartesian coordinates $(3,3)$ to polar coordinates.

GOALS

1. Be able to compute the arc lengths of curves described in polar coordinates.
2. Be able to compute the area of a region described by polar coordinates.

STUDY HINTS

1. Arc length. The derivation of formula (1) on p. 500 is not difficult. Simply use $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, differentiate by the product rule and substitute into $L =$

1. (continued)
$\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$. Don't forget to change $[a,b]$ to $[\alpha,\beta]$. You may find it easier just to memorize $L = \int_\beta^\alpha \sqrt{(r')^2 + r^2}\, d\theta$.

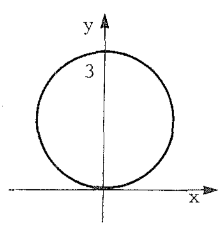
2. Area. You should remember that $A = (1/2)\int_\alpha^\beta r^2 d\theta$. A sketch is usually helpful and noting symmetries will save much work.

3. Choosing limits. In many instances, you will have to find the limits of integration. For arc length problems, be careful the limits are chosen so that the curve is traversed only once. For example, the length of the circle $r = 1$ should be found by integrating from 0 to 2π , not 0 to 4π . For most problems about area, as in Example 3, you will need to find where $f(\theta) = 0$.

4. Area between curves. This is done by computing the larger area and then subtracting the smaller area. Thus, the integrand is $[f(\theta)]^2 - [g(\theta)]^2$, not $[f(\theta) - g(\theta)]^2$.

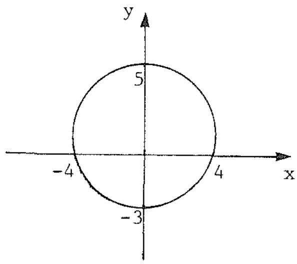
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We have $dr/d\theta = 3 \cos \theta$, so the length of L is $\int_0^{2\pi} \sqrt{9 \cos^2\theta + 9(1 + \sin \theta)^2} d\theta = \int_0^{2\pi} \sqrt{18 + 18 \sin \theta}\, d\theta = \sqrt{18} \times \int_0^{2\pi} \sqrt{1 + \cos(\theta - \pi/2)}\, d\theta$ by the identity $\sin \theta = \cos(\theta - \pi/2)$. By the half-angle formula, this becomes $\sqrt{18}\int_0^{2\pi} \sqrt{2} \cos[(1/2)(\theta - \pi/2)] d\theta = 12 \sin[(\theta - \pi/2)/2] |_0^{\pi/2} = 12\sqrt{2}$.

5. 

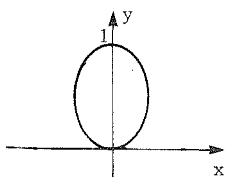
Using the formula $A = (1/2)\int_\alpha^\beta r^2 d\theta$, we get $(1/2)\int_0^\pi (3 \sin \theta)^2 d\theta = (9/2)\int_0^\pi [(1 - \cos 2\theta)/2] d\theta = (9/2)(\theta/2 - \sin 2\theta/4) |_0^\pi = 9\pi/4$.

9.



The area is $A = (1/2)\int_0^{2\pi}(4 + \sin\theta)^2 d\theta = (1/2)\int_0^{2\pi}(16 + 8\sin\theta + \sin^2\theta)d\theta$. Using the half-angle formula yields $(1/2)\int_0^{2\pi}[16 + 8\sin\theta + (1 - \cos 2\theta)/2]d\theta = (1/2) \times [33\theta/2 - 8\cos\theta - \sin 2\theta/4]\Big|_0^{2\pi} = 33\pi/2$.

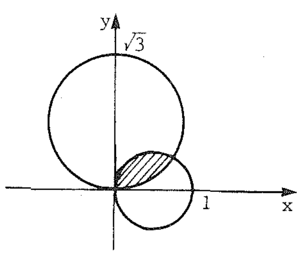
13.



The length is $L = \int_\alpha^\beta\sqrt{r^2 + (r')^2}\,d\theta = \int_{-\pi/2}^{\pi/2}\sqrt{\tan^2(\theta/2) + \sec^2(\theta/2)/4}\,d\theta$. The area is $A = (1/2)\int_\alpha^\beta r^2 d\theta = (1/2)\int_{-\pi/2}^{\pi/2}\tan^2(\theta/2)d\theta = (1/2)\int_{-\pi/2}^{\pi/2}(\sec^2(\theta/2) - 1)d\theta = (\tan(\theta/2) - \theta/2)\Big|_{-\pi/2}^{\pi/2} = 1 - \pi/4 + 1 - \pi/4 = 2 - \pi/2$.

17. The length is $L = \int_0^{\pi/2}\sqrt{(1 + \cos\theta - \theta\sin\theta)^2 + \theta^2(1 + 2\cos\theta + \cos^2\theta)}\,d\theta$. The area is $A = (1/2)\int_0^{\pi/2}(\theta^2 + 2\theta^2\cos\theta + \theta^2\cos^2\theta)d\theta = (1/2)[(\theta^3/3)\Big|_0^{\pi/2} + \int_0^{\pi/2}(2\theta^2\cos\theta + (\theta^2/2)(1 + \cos 2\theta))d\theta] = (1/2)[(\theta^3/2)\Big|_0^{\pi/2} + \int_0^{\pi/2}(2\theta^2\cos\theta + \theta^2\cos 2\theta/2)d\theta]$. Integrate each term by parts to get $A = (1/2)[\pi^3/16 + (2\theta^2\sin\theta + 4\theta\cos\theta - 4\sin\theta + (1/4)\theta^2\sin 2\theta + (1/4)\theta\cos 2\theta - (1/4)\sin 2\theta)]\Big|_0^{\pi/2} = (1/2)[\pi^3/16 + \pi^2/2 - 4 - \pi/8]$.

21.



Since $\cos\theta = \sqrt{3}\sin\theta$ when $\theta = \pi/6$, the length is $L = \int_{\pi/6}^{\pi/2}\sqrt{\sin^2\theta + \cos^2\theta}\,d\theta + \int_0^{\pi/6}\sqrt{3\cos^2\theta + 3\sin^2\theta}\,d\theta = \pi/2 - \pi/6 + \sqrt{3}\pi/6 = (2 + \sqrt{3})\pi/6$. The area is $A = (1/2)[\int_0^{\pi/6}3\sin^2\theta\,d\theta + \int_{\pi/6}^{\pi/2}\cos^2\theta\,d\theta] = (1/2)[(3/2)\int_0^{\pi/6}(1 - \cos 2\theta)d\theta + (1/2) \times \int_{\pi/6}^{\pi/2}(1 + \cos 2\theta)d\theta] = (1/4)[(3\theta - (3/2)\sin 2\theta)\Big|_0^{\pi/6} + (\theta + (1/2) \times \sin 2\theta)\Big|_{\pi/6}^{\pi/2}] = (1/4)[\pi/2 - 3\sqrt{3}/4 + \pi/2 - \pi/6 - \sqrt{3}/4] = (1/4)(5\pi/6 - \sqrt{3})$.

25. Substituting into the formula $L = \int_\alpha^\beta \sqrt{f(\theta) + [f'(\theta)]^2}\, d\theta$, we get $\int_{2n\pi}^{2(n+1)\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \sqrt{2}\int_{2n\pi}^{2(n+1)\pi} e^\theta d\theta = \sqrt{2}[\exp(2(n+1)\pi) - \exp(2n\pi)]$.

SECTION QUIZ

1. (a) Find the area bounded by the curve described by $r = \theta \cos\theta$ and the rays $\theta = 2\pi$ and $\theta = 3\pi$.

   (b) Write the length of the curve as an integral, but do not evaluate it.

2. Find the area and the perimeter of the region enclosed by each of the following:

   (a) $r = \sin\theta + \cos\theta$

   (b) $r = |2\cos\theta|$

3. The region inside both $r = \sin\theta$ and $r = \cos(\theta + \pi/3)$ has area $A = (1/2)[\int_a^b \cos^2(\theta + \pi/3)d\theta + \int_c^d \sin^2\theta\, d\theta]$. Find the limits of integration and compute the area.

4. An astropirate on the surface of Mars was about to accelerate into hyperspace when he noticed that the space patrol seemed confused and was flying circles around him. In reality, the space patrol was relesing a fence described by $r = 1$ and $r = 1 + \cos\theta$.

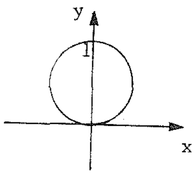
   (a) If the fence has height 1, how much fence material was used? (Assume that the space patrol may penetrate their own fences.)

   (b) The astropirate was caught in one of the regions inside $r = 1$ and outside $r = 1 + \cos\theta$. How much material was needed to make the top and bottom of the space prison?

ANSWERS TO PREREQUISITE QUIZ

1. c

2.



3. $L = \int_0^{\pi} \sqrt{4 \sin^2\theta + \cos^2\theta}\, d\theta$

4. $(3\sqrt{2}, \pi/4)$

ANSWERS TO SECTION QUIZ

1. (a) $19\pi^3/12 + \pi/8$

   (b) $\int_{2\pi}^{3\pi} \sqrt{\theta^2 + \cos^2\theta - \theta \sin 2\theta}\, d\theta$

2. (a) Area = π ; perimeter = $2\sqrt{2}\pi$

   (b) Area = $\pi/2$; perimeter = 2π

3. $a = \pi/12$; $b = \pi/6$; $c = 0$; $d = \pi/12$; area = $\pi/12 - 1/4$

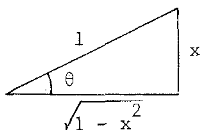
4. (a) $8 + 2\pi$

   (b) $\pi/2 + 4$

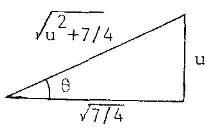
10.R Review Exercises for Chapter 10

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Substitute $u = \sin x$, so $du = \cos x\,dx$, and $\int 3\sin^2 x \cos x\,dx = \int 3u^2 du = u^3 + C = \sin^3 x + C$.

5. 

Let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$ and $\sqrt{1-x^2} = \cos\theta$. Thus, $\int\left(x^3/\sqrt{1-x^2}\right)dx = \int(\sin^3\theta/\cos\theta)\cos\theta\,d\theta = \int\sin^3\theta\,d\theta = \int(1-\cos^2\theta)\sin\theta\,d\theta$. Now, let $u = \cos\theta$, so $-du = \sin\theta\,d\theta$, and we get $-\int(1-u^2)du = -u + u^3/3 + C = \cos^3\theta/3 - \cos\theta + C$. Using the figure, the answer becomes $(1-x^2)^{3/2} - \sqrt{1-x^2} + C$.

9. 

Let $u = x + 1/2$, so $x^2 + x + 2 = (x + 1/2)^2 + 7/4 = u^2 + 7/4$. Now, let $u = \sqrt{7/4}\tan\theta$, so $du = \sqrt{7/4}\sec^2\theta\,d\theta$ and $\sqrt{u^2 + 7/4} = \sqrt{7/4}\sec\theta$. Thus, $\int[dx/(x^2+x+2)] = \int[du/(u^2+7/4)] = \int[\sqrt{7/4}\sec^2\theta\,d\theta/(7/4)\sec^2\theta] = \sqrt{4/7}\int d\theta = \sqrt{4/7}\,\theta + C = \sqrt{4/7}\tan^{-1}(u/\sqrt{7/4}) + C = (2\sqrt{7}/7)\tan^{-1}[(2x+1)/\sqrt{7}] + C$.

13. Let $u = x^2 + 1$, so $x^2 = u - 1$, and $du = 2x\,dx$. Thus $\int[x^3/(x^2+1)^2]dx = \int[(u-1)du/2u^2] = (1/2)\int(1/u - 1/u^2)du = (1/2)[\ln|u| + 1/u] + C = (1/2)[\ln|x^2+1| + 1/(x^2+1)] + C$.

17. Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then $\int\sin\sqrt{x}\,dx = 2\int u\sin u\,du$. Integrate by parts to get $-2u\cos u + 2\sin u + C = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$.

21. Using $\sin^2 x = 1 - \cos^2 x$, we get $\int(\sin^2 x/\cos x)dx = \int(\sec x - \cos x)dx = \ln|\sec x + \tan x| - \sin x + C$. The technique for integrating $\sec x$ is shown in Example 6(b), Section 10.1.

25. Let $I = \int [x/(x^3 - 9)]\,dx$ and let $a = \sqrt[3]{9}$. Then $I = \int [x/((x - a)\times (x^2 + ax + a^2))]\,dx = \int [B/(x - a) + (Cx + D)/(x^2 + ax + a^2)]\,dx$, where $B(x^2 + ax + a^2) + (Cx + D)(x - a) = x = (B + C)x^2 + (a(B - C) + D)x - a(aB - D)$. Equating coefficients, we get $B + C = 0$; $a(B - C) + D = 1$; and $aB - D = 0$. Substitute $C = -B$ and $D = aB$ into $a(B - C) + D = 1 = a(2B) + aB = 1$, so $B = 1/3a$. Therefore, $C = -1/3a$ and $D = 1/3$. Thus, $I = (1/3)\int [(1/a)/(x - a) + (-x/a + 1)/(x^2 + ax + a^2)]\,dx = (1/3a)\{\ln|x - a| - (1/2)\int [(2x + a - 3a)/(x^2 + ax + a^2)]\,dx\} = (1/3a)\times \{\ln|x - a| - (1/2)[\ln(x^2 + ax + a^2) - \int (3a/(x + a/2)^2 + 3a^2/4)dx]\} = (1/3a)\{\ln|x - a| - (1/2)[\ln(x^2 + ax + a^2) - (3a)(2/a\sqrt{3})\tan^{-1}[2(x + a/2)/(a\sqrt{3})]]\} + C = (1/3a)[\ln|x - a| - \ln\sqrt{x^2 + ax + a^2} + \sqrt{3}\tan^{-1}((2x/a + 1)/\sqrt{3})] + C = (1/3\sqrt[3]{9})[\ln|x - \sqrt[3]{9}| - \ln\sqrt{x^2 + \sqrt[3]{9}\,x + 3\sqrt[3]{3}} + \sqrt{3}\tan^{-1}((2x/\sqrt[3]{9} + 1)/\sqrt{3})] + C$.

29. $\int (1 + e^x)^{-1}dx = \int [1 - e^x(e^x + 1)^{-1}]\,dx = x - \ln(e^x + 1) + C$, by substituting $u = e^x + 1$.

33. Use $\sin x \cos y = (1/2)[\sin(x + y) + \sin(x - y)]$ to get $\int \sin 3x \cos 2x\,dx = (1/2)\int (\sin 5x + \sin x)dx = -(1/10)\cos 5x - (1/2)\times \cos x + C$.

37. Let $u = \sqrt{x}$, so $du = (1/2\sqrt{x})dx$. Then $\int (e^{\sqrt{x}}/\sqrt{x})dx = 2\int e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

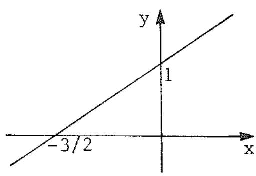
41. Let $u = x^2 + 3$, so $du = 2x\,dx$. Then $\int [x/(x^2 + 3)]\,dx = (1/2)\int (du/u) = (1/2)\ln|u| + C = (1/2)\ln(x^2 + 3) + C = \ln\sqrt{x^2 + 3} + C = (1/2)\ln(x^2 + 3) + C$.

45. Let $u = (\ln 3x) + 5$, so $du = (3/3x)dx = dx/x$. Then $\int_1^2 [(\ln 3x + 5)^3/x]\,dx = \int_{\ln 3+5}^{\ln 6+5} u^3 du = (u^4/4)\Big|_{\ln 3+5}^{\ln 6+5} = (1/4)[(\ln 6 + 5)^4 - (\ln 3 + 5)^4] \approx 186.1$.

49. Let $u = \cos\theta$, so $-du = \sin\theta\,d\theta$. Then $\int_0^{2\pi} [\sin\theta/(1 + \cos\theta + \cos^2\theta)]\,d\theta = -\int_1^1 [1/(1 + u + u^2)]\,du$. Since the limits of integration are both 1, the integral is 0.

53. Use $L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$, where $f'(x) = x^2 - 1/4x^2$. Thus, $L = \int_1^2 \sqrt{1 + (x^2 - 1/4x^2)^2}dx = \int_1^2 \sqrt{1 + (x^4 - 1/2 + 1/16x^4)}dx = \int_1^2 (x^2 + 1/4x^2)dx = (x^3/3 - 1/4x)\Big|_1^2 = 7/3 - (1/8 - 1/4) = 59/24$.

57. Since the graph is revolved around the y-axis, use the equation $A = 2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}\, dx$. Here, $f'(x) = 1/x \ln 10$, so $A = 2\pi\int_{10}^{100} x\sqrt{1 + (1/x \ln 10)^2}\, dx = 2\pi\int_{10}^{100}\sqrt{x^2 + 1/(\ln 10)^2}\, dx$. Integrate as in Example 3, Section 10.3 to get $A = 2\pi[(x/2)\sqrt{x^2 + 1/(\ln 10)^2} + (1/2(\ln 10)^2)\ln|x + \sqrt{x^2 + 1/(\ln 10)^2}|]\,\Big|_{10}^{100} = 2\pi[50\sqrt{10000 + 1/(\ln 10)^2} - 5\sqrt{100 + 1/(\ln 10)^2} + (1/2(\ln 10)^2)\ \ln[(100 + \sqrt{10000 + 1/(\ln 10)^2})/(10 + \sqrt{100 + 1/(\ln 10)^2})] \approx 31103$.

61. 

Solving for t , we get $t = x/3$. Substitute into y to get $y = 2x/3 + 1$. This is a line with slope $2/3$ and y-intercept 1 .

65. For parametric equations, the slope of the graph is $dy/dx = (dy/dt)/(dx/dt)$. In this case, it is $3t^2/4t^3$. When $t = 1$, $dy/dx = 3/4$, $x = 1$, and $y = 2$. Thus, the tangent line is $(y - 2)/(x - 1) = 3/4$, i.e., $y = 3x/4 + 5/4$.

69. The length L is given by $L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$. In this case, $r^2 + (r')^2 = \theta^4 + 4\theta^2 = \theta^2(\theta^2 + 4)$, and $\sqrt{r^2 + (r')^2}$ becomes $\theta\sqrt{\theta^2 + 4}$. Let $u = \theta^2 + 4$, so $du = 2\theta\, d\theta$, and $L = (1/2) \times \int_4^{\pi^2/4+4}\sqrt{u}\, du = (1/3)u^{3/2}\Big|_4^{\pi^2/4+4} = (1/3)[(\pi^2/4 + 4)^{3/2} - 8]$.

The area A is given by $A = (1/2)\int_\alpha^\beta [r(\theta)]^2 d\theta$. In this case, $A = (1/2)\int_0^{\pi/2}\theta^4 d\theta = (\theta^5/10)\Big|_0^{\pi/2} = \pi^5/320$.

73. The length L is given by $L = \int_\alpha^\beta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}d\theta$. Here, $r' = 3\cos^3(\theta/4)\sin(\theta/4)$, so $r^2 + (r')^2 = 9\cos^8(\theta/4) + 9\cos^6(\theta/4)\sin^2(\theta/4) = 9\cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = 9\cos^6(\theta/4) = 9\cos^2(\theta/4)(1 -$

73. (continued)

$\sin^2(\theta/4))^2$. Then $L = 3\int_0^\pi [\cos(\theta/4) - \cos(\theta/4)\sin^2(\theta/4)]\,d\theta$. Let $u = \sin(\theta/4)$, so $du = (1/4)\cos(\theta/4)d\theta$, and $L = 12\int_0^{\sqrt{2}/2}(1 - u^2)du = 12(u - u^3/3)\Big|_0^{\sqrt{2}/2} = 12(\sqrt{2}/2 - \sqrt{2}/12) = 5\sqrt{2}$.

The area A is given by $A = (1/2)\int_\alpha^\beta [r(\theta)]^2 d\theta$. Thus, $A = (1/2) \times \int_0^\pi 9\cos^8(\theta/4)d\theta = (9/2)\int_0^\pi(1 + \cos(\theta/2))^4 d\theta/16 = (9/32)\int_0^\pi [1 + 4\cos(\theta/2) + 6\cos^2(\theta/2) + 4\cos^3(\theta/2) + \cos^4(\theta/2)]\,d\theta = (9/32)[\theta + 8\sin(\theta/2)]\Big|_0^\pi + (9/32)\int_0^\pi [6(1 + \cos\theta)/2 + 4\cos(\theta/2)(1 - \sin^2(\theta/2)) + (1 + \cos\theta)^2/4]\,d\theta = 9\pi/32 + 9/4 + (9/32)[3\theta + 3\sin\theta + 8\sin(\theta/2) - 8\sin^3(\theta/2)]\Big|_0^\pi + (9/128)\int_0^\pi(1 + 2\cos\theta + \cos^2\theta)d\theta = 9\pi/32 + 9/4 + 27\pi/32 + 9/4 - 9/4 + (9/128)(\theta + 2\sin\theta)\Big|_0^\pi + (9/256)\int_0^\pi(1 + \cos 2\theta)d\theta = 9\pi/8 + 9/4 + 9\pi/128 + (9/256)(\theta + \sin 2\theta/2)\Big|_0^\pi = 9\pi(1/8 + 1/128 + 1/256) + 9/4 = 315\pi/256 + 9/4$.

77. This exercise uses the formulas $\sin x\cos y = (1/2)[\sin(x + y) + \sin(x - y)]$ and $\cos x\cos y = (1/2)[\cos(x + y) + \cos(x - y)]$. $a_m = (1/\pi)\int_0^{2\pi}\cos 3x\cos mx\,dx = (1/2\pi)\int_0^{2\pi}[(\cos(3 + m)x + \cos(3 - m)x]\,dx = -(1/2\pi)[(1/(3 + m))\sin(3 + m)x + (1/(3 - m))\sin(3 - m)x]\Big|_0^{2\pi} = 0$ unless $m = 3$. Then $a_3 = (1/2\pi)\int_0^{2\pi}(\cos 6x + 1)dx = (1/2\pi)((1/6)\sin 6x + x)\Big|_0^{2\pi} = 1$. Also, $b_m = (1/\pi)\int_0^{2\pi}\cos 3x\sin mx\,dx = (1/2\pi)\int_0^{2\pi}[\sin(m + 3)x + \sin(m - 3)x]\,dx = -(1/2\pi)[(1/(m + 3))\cos(m + 3)x + (1/(m - 3))\cos(m - 3)x]\Big|_0^{2\pi} = 0$.

81. We use the product formulas listed on p. 460 and the half-angle formula $\sin^2 x = (1 - \cos 2x)/2$. $a_m = (1/\pi)\int_0^{2\pi}\sin^2 x\cos mx\,dx = (1/2\pi) \times \int_0^{2\pi}[\cos mx - \cos 2x\cos mx]\,dx = (1/2\pi)\int_0^{2\pi}[\cos mx - (1/2)(\cos(2 + m)x + \cos(2 - m)x)]\,dx = (1/4\pi)[(2/m)\sin mx - (1/(2 + m))\sin(2 + m)x - (1/(2 - m))\sin(2 - m)x]\Big|_0^{2\pi} = 0$, unless $m = 0$ or $m = 2$. Then $a_2 = -(1/4\pi)\times \int_0^{2\pi}[\cos 4x + 1]\,dx + 0 = -(1/4\pi)((1/4)\sin 4x + x)\Big|_0^{2\pi} = -1/2$; and $a_0 = (1/2\pi)\int_0^{2\pi}1\,dx + 0 = 1$. Also, $b_m = (1/\pi)\int_0^{2\pi}\sin^2 x\sin(mx)dx = (1/2\pi) \times$

81. (continued)

$\int_0^{2\pi}[\sin(mx) - \cos 2x \sin(mx)]\,dx = (1/2\pi)\int_0^{2\pi}[\sin(mx) - (1/2)(\sin(2+m)x + \sin(m-2)x]\,dx = (1/4\pi)[(2/m)\cos(mx) + (1/(2+m))\cos(2+m)x + (1/(m-2))\cos(m-2)x]\big|_0^{2\pi} = 0$.

85. Substituting the given formulas yields $T = (4/\pi)\int_0^{\phi_m}(1 - 2\sin^2\phi - 1 + 2\sin^2\phi_m)^{-1/2}d\phi = (4/\pi)\int_0^{\phi_m}(-2\sin^2\phi_m\sin^2\beta + 2\sin^2\phi_m)^{-1/2}d\phi = (2\sqrt{2}/\pi)\times\int_0^{\phi_m}(1/\sin\phi_m)(1-\sin^2\beta)^{-1/2}d\phi = (2\sqrt{2}/\pi)\int_0^{\phi_m}[1/(\sin\phi_m\cos\beta)]\,d\phi$. Differentiate $\sin\phi = \sin\phi_m\sin\beta$ to get $\cos\phi\, d\phi = \sin\phi_m\cos\beta\, d\beta$. Since $\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-k^2\sin^2\beta}$, where $k^2 = \sin^2\phi_m$, we get $d\phi = \sin\phi_m\cos\beta\, d\beta/\sqrt{1-k^2\sin^2\beta}$. When $\phi = \phi_m$, $\sin\beta = 1$ so $\beta = \pi/2$; and when $\phi = 0$, $\beta = 0$. Substitute all this and cancel $\sin\phi_m\cos\beta$ to get $T = (4/\pi\sqrt{2})\int_0^{\pi/2}[1/\sqrt{1-k^2\sin^2\beta}]\,d\beta$.

89. (a) The area beneath the graph is $A = \int_a^b(1+x^n)dx = b - a + (b^{n+1} - a^{n+1})/(n+1)$, unless $n = -1$. If $n = -1$, $A = \int_a^b(1+x^{-1})dx = b - a + \ln(b/a)$.

(b) The length of this graph is $L = \int_a^b\sqrt{1+n^2x^{2n-2}}\,dx$. If $n = 0$, then the length is $L = \int_a^b dx = b - a$. If $n = 1$, then $L = \int_a^b\sqrt{1+1}\,dx = \int_a^b\sqrt{2}\,dx = \sqrt{2}(b-a)$. If $n = 2$, then $L = \int_a^b\sqrt{1+4x^2}\,dx = [x\sqrt{1+4x^2} + (1/2)\ln|2x + \sqrt{1+4x^2}|]\big|_a^b$. The $n = 2$ case is Example 3 of Section 10.3 .

For the case $n = 3/2$, we have $L = \int_a^b\sqrt{1+(9/4)x}\,dx$. Substituting $u = 1 + (9/4)x$, we get $L = (8/27)(1+9x/4)^{3/2}\big|_a^b = (1/27)(4+9x)^{3/2}\big|_a^b$. For the more general case where $n = (2k+3)/(2k+2)$ and k is a nonnegative integer, we have $L = \int_a^b\sqrt{1+[(2k+3)/(2k+2)]^2x^{1/(k+1)}}\,dx$. Now, let $u = n^2x^{2n-2}$ so $x = n^{1/(1-n)}(u-1)^{1/(2n-2)}$ and $dx = n^{1/(1-n)}du/[(2n-2)\times(u-1)^{(3-2n)/(2n-2)}]$. Then $L = \int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}}[\sqrt{u}n^{1/(1-n)}du/$

89. (b) (continued)

$(2n - 2)(u - 1)^k]$. Recall that the binomial expansion for $(u - 1)^k$ is $\Sigma_{i=0}^{k}\binom{k}{i}u^i$ and this gives us $L = (n^{1/(1-n)}/(2n-2))\times$ $\int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}} \Sigma_{i=0}^{k}\binom{k}{i}(-1)^{k-i}u^{i+1/2}du = (n^{1/(1-n)}/(2n-2))\times$ $\Sigma_{i=0}^{(3-n)/(2n-2)}\binom{(3-n)/(2n-2)}{i}(-1)^{(3-n)/(2n-2)-i}[(1 + n^2b^{2n-2})^{i+3/2} - (1 + n^2a^{2n-2})^{i+3/2}]/(i + 3/2)$. (Note: recall that $\binom{k}{i} = k!/i!(k-i)!$) .

(c) Around the x-axis, $V_x = \pi\int_a^b(1 + 2x^n + x^{2n})dx = \pi[b - a + 2(b^{n+1} - a^{n+1})/(n + 1) + (b^{2n+1} - a^{2n+1})/(2n + 1)]$, unless $n = -1$ or $n = -1/2$. If $n = -1$, $V_x = \pi\int_a^b(1 + 2x^{-1} + x^{-2})dx = \pi[b - a + 2\ln(b/a) - (a^{-1} - b^{-1})]$. If $n = -1/2$, $V_x = \pi\int_a^b(1 + 2x^{-1/2} + x^{-1})dx = \pi[b - a + 4\sqrt{b} - 4\sqrt{a} + \ln(b/a)]$. Around the y-axis, $V_y = 2\pi\int_a^b(x + x^{n+1})dx = \pi[b^2 - a^2 + 2(b^{n+2} - a^{n+2})/(n + 2)]$ unless $n = -2$. If $n = -2$, $V_y = 2\pi\int_a^b(x + x^{-1})dx = \pi[b^2 - a^2 + 2\ln(b/a)]$.

89. (d)

Around the x-axis, $A_x = 2\pi\int_a^b(1 + x^n)\sqrt{1 + n^2x^{2n-2}}\,dx = 2\pi\int_a^b\sqrt{1 + n^2x^{2n-2}}\,dx + 2\pi\int_a^b x^n\sqrt{1 + n^2x^{2n-2}}\,dx$. Note that $2\pi\int_a^b\sqrt{1 + n^2x^{2n-2}}\,dx$ is 2π times the arc length which was analyzed in part (b). Thus, we only need to look at $a_x = 2\pi\int_a^b x^n\sqrt{1 + n^2x^{2n-2}}\,dx$.

If $n = 0$, then $a_x = 2\pi\int_a^b dx = 2\pi(b - a)$. If $n = 1$, then $a_x = 2\pi\int_a^b\sqrt{2}x\,dx = \sqrt{2}\pi(b^2 - a^2)$.

If $n = 2$, then $a_x = 2\pi\int_a^b(x^2\sqrt{1 + 4x^2})dx$. The integral $\int_a^b(x^2\sqrt{1 + 4x^2})dx$ is equal to $\int_a^b[(x^2 + 4x^2)/\sqrt{1 + 4x^2}]dx = \int_a^b(x^2/\sqrt{1 + 4x^2})dx$ $\int_a^b(4x^4/\sqrt{1 + 4x^2})dx$. The latter integral is integrated by parts with $u = x^3$, $dv = 4x/\sqrt{1 + 4x^2}\,dx$, $du = 3x^2\,dx$, and $v = \sqrt{1 + 4x^2}$. Therefore,

89. (d) (continued)

$\int_a^b (x^2\sqrt{1+4x^2})dx = \int_a^b (x^2/\sqrt{1+4x^2})dx + [x^3\sqrt{1+4x^2}\,|_a^b - 3\int_a^b (x^2\sqrt{1+4x^2})dx]$. Rearrangement yields $4\int_a^b (x^2\sqrt{1+4x^2})dx = b^3\sqrt{1+4b^2} - a^3\sqrt{1+4a^2} + \int_a^b (x^2/\sqrt{1+4x^2})dx$. The latter integral can be done with a trigonometric substitution. Let $x = (1/2)\tan\theta$ so $dx = (1/2)\sec^2\theta d\theta$. This yields $(1/8)\int_{\tan^{-1}2a}^{\tan^{-1}2b} \tan^2\theta \sec\theta\, d\theta = (1/8)\int_{\tan^{-1}2a}^{\tan^{-1}2b} (\sec^3\theta - \sec\theta)d\theta$. The integration of $\sec\theta$ and $\sec^3\theta$ are shown in detail in Example 6(b), Section 10.1 and Example 3, Section 10.3, respectively. Therefore, $\int_a^b (x^2/\sqrt{1+4x^2})dx = (1/16)[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - 2\ln|\sec\theta + \tan\theta|]\,\Big|_{\tan^{-1}2a}^{\tan^{-1}2b}$. Now if $\tan\theta = 2b$, then $\sec\theta = \sqrt{1+2b^2}$ and similarly, if $\tan\theta = 2a$, then $\sec\theta = \sqrt{1+2a^2}$. This substitution gives us $(1/16)(2b\sqrt{1+4b^2} - \ln|\sqrt{1+4b^2} + 2b| - 2a\sqrt{1+4a^2} + \ln|\sqrt{1+4a^2} + 2a|)$. Finally, we get $a_x = (\pi/2)[(1/16)\ln|(\sqrt{1+4a^2} + 2a)/(\sqrt{1+4b^2} + 2b)| + b\sqrt{1+4b^2}(b^2 + 1/8) - a\sqrt{1+4a^2}(a^2 + 1/8)]$.

If $n = (2k+3)/(2k+2)$ where k is any nonnegative integer, let $x = ((u-1)/n^2)^{1/(2n-2)}$ so $dx = 1/[((2n-2)n^{1/(n-1)}) \times (u-1)^{(3-2n)/(2n-2)}du$. Substitute this into $a_x = 2\pi\int_a^b x^n\sqrt{1+n^2x^{2n-2}}dx = [\pi/(n-1)n^{(n+1)/(n-1)}]\int_{x=a}^{x=b} (u-1)^{(3-n)/(2n-2)}\, u\,du$. Substitute for n in $(3-n)/(2n-2) = ((3-1-2)/(2k+1))/(2+4/(2k+1)-2) = (4k+2-2)/4 = k$. Hence $a_x = [\pi n^{(n+1)/(1-n)}/(n-1)]\int_{x=a}^{x=b}(u-1)^k\sqrt{u}\,du$. From the binomial expansion formula, $(u-1)^k = \sum_{i=0}^k \binom{k}{i}(-1)^{k-i}u^i$. Let $I = \int (u-1)^k\sqrt{u}\,du$. Then $I = \int\sum_{i=0}^k \binom{k}{i}(-1)^{k-i}u^{i+1/2}du = \sum_{i=0}^k \binom{k}{i}(-1)^{k-i}u^{i+3/2}/(i+3/2) + C$. Substitute for u and k to get $a_x = \pi n^{(n+1)/(1-n)}/(n-1)\left|\sum_{i=0}^{(3-n)/(2n-2)}\binom{(3-n)/(2n-2)}{i} \times (-1)^{(3-n)/(2n-2)-i}(1+n^2x^{2n-2})^{i+3/2}/(i+3/2)\right|\Big|_a^b = [\pi n^{(n+1)/(1-n)}/(n-1)]\sum_{i=0}^{(3-n)/(2n-2)}\binom{(3-2n)/(2n-2)}{i}(-1)^{(3-n)/(2n-2)-i} \times ((1+n^2b^{2n-2})^{i+3/2} - (1+n^2a^{2n-2})^{i+3/2})/(i+3/2)$.

89. (d) (continued)

Around the y-axis, $A_y = 2\pi\int_a^b x\sqrt{1 + n^2x^{2n-2}}\,dx$. If $n = 0$, then $A_y = 2\pi\int_a^b x\,dx = \pi(b^2 - a^2)$. If $n = 1$, then $A_y = 2\pi\int_a^b\sqrt{2}x\,dx = \pi\sqrt{2}(b^2 - a^2)$. If $n = 2$, then substitution of $u = 1 + 4x^2$ yields $A_y = 2\pi\int_a^b x\sqrt{1 + 4x^2}\,dx = (\pi/6)(1 + 4x^2)^{3/2}\big|_a^b = (\pi/6)[(1 + 4b^2)^{3/2} - (1 + 4a^2)^{3/2}]$.

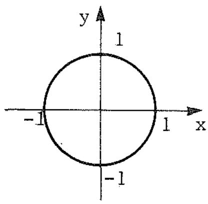
If $n = 3$, then $A_y = 2\pi\int_a^b x\sqrt{1 + 9x^4}\,dx$. Let $u = 3x^2$ so $du = 6x\,dx$. Then $A_y = (\pi/3)\int_{3a^2}^{3b^2}\sqrt{1 + u^2}\,du$ Integrate as in Example 3, Section 10.3 to get $A_y = (\pi/3)[(u/2)\sqrt{u^2 + 1} + (1/2)\ln|u + \sqrt{u^2 + 1}|]\big|_{3a^2}^{3b^2} = (\pi/6)(3b^2\sqrt{9b^4 + 1} + \ln[(3b^2 + \sqrt{9b^4 + 1})/(3a^2 + \sqrt{9a^4 + 1})] - 3a^2\sqrt{9a^4 + 1}$.

If $n = (k + 3)/(k + 2)$ where k is any nonnegative integer, then $(2 - n)/(n - 1) = (2 - 1 - 1/(k + 1))/(1/(k + 1)) = (k + 1 - 1)/1 = k$. Let $u = 1 + n^2x^{2n-2}$, so $x = (u - 1)^{1/(2n-2)}n^{1/(1-n)}$ and $dx = (n^{1/(1-n)}(u - 1)^{(3-2n)/(2n-2)}/(2n - 2))\,du$. Then $A_y = [\pi n^{2/(1-n)}/(n - 1)]\int_{1+n^2a^{2n-2}}^{1+n^2b^{2n-2}}(u - 1)^{(2-n)/(n-1)}\sqrt{u}\,du$. Since $(2 - n)/(n - 1) = k$, expand $(u - 1)^k$ with the binomial formula: $(u - 1)^k$ $\Sigma_{i=0}^{k}\binom{k}{i}(-1)^{k-i}u^i$. Substitute this into the integral to get $A_y = \pi[n^{2/(1-n)}/(n - 1)]\Sigma_{i=0}^{(2-n)/(n-1)}\binom{(2-n)/(n-1)}{i}[(-1)^{(2-n)/(n-1)-1}/(i + 3/2)]\times[(1 + n^2b^{2n-2})^{i+3/2} - (1 + n^2a^{2n-2})^{i+3/2}]$.

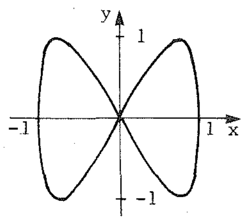
93. (a) Plot the curves point-by-point, but notice that if $m = 1$ and n is any integer, then for $t = 0$, the point P is $(1,0)$; for $t = \pi$, P is $(-1,0)$. Also, since $\cos(\pi - \alpha) = \cos(\pi + \alpha)$ and $\sin(\pi - \alpha) = -\sin(\pi + \alpha)$ for all α , the curves are symmetric about the x-axis. Since $\cos(\pi - \alpha) = -\cos\alpha$ and $\sin(\pi - \alpha) = \sin\alpha$, the curves are also symmetric about the y-axis and there-

93. (a) (continued)

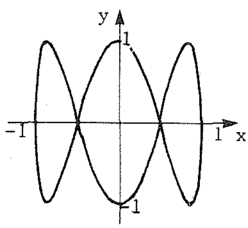
fore, the origin. Finally, since $y(\pi/n) = 0$, each curve crosses the x-axis at $n + 1$ points. (Actually, each curve crosses $2n + 2$ times, but as t goes from π to 2π the curve will recross the x-axis at the same points it crossed when t went from 0 to π.)



