Review questions

1. Find the gradient for the following functions

   (1) $f(x, y, z) = \sin(xyz)$

   $\nabla f = \text{grad} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \cos(xyz)\langle yz, xz, xy \rangle.$

   (2) $f(x, y, z) = xe^y\cos z$

   $\nabla f = e^y \langle \cos z, x\cos z, -x\sin z \rangle$

   (3) $f(x, y, z) = y^2e^{-2z}$

   $\nabla f = 2ye^{-2z}\langle 0, 1, -y \rangle.$

2. Find div$\boldsymbol{F}$ and curl$\boldsymbol{F}$

   (1) $\boldsymbol{F} = < x^2, -2xy, yz^2 >$ $\quad \text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x - 2x + 2yz = 2yz.$

   $$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xy & yz^2 \end{vmatrix} = \vec{i}(z^2 - 0) - \vec{j}(0 - 0) + \vec{k}(-2y - 0)$$
   $$= \langle z^2, 0, -2y \rangle.$$

   (2) $\boldsymbol{F} = < e^x\cos y, e^x\sin y, z >$

   $$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = e^x\cos y + e^x\cos y + 1 = 2e^x\cos y + 1.$$

   $$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y & e^x\sin y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(e^x\sin y + e^x\sin y)$$
   $$= \langle 0, 0, 2e^x\sin y \rangle.$$

3. Assuming that the required partial derivatives exist and are continuous, show that

   (1) div(curl$\boldsymbol{F}$)=0 $\quad$ Assume $\vec{F} = \langle F_1, F_2, F_3 \rangle$

   $$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$
   $$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

   $$\text{So } \text{div}(\text{curl}\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

   (2) curl(grad$f$)=0

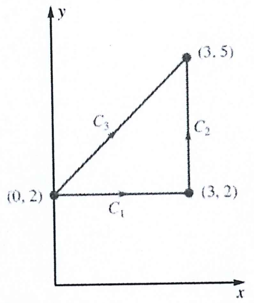
   $$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}} = 0.$$

   $$\text{grad} f = \nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle.$$

   $$\therefore \text{curl}(\text{grad} f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \vec{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$
   $$= \vec{0}.$$

4. (1) Evaluate $\int_C xy^2dx + xy^2dy$ along the path $C = C_1 \cup C_2$.

(2) Evaluate $\int_{C_3} xy^2dx + xy^2dy$ along C_3.



(1). $\int_C xy^2dx + xy^2dy = \int_{C_1} xy^2dx + xy^2dy + \int_{C_2} xy^2dx + xy^2dy$.

on C_1: $y=2$, $dy=0$, $0\le x\le 3$, so

$\int_{C_1} xy^2dx + xy^2dy = \int_0^3 x\cdot 4\cdot dx + 0 = \int_0^3 4x dx = [2x^2]_0^3 = 18$.

on C_2: $x=3$, $dx=0$, $2\le y\le 5$, so

$\int_{C_2} xy^2dx + xy^2dy = \int_2^5 0 + 3\cdot y^2dy = \int_2^5 3y^2dy = [y^3]_2^5 = 117$.

So $\int_C xy^2dx + xy^2dy = 18+117 = 135$.

(2). On C_3: $y = x+2$, $dy = dx$, $0\le x\le 3$, so

$\int_{C_3} xy^2dx + xy^2dy = \int_0^3 [x\cdot(x+2)^2 + x(x+2)^2]dx = 2\int_0^3 x(x+2)^2dx = 2\int_0^3 x^3+4x^2+4x\, dx$

$= 2[\frac{x^4}{4} + \frac{4x^3}{3} + 2x^2]_0^3 = \frac{29}{2}$

5. Evaluate $\int_C y^3dx + x^3dy$; C is the curve $x = 2t, y = t^2 - 3, -2 \le t \le 1$.

Method I: $x=2t \Rightarrow dx = 2dt$, $y = t^2-3 \Rightarrow dy = 2t\,dt$.

$\therefore \int_C y^3dx + x^3dy = \int_{-2}^1 (t^2-3)^3\cdot 2dt + (2t)^3\cdot 2t\,dt = \int_{-2}^1 2t^6 - 18t^4 + 54t^2 - 54 + 16t^4 dt$

$= \int_{-2}^1 2t^6 - 2t^4 + 54t^2 - 54 dt = [\frac{2}{7}t^7 - \frac{2}{5}t^5 + 18t^3 - 54t]_{-2}^1 = \frac{828}{35}$.

Method II: $\vec{F} = \langle y^3, x^3\rangle$. $\vec{r}(t) = \langle 2t, t^2-3\rangle$. $\therefore \vec{r}'(t) = \langle 2, 2t\rangle$, $\vec{F}(\vec{r}) = \langle (t^2-3)^3, (2t)^3\rangle$.

