EEE319 Optimisation
Lecture 4 Linear Programming (1)

Prof. Xinheng Wang

xinheng.wang@xjtlu.edu.cn

Office: EE512

Outline

- Last week
  - Constraints
  - Graphical optimisation
    - Iso-profit/iso-cost
    - Extreme points
    - Two-variable linear function
- Linear programming
  - Optimisation of general linear functions
  - Algorithms
    - Simplex Method

Graphical optimisation

- Limited to two-variable function
- Both iso-profit/iso-cost and extreme points methods are manual

- Question: is it possible to use a method to find out the solutions by a computer?

Origin of Linear Programming

- Linear programming:
- The word programming was borrowed from military, which refers to activities such as planning schedules efficiently or deploying men optimally.
- Linear programming was invented during WWII, by George Dantzig, a member of US Air Force and developed Simplex Method.
- Linear programming requires that all the mathematical functions in the model are linear functions.

Simplex Method

- Simplex method is an iterative algorithm to solve linear programming problem
  - Algorithm: a set of mathematical instructions or rules that, especially if given to a computer, will help to calculate an answer to a problem.
  - Iterative algorithm: an iterative method/algorithm is a mathematical procedure that uses an initial guess to generate a sequence of improving approximate solutions for a class of problems, in which the *n*-th approximation is derived from the previous ones. (Wikipedia)
    - Recall from Lecture 2 Gradient Descent algorithm

$$x_{i+1} = x_i - a\nabla f(x_i)$$

  - Question: where and how to terminate the iterations?

Simplex Method

- Algorithm termination
  - Three aspects of a termination criterion:
    1. Error is small enough to stop;
    2. Stop if the error is no longer decreasing or decreasing slowly, and
    3. Set a maximum iteration number.

  - It is also called convergence criteria. If all the conditions are satisfied, we say that the algorithm is converged. The speed of the convergence is called Convergence Rate.

Simplex Method

- Simplex method – a method to find the optimal values from extreme points
  - Recall from last week, an optimisation problem