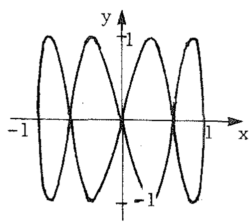
$m=1$, $n=1$



$m=1$, $n=2$



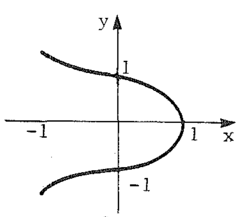
$m=1$, $n=3$



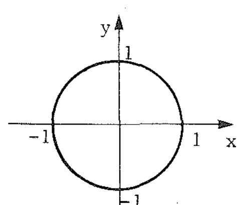
$m=1$, $n=4$

(b) Consequently, each curve will consist of n loops, for n odd or even.

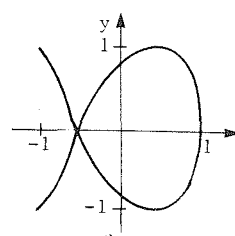
(c)



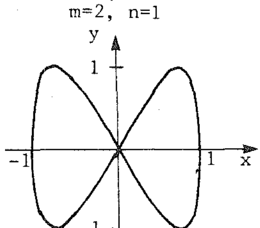
$m=2$, $n=1$



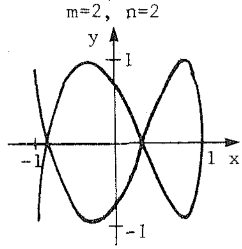
$m=2$, $n=2$



$m=2$, $n=3$

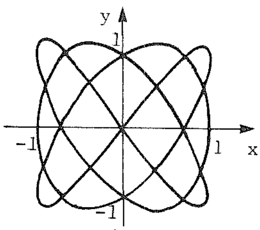


$m=2$, $n=4$

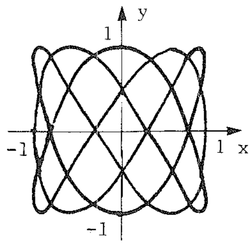


$m=2$, $n=5$

93. (d)



m=3, n=4



m=3, n=5

TEST FOR CHAPTER 10

1. True or false.

(a) Every polynomial is a product of linear and/or quadratic factors.

(b) The area between $r = f(\theta)$ and $r = g(\theta)$, where $f(\theta) \geqslant g(\theta)$ on the interval $[\alpha,\beta]$, is $(1/2)\int_\alpha^\beta [f(\theta) - g(\theta)]^2$.

(c) Using the method of partial fractions, $1/(x-1)^2x^2 = A/(x-1) + B/(x-1)^2 + (Cx+D)/x^2$ for some constants A, B, C, and D.

(d) If $f(x) \geqslant 0$ on $[a,b)$, the surface area obtained by revolution around the x-axis is $2\pi\int_a^b \sqrt{[f(x)]^2 + [f'(x)f(x)]^2}\,dx$.

(e) In polar coordinates, the arc length of $r = f(\theta)$ over the interval $[\alpha,\beta]$ is $\int_\alpha^\beta \sqrt{1 + [f'(\theta)]^2}\,d\theta$.

2. The line $y = -x + 1$ on $[0,3]$ is revolved to form a surface of revolution. Find the surface area if the line is revolved around:

(a) the y-axis

(b) the x-axis

3. Find the arc length of the curves described by the following equations on the given intervals:

(a) $y = (2/3)x^{3/2}$ on $[0,8]$

(b) $r = \sin^2\theta$ in polar coordinates for $0 \leqslant \theta \leqslant 2\pi$

(c) $y = t^2 + 3$ and $x = 2t + 1$ for $0 \leqslant t \leqslant 1$

4. Evaluate the following integrals:

(a) $\int_0^1 [(x^3 + 7x^2 - 8)/(x^2 - 4)]\,dx$

(b) $\int [dt/t(t^2 - t + 1)]$

5. Show that $\int_0^{n\pi/2} \sin^2\theta\, d\theta = \int_0^{n\pi/2} \cos^2\theta\, d\theta = n\pi/4$ for all integers $n \geqslant 0$.

6. Suppose $y_1 = f(x)$ and $y_2 = f(x) + 3$. If $f(x) \geqslant 0$ on $[a,b]$, where $b > a > 0$, then which of the following are equal, if any?

(a) The arc lengths of y_1 and y_2 on $[a,b]$.

(b) The surface areas of revolution obtained by revolving y_1 and y_2 on $[a,b]$ around the x-axis.

(c) The surface areas of revolution obtained by revolving y_1 and y_2 on $[a,b]$ around the y-axis.

7. Compute the average value of $f(t)$ for the given interval:

(a) $f(t) = 1/(t - 1)^2 t$ on the interval $2 \leqslant t \leqslant 3$.

(b) $f(t) = \sin^2(t/2)\cos^3(t/2)$ on the interval $0 \leqslant t \leqslant \pi/2$.

8. Find the area of the region inside $r = \sin\theta\cos\theta$ and outside $r = \cos 2\theta$.

9. Find the area between the following curves and the x-axis in the given intervals:

(a) $y = \sin^4 x$ on $[0, \pi/8]$

(b) $y = x\sin^2 x\cos x$ on $[\pi/4, \pi/2]$

10. Dragster Debbie, the 80-year-old grandmother, needed some excitement in her life. She decided to take her Ferrari for a little spin down at the speedway. As she floors the gas petal, her position can be described parametrically by $x = 2\cos t$ and $y = 4\sin t\cos t$.

(a) What is her speed at time t_0 ?

(b) At $t = 8\pi$, she realized she had to hurry to her karate lessons, so she sped off along the tangent line. Describe the tangent line with a set of parametric equations.

ANSWERS TO CHAPTER TEST

1. (a) True

   (b) False; it is $(1/2)\int_{\alpha}^{\beta}\{[f(\theta)]^2 - [g(\theta)]^2\}d\theta$.

   (c) False; $(Cx + D)/x^2$ should be $C/x + D/x^2$.

   (d) True

   (e) False; the given formula is for cartesian coordinates

2. (a) $9\pi\sqrt{2}$

   (b) $5\pi\sqrt{2}$

3. (a) $52/3$

   (b) $4 + (2\sqrt{3}/3)\ln(\sqrt{3} + 2)$

   (c) $\sqrt{2} + \ln(1 + \sqrt{2})$

4. (a) $15/2 - 3\ln 3 - 4\ln 2$

   (b) $\ln|t/\sqrt{t^2 - t + 1}| + (1/\sqrt{3})\tan^{-1}[(2t - 1)/3] + C$

5. $\int_0^{n\pi/2}\sin^2\theta = \int(1/2 - \cos 2\theta/2)d\theta = (\theta/2 - \sin 2\theta/4)\Big|_0^{n\pi/2} = n\pi/4$;
   $\int_0^{n\pi/2}\cos^2\theta = \int(1/2 + \cos 2\theta/2)d\theta = (\theta/2 + \sin 2\theta/4)\Big|_0^{n\pi/2} = n\pi/4$.

6. (a) and (c)

7. (a) $\ln(3/4) + 1/2$

   (b) $7\sqrt{2}/30\pi$

8. $\pi - 4/5$

9. (a) $3\pi/64 - \sqrt{2}/8 + 1/32$

   (b) $(3\pi\sqrt{2} + 20\sqrt{2} - 24\pi)/144$

10. (a) $2[\sin^2 t_0 + 4\sin^4 t_0 + 4\cos^4 t_0 - 2\cos^2 t_0 \sin^2 t_0]^{1/2}$

    (b) $x = 2$ and $y = 4(t - 8\pi)$

CHAPTER 11

LIMITS, L'HÔPITAL'S RULE, AND NUMERICAL METHODS

11.1 Limits of Functions

PREREQUISITES

1. Recall the basic properties of limits (Section 1.2).
2. Recall the concept of horizontal and vertical asymptotes (Sections 1.2 and 3.4).
3. Recall how to compute limits at infinity or infinite limits (Section 1.2).

PREREQUISITE QUIZ

1. Complete the following statements (assuming all limits are finite):

   (a) $\lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = \lim_{x\to x_0}$ ________________ .

   (b) $\lim_{x\to x_0} 3g(x) = [\lim_{x\to x_0} g(x)] \times$ ________________ .

   (c) $\lim_{x\to\infty} [f(x)/g(x)] =$ ____________ $/ \lim_{x\to\infty} g(x)$, provided $\lim_{x\to\infty} g(x) \neq 0$.

2. What is $\lim_{x\to 0}(1/x)$? Explain your answer.
3. Does $\lim_{x\to\infty}[(x + 1)/x^2]$ exist? If yes, what is it?
4. Sketch the graph of $y = x^2/(1 - x^2)$. Discuss the asymptotes.

GOALS

1. Be able to state the $\varepsilon - \delta$ definition of the limit and apply it to prove basic general properties of limits.

2. Be able to evaluate simple specific limits.

3. Be able to compute limits at infinity.

4. Be able to compute one-sided limits.

STUDY HINTS

1. Limit definition. The $\varepsilon - \delta$ definition of the limit is not useful for computing limits. It is, however, very important in theoretical work. Example 1 demonstrates how the definition is used. We write $f(x) - \ell$ in terms of $x - x_0$. This will permit us to find a relationship between δ and ε so that $|x - x_0| < \delta$ ensures that $|f(x) - \ell| < \varepsilon$. Remember that ε is given and your job is to find a δ which fits the definition. Both ε and δ are positive numbers.

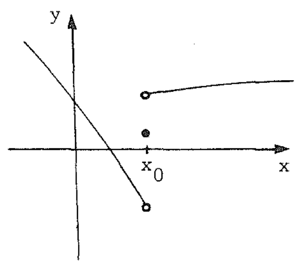
2. Basic properties. The box on p. 511 was already presented in Chapter 1. Again, you are reminded that "common sense" should help you recall all of the properties.

3. Important methods. Example 3 shows two important tricks used in limit computations. If the denominator at x_0 is zero, try to factor out $x - x_0$. A second trick is rationalization. If a radical appears in the denominator, use $a^2 - b^2 = (a - b)(a + b)$ to eliminate the radical. Also $a^3 - b^3 = (a - b)(a^2 + 2ab + b^2)$ may be used in some situations.

4. Limits at infinity. To compute limits, divide numerator and denominator by the largest degree of x in the numerator. Then use the fact that $\lim_{x\to\infty}(1/x) = 0$. A horizontal asymptote, if it exists, is simply the limit at $-\infty$ or $+\infty$.

5. Infinite limits. These have already been discussed when vertical asymptotes were introduced in Chapter 3.

6. Reciprocal test for infinite limits. If the limit of the reciprocal is zero, then the limit of the original function may be $+\infty$, $-\infty$, or neither. Which occurs depends on how $f(x)$ behaves about x_0. If the sign is positive on one side of x_0 and negative on the other side, then no limit exists.

7. 

One-sided limits. The definition is just like that for ordinary limits except that we now consider the intervals $(x_0, x_0 + \delta)$ or $(x_0 - \delta, x_0)$ rather than $(x_0 - \delta, x_0 + \delta)$. The concept of one-sided limits is important when the value of a function changes from $-\infty$ to $+\infty$. This is also important when absolute values are involved. The figure demonstrates that one-sided limits may exist even though the (two-sided) limit does not exist at x_0.

8. Comparison test. Thinking of this as the "sandwich principle" makes the test simple to understand. You should understand why the nickname is appropriate.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $\varepsilon > 0$ be given. We have to find $\delta > 0$ such that $|x^2 - a^2| < \varepsilon$ if $|x - a| < \delta$. We note that $x^2 - a^2 = (x - a)^2 + 2a(x - a)$. If $|x - a| < \delta$, we have $|x^2 - a^2| = |(x - a)^2 + 2a(x - a)| \leqslant |x - a|^2 + 2|a||x - a| < \delta^2 + 2|a|\delta$. Thus, we want δ to be such that $\delta^2 + 2|a|\delta \leqslant \varepsilon$. We may choose δ such that $\delta \leqslant 1$ (so $\delta^2 \leqslant \delta$) and $\delta \leqslant \varepsilon/(1 + 2|a|)$. This choice (or a smaller one) guarantees that $\delta^2 + 2|a|\delta \leqslant (1 + 2|a|)\delta \leqslant \varepsilon$, and so $|x^2 - a^2| < \varepsilon$ if $|x - a| < \delta$.

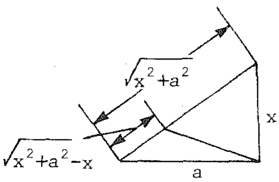
5. Note that the exponential function is continuous at 0, so $\lim_{\theta\to 0} \exp[(3\tan\theta)/\theta] = \exp \lim_{\theta\to 0}[3\tan\theta/\theta] = \exp[3 \lim_{\theta\to 0}(\tan\theta/\theta)] = e^3$.

9. Using the replacement rule, we get $\lim_{x\to 2}[(x^2 - 4)/(x^2 - 5x + 6)] = \lim_{x\to 2}[(x - 2)(x + 2)/(x - 2)(x - 3)] = \lim_{x\to 2}[(x + 2)/(x - 3)] = -4$.

13. We need to find A such that $|f(x) - \ell| < \varepsilon$ whenever $x > A$. We have $|f(x) - \ell| = |(1 + x^3)/x^3 - 1| = |(1 + x^3 - x^3)/x^3| = |1/x^3|$. To make this less than ε, we note that $1/x^3 < \varepsilon$ whenever $x > 1/\sqrt[3]{\varepsilon}$, so we may choose $A = 1/\sqrt[3]{\varepsilon}$.

17. Since $\lim_{x\to\infty}(1/x) = 0$, $\lim_{x\to\infty}(3/x + 5/x^2 - 2) = 3\lim_{x\to\infty}(1/x) + 5\left(\lim_{x\to\infty}(1/x)\right) \times \lim_{x\to\infty}(1/x) - 2\lim_{x\to\infty}(1) = 0 + 0 - 2 = -2$.

21. Divide through by x^2/x^2 to get $\lim_{x\to\infty}[(3x^2 + 2x + 4)/(5x^2 + x + 7)] = \lim_{x\to\infty}[(3 + 2/x + 4/x^2)/(5 + 1/x + 7/x^2)]$. Using the fact that $\lim_{x\to\infty}(1/x) = 0$ and the other rules for limits, this becomes $(3 + 0 + 0)/(5 + 0 + 0) = 3/5$.

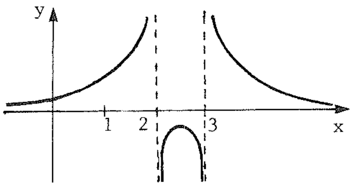
25. 

Upon rationalizing, we have $\lim_{x\to\infty}(\sqrt{x^2 + a^2} - x) = \lim_{x\to\infty}[(\sqrt{x^2 + a^2} - x)(\sqrt{x^2 + a^2} + x)/(\sqrt{x^2 + a^2} + x)] = \lim_{x\to\infty}[(x^2 + a^2 - x^2)/(\sqrt{x^2 + a^2} + x)] = a^2 \lim_{x\to\infty}[1/(\sqrt{x^2 + a^2} + x^2)] = 0$. Geometrically, we can look at $\sqrt{x^2 + a^2} - x$ as the difference between the hypotenuse and one leg of a right triangle. As x gets larger and larger with a fixed, the difference is getting smaller and smaller.

29. We have $\lim_{x\to 2}(x - 2)^2 = 0$ and $1/(x - 2)^2 \geqslant 0$ for all x. Thus, by the reciprocal test, $\lim_{x\to 2}[1/(x - 2)^2] = +\infty$.

33. Factoring out $x - 2$ yields $\lim_{x\to 2+}[(x + 2)/(x - 2)]$. This approaches $4/0$; however, $x - 2 > 0$ for $x > 2$. Therefore, the limit is $+\infty$.

37. Note that $|x| > 0$ and $x > 0$ as $x \to 0+$. Thus, $\lim_{x\to 0+}(|x|/x) = 1$ and $\lim_{x\to 0+}[(x^3 - 1)|x|/x] = \lim_{x\to 0+}(x^3 - 1) = -1$.

41. 

Vertical asymptotes occur where $\lim_{x\to x_0} f(x) = \pm\infty$ and horizontal asymptotes occur at $y = \lim_{x\to\pm\infty} f(x)$. We find that $f(x) = 1/(x^2 - 5x + 6) = 1/(x - 3)(x - 2)$. Thus, $f(x) \longrightarrow \pm\infty$ as $x \longrightarrow 3\pm$ and $f(x) \longrightarrow \mp\infty$ as $x \longrightarrow 2\pm$, so $x = 3$ and $x = 2$ are vertical asymptotes. Also, $\lim_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is the horizontal asymptote.

45. (a) Given $\varepsilon > 0$, there is an $A > 0$ such that $|f(x)| < \varepsilon$ whenever $x > A$ by the assumption that $\lim_{x\to\infty} f(x) = 0$. Given $\varepsilon > 0$, this same δ also gives $|g(x)| < \varepsilon$ whenever $x > A$ since $|g(x)| < |f(x)|$. Hence, g has limit 0 as $x \longrightarrow \infty$, as well.

(b) Let $g(x) = \sin(1/x)/x$ and $f(x) = 1/x$. Then $|g(x)| \leqslant |f(x)|$ and the sandwich principle applies. $\lim_{x\to\infty} f(x) = 0$, so $\lim_{x\to\infty} g(x) = 0$ also.

49. The numerator is $(x - 1)(x^2 + x + 1)$ and the denominator is $(x - 1) \times (x + 1)$. Therefore, if $x^2 - 1 \neq 0$, then $\lim_{x\to 1}[(x^3 - 1)/(x^2 - 1)] = \lim_{x\to 1}[(x^2 + x + 1)/(x + 1)] = 3/2$.

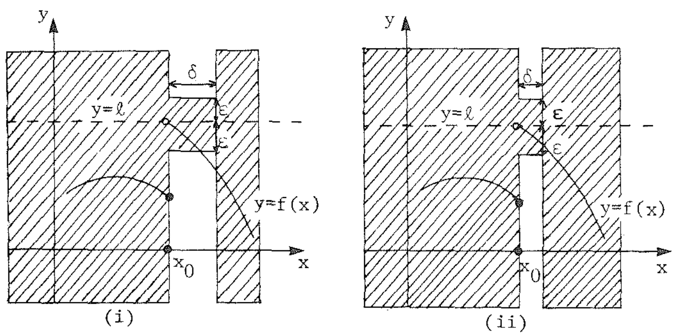
53. If $x \neq -1$, then long division shows us that $\lim_{x\to -1}[(x^{2n+1} + 1)/(x + 1)] = \lim_{x\to -1}(x^{2n} - x^{2n-1} + x^{2n-2} - x^{2n-3} + \ldots + x^4 - x^3 + x^2 - x + 1)$. The limit of the terms with even exponents is 1, and the limit of the terms with odd exponents is $-(-1) = 1$. There are $2n + 1$ terms, so the answer is $2n + 1$.

57. The numerator approaches $e - 1$, which is finite and nonzero, while the denominator approaches 0 on the positive side; therefore, the limit is $+\infty$.

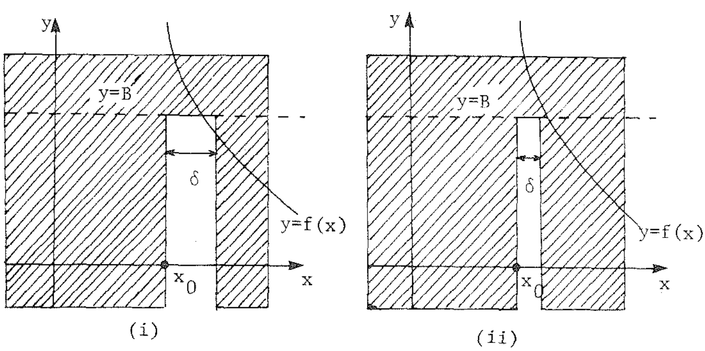
61. By definition, when $\lim_{x\to\infty} f(x) = \ell$ or $\lim_{x\to-\infty} f(x) = \ell$, the line $y = \ell$ is called a horizontal asymptote. A vertical asymptote occurs where a one-sided limit of $f(x)$ at x_0 is $+\infty$ or $-\infty$. Vertical asymptotes usually occur where the denominator is 0. Division by x^2/x^2 yields $y = (1/x)/(1 - 1/x^2)$, so $\lim_{x\to\infty} f(x) = 0$, which is the horizontal asymptote. The vertical asymptotes are $x = -1$ and $x = 1$.

65. Let $f(x) = \Sigma_{i=0}^{N} a_i x^i$ and let $g(x) = \Sigma_{i=0}^{M} b_i x^i$. If $N < M$, then we can divide by x^M/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N x^N/x^M + a_{N-1}x^{N-1}/x^M + \ldots + a_0/x^M] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. Since $N < M$, the limit of the numerator is 0 because $\lim_{x\to\infty}(1/x) = 0$. The limit of the denominator is b_M, so $\lim_{x\to\infty}[f(x)/g(x)] = 0$ if $N < M$. We also know that $\lim_{x\to-\infty}(1/x) = 0$, so $\lim_{x\to-\infty}[f(x)/g(x)] = 0/b_M = 0$ by a similar method. If $N = M$, then we can divide by x^N/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N + a_{N-1}/x + a_{N-2}/x^2 + \ldots + a_0/x^N] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. Since $\lim_{x\to\infty}(1/x) = 0$, we have $\lim_{x\to\infty}[f(x)/g(x)] = a_N/b_M$. By a similar method, $\lim_{x\to-\infty}[f(x)/g(x)] = a_N/b_M$. If $N > M$, then we can divide by x^M/x^M to get $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to\infty}[a_N x^N/x^M + a_{N-1}x^{N-1}/x^M + \ldots + a_0/x^M] / \lim_{x\to\infty}[b_M + b_{M-1}/x + b_{M-2}/x^2 + \ldots + b_0/x^M]$. The limit of the denominator is b_M. The numerator is $\lim_{x\to\infty} a_N x^{N-M}$. If $a_N < 0$, then $\lim_{x\to\infty}[f(x)/g(x)] = -\infty$ and the limit is $+\infty$ if $a_N > 0$. $\lim_{x\to\infty}[f(x)/g(x)]$ depends on $N - M$. If $N - M$ is even, then $\lim_{x\to\infty}[f(x)/g(x)] = \lim_{x\to-\infty}[f(x)/g(x)]$. If $N - M$ is odd, then $\lim_{x\to\infty}[f(x)/g(x)] = -\lim_{x\to-\infty}[f(x)/g(x)]$.

69. (a) To satisfy the definition, the graph of $f(x)$ must be between $\ell - \varepsilon$ and $\ell + \varepsilon$ in the interval $[x_0, x_0 + \delta]$. (i) is an example of a poorly chosen δ. (ii) shows an example of a well-chosen δ.



(b) The definition of the statement requires the graph of $f(x)$ to lie above B in the interval $[x_0, x_0 + \delta]$. (i) shows an example of a poorly chosen δ. (ii) shows an example of a well-chosen δ.



73. Example 5 tells us that $\lim\limits_{x\to\infty} e^{-x} = 0$, so $\lim\limits_{t\to\infty} e^{-\mu t} = 0$ for $\mu > 0$. For fixed x, $\lim\limits_{t\to\infty} T(x,t) = [\lim\limits_{t\to\infty} B_1 \sin\lambda_1 x][\lim\limits_{t\to\infty} e^{-\mu_1 t}] + [\lim\limits_{t\to\infty} B_2 \sin\lambda_2 x] \times [\lim\limits_{t\to\infty} e^{-\mu_2 t}] + [\lim\limits_{t\to\infty} B_3 \sin\lambda_3 x][\lim\limits_{t\to\infty} e^{-\mu_3 t}]$. Since B, λ, and x are

73. (continued)

constants, the limit is $(B_1\sin\lambda_1x)(0) + (B_2\sin\lambda_2x)(0) + (B_3\sin\lambda_3x)(0) = 0$

77. The composite function rule states that if g is continuous at $\lim_{x\to x_0} f(x)$, then $\lim_{x\to x_0} g(f(x)) = g\left(\lim_{x\to x_0} f(x)\right)$. Let $\lim_{x\to x_0} f(x) = L$ and $\varepsilon > 0$ be given. We need to find δ such that $|g(f(x)) - g(L)| < \varepsilon$ whenever $|x - x_0| < \delta$. Since g is continuous at L, there is a $\rho > 0$ such that, whenever $|y - L| < \rho$, $g(y)$ is defined and $|g(y) - g(L)| < \varepsilon$. By the definition of $\lim_{x\to x_0} f(x) = L$, we can find δ such that, whenever $|x - x_0| < \delta$, we also have $|f(x) - L| < \rho$. For such x, we apply the previously obtained property of ρ, with $y = f(x)$, to conclude that $g(f(x))$ is defined and that $|g(f(x)) - g(L)| < \varepsilon$.

81. By the definition of $\lim_{x\to\infty} f(x) = \ell$, we have, whenever $x > A$, $|f(x) - \ell| < \varepsilon$, i.e., $-\varepsilon < f(x) - \ell < \varepsilon$, i.e., $-\varepsilon + \ell < f(x) < \varepsilon + \ell$. Similarly, $\lim_{y\to 0+} f(1/y) = \ell$ means that whenever $0 < y < \delta$, $|f(1/y) - \ell| < \varepsilon$, i.e., $-\varepsilon < f(1/y) < \ell < \varepsilon$, i.e., $-\varepsilon + \ell < f(1/y) < \varepsilon + \ell$. Replacing $1/y$ by x gives $-\varepsilon + \ell < f(x) < \varepsilon + \ell$ whenever $0 < 1/x < \delta$, i.e., $\infty > x > 1/\delta$. This means that $\lim_{x\to\infty} f(x) = \ell$ if $1/\delta = A$. Now, substituting x by $1/y$ gives $-\varepsilon + \ell < f(1/y) < \varepsilon + \ell$ whenever $\infty > 1/y > A$, i.e., $0 < y < 1/A$. This means that $\lim_{x\to 0+} f(1/y) = \ell$ if $1/A = \delta$.

SECTION QUIZ

1. Evaluate the following limits: (State $+\infty$ or $-\infty$, if appropriate.)

   (a) $\lim_{x\to 5+}(|2x - 10|/(x - 5))$

   (b) $\lim_{x\to 5-}(|x - 5|/(2x - 10))$

   (c) $\lim_{t\to -\infty}(|t|/t))$

1. (d) $\lim_{r\to\infty}[(2r^3 - 5r^2 + 1000)/(3r^3 - 1)]$

   (e) $\lim_{x\to 0}(1/x^5)$

   (f) $\lim_{x\to 0}(1/x^6)$

   (g) $\lim_{x\to 2-}[(x^2 - 5x + 6)/(x - 2)]$

   (h) $\lim_{x\to 3} 1$

   (i) $\lim_{t\to 9}[(81 - t^2)/(3 - \sqrt{t})]$

2. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x\to 4}(x^3 - x^2) = 48$.

3. Use the $\varepsilon - A$ definition of the limit to prove that $\lim_{x\to\infty}[(x^3 + 1)/x^3] = 1$.

4. City ordinance limits the height of residential buildings to 50 feet. Modern Architect, Inc. wants to build a home whose height is $30 + (20x^3 + x)/(x^3 + x^2 + 2)$ feet, $x \geqslant 0$, where x is the number of centimeters from a designated boundary.

   (a) If a piece of property extends forever, how high will the building be very far from the designated boundary?

   (b) Will Modern Architect violate the building laws? Explain.

   (c) Use the definition of limits to prove your answer in part (a).

ANSWERS TO PREREQUISITE QUIZ

1. (a) $[f(x) + g(x)]$

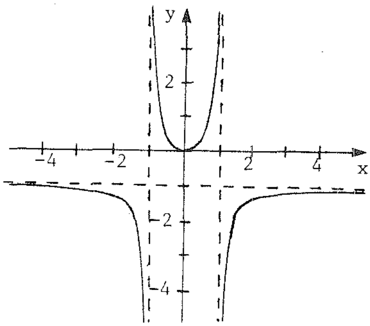
   (b) 3

   (c) $\lim_{x\to\infty} f(x)$

2. The limit doesn't exist. $\lim_{x\to 0-}(1/x) = -\infty$ and $\lim_{x\to 0+}(1/x) = +\infty$.

3. Yes; zero.

4.



The horizontal asymptote is $y = -1$. The vertical asymptotes are $x = -1$ and $x = 1$.

ANSWERS TO SECTION QUIZ

1. (a) 2
   (b) $-1/2$
   (c) -1
   (d) $2/3$
   (e) Does not exist
   (f) $+\infty$
   (g) -1
   (h) 1
   (i) 108
2. Choose $\delta = \sqrt[3]{\varepsilon}$ and show that $|f(x) - L| < \varepsilon$ whenever $|x - 4| < \delta$.
3. Choose $A > \sqrt[3]{1/\varepsilon}$ and show that $|f(x) - L| < \varepsilon$ whenever $x > A$.
4. (a) $\lim_{x\to\infty}[30 + (20x^3 + x)/(x^3 + x^2 + 2)] = 50$
   (b) No, the function is strictly increasing.
   (c) Choose $A > 61/\varepsilon$ and show that $|f(x) - L| < \varepsilon$ whenever $x > A$.

11.2 L'Hôpital's Rule

PREREQUISITES

1. Recall how to compute a limit (Sections 1.2 and 11.1).

2. Recall the various rules for differentiation (Chapters 1 and 2, Sections 5.2, 5.3, and 6.3).

3. Recall the mean value theorem for differentiation (Section 3.6).

PREREQUISITE QUIZ

1. Let $f(x) = (x - 3)/(x^2 - 5x + 6)$.

   (a) What is $\lim_{x\to 2+} f(x)$?

   (b) What is $-\lim_{x\to 2} |f(x)|$?

   (State $+\infty$ or $-\infty$, if appropriate.)

2. Differentiate $g(x) = (x^2 + e^x)/(3^x - \cos x + \ln x)$.

3. Differentiate $f(x) = \sin(\log_{10} x)$.

4. Let f be differentiable on $[2,3]$, $f(2) = -1$ and $f(3) = 3$. What does the mean value theorem tell you about the slope of the graph on $[2,3]$?

GOALS

1. Be able to know when and how to use l'Hôpital's rule for computing limits of functions.

STUDY HINTS

1. L'Hôpital's rule. This can only be used if the function has the form $0/0$ or ∞/∞ . The process may be repeated as often as necessary, but be sure the function has the appropriate form. You should be concerned about applying this rule, rather than learning its proof, unless you are in an honors class.

2. Trick number 1. Example 6 shows how to convert a product into a form for which l'Hôpital's rule applies. The trick is to divide by the reciprocal of one of the factors: $uv = u/(1/v)$.

3. Trick number 2. Many times, the limit of an exponential form can be determined by applying l'Hôpital's rule to the logarithm of the desired function. See Example 7. The composite function rule for limits and the continuity of the exponential function permits us to use this trick.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This has the form $0/0$, so $\lim_{x\to3}[(x^4 - 81)/(x - 3)] = \lim_{x\to3}(4x^3/1) = 108$.

5. This has the form $0/0$, so $\lim_{x\to0}[(\cos 3x - 1)/5x^2] = \lim_{x\to0}(-3 \sin 3x/10x)$ which has the form $0/0$. Applying l'Hôpital's rule again yields $\lim_{x\to0}(-9 \cos 3x/10) = -9/10$.

9. This has the form ∞/∞ , so $\lim_{x\to\infty}(e^x/x^{375}) = \lim_{x\to\infty}(e^x/375x^{374})$, which again has the form ∞/∞ . Continuing with l'Hôpital's rule yields $\lim_{x\to\infty}(e^x/375\cdot374x^{373})$. Using l'Hôpital's rule 373 more times gives us $\lim_{x\to\infty}(e^x/375!) = \infty$, where $375! = 375\cdot374\cdot\ldots\cdot1$.

13. $x^4\ln x$ does not have a form to which l'Hôpital's rule applies, but $\lim_{x\to0}(x^4\ln x) = \lim_{x\to0}(\ln x/x^{-4})$, which has the form ∞/∞ . By l'Hôpital's rule, this is $\lim_{x\to0}[(1/x)/(-4x^{-5})] = \lim_{x\to0}(-x^4/4) = 0$.

17. We need to transform the function to a form for which l'Hôpital's rule applies. $(\tan x)^x = \exp[x \ln(\tan x)]$ and $x \ln(\tan x) = \ln(\tan x)/(1/x)$ has the form ∞/∞ . Applying l'Hôpital's rule, we have $\lim_{x\to0}[\ln(\tan x)/(1/x)] = \lim_{x\to0}[(\sec^2x/\tan x)/(-x^{-2})] = \lim_{x\to0}(-x^2/\cos x \sin x)$, which has the form $0/0$. Using l'Hôpital's rule again, we get $\lim_{x\to0}[-2x/(\cos^2x - \sin^2x)] = 0/1$. By the continuous function rule, $\lim_{x\to0}[(\tan x)^x] = e^0 = 1$.

21. This has the form $0/0$, so $\lim_{x\to 0}\left[\left(\sqrt{1+x^2}-1\right)/\sin 2x\right] = \lim_{x\to 0}\left[\left(x/\sqrt{1+x^2}\right)/2\cos 2x\right] = 0/2 = 0$.

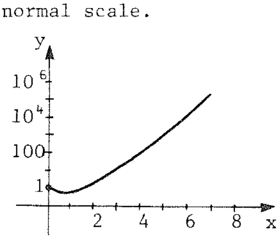
25. This has the form ∞/∞ , so $\lim_{x\to\infty}[x/(x^2+1)] = \lim_{x\to\infty}[1/2x] = 0$.

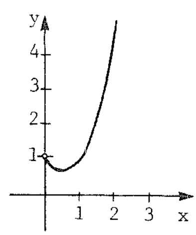
29. This has the form $0/0$, so $\lim_{x\to -1}[(x^2+2x+1)/(x^2-1)] = \lim_{x\to -1}[(2x+2)/2x] = 0/(-2) = 0$.

33. This has the form $0/0$, so $\lim_{x\to\pi}[(1+\cos x)/(x-\pi)] = \lim_{x\to\pi}(-\sin x/1) = 0/1 = 0$.

37. Write $x^p \ln x$ in the form ∞/∞ to use l'Hôpital's rule. $\lim_{x\to 0} x^p \ln x = \lim_{x\to 0}(\ln x/x^{-p}) = \lim_{x\to 0}(x^{-1}/-px^{-p-1}) = \lim_{x\to 0}(x^p/-p) = 0$ because $p > 0$.

41. By Example 7(a), $\lim_{x\to 0+} x^x = 1$. By logarithmic differentiation, we differentiate $\ln y = x \ln x$, so $y'/y = \ln x + 1$. Thus $y' = x^x(\ln x + 1)$, which implies $1/e$ is the critical point. The graph on the left is drawn on a logarithmic scale. The one on the right is drawn on a normal scale.





SECTION QUIZ

1. Use l'Hopital's rule to compute the following limits:

(a) $\lim_{x\to 0+} x(\ln x)^2$

(b) $\lim_{x\to 0+} (1/x)^{(x^2)}$

(c) $\lim_{x\to -\infty} xe^{2x}$

(d) $\lim_{x\to 0} (x^2 \cos x/x \sin x)$

(e) $\lim_{x\to 0+} (1/2x - 1/\tan x)$

(f) $\lim_{x\to 0+} (1/x - 1/\tan x)$

2. The function $f(x) = (\cos x - \sqrt{2}/2)/(x^2 - \pi x/4)$ has the form 0/0 when $x = \pi/4$. Thus, we apply l'Hôpital's rule to get $\lim_{x\to\pi/4}[(-\sin x)/(2x - \pi/4)] = \lim_{x\to\pi/4}(-\cos x/2) = -\sqrt{2}/4$. This is not correct.

   (a) Why isn't the calculation correct?

   (b) What is the limit $\lim_{x\to\pi/4} f(x)$?

3. Can we use l'Hôpital's rule to find $\lim_{x\to\infty}(\sin x/x)$? Why or why not?

4. Deep in the jungles of Brazil, archaeologists have found a headhunter's recipe for shrinking heads. In addition, the notes of a famous mathematician were found nearby. His observations determined that after t seconds of shrinking, the head was shrunk to x percent of the original size, where x is $100 - (25e^t - t\cos t)/(e^t + 1 - t)$. How effective is the recipe, i.e., how much can a head be shrunk after a very long time?

ANSWERS TO PREREQUISITE QUIZ

1. (a) $+\infty$

   (b) $-\infty$

2. $[(2x + e^x)(3^x - \cos x + \ln x) - (x^2 + e^x)(3^x \ln 3 + \sin x + 1/x)]/(3^x - \cos x + \ln x)^2$

3. $\cos(\log_{10}x)/x \ln 10$

4. Somewhere in $(2,3)$, the slope is 4.

ANSWERS TO SECTION QUIZ

1. (a) 0

   (b) 1

   (c) 0

   (d) 1

1. (e) $-\infty$

   (f) 0

2. (a) $-\sin x/(2x - \pi/4)$ does not have the form $0/0$ or ∞/∞ .

   (b) $2\sqrt{2}/\pi$.

3. No; $\sin(x)$ does not approach 0 nor ∞ as x approaches ∞ .

4. 75% of the original size.

11.3 Improper Integrals

PREREQUISITES

1. Recall how to compute limits at infinity (Section 1.2).
2. Recall how to compute infinite limits (Section 1.2).
3. Recall the techniques of integration (Chapter 7, Sections 10.1 and 10.2).

