1. $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix} = \langle -4, 5, 2\rangle$. 2. $\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & y^2-x^2 & 2z^2 \end{vmatrix} = \langle 0-0, 0-0, -2x-2y\rangle = \langle 0, 0, -2x-2y\rangle$

3. $\text{div}\vec{F} = e^x\cos y - e^x\cos y + a + 3 = 0 \quad \therefore a = -3$

5. $A=3,\ B=\frac{1}{2},\ C=1 \quad \therefore AC-B^2 = 3-\frac{1}{4} = \frac{11}{4} > 0$ $\therefore$ it is elliptic.

4. $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1\rangle$

1. If $\boldsymbol{a} = \langle 1,0,2 \rangle$ and $\boldsymbol{b} = \langle 3,2,1 \rangle$, then $\boldsymbol{a}\times\boldsymbol{b} =$ $\langle -4, 5, 2\rangle$.
2. For a given $\boldsymbol{F} = \langle y^2,\ y^2-x^2,\ 2z^2 \rangle$, curl$\boldsymbol{F}$= $\langle 0, 0, -2x-2y\rangle$.
3. If $\boldsymbol{F} = \langle e^x\cos y,\ -e^x\sin y + ay,\ 3z \rangle$ and div$\boldsymbol{F}$=0, then a= -3.
4. A normal vector for surface $\boldsymbol{r}(u,v) = \langle u,\ v,\ 1-u^2-v^2 \rangle$ is $\langle 2u, 2v, 1\rangle$.
5. The type of the PDE $3u_{xx} + u_{xy} + u_{yy} = 0$ is elliptic.
6. If $\boldsymbol{F} = \langle x,\ 2y+3 \rangle$ and the curve C is $\boldsymbol{r}(t) = \langle t,\ t^2-1 \rangle$, $0 \le t \le 1$, then the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} =$ $\frac{5}{2}$. $\vec{F}(\vec{r}(t)) = \langle t, 2t^2+1\rangle$, $\vec{r}'(t) = \langle 1, 2t\rangle$. $\therefore \vec{F}\cdot\vec{r}'(t) = t + 4t^3 + 2t = 4t^3 + 3t \quad \therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 4t^3 + 3t\,dt = \left[t^4 + \frac{3}{2}t^2\right]_0^1 = 1 + \frac{3}{2} = \frac{5}{2}$
7. Show that the line integral

8

$$\int_{(0,0,0)}^{(1,1,1)} (6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz+1)dz$$

is independent of path and evaluate it.

Assume $\vec{F} = \langle 6xy^3 + 2z^2, 9x^2y^2, 4xz+1\rangle = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$.

$\therefore \begin{cases} \frac{\partial f}{\partial x} = 6xy^3 + 2z^2 \\ \frac{\partial f}{\partial y} = 9x^2y^2 \\ \frac{\partial f}{\partial z} = 4xz+1 \end{cases}$ $\Rightarrow f(x,y,z) = 3x^2y^3 + 2xz^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = 9x^2y^2 + \frac{\partial g}{\partial y} = 9x^2y^2$.

$\therefore \frac{\partial g}{\partial y} = 0 \quad \therefore g(y,z) = h(z)$. $\therefore f(x,y,z) = 3x^2y^3 + 2xz^2 + h(z)$.

$\therefore \frac{\partial f}{\partial z} = 4xz + h'(z) = 4xz + 1$.

$\therefore h'(z) = 1 \quad \therefore h(z) = z + \text{Constant}$.

$\therefore f(x,y,z) = 3x^2y^3 + 2xz^2 + z + \text{Constant}$.

$\therefore \int_{(0,0,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = f(1,1,1) - f(0,0,0) = 3 + 2 + 1 + \text{Constant} - 0 - \text{Constant} = 6$.