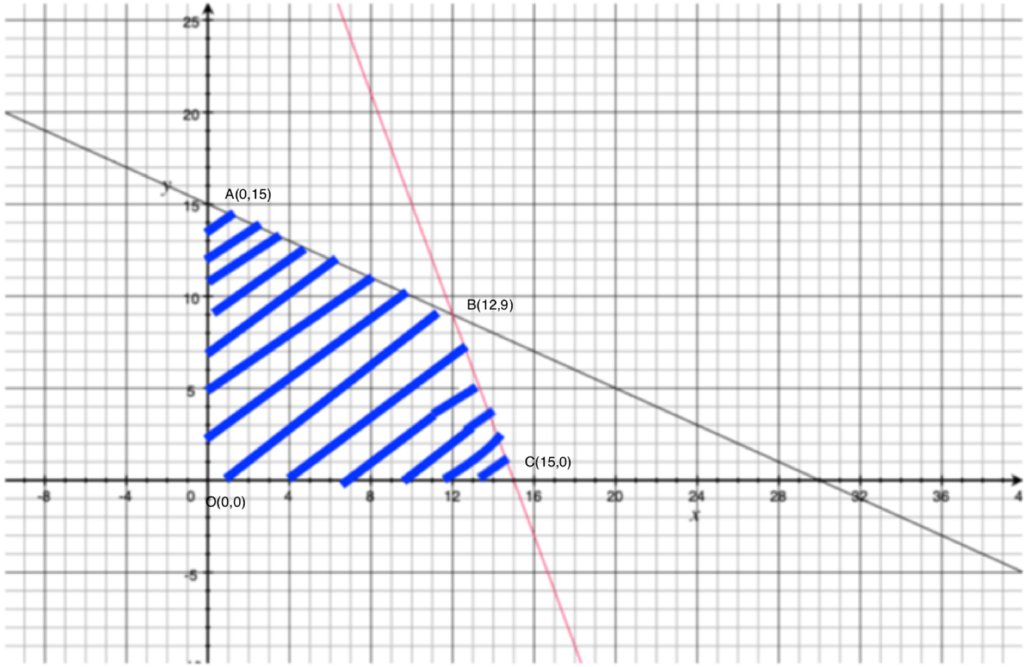
    $\max Z = 4x + 3y$

    Subject to $x + 2y \leq 30$

    $3x + y \leq 45$, and

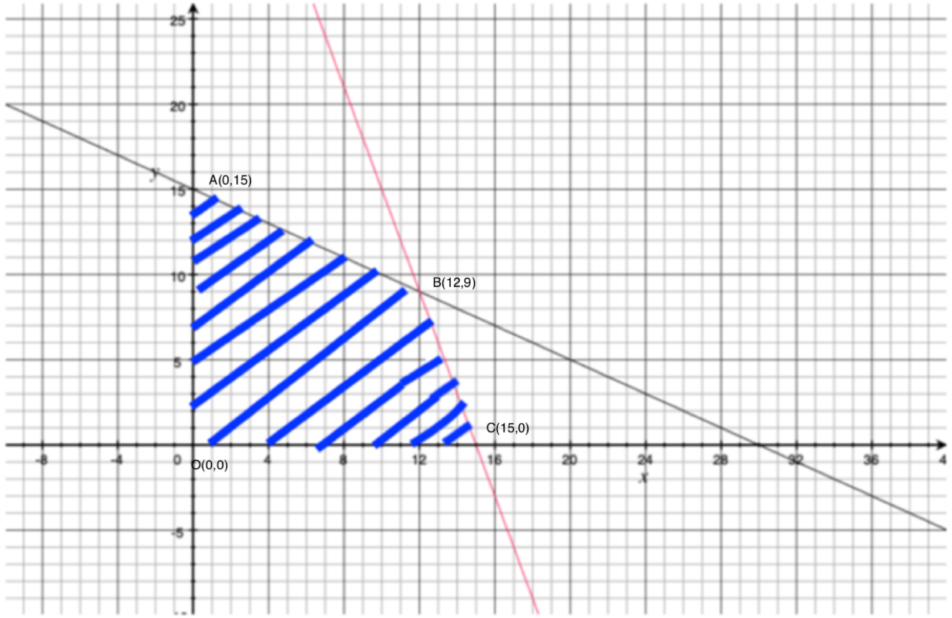
    $x \geq 0, y \geq 0$

  - The optimal value could be at one of the extreme points in the feasible region



Simplex Method

- Simplex method – a method to find the optimal values from extreme points
  - Extreme points are the intersections of the constraints (inequality to equality). They are also called vertex (vertices).
  - Key idea of Simplex Method: At each iteration of the simplex method, the algorithm starts at one vertex of the feasible region and moves along an edge to the next vertex.
- Question: how to find out the starting point?



Simplex Method

- A few important concepts used for Simplex Method
  - Extra variables
    - Slack variable
    - Surplus variable
    - Basic variable
    - Non-basic variable

Simplex Method - Steps

- Steps from an Example

$\max\ 5x_1+4x_2+3x_3$

s.t. $2x_1+3x_2+x_3 \leq 5$

$4x_1+x_2+2x_3 \leq 11$

$3x_1+4x_2+2x_3 \leq 8$

$x_1, x_2, x_3 \geq 0$

Simplex Method - Steps

- Step 1 transform inequality into equalities

$z = 5x_1 + 4x_2 + 3x_3$

$2x_1 + 3x_2 + x_3 = 5$

$4x_1 + x_2 + 2x_3 = 11$

$3x_1 + 4x_2 + 2x_3 = 8$

Simplex Method - Steps

- Step 2 insert a non-negative variable into constraints

$2x_1+3x_2+x_3 + x_4 = 5$

$4x_1+x_2+2x_3 + x_5 = 11$

$3x_1+4x_2+2x_3 + x_6 = 8$

- Another form:

$$
\begin{aligned}
2x_1+3x_2+x_3 + x_4 \qquad\qquad\quad &= 5 \\
4x_1+x_2+2x_3 \qquad + x_5 \quad &= 11 \\
3x_1+4x_2+2x_3. \qquad\quad +x_6 &= 8
\end{aligned}
$$

Variables x_4, x_5, x_6 are called slack variables because they take up the slack in the equations.

Slack variables are for $\leq$ constraints

Simplex Method - Steps

- Step 3 Rearrange the equations, including objective function

$z = 5x_1 + 4x_2 + 3x_3$

$x_4 = 5 - 2x_1 - 3x_2 - x_3$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

z, x_4, x_5, and x_6 are the linear combinations of variables x_1, x_2 and x_3. z, x_4, x_5, and x_6 in this form are called basic variables and x_1, x_2 and x_3 are called non-basic variables.

Simplex Method - Steps

- Step 4 Set the non-basic variables to zero (0), one solution could be obtained:

$z = 5x_1 + 4x_2 + 3x_3 = 0$

$x_4 = 5 - 2x_1 - 3x_2 - x_3 = 5$

$x_5 = 11 - 4x_1 - x_2 - 2x_3 = 11$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3 = 8$

This solution is called basic feasible solution.