PREREQUISITE QUIZ

1. Compute the following limits: (State $\pm\infty$ if appropriate.)
   (a) $\lim_{x\to 1}|1/(x-1)|$
   (b) $\lim_{x\to\infty}[x^2/(x-1)]$
   (c) $\lim_{x\to 5-}[(5+x)/(25-x^2)]$
2. Evaluate the following integrals:
   (a) $\int(dx/x)$
   (b) $\int(\cos x + x^2)dx$
   (c) $\int xe^x dx$

GOALS

1. Be able to demonstrate the convergence or divergence of improper integrals, and evaluate them.

STUDY HINTS

1. Improper integrals. Integrals may be improper because the integrand approaches $\pm\infty$ for some x_0 or the interval goes to $\pm\infty$. Note that if an integral is the sum of several improper integrals, all of them must separately converge for the entire integral to converge.
2. Comparison test. The important part which needs to be compared is the region near $\pm\infty$. As long as $f(x)$ is finite in the interval $[a,x_0]$, we only need to apply the comparison test on the interval $[x_0,\infty)$ to determine convergence.

3. Infinite at midinterval. $\int_{-3}^{1}(1/x^2)dx$ may appear to be proper since the integrand is finite at the endpoints. However, you should notice that the integral becomes improper at $x_0 = 0$. You should always look for points where an integrand becomes infinite before beginning to evaluate an integral.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $\int_1^\infty(3/x^2)dx = \lim_{b\to\infty}(-3/x)\Big|_1^b = \lim_{b\to\infty}(3 - 3/b) = 3$.

5. Using the method of partial fractions, $\int_2^\infty[dx/(x^2 - 1)] = (1/2)\int_2^\infty[dx/(x - 1) - dx/(x + 1)] = \lim_{b\to\infty}(1/2)\ \ln|(x - 1)/(x + 1)|\Big|_2^b = (1/2) \times \{\lim_{b\to\infty} \ln[(b - 1)/(b + 1)] + \ln 3\}$. By l'Hôpital's rule, $\lim_{b\to\infty} \ln[(b - 1)/(b + 1)] = \lim_{b\to\infty} \ln(1) = 0$. Thus, $\int_2^\infty[dx/(x^2 - 1)] = \ln 3/2$.

9. We want to find $g(x)$ so that $|f(x)| < g(x)$ and that $\int_a^\infty g(x)dx$ converges. If this is possible, then the comparison test states that $\int_a^\infty f(x)dx$ also converges. $f(x) = 1/(3 + x^3) < 1/x^3 = g_1(x)$ on $[1,\infty)$ and $f(x) < 1/3 = g_2(x)$ on $[0,1]$. Thus, $\int_0^\infty[dx/(3 + x^3)] = \int_0^1[dx/(3 + x^3)] + \int_1^\infty[dx/(3 + x^3)] < \int_0^1(dx/3) + \int_1^\infty(dx/x^3) = (x/3)\Big|_0^1 + \lim_{b\to\infty}(-1/2x^2)\Big|_1^b = 1/3 + 1/2 - \lim_{b\to\infty}(1/2b^2) = 5/6$. Thus, $\int_0^\infty[dx/(3 + x^3)]$ converges.

13. Use the method of Example 6. Find $g(x)$ less than the given $f(x)$ and then, show that $g(x)$ diverges. Choose $g_1(x) = 1/2 < 1/\sqrt{2 + x^2} = f(x)$ on $[0,1]$ and choose $g_2(x) = 1/x = 1/\sqrt{x^2} < f(x)$ on $[1,\infty)$. Thus, $\int_0^\infty\left(dx/\sqrt{2 + x^2}\right) = \int_0^1\left(dx/\sqrt{2 + x^2}\right) + \int_1^\infty\left(dx/\sqrt{2 + x^2}\right) > \int_0^1(dx/2) + \int_1^\infty(dx/x) = (x/2)\Big|_0^1 + \lim_{b\to\infty} \ln x\Big|_1^b$. Since $\lim_{b\to\infty} \ln b$ diverges, $\int_0^\infty\left(dx/\sqrt{2 + x^2}\right)$ diverges.

17. $\int_0^{10}(dx/x^{2/3}) = \lim_{a\to 0+}\int_a^{10}(dx/x^{2/3}) = \lim_{a\to 0+} 3x^{1/3}\Big|_a^{10} = 3 \lim_{a\to 0+}\left(\sqrt[3]{10} - \sqrt[3]{a}\right) = 3\sqrt[3]{10}$.

21. The integral is improper at $x = 0$, so $\int_{-1}^{1}[dx/(x^2 + x)] = \int_{-1}^{0}[dx/(x^2 + x)] + \int_{0}^{1}[dx/(x^2 + x)]$. Since $x^2 \leqslant x$ on $(0,1)$, we have $\int_0^1[dx/(x^2 + x)] \geqslant \int_0^1(dx/2x) = \lim_{a\to 0+}\int_a^1(dx/2x) = \lim_{a\to 0+}(\ln x/2)\big|_a^1$, which approaches $-\infty$. Since part of the original integral diverges, the entire integral diverges.

25. $\tan^{-1}x$ lies between $-\pi/2$ and $\pi/2$ for all x, so we can apply the comparison test with $f(x) = \tan^{-1}x/(2 + x)^3$ and $g(x) = (\pi/2)/(2 + x)^3$. $\int_{-1}^{\infty}g(x)dx = (\pi/2)[\lim_{b\to\infty}\int_1^b(2 + x)^{-3}dx] = (\pi/2)\lim_{b\to\infty}-(2 + x)^{-2}/2\big|_1^b = (\pi/2)(-1/2 + \lim_{b\to\infty}[1/2(2 + b)^2]) = (\pi/2)(-1/2) = -\pi/4$. Since $\int_{-1}^{\infty}g(x)dx$ converges, so does $\int_{-1}^{\infty}f(x)dx$.

29. Let $f(x) = 1/(5x^2 + 1)^{2/3}$ and $g(x) = 1/(5x^2)^{2/3} = 1/\sqrt[3]{25}x^{4/3}$. $\int_1^{\infty}|f(x)|dx \leqslant \int_1^{\infty}g(x)dx = (1/\sqrt[3]{25})\int_1^{\infty}(dx/x^{4/3}) = \lim_{b\to\infty}(-3/x^{1/3})\big|_1^b = 3$. Thus, $\int_1^{\infty}f(x)dx$ converges by the comparison test.

33. Note that $\int_{-\infty}^{-2}(1/x^{6/5} - 1/x^{4/3})dx = -\int_{\infty}^{2}[1/(-u)^{6/5} - 1/(-u)^{4/3}]du = \int_2^{\infty}[1/(-u)^{6/5} - 1/(-u)^{4/3}]du = \lim_{b\to\infty}\{\int_2^b[(-u)^{-6/5} - (-u)^{-4/3}]du\}$. Since each exponent is less than -1, by the result of Example 2, both terms converge. Thus, $\int_{-\infty}^{-2}(1/x^{6/5} - 1/x^{4/3})dx$ converges.

37. The integrand is unbounded at 0, so $\int_{-\infty}^{2}(1/x^{5/3} - 1/x^{4/3})dx = \lim_{a\to-\infty}\lim_{b\to 0-}\int_a^b(1/x^{5/3} - 1/x^{4/3})dx + \lim_{c\to 0+}\int_c^2(1/x^{5/3} - 1/x^{4/3})dx = \lim_{a\to-\infty}\lim_{b\to 0-}(-3/2x^{2/3} + 3/x^{1/3})\big|_a^b + \lim_{c\to 0+}(-3/2x^{2/3} + 3/x^{1/3})\big|_c^2$. This diverges toward $-\infty$, so the integral is divergent.

41. The parametric equations $\theta = t$, $r = t^{-k}$ are equivalent to $r = \theta^{-k}$, so $r'(\theta) = -k\theta^{-k-1}$. From Section 10.5, the arc length in polar coordinates is $L = \int_{\pi/2}^{\infty}\sqrt{(-k\theta^{-k-1})^2 + (\theta^{-k})^2}\,d\theta = \int_{\pi/2}^{\infty}(\sqrt{k^2 + \theta^2}/\sqrt{\theta^{2k+2}})d\theta \geqslant \int_{\pi/2}^{\infty}(\sqrt{\theta^2}/\theta^{k+1})d\theta = \int_{\pi/2}^{\infty}(d\theta/\theta^k)$. Since $\theta > 1$, we have $\int_{\pi/2}^{\infty}(\sqrt{k^2 + \theta^2}/\sqrt{\theta^{2k+2}})d\theta \leqslant \int_{\pi/2}^{\infty}(\sqrt{k^2\theta^2 + \theta^2}/\theta^{k+1})d\theta = \sqrt{k^2 + 1}\int_{\pi/2}^{\infty}(\theta/\theta^{k+1})d\theta = \sqrt{k^2 + 1} \times \int_{\pi/2}^{\infty}(d\theta/\theta^k)$. Thus, we have $\sqrt{k^2 + 1}\int_{\pi/2}^{\infty}(d\theta/\theta^k) \geqslant L \geqslant \int_{\pi/2}^{\infty}(d\theta/\theta^k)$, so

41. (continued)

L acts like $\int_{\pi/2}^{\infty}(d\theta/\theta^k)$. By a method similar to that of Example 2, this converges for $k > 1$.

The case $k = 0$ is special because the spiral self-intersects forming a circle of radius 1 . Thus, the arc length is finite for $k = 0$ or $k > 1$.

45. In Example 11, Luke travels 10 million miles. In this problem, he must wake up before he gets within 1 million miles of the sun, so the distance travelled is 9 million miles. Thus, the time to travel 9 million miles is $[(9 \times 10^6)/(\sqrt{3}/2)](1/10^6) = 6\sqrt{3}$ hours ≈ 10.39 hours.

49. The limit is $\lim_{A\to 0+}[\int_{-3}^{-A}(dx/x) + \int_A^2(dx/x)]$. Recall that an improper integral converges only if each part separately converges. Since $\lim_{A\to 0+}\int_A^2(dx/x) = \lim_{A\to 0+}\ln|x|\,\Big|_A^2$ diverges, the entire integral diverges.

53. (a) If $x = (\tau - \mu)/\sigma$, then $x = x_1$ when $\tau = a$ and x approaches $-\infty$ as τ approaches $-\infty$. $dx = d\tau/\sigma$, so substitution yields $p = \int_{-\infty}^{x_1}(1/\sqrt{2\pi}\sigma)e^{-x^2/2}(\sigma dx) = \int_{-\infty}^{x_1}(1/\sqrt{2\pi})e^{-x^2/2}dx$.

(b) If $|x| > 1$, then $e^{-x^2/2} < e^{-x/2}$ and if $|x| < 1$, then $e^{-x^2/2} < 1$. Since the integrand is symmetric to the y-axis, $\int_{-\infty}^{\infty}e^{-x^2/2}dx = 2\int_0^{\infty}e^{-x^2/2}dx < 2\int_0^1 dx + 2\int_1^{\infty}e^{-x/2}dx = 2 + 2\lim_{b\to\infty}\int_1^b e^{-x/2}dx =$ $2 + \lim_{b\to\infty}(-2e^{-x/2})\Big|_1^b = 2 - 2(\lim_{b\to\infty}e^{-b/2} + 2/\sqrt{e}) = 2 + 4/\sqrt{e} < \infty$. Thus, the comparison test tells us that $\int_{-\infty}^{\infty}e^{-x^2/2}dx < \infty$.

57. If $0 < f'(x) < 1/x^2$ on $[0,\infty)$, the same is true on $[a,\infty)$ for $a > 1$. By the comparison test, we have $\int_a^{\infty}0\,dx < \int_a^{\infty}f'(x)dx < \int_a^{\infty}(dx/x^2)$, i.e., $0 < \lim_{b\to\infty}f(x)\Big|_a^b < \lim_{b\to\infty}(-1/x)\Big|_a^b$ by the fundamental theorem of calculus. Thus, $0 < \lim_{b\to\infty}f(b) - f(a) < 1/a$, i.e., $f(a) < \lim_{b\to\infty}f(b) < 1/a + f(a)$. Since $f'(a)$ exists, $f(a)$ also exists. Now, since $f'(x) > 0$, f is an increasing function. Therefore, $f(x)$ is a strictly increasing

57. (continued)

function and $\lim_{b\to\infty} f(b)$ is bounded above by $1/a + f(a)$. Thus, the limit exists.

SECTION QUIZ

1. Evaluate the following improper integrals, if they converge:
   (a) $\int_{-\infty}^{-3}[x/(x^2 - 1)]\,dx$
   (b) $\int_{-\infty}^{10} e^{4x}\,dx$
   (c) $\int_0^3 x^2 \ln x\,dx$
2. The integral $\int_{-10}^{10}[(x - 3)/(x + 5)^2]\,dx$ can be evaluated by substituting $u = x + 5$ to get $\int_{-5}^{15}[(u - 8)/u^2]\,du = (\ln|u| + 8/u)\Big|_{-5}^{15} = \ln 3 + 8/15 + 8/5$. Review these calculations and explain the errors, if any.
3. Discuss the convergence or divergence of the following integrals:
   (a) $\int_0^{50}[dx/(x^4 + x^3 + x^2 + 1)]$
   (b) $\int_0^{\infty}[dx/(x^{3/2} + 1)]$
   (c) $\int_{-2}^{1}[x^3/(x^4 - 1)]\,dx$
4. Ever since old man Anderson passed away 25 years ago, no one has set foot in his mansion. Each year, on the 13th of March, neighbors have reported strange, screeching noises from inside the mansion. This year, Nasty Nicholas wanted to prove his manliness by spending the night in the mansion. On the morning of the 14th, Nasty Nicholas emerged shaking with fear. One hour later, his hair began turning white at a rate of $25x^2/(x^4 + x)$, in strands of hair per hour. Will all of Nasty Nick's hair eventually turn white due to the big scare? Explain.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $+\infty$

   (b) $+\infty$

   (c) $+\infty$

2. (a) $\ln|x| + C$

   (b) $\sin x + x^3/3 + C$

   (c) $xe^x - e^x + C$

ANSWERS TO SECTION QUIZ

1. (a) Diverges

   (b) $e^{40}/4$

   (c) $9 \ln 3 - 3$

2. The integral is improper at $x = -5$, so $\int_{-10}^{10}[(x - 3)/(x + 5)^2]\,dx = \int_{-10}^{-5}[(x - 3)/(x + 5)^2]\,dx + \int_{-5}^{10}[(x - 3)/(x + 5)^2]\,dx$. Then substituting $u = x + 5$ gives $\int_{-5}^{0}[(u - 8)/u^2]\,du + \int_{0}^{15}[(u - 8)/u^2]\,du$, which diverges.

3. (a) Converges; it is not even improper on $[0,50]$.

   (b) Converges; it is less than $\int_0^1 dx + \int_1^\infty (dx/x^{3/2})$.

   (c) Diverges, $\int_{-2}^{-1}|x^3/(x^4 - 1)|\,dx > \int_{-2}^{-1}|(x^4 - 1)^{-1}|\,dx$.

4. By comparison, $\int_1^\infty [25x^2/(x^4 + x)]\,dx \leqslant \int_1^\infty (25/x^2)\,dx = (-25/x)\big|_1^\infty = 25$. Thus, the big scare will cause less than 25 strands of hair to turn white.

11.4 Limits of Sequences and Newton's Method

PREREQUISITES

1. Recall how to compute a limit by using basic properties (Sections 1.2 and 11.1).
2. Recall how to compute a limit by using l'Hôpital's rule (Section 11.2).
3. Recall the formula used for the linear approximations (Section 1.6).

PREREQUISITE QUIZ

1. Compute $\lim_{x\to 2}(2/x - x)$.
2. Compute $\lim_{x\to\infty}(2/x)$.
3. Compute $\lim_{x\to\pi}[(\cos x + 1)/(\pi^2 - \pi x)]$.
4. If f is differentiable, write down the linear approximation to $f(x + \Delta x)$.

GOALS

1. Be able to find the limit of a sequence.
2. Be able to find roots by using Newton's method.

STUDY HINTS

1. Limit of sequences. A sequence is merely a list of numbers. For almost all sequences that we will consider, a definite pattern, which may be expressible by a formula, will exist. A limit ℓ exists if for any $\varepsilon > 0$, a_n is within $(\ell - \varepsilon, \ell + \varepsilon)$ whenever n is large enough. Example 3 shows how to use the precise definition.
2. Limit of "functional" sequences. If a sequence a_n is given by a formula, i.e., $a_n = f(n)$ for a function $f(x)$, then the limit of the sequence as n approaches ∞ is the same as the limit of the function

2. (continued)

as x approaches ∞. Notice that since n is restricted to be an integer, a sequence may have a limit even though the function does not have a limit. For example, the sequence $\lim_{n\to\infty} \sin \pi n$ has a limit, but $\lim_{x\to\infty} \sin \pi x$ does not have a limit.

3. Limit of powers. The box located at the bottom of p. 542 is obvious if you recall the discussion about exponents in Chapter 6. Alternatively, you may remember simple examples: 2^n gets large with n, while $(1/2)^n$ gets small.

4. Comparison test. Notice how this statement is very similar to the comparison test of Section 11.1. Again, there is a "sandwich" effect.

5. Newton's method. This procedure uses the linear approximation. You should either memorize the iteration formula, which is $x_{n+1} = x_n - f(x_n)/f'(x_n)$, or learn to derive it. Since we are looking for a root, we seek an approximation so that $f(x_n + \Delta x) \approx f(x_n) + f'(x_n)\Delta x = 0$. Thus, rearrangement yields $\Delta x = -f(x_n)/f'(x_n)$, and so $x_{n+1} = x_n + \Delta x = x_n - f(x_n)/f'(x_n)$. The formula is used repeatedly until $x_{n+1} \approx x_n$, i.e., $\Delta x \approx 0$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This exercise is analogous to Example 1. $10a_n = 1 + 1/10 + \dots + 1/10^{n-1}$, so $10a_n - a_n = 9a_n = 1 - 1/10^n$, i.e., $a_n = 1/9 - (1/9)(1/10^n)$, i.e., $1/9 - a_n = (1/9)(1/10^n)$, which we want to be less than $1/10^6$. Thus, n must be at least 6.

5. The sequence is determined by substituting $n = 0, 1, 2, \dots$ into $k_n = n^2 - 2\sqrt{n}$. Hence, $k_0 = (0)^2 - 2\sqrt{0} = 0$; $k_1 = (1)^2 - 2\sqrt{1} = -1$; $k_2 = (2)^2 - 2\sqrt{2} = 4 - 2\sqrt{2}$; $k_3 = (3)^2 - 2\sqrt{3} = 9 - 2\sqrt{3}$; $k_4 = (4)^2 -$

5. (continued)

$2\sqrt{4} = 12$; $k_5 = (5)^2 - 2\sqrt{5} = 25 - 2\sqrt{5}$. Thus, the sequence is 0 , -1 , $4 - 2\sqrt{2}$, $9 - 2\sqrt{3}$, 12 , $25 - 2\sqrt{5}$, ...

9. The first term a_0 is given as $1/2$. The next term is $a_1 = a_{0+1} = [1/(0 + 1)]\Sigma_{i=0}^{0}(1/2) = (1/1)(1/2) = 1/2$; $a_2 = a_{1+1} = (1/2)(a_0 + a_1) = (1/2)(1) = 1/2$; $a_3 = a_{2+1} = (1/3)(3/2) = 1/2$; $a_4 = a_{3+1} = (1/4)(2) = 1/2$; $a_5 = a_{4+1} = (1/5)(5/2) = 1/2$. Thus, the sequence is $1/2$ for all n .

13. We need to choose N such that $|a_n - \ell| < \varepsilon$ for all $n \geqslant N$. Note that if $3/2n < \varepsilon$, then $3/(2n + 1) < \varepsilon$ and $1/n < 2\varepsilon/3$. If we choose $N \geqslant 3/2\varepsilon$, then for $n > N$, we have $|3/(2n + 1) - 0| < 3/2n < 3/2N < (3/2)/(3/2\varepsilon) = \varepsilon$.

17. Divide by n^2/n^2 to get $\lim_{n\to\infty}[(1/n - 3)/(1 + 1/n^2)] = (0 - 3)/(1 - 0) = -3$.

21. Note that $(\sin n)^2 \leqslant 1$ for all n and $\lim_{n\to\infty}[1/(n + 2)] = \lim_{n\to\infty}[(1/n)/(1 + 2/n)] = 0$; therefore, the absolute value of the limit is less than or equal to $\lim_{n\to\infty}[1/(n + 2)] = 0$. Thus, we conclude that the limit must be 0 .

25. For $a_n = \sqrt[n]{n/2}$, we calculate $a_{100} \approx 1.03999$, $a_{1000} \approx 1.00623$, and $a_{1000000} \approx 1.00001$. Thus, we guess that $\lim_{n\to\infty} a_n = 1$. By l'Hôpital's rule, $\lim_{x\to\infty}(x/2)^{1/x} = \exp[\lim_{x\to\infty}(\ln(x/2)/x)] = \exp[\lim_{x\to\infty}(1/2x)/1] = e^0 = 1$.

29. We use the fact that $\lim_{n\to\infty} r^n = 0$ if $0 \leqslant r < 1$. Here $r = 1/8$, so $\lim_{n\to\infty}(1/8)^n = 0$.

33. The limit is $\{[\lim_{n\to\infty} 3b + \lim_{n\to\infty} (1/4)^n]/[\lim_{n\to\infty}(n^2 - 1)]\}^3 = \lim_{n\to\infty}(3b/(n^2 - 1))$. Divide by $(n^2/n^2)^3$ to get $\lim_{n\to\infty}[(3b/n^2)/(1 - 1/n^2)]^3 = (0/1)^3 = 0$.

37. The iteration formula is $x_{n+1} = x_n - (x_n^5 + x_n^2 - 3)/(5x_n^4 + 2x_n)$. Starting with $x_0 = 0$, we get $x_1 = 0 - (-3)/0$, so this method does not work. Starting with $x_0 = 2$, we get $x_1 = 1.607143$; $x_2 = 1.325370$; $x_3 = 1.167936$; $x_4 = 1.121778$; $x_5 = 1.118357$; $x_6 = 1.118340$; $x_7 = 1.118340$. $x_6 = x_7$ to 6 decimal places, so one root is 1.118340 . The critical points are $\sqrt[3]{-2/5}$ and 0 , and sketching the graph reveals only one root, so our answer is 1.118340 .

41. Analysis of the graphs of $\tan x$ and αx shows that λ_1 is near $\pi/2$; λ_2 is near $3\pi/2$; λ_3 is near $5\pi/2$. The iteration formula we will be using is $x_{n+1} = x_n - (\tan x_n - \alpha x_n)/(\sec^2 x_n - \alpha)$. Since we want $\tan x_n - \alpha x_n = 0$, a good choice for x_0 is some number just larger than $\tan^{-1}(\alpha n\pi/2)$, where $n = 1$, 3 , and 5 .

For $\alpha = 2$, we start with $x_0 = 1.263$, then $x_3 = 1.1655820$ and $x_4 = 1.1655612$. Starting with $x_0 = 4.607$, we have $x_2 = 4.6042168$ and $x_3 = 4.6042168$. Starting with $x_0 = 7.791$, we get $x_2 = 7.7898996$ and $x_3 = 7.7898838$.

For $\alpha = 3$, we start with $x_0 = 1.362$, then $x_3 = 1.3241947$ and $x_4 = 1.3241945$. Starting with $x_0 = 4.642$, we have $x_2 = 4.6406836$ and $x_3 = 4.6406836$. Starting with $x_0 = 7.812$, we get $x_2 = 7.8113345$ and $x_3 = 7.8113345$.

For $\alpha = 5$, we start with $x_0 = 1.444$, then $x_2 = 1.432042$ and $x_3 = 1.4320322$. Starting with $x_0 = 4.670$, we have $x_1 = 4.6695888$ and $x_2 = 4.6695848$. Starting with $x_0 = 7.829$, we get $x_2 = 7.8284393$ and $x_3 = 7.8284393$.

(continued on next page)

41. (continued)

λ \ α	2	3	5
λ_1	1.1656	1.3242	1.4320
λ_2	4.6042	4.6407	4.6696
λ_3	7.7899	7.8113	7.8284

45. The first term is $a_1 = 2^1$. The second term is $a_2 = 2^2$; the third is $(2)^2 = 2^4$; $a_4 = (2^4)^2 = 2^8$; $a_5 = (2^8)^2 = 2^{16}$. Note that the exponent of 2 is doubling each time we press the "x^2" key, so $a_n = 2^{(2^{n-1})}$.

49. After 1 half-life, only half of the substance is left. After 2 half-lives, half of the half, or 1/4 is left. In general, after n half-lives, $1/2^n$ is left; therefore, the sequence, beginning with a_0, is 1, 1/2, 1/4, 1/8, 1/16, ..., $1/2^n$. $\lim_{n\to\infty} a_n = 0$ taking $r = 1/2$ in the limits of powers.

53. By the sum rule for limits, $\lim_{n\to\infty}[3n/(4n+1) + (-1)^n \sin n/(n+1)] = \lim_{n\to\infty}[3n/(4n+1)] + \lim_{n\to\infty}[(-1)^n \sin n/(n+1)]$. Dividing by n/n yields $\lim_{n\to\infty}[3/(4+1/n)] + \lim_{n\to\infty}[(-1)^n(\sin n/n)/(1+1/n)] = 3/4 + \lim_{n\to\infty}(-1)^n(\sin n/n)$. Since $|(-1)^n \sin n| < 1$, $\lim_{n\to\infty}[(-1)^n \sin n/n] = 0$. Thus, the limit is 3/4.

57. (a) Suppose $\lim_{n\to\infty} b_n < L$, then $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n < L - L = 0$. This says that $\lim_{n\to\infty}(b_n - a_n) < 0$, but the hypothesis that $a_n < b_n$ implies $\lim_{n\to\infty}(b_n - a_n) > 0$, so we have a contradiction. Thus, the original assumption is false, and it must be the case that $\lim_{n\to\infty} b_n \geqslant L$.

57. (b) According to the definition of the limit, $\lim_{n\to\infty} a_n = L$ means that $|a_n - L| < \varepsilon$ or $-\varepsilon + L < a_n < \varepsilon + L$ whenever $n > N_1$. Likewise, $\lim_{n\to\infty} c_n = L$ means that $-\varepsilon + L < c_n < \varepsilon + L$ whenever $n > N_2$. By hypothesis, $-\varepsilon + L < a_n < b_n < c_n < \varepsilon + L$ or $|b_n - L| < \varepsilon$ whenever $n > N_3 = \max(N_1, N_2)$. Thus, we have $\lim_{n\to\infty} b_n = L$.

SECTION QUIZ

1. Use Newton's method to approximate a root of $f(x) = x^3/3 - x^2 - 1$ to three decimal places. Start with $x_0 = -3$. If your calculator is programmable, try starting with $x_0 = 1$. (Hint: 30 steps are required if $x_0 = 1$ on a TI-58C. Your calculator may be different.)

2. Using Newton's method, find the solution of $y^5 + y^3 + y = 2$ accurate to two decimal places.

3. (a) If $f(x) = \cos^2 x\pi$ is a function, what is $\lim_{x\to\infty} f(x)$?

   (b) If $a_n = \cos^2 n\pi$ is a sequence, what is $\lim_{n\to\infty} a_n$?

   (c) Comment about your answers to parts (a) and (b).

4. Find the limit of the following sequences if they exist:

   (a) $\lim_{n\to\infty}[(-1)^{2n}n/(n^2 - n + 5)]$

   (b) $\lim_{n\to\infty}[(2n^3 - n + 1)/(3n^3 - 50n^2)]$

   (c) $\lim_{n\to\infty}[((-1)^n \sin(\pi n/4))/(n^2 + 2)]$

5. Bouncing Bobby, the aspiring four-year-old trampolinist, was practicing on his parent's bed. The ceiling is eight feet above the floor. The bed is 24 inches off the ground. On his n^{th} jump, Bouncing Bobby's height above the floor is $96 - (1/2)^n(72)$ in inches.

   (a) How high does Bobby bounce up to after each of his first six jumps?

5. (b) If Bouncing Bobby continues to jump for a long time before his parents discover his joyful playing, how high will he be jumping when he is discovered?

ANSWERS TO PREREQUISITE QUIZ

1. -1
2. 0
3. 0
4. $f(x + \Delta x) \approx f(x) + [f'(x)]\Delta x$

ANSWERS TO SECTION QUIZ

1. 3.279
2. 0.87
3. (a) Does not exist.
   (b) 1
   (c) The variable for sequences only takes on integer values, so the limit of a sequence may exist even if the limit of the corresponding function does not exist.
4. (a) 0
   (b) 2/3
   (c) 0
5. (a) 60 ; 78 ; 81 ; 91.5 ; 93.75 ; and 94.875 inches
   (b) 96 inches

11.5 Numerical Integration

PREREQUISITES

1. Recall how to express an integral as a Riemann sum (Section 4.3).
2. Recall the relationship between integrals and areas (Section 4.2 and 4.6).

PREREQUISITE QUIZ

1. Using 10 intervals and $f(c_i)$ where c_i is the midpoint of each interval, approximate $\int_0^1 x^2\,dx$ as a Riemann sum. Express your answer using summation notation.
2. What is the relationship between $\int f(x)dx$ and the area under $f(x)$?

GOALS

1. Be able to integrate numerically by using Riemann sums, the trapezoidal rule, or Simpson's rule.
2. Be able to estimate the errors incurred by using Riemann sums, the trapezoidal rule, or Simpson's rule.

STUDY HINTS

1. Riemann sums. After division into equal subintervals, evaluate at one chosen point in each interval. Add up each evaluation and then multiply by $(b - a)/n$, the length of a subinterval. Often, the integrand is evaluated at either the right endpoint or the left endpoint.
2. Riemann error. The error made by using Riemann sums depends on the first derivative of the integrand and the length of the subintervals to the first power.

3. Trapezoidal rule. In this method, the interval of integration is divided into n equal parts. At each division point, the integrand is evaluated and counted twice except at the endpoints. This is then multiplied by $(b - a)/2n$, one-half of a subinterval's length.

4. Trapezoidal error. The error in the trapezoidal rule estimate depends on the second derivative of the integrand and the square of the subinterval's length.

5. Simpson's rule. The method requires an even number of subdivisions. Notice the pattern of the coefficients at the evaluated points; it is $1, 4, 2, 4, 2, \ldots, 2, 4, 1$. 2 and 4 alternate, while the endpoints are considered only once. The weighted sum is then multiplied by $(b - a)/3n$, one-third of a subinterval's length.

6. Simpson's error. The error in the estimate made by Simpson's rule depends on the fourth derivative of the integrand and the fourth power of the length of the subintervals.

7. Equal subdivisions unnecessary. Until this point, the methods presented have used equal subdivisions as a convenience for deriving the special formulas. Be aware that none of the numerical integration methods presented here require equal subdivisions. However, Simpson's method does require an even number of intervals and that the length of the i^{th} interval equal the length of the $(i + 1)^{th}$ interval for odd i.

8. Error estimates. Most beginning calculus courses will not require memorization of the error estimates; however, you should consult your instructor.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. For $\int_{-1}^{1}(x^2 + 1)dx$, with $n = 10$, we have $(b - a)/n = 1/5$ and $x_i = -1 + i/5$. Thus, the Riemann sum is: $(1/5)\Sigma_{i=1}^{10}(x_i^2 + 1) = (1/5)\Sigma_{i=1}^{10}[(-1 + i/5)^2 + 1] = [(1/5)\Sigma_{i=1}^{10}(1 - 2i/5 + i^2/25)] + 2 = 4 - (2/25)\Sigma_{i=1}^{10}i + (1/125)\Sigma_{i=1}^{10}i^2 = 4 - (2/25)[10(11)/2] + (1/125) \times [10(11)(21)/6] = 2.68$. The actual value of $\int_{-1}^{1}(x^2 + 1)dx = (x^3/3 + x)|_{-1}^{1} = 8/3 \approx 2.6667$.

5. With $n = 10$, Simpson's rule tells us that $\int_0^1\left(x/\sqrt{x^3 + 2}\right)dx \approx (1/30)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_8) + 4f(x_9) + f(x_{10})]$, where $f(x) = x/\sqrt{x^3 + 2}$ and $x_i = i/10$. Thus, the integral is approximately $(1/30)[0/\sqrt{2} + 0.4/\sqrt{2.001} + 0.4/\sqrt{2.008} + 1.2/\sqrt{2.027} + 0.8/\sqrt{2.064} + 2/\sqrt{2.125} + 1.2/\sqrt{2.216} + 2.8/\sqrt{2.343} + 1.6/\sqrt{2.512} + 3.6/\sqrt{2.729} + 1/\sqrt{3}] \approx 0.3246$.

9. Simpson's rule needs adjacent equidistant intervals, so let $\Delta x_1 = 0.2$ in $[0,1.2]$ and let $\Delta x_2 = 0.1$ in $[1.2,1.8]$. $n = 6$ for both subdivisions fulfilling the requirement that n be even. Write the integral as a sum of two integrals and apply Simpson's rule to each. $\int_0^{1.8}f(x)dx = \int_0^{1.2}f(x)dx + \int_{1.2}^{1.8}f(x)dx = (1.2/18)[2.037 + 4(1.980) + 2(1.843) + 4(1.372) + 2(1.196) + 4(0.977) + 0.685] + (0.6/18)[0.685 + 4(0.819) + 2(1.026) + 4(0.799) + 2(0.662) + 4(0.538) + 0.555] = (0.2/3)(26.116) + (0.1/3)(13.24) = 2.1824$.

13. If $f(x) = x + \sin x$, then $f'(x) = 1 + \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and $f^{iv}(x) = \sin x$. The maximum error from the trapezoidal rule is $[(b - a)/12]M_2(\Delta x)^2$, where M_2 is the maximum of $|f''(x)|$ in $[a,b]$. We have $M_2 = 1$ and $b - a = \pi/2$, so we want $(\pi/24)(\Delta x)^2 \leqslant 10^{-5}$ or $\Delta x \leqslant (10^{-5}\cdot 24/\pi)^{1/2} \approx 0.00874$. Therefore, we need $n = (\pi/2)/\Delta x \geqslant 180$. (Continued next page.)

13. (continued)

For Simpson's rule, the maximum error is $[(b-a)/180]M_4(\Delta x)^4$, where M_4 is the maximum of $|f^{iv}(x)|$ in $[a,b]$. We have $M_4 = 1$ and $b - a = \pi/2$, so we want $(\pi/360)(\Delta x)^4 \leqslant 10^{-5}$ or $\Delta x \leqslant (10^{-5}\cdot 360/\pi)^{1/4} \approx 0.1840$. Therefore, we need $n = (\pi/2)/\Delta x \geqslant 9$.

17. The maximum error in the trapezoidal rule is $[(b-a)/12]M_2(\Delta x)^2$ where M_2 is the maximum of $f''(x)$ on $[a,b]$. So the maximum error is $(1/12)M_2(1/10)^2 = M_2/1200$. $f'(x) = \Big\{\Big[\sqrt{(1+x^2)^4+(1-x^2)^2} + x(4(1+x^2)(2x)+2(1-x^2)(-2x))/2\sqrt{(1+x^2)^4+(1-x^2)^2}\Big](1+x^2)^3 - x\sqrt{(1+x^2)^4+(1-x^2)^2}(3)(1+x^2)^2(2x)\Big\}/(1+x^2)^6 = \{[((1+x^2)^4 + (1-x^2)^2) + 2x^2 + 6x^4](1+x^2) - 6x^2[(1+x^2)^4+(1-x^2)^2]\}/(1+x^2)^4\sqrt{(1+x^2)^4+(1-x^2)^2} = (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})/(1+x^2)^4\sqrt{(1+x^2)^4+(1-x^2)^2}$.

$f''(x) = \Big\{(-12x + 20x^3 - 150x^5 - 152x^7 - 50x^9)(1+x^2)^4 \times \sqrt{(1+x^2)^4+(1-x^2)^2} - (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})\Big[(2x^2 + 6x^4)(1+x^2)^4/\sqrt{(1+x^2)^4+(1-x^2)^2} + \sqrt{(1+x^2)^4+(1-x^2)^2}(4)(1+x^2)^3 \times (2x)\Big]\Big\}/(1+x^2)^8[(1+x^2)^4+(1-x^2)^2] = (-12x + 20x^3 - 150x^5 - 152x^7 - 50x^9)[(1+x^2)^4+(1-x^2)^2](1+x^2) - (2 - 6x^2 + 5x^4 - 25x^6 - 19x^8 - 5x^{10})[(2x^2+6x^4)(1+x^2) + ((1+x^2)^4 + (1+x^2)^2)8x]\Big/\Big[(1+x^2)^5 \times \Big(\sqrt{(1+x^2)^4+(1-x^2)^2}\Big)^3\Big] = (-56x - 4x^2 + 56x^3 - 4x^4 - 488x^5 + 26x^6 - 264x^7 + 46x^8 - 1298x^9 + 208x^{10} - 1458x^{11} + 312x^{12} - 948x^{13} + 154x^{14} - 372x^{15} + 30x^{16} - 90x^{17} - 16x^{19})/(1+x^2)^5\Big(\sqrt{(1+x^2)^4+(1-x^2)^2}\Big)^3$.

$f''(1) = -2.03$; $f''(0) = 0$; $f''(1/2) = -2.29$; $f''(1/4) = -3.2$; $f''(3/4) = -2.48$. From this data, we can guess that $M_2 < 6$, so the error is $< 6/1200 = 0.005$. Therefore, the first 2 digits are correct.

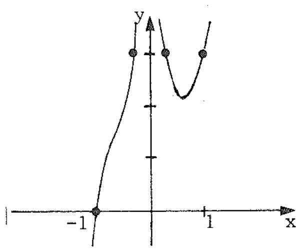
SECTION QUIZ

1. Use each of the methods of Riemann sums (evaluating at the right endpoint), the trapezoidal method, and Simpson's method to estimate $\int_0^1 f(y)dy$ from the given information. If not possible, state why.

   (a) $f(0) = 2$; $f(0.3) = 1$; $f(0.6) = -0.5$; $f(0.7) = -2$; $f(1) = 1$

   (b) $f(0) = 3$; $f(0.25) = 0$; $f(0.5) = -0.5$; $f(0.75) = 0.3$; $f(1) = 0$

   (c) $f(0) = -1$; $f(1/3) = -1$; $f(2/3) = 0$; $f(1) = 0.5$

2. 

   For the figure at the left, $f(-1) = 0$; $f(-1/3) = 3$; $f(1/3) = 3$; and $f(1) = 3$. Can the trapezoidal rule be used to estimate the area under the curve on $[-1,1]$? If yes, estimate it. If no, explain why.

3. Santa Claus' elves produce $x^3 + x^2$ dollars worth of toys during the x^{th} week of the new working season. They begin work 12 weeks after Christmas and work continuously for 40 weeks. Thus, the total toy production in one year is $\int_0^{40}(x^3 + x^2)dx$.

