Solutions to final review questions.

1. $\operatorname{div}\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} = 2xy + 2xyz + xy = xy(2z+3)$

$\therefore \operatorname{div}\vec{A}(1,-1,2) = -7.$

2. $\operatorname{div}\vec{V} = 2y + 3x^2 - 3py.$

At $(1,1,1)$ $\operatorname{div}\vec{V} = 5-3p = 0$ $\therefore p = \frac{5}{3}.$

3. $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2z^3 - 6xyz, -3x^2z, 6xz^2 - 3x^2y \rangle$

At $(2,2,-1)$ $\nabla f(2,2,-1) = \langle 22, 12, -12 \rangle.$

$$\|\nabla f(2,2,-1)\| = \sqrt{22^2 + 12^2 + (-12)^2} = 2\sqrt{193}.$$

4. $f(x,y,z) = 2xyz,\ g(x,y,z) = x^2y + z.$

$\therefore f+g = 2xyz + x^2y + z.$ $\qquad fg = 2x^3y^2z + 2xyz^2.$

$\therefore \nabla(f+g) = \langle 2yz + 2xy, 2xz + x^2, 2xy + 1 \rangle.$

$\therefore \nabla(f+g)(1,-1,0) = \langle -2, 1, -1 \rangle.$

$\therefore \nabla(fg) = \langle 6x^2y^2z + 2yz^2, 4x^3yz + 2xz^2, 2x^3y^2 + 4xyz \rangle$

$\therefore \nabla(fg)(1,-1,0) = \langle 0, 0, 2 \rangle.$

5. $$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix} = \langle -1, 1, -1 \rangle.$$

$\therefore \vec{F} \cdot \operatorname{curl}\vec{F} = \langle x+y+1, 1, -x-y \rangle \cdot \langle -1, 1, -1 \rangle = -x-y-1+1+x+y = 0.$

6. $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$

$\therefore\ a=4,\ b=2,\ c=-1.$

7. (a) Substitute $x=\sin t$, $y=\sin 2t$ into the equation of the curve $y^2+4x^4-4x^2=0$ we find they satisfy the equation. Therefore

$$\vec{r}(t) = \langle \sin t, \sin 2t \rangle$$

is a parametrization of the curve.

(b). $\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$\vec{r}\,'(t) = \langle \cos t,\ 2\cos 2t \rangle.$

$\vec{F}(\vec{r}(t)) = \langle \sin t + \sin 2t,\ -\sin t \rangle.$

$\therefore \vec{F}(\vec{r})\cdot\vec{r}\,'(t) = \langle \sin t+\sin 2t, -\sin t\rangle \cdot \langle \cos t, 2\cos 2t\rangle$

$= \sin t\cos t + \sin 2t\cos t - 2\sin t\cos 2t.$

$= \frac{1}{2}\sin 2t + 2\sin t\cos^2 t - 2\sin t\,(2\cos^2 t - 1).$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} \frac{1}{2}\sin 2t + 2\int_0^{\pi} \sin t\cos^2 t\,dt - 2\int_0^{\pi} \sin t\,(2\cos^2 t-1)\,dt.$

$= -\frac{1}{4}\Big[\cos 2t\Big]_0^{\pi} - 2\int_0^{\pi} \cos^2 t\,d(\cos t) + 2\int_0^{\pi} (2\cos^2 t - 1)\,d(\cos t).$

$= 0 - 2\cdot\frac{1}{3}\Big[\cos^3 t\Big]_0^{\pi} + 2\cdot\Big[\frac{2}{3}\cos^3 t - \cos t\Big]_0^{\pi}$

$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$

8. The work is

$$\text{work} = \int_C \vec{F}(\vec{r}) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^1 \langle 1, -y, xyz\rangle \cdot \langle 1, -2t, 1\rangle dt$$

$$= \int_0^1 \langle 1, t^2, -t^4\rangle \cdot \langle 1, -2t, 1\rangle dt$$

$$= \int_0^1 1 - 2t^3 - t^4 dt = \frac{3}{10}.$$

9. (a). $F_1 = 2x\cos 2y$, $F_2 = -2x^2 \sin 2y$.

$\therefore \frac{\partial F_2}{\partial x} = -4x\sin 2y$, $\frac{\partial F_1}{\partial y} = -4x\sin 2y$ $\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

$\therefore \operatorname{curl}\vec{F} = 0$ $\therefore$ $\vec{F}\cdot d\vec{r}$ is exact.

(b). $$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y + yz & xz - e^x\sin y & xz \end{vmatrix} = \vec{i}(0-x) - \vec{j}(z-y) + \vec{k}(z - e^x\sin y - z + e^x\sin y) \neq \vec{0}.$$

So $\vec{F}\cdot d\vec{r}$ is not exact.