Strategy of Simplex Method: Increase the value of one non-basic variable, and simultaneously increase the value of the objective function. At the same time, to keep the value of all variables to positive. So it is also called integer programming.

Question: Which variable goes for the first?

Simplex Method - Steps

- Step 5 Select first non-basic variable:
  - Have a look at the objective function $z = 5x_1$ +$4x_2$+$3x_3$, the coefficient of x_1 is the greatest, we can increase *z* the fastest by increasing x_1.
  - Set two other variables to zero.

x_4 = 5- $2x_1$-$3x_2$-x_3 = 5 −$2x_1$

x_5 = 11- $4x_1$-x_2-$2x_3$ = 11 − $4x_1$

x_6 = 8- $3x_1$-$4x_2$-$2x_3$ = 8 −$3x_1$

- Increase the value x_1, but maintain the variables x_4, x_5, and x_6 positive.

Simplex Method - Steps

- Step 5 Select first non-basic variable:

$x_4 = 5\text{-}\ 2x_1\text{-}3x_2\text{-}x_3 = 5 - 2x_1$

$x_5 = 11\text{-}\ 4x_1\text{-}x_2\text{-}2x_3 = 11 - 4x_1$

$x_6 = 8\text{-}\ 3x_1\text{-}4x_2\text{-}2x_3 = 8 - 3x_1$

- Increase the value x_1, but maintain the variables x_4, x_5, and x_6 positive.
- In this case, x_1 cannot be greater than 5/2 for x_4; 11/4 for x_5; and 8/3 for x_6.
- Take the minimum [5/2, 11/4, and 8/3], which is 5/2.

Simplex Method - Steps

- Step 6 Swap the variables:
- If we increase the value x_1 to 5/2, the variable x_4 is driven to 0. Think it the other way is that x_1 enters basis whist x_4 leaves the basis. We will have a new equation of x_1 being a basic variable and x_4 being a non-basic variable. Thus equation $x_4 = 5$- $2x_1$-$3x_2$-x_3 becomes

$$2x_1 = 5 - x_4 - 3x_2 - x_3$$
$$x_1 = 5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

- This process is also called pivoting

Simplex Method - Steps

- Step 7 Replace x_1 in two other constraint functions and the objective function:

$$x_1 = 5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

- $x_5 = 11 - 4x_1 - x_2 - 2x_3 = 11 - 4(5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - x_2 - 2x_3 = 1 + 2x_4 + 5x_2$
- $x_6 = 8 - 3x_1 - 4x_2 - 2x_3 = 8 - 3(5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3) - 4x_2 - 2x_3 = \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}x_3$
- $z = 5x_1 + 4x_2 + 3x_3 = 5(5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3) + 4x_2 + 3x_3$

$$= \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

Simplex Method - Steps

- Step 7 Replace x_1 in two other constraint functions and the objective function
  - Rewrite the equations as

$$x_1 = 5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

$$x_5 = 1 + 2x_4 + 5x_2$$

$$x_6 = \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}\ x_3$$

$$z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$$

The basic feasible solution is 25/2.

Simplex Method - Steps

- Step 8 Repeat the process. This time the coefficient of x_3 is positive and shall make the greatest impact. Set two other variables to zero. The possibility to keep variables positive is $x_3 = 5$ for x_1, any value for x_5, and $x_3 = 1$ for x_6. The minimum [5, 1] is 1.
- $x_1 = 5/2 - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3$
- $x_5 = 1 + 2x_4 + 5x_2$
- $x_6 = \frac{1}{2} + \frac{3}{2}x_4 + \frac{1}{2}x_2 - \frac{1}{2}\ x_3$
- $z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$

Simplex Method - Steps

- Step 8 Repeat the process. This time the coefficient of x_3 is positive and shall make the greatest impact. Set two other variables to zero. The possibility to keep variables positive is $x_3 = 5$ for x_1, any value for x_5, and $x_3 = 1$ for x_6. The minimum [5, 1] is 1.
- $x_6 = \frac{1}{2} + \frac{3}{2} x_4 + \frac{1}{2} x_2 - \frac{1}{2}\ x_3$
- $x_3 = 1 - 2x_6 + 3x_4 + x_2$

Simplex Method - Steps

- Substitute x_3 into other equations
  - $x_3 = 1 - 2x_6 + 3x_4 + x_2$
  - $x_1 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2}x_3 = \frac{5}{2} - \frac{1}{2}x_4 - \frac{3}{2}x_2 - \frac{1}{2} + x_6 - \frac{3}{2}x_4 - \frac{1}{2}x_2 = 2 - 2x_4 + x_6 - 2x_2$
  - $x_5 = 1 + 2x_4 + 5x_2$
  - $z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3 = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}(1 - 2x_6 + 3x_4 + x_2) = 13 - x_4 - x_6 - 3x_2$

The basic feasible solution is 13.

