STAT 620: Asymptotic Statistics **Spring 2022**

Lecture: Apr 12

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1 Asymptotically equicontinuous: ASEC

We first recall the definition of ASEC. Let $X_n \in L^\infty(T)$. We say that X_n's are asymptotically equicontinuous if for all $\epsilon, \eta > 0$, there is a finite partition $T_1, \ldots, T_k$ of T such that

$$\limsup_{n\to\infty} P\left(\max_{1\le i\le k} \sup_{s,t\in T_i} |X_{n,s} - X_{n,t}| \ge \epsilon\right) \le \eta.$$

2 Weak convergence in $L^\infty(T)$

For $X_n \in L^\infty(T)$, we have

$$X_n \xrightarrow{d} X$$

where $X \in UC(T,\rho)$ is tight if and only if

(a) **finite dimensional convergence**:

$$(X_{n,t_1}, \ldots, X_{n,t_k}) \xrightarrow{d} \text{some limit}$$

for any $k < \infty$ and $t_1, \ldots, t_k \in T$.

(b) **ASEC**: X_n is asymptotically stochastically equicontinuous.

Note: in the proof below we shall write $X_{n,t} = X_n(t)$ and $X_t = X(t)$.

3 Proof

We only prove that (a) and (b) imply the weak convergence in $L^\infty(T)$. We divide the proof into several steps.

3.1 Step 1: Construct a dense subset of T

For any $m \in N$, construct a sequence of partitions of T as $T_1^m, T_2^m, \ldots T_{k_m}^m$ such that

$$\limsup_{n\to\infty} P\left[\max_{1\le i\le k_m} \sup_{s,t\in T_i^m} \mid X_{n,t} - X_{n,s} \mid\ge 2^{-m}\right] \le 2^{-m}.$$

Here we assume that $\{T_i^m\}_m$ are nested partitions of T. For each $m \in \mathbb{N}$, define the distance

$$\rho_m(s,t) = \begin{cases} 0 & \text{if there exists } i \text{ such that } s,t \in T_i^m, \\ 1 & \text{otherwise,} \end{cases}$$

and let

$$\rho(s,t) = \sum_{m=1}^{\infty} 2^{-m} \rho_m(s,t)$$

for any $s, t \in T$. Note that for $s, t \in T_i^m$, $\rho_m(s,t) = 0$. Since the partitions are nested, we have $s, t \in T_i^k$ for $k \leq m$ and some i, and thus $\rho_k(s,t) = 0$. Then for $s, t \in T_i^m$,

$$\begin{aligned}
\rho(s,t) &= \sum_{i=1}^{m} 2^{-i} \rho_i(s,t) + \sum_{i=m+1}^{\infty} 2^{-i} \rho_i(s,t) \\
&\leq 0 + \sum_{i=m+1}^{\infty} 2^{-i} \\
&= 2^{-m}.
\end{aligned}$$

Pick t_i^m from T_i^m for all m and i. Define

$$T_0 = \cup_{m=1}^{\infty} \cup_{i=1}^{k_m} \{t_i^m\}.$$

Then T_0 is countable and for any $t \in T$, $\exists$ i and m such that $\rho(t_i^m, t) \leq 2^{-m}$. Hence T_0 is a dense subset of T.

3.2 Step 2: Find a limit process

By the finite-dimensional convergence, we have

$$(X_n(t_1), X_n(t_2), \ldots, X_n(t_k)) \xrightarrow{d} \text{some limit.}$$

By the Kolmogorov's Extension theorem, we have $\{X(t)\}_{t \in T_0}$ such that

$$(X_n(t_1), X_n(t_2), \ldots, X_n(t_k)) \xrightarrow{d} (X(t_1), X(t_2), \ldots, X(t_k))$$

for any k and $t_1, t_2, \ldots, t_k \in T_0$. Let $\tilde{T_0}$ be a finite subset of T_0. We have