10. (a) Assume $\vec{F} = \langle y+z, x+z, x+y\rangle = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$.

$$\therefore \begin{cases} \frac{\partial f}{\partial x} = y+z & ① \\ \frac{\partial f}{\partial y} = x+z & ② \\ \frac{\partial f}{\partial z} = x+y & ③ \end{cases}$$

① $\xrightarrow{\text{Integrate }(x)}$ $f(x,y,z) = xy + xz + g(y,z)$ $\xrightarrow{D(y)}$ $\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}$. ④

Compare ② and ④, we find $\frac{\partial g}{\partial y} = z$ $\therefore g(y,z) = yz + h(z)$

$\therefore f(x,y,z) = xy + xz + yz + h(z)$.

$\therefore \frac{\partial f}{\partial z} = x + y + h'(z)$ ⑤. Compare ③ and ⑤, we find $h'(z) = 0$.

$\therefore h(z) = \text{Constant}$ $\therefore$ $f(x,y,z) = xy + xz + yz + \text{Constant}$.

$$\therefore \int_{(0,0,0)}^{(-1,0,\pi)} (y+z)dx + (x+z)dy + (x+y)dz = f(-1,0,\pi) - f(0,0,0) = -\pi.$$

(b). Assume $\vec{F} = \langle y - \frac{1}{x^2},\ x - \frac{1}{y^2} \rangle = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

$\therefore \begin{cases} \frac{\partial f}{\partial x} = y - \frac{1}{x^2} \quad ① \\ \frac{\partial f}{\partial y} = x - \frac{1}{y^2} \quad ② \end{cases}$

Integrate ① with respect to x: $f(x,y) = xy + \frac{1}{x} + h(y)$ ③.

Differentiate ③, we get $\frac{\partial f}{\partial y} = x + h'(y) = x - \frac{1}{y^2}$.

$\therefore h'(y) = -\frac{1}{y^2} \quad \therefore h(y) = \frac{1}{y} + \text{Constant}$.

$\therefore f(x,y) = xy + \frac{1}{x} + \frac{1}{y} + \text{constant}$.

$\therefore \int_{(-1,1)}^{(4,2)} (y - \frac{1}{x^2})dx + (x - \frac{1}{y^2})dy = f(4,2) - f(-1,1) = \frac{39}{4}$.

11. Green's theorem is $\int_C F_1 dx + F_2 dy = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$,

where C is the boundary of S. Here C is $x^2 + y^2 = 1$.

At first, we compute the left hand side: we write C as: $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$

$\therefore \int_C F_1 dx + F_2 dy = \int_C y^2 dx + x\, dy = \int_0^{2\pi} \sin^2 t \cdot (-\sin t)\, dt + \cos t \cdot \cos t\, dt$.

$= \int_0^{2\pi} \cos^2 t - \sin^3 t\, dt = \int_0^{2\pi} \frac{\cos 2t + 1}{2} - (1 - \cos^2 t)\sin t\, dt$.

$= \frac{1}{2}\left[\frac{1}{2}\sin 2t + t\right]_0^{2\pi} + \int_0^{2\pi} 1 - \cos^2 t\, d(\cos t)$.

$= \pi + \left[\cos t - \frac{1}{3}\cos^3 t\right]_0^{2\pi} = \pi$.

Now we compute the right hand side:

$\iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy = \iint_S 1 - 2y\, dx\, dy$, where S is: $\{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 < 1\}$.

$= \int_0^{2\pi} \int_0^1 (1 - 2r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{2}{3}r^3 \sin\theta\right]_0^1 d\theta$

$= \int_0^{2\pi} \frac{1}{2} - \frac{2}{3}\sin\theta\, d\theta = \left[\frac{1}{2}\theta + \frac{2}{3}\cos\theta\right]_0^{2\pi} = \pi$.

$\therefore$ Left hand side = right hand side $\therefore$ Green's theorem is verified.

12. Green's theorem: $\int_C F_1 dx + F_2 dy = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx dy$,

where C is the boundary of S: $x^2 + (y-1)^2 = 1$.

Let $\begin{cases} x = \cos t \\ y-1 = \sin t \end{cases}$, then C can be represented by $\begin{cases} x = \cos t \\ y = \sin t + 1 \end{cases}$ $0 \le t \le 2\pi$,