   (a) Estimate toy production to the nearest cent by the trapezoidal rule with $n = 4$.

   (b) Estimate toy production to the nearest cent by Simpson's rule with $n = 4$.

   (c) What is the exact amount of toys produced in one year?

ANSWERS TO PREREQUISITE QUIZ

1. $\Sigma_{i=0}^{9}(2i + 1)^2/400$

2. They are equal.

ANSWERS TO SECTION QUIZ

1. (a) Riemann, 0.25 ; Trapezoidal, 0.25 ; Simpson's can't be used since the third interval does not equal the fourth interval.

   (b) Riemann, -0.050 ; Trapezoidal, 0.325 ; Simpson's, $4/15 \approx 0.267$.

   (c) Riemann, $-1/6 \approx -0.167$; Trapezoidal, $-5/12 \approx -0.417$; Simpson's can't be used since there is an odd number of intervals.

2. Yes; 5

3. (a) \$702,000.00

   (b) \$661,333.33

   (c) \$661,333.33

11.R Review Exercises for Chapter 11

SOLUTIONS TO EVERY OTHER ODD EXERCISE

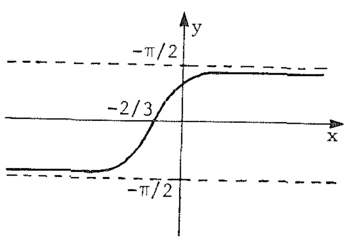
1. Use the method of Example 1, Section 11.1. $f(x) - \ell = (x^2 + x - 1) - 1 = (x - 1 + 1)^2 + (x - 1 + 1) - 2 = (x - 1)^2 + 2(x - 1) + 1 + (x - 1) + 1 - 2 = (x - 1)^2 + 3(x - 1)$. By the properties of absolute value, $|f(x) - \ell| \leqslant |x - 1|^2 + 3|x - 1| = \delta^2 + 3\delta$. Now, assuming that $\delta < 1$, $|f(x) - \ell| \leqslant 4\delta < \varepsilon$. Therefore, if we choose δ to be the minimum of 1 or $\varepsilon/4$, then the definition of the limit is satisfied.

5. By the rational function and the composite function rules, $\lim_{x\to 0} \tan[(x + 1)/(x - 1)] = \tan(-1)$.

9. Multiply numerator and denominator by $\sqrt{2x^2 + 1} + \sqrt{2}x$ to get $\lim_{x\to\infty}\left[(2x^2 + 1 - 2x^2)/(\sqrt{2x^2 + 1} + \sqrt{2}x)\right] = \lim_{x\to\infty}\left[1/(\sqrt{2x^2 + 1} + \sqrt{2}x)\right]$. The denominator tends to ∞ as x tends to ∞ and the numerator is finite, so the entire limit is 0 .

13. $\lim_{x\to 0}[x \sin(3/x^2)] = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \sin(3/x^2)\right)$. $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0} \sin(3/x^2)$ is finite, oscillating between -1 and 1 . Thus, the limit is 0 .

17.



The horizontal asymptotes occur at $y = \lim_{x\to\pm\infty} f(x)$. $\lim_{x\to\infty} \tan^{-1}(3x + 2) = \pi/2$ and $\lim_{x\to-\infty} \tan^{-1}(3x + 2) = -\pi/2$, so $y = \pm\pi/2$ are the horizontal asymptotes. The x-intercept occurs where $0 = \tan^{-1}(3x + 2)$, i.e., $x = -2/3$. The y-intercept is $\tan^{-1}2$.

21. This has the form ∞/∞ , so l'Hôpital's rule gives $\lim_{x\to\infty}[(3x^2 + 8)/(12x^2 - 18x)] = \lim_{x\to\infty}[6x/(24x - 18)] = \lim_{x\to\infty}(6/24) = 1/4$.

25. This has the form $0/0$, so l'Hôpital's rule gives $\lim_{x\to 0}\left[\left(\sqrt{x^2+9}-3\right)/\sin x\right] = \lim_{x\to 0}[(1/2)(x^2+9)^{-1/2}(2x)/\cos x] = 0$.

29. This has the form $0/0$, so l'Hôpital's rule gives us $\lim_{x\to 2}\{[\cos(x-2)-1]/3(x-2)^2\}$. We can apply l'Hôpital's rule again to get $\lim_{x\to 2}[-\sin(x-2)/6(x-2)]$. Applying the same rule gives us $\lim_{x\to 2}[-\cos(x-2)/6] = -1/6$.

33. The function $x \cot x$ does not have the appropriate form, so it must be transformed before l'Hôpital's rule can be used. Thus, $\lim_{x\to 0} x \cot x = \lim_{x\to 0}(x/\tan x)$ and has the form $0/0$; therefore, l'Hôpital's rule gives us $\lim_{x\to 0}(1/\sec^2 x) = 1$.

37. $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty}(x^2/e^x)$ and has the form ∞/∞; therefore, l'Hôpital's rule gives us $\lim_{x\to\infty}(2x/e^x)$. Applying l'Hôpital's rule again yields $\lim_{x\to\infty}(2/e^x) = 0$.

41. Find the limit of the logarithm and then exponentiate. $\lim_{x\to 0+}[\ln(1+\sin 2x)^{1/x}] = \lim_{x\to 0+}[(1/x)\ln(1+\sin 2x)] = \lim_{x\to 0+}[\ln(1+\sin 2x)/x]$, which has the form $0/0$. By l'Hôpital's rule, we get $\lim_{x\to 0+}[(2\cos 2x/(1+\sin 2x))/1] = 2$. Thus, $\lim_{x\to 0+}[(1+\sin 2x)^{1/x}] = e^2$.

45. The integral is $\int_1^\infty (1/x^2)dx = \lim_{b\to\infty}\int_1^b x^{-2}dx = \lim_{b\to\infty}(-1/x)\big|_1^b = \lim_{b\to\infty}(-1/b) + 1 = 1$. Thus, the integral converges to 1.

49. Substitute $u = x - 1$, so $\int_1^2 (dx/\sqrt{x-1}) = \int_0^1 (du/\sqrt{u}) = \lim_{a\to 0+}\int_a^1 u^{-1/2}du = \lim_{a\to 0+} 2\sqrt{u}\big|_a^1 = 2 - 0 = 2$. Thus, the integral converges to 2.

53. Integration by parts gives us $\int x \ln x\, dx = (\ln x)(x^2/2) - \int (x^2/2)(dx/x) = x^2\ln x/2 - \int x\, dx/2 = x^2 \ln x/2 - x^2/4 + C$. Therefore, $\int_0^1 x \ln x\, dx = \lim_{a\to 0+}\int_a^1 x\ln x\, dx = \lim_{a\to 0+}(x^2\ln x/2 - x^2/4)\big|_a^1$. $\lim_{a\to 0+} a^2 \ln a/2 = (1/2)\lim_{a\to 0+}[\ln a/(1/a^2)] = (1/2)\lim_{a\to 0+}[(1/a)/(-1/a^3)] = (1/2)\lim_{a\to 0+}(-a^2) = 0$ by l'Hôpital's rule, so the integral converges to $(0 - 1/4) - (0 - 0) = -1/4$.

57. From Section 9.1, the volume is $\pi\int_0^\infty (xe^{-x})^2 dx$. Integrate by parts as follows: $\int x^2 e^{-2x} dx = x^2 e^{-2x}/(-2) + \int xe^{-2x} dx = -x^2 e^{-2x}/2 + xe^{-2x}/(-2) + \int e^{-2x} dx/2 = -x^2 e^{-2x}/2 - xe^{-2x}/2 - e^{-2x}/4 + C$. By l'Hôpital's rule, $\lim_{x\to\infty}(-x^2/2e^{2x}) = \lim_{x\to\infty}(-2x/4e^{2x}) = \lim_{x\to\infty}(-2/8e^{2x}) = 0$. Also, $\lim_{x\to\infty}(-x/2e^{2x}) = \lim_{x\to\infty}(-1/4e^{2x}) = 0$, and $\lim_{x\to\infty}(-e^{-2x}/4) = 0$. Thus, $\pi\int_0^\infty (xe^{-x})^2 dx = \pi[\lim_{b\to\infty}(-x^2/2e^{2x} - x/2e^{2x} - 1/4e^{2x})]\,|_0^\infty = \pi/4$.

61. Find the limit of the logarithm and then exponentiate. $\lim_{n\to\infty}[n \ln(1 + 8/n)] = \lim_{n\to\infty}[\ln(1 + 8/n)/(1/n)]$. This has the form $0/0$ and l'Hôpital's rule gives us $\lim_{n\to\infty}[((-8n^{-2})/(1 + 8/n))/(-n^{-2})] = \lim_{n\to\infty}[8/(1 + 8/n)] = 8$. Thus, the limit of the original sequence is e^8 .

65. Divide by n/n to get $\lim_{n\to\infty}[1/(1 + 2/n)] = 1/1 = 1$.

69. For any integer n , $\sin \pi n = 0$. Thus, l'Hôpital's rule gives us $\lim_{n\to\infty}(\sin \pi n/n^{-3}) = \lim_{n\to\infty}(\pi \cos \pi n/(-3n^{-4}))$. Now, for any integer n , $\cos \pi n = \pm 1$, so the sequence becomes $\pm(\pi/3)\lim_{n\to\infty} n^4$. Thus, the sequence oscillates between $+\infty$ and $-\infty$. Therefore, the limit does not exist.

73. By l'Hôpital's rule, $\lim_{n\to\infty} \sqrt[2n]{3n} = \exp[\lim_{n\to\infty}((\ln 3n)/2n)] = \exp[\lim_{n\to\infty}(1/n)/2] = e^0 = 1$.

77. Note that $f(-2) = -12$ and $f(-1) = 4$, so let $x_0 = -1$. We use the iteration formula $x_{n+1} = x_n - (x_n^3 - 3x_n^2 + 8)/(3x_n^2 - 6x_n)$ to get $x_1 = -1.44444$, $x_2 = -1.35916$, $x_3 = -1.35531$, $x_4 = -1.35530$, $x_5 = -1.35530$. Thus, one root is -1.35530 . By long division, $(x^3 - 3x^2 + 8)/(x + 1.35530) = x^2 - 4.35530x + 5.90274$. The quadratic equation gives us $(4.35530 \pm \sqrt{-4.64232})/2$. Therefore, the only root is $-1.35530\ldots$.

81. We have $(b - a)/2n = 1/20$ and $f(x) = x^2/\sqrt{x^2 + 1}$. The approximation of $\int_2^3 \left(x^2/\sqrt{x^2 + 1}\right)dx$ by the trapezoidal rule is $(1/20)[f(2) + f(3) + 2\Sigma_{i=1}^{9} f(2 + i/10)] = (1/20)(46.39836) = 2.31992$.

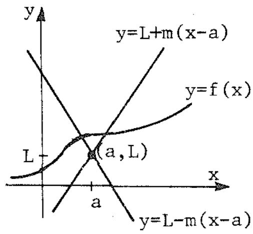
85. For continuity, we must have $\lim_{x\to 0} f(x) = f(0) = \cos(0) = 1$. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \cos x = 1$ and $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 1 = 1$; therefore, the function is continuous. For differentiability, $\lim_{\Delta x\to 0}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\}$ must exist. $f(x_0) = f(0) = 1$. If $x < 0$, then $f(x_0 + \Delta x) = f(\Delta x) = 1$ because $f(x) = 1$ if $x < 0$. Therefore, $\lim_{\Delta x\to 0-} \{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\} = \lim_{\Delta x\to 0-} [f'(x_0 + \Delta x)/1] = 0/1$ because l'Hôpital's rule applies and $f'(\Delta x) = (d/dx)(1) = 0$. If $\Delta x > 0$, $f(x_0 + \Delta x) = \cos(x_0 + \Delta x)$. Thus $\lim_{x\to 0+}\{[f(x_0 + \Delta x) - f(x_0)]/\Delta x\} = \lim_{\Delta x\to 0+} [f'(x_0 + \Delta x)/1] = \lim_{\Delta x\to 0+} \sin(x_0 + \Delta x) = 0$ because l'Hôpital's rule applies again. Since both one-sided limits are equal, the function is both continuous and differentiable.

89. (a) The right-hand side has the form $0/0$, so by l'Hôpital's rule we get $f''(x_0) = \lim_{h\to 0}\{[f'(x_0 + h) - 0 + f'(x_0 - h)(-1)]/2h\}$. This limit also has the form $0/0$, so l'Hôpital's rule yields $\lim_{h\to 0}\{[f''(x_0 + h) - f''(x_0 - h)(-1)]/2 = [f''(x_0) + f''(x_0)]/2 = f''(x_0)$.

(b) By definition, $f'''(x_0) = \lim_{h\to 0}\{[f''(x_0 + h) - f''(x_0)]/h\}$. Substituting the formula from part (a) yields $f'''(x_0) = \lim_{h\to 0}\{[(f(x_0 + 2h) - 2f(x_0 + h) + f(x_0))/h^2 - (f(x_0 + h) - 2f(x_0) + f(x_0 - h))/h^2]/h\} = \lim_{h\to 0}\{[f(x_0 + 2h) - 3f(x_0 + h) + 3f(x_0) - f(x_0 - h)]/h^3\}$. This can be verified by applying l'Hôpital's rule three times.

93. S_n is the Riemann sum for $f(x) = x + x^2$, where $\Delta t_i = 1/n$ and the partition is $a = t_0 = 0$, $t_1 = 1/n$, $t_2 = 2/n$, ... , $t_n = b = 1$. c_i has been chosen to be t_i , so as $n \to \infty$, $\Delta t_i \to 0$ and $S_n \to \int_0^1 (x + x^2)dx = (x^2/2 + x^3/3)\big|_0^1 = 1/2 + 1/3 = 5/6$.

97. (a)



(b) $f(x)$ lies between the two lines means that $L - m(x - a) \leqslant f(x) \leqslant L + m(x - a)$. Subtracting L yields $-m(x - a) \leqslant f(x) - L \leqslant m(x - a)$, which implies $|f(x) - L| \leqslant m|x - a|$. Finally, division gives us $|[f(x) - L]/(x - a)| \leqslant m$.

(c) Given $\varepsilon > 0$, choose $\delta = \varepsilon/2m$. Then, the Lipschitz condition implies $|f(x) - L| \leqslant m|x - a| \leqslant mh = m(\varepsilon/2m) = \varepsilon/2 < \varepsilon$. Thus, we have shown that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| \leqslant \delta$, which is the definition of $\lim\limits_{x\to a} f(x) = L$.

101. (a) If $f(x) = 0$ and $f'(x) \neq 0$, then $N(x) = x - f(x)/f'(x)$ becomes $N(x) = x - 0/f'(x) = x$. Now, if $N(x) = x$, then $N(x) = x - f(x)/f'(x)$ becomes $x = x - f(x)/f'(x)$, i.e., $0 = f(x)/f'(x)$, i.e., $0 = f(x)$.

(b) By the sum and quotient rules, $N'(x) = 1-[f'(x)f'(x) - f(x)f''(x)]/[f'(x)]^2$. This simplifies to $[(f'(x))^2 - (f'(x))^2 + f(x)f''(x)]/[f'(x)]^2 = f(x)f''(x)/[f'(x)]^2$.

(c) By the mean value theorem, we have $N'(\xi) = [N(x) - N(\bar{x})]/(x - \bar{x})$ for some ξ in $(x,\bar{x})$. Since $\bar{x}$ is a root of f , we have $N(\bar{x}) = \bar{x} - f(\bar{x})/f'(\bar{x}) = \bar{x} - 0/f'(\bar{x}) = \bar{x}$. Thus, $N'(\xi) = [N(x) - \bar{x}]/(x - \bar{x})$ and rearrangement yields $N(x) - \bar{x} = N'(\xi)[x - \bar{x}]$. Now, by part (b), $|N'(\xi)| = |f(\xi)f''(\xi)/(f'(\xi))^2| \leqslant |f(\xi)|M/p^2$.

101. (c) (continued)

By Consequence 1 of the mean value theorem, $p \leqslant f'(x) \leqslant q$ implies $p \leqslant |[f(\xi) - f(\bar{x})]/(\xi - \bar{x})| \leqslant q$ or $p|\xi - \bar{x}| \leqslant |f(\xi)| \leqslant q|\xi - \bar{x}|$, since $f(\bar{x}) = 0$. And since $|x - \bar{x}| > |\xi - \bar{x}|$, we have $|f(\xi)| \leqslant q|x - \bar{x}|$. Therefore $|N'(\xi)| \leqslant q|x - \bar{x}|M/p^2$. Finally, $|N(x) - \bar{x}| = |N'(\xi)||x - \bar{x}| \leqslant (qM/p^2)|x - \bar{x}|^2$, so $C = qM/p^2$.

(d) Using the formula, $|N(x) - \bar{x}| \leqslant (qM/p^2)|x - \bar{x}|^2$, we substitute $p = 2.8$, $q = 3.0$, and $M = 2$ since $f'(x) = 2x$ and $f''(x) = 2$. In $[1.4,1.5]$, the maximum of $x - \bar{x}$ is 0.1, so $|N(x) - \bar{x}| \leqslant 0.00765$ for the first iteration. For the second iteration, $|x - \bar{x}| \leqslant 0.00765$, so $|N(x) - \bar{x}| \leqslant 0.000045$. For the third iteration, $|x - \bar{x}| \leqslant 0.000045$, so $|N(x) - \bar{x}| \leqslant 1.538 \times 10^{-9}$. For the fourth iteration $|x - \bar{x}| \leqslant 1.5 \times 10^{-9}$, so $|N(x) - \bar{x}| \leqslant 1.8 \times 10^{-18}$. For the fifth iteration, $|x - \bar{x}| \leqslant 1.8 \times 10^{-18}$, so $|N(x) - \bar{x}| \leqslant 2.5 \times 10^{-36}$. Thus, five iterations are needed to guarantee 20 decimal place accuracy.

TEST FOR CHAPTER 11

1. True or false.

(a) $\lim_{x\to\infty} \sin x = 0$.

(b) If Simpson's method is applicable, either the trapezoidal rule or the method of Riemann sums may also be used.

(c) If an interval is subdivided into an odd number of equal subintervals, Simpson's method may be used even if a discontinuity exists.

(d) The integral $\int_{-1}^{1}(dx/x^2) = -x^{-1}\Big|_{-1}^{1}$ converges.

(e) By l'Hôpital's rule, $\lim_{x\to 0}[(x + 1)/\sin x] = \lim_{x\to 0}(1/\cos x)$.

2. Let $f(x) = x(2x + 4)(x - 7)^2/(x + 3)(x - 7)(7x - 2)^2$.

(a) Find the vertical and horizontal asymptotes of $f(x)$.

(b) For the vertical asymptotes $x = x_0$, compute $\lim_{x\to x_0-} f(x)$ and $\lim_{x\to x_0+} f(x)$ for each x_0 found in part (a).

3. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x\to -1} f(x) = 5$ if $f(x) = \begin{cases} x^2 - x + 3 & \text{if } x \neq -1 \\ 0 & \text{if } x = -1 \end{cases}$.

4. Compute the following limits:

(a) $\lim_{x\to 0}(\tan x/x^3 - 1/x^2)$

(b) $\lim_{x\to 0+}(1 + e^x)^x$

(c) $\lim_{x\to\infty} e^{-x}\sqrt{x + 5}$

(d) $\lim_{x\to 3}[(x + 3)/|\sqrt{x} - \sqrt{3}|]$

5. Evaluate the following improper integrals:

(a) $\int_0^\infty xe^{-x}dx$

(b) $\int_0^3 (dx/\sqrt[3]{x - 2})$

(c) $\int_0^\infty [e^t/(1 + e^{2t})]\,dt$

6. Solve the equation $\sqrt{x} + x = 3$ accurate to three decimal places by using Newton's method.

7. Using the following methods with the partition (0 , 0.5 , 1.0 , 1.25 , 1.5 , 1.75 , 2.0) , approximate the arc length of

$$f(x) = \begin{cases} x^4 & \text{if } 0 \leqslant x \leqslant 1 \\ x^3 & \text{if } 1 \leqslant x \leqslant 2 \end{cases} .$$

(a) Use Simpson's method.

(b) Use the trapezoidal method.

8. Do the following integrals converge or diverge? Justify your answer.

(a) $\int_{-\infty}^\infty x\exp(-x^4)dx$

(b) $\int_{-3}^2 [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})]$

8. (c) $\int_0^2 x(\ln x)^4 dx$

9. Functions of the form $\infty - \infty$ may have any value (if the limit exists). Give a simple example of a limit having the form $\infty - \infty$ whose value is:

(a) $+\infty$

(b) $-\infty$

(c) 3 (Hint: consider $1/\ln x - 1/(x - 1)$.)

10. Krazy Karen's Kookie Kangaroos is a unique rent-a-kangaroo firm. People moving across town can rent a kangaroo, fill its pouch, and ride the kangaroos with their belongings across town. Krazy Karen, who used to work for a mad scientist, has developed energizing tablets for her kookie kangaroos. The average kangaroo can hop for $100n/(n + 3)$ kilometers after consuming n energizing tablets. The effects of the tablets last only one day.

(a) Plot the kilometerage, km, vs. the number of tablets, n , for $n = 1, 2, 3, 4, 5, 6$.

(b) What is the limit of the sequence as $n \to \infty$.

(c) Prove your answer in (b) by using the definition of the limit of sequences.

ANSWERS TO CHAPTER TEST

1. (a) False; limit doesn't exist.

(b) True

(c) False; Simpson's method requires an even number of subintervals.

(d) False; $\int_{-1}^{1}(dx/x^2) = \int_{-1}^{0}(dx/x^2) + \int_0^1(dx/x^2) = (-x^{-1})\big|_{-1}^{0} + (-x^{-1})\big|_0^1$.

(e) False; l'Hôpital's rule doesn't apply to functions with the form $1/0$.

2. (a) Vertical asymptotes at $x = -3$ and $x = 2/7$; horizontal asymptotes at $y = 2/49$.

(b) $\lim_{x\to -3-} f(x) = +\infty$; $\lim_{x\to -3+} f(x) = -\infty$; $\lim_{x\to 2/7-} f(x) = -\infty$; $\lim_{x\to 2/7+} f(x) = -\infty$.

3. Choose $\delta = \varepsilon/2$, so $|f(x) - L| = |x^2 - x + 3 - 5| = |(x + 1 - 1)^2 - (x + 1 - 1) - 2| < |x + 1|^2 + |x + 1| < \delta^2 + \delta < 2\delta$ if $\delta < 1$. Thus, if $|x + 1| < \delta$, then $|f(x) - L| < 2\delta = \varepsilon$.

4. (a) 1/3

(b) 1

(c) 0

(d) $+\infty$

5. (a) 1

(b) $(3/2)(1 - \sqrt[3]{4})$

(c) $\pi/4$

6. 1.697

7. (a) 8.682

(b) 8.955

8. (a) $\int_{-\infty}^{\infty} x \exp(-x^4)dx \leqslant \int_{-\infty}^{\infty} x \exp(-x^2)dx = \exp(-x^2)/2\big|_{-\infty}^{\infty} = 0$. Therefore, $\int_{-\infty}^{\infty} x \exp(-x^4)dx$ converges.

(b) The integral is improper around $x = 0$. If $-1 \leqslant x \leqslant 1$, then $\int_{-1}^{1} [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})] \leqslant \int_{-1}^{1} [dx/3\sqrt[3]{x}] = \int_{-1}^{0} (dx/3\sqrt[3]{x}) + \int_{0}^{1} (dx/3\sqrt[3]{x}) = x^{2/3}/2\big|_{-1}^{0} + x^{2/3}/2\big|_{0}^{1} = 1$. Therefore, $\int_{-1}^{1} [dx/(\sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x})]$ converges.

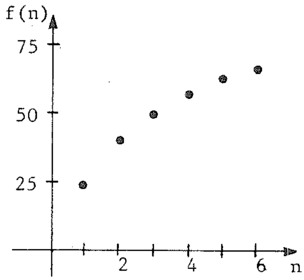
(c) The integral converges because $\lim_{x\to 0+} x(\ln x)^4 = 0$.

9. (a) $\lim_{x\to 0+}(2/x - 1/x)$

(b) $\lim_{x\to 0+}(1/x - 2/x)$

(c) $\lim_{x\to 1+}(6/\ln x - 6/(1 - x))$

10. (a)



(b) 100

(c) Choose $N > (300 - 3\varepsilon)/\varepsilon$

CHAPTER 12

INFINITE SERIES

12.1 The Sum of an Infinite Series

PREREQUISITES

1. Recall the concept of convergence and divergence for infinite sequences and improper integrals (Sections 11.3 and 11.4).
2. Recall how to compute a limit at infinity (Sections 1.2, 11.1 and 11.2).

PREREQUISITE QUIZ

1. Show that $\int_0^\infty [dx/(x^2 + 2)]$ converges.
2. Does the sequence $(-1)^n$ converge as $n \to \infty$? Explain.
3. Compute the following limits:
   (a) $\lim_{x\to\infty}(e^x/x^4)$
   (b) $\lim_{x\to\infty}(\cos x/x^2)$

GOALS

1. Be able to compute the sum of a geometric series.
2. Be able to use the i^{th} term test to show divergence.

STUDY HINTS

1. Infinite series, partial sums, and sequences. Suppose we have a sequence of numbers. The sum of the first n numbers is called the nth partial sum. The partial sums form a sequence. An infinite series is the sum of an infinite number of terms. If the partial sums have a limit, it is the sum of the series.

2. Convergence vs. divergence. If a series has a finite sum, it converges; otherwise, it diverges. Convergence requires a finite sum; that is, the limit of the partial sums must exist. If the limit of the partial sums does not exist, the series diverges. For example, $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ has partial sums alternating between 1 and 0. Thus, the limit of the partial sums does not exist, and the series diverges. This example shows that convergence implies that a series has a finite value, but divergence does not imply that a series tends to infinity.

3. Notation. a_i is the symbol used for the i^{th} term of a series.. S_i is the symbol denoting the i^{th} partial sum.

4. Properties of limits of sequences. Most of the box on p. 563 is common sense. Property 9 is mostly review from p. 542. If $|r| < 1$ or $|r| > 1$, you should understand the result from your work with exponents. If $r = 1$, then $\lim_{n\to\infty} r^n = 1$. If $r = -1$, r^n alternates between 1 and -1, so no limit exists.

5. Geometric series. You should memorize the fact that $\Sigma_{i=0}^{\infty} ar^i = a/(1 - r)$ (first term)/(1 - ratio), provided $|r| < 1$. The series begins at $i = 0$, so a is the first term.

6. Algebraic rules. Again, common sense should tell you the validity of these statements. Note that the statements apply only for convergent series.

7. Important tails. Any changes which occur at the beginning of a series do not affect the convergence or divergence. What is important for convergence is the behavior as i approaches infinity. Note how this example applies to Example 7.

8. i^{th} term test. Divergence is guaranteed if a_i does not approach 0 as i approaches ∞. Note that a_i approaching 0 does not guarantee convergence. The harmonic series is a counterexample.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By definition, $S_n = \Sigma_{i=1}^{n} a_i$. Thus, $S_1 = 1/2$; $S_2 = 1/2 + 1/3 = 5/6$; $S_3 = 1/2 + 1/3 + 1/4 = 13/12$; $S_4 = 1/2 + 1/3 + 1/4 + 1/5 = 77/60$.

5. This is a geometric series, so we use $\Sigma_{i=0}^{\infty} ar^i = a/(1 - r)$. The formula may be used since $r = 1/7 < 1$. The first term is $a = 1$, so the sum is $1/(1 - 1/7) = 1/(6/7) = 7/6$.

9. During the first year, you draw out \$10,000. Next year, you draw out $(3/4)(\$10,000)$, then $(3/4)^2(\$10,000)$, then $(3/4)^3(\$10,000)$, and so forth. For an arbitrarily large life span, the total amount to be drawn out is $\Sigma_{i=0}^{\infty}(3/4)^i(10,000) = 10,000/(1 - 3/4) = \$40,000$.

13. The series is $\Sigma_{i=0}^{\infty}(2^{3i+4}/3^{2i+5}) = 2^4/3^5 + 2^7/3^7 + 2^{10}/3^9 + \ldots$. This is a geometric series beginning with $2^4/3^5$ and having a ratio $r = 2^3/3^2 = 8/9 < 1$. Thus, the sum is $(2^4/3^5)/(1 - 8/9) = 9(2^4/3^5) = 2^4/3^3 = 16/27$.

17. We have $(2^n + 3^n)/6^n = (2/6)^n + (3/6)^n = (1/3)^n + (1/2)^n$. Thus, $\Sigma_{n=1}^{\infty}[(2^n + 3^n)/6^n]$ is the sum of two geometric series and it equals $\Sigma_{n=1}^{\infty}(1/3)^n + \Sigma_{n=1}^{\infty}(1/2)^n = \Sigma_{n=0}^{\infty}(1/3)(1/3)^n + \Sigma_{n=0}^{\infty}(1/2)(1/2)^n = (1/3)/(1 - 1/3) + (1/2)/(1 - 1/2) = (1/3)/(2/3) + (1/2)/(1/2) = 3/2$.

21. Using the method of Example 6, we have $\Sigma_{i=1}^{\infty}(1 + 1/2^i) = \Sigma_{i=1}^{\infty}1^i + \Sigma_{i=1}^{\infty}(1/2)^i$. $\Sigma_{i=1}^{\infty}(1/2)^i$ converges since $r = 1/2$; however, $r = 1$ in $\Sigma_{i=1}^{\infty}1^i$, so it diverges. Thus, the entire integral diverges.

25. Consider the terms $i/\sqrt{i-1}$. Divide by $\sqrt{i}/\sqrt{i}$ to get $\sqrt{i}/\sqrt{1 + 1/i}$. As $i \to \infty$, the denominator tends to 1 and the numerator tends to ∞ . Since $\lim_{i\to\infty} a_i \neq 0$, the i^{th} term test tells us the series diverges.

29. This series is equivalent to $1 + 1/2 + 1/2 + 1/2 + \ldots$. Since $\lim_{i\to\infty} a_i = 1/2 \neq 0$, the series diverges.

33. Let $a_i = 1$ and $b_i = -1$ for all i . Then $a_i + b_i = 0$ and $\Sigma_{i=0}^{\infty}(a_i + b_i) = 0$ converges, but both $\Sigma_{i=0}^{\infty}1$ and $\Sigma_{i=0}^{\infty}(-1)$ diverge.

37. (a) Let t_k be the carriage transit time for trip k . Let d_k be the distance between the crews at the beginning of trip k . Then $t_{2n+1} = d_{2n+1}/(20 + 7)$ and $t_{2n+2} = d_{2n+2}/(20 + 5)$, since the speed is a sum of the riding and working speeds. Also, $d_{n+1} = d_n - (5 + 7)t_n = d_1(5 + 7)\Sigma_{i=1}^{n}t_i = 12(1 - \Sigma_{i=1}^{n}t_i)$. Therefore, $t_{2n+1} = (12/27)(1 - \Sigma_{i=1}^{2n}t_i)$, and $t_{2n+2} = (12/25)(1 - \Sigma_{i=1}^{2n+1}t_i) = (12/25)(1 - \Sigma_{i=1}^{2n}t_i - t_{2n+1}) = (12/25)[1 - \Sigma_{i=1}^{2n}t_i - (12/27)(1 - \Sigma_{i=1}^{2n}t_i)] = (12/25)(15/27)(1 - \Sigma_{i=1}^{2n}t_i) = (15/25)t_{2n+1}$. Similarly, $t_{2n+1} = (12/27)[1 - \Sigma_{i=1}^{2n-1}t_i - (12/25)(1 - \Sigma_{i=1}^{2n-1}t_i)] = (12/27) \times (13/25)(1 - \Sigma_{i=1}^{2n-1}t_i) = (13/27)t_{2n}$. Therefore, $t_{2n+1} = (13/27) \times (12/25)t_{2n-1} = r^n t_1 = r^n \cdot (12/27)$. Also, $t_{2n+2} = (15/25)t_{2n+1} = r^n \cdot (15/25)(12/27) = r^{n+1} \cdot (12/13)$.

(b) The total time for carriage travel is $\lim_{n\to\infty}\Sigma_{i=1}^{n}t_i = \lim_{n\to\infty}(\Sigma_{i=0}^{n}t_{2i+1} + \Sigma_{i=0}^{n}t_{2i+2}) = \lim_{n\to\infty}[\Sigma_{i=0}^{n}r^i \cdot (12/27) + \Sigma_{i=0}^{n}r^{i+1} \cdot (12/13)]$. Using the fact that $1 + r + r^2 + \ldots + r^n = (r^{n+1} - 1)/(r - 1)$, we get $\lim_{n\to\infty}(12/27)[(r^{n+1} - 1)/(r - 1)] + \lim_{n\to\infty}r \cdot (12/13) \cdot [(r^{n+2} - 1)/(r - 1)] =$

37. (b) (continued)

$(12/27)[(-1)/(r - 1)] + r\cdot(12/13)[-1/(r - 1)]$ (since $|r| < 1$) $=$ $[(12/27) + r\cdot(12/13)]/(1 - r) = [(12/27) + (12/27)\cdot(15/25)]/(1 - r) =$ $(12/27)\cdot(40/25)/\{[25\cdot 27 - (13\cdot 15)]/25\cdot 27\} = [(12\cdot 40)/(25\cdot 27)] \times$ $[25\cdot 27/480] = 1$.

SECTION QUIZ

1. (a) Discuss the convergence or divergence of $(1 + 0 - 1) + (1 + 0 - 1) + (1 + 0 - 1) + \dots$.

   (b) What happens if the parentheses are removed from the series in part (a)?

2. The series $1 + 3/2 + 9/4 + 27/8 + \dots$ is a geometric series with ratio $3/2$. Thus, the sum is $1/(1 - 3/2) = -2$. The sum of positive numbers can't be negative. What's wrong?

3. The sum of a geometric series is 5 and the first term of the series is 2 . Write the series in the form $\Sigma_{i=0}^{\infty} ar^i$.

4. Consider the series $\Sigma_{i=1}^{\infty} 3$.

   (a) Use the i^{th} term test to analyze the series.

   (b) Does part (a) tell you anything about the tenth partial sum? Explain.

5. Mindless Marvin, the mixed-up medical student, wanted to start one of his patients on a drug which would be used for the rest of her life. Mindless Marvin knew that drugs obeyed exponential decay in the body so that $A \exp(-kt)$ would be left after the first dose. After the second dose, $A[\exp(-kt) + \exp(-2kt)]$ is left. Even Mindless Marvin could see that after n doses, the amount in the body would be $A[\exp(-kt) + \exp(-2kt) + \dots + \exp(-nkt)]$. Seeing this, Mindless

5. (continued)

Marvin runs to the pharmacist and asks, "Won't the drug level become infinite and kill the patient after several doses?" The pharmacist responds, "No, Mindless Marvin." Explain this if A , k , and t are constants and $|\exp(-kt)| < 1$.

ANSWERS TO PREREQUISITE QUIZ

1. $\int_0^\infty [dx/(x^2 + 2)] = \int_0^1 [dx/(x^2 + 2)] + \int_1^\infty [dx/(x^2 + 2)]$. $\int_0^1 [dx/(x^2 + 2)]$ is proper, so it converges. $\int_1^\infty [dx/(x^2 + 2)] \leqslant \int_1^\infty (dx/x^2) = -x^{-1}|_1^\infty = 1$. Therefore, the entire integral converges.

2. It diverges; it oscillates between -1 and 1 .

3. (a) $+\infty$

   (b) 0

ANSWERS TO SECTION QUIZ

1. (a) The partials sums are $0, 0, 0, \ldots$. Thus, the series converges to zero.

   (b) Without the parentheses, the partial sums are $1, 0, 0, 1, 0, 0, \ldots$. Thus, the series diverges since the partial sums have no limit.

2. The formula $a/(1 - r)$ applies only if $|r| < 1$.

3. $\Sigma_{i=0}^\infty 2(3/5)^i$

4. (a) It diverges because $a_i = 3$ for all i and $\lim_{i\to\infty} a_i = 3 > 1$.

   (b) Partial sums always exist; here $S_{10} = 30$. The i^{th} term test only applies to infinite series, not to partial sums.

5. This is a geometric series where $r = \exp(-kt)$. Thus, the maximum drug level is $A\exp(-kt)/[1 - \exp(-kt)]$.

12.2 The Comparison Test and Alternating Series

PREREQUISITES

1. Recall how to use the comparison test for studying improper integrals (Sections 11.3).
2. Recall that the harmonic series diverges (Section 12.1).

PREREQUISITE QUIZ

1. Discuss the convergence or divergence of $\Sigma_{i=5}^{\infty}[i/(i^2 - 4i)]$.
2. Show that $\int_1^{\infty}[1/(x^5 + x^3 + 2x^2)]\,dx$ converges.
3. Show that $\int_1^{\infty}[1/(\sqrt{x} + \sqrt[3]{x})]\,dx$ diverges.

GOALS

1. Be able to demonstrate convergence or divergence by using the comparison test.
2. Be able to show convergence or divergence by using the ratio comparison tests.
3. Be able to distinguish between absolute and conditional convergence.
4. Be able to apply the alternating series test to show convergence.

STUDY HINTS

1. Comparison test. You can only draw a conclusion if the terms of a series are smaller than a known convergent series or larger than a known divergent series. If a series is shown to be termwise larger than a convergent series, you have no information; it may converge or diverge. Similarly, showing that a series is termwise smaller than a divergent series tells you nothing. Example: $\Sigma_{i=1}^{\infty}(1/n^2)$ and $\Sigma_{i=1}^{\infty}(1/n)$ are both less than $\Sigma_{i=1}^{\infty}1$, but this comparison tells you nothing about the convergence

1. (continued)

or divergence of $\Sigma_{i=1}^{\infty}(1/n^2)$ or $\Sigma_{i=1}^{\infty}(1/n)$. Note the use of absolute values in the test. All of the terms being compared must be positive.

2. Using the comparison test. The first step is to guess convergence or divergence by resemblance to a known series. If one wishes to show convergence, one can make the numerator larger and the denominator smaller. If the new series is less than a known convergent series, then the original series converges. As an example, consider the series $\Sigma_{i=1}^{\infty}[(4i^4 + i^3 - i^2)/(i^6 + 7i^8)]$. Note that as i becomes very large, the numerator may be made larger by increasing the exponent of positive terms or by deleting negative terms. Similarly, the denominator may be made smaller by deleting the positive terms with large exponents. Thus, $(4i^4 + i^3 - i^2)/(i^6 + 7i^8) \leqslant (4i^4 + i^4)/i^6 = 4/i^2$ for large i and so, the series in question converges. Similar techniques may be used if divergence is suspected.