$\therefore \int_C y^3dx + x^3dy = \int_{-2}^1 \vec{F}\cdot\vec{r}'(t)dt = \int_{-2}^1 \langle (t^2-3)^3, (2t)^3\rangle\cdot\langle 2, 2t\rangle dt = \int_{-2}^1 2(t^2-3)^3 + 2t(2t)^3 dt = \frac{828}{35}$

6. Evaluate $\int_C xzdx + (y+z)dy + xdz$; C is the curve $x = e^t, y = e^{-t}, z = e^{2t}, 0 \le t \le 1$.

$x = e^t$, $y = e^{-t}$, $z = e^{2t}$, so $dx = e^t dt$, $dy = -e^{-t}dt$, $dz = 2e^{2t}dt$.

$\therefore \int_C xzdx + (y+z)dy + xdz = \int_0^1 e^t\cdot e^{2t}\cdot e^t dt + (e^{-t}+e^{2t})(-e^{-t})dt + e^t\cdot 2e^{2t}dt$

$= \int_0^1 e^{4t} + 2e^{3t} - e^t - e^{-2t} dt = [\frac{1}{4}e^{4t} + \frac{2}{3}e^{3t} - e^t + \frac{1}{2}e^{-2t}]_0^1$

$= \frac{1}{4}e^4 + \frac{2}{3}e^3 - e + \frac{1}{2}e^{-2} - \frac{1}{4} - \frac{2}{3} + 1 - \frac{1}{2} = \frac{1}{4}e^4 + \frac{2}{3}e^3 - e + \frac{1}{2e^2} - \frac{5}{12}$.

7. Find the work done by the force $\mathbf{F}$ in moving a particle along the curve C.

(1) $\mathbf{F}(x,y) = (x^3 - y^3)\mathbf{i} + xy^2\mathbf{j}$; C is the curve $x = t^2, y = t^3, -1 \le t \le 0$.

$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}'(t)dt$, where $\vec{r}(t) = \langle t^2, t^3\rangle$. $\therefore \vec{r}'(t) = \langle 2t, 3t^2\rangle$.

And $\vec{F}(\vec{r}) = \langle x^3-y^3, xy^2\rangle = \langle t^6 - t^9, t^2\cdot t^6\rangle = \langle t^6 - t^9, t^8\rangle$.

$\therefore \vec{F}(\vec{r})\cdot\vec{r}'(t) = \langle t^6 - t^9, t^8\rangle\cdot\langle 2t, 3t^2\rangle = 2t^7 - 2t^{10} + 3t^{10} = t^{10} + 2t^7$.

$\therefore W = \int_{-1}^0 \vec{F}\cdot\vec{r}'(t)dt = \int_{-1}^0 t^{10} + 2t^7 dt = [\frac{1}{11}t^{11} + \frac{1}{4}t^8]_{-1}^0$

$= 0 + \frac{1}{11} - \frac{1}{4} = -\frac{7}{44}$.

(2) $\boldsymbol{F}(x,y,z) = \langle 2x-y, 2z, y-z \rangle$; C is the curve $x=\sin\frac{\pi t}{2}, y=\sin\frac{\pi t}{2}, z=t, 0\le t\le 1$.

$\vec{r}(t) = \langle \sin\frac{\pi t}{2}, \sin\frac{\pi t}{2}, t \rangle$. $\therefore \vec{r}'(t) = \langle \frac{\pi}{2}\cos\frac{\pi t}{2}, \frac{\pi}{2}\cos\frac{\pi t}{2}, 1 \rangle$.

$\vec{F}(\vec{r}) = \langle 2\sin\frac{\pi t}{2} - \sin\frac{\pi t}{2}, 2t, \sin\frac{\pi t}{2} - t \rangle = \langle \sin\frac{\pi t}{2}, 2t, \sin\frac{\pi t}{2} - t \rangle$.

$\therefore \vec{F}(\vec{r})\cdot\vec{r}'(t) = \sin\frac{\pi t}{2}\cdot\frac{\pi}{2}\cos\frac{\pi t}{2} + 2t\,\frac{\pi}{2}\cos\frac{\pi t}{2} + \sin\frac{\pi t}{2} - t$.