We first calculate the left hand side:

$$F_1 dx + F_2 dy = -x^2 y \, dx + x y^2 dy = -\cos^2 t (\sin t + 1)(-\sin t) dt + \cos t (\sin t + 1)^2 \cos t \, dt.$$

$$= [\sin^2 t \cos^2 t + \cos^2 t \sin t + \cos^2 t (\sin^2 t + 2\sin t + 1)] dt.$$

$$= (2\cos^2 t \sin^2 t + 3\cos^2 t \sin t + \cos^2 t) dt.$$

$$= [\tfrac{1}{2} \sin^2 2t + 3\cos^2 t \sin t + \tfrac{1}{2}(\cos 2t + 1)] dt.$$

$$= [\tfrac{1}{4}(1 - \cos 4t) + 3\cos^2 t \sin t + \tfrac{1}{2}(\cos 2t + 1)] dt.$$

$$\therefore \int_C F_1 dx + F_2 dy = \int_0^{2\pi} \frac{1}{4}(1 - \cos 4t) dt + 3\int_0^{2\pi} \cos^2 t \, (-1) d(\cos t) + \frac{1}{2}\int_0^{2\pi} \cos 2t + 1 \, dt$$

$$= \frac{1}{4} \cdot \left[t - \frac{1}{4}\sin 4t\right]_0^{2\pi} - \left[\cos^3 t\right]_0^{2\pi} + \frac{1}{2} \cdot \left[\frac{1}{2}\sin 2t + t\right]_0^{2\pi}.$$

$$= \frac{2\pi}{4} - 0 + \pi = \frac{3}{2}\pi.$$

The right hand side is:

$$\iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dx dy = \iint_S y^2 + x^2 \, dx dy,$$ where $S = \{(x, y) \in \mathbb{R}^2, \ x^2 + (y-1)^2 < 1\}$.

Let $\begin{cases} x = r\cos t \\ y-1 = r\sin t \end{cases} \Rightarrow \begin{cases} x = r\cos t \\ y = r\sin t + 1, \end{cases}$ $0 \le r \le 1$, $0 \le t \le 2\pi$.

$$= \int_0^{2\pi} \int_0^1 [(r\sin t + 1)^2 + r^2 \cos^2 t] \, r \, dr \, d\theta.$$

$$= \int_0^{2\pi} \int_0^1 r^3 + 2r^2 \sin\theta + r \, dr d\theta.$$

$$= \int_0^{2\pi} \left[\frac{1}{4} r^4 + \frac{2}{3} r^3 \sin\theta + \frac{1}{2} r^2\right]_0^1 d\theta = \int_0^{2\pi} \frac{3}{4} + \frac{2}{3}\sin\theta \, d\theta$$

$$= \left[\frac{3}{4}\theta - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \frac{3}{2}\pi.$$

So left hand side = right hand side. Green's theorem is verified.

13. At first we represent the surface in parametric equation:

$$\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle. \quad 0 \le u \le 2\pi,\ 0 \le v \le 1.$$

So $\iint_S f(x,y,z)\, dA = \iint_S f(\vec{r}(u,v)) \|\vec{N}(u,v)\|\, du\, dv$,

where $\vec{N}(u,v)$ is the normal vector of S.

$$\therefore \vec{N} = \vec{r}_u \times \vec{r}_v = \langle -\sin u, \cos u, 0 \rangle \times \langle 0,0,1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0 \rangle.$$

$\therefore \|\vec{N}\| = \sqrt{\cos^2 u + \sin^2 u} = 1$. $f(\vec{r}(u,v)) = \cos^2 u + \sin^2 u + 2v$.

$$\therefore \iint_S f(x,y,z)\, dA = \int_0^1 \int_0^{2\pi} (\cos^2 u + \sin^2 u + 2v) \cdot 1\, du\, dv.$$

$$= \int_0^1 \int_0^{2\pi} 1 + 2v\, du\, dv.$$

$$= \int_0^1 [u + 2uv]_0^{2\pi} dv = \int_0^1 2\pi + 4\pi v\, dv = \left[2\pi v + 2\pi v^2\right]_0^1 = 4\pi.$$

14. The equation of the surface S can be written into $\vec{r}(x,y) = \langle x, y, 2-x-y \rangle$.

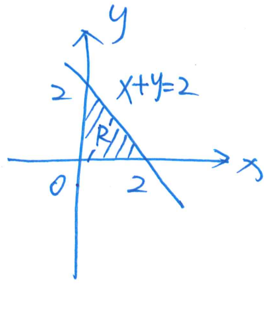
The flux across the surface is $\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R \vec{F}(\vec{r}(x,y)) \cdot \vec{N}(x,y)\, dx\, dy$,

where R is the projection of S in xy-plane and $\vec{N}$ is the outgoing normal vector of S.

$$\therefore \vec{N} = \vec{r}_x \times \vec{r}_y = \langle 1,0,-1 \rangle \times \langle 0,1,-1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,1,1 \rangle.$$

$$\therefore \vec{F}(\vec{r}(x,y)) \cdot \vec{N} = \langle 0, 2-x-y, 2-x-y \rangle \cdot \langle 1,1,1 \rangle = 4-2x-2y.$$