3. Ratio convergence tests. These are usually easier to use than the comparison test. Convince yourself of their validity. If $\lim_{i\to\infty}(|a_i|/b_i) < \infty$, then eventually, $|a_i| < cb_i$, where c is some finite constant. Again, convergence is implied. Similar reasoning should convince you of the second part of the test.

4. Error estimates. Often, we wish to estimate the sum of a series; however, an estimate is useless if it is far from its true answer. Error analysis helps us to decide if an estimate is useful. Study Example 5 to see how upper bounds of errors are estimated.

5. Alternating series. All alternating series converge with error no greater than $|a_{n+1}|$ if the partial sum, $S_n = a_1 + a_2 + \ldots + a_n$, is the estimate of the sum. The three conditions in the top box on p. 573

5. Alternating series (continued).

need to be shown before a series can be called alternating.

6. Absolute vs. conditional convergence. Absolute convergence means a series would still converge if all the minus signs were removed. For example, $\Sigma_{n=1}^{\infty}(-1)^n(3/5)^n$ converges absolutely as a geometric series. Conditional convergence means that the minus signs are necessary. For example, $\Sigma_{i=1}^{\infty}[(-1)^i/i]$ converges, but if the minus signs are removed, it would diverge.

7. Increasing sequence property. This is theoretical material and is optional for most classes. Ask your instructor. Simply stated, a sequence increasing toward an upper bound has a limit. No statement is made about its associated series.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. For $i \geqslant 1$, $8/(3^i + 2) \leqslant 8/3^i$. Also, $\Sigma_{i=1}^{\infty}(8/3^i) = 8\Sigma_{i=1}^{\infty}(1/3)^i$, which is a convergent geometric series. Thus, $\Sigma_{i=1}^{\infty}[8/(3^i + 2)]$ converges by the comparison test.

5. For $i \geqslant 1$, $|a_i| = 1/(3^i + 2) \leqslant 1/3^i$ and $\Sigma_{i=1}^{\infty}(1/3)^i$ is a convergent geometric series. Thus, $\Sigma_{i=1}^{\infty}[(-1)^i/(3^i + 2)]$ converges by the comparison test.

9. For $i \geqslant 1$, $3/(2 + i) \geqslant 3/(2i + i) = 1/i$, and $\Sigma_{i=1}^{\infty}(1/i)$ is the divergent harmonic series. Thus, $\Sigma_{i=1}^{\infty}[3/(2 + i)]$ diverges by the comparison test.

13. For $n \geqslant 1$, $|a_n| = 3/(4^n + 2) \leqslant 3/4^n$ and $\Sigma_{n=1}^{\infty}(3/4^n) = 3\Sigma_{n=1}^{\infty}(1/4^n)$ is a convergent geometric series. Thus, $\Sigma_{n=1}^{\infty}[3/(4^n + 2)]$ converges by the comparison test.

17. Note that $1/(3i + 1/i) > 1/(3i + 3) = (1/3)(1/(i + 1))$. Since $(1/3)\Sigma_{i=1}^{\infty}[1/(i + 1)]$ is a harmonic series, it diverges. Then, by comparison, $\Sigma_{i=1}^{\infty}[1/(3i + 1/i)]$ diverges also.

21. Let $a_i = [1 + (-1)^i]/(8i + 2^{i+1})$ and $b_i = 2/2^{i+1} = 1/2^i$. Since $1 + (-1)^i$ equals either 0 or 2, $|1 + (-1)^i| \leqslant 2$. Since $8i + 2^{i+1} > 2^{i+1}$, $|a_i| = [1 + (-1)^i]/(8i + 2^{i+1}) < 2/2^{i+1} = b_i$. Therefore, since $\Sigma_{i=1}^{\infty} b_i$ converges, so does $\Sigma_{i=1}^{\infty} a_i$.

25. Let $a_i = 3i/2^i$ and $b_i = 3(3/4)^i$. Since $i < (3/2)^i$, we have $|a_i| = 3i/2^i < 3(3/4)^i = b_i$. Therefore, since $\Sigma_{i=1}^{\infty} b_i$ converges, so does $\Sigma_{i=1}^{\infty} a_i$.

29. Let $a_j = \sin j/2^j$ and $b_j = 1/2^j$. Since $|\sin j| \leqslant 1$ for all j, we have $|a_j| \leqslant 1/2^j = b_j$. Thus, since $\Sigma_{j=1}^{\infty} b_j$ converges as a geometric series, $\Sigma_{j=1}^{\infty} a_j$ also converges by comparison.

33. Let $a_n = 1/(2^n + 1)$ and $b_n = 1/2^n$. Then $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty}[2^n/(2^n + 1)] = 1$. Therefore, since $\Sigma_{n=1}^{\infty} b_i$ converges, so does $\Sigma_{n=1}^{\infty} a_i$.

37. Let $a_n = (2^n - 1)/(5^n + 1) < (2/5)^n$. The error in approximating the series by $\Sigma_{n=1}^{k} a_n$ is less than or equal to $(2/5)^{k+1}/(1 - 2/5) = 3(2^{k+1})/5^k < 0.01$ when $n = 10$. Thus, $\Sigma_{n=1}^{10} a_n \approx 0.37$.

41. Note that $\lim_{k\to\infty}[k/(k + 1)] = \lim_{k\to\infty}[1/(1 + 1/k)] = 1 \neq 0$. Thus, $\Sigma_{k=1}^{\infty}[k/(k + 1)]$ diverges by the i^{th} term test.

45. Let $a_1 = 1$, and for $i > 1$, $a_i = (-1)^{i+1}(i - 1)/i$. Then $\lim_{i\to\infty}|a_i| = 1 \neq 0$. Therefore, $\Sigma_{i=1}^{\infty} a_i$ diverges, by the i^{th} term test.

49. The derivative of $\ln[(n + 1)/n]$ is $1/n(n + 1)$, so it is a decreasing function for $n > 1$. The signs alternate and $\lim_{n\to\infty}\ln[(n + 1)/n] = \ln(1) = 0$. Therefore, the series converges conditionally as an alternating series.

49. (continued)

Absolutely, $\Sigma_{n=1}^{\infty} \ln[(n+1)/n] = \Sigma_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$. Since this is a telescoping sum, $\Sigma_{n=1}^{\infty} = \lim_{n\to\infty} \ln(n+1) - \ln(1) = \infty$. Thus, the series $\Sigma_{n=1}^{\infty}(-1)^n \ln[(n+1)/n]$ only converges conditionally.

53. The error in estimating $\Sigma_{n=1}^{\infty}(1/5)^n$ is $(1/5)(1/5)^{n+1}/(1 - 1/5) = (1/4)(1/5)^{n+1}$. The error in estimating $\Sigma_{n=1}^{\infty}[(-1)^n/2n]$ is less than $1/2n$. Therefore, $(1/4)(1/5)^{n+1} + 1/2n < 1/50$ for $n \geqslant 26$, and $\Sigma_{n=1}^{\infty}[(-1)^n/2n + 1/5^n] \approx \Sigma_{n=1}^{26}[(-1)^n/2n + 1/5^n] \approx -0.087$.

57. (a) $a_1 = \sqrt{4} = 2$; $a_2 = \sqrt{4+2} = \sqrt{6}$; $a_3 = \sqrt{4 + \sqrt{6}}$.

(b) $a_1 = 2$; $a_2 = 2.44949$; $a_3 = 2.53958$; $a_4 = 2.55726$; $a_5 = 2.56071$; $a_6 = 2.56139$; $a_7 = 2.56152$; $a_8 = 2.56155$; $a_9 = 2.56155$; $a_{10} = 2.56155$; $a_{11} = 2.56155$; $a_{12} = 2.56155$. We guess that $\lim_{n\to\infty} a_n = 2.56155\ldots$.

61. This is an increasing sequence which is bounded above. $a_{n+1} = 1/2(n+1) - 1/[(n+1)+1] = 1/(2n+2) - 1/(n+2) = -n/[(2n+2) \times (n+2)]$. $a_n = (1-n)/(2n)(n+1)$, so $a_{n+1} \geqslant a_n$ means $-2n^2 \times (n+1) \geqslant (1-n)(2n+2)(n+2)$ or $-2n^3 - 2n^2 \geqslant -2n^3 - 4n^2 + 2n + 4$ or $2n^2 \geqslant 2n + 4$. This is true if $n \geqslant 2$, so the sequence is increasing. For positive n , $1 - n \leqslant 0$ and $(2n)(n+1) \geqslant 0$, so $a_n \leqslant 0$; therefore, a_n is bounded above by 0 .

65. To show that a_n is increasing, we will show that $a_{n+1} \geqslant a_n$ or $a_n/2 + \sqrt{a_n} \geqslant a_n$, which implies $\sqrt{a_n} \geqslant a_n/2$. Rearrangement and squaring implies $4a_n \geqslant a_n^2$. Thus, if $0 \leqslant a_n \leqslant 4$, then a_n is increasing. Since $a_0 = 1$, a_{n+1} is always a sum of positive terms, so $a_n \geqslant 0$. We will now show a_n is bounded above by 4 using mathematical induction. For $n = 0$, $a_1 = 1/2 + 1 = 3/2 < 4$, which is true. Now, assume the statement holds for $n - 1$, i.e., $a_{n-1} \leqslant 4$. Then, $a_n \leqslant 4/2 + \sqrt{4} = 4$; therefore, a_n is increasing and bounded

65. (continued)

above by 4. By the increasing sequence property, $\lim_{n\to\infty} a_n$ converges to a limit, ℓ. To find ℓ, we solve $\lim_{n\to\infty} a_{n+1} = \ell = \ell/2 + \sqrt{\ell} = a_n/2 + \sqrt{a_n}$. The solution is $\ell = \lim_{n\to\infty} a_n = 4$.

69. For any x and any n, $\phi(4^n - x) \leqslant 1$. Let $b_n = (3/4)^n$ and let $a_n = (3/4)^n\phi(4^n x)$. Then we have $|a_n| = (3/4)^n\phi(4^n x) \leqslant (3/4)^n = b_n$. Therefore, since $\Sigma_{n=0}^{\infty} b_n$ converges as a geometric series, so does $\Sigma_{n=0}^{\infty} a_n$ by the comparison test.

SECTION QUIZ

1. Let $\Sigma_{i=1}^{\infty} a_i$ be the series to be analyzed and let $\Sigma_{i=1}^{\infty} b_i$ be a series whose convergence or divergence is known. What does the comparison test say? (More than one may be correct.)

   (a) If $|a_i| < |b_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ converges, then $\Sigma_{i=1}^{\infty} a_i$ also converges.

   (b) If $|b_i| < |a_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ converges, then $\Sigma_{i=1}^{\infty} a_i$ also converges.

   (c) If $|a_i| > |b_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ diverges, then $\Sigma_{i=1}^{\infty} a_i$ also diverges.

   (d) If $|b_i| > |a_i|$ and $|\Sigma_{i=1}^{\infty} b_i|$ diverges, then $\Sigma_{i=1}^{\infty} a_i$ also diverges.

2. Does $3 + 2 + 1 + 1/2 + 1/3 + \ldots + 1/100$ converge? Explain.

3. Can $\Sigma_{n=10}^{\infty}[(-1)^n(n - 5)/n]$ be analyzed by the alternating series test? Explain.

4. Determine if the following series converge conditionally, absolutely, or not at all. Justify your answer.

   (a) $\Sigma_{n=3}^{\infty}[-1/(3^n - 45)]$

4. (b) $\Sigma_{n=0}^{\infty}[(-1)^n(n^2+5)/(n^3+3n^2+1)]$

(c) $\Sigma_{n=-\infty}^{-1}(2^n/n^2)$

(d) $\Sigma_{n=0}^{\infty}(1/n(3/2)^n)$

5. Little Lisa was a very bad girl. She went out last night and slept today rather than doing her homework. Because Little Lisa has a little brain, she can only remember $1/4$ of what she learned yesterday, $4/6$ of what she learned 2 days ago and so forth, i.e., she retains $n^2/(2+2^n)$ of what she learned n days ago. Explain in terms of convergence or divergence of infinite series whether Little Lisa's little brain is capacity-limited, i.e., does she have a bird-sized brain?

ANSWERS TO PREREQUISITE QUIZ

1. For $i > 5$, $i/(i^2-4i) \geqslant i/i^2 = 1/i$. Thus, $\Sigma_{i=5}^{\infty}[i/(i^2-4i)]$ diverges by the comparison test and by the divergence of the harmonic series.

2. $\int_1^{\infty}[dx/(x^5+x^3+2x^2)] \leqslant \int_1^{\infty}[dx/2x^2] = (-1/2x)\Big|_1^{\infty}$, which converges.

3. $\int_1^{\infty}[dx/(\sqrt{x}+\sqrt[3]{x})] \geqslant \int_1^{\infty}[dx/(\sqrt{x}+\sqrt{x})] = \sqrt{x}\Big|_1^{\infty}$, which diverges.

ANSWERS TO SECTION QUIZ

1. a and c

2. This is not an infinite series, so it does converge. If the series continued as $+1/101+1/102+\ldots$, then it would diverge.

3. No, $\lim_{n\to\infty}[(n-5)/n] = 1 \neq 0$.

4. (a) Absolute convergence; $1/(3^n-45) < 1/2^n$ if $n \geqslant 4$, and $\Sigma_{i=4}^{\infty}(1/2^n)$ is a geometric series.

4. (b) Converges conditionally; $(n^2 + 5)/(n^3 + 3n^2 + 1) \geqslant n^2/n^3 = 1/n$ and $\Sigma_{n=0}^{\infty}(1/n)$ is the harmonic series. The original series converges by the alternating series test.

(c) Absolute convergence; $\Sigma_{n=-\infty}^{-1}(2^n/n^2) = \Sigma_{n=1}^{\infty}(1/2^n n^2) < \Sigma_{n=0}^{\infty}(1/2^n)$, which converges as a geometric series.

(d) Diverges; the $n = 0$ term is infinite. Note that $\Sigma_{n=1}^{\infty}[1/n(3/2)^n]$ does converge.

5. Her brain is capacity-limited; $\Sigma_{n=1}^{\infty}[n^2/(2 + 2^n)] \leqslant \Sigma_{n=1}^{\infty}(n^2/2n)$, which converges by the ratio test.

12.3 The Integral and Ratio Tests

PREREQUISITES

1. Recall how to evaluate improper integrals (Section 11.3).
2. Recall that the geometric series converges (Section 12.1).
3. Recall how to use the i^{th} term test to demonstrate divergence (Section 12.1).
4. Recall how to use the comparison test to show convergence or divergence (Section 12.2).
5. Recall how to find limits by using l'Hôpital's rule (Section 11.2).

PREREQUISITE QUIZ

1. Evaluate $\int_0^\infty (dx/e^x)$.
2. Compute $\Sigma_{i=1}^{\infty}(2/5)^i$.
3. Compute $\lim_{x\to 1}[(x^5 + x^4 - 2)/(x - 1)]$.
4. Compute $\lim_{x\to\infty}[(x^5 + x^4 - 2)/(3x^5 + 2x)]$.
5. Discuss the convergence or divergence of the following series:

   (a) $\Sigma_{n=1}^{\infty}[(n + 1)/2n]$

   (b) $\Sigma_{i=1}^{\infty}[(i^5 + i^4 + 2)/(i^3 + i)]$.

GOALS

1. Be able to use either the integral test, the p-series test, the ratio test, or the root test for determining the convergence or divergence of a series.
2. Be able to estimate the error when an infinite series is approximated by a partial sum .

STUDY HINTS

1. Integral test. Just understand the conclusion. If the integral converges, so does the series. If the integral diverges, so does the series. Although the test asks you to integrate from 1 to ∞, 1 is just the arbitrary starting point. Again, we are concerned mostly with the tail. Be sure that the initial terms are finite. For example, $\Sigma_{i=0}^{\infty}(1/n^2)$ diverges since the $i = 0$ term is infinite.
2. p-series. This test follows directly from the integral test. Know this statement well. A good way to remember this statement is to realize that any exponent less than one makes the series larger than the harmonic series. In estimating the sum by S_N, the error is less than $1/(p - 1)N^{p-1}$.
3. Ratio test. Do not confuse this with the ratio comparison test in Section 12.2. The idea is to compare a term in the tail of the series with the preceding term. It makes sense that if the ratio is absolutely less than one, then the terms are getting smaller in geometric progression, so we expect absolute convergence (at least, intuitively). If the terms are getting larger, then the ratio is greater than one, and divergence occurs. Consider the harmonic series and the alternating harmonic series to note that the test fails if the ratio is one. The ratio test is easy to use if factorials occur. See Example 7. Remember, the magic number for the ratio test is one. An upper bound for the error estimate is $|a_N|r/(1 - r)$.
4. Root test. Essentially, the test uses the following facts: If $0 < a_i < 1$, then any root of a_i is less than one. If $a_i > 1$, then any of its roots is greater than one. This is not a proof that the test works, but it should help you to remember it. Again, we are only interested in the tailing terms for the convergence issue.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. We need to evaluate $\int_1^\infty [x/(x^2 + 1)]\,dx$. By letting $u = x^2 + 1$, we get $(1/2)\int_2^\infty (du/u) = \lim_{b\to\infty}(\ln u/2)\big|_2^b$. The integral is ∞ , so according to the integral test, $\Sigma_{i=1}^\infty [i/(i^2 + 1)]$ diverges.

5. Since $|\cos n| \leqslant 1$ for all n , we have $\Sigma_{n=1}^\infty |\cos n/n^2| \leqslant \Sigma_{n=1}^\infty |1/n^2|$. Now, $p = 2 > 1$, so $\Sigma_{n=1}^\infty (\cos n/n^2)$ converges (absolutely).

9. The error in estimating a p-series is $\Sigma_{n=N+1}^\infty (1/n^p)$, which is less than or equal to $1/(p - 1)N^{p-1}$. The error is $\Sigma_{n=N+1}^\infty (\cos n/n^3) \leqslant \Sigma_{n=N+1}^\infty (1/n^3)$ since $|\cos n| \leqslant 1$. Thus, we need $1/2N^2 < 0.05$, i.e., $N \geqslant 4$. Therefore, $\Sigma_{n=1}^\infty (\cos n/n^3) \approx 0.44$.

13. The ratio $|a_n/a_{n-1}|$ is $(2\sqrt{n}/3^n)\cdot(3^{n-1}/2\sqrt{n - 1}) = (1/3)\sqrt{n/(n - 1)}$. Thus, $\lim_{n\to\infty}|a_n/a_{n-1}| = 1/3$, which is less than 1 . Therefore, the series $\Sigma_{n=1}^\infty (2\sqrt{n}/3^n)$ converges.

17. The error is less than $|a_N|r/(1 - r)$, where $|a_n/a_{n-1}| < r < 1$. The ratio is $r = \pi^2/(2n)(2n + 1)$, so the error is less than $[\pi^{2N+1}/(2N + 1)!][\pi^2/(2N)(2N + 1)]/[((2N)(2N + 1) - \pi^2)/(2N)(2N + 1)] = \pi^{2N+3}/(4N^2 + 2N - \pi^2)(2N + 1)!$. The error estimate is approximately 0.013 for $N = 4$, so $\Sigma_{n=0}^\infty[\pi^{2n+1}/(2n + 1)!] \approx \Sigma_{n=0}^4[\pi^{2n+1}/(2n + 1)!] = \pi + \pi^3/6 + \pi^5/120 + \pi^7/5040 + \pi^9/362880 \approx 11.54$.

21. $a_n = 3^n/n^n$, so $|a_n|^{1/n} = 3/n$, and the limit $\lim_{n\to\infty}|a_n|^{1/n}$ is 0 . Therefore, $\Sigma_{n=1}^\infty (3^n/n^n)$ converges by the root test.

25. The series $\Sigma_{i=1}^\infty (1/i^4)$ converges by the p-series test with $p = 4 > 1$.

29. When k is even, $\cos k\pi = 1 > 0$. When k is odd, $\cos k\pi = -1 < 0$. Therefore, the terms of $(\cos k\pi)/\ln k$ alternate in sign. Since $\ln(k + 1) > \ln k$, $1/\ln(k + 1) < 1/\ln k$. $\lim_{k\to\infty}|(\cos k\pi)/\ln k| = \lim_{k\to\infty}(1/\ln k) = 0$. Therefore, the absolute values of the terms of this

29. (continued)

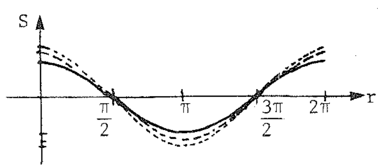
series decrease to 0 . Hence $\Sigma_{k=0}^{\infty}[(\cos k\pi)/\ln k]$ converges by the alternating series test.

33. The series $\Sigma_{r=0}^{\infty}[2^r/(2^r + 3^r)] \geqslant \Sigma_{r=0}^{\infty}(2^r/3^r)$, which converges because of the root test: $\lim_{r\to\infty}(2^r/3^r)^{1/r} = \lim_{r\to\infty}(2/3) = 2/3 < 1$. By the comparison test, the original series converges.

37. If $\lim_{n\to\infty}|a_n|^{1/n} > 1$, then by the definition of the limit, there exists an N such that $|a_n|^{1/n} > 1$ for all $n > N$. Then, we also have $|a_n| > 1$ for all $n > N$. Since $\lim_{n\to\infty}|a_n| \neq 0$, the series $\Sigma_{n=1}^{\infty}a_n$ diverges by the i^{th} term test.

41. If $p > 1$, then $1/n^p \ln n < 1/n^p$ for $n \geqslant 3$ because $\ln n > 1$ for $n \geqslant 3$. Then, by the comparison test and the p-series test, $\Sigma_{n=2}^{\infty}(1/n^p \ln n)$ converges if $p > 1$. If $p = 1$, the integral test gives $\int_2^{\infty}(dx/x \ln x) = \int_{\ln 2}^{\infty}(du/u) = \lim_{b\to\infty} \ln u|_{\ln 2}^{\infty}$, so the series diverges if $p = 1$. If $p < 1$, then $1/n^p \ln n > 1/n \ln n$, whose series diverges. Therefore, $\Sigma_{n=2}^{\infty}(1/n^p \ln n)$ converges for $p > 1$.

45. (a) Let $a_n = \{\cos[(2n + 1)^2 r]\}/(2n + 1)^4$; then $a_0 = \cos r$, $a_1 = \cos(9r)/3^4$, and $a_2 = \cos(25 r)/5^4$.

(b) 

$S_1(r)$ —— $S_1(0) = 1$
$S_2(r)$ - - $S_2(0) = 82/81 = 1.012$
$S_3(r)$ ··· $S_3(0) = 51331/50625 = 1.014$

45. (c) Since $|\cos[(2n+1)^2 r]| \leqslant 1$, $|a_n| \leqslant 1/(2n+1)^4$. Let $f(x) = 1/(2x+1)^4$. Since f is decreasing, the integral test may be applied. $\int_1^\infty f(x)dx = \lim_{b\to\infty}[1/6(2x+1)^3]\,|_1^b < \infty$. Therefore $\Sigma_{n=0}^\infty [1/(2n+1)^4]$ converges, and so does $\Sigma_{n=0}^\infty a_n$.

SECTION QUIZ

1. Show that the ratio test cannot be used to analyze a p-series.

2. Discuss the convergence (conditional or absolute) or divergence of the following series:

   (a) $\Sigma_{n=-\infty}^\infty [\tan^{-1} n/(1+n^2)]$

   (b) $\Sigma_{n=0}^\infty [(-1)^n (3r)!/(2n+8)!]$

   (c) $\Sigma_{n=1}^\infty n^{-1-1/n}$

   (d) $\Sigma_{n=1}^\infty (\sqrt[n]{n} - 1)^n$

   (e) $\Sigma_{n=0}^\infty (4^n \cos n/2^n n!)$

   (f) $\Sigma_{n=-5}^\infty \left(1/\sqrt[3]{n^2 + 3n + 5}\right)$

   (g) $\Sigma_{n=7}^\infty (8^n/(n+3)^n)$

   (h) $\Sigma_{n=-3}^\infty (n/\exp(n^2))$

3. A certain planet in a distant galaxy was on the verge of destruction from radioactivity. The survivors have protected themselves with a special serum which lines their entire body surface with lead. However, lead is not an unlimited resource on their planet. Suppose that the species is shrinking as an adaptive mechanism so that the total surface area of the n^{th} generation is proportional to $3^n/n!$. Will a finite amount of lead protect the population forever, i.e., is $\Sigma_{n=0}^\infty (3^n/n!)$ finite? Justify your answer.

ANSWERS TO PREREQUISITE QUIZ

1. 1

2. 2/3

3. 9

4. 1/3

5. (a) $\lim_{n\to\infty}[(n + 1)/2n] = 1/2$, so the series diverges by the i^{th} term test.

   (b) $\lim_{i\to\infty}[(i^5 + i^4 + 2)/(i^3 + i)] = +\infty$, so the series diverges by the i^{th} term test.

ANSWERS TO SECTION QUIZ

1. $\Sigma_{i=1}^{\infty} i^p$ has ratio $(i - 1)^p/i^p = [(i - 1)/i]^p$, whose limit is one.

2. (a) Converges absolutely; use integral test.

   (b) Diverges; use i^{th} term test.

   (c) Converges absolutely; use root test.

   (d) Converges absolutely; use root test.

   (e) Converges absolutely; compare to $\Sigma_{n=0}^{\infty}(2^n/n!)$ and use ratio test.

   (f) Diverges; compare to $\Sigma_{n=1}^{\infty}(1/n^{2/3})$ and use p-series.

   (g) Converges; use root test.

   (h) Converges absolutely; use integral test.

3. Yes; by ratio test, $\lim_{n\to\infty}(3/n) = 0 < 1$.

12.4 Power Series

PREREQUISITES

1. Recall how to apply the ratio test to demonstrate convergence or divergence (Section 12.3).
2. Recall how to apply the root test for analyzing infinite series (Section 12.3).
3. Recall how to differentiate and integrate polynomials (Chapters 1, 2, and 4).

PREREQUISITE QUIZ

1. Differentiate the following:
   (a) $x^5 - 3x^2 + x$
   (b) $6x^2 + x^4$
2. Perform the following integrations:
   (a) $\int_{-1}^{0}(x^3 + x)dx$
   (b) $\int(t^5 + t^2 + 3)dt$
3. Do the following series converge or diverge? Justify your answers.
   (a) $\Sigma_{n=1}^{\infty}(3/2n)^n$
   (b) $\Sigma_{n=1}^{\infty}(100n/n!)$

GOALS

1. Be able to find the radius of convergence of a power series.
2. Be able to differentiate, integrate, and algebraically manipulate convergent power series.

STUDY HINTS

1. Power series. These series have the form $\Sigma_{i=0}^{\infty}a_i(x - x_0)^i$. They behave like regular functions in that they can be added, subtracted, multiplied, divided, differentiated, and integrated, but only in the regions where the power series converges.

2. Ratio test for power series. Learn the test for the general power series by replacing x with $x - x_0$. As with the ratio test of Section 12.3, you need to find the limit of a_i/a_{i-1} . For convergence, we want $[\lim_{i\to\infty}(a_i/a_{i-1})]\cdot|x - x_0| < 1$ or $|x - x_0| < 1/[\lim_{i\to\infty}(a_i/(a_{i-1})]$. Thus, the radius of convergence, R , is the reciprocal of the limit. The series converges if $|x - x_0| < R$, i.e., $-R < x - x_0 < R$, i.e., $-R + x_0 < x < R + x_0$. To determine convergence at $-R + x_0$ and $R + x_0$, substitute into the original series and apply the tests presented earlier in the chapter. The ratio test will give a ratio of one at the endpoints, so it is mandatory to apply other tests to analyze the convergence or divergence at $x_0 \pm R$.

3. Root test for power series. As with the root test of Section 12.3, $\lim_{i\to\infty}|a_i|^{1/i}$ needs to be computed. Again, the radius of convergence is the reciprocal of the limit.

4. Differentiation and integration. In the differentiation formula, $(d/dx)\Sigma_{i=0}^{\infty}a_i(x - x_0)^i = \Sigma_{i=1}^{\infty}ia_i(x - x_0)^{i-1}$, we would like the derivative to converge at $x = x_0$. Thus, the exponent of $x - x_0$ should be nonnegative. Hence, in the differentiation process, the index on the right begins at the next higher integer in most cases. The index starting point does not change during the integration process. For both processes, the radius of convergence does not change.

5. Algebraic manipulations. When two series are added, subtracted, or multiplied, the new radius of convergence is at least as big as the smaller of the original radii. The radius of convergence for a quotient must be determined after the new series is determined. Example 10 shows how tedious division can be.

6. Applications of power series. Starting with a known power series, manipulations may be performed to derive power series for new expressions. Differentiation and integration are commonly used. See Example 8.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Here, $\ell = \lim_{i\to\infty}|[2/(i+1)][i/2]| = 1$. Thus, the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $x = 1$, $\Sigma_{i=0}^{\infty}[2/(i+1)] = 2 \times \Sigma_{i=1}^{\infty}(1/i)$ is a harmonic series; thus, it diverges. If $x = -1$, $2\Sigma_{i=0}^{\infty}[(-1)^i/(i+1)]$ is an alternating series; thus, it converges. Therefore, the series converges for $-1 \leqslant x < 1$.

5. We have $\ell = \lim_{i\to\infty}|[(5i+1)/i][(i-1)/(5(i-1)+1)]| = 1$, so $R = 1/\ell = 1$. Thus, the series converges absolutely for all x such that $|x-1| < 1$, i.e., $0 < x < 2$. Since $\lim_{i\to\infty}[(5i+1)/i] \neq 0$, the series diverges at the endpoints. Thus, $\Sigma_{i=1}^{\infty}[(5i+1)(x-1)^i/i]$ converges for $0 < x < 2$.

9. Here, $\ell = \lim_{n\to\infty}|[1/(2^n+4^n)][2^{n-1}+4^{n-1}]|$. Now, $2^{n-1}/(2^n+4^n) = 2^n/2(2^n+4^n) = 1/2(1+2^n)$ and $4^{n-1}/(2^n+4^n) = 4^n/4(2^n+4^n) = 1/4((1/2)^n+1)$; therefore, $\ell = \lim_{n\to\infty}[1/2(1+2^n) + 1/4((1/2)^n+1)] = 1/4$. Thus, the series converges if $|x| < 4$. If $x = -4$, then $x^n/(2^n+4^n) = (-4)^n/(2^n+4^n) = (-1)^n/((1/2)^n+1)$. The limit $\neq 0$, so the series diverges if $x = -4$. Similarly, the series diverges at $x = 4$. Therefore, $\Sigma_{n=1}^{\infty}[x^n/(2^n+4^n)]$ converges for $-4 < x < 4$.

13. Here, $a_i = 5i/2^i$, so $\ell = \lim_{i\to\infty}(5i/2^i)(2^{i-1}/5(i-1)) = \lim_{i\to\infty}(5i/2\cdot 5(i-1))$. Since the limit is $1/2$, the radius of convergence for $\Sigma_{i=1}^{\infty}(5ix^i/2^i)$ is 2. If $x = \pm 2$, the series is $\Sigma_{i=1}^{\infty}(-1)^i 5i$ or $\Sigma_{i=1}^{\infty} 5i$, which both diverge. Therefore, $\Sigma_{i=1}^{\infty}(5ix^i/2^i)$ converges if $-2 < x < 2$.

17. For $a_n = (n^2 + n^3)/(1+n)^5$, $\ell = \lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}[(n^2+n^3)/(1+n)^5]\{n^5/[(n-1)^2 + (n-1)^3]\} = 1$ by l'Hôpital's rule. Thus, $R = 1/\ell = 1$. When $x = \pm 1$, $\Sigma_{n=0}^{\infty}|a_n x^n| = \Sigma_{n=0}^{\infty}[(n^2+n^3)/(1+n)^5] = \Sigma_{n=0}^{\infty} a_n$. Let $b_n = n^3/n^5$. Then $\lim_{n\to\infty}(|a_n|/b_n) = 1 < \infty$, and since $\Sigma_{n=0}^{\infty} b_n$ converges, $\Sigma_{n=0}^{\infty} a_n$ is absolutely convergent. Thus, $\Sigma_{n=0}^{\infty} a_n x^n$ converges for $x = 1$ and -1.

21. The $\lim_{n\to\infty}|(-1)^n n^n|^{1/n}$ is $\lim_{n\to\infty} n$, which is ∞. Therefore, the series converges only for $x = 0$, and the radius of convergence is 0.

25. (a) $\ell = \lim_{i\to\infty}[(i+1)/i] = 1$, so the radius of convergence is $R = 1/\ell = 1$.

(b) $\int_0^x f(t)dt = \int_0^x \Sigma_{i=1}^{\infty}(i+1)t^i dt = \Sigma_{i=1}^{\infty}\int_0^x (i+1)t^i dt = \Sigma_{i=1}^{\infty} x^{i+1}$.

(c) Using the geometric series $1/(1-x) = 1 + x + x^2 + \ldots$, note that $\Sigma_{i=1}^{\infty} x^{i+1} = \Sigma_{i=0}^{\infty} x^2 x^i = x^2/(1-x)$. Also $f(x) = (d/dx)\int_0^x f(t)dt = (d/dx)(x^2/(1-x)) = x(2-x)/(1-x)^2$ for $|x| < 1$.

(d) $2/2 + 3/4 + 4/8 + 5/16 + \ldots = \Sigma_{i=1}^{\infty}[(i+1)(1/2)^i] = f(1/2)$. Thus, using the result of part (c), the sum is $(1/2)(2 - 1/2)/(1 - 1/2)^2 = 3$.

29. Note that $(d/dx)\tan^{-1}x = 1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \ldots$ by long division. So $\tan^{-1}x = \int_0^x[1/(1+t^2)]dt = x - x^3/3 + x^5/5 - x^7/7 + \ldots$, i.e., $\tan^{-1}x = \Sigma_{n=0}^{\infty}[(-1)^n x^{2n+1}/(2n+1)]$ and $(d/dx)\tan^{-1}x = \Sigma_{n=0}^{\infty}(-1)^n x^{2n}$.

33. From Exercise 23, $\sin x = x - x^3/3! + x^5/5! - \ldots$. Therefore, we have $\sin^2 x = (x - x^3/3! + x^5/5! - \ldots)(x - x^3/3! + x^5/5! - \ldots) = x^2 - x^4/3 + (2/5! + 1/(3!)^2)x^6 - \ldots = x^2 - x^4/3 + 2x^6/45 \ldots$.

37. (a) Using long division with $\sin x = x - x^3/3! + x^5/5! - \ldots$ and $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$, we get $\tan x = \sin x/\cos x = x + (1/3)x^3 + (2/15)x^5 + \ldots$.

(b) From the result of part (a) and differentiating term by term, we get $\sec^2 x = (d/dx)\tan x = 1 + x^2 + (2/3)x^4 + \ldots$.

(c) Using the result of part (b) and long division, we get $1/\sec^2 x = 1/[1 + x^2 - (2/3)x^4 + \ldots] = 1 - x^2 + (1/3)x^4 - \ldots$.

41. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ have radius of convergence R_f . Thus, if $|x_0| < R$, this series converges. By the theorem, $\sqrt[i]{|a_i|} < 1/|x_0|$ for $i \geqslant N$. Now $\sqrt[i]{2} \to 1$ as $i \to \infty$ (for example, write $\sqrt[i]{i} = i^{1/i}$ and use l'Hôpital's rule, as in Example 7(a), p. 525 to show that $\lim_{x\to\infty} x^{1/x} = 1$). Thus, $\sqrt[i]{i}\sqrt[i]{|a_i|} < 1/|x_0|$ if i is large enough. By the root test, $\sum_{i=1}^{\infty} i a_i x_0^i$ converges. Thus $g(x_0) = \sum_{i=1}^{\infty} i a_i x_0^{i-1}$ converges as well. Therefore, $g(x)$ has radius of convergence $R_g \geqslant R_f$. A similar argument, interchanging f and g shows that $R_f \geqslant R_g$, so $R_f = R_g$. Likewise, one shows that $R_f = R_h$.

45. Let $|x| < R$. From Exercise 44, $\int_0^x g(t)dt = f(x)$. Therefore, using the alternative version of the fundamental theorem of calculus, $f'(x) = g(x)$.

SECTION QUIZ

1. Suppose $f(x) = \Sigma_{i=0}^{\infty}(2/3^i)x^i$ and $g(x) = \Sigma_{i=2}^{\infty}(4/3^{i+1})x^i$.
   (a) Does $f(x) - g(x)$ converge? Explain.
   (b) If it converges, write an expression for the difference and find its domain.
2. True or false:
   (a) The derivative of $\Sigma_{i=0}^{\infty}(x^i/i!)$ is $\Sigma_{i=0}^{\infty}(ix^{i-1}/i!) = \Sigma_{i=0}^{\infty}[x^{i-1}/(i-1)!]$.
   (b) The integral of $\Sigma_{i=0}^{\infty}(x^i/i!) = \Sigma_{i=0}^{\infty}[x^{i+1}/(i+1)i!] = \Sigma_{i=0}^{\infty}[x^{i+1}/(i+1)!]$.
3. Find all x for which the following power series converge:
   (a) $\Sigma_{i=1}^{\infty}[(x-3)^i/i]$
   (b) $\Sigma_{n=1}^{\infty}(1+1/n)^{(n^2)}x^n$
   (c) $\Sigma_{n=1}^{\infty}[(x+2)^n \cos \pi n/4^n]$
4. The ancient Greeks believed that an average centaur's speed was given by $f(t) = \Sigma_{n=1}^{\infty}[2n^2/(n^3+n)]t^n$, where t is a unit of time.
   (a) For what t does $f(t)$ converge?
   (b) Within the radius of convergence, find a formula for the distance travelled by the centaur.
   (c) Find a formula for the centaur's acceleration and determine the radius of convergence for the acceleration.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $5x^4 - 6x + 1$

   (b) $12x + 4x^3$

2. (a) $-3/4$

   (b) $t^6/6 + t^3/3 + 3t + C$

3. (a) $\lim_{i\to\infty}|(3/2i)^i|^{1/i} = \lim_{i\to\infty}(3/2i) = 0 < 1$, so $\Sigma_{n=1}^{\infty}(3/2n)^n$ converges by the root test.