8. Use Green's theorem to evaluate the line integral

5

$$\oint_C (x^3 + 2y)dx + (4x - 3y^2)dy,$$

Where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $F_1 = x^3 + 2y$, $F_2 = 4x - 3y^2$, then by Green's theorem

$\oint_C F_1dx + F_2dy = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dxdy = \iint_S 4 - 2\,dxdy = 2\iint_S dxdy$

$= 2 \times$ Area of the ellipse $= 2\pi ab$.

9. Find the general solution $u(x,y)$ of the PDE

$$u_y + 2yu = 0.$$

At first, consider the PDE as an ODE: $u' + 2yu = 0$, which can be written as $\frac{u'}{u} = -2y$. Integrate this equation both hand sides, with respect to y, we have

$\int \frac{u'}{u} dy = \int -2y\,dy \Rightarrow \int \frac{du}{u} = -y^2 + C$, where C is an arbitrary constant.

$\Rightarrow \ln|u| = e^{-y^2+C}$ $\therefore u(x,y) = \pm e^{C} \cdot e^{-y^2} = Ae^{-y^2}$, where A is an arbitrary constant.

So the general solution for the PDE is $u(x,y) = A(x)e^{-y^2}$, where $A(x)$ is an arbitrary function of x.

10. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dA$.

10

(a) $\boldsymbol{F} =< x^2,\ y^2,\ 1 >,\ \boldsymbol{r}(u,v) =< u,\ v,\ 2u - 3v >, 0 \le u \le 1, 0 \le v \le 2.$
(b) $\boldsymbol{F} =< yz,\ x + y,\ e^x\cos y + z >$, S is the surface of $2 \le x^2 + y^2 + z^2 \le 4$.

(a). $\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R \vec{F} \cdot \vec{N}\, du\, dv$, where $\vec{N} = \vec{r}_u \times \vec{r}_v$.

$\vec{r}_u = \langle 1, 0, 2 \rangle$, $\vec{r}_v = \langle 0, 1, -3 \rangle$ $\therefore \vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix} = \langle -2, 3, 1 \rangle.$

$\therefore \vec{F} \cdot \vec{N} = \langle u^2, v^2, 1 \rangle \cdot \langle -2, 3, 1 \rangle = -2u^2 + 3v^2 + 1.$

$\therefore \iint_S \vec{F} \cdot \vec{n}\, dA = \int_0^2 \int_0^1 -2u^2 + 3v^2 + 1\, du\, dv = \int_0^2 \left[-\frac{2}{3}u^3 + 3uv^2 + u \right]_0^1 dv$

$= \int_0^2 3v^2 + \frac{1}{3} dv = \left[v^3 + \frac{1}{3}v \right]_0^2 = 8 + \frac{2}{3} = \frac{26}{3}.$

(b). By the divergence theorem,

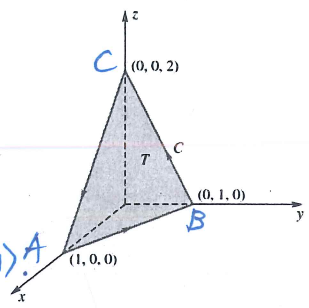
$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \iiint_T 0 + 1 + 1\, dV = 2 \iiint_T dV$

$= 2 \cdot \left[\frac{4}{3}\pi \cdot 2^3 - \frac{4}{3}\pi (\sqrt{2})^3 \right] = 2 \cdot \left(\frac{32\pi}{3} - \frac{8\sqrt{2}\pi}{3} \right) = \frac{64 - 16\sqrt{2}}{3}\pi.$

8 11. Using Stokes's theorem to evaluate $\oint_C \mathbf{F}\cdot\mathbf{n}ds$, where $F =< 2z,\ 8x-3y,\ 3x+y >$ and C is the triangular curve in the figure.

By Stokes's theorem, $\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_T \text{curl}\,\vec{F}\cdot\vec{n}\,dA$,

where $\vec{n}$ is the unit normal vector of planar surface T.