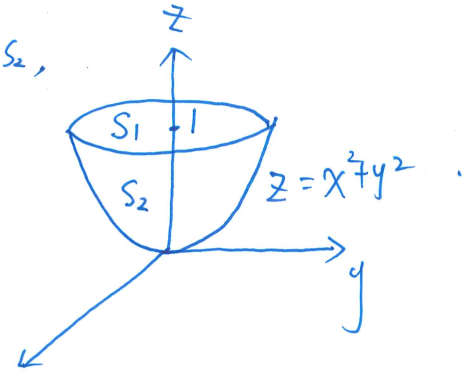
$\therefore$ The flux is $\iint_R 4-2x-2y\, dx\, dy = \int_0^2 \int_0^{2-x} 4-2x-2y\, dy\, dx$



$$= \int_0^2 \left[4y - 2xy - y^2\right]_0^{2-x} dx = \int_0^2 8-4x-2x(2-x)-(2-x)^2\, dx.$$

$$= \int_0^2 x^2 - 2x + 4\, dx = \left[\tfrac{1}{3}x^3 - x^2 + 4x\right]_0^2 = \frac{8}{3} - 4 + 8 = \frac{20}{3}.$$

15. The surface of T is composed of two parts: S_1 and S_2, where S_1 is: $x^2+y^2\le 1$, $z=1$, so the area is $\pi\cdot 1^2=\pi$.



Now we compute the area of $S_2 = \iint_{S_2} dA$.

$= \iint_{S_2} \sqrt{f_x^2+f_y^2+1}\, dx\,dy$, where $f(x,y,z)$ is the equation of S_2:

$z = x^2+y^2 = f(x,y)$.

$\therefore f_x = 2x,\quad f_y = 2y$.

$= \iint_{S_2} \sqrt{4x^2+4y^2+1}\, dx\,dy$.

The equation of S_2 can be written into: $\vec{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle$. $0\le u\le 1$, $0\le v\le 2\pi$.

$\therefore \text{Area}(S_2) = \int_0^{2\pi}\int_0^1 \sqrt{4u^2+1}\, u\,du\,dv = 2\pi\cdot\int_0^1 \sqrt{4u^2+1}\, u\,du$. $(*)$

Let $w = 4u^2+1$ then $dw = 8u\,du$, $\therefore u\,du = \frac{1}{8}dw$.

As $u=0$, $w=1$, as $u=1$, $w=5$.

$\therefore (*) = 2\pi\cdot\int_1^5 \sqrt{w}\cdot\frac{1}{8}dw = \frac{\pi}{4}\cdot\int_1^5 \sqrt{w}\,dw = \frac{\pi}{4}\cdot\frac{2}{3}\left[w^{\frac{3}{2}}\right]_1^5 = \frac{\pi}{6}\left[5^{\frac{3}{2}}-1\right]$.

16. As the surface of T is a closed surface, we may apply the divergence theorem.

$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dV = \iiint_T y+z+x\,dV$.

$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} x+y+z\,dz\,dy\,dx$.

$= \int_0^1\int_0^{1-x}\left[(x+y)z+\frac{1}{2}z^2\right]_0^{1-x-y} dy\,dx$

$= \int_0^1\int_0^{1-x} (x+y)(1-x-y)+\frac{1}{2}(1-x-y)^2\,dy\,dx$.

$$= \int_0^1 \int_0^{1-x} x+y-(x+y)^2+\frac{1}{2}+\frac{1}{2}(x+y)^2-(x+y)\,dy\,dx \cdot$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}-\frac{1}{2}(x+y)^2\,dy\,dx = \frac{1}{2}\int_0^1\int_0^{1-x} 1-x^2-y^2-2xy\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x^2)y-\frac{1}{3}y^3-xy^2\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x^2)(1-x)-\frac{1}{3}(1-x)^3-x(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1 1-x-x^2+x^3+\frac{1}{3}(x^3-3x^2+3x-1)-x^3+2x^2-x\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{3}x^3-x+\frac{2}{3}\,dx = \frac{1}{2}\left[\frac{1}{12}x^4-\frac{1}{2}x^2+\frac{2}{3}x\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{12}-\frac{1}{2}+\frac{2}{3}\right) = \frac{1}{8}.$$

17. As the surface of T is closed, we apply the divergence theorem.

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dV$$

As T is a closed cylinder, we may us the cylindrical coordinates:

$$T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \quad 0<r<2,\quad 0\le\theta\le 2\pi,\quad 0<z<2.$$

$$\operatorname{div}\vec{F} = 2x-2y+2z = 2r\cos\theta-2r\sin\theta+2z \cdot$$

$$\therefore \iiint_T \operatorname{div}\vec{F}\,dV = \int_0^{2\pi}\int_0^2\int_0^2 [2r(\cos\theta-\sin\theta)+2z]\,r\,dz\,dr\,d\theta \cdot$$