   (b) $\lim_{i\to\infty}|(100i/i!)/[100(i-1)/(i-1)!]| = \lim_{i\to\infty}[1/(i-1)] = 0$, so $\Sigma_{n=1}^{\infty}(100n/n!)$ converges by the ratio test.

ANSWERS TO SECTION QUIZ

1. (a) $f(x) - g(x)$ converges if $f(x)$ and $g(x)$ both converge. In this case, $f(x)$ and $g(x)$ both converge if $-3 < x < 3$.

   (b) $f(x) - g(x) = 2 + 2x/3 + \Sigma_{i=2}^{\infty}(2/3^i - 4/3^{i+1})x^i = 2 + 2x/3 - \Sigma_{i=2}^{\infty}(2/3^{i+1})x^i$. Its radius of convergence is $-3 < x < 3$.

2. (a) False; the index for the derivative should go from $i = 1$ to ∞ .

   (b) True

3. (a) $2 \leqslant x \leqslant 4$

   (b) $-1/e < x < 1/e$

   (c) $-6 < x < 2$

4. (a) $-1 \leqslant t < 1$

   (b) Distance $= \int f(t)dt = \Sigma_{n=1}^{\infty}[2n^2/(n^3 + n)][t^{n+1}/(n+1)]$

   (c) Acceleration $= f'(t) = \Sigma_{n=1}^{\infty}[2n^2/(n^3 + n)]nt^{n-1}$; radius of convergence $= 1$.

12.5 Taylor's Formula

PREREQUISITES

1. Recall how to differentiate and integrate power series (Section 12.4).
2. Recall how to calculate the radius of convergence of a power series (Section 12.4).
3. Recall how to use the linear approximation (Section 1.6).

PREREQUISITE QUIZ

1. Find all x for which the power series $\Sigma_{i=1}^{\infty}[(-1)^i(x-2)^i/i]$ converges.
2. Let $f(x) = \Sigma_{n=0}^{\infty}[(x-3)^n/n!]$.
   (a) Compute $\int f(x)dx$.
   (b) Compute $(d/dx)f(x)$.
3. Use the linear approximation to estimate $(2.11)^3$.

GOALS

1. Be able to find the Taylor series for a given function.
2. Be able to estimate the error in using the Taylor approximation.
3. Be able to state the Taylor series expansions of commonly used functions.

STUDY HINTS

1. Definitions. $\Sigma_{i=0}^{\infty}[f^{(i)}(x_0)/i!](x-x_0)^i$ is called the Taylor series of f about the point $x = x_0$. $f^{(i)}(x_0)$ is the i^{th} derivative of f computed at x_0 . If $x_0 = 0$, the series is known as a Maclaurin series.

2. Error estimates. There are two forms of the remainder for Taylor's series. The integral form is $\int_{x_0}^{x} [(x - t)^n/n!] f^{(n+1)}(t)dt$. The derivative form is $[f^{(n+1)}(c)/(n + 1)!](x - x_0)^{n+1}$, where c is some number between x and x_0 . You estimate the remainder by inserting an inequality for f^{n+1} .

3. Proving convergence. Study Examples 4(a) and 4(b) well. They demonstrate how convergence is often proven for Taylor's series.

4. Taylor series limitations. The paragraph following Example 4 describes interesting limitations of Taylor's series. The first problem is that we may be interested in a point outside the radius of convergence. The second problem is that all of the derivatives may be zero for the Maclaurin series, yet the function isn't zero.

5. Usefulness of Taylor's series. You may be wondering why you should bother with learning about Taylor's series. It can be used to approximate functions whose values could only be found in tables before the invention of calculators. The Taylor approximation improves the linear one.

6. Important series. The box on p. 600 lists the most important Taylor series that you will encounter. Although you should be able to derive each one, it is recommended that you memorize the series in order to save time, except perhaps the series for $\ln(1 + x)$ which can be rapidly derived by integrating $1/(1 + x) = 1 - x + x^2 - x^3 + \ldots$ [$1/(1 + x)$ is a geometric series whose ratio is $-x$.] .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Since $\sin x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i+1}/(2i + 1)!]$, we replace x with $3x$ to get $\sin 3x = \Sigma_{i=0}^{\infty}[(-1)^i(3x)^{2i+1}/(2i + 1)!] = 3x - 9x^3/2 + 81x^5/80 - 243x^7/1120 + \ldots$.

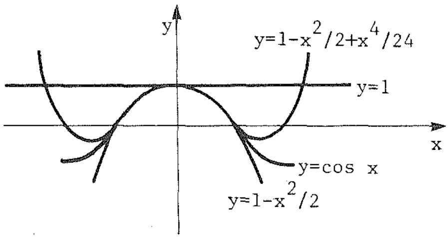
5. Use the method of Example 2. $f(x) = (1 + x^2 + x^4)^{-1}$, $f'(x) = -1(1 + x^2 + x^4)^{-2}(2x + 4x^3)$, $f''(x) = 2(1 + x^2 + x^4)^{-3}(2x + 4x^3)^2 - (1 + x^2 + x^4)^{-2}(2 + 12x^2)$, and $f'''(x) = -6(1 + x^2 + x^4)^{-4}(2x + 4x^3)^3 + 4(1 + x^2 + x^4)^{-3}(2x + 4x^3)(2 + 12x^2) + 2(1 + x^2 + x^4)^{-3}(2x + 4x^3)(2 + 12x^2) - (1 + x^2 + x^4)^{-2}(24x)$. Evaluating at $x_0 = 1$, we have $f(1) = 1/3$, $f'(1) = -2/3$, $f''(1) = 2(36)/27 - 14/9 = 10/9$, and $f'''(1) = -6(216)/81 + 4(6)(14)/27 + 2(6)(14)/27 - 24/9 = -16 + 56/3 - 24/9 = 0$. Thus, $1/(1 + x^2 + x^4) \approx 1/3 + (-2/3)(x - 1) + (10/9)(x - 1)^2/2 + 0(x - 1)^3 = 1/3 - 2(x - 1)/3 + 5(x - 1)^2/9 + 0(x - 1)^3$.

9. (a) Substituting $-x^2 - x^4$ for x in the formula for $1/(1 - x)$ yields $1 + (-x^2 - x^4) + (-x^2 - x^4)^2 + (-x^2 - x^4)^3 + \ldots = 1 - x^2 + x^6 + \ldots$.

(b) By using the formula for the Maclaurin series, we know that $1 = f''''''(0)/6!$; therefore, $f''''''(0) = 6! = 720$.

13. $f(x) = x^{1/2}$, so $f'(x) = (1/2)x^{-1/2}$, $f''(x) = (1/2)(-1/2)x^{-3/2}$, $f'''(x) = (1/2)(-1/2)(-3/2)x^{-5/2}$, ... , $f^{(n)}(x) = (-1)^{n+1}[(2n - 3)!/2(n - 2)!]x^{-(2n-1)/2}/2^n$. Thus, f is infinitely differentiable. At $x_0 = 1$, $f(1) = 1$, $f'(1) = 1/2$, $f''(1) = -1/4$, and $f'''(1) = 3/8$. Thus, $\sqrt{x} = 1 + (1/2)(x - 1) - (1/8)(x - 1)^2 + (1/16)(x - 1)^3 - \ldots$. We need to show that $|[(2n - 3)!/2(n - 2)!]x^{-(2n-1)/2}/2^n| \leqslant CM^n$. Since $(2n - 3)! \leqslant (2n)^n$, choose $M = (2n)/2 = n$ and C to be the maximum of $x^{-(2n-1)/2}$ on I .

Alternatively, we can assume that x is in the interval $(0,2)$ and use the binomial series $(1 + u)^\alpha = \Sigma_{i=0}^{\infty} [\alpha(\alpha - 1)\cdots(\alpha - i + 1)u^i/i!]$. Then let $\alpha = 1/2$ and $u = x - 1$ to get $(1 + x - 1)^{1/2} = x^{1/2} = 1 + (x - 1)/2 + [(1/2)(-1/2)/2](x - 1)^2 + [(1/2)(-1/2)(-3/2)/3!](x - 1)^3 + \ldots = 1 + (x - 1)/2 - (x - 1)^2/8 + (x - 1)^3/16 - \ldots$.

17. 

The Maclaurin series for $\cos x$ is $1 - x^2/2 + x^4/4! - \ldots$. Thus, $f_0(x) = f_1(x) = 1$, $f_2(x) = f_3(x) = 1 - x^2/2$, and $f_4(x) = 1 - x^2/2 + x^4/24$.

21. Since $2.5 = (3/2)\cdot(5/3)$, we have $\ln(2.5) = \ln(3/2) + \ln(5/3) = \Sigma_{i=1}^{\infty}[(-1)^{i+1}(1/2)^i/i] + \Sigma_{i=1}^{\infty}[(-1)^{i+1}(2/3)^i/i]$. The error is $R_n(x) = [f^{(n+1)}(c)/(n+1)!](x - x_0)^{n+1}$, where $f(x) = \ln(1 + x)$ *and* $x_0 = 0$. We have $f'(x) = (1 + x)^{-1}$, $f''(x) = -(1 + x)^{-2}$, $f'''(x) = 2(1 + x)^{-3}$, $\ldots$, $f^{(n)}(x) = (-1)^{n+1}(n - 1)!(1 + x)^{-n}$. On $[0,2/3]$, $R_4 < 0.007$ and on $[0,1/2]$, $R_4 < 0.002$, so for $n = 4$, the total error is less than 0.009 . Therefore, $\ln(2.5) \approx (1/2 - 1/8 + 1/24 - 1/64) + (2/3 - 2/9 + 8/81 - 4/81) \approx 0.9$.

25. (a) $f(x) \approx f(x_0) + [f'(x_0)](x - x_0) + [f''(x_0)](x - x_0)^2/2$. $\int_{x_0-R}^{x_0+R} f(x)dx = \int_{x_0-R}^{x_0+R}\{f(x_0) + [f'(x_0)](x - x_0) + [f''(x_0)](x - x_0)^2/2\}dx = \{[f(x_0)](x - x_0) + [f'(x_0)](x - x_0)^2/2 + [f''(x_0)](x - x_0)^3/6\}\Big|_{x_0-R}^{x_0+R} = f(x_0)R + f'(x_0)R^2/2 + f''(x_0)R^3/6 - [f(x_0)(-R) + f'(x_0)R^2/2 + f''(x_0)(-R)^3/6] = 2Rf(x_0) + 2f''(x_0)R^3/3!$. By equation (4) in the text, the remainder is $\left|\int_{x_0-R}^{x_0+R}[(x - t)^3 f'''(t)/3!]dt\right| \leqslant \left|\int_{x_0}^{x_0+R}\{[(x_0 + R) - t]^3 f'''(t)/3!\}dt\right| + \left|\int_{x_0}^{x_0-R}\{[(x_0 - R) - t]^3 f'''(t)/3!\}dt\right| \leqslant \left|\int_{x_0}^{x_0+R}\{[(x_0 + R) - t]^3 M_3/3!\}dt\right| + \left|\int_{x_0}^{x_0-R}\{[(x_0 - R) - t]^3 M_3/3!\}dt\right| = \left|(x_0 + R - t)^4 M_3/4!\Big|_{x_0}^{x_0+R}\right| + \left|(x_0 - R - t)^4 M_3/4!\Big|_{x_0}^{x_0-R}\right| = R^4M_3/4! + R^4M_3/4! = R^4M_3/12$, where M_3 is the maximum value $|f'''(x)|$ takes on for x in $[x_0 - R, x_0 + R]$.

25. (b) Let $x_0 = 0$, $R = 1/2$, and $f(x) = 1/\sqrt{1 + x^2}$. Then $f'(x) = -x(1 + x^2)^{-3/2}$ and $f''(x) = -(1 + x^2)^{-3/2} - 3x^2(1 + x^2)^{-5/2}$. $f(0) = 1$ and $f''(0) = -1$. Therefore, $\int_{-1/2}^{1/2}\left[dx/\sqrt{1 + x^2}\right] \approx 2(1/2)1 + 2(-1)(1/2)^3/6 = 1 - 1/24 = 23/24 \approx 0.958$. Simpson's rule for $n = 4$, gives $(1/12)[f(-1/2) + 4f(-1/4) + 2f(0) + 4f(1/4) + f(1/2)] = (1/12)[4/\sqrt{5} + 32/\sqrt{17} + 1] = 0.879$.

29. $\lim_{x\to 0}(1/x \sin x - 1/x^2) = \lim_{x\to 0}[(x - \sin x)/x^2 \sin x]$. Now, $-\sin x = -x + x^3/3! - x^5/5! + x^7/7! - \dots$ and $x - \sin x = x^3/3! - x^5/5! + x^7/7! - \dots$. The denominator is $x^3 - x^5/3! + x^7/5! - \dots$, so $(x - \sin x)/x^2 \sin x = (x^3/3! - x^5/5! + x^7/7! - \dots)/(x^3 - x^5/3! + x^7/5! - \dots) = (1/6 - x^2/5! + \dots)/(1 - x^2/3! + \dots)$ by dividing by x^3/x^3. Since the term involving x^n tends to 0 as x approaches 0, we get the limit as $1/6$.

33. By the sum rule, $1/(1 - x) - 1/(1 + x) = \Sigma_{n=0}^{\infty} x^n - \Sigma_{n=0}^{\infty}(-1)^n x^n = (1 + x + x^2 + x^3 + \dots) - (1 - x + x^2 - x^3 + \dots) = 2x + 2x^3 + \dots = \Sigma_{n=0}^{\infty} 2x^{2n+1}$ for $|x| < 1$.

37. (a) Multiplying out, $f(x) = (1 + x^2)^2 = 1 + 2x^2 + x^4$.

(b) Since $f(x) = (1 + x^2)^2$, $f'(x) = 2(1 + x^2)\cdot 2x$, $f''(x) = 4(1 + x^2) + 8x^2$, $f'''(x) = 24x$, $f^{(4)}(x) = 24$, and $f^{(k)}(x) = 0$ for all $k > 4$. Thus $f(0) = 1$, $f'(0) = 0$, $f''(0) = 4$, $f'''(0) = 0$, $f^{(4)}(0) = 24$, and $f^{(k)}(0) = 0$ for all $k > 4$. Hence, the Maclaurin series for $f(x) = (1 + x^2)^2$ is $f(x) = f(0) + f'(0)(x - 0) + f''(0)(x - 0)^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + \dots = 1 + 0 + (4/2)x^2 + 0 + (24/4!)x^4 + 0 + 0 + \dots = 1 + 2x^2 + x^4$.

41. Since $f(x) = \sec x$, $f(0) = 1$. $f'(x) = \sec x \tan x$, so $f'(0) = 0$. $f''(x) = \sec x \tan^2 x + \sec^3 x$, so $f''(0) = 1$, $f'''(x) = \sec x \tan^3 x + 2 \sec^3 x \tan x + 3 \sec^3 x \tan x$, so $f'''(0) = 0$. Thus, $a_0 = f(0) = 1$; $a_1 = f'(0) = 0$; $a_2 = f''(0)/2 = 1/2$; $a_3 = f'''(0)/3! = 0$.

45. We know that $\cos x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i}/(2i)!]$. Therefore, $1 - \cos x = \Sigma_{i=1}^{\infty}[(-1)^{i+1}x^{2i}/(2i)!]$. Thus, $(1 - \cos x)/x^2 = \Sigma_{i=1}^{\infty}[(-1)^{i+1}x^{2i-2}/(2i)!] = 1/2 - x^2/4! + x^4/6! - \ldots$.

49. Let $f(x) = \ln x$. Then, $f'(x) = x^{-1}$; $f''(x) = -x^{-2}$; $f'''(x) = 2x^{-3}$; $f''''(x) = -6x^{-4}$. *The Taylor expansion of degree* 4 *for* $\ln x$ is $\ln x_0 + x_0^{-1}(x - x_0) - x_0^{-2}(x - x_0)^2/2! + 2x_0^{-3}(x - x_0)^3/3! - 6x_0^{-4}(x - x_0)^4/4! = \ln x_0 + (x - x_0)/x_0 - (x - x_0)^2/2x_0^2 + (x - x_0)^3/3x_0^3 - (x - x_0)^4/4x_0^4$.

(a) For $x_0 = 1$, the polynomial is $(x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - (x - 1)^4/4$.

(b) For $x_0 = e$, the polynomial is $1 + (x - e)/e - (x - e)^2/2e^2 + (x - e)^3/3e^3 - (x - e)^4/4e^4$.

(c) For $x_0 = 2$, the polynomial is $\ln 2 + (x - 2)/2 - (x - 2)^2/8 + (x - 2)^3/24 - (x - 2)^4/64$.

53. Since $f(x) = \sin e^x$, $f(0) = \sin 1$; $f'(x) = e^x \cos e^x$, $f'(0) = \cos 1$; $f''(x) = e^x \cos e^x - e^{2x} \sin e^x$, $f''(0) = \cos 1 - \sin 1$; $f'''(x) = e^x \cos e^x - e^{2x} \sin e^x - 2e^{2x} \sin e^x - e^{3x} \cos e^x$, $f'''(0) = -3 \sin 1$. The first four terms are $\sin 1 + (\cos 1)x + [(\cos 1 - \sin 1)/2]x^2 - [(\sin 1)/2]x^3$.

57. (a) $f'(0) = \lim_{\Delta x\to 0}\{[f(\Delta x) - f(0)]/\Delta x = \lim_{\Delta x\to 0}\{[(\sin \Delta x)/\Delta x - 1]/\Delta x\} = \lim_{\Delta x\to 0}[(\sin \Delta x - \Delta x)/(\Delta x)^2] = \lim_{\Delta x\to 0}[(\cos \Delta x - 1)/2(\Delta x)] = \lim_{\Delta x\to 0}[-\sin \Delta x/2] = 0$ (l'Hôpital's rule). For $x \neq 0$, $f'(x) = (x \cos x - \sin x)/x^2$. Therefore, $f''(0) = \lim_{\Delta x\to 0}\{[f'(\Delta x) - f'(0)]/\Delta x\} = \lim_{\Delta x\to 0}\{[(\Delta x \cos \Delta x - \sin \Delta x)/(\Delta x)^2]/\Delta x\} = \lim_{\Delta x\to 0}[(\Delta x \cos \Delta x - \sin \Delta x)/(\Delta x)^3] = \lim_{\Delta x\to 0}[(\cos \Delta x - \Delta x \sin \Delta x - \cos \Delta x)/3(\Delta x)^2] = \lim_{\Delta x\to 0}[(-\sin \Delta x - \Delta x \cos \Delta x)/6(\Delta x)] = \lim_{\Delta x\to 0}[(-\cos \Delta x - \cos \Delta x + \Delta x \sin \Delta x)/6] = -1/3$ (l'Hôpital's rule). For $x \neq 0$, $f''(x) = [x^2(\cos x - x \sin x - \cos x) - (x \cos x - \sin x)2x]/x^4 = (-x^2 \sin x - 2x \cos x + 2 \sin x)/x^3$. Therefore, $f'''(0) = \lim_{\Delta x\to 0}\{[f''(\Delta x) - f''(0)]/\Delta x\} = \lim_{\Delta x\to 0}\{[-(\Delta x)^2 \sin \Delta x - 2\Delta x \cos \Delta x + 2 \sin \Delta x + (\Delta x)^3/3]/(\Delta x)^4\} = \lim_{\Delta x\to 0}\{[-2\Delta x \sin \Delta x - (\Delta x)^2 \cos \Delta x - 2 \cos \Delta x + 2\Delta x \sin \Delta x + 2 \cos \Delta x + (\Delta x)^2]/4(\Delta x)^3\} = \lim_{\Delta x\to 0}\{[-2\Delta x \cos \Delta x + (\Delta x)^2 \sin \Delta x + 2\Delta x]/12(\Delta x)^2\} = \lim_{\Delta x\to 0}\{[-2 \cos \Delta x + 2\Delta x \sin \Delta x + 2\Delta x \sin \Delta x + (\Delta x)^2 \cos \Delta x + 2]/24\Delta x\} = \lim_{\Delta x\to 0}\{[-2 \sin \Delta x + 4 \sin \Delta x + 4 \Delta x \cos \Delta x + 2\Delta x \cos \Delta x - (\Delta x)^2 \sin \Delta x]/24\} = 0$.

(b) $\sin x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i+1}/(2i + 1)!]$, so $\sin x/x = \Sigma_{i=0}^{\infty}[(-1)^i x^{2i}/(2i + 1)!] = 1 - x^2/3! + x^4/5! - x^6/7! + \ldots$.

SECTION QUIZ

1. (a) What is the value of $\Sigma_{i=0}^{\infty}[(-1)^i/i!]$?

(b) What is the value of $\Sigma_{i=0}^{\infty}[(-1)^i \pi^{2i+1}/(2i)!]$?

2. Use a Taylor series to prove that $\lim_{x\to 0}(\sin x/x) = 1$.

3. Find the Taylor series expansion of $f(x) = x^3 + 2x - 2$ about $x = 2$ and simplify your answer.

4. Find the sum of $\Sigma_{n=0}^{\infty}[(-1)^n/4^n(2n + 1)]$. (Hint: If $f(x) = \tan^{-1}(x/2)$, what is $f'(x)$?)

5. Find the Taylor series expansion of the following functions around the indicated point:

   (a) $y = \sin^{-1}x$ about $x = 0$

   (b) $y = 1/\sqrt[3]{x - 2}$ about $x = 3$

6. A frustrated calculus student was overheard saying, "Taylor's formula is useless." That evening, the student was visited by Taylor's ghost. The ghost explained that he travels throughout the night with speed $f(t) = t \cos t$ at time t to haunt nonbelievers of Taylor's formula. He will return to haunt calculus students who can't find a fourth-order approximation to his speed at $t = 0.2$. He will return tomorrow for the answer and an upper bound for the error. Find the answers which will prevent return visits by Taylor's ghost.

ANSWERS TO PREREQUISITE QUIZ

1. $1 < x \leqslant 3$

2. (a) $\Sigma_{n=0}^{\infty}[(x - 3)^{n+1}/(n + 1)!]$

   (b) $\Sigma_{n=1}^{\infty}[n(x - 3)^{n-1}/n!]$

3. 9.32

ANSWERS TO SECTION QUIZ

1. (a) $1/e$

   (b) $-\pi$

2. The Taylor expansion of $\sin x$ about $x = 0$ is $x - x^3/3! + x^5/5! - \ldots$, so $\sin x/x = 1 - x^2/3! + x^4/5! - \ldots$, and the limit is 1.

3. $f(x) = 10 + 14(x - 2) + 6(x - 2)^2 + (x - 2)^3 = x^3 + 2x - 2$.

4. $\tan^{-1}(1/2)$

5. (a) $x + x^3/6 + \Sigma_{n=2}^{\infty}[(2n - 1)!/2^{n+1}n!(n - 1)!(2n + 1)]x^{2n+1}$

(b) $\Sigma_{n=0}^{\infty}[(-1)^n(2n + 1)!(1)(4)(7) \ldots (2n + 1)/3^n n!](x - 3)^n$

6. Expand around $t = 0$; $t \cos t \approx t - t^3/2$, which is 0.196 at $t = 0.2$. Error $\leqslant |-3 \cos(0.2) - (0.2) \sin(0.2)|/120(5)^5 <$ $1/120(5)^4$.

12.6 Complex Numbers

PREREQUISITES

1. Recall the power series expansions for $1/(1 + x)$, $\cos x$, $\sin x$, and e^x (Section 12.5).
2. Recall how to use polar coordinates (Section 5.1).

PREREQUISITE QUIZ

1. Plot the polar coordinates $(2,-5\pi/8)$ in the xy-plane.
2. Convert the polar coordinates $(-1, 3\pi/4)$ into cartesian coordinates.
3. Convert the cartesian coordinates $(2,-2)$ into polar coordinates.
4. Write down the power series expansions for the following:

   (a) $\cos 2x$

   (b) e^{-x}

GOALS

1. Be able to explain the relationship between trigonometry and the imaginary numbers.
2. Be able to do basic algebra with complex numbers.
3. Be able to write down the polar representation of a complex number and use this form to extract roots.

STUDY HINTS

1. Imaginary numbers defined. The basic definition is $\sqrt{-1} = i$. You should know that $e^{ix} = \cosh ix + \sinh ix = \cos x + i \sin x$. Therefore, $\cosh ix = \cos x$ and $\sinh ix = i \sin x$. In addition, by letting $x = \pi$, we get $e^{i\pi} = -1$.

2. Algebra of complex numbers. When you add and mulitply, i acts just like an ordinary number except that you must remember that $i^2 = -1$.

3. Terminology. As with any new subject, an essential starting point is learning the vocabulary. Therefore, learning the vocabulary presented on p. 611 will aid you in understanding the discussions about complex numbers.

4. Notation. Many times, (a,b) will denote a complex number. It is understood to stand for $z = a + bi$.

5. Removing i from denominator. To standardize answers, it is desirable to remove i from the denominator just as we desired to have denominators without radicals. If $a + bi$ occurs in the denominator, multiplying by $(a - bi)/(a - bi)$ yields $a^2 + b^2$ in the denominator and removes the symbol i from the denominator.

6. Properties of complex numbers. The properties presented on p. 612 are helpful in manipulations. Note their similarities to properties of real numbers.

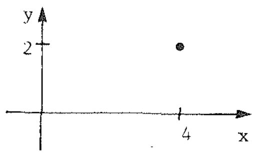
7. Exponential properties. Multiplication and division of exponents are done exactly as they are done with real numbers.

8. Polar representation. $z = re^{i\theta}$ is called the polar representation of $z = a + ib$. (r,θ) are the polar coordinates of (a,b) . Notice how easy it is to mulitply two imaginary numbers which are written in polar representation.

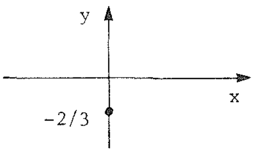
9. DeMoivre's formula. The n^{th} roots of an imaginary number $\rho e^{i\phi}$ are located on a circle centered at the origin with radius $\sqrt[n]{\rho}$. One root is located at an angle of ϕ/n from the positive real axis. There are $n - 1$ other roots equally spaced on the circle, separated by angles of $2\pi/n$. The validity of DeMoivre's formula is easily demonstrated using the polar representation to raise the claimed roots to the n^{th} power.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using $e^{ix} = \cos x + i \sin x$, we get $e^{-\pi i/2} = \cos(-\pi/2) + i \sin(-\pi/2) = -i$.

5. 

The complex number $x + yi$ is plotted at the point (x,y) in the xy-plane. Thus, $4 + 2i$ is plotted at $(4,2)$.

9. 

The complex number $x + yi$ is plotted at the point (x,y) in the xy-plane. Thus, $(-2/3)i$ is plotted at $(0,-2/3)$.

13. We need to add the real and imaginary parts separately. $(1 + 2i) - 3(5 - 2i) = (1 + 2i) + (-15 + 6i) = -14 + 8i$.

17. Mulitply by $(5 + 3i)/(5 + 3i)$ to get $(5 + 3i)/(25 - 9i^2) = (5 + 3i)/34$.

21. If $z^2 + 3 = 0$, then $z^2 = -3$ or $z = \pm\sqrt{-3} = \pm\sqrt{3}i$.

25. By the quadratic formula, $z = (7 \pm \sqrt{49 + 4})/2 = (7 \pm \sqrt{53})/2$.

29. From Example 4(d) , $\sqrt{i} = \pm(\sqrt{2}/2)(1 + i)$, so $\sqrt{-16i} = \sqrt{16}\sqrt{i}\sqrt{-1} = 4i\sqrt{i} = \pm 4i(\sqrt{2}/2)(1 + i) = \pm(\sqrt{2}/2)(4i - 4) = \pm(2\sqrt{2}i - 2\sqrt{2})$.

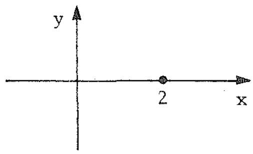
33. Since $(1 + 2i)^2 = (-3 + 4i)$, $(10 + 5i)/(1 + 2i)^2 = (10 + 5i)/(-3 + 4i)$. Multiply by $(-3 - 4i)/(-3 - 4i)$ to get $(-30 - 55i - 20i^2)/(9 - 16i^2) = (-10 - 55i)/25$, so the imaginary part is $-55/25 = -11/5$.

37. The complex conjugate of $a + bi$ is $a - bi$, so $\bar{z} = 5 - 2i$.

41. Multiply by i/i to get $(2 - i)/3i = (2i + 1)/(-3)$, so $\bar{z} = -1/3 + 2i/3$.

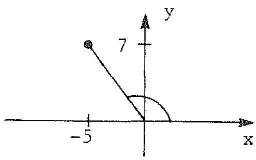
45. Here, $z = 3 + 0i$, so the complex conjugate is $\bar{z} = 3 - 0i = 3$.

49.



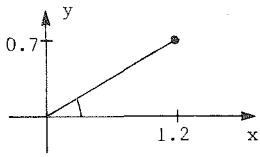
$z = 2$ is plotted at the left. $|z|$ is the distance from the origin, which is 2 in this case. θ is the angle from the positive x-axis, which is 0 in this case.

53.



$z = -5 + 7i$ is plotted at the left. $|z| = \sqrt{a^2 + b^2} = \sqrt{25 + 49} = \sqrt{74}$. The argument, θ, is the angle from the positive x-axis. In this case, $\theta = \cos^{-1}(-5/\sqrt{74}) \approx 2.19$.

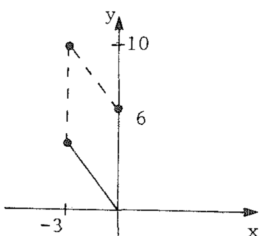
57.



$z = 1.2 + 0.7i$ is plotted at the left. Here, $|z| = \sqrt{a^2 + b^2} = \sqrt{1.44 + 0.49} = \sqrt{1.93}$. The argument is $\theta = \cos^{-1}(1.2/\sqrt{1.93}) \approx 0.53$.

61. From Exmaple 7(b), we know that $\overline{z^n} = \bar{z}^n$, so $\overline{(8 - 3i)^4} = \overline{(8 - 3i)}^4 = (8 + 3i)^4$.

65.



Two sides of a parallelogram are determined by line segments joinging the origin to z_1 and z_2. The sum is the fourth vertex of the parallelogram. In this case, the sum is $-3 + 10i$.

69. We know that $e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$. Thus, $|e^z| = (e^{2x}\cos^2 y + e^{2x}\sin^2 y)^{1/2} = [e^{2x}(\cos^2 y + \sin^2 y)]^{1/2} = e^x$. The argument is $\theta = \cos^{-1}(a/r)$, where $z = a + bi$ and $r = |z|$. Therefore, $\theta = \cos^{-1}(e^x \cos y/e^x) = \cos^{-1}(\cos y) = y$.

73. Use the fact that $e^{ix} = \cos x + i \sin x$. Thus, $e^{1-\pi i/2} = e \cdot e^{-\pi i/2} = e[\cos(-\pi/2) + i \sin(-\pi/2)] = e(-i) = -ei$.

77. We have $f(z) = 1/(2 + i)^2 = 1/(3 + 4i)$. Multiply by $(3 - 4i)/(3 - 4i)$ to get $(3 - 4i)/(9 - 16i^2) = (3 - 4i)/25$.

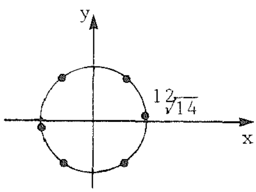
81. By the law of exponents, $e^{i(\theta+3\pi/2)} = e^{i\theta} \cdot e^{3\pi i/2}$. Using $e^{ix} = \cos x + i \sin x$, we get $e^{i\theta} \cdot e^{3\pi i/2} = (\cos\theta + i\sin\theta)(\cos(3\pi/2) + i\sin(3\pi/2)) = (\cos\theta + i\sin\theta)(-i) = \sin\theta - i\cos\theta$.

85. If $z = a + ib$, then $r = \sqrt{a^2 + b^2}$ and $\theta = \cos^{-1}(a/r) = \sin^{-1}(b/r)$. Here, $z = 1 + i$, so $r = \sqrt{1 + 1} = \sqrt{2}$ and $\theta = \cos^{-1}(1/\sqrt{2}) = \pi/4$. Thus, $1 + i = \sqrt{2}e^{i\pi/4}$.

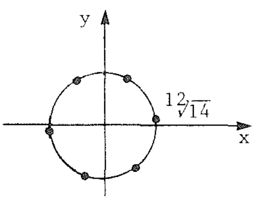
89. If $z = a + ib$, then $r = \sqrt{a^2 + b^2}$ and $\theta = \cos^{-1}(a/r) = \sin^{-1}(b/r)$. Here, $z = 7 - 3i$, so $r = \sqrt{49+9} = \sqrt{58}$ and $\theta = \cos^{-1}(7/\sqrt{58}) = \sin^{-1}(-3/\sqrt{58}) \approx -0.40$. Thus, $7 - 3i = \sqrt{58}\, e^{i(-0.40)}$.

93. Squaring gives $(3 + 4i)^2 = 9 + 24i + 16i^2 = -7 + 24i$. So $r = \sqrt{7^2 + 24^2} = 25$ and $\theta = \cos^{-1}(-7/25) \approx 1.85$. Thus, $(3 + 4i)^2 = 25e^{i(1.85)}$.

97.



Write the number as $\rho e^{i\phi}$. Then the n^{th} roots of $\rho e^{i\phi}$ are given by $\sqrt[n]{\rho}\, e^{i(\phi/n+2\pi k/n)}$, where $k = 0, 1, 2, \ldots, n - 1$. For $\sqrt{5} + 3i$, $\rho = \sqrt{5 + 9} = \sqrt{14}$ and $\phi = \cos^{-1}(\sqrt{5/14}) = \sin^{-1}(3/\sqrt{14}) \approx 0.93$. Thus, $\sqrt{5} + 3i = \sqrt{14}\, e^{i(0.93)}$ and the sixth roots are $\sqrt[12]{14}\, e^{i(0.155+2\pi k/6)}$, where $k = 0, 1, 2, 3, 4,$ and 5 .



For $3 + \sqrt{5}i$, $\rho = \sqrt{9 + 5} = \sqrt{14}$ and $\phi = \cos^{-1}(3/\sqrt{14}) = \sin^{-1}(\sqrt{5/14}) \approx 0.64$. Thus, $3 + \sqrt{5}\, i = \sqrt{14}\, e^{i(0.64)}$ and the sixth roots are $\sqrt[12]{14}\, e^{i(0.107+ 2\pi k/6)}$, where $k = 0, 1, 2, 3, 4,$ and 5 .

101. For $z = 1$, $w = e^0$, so the sixth roots are $e^{\pi ik/3}$. Thus, the roots are located on the unit circle at angles of 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$. The tenth roots of e^0 are $e^{\pi ik/5}$, which are located on the unit circle at angles of 0, $\pi/5$, $2\pi/5$, $3\pi/5$, $4\pi/5$, π, $6\pi/5$, $7\pi/5$, $8\pi/5$, and $9\pi/5$. The sixth and tenth roots of 1 share common points at 0 and π, which correspond to $z = \pm 1$.

105. The quadratic formula gives us the roots $z = (-2 \pm \sqrt{4 - 4i})/2 = -1 \pm \sqrt{1 - i}$. Now, if $\sqrt{1 - i} = a + bi$, then $(a + bi)^2 = 1 - i$ or $a^2 - b^2 = 1$ and $2ab = -1$. Since $b = -1/2a$, $a^2 - b^2 = 1$ becomes $a^2 - 1/4a^2 = 1$ or $4a^4 - 4a^2 - 1 = 0$. Therefore, $a^2 = (4 \pm \sqrt{32})/8 = (1 \pm \sqrt{2})/2$. Also, $b^2 = a^2 - 1 = (-1 \pm \sqrt{2})/2$. Since a and b are real numbers, we get $\sqrt{1 - i} = (1/\sqrt{2})[(1 + \sqrt{2})^{1/2} + (-1 + \sqrt{2})^{1/2}i]$ or $\sqrt{1 - i} = (1/\sqrt{2})[-(1 + \sqrt{2})^{1/2} - (-1 + \sqrt{2})^{1/2}i]$. Therefore, $z^2 + 2z + i = (1/\sqrt{2})[z - (1 + \sqrt{2})^{1/2} - (-1 + \sqrt{2})^{1/2}i] \times [z + (1 + \sqrt{2})^{1/2} + (-1 + \sqrt{2})^{1/2}i]$.

109. (a) Using the result of Example 2, we have $\cos i\theta = [e^{i(i\theta)} + e^{-i(i\theta)}]/2$ and $\sin i\theta = [e^{i(i\theta)} - e^{-i(i\theta)}]/2i$. Also, $\tan i\theta = \sin i\theta/\cos i\theta = (e^{-\theta} - e^{\theta})/(e^{-\theta} + e^{\theta})i$. We recognize this as $-\tanh \theta/i$. Thus, $\tan i\theta$ is the function $i \tanh \theta$.

(b) Using part (a), we have $r = \tanh \theta$ and $\phi = \pi/2$, so $\tan i\theta = (\tanh \theta)e^{i\pi/2}$.

113. (a) Note that $(z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1) = z^n - z^{n-1} + z^{n-1} - z^{n-2} + z^{n-2} - \ldots - z^2 + z^2 - z + z - 1 = z^n - 1$. Therefore, if $z^n - 1 = 0$, either $z - 1 = 0$ or $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$.

(b) If $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$, then $z(z^{n-1} + z^{n-2} + \ldots + z + 1) = z^n + (z^{n-1} + z^{n-2} + \ldots + z^2 + z) = 0$ or $z^n = -z^{n-1} - z^{n-2} - \ldots - z^2 - z = 1$, from the original equation.

113. (c) The four roots of $z^4 = 1$ are 1 , -1 , i , $-i$. Therefore, the three roots of $z^3 + z^2 + z + 1 = 0$ are -1 , i , $-i$, by part (b).

117. (a) If $z = x + iy$, then $e^z = e^x(\cos y + i \sin y) = -1$ when $x = 0$ and when $y = \pi + 2n\pi$, where n is an integer.

(b) You might define $\ln(-1) = i\pi$, though there are many values of z such that $e^z = -1$.

SECTION QUIZ

1. Let $u = i/(a + ib)$, where a and b are real numbers.

   (a) What is the imaginary part of u ?

   (b) What is the real part of u ?

   (c) What is $u \cdot \bar{u}$?

   (d) What is the polar representation of u ?