One normal vector $\vec{N}$ of the planar triangle T is:

$$\vec{N} = \vec{AB}\times\vec{AC} = \langle -1,1,0\rangle \times \langle -1,0,2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \langle -2,-2,-1\rangle$$

As the direction of C is anti-clockwise, we should take $\vec{n}$ as upward.

$$\therefore \vec{n} = -\frac{\langle -2,-2,-1\rangle}{\sqrt{4+4+1}} = \langle \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}\rangle$$

And the curl of $\vec{F}$ is $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 8x-3y & 3x+y \end{vmatrix} = \langle 1,-1,8\rangle$

$$\therefore \iint_T \text{curl}\,\vec{F}\cdot\vec{n}\,dA = \iint_T \langle 1,-1,8\rangle\cdot\langle \tfrac{2}{3},\tfrac{2}{3},\tfrac{1}{3}\rangle dA = \tfrac{8}{3}\times \text{Area of } T = \tfrac{8}{3}\times\tfrac{3}{2} = 4.$$

12. A period function of period $2L$ is defined by

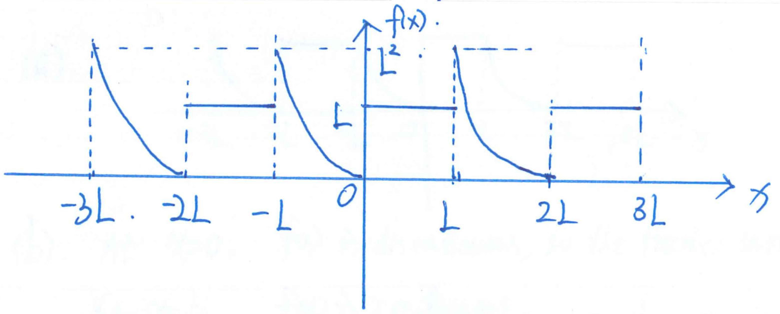
$$f(x) = \begin{cases} x^2, & -L \le x < 0 \\ L, & 0 \le x < L \end{cases}.$$

2 (a) Sketch the graph of $f(x)$ in the range $-3L \le x \le 3L$.

4 (b) State the values the Fourier series will converge to at $x = 0, \frac{L}{2}, L, \frac{3L}{2}$.

14 (c) Find the Fourier series of $f(x)$ in $-L \le x < L$ and give the first three non-zero terms.

(a).



(b) At $x=0$, $f(x)$ is discontinuous, so the Fourier series converges to $\frac{1}{2}[f(0^+)+f(0^-)] = \frac{L}{2}$

At $x=\frac{L}{2}$, $f(x)$ is continuous, ... $f(\frac{L}{2}) = L$

At $x=L$, $f(x)$ is discontinuous, ... $\frac{1}{2}[f(L^+)+f(L^-)] = \frac{L^2+L}{2}$

At $x=\frac{3L}{2}$, $f(x)$ is continuous, ... $f(\frac{3}{2}L) = \frac{L^2}{4}$

(c). $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{0} x^2 dx + \frac{1}{2L}\int_{0}^{L} L\, dx = \frac{1}{2L}\cdot\left[\frac{1}{3}x^3\right]_{-L}^{0} + \frac{1}{2L}\cdot[Lx]_0^L$.

$= \frac{L^2}{6} + \frac{L}{2}$.

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_{-L}^{0} x^2\cos\frac{n\pi}{L}x\,dx + \frac{1}{L}\int_0^L L\cos\frac{n\pi}{L}x\,dx = (I_1 + I_2)\frac{1}{L}$.

$\therefore I_1 = \int_{-L}^{0} x^2\cos\frac{n\pi}{L}x\,dx$. Let $u = x^2$, $v' = \cos\frac{n\pi}{L}x$ $\therefore u' = 2x$, $v = \frac{L}{n\pi}\sin\frac{n\pi}{L}x$

$= \left[x^2\cdot\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_{-L}^{0} - \int_{-L}^{0} 2x\cdot\frac{L}{n\pi}\sin\frac{n\pi}{L}x\,dx = -\frac{2L}{n\pi}\int_{-L}^{0} x\sin\frac{n\pi}{L}x\,dx$.