$$= \int_0^{2\pi}\int_0^2 2r^2(\cos\theta-\sin\theta)+2rz\,dz\,dr\,d\theta$$

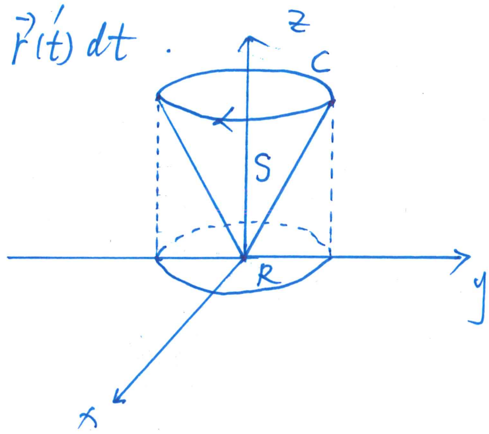
$$= \int_0^{2\pi}\int_0^2 \left[2r^2(\cos\theta-\sin\theta)z+rz^2\right]_0^2 dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 [4r^2(\cos\theta-\sin\theta)+4r]\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{4}{3} r^3 (\cos\theta - \sin\theta) + 2r^2\right]_0^2 d\theta = \int_0^{2\pi} \frac{32}{3} (\cos\theta - \sin\theta) + 8 \, d\theta$$

$$= \frac{32}{3} \left[\sin\theta + \cos\theta\right]_0^{2\pi} + \left[8\theta\right]_0^{2\pi} = 16\pi .$$

18. The stoke's theorem is $\iint_S \operatorname{curl}\vec{F} \cdot \vec{n} \, dA = \oint_C \vec{F} \cdot \vec{r}'(t) \, dt$.



The left hand side $= \iint_R \operatorname{curl}\vec{F} \cdot \vec{N} \, dx dy$,

where $\vec{N}$ is the normal vector of S.

$\because 0 < Z < 1 \quad \therefore Z = \sqrt{x^2 + y^2}$.

$\therefore \vec{N} = \langle Z_x, Z_y, -1 \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y}}, -1 \right\rangle$.

$R = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 \leq 1\}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \langle 1, 1, 1 \rangle .$$

$$\therefore \operatorname{curl}\vec{F} \cdot \vec{N} = \langle 1, 1, 1 \rangle \cdot \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle = \frac{x+y}{\sqrt{x^2+y^2}} - 1 .$$

$$\therefore \iint_R \operatorname{curl}\vec{F} \cdot \vec{N} \, dx dy = \iint_R \frac{x+y}{\sqrt{x^2+y^2}} - 1 \, dx dy$$

$$= \int_0^{2\pi} \int_0^1 \frac{r\cos\theta + r\sin\theta}{r} - 1 \; r \, dr \, d\theta .$$

$$= \int_0^{2\pi} \int_0^1 (\cos\theta + \sin\theta - 1) r \, dr \, d\theta$$

$$= \int_0^{2\pi} (\cos\theta + \sin\theta - 1) \left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \cos\theta + \sin\theta - 1 \, d\theta$$

$$= \frac{1}{2} \left[\sin\theta - \cos\theta - \theta\right]_0^{2\pi} = \frac{1}{2}(-2\pi) = -\pi .$$

Now let's calculate the right hand side. $\oint_C \vec{F}\cdot\vec{r}'(t)\,dt$.

As the normal vector $\vec{N}$ is pointed downward, the direction of the boundary C of the surface S is clockwise, so we should write the equation of curve C as:

$$C: \vec{r}(t) = \langle \cos t, \sin t, 1\rangle, \quad t: 2\pi \to 0.$$

$$\therefore \vec{r}'(t) = \langle -\sin t, \cos t, 0\rangle.$$

$$\therefore \oint_C \vec{F}\cdot\vec{r}'(t)\,dt = \int_{2\pi}^{0} \langle 1, \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_{2\pi}^{0} -\sin t + \cos^2 t\, dt$$

$$= [\cos t]_{2\pi}^{0} + \int_{2\pi}^{0} \frac{\cos 2t + 1}{2}\, dt$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2t + t\right]_{2\pi}^{0} = \frac{1}{2}(0-2\pi) = -\pi.$$

So left hand side = right hand side.

Stoke's theorem is verified.

19. Here we take S as the part enclosed by C on the plane $y+z=2$.

$$\therefore \oint_C -y^2dx + xdy + z^2dz = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dA = \iint_R \operatorname{curl}\vec{F}\cdot\vec{N}\,dxdy. \quad (*)$$

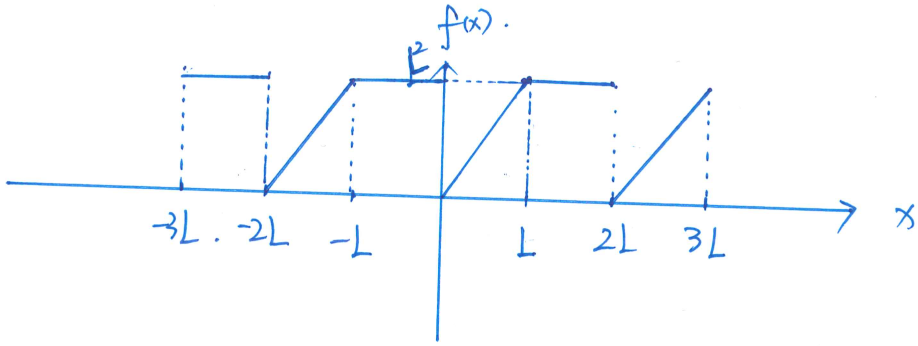
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y\rangle.$$