2. What is e^{ix} in terms of sines and cosines?

3. Find all of the solutions of $z^4 + 1 = 0$.

4. Let $z_1 = 2 + 2i$ and $z_2 = 2\sqrt{3} - 2i$.

   (a) Convert z_1 and z_2 to their polar representation.

   (b) Compute $z_1 \cdot z_2$. Express your answer in both the $a + bi$ form and the polar representation.

   (c) Explain how multiplication of complex numbers can be related to the lengths and arguments of complex numbers.

5. Compute the following:

   (a) $(3 + 2i)^2$

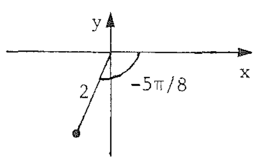
   (b) $\bar{z}$ if $z = 1/(i - 2)$

   (c) $(5 + 3i)i - 2e$

6. Ronnie was struggling through his calculus assignment when his fairy godfather appeared to offer his assistance. Asked what the problem was, Ronnie explained that he needed to do some root extractions. When the dumb fairy godfather heard this, he sent for the tooth fairy. Ronnie explained that he had to find the sixth roots of $2 + 3i$.
   (a) Fortunately, the tooth fairy understood Ronnie's problem. What answer did she give him? Express the answer in the form $a + bi$.
   (b) Plot all of the points representing $\sqrt[6]{2 + 3i}$ in the xy-plane.
   (c) Of all the answers in (a), find the complex conjugate of the one with the smallest positive argument.

ANSWERS TO PREREQUISITE QUIZ

1.



2. $(\sqrt{2}/2, -\sqrt{2}/2)$

3. $(2\sqrt{2}, -\pi/4)$

4. (a) $1 - 4x^2/2! + 16x^4/4! - \dots = \Sigma_{i=0}^{\infty}[(-1)^i(2x)^{2i}/(2i)!]$
   (b) $1 - x + x^2/2! - x^3/3! + \dots = \Sigma_{i=0}^{\infty}[(-x)^i/i!]$

ANSWERS TO SECTION QUIZ

1. (a) $a/(a^2 + b^2)$
   (b) $b/(a^2 + b^2)$
   (c) $1/(a^2 + b^2)$
   (d) $re^{i\theta}$, where $r = 1/\sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(a/b)$

2. $e^{ix} = \cos x + i \sin x$

3. $e^{i\pi/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$

4. (a) $z_1 = 2\sqrt{2}\ \exp(\pi i/4)$ and $z_2 = 4\ \exp(-\pi i/6)$

(b) $(4\sqrt{3} + 4) + (4\sqrt{3} - 4)i = 8\sqrt{2}\ \exp(\pi i/12)$

(c) $(r_1 \exp(i\theta_1))\ (r_2 \exp(i\theta_2)) = r_1 r_2 \exp[(\theta_1 + \theta_2)i]$. The lengths are multiplied and the arguments are added.

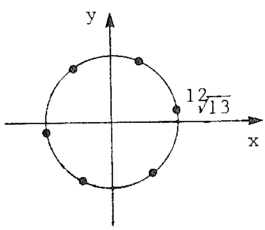
5. (a) $5 + 12i$

(b) $(-2 + i)/5$

(c) $-3 - 2e + 5i$

6. (a) $1.2 + 0.2i$; $0.4 + 1.2i$; $-0.8 + 1.0i$; $-1.2 - 0.2i$; $-0.4 - 1.2i$; $0.8 - 1.0i$

(b)



(c) $1.2 - 0.2i$

12.7 Second-Order Linear Differential Equations

PREREQUISITES

1. Recall the quadratic formula (Section R.1).
2. Recall the polar representation for complex numbers (Section 12.6).
3. Recall basic differentiation and integration formulas (Section 7.1).
4. Recall how to solve the spring equation (Section 8.1).

PREREQUISITE QUIZ

1. What is the solution of $ax^2 + bx + c = 0$ if a, b, c are real and $a \neq 0$?
2. Let z be the complex number $5 + 5i$. What is the polar representation of z ?
3. In terms of trigonometric functions, what is e^{ix} ?
4. Solve $d^2y/dx^2 + 4y = 0$, $y'(0) = 2$, $y(0) = 3$.
5. Evaluate the following:

   (a) $\int e^{3t}dt$

   (b) $(d/dy)(\cos y + \sin y)$

   (c) $\int (x^2 + 2x)dx$

GOALS

1. Be able to solve differential equations of the form $ay'' + by' + cy = 0$.
2. Be able to find a solution for nonhomogeneous equations using the methods of variation of parameters or undetermined coefficients.

STUDY HINTS

1. Characteristic equation. The key to solving $ay'' + by' + cy = 0$ is finding the roots of the characteristic equation $ar^2 + br + c = 0$. This is easily solved using the quadratic formula.

2. Distinct characteristic roots. If r_1 and r_2 are distinct characteristic roots, then the solution of $ay'' + by' + cy = 0$ is $y = c_1\exp(r_1x) + c_2\exp(r_2x)$; c_1 and c_2 are determined by the initial conditions. If r_1 and r_2 are complex, then the exponential part of the solution may be rewritten using $e^{ix} = \cos x + i \sin x$.
3. Repeated characteristic roots. If r_1 is the only characteristic root, i.e., $b^2 - 4ac = 0$, then the solution of $ay'' + by' + cy = 0$ is $(c_1 + c_2x)\exp(r_1x)$.
4. Method of reduction of order. When the roots repeat, simply look for a solution of the form $y = v \exp(r_1x)$, where v is a function of x .
5. Method of root splitting. This is an alternative derivation of the formula for repeated roots given in item 3 above (formula (5) on p. 619). You shouldn't worry if you don't understand it, as it won't be used later.
6. Damping. The equation $d^2x/dt^2 + \beta(dx/dt) + \omega^2x = 0$ describes damped harmonic motion. Whether we are dealing with an overdamped, critically damped, or underdamped case depends upon whether the characteristic equation has two, one, or no real roots. The solution can be written in terms of sine and cosine if no real roots exist and the solution is oscillatory. See Fig. 12.7.2.
7. Nonhomogeneous equations. $ay'' + by' + cy = F(x)$ is a nonhomogeneous equation. Notice that the right-hand side is a function of x only. It is a homogeneous equation if $F(x) = 0$. The general solution of the homogeneous equation (containing two arbitrary constants) is denoted by y_h . A specific solution of the nonhomogeneous equation (with no arbitrary constants) is called a particular solution and is denoted by y_p . Then $y_p + y_h$ is the general solution of the nonhomogeneous equation. Note

7. (continued)

that y_p alone solves it and y_h adds zero to $F(x)$.

8. Method of undetermined coefficients. Depending on the form of the right-hand side, guess that the particular solution is a linear combination of sines, cosines, exponentials, and polynomials. Differentiate and substitute into the left-hand side. Then, solve for the constants. See Example 5. Make up an equation with $F(x)$ as a polynomial. Your guess for a particular solution should be a polynomial of the same degree.

9. Variation of parameters. If y_1 and y_2 are solutions of the homogeneous equation, we look for a solution of the form $y_p = v_1y_1 + v_2y_2$, where v_1 and v_2 are also functions of x . Differentiate and substitute into the original equation. We set $v_1'y_1 + v_2'y_2 = 0$ and then the equation becomes $v_1'y_1' + v_2'y_2' = F/a$. These can be solved simultaneously for v_1' and v_2' and integration of v_1' and v_2' yields a particular solution.

10. Damped forced oscillations. The box on p. 628 should not be memorized. If you encounter a forced oscillation question on an exam, look for a particular solution of the form $A\cos(\Omega t - \delta)$, where Ω is the forcing frequency, and add it to the general solution of the homogeneous equation.

11. Wronksians. If y_1 and y_2 are solutions of the homogeneous second-order linear differential equation, and $y_1y_2' - y_2y_1' \neq 0$, then $y = c_1y_1 + c_2y_2$ is a general solution of the differential equation. This is the gist of the supplement. The expression $y_1y_2' - y_2y_1'$ is called a Wronksian.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The general solution of $ay'' + by' + cy = 0$ is $y = c_1\exp(r_1x) + c_2\exp(r_2x)$, where r_1 and r_2 are the roots of $ar^2 + br + c = 0$ and c_1 and c_2 are constants. Here, $r = (4 \pm \sqrt{16 - 12})/2 = (4 \pm 2)/2 = 3$ or 1. Thus, the solution is $y = c_1\exp(3x) + c_2\exp(x)$.

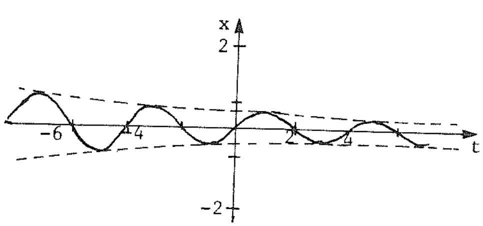
5. Differentiating the solution to Exercise 1 gives $y' = 3c_1\exp(3x) + c_2\exp(x)$. Thus, $y(0) = 0$ and $y'(0) = 1$ gives $c_1 + c_2 = 0$ and $3c_1 + c_2 = 1$. This yields $c_1 = 1/2$ and $c_2 = -1/2$. The particular solution is $y = (1/2)\exp(3x) - (1/2)\exp(x)$.

9. We have $r = (4 \pm \sqrt{16 - 20})/2 = (4 \pm 2i)/2 = 2 \pm i$. Thus, the solution is $y = c_1\exp[(2 + i)x] + c_2\exp[(2 - i)x] = e^{2x}[c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)] = e^{2x}[c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)] = e^{2x}(a_1\cos x + a_2\sin x)$, where $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$.

13. The general solution is $y = (c_1 + c_2x)\exp(r_1x)$, where r_1 is the repeated root of the characteristic equation. Here, we have $r^2 - 6r + 9 = (r - 3)^2 = 0$, so $r = 3$ and the general solution is $y = (c_1 + c_2x)\exp(3x)$. Differentiation gives $y' = c_2\exp(3x) + (c_1 + c_2x) \times 3\exp(3x)$. Substituting $y(0) = 0$ and $y'(0) = 1$ yields $0 = c_1$ and $1 = c_2$. Thus, the particular solution is $y = \exp(3x)$.

17. (a) $\beta^2 - 4\omega^2 = \pi^2/256 - 4(\pi^2/4) < 0$, so the spring is underdamped.

(b)



The general solution is $x = (c_1\cos \bar{\omega}t + c_2\sin \bar{\omega}t)\exp(-\pi t/32)$ where $\bar{\omega} = \sqrt{4\omega^2 - \beta^2}/2 = \sqrt{255}\pi/32$. $x' = (-c_1\bar{\omega}\sin \bar{\omega}t + c_2\bar{\omega}\cos \bar{\omega}t)\exp(-\pi t/32) + (c_1\cos \bar{\omega}t + c_2\sin \bar{\omega}t)(-\pi/32)\exp(-\pi t/32)$.

17. (b) (continued)

Substituting $x(0) = 0$ and $x'(0) = 1$ yields $0 = c_1$ and $1 = c_2\bar{\omega}$, so $c_2 = 1/\bar{\omega}$. Thus, $x = (1/\bar{\omega})(\sin \bar{\omega}t)\exp(-\pi t/32)$.

21. The general solution is $y = y_p + y_h$, where y_h is the general solution of the homogeneous equation and y_p is a particular solution of the nonhomogeneous equation. The characteristic equation is $r^2 - 4r + 3 = (r - 3)(r - 1) = 0$, so the general solution of the homogeneous equation is $y = c_1\exp(3x) + c_2\exp(x)$. A particular solution should have the form $y = Ax + B$, so $y' = A$ and $y'' = 0$. Substitution gives $-4A + 3(Ax + B) = 3Ax + (-4A + 3B) = 6x + 10$; therefore, $A = 2$ and $B = 6$. Hence, the solution is $y = c_1\exp(3x) + c_2\exp(x) + 2x + 6$.

25. The characteristic equation is $r^2 - 4r + 5$. The roots are given by $(4 \pm \sqrt{16 - 20})/2 = 2 \pm 2i$, so the general solution of the characteristic equation is $y = c_1\exp[(2 + 2i)x] + c_2\exp[(2 - 2i)x] = e^{2x} \times [c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x)] = e^{2x}(C_1\cos 2x + C_2\sin 2x)$, where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$. A particular solution should have the form $y = Ax^2 + Bx + D$, so $y' = 2Ax + B$ and $y'' = 2A$. Substitution yields $2A - 4(2Ax + B) + 5(Ax^2 + Bx + D) = 5Ax^2 + (-8A + 5B)x + (2A - 4B + 5D) = x^2 + x$, i.e., $A = 1/5$, $B = 13/25$, and $D = 42/125$. Hence, the solution is $y = e^{2x}(C_1\cos 2x + C_2\sin 2x) + x^2/5 + 13x/25 + 42/125$.

29. The method of variation of parameters gives a particular solution, $y_p = v_1y_1 + v_2y_2$, where y_1 and y_2 are solutions of the homogeneous equation, and v_1 and v_2 are found by solving $v_1'y_1 + v_2'y_2 = 0$ and $v_1'y_1' + v_2'y_2' = F/a$. From Exercise 21, $y_1 = e^{3x}$ and $y_2 = e^x$, so we get $v_1'e^{3x} + v_2'e^x = 0$ and $v_1'(3e^{3x}) + v_2'e^x = (6x + 10)/1$. Subtraction

29. (continued)

gives $2v_1'e^{3x} = 6x + 10$, i.e., $v_1' = (3x + 5)e^{-3x}$. Thus, by letting $u = 3x + 5$ and integrating by parts, we get $v_1 = -(3x + 5)e^{-3x}/3 + \int e^{-3x}dx = -(3x + 5)e^{-3x}/3 - e^{-3x}/3$. Similarly, $2v_2'e^{x} = -(6x + 10)$, i.e., $v_2' = -(3x + 5)e^{-x}$. Integration by parts with $u = -(3x + 5)$ yields $(3x + 5)e^{-x} - \int 3e^{-x}dx = (3x + 5)e^{-x} + 3e^{-x}$. Therefore, the general solution is $y = c_1\exp(3x) + c_2\exp(x) + [(-x - 2)\exp(-3x)] \times \exp(3x) + [(3x + 8)\exp(-x)]\exp(x) = c_1\exp(3x) + c_2\exp(x) + 2x + 6$.

33. From Exercise 21, $y_1 = \exp(3x)$ and $y_2 = \exp(x)$, so $v_1'\exp(3x) + v_2'\exp(x) = 0$ and $3v_1'\exp(3x) + v_2'\exp(x) = \tan x$. Subtraction yields $2v_1'\exp(3x) = \tan x$, i.e., $v_1' = (\tan x)\exp(-3x)/2$, and so $v_1 = \int(\tan x)\exp(-3x)dx/2$. Similarly, we get $2v_2'\exp(x) = -\tan x$, i.e., $v_2' = (-\tan x)\exp(-x)/2$, and so $v_2 = \int(\tan x)\exp(-x)dx/2$. Thus, the solution is $y = c_1\exp(3x) + c_2\exp(x) + [\exp(3x)/2]\int(\tan x)\exp(-3x)dx - [\exp(x)/2]\int(\tan x)\exp(-x)dx$.

37. Use the method of Example 7. Try a particular solution of the form $x = C\cos t$, so $x' = -C\sin t$, and $x'' = -C\cos t = -x$. Substitution yields $3C\cos t = 3\cos t$, so $C = 1$. The solution of the homogeneous equation is $x = A\cos 2t + B\sin 2t$, so a general solution is $x = A\cos 2t + B\sin 2t + \cos t$. Differentiation yields $x' = -2A\sin 2t + 2B\cos 2t - \sin t$; therefore, $x(0) = 0$ gives $0 = A + 1$ and $x'(0) = 0$ gives $0 = 2B$, i.e., $A = -1$ and $B = 0$. Hence, the solution is $x = -\cos 2t + \cos t = 2\sin(3t/2)\sin(t/2)$ by the product formula.

41. (a) Here, $r^2 + 4r + 25 = 0$ implies $r = (-4 \pm \sqrt{16-100})/2 = -2 \pm i\sqrt{21}$, and $F_0 = 2$, $m = 1$, $k = 25$, $\omega = 5$, $\Omega = 2$, $\gamma = 4$, and $\delta = \tan^{-1}(8/21)$. In terms of sine and cosine, the solution is $x(t) = \exp(-4t)[A\sin\sqrt{21}t + B\cos\sqrt{21}t] + (2/\sqrt{505})\cos(2t + \delta)$.

41. (a) (continued)

Differentiation gives $x'(t) = -4\exp(-4t)[A \sin \sqrt{21}t + B\cos\sqrt{21}t] + \exp(-4t)[\sqrt{21}\ A \cos \sqrt{21}t - \sqrt{21}\ B \sin \sqrt{21}t] - 2(2/\sqrt{505})\sin(2t - \delta)$. $x(0) = 0$ gives $0 = B + (2/\sqrt{505})\cos \delta = B + (2/\sqrt{505})(21/\sqrt{505}) = B + 42/505$, i.e., $B = -42/505$; $x'(0) = 0$ gives $0 = -4B + \sqrt{21}\ A + (4/\sqrt{505})(8/\sqrt{505}) = 168/505 + 32/505 + \sqrt{21}\ A$, i.e., $A = -200/505\sqrt{21} = -40/101\sqrt{21}$. Thus, $x(t) = \exp(-4t)[(-40/101\sqrt{21}) \times \sin\sqrt{21}\ t - (42/505)\cos\sqrt{21}\ t] + (2/\sqrt{505})\cos(2t - \tan^{-1}(8/21))$.

(b) For large t , the graph approximates $(2/\sqrt{505})\cos(2t - \tan^{-1}(8/21))$.

45. *If the Wronksian does not vanish, then* y_1 *and* y_2 *form a fundamental set.* $y_1'(x) = r_1\exp(r_1x)$ and $y_2'(x) = \exp(r_1x) + xr_1\exp(r_1x)$. Thus, the Wronksian is $y_1(x)y_2'(x) - y_2(x)y_1'(x) = (1 + r_1x)\exp(2r_1x) - r_1x\ \exp(2r_1x) = \exp(2r_1x) \neq 0$. Therefore, y_1 and y_2 form a fundamental set and the supplement tells us that $c_1y_1 + c_2y_2$ is the general solution.

49. (a) If $y(x)$ is a solution, then, by the method of reduction of order, $v(x)y(x)$ is also a solution. By expression (18), with $y_1(x) = y(x)$ and $y_2(x) = v(x)y(x)$, we get $W(x) = y(x) \times [v'(x)y(x) + v(x)y'(x)] - v(x)y(x)y'(x) = [y(x)]^2v'(x)$. We assumed that $y(x) \neq 0$, so $W(x) = 0$ if and only if $v'(x) = 0$. However, if $v'(x) = 0$, then $v(x)$ is a constant and $v(x)y(x)$ is simply a mulitple of $y(x)$. Thus, $v'(x) \neq 0$ if $v(x)y(x)$ is another solution. Therefore, $[y(x)]^2v'(x) \neq 0$ and the fundamental set is $y(x)$ and $v(x)y(x)$.

49. (b) We substitute $y_1(x) = x^r$ and $y_2(x) = (\ln x)x^{(1-\alpha)/2}$ into expression (18) to get $W(x) = x^r\{(1/x)x^{(1-\alpha)/2} + (\ln x) \times [(1-\alpha)/2]x^{(-1-\alpha)/2}\} - (\ln x)x^{(1-\alpha)/2}(rx^{r-1}) = [(\ln x)(1-\alpha)/2 + (1 - r\ln x)]x^{r+(-1-\alpha)/2}$. Now, by assumption, $r^2 + (\alpha - 1)r + \beta = 0$, so $r = ((1-\alpha) \pm \sqrt{(\alpha-1)^2 - 4\beta})/2$, and so $W(x) = [1 \pm (\ln x)\sqrt{(\alpha-1)^2 - 4\beta/2}]x^{-\alpha\pm\sqrt{(\alpha-1)^2-4\beta/2}} \neq 0$. Therefore, x^r and $(\ln x)x^{(1-\alpha)/2}$ form a fundamental set.

The general solution for Euler's equation is $y(x) = c_1x^r + c_2(\ln x)x^{(1-\alpha)/2}$. The general solution for part (a) is $c_1y(x) + c_2v(x)y(x)$.

53. We expect a solution of the form $y = e^{rx}$, so $y' = re^{rx}$, $y'' = r^2e^{rx}$, $y''' = r^3e^{rx}$, and $y'''' = r^4e^{rx}$. Therefore, $y'''' + y = r^4e^{rx} + e^{rx} = 0 = (r^4 + 1)e^{rx}$. Solving $r^4 + 1 = 0$ yields $r = (1+i)/\sqrt{2}$, $(1-i)/\sqrt{2}$, $(-1+i)/\sqrt{2}$, and $(-1-i)/\sqrt{2}$. Therefore, the general solution is $e^{x/\sqrt{2}}(C_1\cos(x/\sqrt{2}) + C_2\sin(x/\sqrt{2})) + e^{x/\sqrt{2}}(C_3\cos(-x/\sqrt{2}) + C_4\sin(-x/\sqrt{2})) + e^{-x/\sqrt{2}}(C_5\cos(x/\sqrt{2}) + C_6\sin(x/\sqrt{2})) + e^{-x/\sqrt{2}}(C_7\cos(-x/\sqrt{2}) + C_8\sin(-x/\sqrt{2}))$. Since $\cos(x/\sqrt{2}) = \cos(-x/\sqrt{2})$ and $\sin(x/\sqrt{2}) = -\sin(x/\sqrt{2})$, the solution reduces to $y = e^{x/\sqrt{2}}(c_1\cos(x/\sqrt{2}) + c_2\sin(x/\sqrt{2})) + e^{-x/\sqrt{2}}(c_3\cos(x/\sqrt{2}) + c_4\sin(x/\sqrt{2}))$.

SECTION QUIZ

1. Find a general solution to each of the following differential equations:

(a) $2y'' - 2y' + y = 0$

(b) $y'' + 6y' + 9y = 0$

(c) $-2y'' + 3y' + 2y = 0$

2. For each of the following differential equations, guess the general form of the particular equation:

   (a) $y'' + y' + y = \cos 2x + 3$

   (b) $y'' + y' + y = x^5 - x^3 + \tan x$

   (c) $y'' + y' + y = e^{-2x} + x^2 - 2$

3. Find the general solution of $y''' - y = x$. [Hint: one solution of the homogeneous equation should be $y = \exp(rx)$.]

4. Solve $y'' - 3y' + 2y = (1 - x^2)^{-3}$ and leave your answer as an integral.

5. Francis, the fruit fly, expends energy when he accelerates; however, he gains energy by speeding between fruit trees and covering more distance since he gets more fruit juices more rapidly. Thus, his distance can be described by $-y'' + 2y' + y = 0$, where $y(0) = 0$ and $y'(0) = \sqrt{2}$. If the distance function y is a function of time t, and the average tree is 0.5 units apart, how many trees has Francis the fruit fly visited after one unit of time (assuming he begins at $t = 0$)?

ANSWERS TO PREREQUISITE QUIZ

1. $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

2. $5\sqrt{2}\exp(i\pi/4)$

3. $\cos x + i \sin x$

4. $y = 3\cos 2t + \sin 2t$

5. (a) $e^{3t}/3 + C$

   (b) $-\sin y + \cos y$

   (c) $x^3/3 + x^2 + C$

ANSWERS TO SECTION QUIZ

1. (a) $[C_1 \sin(x/2) + C_2 \cos(x/2)] \exp(x/2)$

   (b) $(c_1 + c_2 x)\exp(3x)$

   (c) $c_1 \exp(2x) + c_2 \exp(-x/2)$

2. (a) $A \cos 2x + B \sin 2x + C$

   (b) $Ax^5 + Bx^4 + Cx^3 + Dx^2 + E + F \cos x + G \sin x$

   (c) $A \exp(-2x) + Bx^2 + Cx + D$

3. $c_1 e^x + [c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2)] e^{-x/2} - x$.

4. $c_1 e^{2x} + c_2 e^x + e^{2x}\int [dx/e^{2x}(1 - x^2)^3] - e^x \int [dx/e^x(1 - x^2)^3]$

5. 42 trees

12.8 Series Solutions of Differential Equations

PREREQUISITES

1. Recall how to solve a second-order linear differential equation of the form $ay'' + by' + cy = 0$ (Section 12.7).
2. Recall how to differentiate a power series (Section 12.4).

PREREQUISITE QUIZ

1. Find the general solution $y(x)$ of $y'' + 3y' + 2y = 0$.
2. Solve $y'' - 2y' + y = 0$, $y'(0) = 0$, $y(0) = 1$.
3. Find $(d/dx)\Sigma_{i=0}^{\infty}[(2x)^i/i!]$.
4. Find $(d/dy)\Sigma_{i=0}^{\infty}[(-y)^i/(i + 2)]$.

GOALS

1. Be able to solve differential equations by using power series.

STUDY HINTS

1. Series solution. Here we seek the solution of a differential equation in the form $y = \Sigma_{i=0}^{\infty}a_i x^i$. The a_i's need to be determined. Differentiate the series and substitute into the original equation. Be sure you have changed the summation index when you differentiated. At this point, write out the first few terms and look for a pattern. Finally, the ratio test may be used to show convergence.
2. Special equations. Legendre's and Hermite's equation are special in that they are specific equations that arise from physical problems. However, the method of solution is the same.

3. Frobenius method. If the coefficient of y'' vanishes at $x = 0$, then look for a solution of the form $y = x^r\Sigma_{i=0}^{\infty}a_ix^i$. The method of solution is the same except that we now solve for the coefficient of x^{r-1}, x^r, x^{r+1}, etc., rather than x, x^2, x^3, etc. We solve for r, which is generally not an integer. See Example 5.

4. Indicial equation. The values of r in the Frobenius method is determined by setting the coefficient of the lowest power of x equal to zero. This is called the indicial (pronounced "in dish al") equation.

5. Repeated indicial roots. If the roots repeat, then the solutions have the form $y_1(x) = x^r\Sigma_{i=0}^{\infty}a_ix^i$ and $y_2(x) = y_1(x)\ln x + x^r\Sigma_{i=0}^{\infty}b_ix^i$.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Let $y = \Sigma_{i=0}^{\infty}a_ix^i$, so $y' = \Sigma_{i=1}^{\infty}ia_ix^{i-1}$ and $y'' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2}$. Then equate the coefficients of x^i to 0. $y'' - xy' - y = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2} - \Sigma_{i=1}^{\infty}ia_ix^i - \Sigma_{i=0}^{\infty}a_ix^i = \Sigma_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i - \Sigma_{i=1}^{\infty}ia_ix^i - \Sigma_{i=0}^{\infty}a_ix^i = 0$. Thus, $2a_2 - a_0 = 0$ (constant term), $6a_3 - a_1 - a_1 = 0$ (x term), $12a_4 - 2a_2 - a_2 = 0$ (x^2 term), and in general, $(i+2)(i+1)a_{i+2} - ia_i - a_i = 0$ (x^i term). Hence, $a_2 = a_0/2$, $a_3 = a_1/3$, $a_4 = a_2/4 = a_0/8$, $a_5 = a_3/5 = a_1/15$, and in general, $a_{i+2} = (i+1)a_i/(i+2)(i+1) = a_i/(i+2)$, i.e., $a_{2n} = a_0/2\cdot4\cdot\ldots\cdot2n = a_0/2^n(n!)$ and $a_{2n+1} = a_1/3\cdot5\cdot\ldots\cdot(2n+1) = a_12^n(n!)/(2n+1)!$ Therefore, the solution is $y = a_0[\Sigma_{n=0}^{\infty}(x^n/2^n(n!))] + a_1[\Sigma_{n=0}^{\infty}(2^n(n!)x^{2n+1}/(2n+1)!]$.

5. Let $y = \Sigma_{i=0}^{\infty}a_ix^i$, so $y' = \Sigma_{i=1}^{\infty}ia_ix^{i-1}$ and $y'' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2}$. $y(0) = 0$ and $y'(0) = 1$ implies $a_0 = 0$ and $a_1 = 1$. Now, $y'' + 2xy' = \Sigma_{i=2}^{\infty}i(i-1)a_ix^{i-2} + \Sigma_{i=1}^{\infty}2ia_ix^i = \Sigma_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i + \Sigma_{i=1}^{\infty}2ia_ix^i = 0$. Thus, $2a_2 = 0$ (constant term), $6a_3 + 2a_1 = 0$ (x term), $12a_4 + 4a_2 = 0$ (x^2 term), and $20a_5 + 6a_3 = 0$ (x^3 term).

5. (continued)

Hence, $a_2 = 0$, $a_3 = -a_1/3 = -1/3$, $a_4 = -a_2/3 = 0$, $a_5 = -3a_3/10 = 1/10$. Therefore, the solution is $y = x - x^3/3 + x^5/10 - \ldots$.

9. Let $y = \Sigma_{i=0}^{\infty} a_i x^i$, so $y' = \Sigma_{i=1}^{\infty} i a_i x^{i-1}$ and $y'' = \Sigma_{i=2}^{\infty} i(i-1)a_i x^{i-2}$. $y'' - xy = 0 = \Sigma_{i=2}^{\infty} i(i - 1)a_i x^{i-2} - \Sigma_{i=0}^{\infty} a_i x^{i+1} = 2a_2 + \Sigma_{i=1}^{\infty}[(i + 2)\times(i + 1)a_{i+2}x^i - a_{i-1}x^i]$. Thus, $2\cdot 1a_2 = 0$ (constant term) , $3\cdot 2a_3 - a_0 = 0$ (x term) , $4\cdot 3a_4 - a_1 = 0$ (x^2 term) , $5\cdot 4a_5 - a_2 = 0$ (x^3 term) , $6\cdot 5a_6 - a_3 = 0$ (x^4 term) , $7\cdot 6a_7 - a_4 = 0$ (x^5 term) , and in general $(i + 2)(i + 1)a_{i+2} - a_{i-1}$ (x^i term). Hence, $a_2 = 0$, $a_3 = a_0/6$, $a_4 = a_1/12$, $a_5 = a_2/20 = 0$, $a_6 = a_3/30 = a_0/180$, $a_7 = a_4/42 = a_1/504$, and in general, $a_{i+3} = a_i/(i + 3)(i + 2)$. Therefore, the solution is $y = a_0(1 + x^3/6 + x^6/180 + \ldots) + a_1(x + x^4/12 + x^7/504 + \ldots)$. The recursion formula is $a_{i+3} = a_i/(i + 3)(i + 2)$

13. Let $y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \ldots$, so $y' = ra_0x^{r-1} + (r + 1)a_1x^r + (r + 2)a_2x^{r+1} + \ldots$ and $y'' = r(r - 1)a_0x^{r-2} + r(r + 1)a_1x^{r-1} + (r + 2)\times(r + 1)a_2x^r + \ldots$. Thus, $3x^2y'' + 2xy' + y = [3r(r - 1)a_0 + 2ra_0 + a_0]x^r + [3r(r + 1)a_1 + 2(r + 1)a_1 + a_1]x^{r+1} + \ldots = 0$. Setting the coefficient of x^r equal to 0 yields $a_0(3r^2 - r + 1) = 0$, i.e., $r = (1 \pm \sqrt{11}i)/6$. For the x^{r+1} coefficient, we set $a_1(3r^2 + 5r + 3) = 0$, which yields $a_1 = 0$ because r must be $(1 \pm \sqrt{11}i)/6$. Similarly, $a_2 = a_3 = \ldots = 0$. Thus, $y = a_0x^{(1+\sqrt{11}i)/6}$ is one solution and $y = a_0x^{(1-\sqrt{11}i)/6}$ is another solution. The general solution is $y = c_1x^{(1+\sqrt{11}i)/6} + c_2x^{(1-\sqrt{11}i)/6}$. Note that $x^{(1+\sqrt{11}i)/6} = e^{(\ln x)(1+\sqrt{11}i)/6} = e^{(\ln x)/6}e^{(\sqrt{11}\ln x)i/6} = x^{1/6}(\cos(\sqrt{11}\ln x/6) + i\sin(\sqrt{11}\ln x/6))$. Therefore, an equivalent solution is $x^{1/6}[b_1\cos(\sqrt{11}\ln x/6) + b_2\sin(\sqrt{11}\ln x/6)]$.

17. Let $y = \Sigma_{n=0}^{\infty} a_n x^n$, so $y' = \Sigma_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} [n(n-1) \times a_n x^{n-2}]$. Thus, $y'' + \omega^2 y = \Sigma_{j=0}^{\infty} [(j+2)(j+1)a_{j+2}x^j + \omega^2 a_j x^j]$. If $j = 0$, then $2a_2 + \omega^2 a_0 = 0$ or $a_2 = -\omega^2 a_0/2$. If $j = 1$, then $6a_3 + \omega^2 a_1 = 0$ or $a_3 = -\omega^2 a_1/6$. If $j = 2$, then $12a_4 + \omega^2 a_2 = 0$ or $a_4 = -\omega^2 a_2/12 = \omega^4 a_0/24$. If $j = 3$, then $20a_5 + \omega^2 a_3 = 0$ or $a_5 = -\omega^2 a_3/20 = \omega^4 a_1/120$. In general, $a_{2n} = (-1)^{2n}\omega^{2n} a_0/(2n)!$ and $a_{2n+1} = (-1)^{2n}\omega^{2n} a_1/(2n+1)!$. Thus, the general solution is $y = a_0(1 - \omega^2 x^2/2! + \omega^4 x^4/4! - \ldots) + a_1(x - \omega^2 x^3/3! + \omega^4 x^5/5! - \ldots)$. We recognize $1 - \omega^2 x^2/2! + \omega^4 x^4/4! - \ldots$ as the Maclaurin series of $\cos \omega x$, and we recognize $x - \omega^2 x^3/3! + \omega^4 x^5/5! - \ldots$ as the Maclaurin series of $\sin \omega x/\omega$. Thus, the solution is $y = A \cos \omega x + B \sin \omega x$, where $A = a_0$ and $B = a_1/\omega$.

21. In Example 3, we let $y_1 = 1 - (\lambda/2)x^2 - [(6-\lambda)/4\cdot 3\cdot 2]x^4 + \ldots$ and we let $y_2 = x + [(2-\lambda)/3\cdot 2]x^3 + [(12-\lambda)(2-\lambda)/5\cdot 4\cdot 3\cdot 2]x^5 + \ldots$. Thus, $y_1' = -\lambda x - [(6-\lambda)/3\cdot 2]x^3 + \ldots$ and $y_2' = 1 + [(2-\lambda)/2]x^2 + [(12-\lambda)(2-\lambda)/4\cdot 3\cdot 2]x^4 + \ldots$. Since all of the series do converge, they can be multiplied, so the Wronskian is $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 1 + x^2 + (18 - 25\lambda + 7\lambda^2)x^4/24 + \ldots - [-\lambda x^2 + (\lambda^2 - \lambda - 6) \times x^4/6 + \ldots] = 1 + (1+\lambda)x^2 + \ldots$. Note that all of the higher terms have even exponents, so $W(x) \geqslant 1 \neq 0$. Therefore, y_1 and y_2 form a fundamental set.

SECTION QUIZ

1. Find the first few terms of the power series solutions for $y''' - x^2y'' + y = 0$.

2. (a) Find a power series solution for $y'' - 2y' = 0$.

   (b) Use the methods of Section 12.7 to solve $y'' - 2y' = 0$.

   (c) What special equation do you get by equating your answers to parts (a) and (b)?

3. An obnoxious travelling salesman has made you extremely irritated. Consequently, he irks you into slamming the spring door into his face. Due to a defect, the force exerted by the spring is given by $2y'' - 3y' + xy = 0$, where y is a function of x , the door's position. Find a power series for $y(x)$ if $y(0) = y'(0) = 1$

ANSWERS TO PREREQUISITE QUIZ

1. $y = c_1\exp(-2x) + c_2\exp(-x)$.
2. $y = (1 - x)e^x$
3. $\Sigma_{i=1}^{\infty}[2^i(2x)^{i-1}/(i - 1)!]$
4. $\Sigma_{i=1}^{\infty}[i(-1)^i(-y)^{i-1}/(i + 2)]$

ANSWERS TO SECTION QUIZ

1. $a_0(1 - x^3/6 - 5x^6/720 - \ldots) + a_1(x - x^4/24 - 11x^7/5040 - \ldots) + a_2(x^2 + x^5/60 + 19x^8/20160 + \ldots)$

2. (a) $a_0\Sigma_{n=0}^{\infty}[(2x)^n/n!]$

   (b) $y = c_1\exp(2x) + c_2$

   (c) $e^{2x} = \Sigma_{n=0}^{\infty}[(2x)^n/n!]$

3. $1 + x - 3x^2/4 + 7x^3/24 - 87x^4/576 + \ldots$

12.R Review Exercises for Chapter 12

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. This is a geometric series, so it converges. $\Sigma_{i=1}^{\infty}(1/12^i) = \Sigma_{i=0}^{\infty}(1/12)\times(1/12)^i = (1/12)/(1 - 1/12) = 1/11$.

5. $1 + 2 + 1/3 + 1/3^2 + 1/3^3 + \ldots = 2 + \Sigma_{i=0}^{\infty}(1/3)^i = 2 + 1/(1 - 1/3) = 2 + 3/2 = 7/2$, so it converges.

9. $\Sigma_{n=1}^{\infty}5^{-n} = \Sigma_{n=1}^{\infty}(1/5)^n$ converges since it is a geometric series with $r < 1$.

13. Let $a_n = (-1)^n n/3^n$. Then, $\lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}\{[n/(n - 1)]/3\} = 1/3$. Therefore, $\Sigma_{n=1}^{\infty}a_n$ converges by the ratio test.

17. Let $a_n = 2^{(n^2)}/n!$. Then $\lim_{n\to\infty}|a_n/a_{n-1}| = \lim_{n\to\infty}[2^{n^2-(n-1)^2}/n] = \lim_{n\to\infty}(2^{2n-1}/n) = \infty$. Therefore, $\Sigma_{n=1}^{\infty}(2^{(n^2)}/n!)$ diverges by the ratio test.

21. Let $a_n = 1/(\ln n)^{\ln n}$, so $\ln a_n = -\ln n(\ln\ln n)$. Since $\ln n$ is an increasing function, $\ln\ln n \geqslant 2$ if $n \geqslant e^{(e^2)}$. Thus, for such n , $a_n \leqslant 1/n^2$, so the series converges by comparison with $b_n = 1/n^2$.