Let $u = x$, $v' = \sin\frac{n\pi}{L}x$ $\therefore$ $u' = 1$, $v = -\frac{L}{n\pi}\cos\frac{n\pi}{L}x$.

$= -\frac{2L}{n\pi}\cdot\left[x\cdot\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_{-L}^{0} + \frac{2L}{n\pi}\cdot\int_{-L}^{0} -\frac{L}{n\pi}\cos\frac{n\pi}{L}x\,dx$.

$= \frac{2L^2}{n^2\pi^2}[0 + L\cos n\pi] - \frac{2L^2}{n^2\pi^2}\cdot\int_{-L}^{0}\cos\frac{n\pi}{L}x\,dx$

$= \frac{2L^3}{n^2\pi^2}\cos n\pi - \frac{2L^2}{n^2\pi^2}\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi}{L}x\right]_{-L}^{0} = \frac{2L^3}{n^2\pi^2}\cos n\pi$.

$I_2 = \int_0^L L\cos\frac{n\pi}{L}x\,dx = L\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi}{L}x\right]_0^L = 0$

$\therefore a_n = \frac{1}{L}\cdot(I_1 + I_2) = \frac{1}{L}\cdot\frac{2L^3}{n^2\pi^2}\cos n\pi = \frac{2L^2}{n^2\pi^2}\cdot\cos n\pi$.

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_{-L}^{0} x^2\sin\frac{n\pi}{L}x\,dx + \frac{1}{L}\int_0^L L\sin\frac{n\pi}{L}x\,dx = (I_1 + I_2)\frac{1}{L}$.

$I_1 = \int_{-L}^{0} x^2\sin\frac{n\pi}{L}x\,dx$, Let $u = x^2$, $v' = \sin\frac{n\pi}{L}x$ $\therefore u' = 2x$, $v = -\frac{L}{n\pi}\cos\frac{n\pi}{L}x$.

$= \left[x^2\cdot\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_{-L}^{0} - \int_{-L}^{0} 2x\cdot\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\,dx$.

$= 0 - L^2\cdot\frac{-L}{n\pi}\cos n\pi. + \frac{2L}{n\pi}\int_{-L}^{0} x\cos\frac{n\pi}{L}x\,dx$. Let $u = x$, $v' = \cos\frac{n\pi}{L}x$, $\therefore u' = 1$, $v = \frac{L}{n\pi}\sin\frac{n\pi}{L}x$

$= \frac{L^3}{n\pi}\cos n\pi + \frac{2L}{n\pi}\left\{\left[x\cdot\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_{-L}^{0} - \int_{-L}^{0}\frac{L}{n\pi}\sin\frac{n\pi}{L}x\,dx\right\}$.

$= \frac{L^3}{n\pi}\cos n\pi - \frac{2L^2}{n^2\pi^2}\cdot\int_{-L}^{0}\sin\frac{n\pi}{L}x\,dx = \frac{L^3}{n\pi}\cos n\pi - \frac{2L^2}{n^2\pi^2}\,\frac{-L}{n\pi}\left[\cos\frac{n\pi}{L}x\right]_{-L}^{0}$

$= \frac{L^3}{n\pi}\cos n\pi + \frac{2L^3}{n^3\pi^3}(1 - \cos n\pi)$.

$I_2 = \int_0^L L\sin\frac{n\pi}{L}x\,dx = L\cdot\frac{-L}{n\pi}\left[\cos\frac{n\pi}{L}x\right]_0^L = \frac{-L^2}{n\pi}\cdot(\cos n\pi - 1) = \frac{L^2}{n\pi}(1 - \cos n\pi)$.