As $y+z=2$ $\therefore \vec{N} = \langle 0, 1, 1\rangle$. $\therefore \operatorname{curl}\vec{F}\cdot\vec{N} = 1+2y$.

$$\therefore (*) = \iint_R 1+2y\,dxdy = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r+2r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}r^2 + \frac{2}{3}r^3\sin\theta\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}+\frac{2}{3}\sin\theta\,d\theta = \left[\frac{1}{2}\theta - \frac{2}{3}\cos\theta\right]_0^{2\pi} = \pi.$$

20. (a).



(b). At $x=0$, $f(x)$ is discontinuous, the Fourier series converges to $\frac{1}{2}[f(0^+)+f(0^-)]=\frac{1}{2}(0+L^2)=\frac{L^2}{2}$.

At $x=\frac{L}{2}$, $f(x)$ is continuous. . . . . $f(\frac{L}{2})=L\cdot\frac{L}{2}=\frac{L^2}{2}$.

At $x=L$: . . . . $f(L)=L^2$.

At $x=\frac{3L}{2}$, ... $f(\frac{3L}{2})=L^2$.

(c). $a_0=\frac{1}{2L}\int_{-L}^{L} f(x)dx=\frac{1}{2L}\int_{-L}^{0}L^2dx+\frac{1}{2L}\int_0^L Lx\,dx$.

$$=\frac{L^2}{2L}\cdot[x]_{-L}^{0}+\frac{L}{2L}\cdot[\tfrac{1}{2}x^2]_0^L=\frac{L}{2}\cdot L+\frac{1}{2}\cdot\frac{L^2}{2}=L^2.$$

$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}x\,dx=\frac{1}{L}\int_{-L}^{0}L^2\cos\frac{n\pi}{L}x\,dx+\frac{1}{L}\int_0^L Lx\cos\frac{n\pi}{L}x\,dx$

$$=L\cdot\int_{-L}^{0}\cos\frac{n\pi}{L}x\,dx+\int_0^L x\cos\frac{n\pi}{L}x\,dx.$$

$$=L\cdot\frac{L}{n\pi}\cdot\left[\sin\frac{n\pi}{L}x\right]_{-L}^{0}+\int_0^L x\cos\frac{n\pi}{L}x\,dx.$$ Let $u=x$, $v'=\cos\frac{n\pi}{L}x$.

$u'=1$, $v=\frac{L}{n\pi}\cdot\sin\frac{n\pi}{L}x$.

$$=\left[x\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_0^L-\int_0^L\frac{L}{n\pi}\sin\frac{n\pi}{L}x\,dx.$$

$$=\frac{L}{n\pi}\cdot\frac{L}{n\pi}\cdot\left[\cos\frac{n\pi}{L}x\right]_0^L=\frac{L^2}{n^2\pi^2}\cdot[\cos n\pi-1].$$

$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}x\,dx=\frac{1}{L}\int_{-L}^{0}L^2\sin\frac{n\pi}{L}x\,dx+\frac{1}{L}\int_0^L Lx\sin\frac{n\pi}{L}x\,dx.$

$$=L\cdot\frac{-L}{n\pi}\left[\cos\frac{n\pi}{L}x\right]_{-L}^{0}+\int_0^L x\sin\frac{n\pi}{L}x\,dx.$$ Let $u=x$, $v'=\sin\frac{n\pi}{L}x$.

$u'=1$, $v=-\frac{L}{n\pi}\cos\frac{n\pi}{L}x$.

$$=\frac{-L^2}{n\pi}[1-\cos n\pi]+\left[x\cdot\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_0^L-\int_0^L-\frac{L}{n\pi}\cos\frac{n\pi}{L}x\,dx.$$

$$=\frac{L^2}{n\pi}(\cos n\pi-1)+\frac{-L^2}{n\pi}\cos n\pi+\frac{L}{n\pi}\cdot\frac{L}{n\pi}\cdot\left[\sin\frac{n\pi}{L}x\right]_0^L=-\frac{L^2}{n\pi}.$$

Therefore the Fourier Series of $f(x)$ is

$$f(x) = L^2 + \sum_{n=1}^{\infty}\left\{\frac{L^2}{n^2\pi^2}[\cos n\pi - 1]\cos\frac{n\pi}{L}x - \frac{L^2}{n\pi}\sin\frac{n\pi}{L}x\right\}.$$