$$\begin{aligned}
P\left[\sup_{\substack{s,t \in T_0 \\ \rho(s,t) \leq 2^{-m}}} \mid X(t) - X(s) \mid \geq 2^{-m}\right] &\leq P\left[\max_{1 \leq i \leq k_m} \sup_{s,t \in T_0 \cap T_i^m} \mid X(s) - X(t) \mid \geq 2^{-m}\right] \\
&\overset{\text{MCT}}{=} \lim_{\tilde{T_0} \uparrow T_0} P\left[\max_{1 \leq i \leq k_m} \sup_{s,t \in \tilde{T_0} \cap T_i^m} \mid X(s) - X(t) \mid \geq 2^{-m}\right] \\
&= \lim_{\tilde{T_0} \uparrow T_0} \lim_{n \to \infty} P\left[\max_{1 \leq i \leq k_m} \sup_{s,t \in \tilde{T_0} \cap T_i^m} \mid X_n(s) - X_n(t) \mid \geq 2^{-m}\right] \\
&\leq 2^{-m},
\end{aligned}$$

where the second step from the bottom follows from the finite-dimensional convergence and continuous mapping theorem, and the last step follows from the ASEC. By the Borel-Cantelli lemma,

$$P\left(\exists\ M \in \mathbb{N} \text{ such that } \forall\ m \geq M \sup_{\substack{s,t \in T_0 \\ \rho(s,t) \leq 2^{-m}}} |X(t) - X(s)| < 2^{-m}\right) = 1.$$

So $X(t)$ has almost surely uniformly continuous sample path defined on T_0. We can extend X from T_0 to T such that $X \in UC(T, \rho)$ almost surely. Specifically for any $t \in T$, find $t_i \in T_0$ such that $t_i \to t$, define

$$X(t) = \lim_i X(t_i).$$

3.3 Step 3: $X_n \xrightarrow{d} X$

We only need to show $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all bounded and Lipschitz function f. Recall that t_i^m is an element in T_i^m. Let us define $\pi_m(t) = t_i^m$ for any $t \in T$ and $X \circ \pi_m(t) = X(\pi_m(t))$. Remember $\rho(t, \pi_m(t)) \leq 2^{-m}$.

- Claim 1: $\sup_{t\in T} |X \circ \pi_m(t) - X(t)| \xrightarrow{a.s.} 0$ as $m \to \infty$.
- Claim 2: $X_n \circ \pi_m \xrightarrow{d} X \circ \pi_m$ as $n \to \infty$.

Claim 1 follows from the uniform continuity and Claim 2 follows from the finite-dimensional convergence. Note that

$$|\mathbb{E}[f(X_n)]-\mathbb{E}[f(X)]| \leq |\mathbb{E}[f(X_n)]-\mathbb{E}[f(X_n\circ\pi_m)]|+|\mathbb{E}[f(X_n\circ\pi_m)]-\mathbb{E}[f(X\circ\pi_m)]|+|\mathbb{E}[f(X\circ\pi_m)]-\mathbb{E}[f(X)]|.$$

- The third term will go to 0 from claim 1, the Lipshitz continuity of f and DCT.
- The second term is small from claim 2 and portmanteau theorem.

We only need to focus on the first term. We have for some constant $C > 0$ and $\epsilon = 2^{-m}$,

$$\begin{aligned}
|\mathbb{E}[f(X_n)] - \mathbb{E}[f(X_n \circ \pi_m)]| &\leq C\mathbb{E}[1 \wedge ||X_n - X_n \circ \pi_m||_\infty] \\
&\leq C\left(\epsilon + P(||X_n - X_n \circ \pi_m||_\infty \geq \epsilon)\right) = C\left(\epsilon + P(\sup_{t\in T} |X_n(t) - X_n(\pi_m(t))| \geq \epsilon)\right) \\
&\leq C\left(2^{-m} + 2^{-m}\right) \to 0
\end{aligned}$$

as $m \to \infty$ and $n \to \infty$. Thus $|\mathbb{E}[f(X_n)] - \mathbb{E}[f(X)]|$ is small.