25. The error made in estimating the sum of an alternating series is no greater than $|a_{n+1}|$. $1 - 1/4 + 1/16 - 1/32 + \ldots \approx 0.78$. Although there is no formula for the general term, we note that $1/32 < 0.05$, so the sum is approximately 0.78 .

29. $r = |a_n/a_{n-1}| = |(1 - n)3^{n-1}/(2 - n)3^n| = |(1 - n)/(2 - n)3|$. We want to find N such that $|[(1 - N)/3^N]r/(1 - r)| < 0.05$. The error is less than $(1 - N^2)/3^N(5 - 2N)$, which is less than 0.05 for $N \geqslant 5$. Thus, $\Sigma_{n=1}^{\infty}[(1 - n)/3^n] \approx 0 - 1/9 - 2/27 - 3/81 - 4/243 - \ldots \approx -0.24$.

33. False; consider $\Sigma_{n=1}^{\infty}(1/n)$.

37. False; $e^{2x} = \Sigma_{i=0}^{\infty}[(2x)^i/i!] = 1 + 2x + 2x^2 + 4x^3/3 + \ldots$.

41. True; by the sum rule, $\Sigma_{j=1}^{\infty}a_j + \Sigma_{k=0}^{\infty}b_k = b_0 + \Sigma_{j=1}^{\infty}a_j + \Sigma_{k=1}^{\infty}b_k =$ $b_0 + \Sigma_{i=1}^{\infty}(a_i + b_i)$.

45. True; $|a_n| \leqslant |a_n| + |b_n|$. Therefore, by the comparison test, the convergence of $\Sigma_{n=1}^{\infty}(|a_n| + |b_n|)$ implies the convergence of $\Sigma_{n=1}^{\infty}|a_n|$.

49. This is a geometric series, so $\Sigma_{n=1}^{\infty}(1/9^n) = \Sigma_{n=0}^{\infty}(1/9)(1/9)^n =$ $(1/9)/(1 - 1/9) = 1/8$.

53. Let $a_i = (-1)^i x^{2i}/(i + 1)!$. Then $\lim_{i\to\infty}|a_i/a_{i-1}| = \lim_{i\to\infty}[x^2/(i + 1)] =$ 0 . Therefore, the radius of convergence is ∞ by the ratio test.

57. Let $a_n = (-1)^n/2^n$. Then $\ell = \lim_{n\to\infty}|a_n/a_{n-1}| = 1/2$. Therefore, the radius of convergence is $1/\ell = 2$.

61. Since $\ln(1 + x) = \Sigma_{i=1}^{\infty}[(-1)^{i+1}x^i/i]$, we have $\ln(1 + x^4) =$ $\Sigma_{i=1}^{\infty}[(-1)^{i+1}x^{4i}/i]$.

65. We have $e^t = \Sigma_{i=0}^{\infty}(t^i/i!)$, so $e^t - 1 = \Sigma_{i=1}^{\infty}(t^i/i!)$. Then $(e^t - 1)/t = \Sigma_{i=1}^{\infty}(t^{i-1}/i!)$; therefore, $\int_0^x[(e^t - 1)/t]dt =$ $\Sigma_{i=1}^{\infty}[x^i/i(i!)]$.

69. If $f(x) = x^{3/2}$, then $f'(x) = (3/2)x^{1/2}$, $f''(x) = (3/2)(1/2)x^{-1/2}$, $f'''(x) = (3/2)(1/2)(-1/2)x^{-3/2}$, and in general, $f^{(n)}(x) =$ $[(3)(1)(-1)\cdot\ldots\cdot(3 - 2(n - 1))/2^n]x^{(3-2n)/2}$. Evaluating at $x = 1$, we get the coefficients since one to any power is one, i.e., $f(1) =$ 1 ; $f'(1) = 3/2$; $f''(1) = 3/4$; ... $f^{(n)}(1) = (3)(1)(-1) \ldots$ $(5 - 2n)/2^n$. Therefore, the Taylor expansion of $x^{3/2}$ about $x = 1$ is $1 + (3/2)(x - 1) + [(3/4)/2!](x - 1)^2 + [f^{(n)}(1)/n!](x - 1)^n =$ $\Sigma_{n=0}^{\infty}[(3)(1)(-1)\cdot\ldots\cdot(5 - 2n)/2^n n!](x - 1)^n$. Alternatively, one may use the binomial series $(1 + u)^{\alpha} = \Sigma_{i=0}^{\infty}[\alpha(\alpha - 1)\cdots(\alpha - i + 1)u^i/i!]$ with $\alpha = 3/2$ and $u = x - 1$.

By the ratio test, $|a_n/a_{n-1}| = |[(3 - 2n)(5 - 2n)/2^n n!][2^{n-1}(n - 1)!/$

69. (continued)

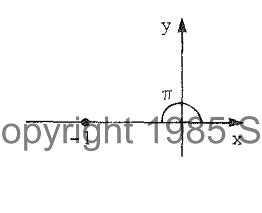
$(3 - 2n)]| = |(5 - 2n)/2n|$. The radius of convergence is the reciprocal of $\lim_{n\to\infty}|a_n/a_{n-1}| = 1$, so $R = 1$.

73. The Maclaurin series for $(1 + x)^{3/2}$ is $1 + 3x/2 + 3x^2/8 - x^3/16 + 3x^4/128 - 3x^5/256 + \ldots$ and $(1 - x)^{3/2}$ is $1 - 3x/2 + 3x^2/8 + x^3/16 + 3x^4/128 + 3x^5/256 + \ldots$. Therefore, $(1 + x)^{3/2} - (1 - x)^{3/2} = 3x - x^3/8 - 3x^5/128 - \ldots$. Dividing by x gives $3 - x^2/8 - 3x^4/128 - \ldots$, so the limit as x approaches 0 is 3 .

77. The real part of $a + bi$ is a ; the imaginary part is b ; the complex conjugate is $a - bi$; and the absolute value is $\sqrt{a^2 + b^2}$. If $\sqrt{2 - i} = a + bi$, then $2 - i = (a + bi)^2 = a^2 - b^2 + 2abi$. Thus, $a^2 - b^2 = 2$ and $2ab = -1$, i.e., $b = -1/2a$ and so $a^2 - 1/4a^2 = 2$, i.e., $4a^4 - 1 = 8a^2$. Rearrangement gives $4a^4 - 8a^2 = 1$ and completing the square gives $4(a^4 - 2a^2 + 1) = 5 = 4(a^2 - 1)^2$, i.e., $\pm\sqrt{5/4} = a^2 - 1$, so $a = \pm(\pm\sqrt{5/4} + 1)^{1/2}$. $b = -1/2a$, so $\sqrt{2 - i} = (\sqrt{5/4} + 1)^{1/2} - (1/2(\sqrt{5/4} + 1)^{1/2})i$ and $(-\sqrt{5/4} + 1)^{1/2} - (1/2(-\sqrt{5/4} + 1)^{1/2})i$.

Thus, if $z = (\sqrt{5/4} + 1)^{1/2} - (1/2(\sqrt{5/4} + 1)^{1/2})i$, then the real part is $(\sqrt{5/4} + 1)^{1/2}$; the imaginary part is $-(1/2(\sqrt{5/4} + 1)^{1/2})$; the complex conjugate is $(\sqrt{5/4} + 1)^{1/2} + (1/2(\sqrt{5/4} + 1)^{1/2})i$; and the absolute value is $\sqrt[4]{5}$.

If $z = (-\sqrt{5/4} + 1)^{1/2} - (1/2(-\sqrt{5/4} + 1)^{1/2})i$, then the real part is $(-\sqrt{5/4} + 1)^{1/2}$; the imaginary part is $-(1/2(-\sqrt{5/4} + 1)^{1/2})i$; the complex conjugate is $(-\sqrt{5/4} + 1)^{1/2} + (1/2(-\sqrt{5/4} + 1)^{1/2})i$; and the absolute value is $\sqrt[4]{5}$.

81. 

$\exp(\pi i/2) = i$ and $i^2 = -1$, so $r = 1$ and $\theta = \pi$. Thus, $z = \exp(\pi i)$.

85. The characteristic equation is $r^2 + 4 = 0$, which has the roots $\pm 2i$. Thus, the general solution is $y = c_1\exp(2ix) + c_2\exp(-2ix) = c_1(\cos 2x + i \sin 2x) + c_2(\cos(-2x) + i \sin(-2x)) = c_1(\cos 2x + i \sin 2x) + c_2(\cos 2x - i \sin 2x) = (c_1 + c_2)\cos 2x + i(c_1 - c_2)\sin 2x = C_1\cos 2x + C_2\sin 2x$, where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

89. The characteristic equation is $r^2 + 3r - 10 = 0$, whose solution is given by $r = (-3 \pm \sqrt{9 + 40})/2 = (-3 \pm 7)/2 = -5$ or 2. Thus, the solution to the homogeneous equation is $y = c_1\exp(-5x) + c_2\exp(2x)$. Let $Ae^x + B \cos x + C \sin x = y$, so $y' = Ae^x - B \sin x + C \cos x$, and $y'' = Ae^x - B \cos x - C \sin x$. Therefore, $y'' + 3y' - 10y = -6Ae^x + (3C - 11B)\cos x + (-3B - 11C)\sin x = e^x + \cos x$, i.e., $-6A = 1$, $3C - 11B = 1$ and $-3B - 11C = 0$. Hence, $A = -1/6$, $C = -3B/11$, so $-9B/11 - 11B = -130B/11 = 1$, i.e., $B = -11/130$ and $C = 33/130$. Thus, the general solution of the nonhomogeneous equation is $y = -e^x/6 - 11 \cos x/130 + 33 \sin x/130 + c_1\exp(-5x) + c_2\exp(2x)$.

93. The characteristic equation for $y'' + 4y = 0$ is $r^2 + 4 = 0$, so $r = \pm 2i$. Thus, the general solution to the homogeneous equation is $y = c_1\cos 2x + c_2\sin 2x$. To obtain a particular solution, we use the method of variation of parameters. Thus, we have $y_1' = \cos 2x$ and $y_2' = \sin 2x$, and we must solve $v_1'\cos 2x + v_2'\sin 2x = 0$ and $-2v_1'\sin 2x + 2v_2'\cos 2x = x/\sqrt{x^2 + 1}$ simultaneously. This yields $v_1 = \int[-x \sin 2x/2\sqrt{x^2 + 1}]dx$ and $v_2 = \int[x \cos 2x/2\sqrt{x^2 + 1}]dx$. Therefore, the general solution is $y = c_1\cos 2x + c_2\sin 2x + \cos 2x\int[-x \sin 2x/2\sqrt{x^2 + 1}]dx + \sin 2x\int[x \cos 2x/2\sqrt{x^2 + 1}]dx$. Since the constants c_1 and c_2 are incorporated into the integrals, the solution simplifies to $y = -\cos 2x\int[x \sin 2x/2\sqrt{x^2 + 1}]dx + \sin 2x\int[x \cos 2x/2\sqrt{x^2 + 1}]dx$.

97. The equation has the form of the damped forced oscillation equation which is discussed in the box on p. 628. Here, we have $m = 1$, $k = 9$, $\gamma = 1$, $F_0 = 1$, and $\Omega = 2$. The solution of the characteristic equation $r^2 + r + 9 = 0$ is $r = (-1 \pm \sqrt{35}\, i)/2$. Thus, the solution of the homogeneous equation is $e^{-t/2}(c_1\cos\sqrt{35}t + c_2\sin\sqrt{35}t)$. This transient part approaches zero as t approaches ∞.

For the particular equation, $\omega = \sqrt{k/m} = 3$ and $\delta = \tan^{-1}[\gamma\Omega/m(\omega^2 - \Omega^2)] = \tan^{-1}(2/5) \approx 0.38$. So $F_0\cos(\Omega t - \delta)/\sqrt{m^2(\omega^2 - \Omega^2)^2 + \gamma^2\omega^2} = \cos(2t - 0.38)/\sqrt{34}$. This is the limiting behavior as $t \to \infty$.

101. If $y = \Sigma_{n=0}^{\infty} a_n x^n$, then $y' = \Sigma_{n=1}^{\infty} na_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} n(n-1) \times a_n x^{n-2}$. Thus, $y'' + 2xy = \Sigma_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\Sigma_{n=0}^{\infty} a_n x^{n+1} = 2a_2 + \Sigma_{j=1}^{\infty}[(j+2)(j+1)a_{j+2} + 2a_{j-1}]x^j = 0$. This gives us $a_2 = 0$. If $j = 1$, then $6a_3 + 2a_0 = 0$ or $a_3 = -a_0/3$. If $j = 2$, then $12a_4 + 2a_1 = 0$ or $a_4 = -a_1/6$. If $j = 3$, then $20a_5 + 2a_2 = 0$ or $a_5 = -a_2/10 = 0$. If $j = 4$, then $30a_6 + 2a_3 = 0$ or $a_6 = -a_3/15 = a_0/45$. Therefore, the general solution is $a_0(1 - x^3/3 + x^6/45 - \ldots) + a_1(1 - x^4/6 + \ldots)$.

105. If $y = \Sigma_{n=0}^{\infty} a_n x^n$, then $y' = \Sigma_{n=1}^{\infty} na_n x^{n-1}$, and $y'' = \Sigma_{n=2}^{\infty} n(n-1) \times a_n x^{n-2}$. Thus, $5x^2y'' + y' + y = 0 = 5\Sigma_{n=2}^{\infty} n(n-1)a_n x^n + \Sigma_{n=1}^{\infty} na_n x^{n-1} + \Sigma_{n=0}^{\infty} a_n x^n = (a_0 + a_1) + (a_1 + 2a_2)x + \Sigma_{j=2}^{\infty}[5j(j-1)a_j + (j+1)a_{j+1} + a_j]x^j$. This gives us $a_1 = -a_0$, $a_2 = -a_1/2 = a_0/2$. If $j = 2$, then $11a_2 + 3a_3 = 0$ or $a_3 = -11a_2/3 = -11a_0/6$. If $j = 3$, then $31a_3 + 4a_4 = 0$ or $a_4 = -31a_3/4 = 343a_0/24$. Thus, one solution is $a_0(1 - x + x^2/2 - 11x^3/6 + 343x^4/24 + \ldots)$.

109. (a) The equation may be rewritten as $L(d^2I/dt^2) = -I/C - R(dI/dt) + dE/dt$. Thus, according to the box on p. 628, this is a damped spring equation if $m = L$, $k = 1/C$, and $\gamma = R$.

109. (b) The characteristic equation of $5(d^2I/dt^2) + 100(dI/dt) + I/0.1 = 0$ is $5r^2 + 100r + 10 = 0 = r^2 + 20r + 2$, so $r = (-20 \pm \sqrt{392})/2 = -10 \pm \sqrt{98}$. Thus, the solution of the homogeneous equation is $I = c_1\exp(-10 - \sqrt{98})t + c_2\exp(-10 + \sqrt{98})t$.

Now, $dE/dt = -120\pi \sin(60\pi t)$, so by the method of undetermined coefficients, we guess that a particular solution is $I_p = A \sin \Omega t + B \cos \Omega t$, where $\Omega = 60\pi$. The original equation is equivalent to $d^2I/dt^2 + 20(dI/dt) + 2I = -24\pi\sin(60\pi t)$. Differentiating I_p and substituting into the differential equation yields $(2A - 20B\Omega - A\Omega^2)\sin \Omega t + (2B + 20A\Omega - B\Omega^2)\cos \Omega t = -(2\Omega/5)\sin \Omega t$. Solving for A and B yields $A = 2\Omega(\Omega^2 - 2)/5[(\Omega^2 - 2)^2 + 400\Omega^2]$ and $B = 8\Omega^2/[(\Omega^2 - 2)^2 + 400\Omega^2]$. Therefore, the solution is $I = c_1\exp(-10 - \sqrt{98})t + c_2\exp(-10 + \sqrt{98})t + A \sin 60\pi t + B \cos 60\pi t$. Using $I(0) = 0$ yields $c_1 + c_2 + B = 0$ and using $I'(0) = 0$ yields $c_1(-10 - \sqrt{98}) + c_2(-10 + \sqrt{98}) + 60\pi A = 0$. Solving for c_1 and c_2 yields $c_1 = [60\pi A + (6 - \sqrt{98})B]/(-2 + 2\sqrt{98})$ and $c_2 = [-60\pi A - (8 + \sqrt{98})B]/(-2 + 2\sqrt{98})$. Therefore, $I(t) = 0.02217 \exp(-19.90t) - 0.02245 \exp(-0.1005t) + 0.002099 \sin 60\pi t + 0.0002227 \cos 60\pi t$.

113. (a) For any fixed value of x and t, let $S_n = \Sigma_{i=1}^{n} A_i \sin(i\pi x/L) \times \cos(i\pi ct/L)$. If the sequence of partial sums $S_1, S_2, \ldots$ converges to a limit, then the series converges for this value of x and t, and $\lim_{n\to\infty} S_n = y(x,t)$.

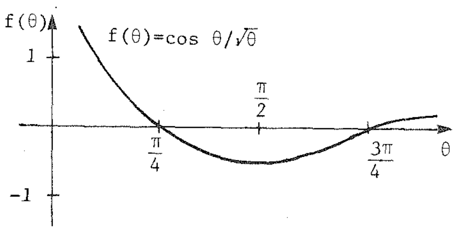
(b) For $t = 0$, $x = L/2$, the deflection of the string is $\Sigma_{n=1}^{\infty} A_n \sin[(n\pi L/2)/L] = \Sigma_{n=1}^{\infty} A_n \sin(n\pi/2) = \Sigma_{k=0}^{\infty} (-1)^k A_{2k+1}$.

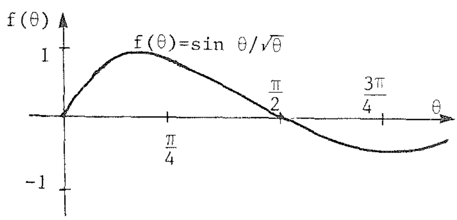
117. (a) Using the binomial series, with $\alpha = 1/2$, $x = 1/4$, $\sqrt{5/4} \approx 1 + (1/2)(1/4) + (1/2)(-1/2)(1/4)^2/2 \approx 1.12$.

(b) $\sqrt{5} = \sqrt{4(5/4)} = 2\sqrt{5/4} \approx 2(1.12) = 2.24$. This is accurate within $2(0.01) = 0.02$.

121. (a) Expanding, $\cos\theta/\sqrt{\theta} = \theta^{-1/2} - (1/2!)\theta^{3/2} + (1/4!)\theta^{7/2} - \ldots$ and $\sin\theta/\sqrt{\theta} = \theta^{1/2} - (1/3!)\theta^{5/2} + (1/5!)\theta^{9/2} - \ldots$. Thus, $\int_0^{\pi/4}(\cos\theta/\sqrt{\theta})d\theta = 2(\pi/4)^{1/2} - (2/5\cdot 2!)(\pi/4)^{5/2} + (2/9\cdot 4!) \times (\pi/4)^{9/2} - \ldots$ and $\int_0^{\pi/4}(\sin\theta/\sqrt{\theta})d\theta = (2/3)(\pi/4)^{3/2} - (2/7\cdot 3!) \times (\pi/4)^{7/2} + (2/11\cdot 5!)(\pi/4)^{11/2} - \ldots$. Taking the ratio and simplifying gives $x/y = (4/\pi)([1 - (1/5\cdot 2!)(\pi/4)^2 + (1/9\cdot 4!) \times (\pi/4)^4 - \ldots]/[1/3 - (1/7\cdot 3!)(\pi/4)^2 + (1/11\cdot 5!)(\pi/4)^4 - \ldots])$.
Using just the leading terms, $x/y \approx (4/\pi)(1/(1/3)) = 12/\pi \approx 3.8$.
Using the first two terms in the numerator and denominator gives $x/y \approx (4/\pi)(1 - (1/10)(\pi/4)^2)/(1/3 - (1/42)(\pi/4)^2) \approx 3.68$.

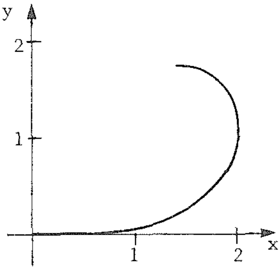
(b) The graphs of $\cos\theta/\sqrt{\theta}$ and $\sin\theta/\sqrt{\theta}$ are shown below. x and y are given parametrically as the area under these curves when $k = 1$.





The transitional spiral is drawn below for $k = 1$ and $0 \leq \phi \leq \pi$. As seen from the graphs above, x will continue to decrease until $\phi = 3\pi/2$. The y-value is just starting to decrease at $\phi = \pi$. Some of the points we have plotted are $(0,0)$ for $\phi = 0$, $(1.23, 0.16)$ for $\phi = \pi/8$, $(1.66, 0.44)$ for $\phi = \pi/4$,

121. (b) (continued)



(1.95,1.09) for $\phi = \pi/2$, (1.75, 1.61) for $\phi = 3\pi/4$, and (1.31, 1.77) for $\phi = \pi$. We approximated the sine and cosine functions with the first five nonzero terms of the Maclaurin series.

125. True; $\Sigma_{n=1}^{\infty}[(a_n^2 + b_n^2)/2]$ converges by the sum rule. Since $a_n^2 - 2|a_n||b_n| + b_n^2 = (|a_n| - |b_n|)^2 \geqslant 0$, $|a_n||b_n| \leqslant (a_n^2 + b_n^2)/2$. Therefore, $\Sigma_{n=1}^{\infty}|a_n b_n|$ converges or $\Sigma_{n=1}^{\infty}a_n b_n$ converges absolutely.

129. Assume that $e = a/b$ for some integers a and b , so e is rational. Now, let $k > b$ and let $\alpha = k!(e - 2 - 1/2! - 1/3! - \ldots - 1/k!) = k!(a/b) - 2k! - k!/2! - k!/3! - \ldots - 1$. Since $k > b$, α is an integer. The quantity $e - 2 - 1/2! - 1/3! - \ldots -1/k!$ is simply $e - \Sigma_{n-0}^{k}(1/n!)$. Using the Maclaurin expansion of e , α becomes $k![1/(k + 1)! + 1/(k + 2)! + \ldots] = (k!/k!)[1/(k + 1) + 1/((k + 2) \times (k + 1)) + \ldots] < \Sigma_{n=1}^{\infty}(k + 1)^{-n}$. The latter is a geometric series whose sum is $[1/(k + 1)]/[1 - 1/(k + 1)] = 1/k$. Thus, $\alpha < 1/k$, so α is not an integer, a contradiction. Therefore, e is irrational.

TEST FOR CHAPTER 12

1. True or false.

   (a) If $\Sigma_{i=0}^{\infty} a_i$ diverges and $\Sigma_{i=0}^{\infty} b_i$ also diverges, then $\Sigma_{i=0}^{\infty}(a_i + b_i)$ also diverges.

   (b) The derivative of $\Sigma_{i=1}^{\infty}(x^{i+1}/i!)$ is $\Sigma_{i=1}^{\infty}[(i + 1)x^i/i!]$.

   (c) Any series that converges absolutely at $x = x_0$ also converges conditionally at that point.

   (d) Any equation of the form $at^2 + b(t + 2) - c = 0$, where a , b , and c are real constants, has at least one solution in the complex number system.

   (e) The series $\Sigma_{j=1}^{\infty}[(-1)^3/j]$ converges as an alternating series.

2. Suppose that $h(t)$ is a third degree polynomial and that $h(0) = -2$, $h'(0) = 8$, $h''(0) = -18$, $h'''(0) = 24$.

   (a) Find the Maclaurin series for $h(x)$.

   (b) Find the Taylor series expansion for h around $x = 1$.

   (c) Find an equivalent expression with the form $a(x - x_0)^3 + b(x - x_0) + c$, where a , b , c and x_0 are constants.

3. Find values of x for which $\Sigma_{n=5}^{\infty} n^4(x + 1)^n$:

   (a) converges absolutely

   (b) converges conditionally

   (c) diverges

4. Discuss the convergence or divergence of $\Sigma_{n=1}^{\infty}(a_n n)^{-1}$ if:

   (a) a_n is the last digit of n , i.e., $a_{13} = 3$, $a_{145} = 5$, etc.

   (b) $a_n = n$

5. (a) Find a solution for $w(x)$ which satisfies $w'' + 4w = \cos 2x + e^x$. (Hint: A particular solution has the form $w = Ax \cos 2x + Bx \sin 2x + Ce^x$.)

   (b) Solve $y''' + 2y'' + y' = 0$ for $y(x)$.

6. Do the following situations imply convergence, divergence, or give no information? Unless otherwise specified, the series we are testing is $\Sigma_{i=1}^{\infty} a_i$, where a_i is assumed to be finite for all $i \geq 1$.

   (a) $\lim_{i\to\infty} a_i = 0$

   (b) $a_i = i^{-1.1}$

   (c) $\lim_{i\to\infty} |a_i/a_{i-1}| = 1$ and the signs are alternating.

   (d) $\lim_{i\to\infty} |a_i/a_{i+1}| > 1$ and every third sign is negative.

   (e) $\lim_{i\to\infty} (1/a_i)^{1/i} = 1/2$

7. (a) Find the fourth degree Taylor polynomial for $f(x) = \tan x$ expanded around $x = \pi$.

   (b) Find the fourth degree Taylor polynomial for $f(x) = \sec^2 x$ expanded around $x = \pi$.

   (c) Find the fourth degree Taylor polynomial for $f(x) = \ln(\cos x)$ expanded around $x = \pi$.

8. (a) Simplify $(1 + i)^2/2$.

   (b) Simplify $\sqrt{-2 + 6i}$. Express your answer in the form $a + bi$.

   (c) If $z = 2e^{i\pi/2}$, what is the polar representation of $\bar{z}$?

   (d) Find all solutions of the equation $y^3 = -1$.

9. Find two power series solutions for $y'' + xy' + y = 0$ and find the radius of convergence for each series.

10. Ol' Baldy appeared to lose 20 years after he bought his new toupee. Since Ol' Baldy received so many compliments on his youthful looks, he purchased a wool toupee to keep his head warm during the winter. Unfortunately for Ol'Baldy, a hungry moth found the toupee. The moth ate 50% of what remained of the hairpiece each day. Unfortunately for the moth, toupees lack nutrients and it died of malnutrition after ten full days. Originally, the wool toupee weighed 60 grams.

10. (a) If the moth could eat forever at the same rate, write the total amount which could be eaten as a geometric series.

(b) Subtract another geometric series to determine how much of the wig was eaten.

ANSWERS TO CHAPTER TEST

1. (a) False; let $a_i = 1$ and $b_i = -1$ for all i.

(b) True

(c) True

(d) True

(e) False; the signs do not alternate.

2. (a) $4x^3 - 9x^2 + 8x - 2$

(b) $4(x - 1)^3 + 3(x - 1)^2 + 2(x - 1) + 1$

(c) $4(x - 3/4)^3 + (37/8)(x - 3/4) + 59/8$

3. (a) $-2 < x < 0$

(b) $-2 < x < 0$

(c) $x < -2$ and $x > 0$

4. (a) Diverges; compare to $\Sigma_{n=1}^{\infty}(1/9n)$

(b) Converges absolutely; $\Sigma_{n=1}^{\infty}(1/n^2)$ is a p-series

5. (a) $c_1 \sin 2x + c_2 \cos 2x + x \sin 2x/4 + e^x/5$

(b) $c_1e^x + c_2(x - 1)e^x + c_3 = C_1e^x + c_2xe^x + c_3$

6. (a) No information

(b) Converges absolutely

(c) No information

(d) Converges absolutely; note subscripts

(e) Diverges

7. (a) $(x - \pi) + (x - \pi)^3/3$

(b) $1 + (x - \pi)^2 + 2(x - \pi)^4/3$

(c) $(x - \pi)^2/2 + (x - \pi)^4/12$

8. (a) i

(b) $a + bi$, where either $a = \sqrt{-1 + \sqrt{10}}$ and $b = \sqrt{1 + \sqrt{10}}$ or $a = -\sqrt{-1 + \sqrt{10}}$ and $b = -\sqrt{1 + \sqrt{10}}$

(c) $2e^{-i\pi/2}$

(d) $e^{-i\pi/3}$, $e^{i\pi/3}$, $e^{i\pi}$

9. $a_0\Sigma_{n=0}^{\infty}[(-1)^n x^{2n}/2^n n!]$ has radius of convergence $= \infty$;

$a_1\Sigma_{n=0}^{\infty}[(-1)^{n+1}2^n(n!)x^{2n+1}/(2n + 1)!]$ has radius of convergence $= \infty$.

10. (a) $\Sigma_{i=1}^{\infty} 60(1/2)^i$

(b) $\Sigma_{i=1}^{\infty} 60(1/2)^i - \Sigma_{i=11}^{\infty} 60(1/2)^i = 60 - 15/256 = 15345/256$ grams .

COMPREHENSIVE TEST FOR CHAPTERS 7-12 (Time limit: 3 hours)

1. True or false. If false, explain why.

   (a) The imaginary part of a complex number is a real number.

   (b) By l'Hôpital's rule, we have $\lim_{x\to 0}(\sin x/e^x) = \lim_{x\to 0}(\cos x/e^x) = 1$.

   (c) Every infinite series must either converge or diverge, but they can not do both.

   (d) An improper integral may have an upper sum and a lower sum, yet diverge.

   (e) One form of the formula for integration by parts is $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$.

   (f) The Taylor series approximation of $e^{-(x^2)}$ for x near zero is $\Sigma_{i=0}^{\infty}(x^{-2i}/(2i)!)$.

   (g) Using an even number of subintervals, Simpson's method gives a better approximation than the trapezoidal method.

   (h) The center of mass of a region may lie on the region's perimeter, assuming uniform density.

   (i) If $f(t) > g(t)$ for all t , then the arc length of $f(t)$ is greater than that of $g(t)$ over the same interval.

   (j) The solution of $x'(t) = -3x(t)$ ia a decreasing function throughout its entire domain.

2. Multiple choice. (Choose the best answer.)

   (a) Which of the following is a solution of $y'' + 4y = 0$?

      (i) $\Sigma_{i=0}^{\infty}[(-1)^i x^{2i}/(2i)!]$

      (ii) $\Sigma_{i=0}^{\infty}[(-1)^i (2x)^{2i}/(2i)!]$

      (iii) $\Sigma_{i=0}^{\infty}[(-2)^i x^{2i}/(2i)!]$

      (iv) $\Sigma_{i=0}^{\infty}[(-1)^i 2x^{2i+1}/(2i + 1)!]$

2. (b) L'Hôpital's rule may be used for all of the following except:

(i) $\lim_{t\to 0^+} (\sin t)(\ln t)$

(ii) $\lim_{x\to 3} [2/(x^2 - 9) - \sqrt{3x^3}/(x^3 - 27)]$

(iii) $\lim_{n\to\infty} (1/n)^{3n+2}$

(iv) None of the above

(c) If one wishes to integrate $g(y) = (y^2 + 3y - 2)/y^2(y^2 + 3)^2(y - 1)$ by the method of partial fractions, the integrand should have the form:

(i) $(Ay + B)/y^2 + (Cy + D)/(y^2 + 3) + (Ey + F)/(y^2 + 3)^2 + G/(y - 1)$

(ii) $(Ay + B)/(y^2 + 3y - 2) + C/y^2 + (Dy + E)/(y^2 + 3)^2 + F/(y - 1)$

(iii) $A/y + B/y^2 + (Cy + D)/(y^2 + 3) + (Ey + F)/(y^2 + 3)^2 + G/(y - 1)$

(iv) None of the above

(d) The complex conjugate of $1 + i$ has polar representation

(i) $\sqrt{2}e^{-i\pi/4}$

(ii) $\sqrt{2}\exp(-\pi/4)$

(iii) $\sqrt{2}\, e^{\pi i/4}$

(iv) $\exp(-i\pi/4)$

(e) The mean value theorem states that for some point x_0 in (a,b) :

(i) $f(x_0) = [1/(b - a)]\int_a^b f'(x)dx$

(ii) $f(b - a) = [1/(b - a)]\int_a^b f(x_0)dx$

(iii) $f[(b + a)/2] = \int_a^b f(x_0)dx$

(iv) $f(x_0) = \int_a^b f(x)dx/(b - a)$

3. Short answers

(a) For Taylor's theorem, what is the remainder in the derivative form?

(b) State the ε-δ definition of the limit.

(c) Define $\sinh x$ in terms of exponentials.

(d) Express the arc length of $f(x) = x^n$ on $[0,1]$ as an integral.

(e) State the alternating series test.

4. Numerical calculations

(a) Estimate $y(1)$ if $(y')^2 - xy = 0$ and $y(0) = 2$. Use a four-step Euler method.

(b) Use Simpson's method with $n = 4$ to estimate $\int_0^4 \exp(-x^2)\,dx$.

(c) State the formula used in Newton's method and use it to solve $x^3 + x^2 = -7$.

5. Fill in the blanks.

(a) According to the ratio test, a series $\Sigma_{n=0}^{\infty} a_n$ converges if ____________________ .

(b) If $(-b \pm \sqrt{b^2 - 4ac})/2a$ equals r_1 and r_2, two distinct real roots,then the solution of $ay'' + by' + cy = 0$ is ____________ ____________________ .

(c) The radius of convergence of $\Sigma_{i=0}^{\infty}(ix^i/3^i)$ is ________________ .

(d) The integration of $\cos^2 x$ is based upon the trigonometric identity $\cos^2 x =$ ____________________ .

(e) The Maclaurin expansion of $1/(1 - x)$ is ____________________ .

6. Integration problems. Evaluate the integrals.

(a) $\int(t^2 + 4t + 7)^{-1/2}dt$

(b) $\int(t^3 + 2t^2 + t)^{-1}dt$

(c) $\int(\Sigma_{n=2}^{\infty}(x^n/n^{8/3}))dx$

(d) $\int\cos^2 y \sin^3 y\ dy$

7. Do the following converge absolutely, converge conditionally, or diverge? Explain why.

(a) $\Sigma_{n=4}^{\infty}[n^2(n - 1)!/(2n + 1)!]$

(b) $4 - 1 + 1/4 - 1/16 + 1/64 - \ldots$

(c) $\int_{-2}^{10}(3e/x \ln|x|)dx$

(d) $\Sigma_{n=1}^{\infty}(3n/n^n)$

(e) $\Sigma_{n=10}^{\infty}[(4n + 5)(5n)/(3n + \sqrt{n})(n^2/5 - 1)](-1)^n$

8. (a) What is the Taylor series expansion of e^{2x} around $x = 2$?

   (b) Use your answer in (a) to estimate e^1 . Use a fourth-order approximation.

   (c) How accurate is your answer in (b)?

9. Miscellaneous calculations.

   (a) Compute $\lim_{x\to 0}[(e^{1/x} - e^{-1/x})/x^2]$.

   (b) Find the general solution of $y'' - y' - 2y = x^2 + e$

   (c) Compute $\lim_{x\to 3+}[(x^2 - 9)/|3 - x|]$.

10. A former mathematician has given up his career to open a bakery. Cake flower decorations have green leaves which can be described in polar coordinates by $r = \cos 3\theta$. He uses red frosting to outline the leaves.

   (a) Write a formula to describe how much red frosting he uses to outline the leaves on each flower.

   (b) How much green frosting does he use for each flower's leaves?

ANSWERS TO COMPREHENSIVE TEST

1. (a) True

   (b) False; $(\sin x/e^x)$ has the form $0/1$, so l'Hôpital's rule can't be used.

   (c) True

   (d) True; consider $\int_0^\infty \sin x\, dx$.

   (e) True

   (f) False; it should be $\Sigma_{n=0}^\infty[(-x^2)^n/n!] = \Sigma_{n=0}^\infty[(-1)^n x^{2n}/n!]$.

   (g) True

   (h) True; consider the graph of $r = |\cos\theta|$ in polar coordinates.

1. (i) False; let $f(t) = 10$ and $g(t) = t$ on $[0,1]$.

   (j) True

2. (a) ii

   (b) iv

   (c) iii

   (d) i

   (e) iv

3. (a) $[f^{(n+1)}(c)](x - x_0)^{n+1}/(n + 1)!$

   (b) $\lim_{x\to x_0} f(x) = L$ if $|f(x) - L| < \varepsilon$ whenever $|x - x_0| < \delta$.

   (c) $\sinh x = (e^x - e^{-x})/2$

   (d) $\int_0^1 \sqrt{1 + n^2x^{2n-2}}\, dx$

   (e) An alternating series converges if (i) the signs are alternating, (ii) the terms are decreasing, and (iii) the limit of the terms is zero.

4. (a) 2.78

   (b) 0.836

   (c) $x_{n+1} = x_n - f(x_n)/f'(x_n)$; -2.31

5. (a) $\lim_{i\to\infty} |a_i/a_{i-1}| < 1$

   (b) $c_1\exp(r_1x) + c_2\exp(r_2x)$

   (c) 3

   (d) $(1 + \cos 2x)/2$

   (e) $\Sigma_{i=0}^{\infty} x^i$

6. (a) $\ln|\sqrt{t^2 + 4t + 7} + t + 2| + C$

   (b) $\ln|t/(t + 1)| + (t + 1)^{-1} + C$

   (c) $\Sigma_{n=2}^{\infty}[x^{n+1}/(n + 1)n^{8/3}]$

   (d) $\cos^5 y/5 - \cos^3 y/3 + C$

7. (a) Converges absolutely; use ratio test.

(b) Converges absolutely; it's a geometric series.

(c) Diverges; $\int (3e/x \ln|x|) = 3e[\ln(\ln|x|)]$, which diverges at $x = 0$.

(d) Converges absolutely; use root test.

(e) Converges conditionally; use alternating series test and comparison test.

8. (a) $\Sigma_{n=0}^{\infty}[2^n e^4 (x - 2)^n / n!]$

(b) $e^1 \approx e^4 - 3e^4 + 9e^4/2 - 9e^4/2 + 27e^4/8 = 11e^4/8$

(c) Error < $81e^4/40$

9. (a) Does not exist

(b) $c_1 \exp(2x) + c_2 \exp(-x) - x^2/2 + x/2 - e/2 - 3/4$

(c) 6

10. (a) $6\int_0^{\pi/6}\sqrt{1 + 8 \sin^2 3\theta}\ d\theta$

(b) $\pi/4$

This book is intended to help the student of Volume II of Marsden and Weinstein's **Calculus**. It runs in parallel with this text, and reinforces the concepts introduced there with exercises, study hints, and quizzes. Each chapter begins with a section detailing the prerequisites for its study, and a quiz on those prerequisites. Detailed solutions of every other odd-numbered exercise and sample examinations have been included.

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