$n=0$: $\quad a_0 = L^2$

$n=1$: $a_1\cos\frac{1\pi}{L}x + b_1\sin\frac{n\pi}{L}x = \frac{L^2}{\pi^2}(-1-1)\cos\frac{\pi}{L}x - \frac{L^2}{\pi}\sin\frac{\pi}{L}x$

$$= \frac{-2L^2}{\pi^2}\cos\frac{\pi}{L}x - \frac{L^2}{\pi}\sin\frac{\pi}{L}x.$$

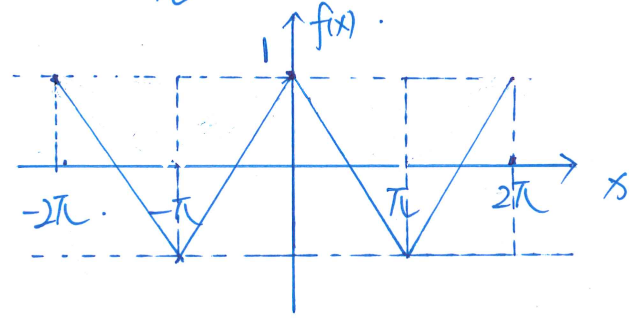
$n=2$: $a_2\cos\frac{2\pi}{L}x + b_2\sin\frac{2\pi}{L}x = 0 - \frac{L^2}{2\pi}\sin\frac{2\pi}{L}x.$

Therefore the first three non-zero terms are

$$L^2 + \frac{-2L^2}{\pi^2}\cos\frac{\pi}{L}x - \frac{L^2}{\pi}\sin\frac{\pi}{L}x - \frac{L^2}{2\pi}\sin\frac{2\pi}{L}x.$$

21. $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x < 0 \\ 1-\frac{2x}{\pi} & 0 < x < \pi. \end{cases}$

(a).



(b). As $f(x)$ is continuous, the Fourier Series will converge to $f(x)$ everywhere.

So at $x=0$, the Fourier series converges to $f(0)=1$.

At $x=\frac{\pi}{2}$, - - - - - - - - $f(\frac{\pi}{2}) = \frac{1}{2}$.

At $x=\pi$, . - - - —— . $f(\pi) = -1$.

(c). Since $f(x)$ is an even function, therefore $b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1-\frac{2x}{\pi}\,dx = \frac{1}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \left(1-\frac{2x}{\pi}\right)\cos nx\,dx \qquad \text{Let } u = 1-\frac{2x}{\pi},\ v' = \cos nx \ldots$$

$$\therefore\ u' = -\frac{2}{\pi}.\quad v = \frac{1}{n}\sin nx$$

$$= \frac{2}{\pi}\left\{\left[\left(1-\frac{2x}{\pi}\right)\cdot\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\cdot\frac{-2}{\pi}\,dx\right\}$$

$$= \frac{2}{\pi}\cdot\frac{2}{n\pi}\cdot\int_0^{\pi}\sin nx\,dx = \frac{4}{n\pi^2}\ \frac{-1}{n}\cdot\left[\cos nx\right]_0^{\pi} = \frac{-4}{n^2\pi^2}(\cos n\pi - 1)$$

Therefore the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(1-\cos n\pi)\cos nx.$$

22. (1). $A=1,\ C=1,\ B=0$, $\therefore AC-B^2 = 1>0$. So it is elliptic.

(2) $A=1,\ C=1,\ B=5$, $\therefore AC-B^2 = 1-25=-24<0$, So it is hyperbolic.

(3) $A=1,\ B=1,\ C=1$, $\therefore AC-B^2 = 1-1=0$, so it is parabolic.

23. (1). The characteristic equation is $\lambda^2+\pi^2=0$ $\therefore \lambda = \pm\pi i$

Therefore the general solution for u is $u(x) = A\cos\pi x + B\sin\pi x$.

Therefore $u(x,y) = A(y)\cos\pi x + B(y)\sin\pi x$.

(2). The characteristic equation is $\lambda^2-\pi^2=0$ $\therefore \lambda = \pm\pi$.

$\therefore$ The general solution for u is $u(x) = Ae^{\pi x} + Be^{-\pi x}$

Therefore $u(x,y) = A(y)e^{\pi x} + B(y)e^{-\pi x}$.

(3). $u_x + xu = 0$.

Consider the following first order homogeneous linear ODE with $u=u(x)$

$$u' + xu = 0 \Rightarrow u' = -xu \Rightarrow \frac{u'}{u} = -x.$$

Integrate both hand sides, we have

$$\ln|u| = \frac{1}{2}x^2 + c_1 \quad \therefore \quad |u| = e^{-\frac{1}{2}x^2 + c_1}$$

$\therefore u = c\,e^{-\frac{1}{2}x^2}$, where $c = \pm e^{c_1}$ is an arbitrary constant.

Then we obtain the solution of the PDE is

$u(x,y) = c(y)e^{-\frac{1}{2}x^2}$, where $c(y)$ is an arbitrary function of y.