Simplex Method - Steps

- Compare feasible solutions
  - $z = 5x_1 + 4x_2 + 3x_3 = 0$
  - $z = \frac{25}{2} - \frac{5}{2}x_4 - \frac{7}{2}x_2 + \frac{1}{2}x_3$
  - $z = 13 - x_4 - x_6 - 3x_2$

Feasible solutions: 0, 25/2 and 13.

Have a look at the last equation, the coefficients of all non-basic variables are all negative and cannot increase the value of *z* anymore. Therefore, 13 is the optimal value and the iteration is terminated here.

Simplex Method

- The simplex algorithm is consistently ranked as one of the ten most important algorithmic discoveries of the 20th century.

- Question: How about minimization problems?

Simplex Method

- One method to solve the minimization problem is to transform the minimization to a maximization problem
  - Method 1 Multiply the objective function by -1
  - Method 2 Duality

Transformations of Minimization and Maximization - Method 1

- Minimize an objective function *f*(*x*) is equivalent to maximize -*f*(*x*). In another word, to multiply -1 to its original function.
- For example

Min $3x_1 - 4x_2$ is equivalent to Max $-3x_1 + 4x_2$

- How about constraints?
  - Keep the same

Transformations of Minimization and Maximization - Method 1

- Example

Min $z = 2x_1 - 3x_2$
s.t. $x_1 + x_2 \leq 4$
$x_1 - x_2 \leq 6$
$x_1, x_2 \geq 0$

is equivalent to

max $w = -2x_1 + 3x_2$
s.t. $x_1 + x_2 \leq 4$
$x_1 - x_2 \leq 6$
$x_1, x_2 \geq 0$

Transformations of Minimization and Maximization - Duality

- Using the previous example

  Min $z = 2x_1 - 3x_2$

  s.t. $x_1 + x_2 \leq 4$

  $x_1 - x_2 \leq 6$

  $x_1, x_2 \geq 0$

Transformations of Minimization and Maximization - Duality

- Using the previous example

  Min $z = 2x_1 - 3x_2$

  s.t. $x_1 + x_2 \leq 4$

  $x_1 - x_2 \leq 6$

  $x_1, x_2 \geq 0$

- Steps of Duality
  - Step1: Put all the coefficients of constraints (go first) and objective function into a matrix.

$$\begin{matrix} 1 & 1 & 4 \\ 1 & -1 & 6 \\ 2 & -3 & 1 \end{matrix}$$

Transformations of Minimization and Maximization - Duality

- Using the previous example

  Min $z = 2x_1 - 3x_2$

  s.t. $x_1 + x_2 \leq 4$

  $x_1 - x_2 \leq 6$

  $x_1, x_2 \geq 0$

- Steps of Duality
  - Step2: Transpose the matrix.

$$\begin{matrix} 1 & 1 & 4 \\ 1 & -1 & 6 \\ 2 & -3 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & 2 \\ 1 & -1 & -3 \\ 4 & 6 & 1 \end{matrix}$$

Transformations of Minimization and Maximization - Duality

- Using the previous example

  Min $z = 2x_1 - 3x_2$

  s.t. $x_1 + x_2 \leq 4$

  $x_1 - x_2 \leq 6$

  $x_1, x_2 \geq 0$

- Steps of Duality
  - Step3: Re-write the new optimisation problem: minimization to maximization and also reverse the sign of the inequalities, by using different variables.

  Max $p = 4y_1 + 6y_2$

  s.t. $y_1 + y_2 \geq 2$

  $y_1 - y_2 \geq -3$

  $y_1, y_2 \geq 0$

$$\begin{matrix} 1 & 1 & 2 \\ 1 & -1 & -3 \\ 4 & 6 & 1 \end{matrix}$$

Summary

- Solved linear maximization problems
- Slack variables
- Transformation from minimization to maximization

- Next week
  - Tableau
  - Mixed constraints
  - Surplus and artificial variabls
  - Big M method

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