$\therefore b_n = \frac{1}{L}(I_1 + I_2) = \frac{L^2}{n\pi}\cos n\pi + \frac{2L^2}{n^3\pi^3}(1 - \cos n\pi) + \frac{L}{n\pi}\cdot(1 - \cos n\pi) = \frac{L^2}{n\pi}\cos n\pi + \left(\frac{2L^2}{n^3\pi^3} + \frac{L}{n\pi}\right)(1 - \cos n\pi)$.

$$\therefore\ f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x$$

$$= \frac{L^2}{6} + \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L^2}{n^2\pi^2}\cos n\pi\cdot\cos\frac{n\pi}{L}x + \left[\frac{L^2}{n\pi}\cos n\pi + \left(\frac{2L^2}{n^3\pi^3}+\frac{L}{n\pi}\right)(1-\cos n\pi)\right]\sin\frac{n\pi}{L}x.$$

$n=0,\quad a_0 = \frac{L^2}{6}+\frac{L}{2}.$

$n=1:\quad a_1\cos\frac{\pi}{L}x + b_1\sin\frac{\pi}{L}x = \frac{2L^2}{\pi^2}\cos(\pi)\cos\frac{\pi}{L}x + \left[\frac{L^2}{\pi}\cos\pi + \left(\frac{2L^2}{\pi^3}+\frac{L}{\pi}\right)(1-\cos\pi)\right]\sin\frac{\pi}{L}x$

$$= -\frac{2L^2}{\pi^2}\cos\frac{\pi}{L}x + \left(\frac{-L^2}{\pi} + \frac{4L^2}{\pi^3} + \frac{2L}{\pi}\right)\sin\frac{\pi}{L}x.$$

$n=2:\quad a_2\cos\frac{2\pi}{L}x + b_2\sin\frac{2\pi}{L}x = \frac{2L^2}{4\pi^2}\cos 2\pi\cos\frac{2\pi}{L}x + \left[\frac{L^2}{2\pi}\cos 2\pi + \left(\frac{2L^2}{8\pi^3}+\frac{L}{2\pi}\right)(1-\cos 2\pi)\right]\sin\frac{2\pi}{L}x.$

$$= \frac{2L^2}{4\pi^2}\cos\frac{2\pi}{L}x + \frac{L^2}{2\pi}\sin\frac{2\pi}{L}x.$$

13. Given the PDE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0, 0 \le x < a, a > 0,$$

Where the function $u(x,t)$ satisfies the boundary conditions

$$u(0,t) = u(a,t) = 0, t \ge 0,$$

and the initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{n\pi x}{a}$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{\sqrt{n^2\pi^2+a^2}}{\sqrt{n}\,a} \sin\frac{n\pi x}{a}.$$

4. (a) Using separation of variables with $u(x,t) = X(x)T(t)$, deduce that $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X''(x) + \alpha_n^2 X(x) = 0$$

and

$$T''(t) + (\alpha_n^2 + 1)T(t) = 0,$$

where α_n is a constant.

6 (b) Solve the first ODE and show that $X(x) = A_n \sin\frac{n\pi}{a}x$, where A_n is a constant.

10. (c) Show that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{n\pi x}{a}\left[\cos\frac{\sqrt{n^2\pi^2+a^2}}{a}t + \sin\frac{\sqrt{n^2\pi^2+a^2}}{a}t\right].$$

(a) As $u(x,t) = X(x)T(t)$ $\therefore$ $\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$, $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$.

$\therefore XT'' - X''T + XT = 0$. $\therefore \frac{T''}{T} - \frac{X''}{X} + 1 = 0$

$\therefore \frac{T''}{T} + 1 = \frac{X''}{X} = \text{constant}$

Assume the constant is $-\alpha_n^2$, then $\begin{cases} X'' + \alpha_n^2 X = 0 \\ T'' + (\alpha_n^2+1)T = 0 \end{cases}$

(b). $X''(x) + \alpha_n^2 X(x) = 0$, the characteristic equation is

$\lambda^2 + \alpha_n^2 = 0$ $\therefore$ $\lambda = \pm \alpha_n i$.

$\therefore X(x) = A_n \cos\alpha_n x + B_n \sin\alpha_n x$.

Now we apply the boundary conditions: $u(0,t) = u(a,t) = 0$,

$\therefore \begin{cases} X(0)T(t) = 0 \\ X(a)T(t) = 0 \end{cases}$ $\therefore X(0) = X(a) = 0$, otherwise zero solution will be obtained.