24. Step 1: Assume $u(x,t) = X(x)T(t) \Rightarrow \frac{\partial u}{\partial t} = X(x)T'(t)$.

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t).$$

So the PDE becomes $(2+t)XT' = X''T \Rightarrow \frac{T'}{T}(t+2) = \frac{X''}{X} = \ell$.

$\therefore$ we obtain two ODEs:

$$\begin{cases} X'' - \ell X = 0 & ① \\ T' - \frac{\ell}{t+2}T = 0. & ② \end{cases}$$

Step 2: Apply the boundary conditions to solve $X(x)$.

$$\begin{cases} u(0,t) = X(0)T(t) = 0 \\ u(\pi,t) = X(\pi)T(t) = 0 \end{cases} \text{ for all } t>0. \quad \therefore X(0) = X(\pi) = 0.$$

The characteristic equation of ODE ① is: $\lambda^2 - \ell = 0 \therefore \lambda^2 = \ell$.

Case I: If $\ell = 0$, $\lambda = 0$. $\therefore$ the general solution of ① is

$$X(x) = Ax + B$$

$\because \begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases} \therefore \begin{cases} B=0 \\ A\pi + B = 0 \end{cases} \therefore A = B = 0. \therefore X(x) \equiv 0.$

$\therefore u(x,t) \equiv 0$ is a zero solution (trivial).

Case II: If $\ell>0$, $\lambda=\pm\sqrt{\ell}$ $\therefore$ the general solution of ① is

$$X(x)=Ae^{\sqrt{\ell}x}+Be^{-\sqrt{\ell}x}.$$

$\because \begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases}$ $\therefore \begin{cases} A+B=0 \\ Ae^{\sqrt{\ell}\pi}+Be^{-\sqrt{\ell}\pi}=0 \end{cases} \Rightarrow Ae^{\sqrt{\ell}\pi}-Ae^{-\sqrt{\ell}\pi}=0 \Rightarrow A(e^{\sqrt{\ell}\pi}-e^{-\sqrt{\ell}\pi})=0$

$\therefore A=0$. $B=-A=0$ $\therefore X(x)\equiv 0$ $\therefore u(x,t)\equiv 0$ is a zero solution.

Case III: $\ell<0$, we assume $\ell=-p^2$.

So the characteristic equation of ① is $\lambda^2=-p^2$ $\therefore \lambda=\pm pi$

$\therefore$ The general solution of ① is

$$X(x)=A\cos px+B\sin px.$$

The boundary conditions are $\begin{cases} X(0)=0 \\ X(\pi)=0 \end{cases}$ $\therefore \begin{cases} A+0=0 \\ A\cos p\pi+B\sin p\pi=0 \end{cases} \Rightarrow B\sin p\pi=0$.

$\therefore B=0$ or $\sin p\pi=0$.

Because $A=0$, so we assume $B\neq 0$ (otherwise $X(x)\equiv 0$, $\therefore u(x,t)\equiv 0$)

$\therefore \sin p\pi=0$ $\therefore p\pi=n\pi$ $\therefore p=n$.

$\therefore$ we get $X(x)=B\sin x$.

As $X(x)$ depends on n, so we write $X_n(x)=B_n\sin nx$.

Step 3: Solve for $T(t)$.

ODE ② is $T'+\frac{p^2}{t+2}T=0$ $\therefore \frac{T'}{T}=-\frac{p^2}{t+2}$.

Integrate both hand sides: $\int\frac{T'}{T}dt=-\int\frac{p^2}{t+2}dt$.

$\therefore \ln|T|=-p^2\ln|t+2| + \text{constant} =\ln(t+2)^{-p^2}+\text{Constant}$.

$\therefore T(t)=D(t+2)^{-n^2}$, where $D=\pm e^{\text{constant}}$.

As $T(t)$ depends on n, we write $T_n(t) = D_n (t+2)^{-n^2}$.

$\therefore U_n(x,t) = X_n(x) T_n(t) = B_n \sin nx \, D_n (t+2)^{-n^2} = B_n D_n \sin nx \, (t+2)^{-n^2}$.

$\therefore U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} B_n D_n \sin nx \cdot (t+2)^{-n^2}$.

Step 4: Apply the initial condition.

$$u(x,0) = \sum_{n=1}^{\infty} B_n D_n \sin nx \, 2^{-n^2} = \sin x + 4 \sin 2x .$$

This implies: $n=1$: $B_1 D_1 \sin x \, 2^{-1} = \sin x \Rightarrow B_1 D_1 = 2$.

$n=2$: $B_2 D_2 \sin 2x \, 2^{-4} = 4 \sin 2x \Rightarrow B_2 D_2 = 2^6 = 64$.

$n \geq 3$: $B_n D_n = 0$.

Therefore the solution $u(x,t)$ is

$$u(x,t) = \frac{2 \sin nx}{t+2} + \frac{64 \sin 2x}{(t+2)^4}.$$