As $X(x) = A_n \cos\alpha_n x + B_n \sin\alpha_n x$, we have

$$\begin{cases} X(0) = A_n + B_n \cdot 0 = 0 & \Rightarrow A_n = 0. \\ X(a) = A_n \cos\alpha_n a + B_n \sin\alpha_n a = 0 \Rightarrow & B_n \sin\alpha_n a = 0 \quad (*). \end{cases}$$

From $(*)$, we know either $B_n = 0$ or $\sin\alpha_n a = 0$. If $B_n = 0$, then $X(x) \equiv 0$.

So $u(x,t) \equiv 0$, this is trivial solution, and we are not interested in it.

Therefore, $\sin\alpha_n a = 0$ $\therefore$ $\alpha_n a = n\pi$, $n = 1, 2, \cdots$ $\therefore$ $\alpha_n = \frac{n\pi}{a}$, $n = 1, 2, \cdots$.

$\therefore X(x) = B_n \sin\frac{n\pi}{a}x$, where B_n is a constant. $n = 1, 2, \cdots$.

(c). Now we solve the second ODE $T'' + (\alpha_n^2 + 1)T = 0$.

The characteristic equation is $\lambda^2 + (\alpha_n^2 + 1) = 0$ $\therefore \lambda = \pm\sqrt{\alpha_n^2 + 1}\, i$.

$$\therefore T(t) = C_n \cos\sqrt{\alpha_n^2+1}\, t + D_n \sin\sqrt{\alpha_n^2+1}\, t = C_n \cos\sqrt{\frac{n^2\pi^2}{a^2}+1}\, t + D_n \sin\sqrt{\frac{n^2\pi^2}{a^2}+1}\, t.$$

$$= C_n \cos\frac{\sqrt{n^2\pi^2+a^2}}{a} t + D_n \sin\frac{\sqrt{n^2\pi^2+a^2}}{a} t.$$ 2.

Therefore the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \cdot \left[C_n \cos\frac{\sqrt{n^2\pi^2+a^2}}{a} t + D_n \sin\frac{\sqrt{n^2\pi^2+a^2}}{a} t \right].$$

$$\therefore \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \cdot \left[-C_n \frac{\sqrt{n^2\pi^2+a^2}}{a} \sin\frac{\sqrt{n^2\pi^2+a^2}}{a} t + D_n \frac{\sqrt{n^2\pi^2+a^2}}{a} \cos\frac{\sqrt{n^2\pi^2+1}}{a} t \right]$$

$$= \sum_{n=1}^{\infty} \left\{ -B_n C_n \frac{\sqrt{n^2\pi^2+a^2}}{a} \sin\frac{\sqrt{n^2\pi^2+a^2}}{a} t \sin\frac{n\pi}{a}x + B_n D_n \frac{\sqrt{n^2\pi^2+a^2}}{a} \cos\frac{\sqrt{n^2\pi^2+1}}{a} t \sin\frac{n\pi}{a}x \right\}.$$

$$\therefore u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x \cdot C_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{n\pi x}{a} \quad \therefore \quad B_n C_n = \frac{1}{\sqrt{n}}.$$

$$u_t(x,0) = \sum_{n=1}^{\infty} 0 + B_n D_n \cdot \frac{\sqrt{n^2\pi^2+a^2}}{a} \sin\frac{n\pi}{a}x = \sum_{n=1}^{\infty} \frac{\sqrt{n^2\pi^2+a^2}}{a} \cdot \frac{1}{\sqrt{n}} \cdot \sin\frac{n\pi x}{a}, \quad \therefore B_n D_n = \frac{1}{\sqrt{n}}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cdot \sin\frac{n\pi x}{a} \cdot \left[\cos\frac{\sqrt{n^2\pi^2+a^2}}{a} t + \sin\frac{\sqrt{n^2\pi^2+a^2}}{a} t \right]$$