So the work done by the force $\vec{F}$ is

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \frac{\pi}{4}\sin(\pi t) + \pi t\cos\frac{\pi t}{2} + \sin\frac{\pi t}{2} - t\,dt$

$= \left[-\frac{\pi}{4}\frac{1}{\pi}\cos(\pi t)\right]_0^1 + \frac{2\pi}{\pi}\left(1-\frac{2}{\pi}\right) - \frac{2}{\pi}\cdot\left[\cos\frac{\pi t}{2}\right]_0^1 - \left[\frac{1}{2}t^2\right]_0^1 = -\frac{1}{4}(-1-1) + 2 - \frac{4}{\pi} + \frac{2}{\pi} - \frac{1}{2} = 2 - \frac{2}{\pi}$.

$$\boxed{\begin{aligned}&\int_0^1 t\cos\frac{\pi t}{2}dt = \int_0^1 t\, d\left(\sin\frac{\pi t}{2}\right)\cdot\frac{2}{\pi}.\\ &= \frac{2}{\pi}\left\{\left[t\sin\frac{\pi t}{2}\right]_0^1 - \int_0^1 \sin\frac{\pi t}{2}dt\right\}.\\ &= \frac{2}{\pi}\left\{1 + \frac{2}{\pi}\left[\cos\frac{\pi t}{2}\right]_0^1\right\} = \frac{2}{\pi}\left(1-\frac{2}{\pi}\right).\end{aligned}}$$

8. Determine whether the given field $\boldsymbol{F}\cdot d\boldsymbol{r}$ is exact. If so, find f such that $\boldsymbol{F} = \nabla f$.

(1) $\boldsymbol{F}(x,y) = \langle 12x^2+3y^2+5y,\ 6xy-3y^2+5x \rangle$.

$F_1 = 12x^2+3y^2+5y$, $F_2 = 6xy-3y^2+5x$. So $\frac{\partial F_1}{\partial y} = 6y+5 = \frac{\partial F_2}{\partial x}$. So $\vec{F}d\vec{r}$ is exact.

Assume $\vec{F} = \nabla f$, then ① $\frac{\partial f}{\partial x} = 12x^2+3y^2+5y$, $\frac{\partial f}{\partial y} = 6xy-3y^2+5x$ ②.

From ①, $f(x,y) = 4x^3+3xy^2+5xy+G(y)$. $\Rightarrow \frac{\partial f}{\partial y} = 6xy+5x+G'(y) = 6xy-3y^2+5x$.

$\therefore G'(y) = -3y^2$ $\therefore G(y) = -y^3 +$ Constant.

So $f(x,y) = 4x^3+3xy^2+5xy-y^3+$ Constant.

(2) $\boldsymbol{F}(x,y) = \langle 4y^2\cos(xy^2), 8x\cos(xy^2) \rangle$.

$F_1 = 4y^2\cos(xy^2)$, $F_2 = 8x\cos(xy^2)$.

$\frac{\partial F_1}{\partial y} = 8y\cos(xy^2) + 4y^2(-2xy)\sin(xy^2)$. $\frac{\partial F_2}{\partial x} = 8\cos(xy^2) + 8x(-y^2)\sin(xy^2)$.

So $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, So $\vec{F}\cdot d\vec{r}$ is not exact.

(3) $\boldsymbol{F}(x,y,z) = \langle 3x^2, 6y^2, 9z^2 \rangle$.

$F_1 = 3x^2$, $F_2 = 6y^2$, $F_3 = 9z^2$. So $\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}$, $\frac{\partial F_3}{\partial x} = 0 = \frac{\partial F_1}{\partial z}$.

therefore $\vec{F}\cdot d\vec{r}$ is exact.

Assume $\vec{F} = \nabla f$, then $\frac{\partial f}{\partial x} = 3x^2$ ①, $\frac{\partial f}{\partial y} = 6y^2$ ②. $\frac{\partial f}{\partial z} = 9z^2$ ③.

From ①, $f(x,y,z) = x^3 + g(y,z) \rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 6y^2 \rightarrow g(y,z) = 2y^3 + h(z)$.

$\therefore f(x,y,z) = x^3+2y^3+h(z) \rightarrow \frac{\partial f}{\partial z} = h'(z) = 9z^2 \rightarrow h(z) = 3z^3 +$ Constant.

$\therefore f(x,y,z) = x^3+2y^3+3z^3+$ Constant.

9. (1) $\int_{(-1,2)}^{(3,1)} (y^2+2xy)dx + (x^2+2xy)dy$

Solution: $F_1 = y^2+2xy$, $F_2 = x^2+2xy$, so $\frac{\partial F_1}{\partial y} = 2y+2x = \frac{\partial F_2}{\partial x}$, therefore the integral is independent of path. So we want to find a f such that $\vec{F} = \nabla f$, then

$\int_{(-1,2)}^{(3,1)} (y^2+2xy)dx + (x^2+2xy)dy = f(3,1) - f(-1,2)$.

$$\begin{cases} \frac{\partial f}{\partial x} = y^2+2xy \quad ① \rightarrow f(x,y) = xy^2 + x^2y + g(y) \rightarrow \frac{\partial f}{\partial y} = 2xy + x^2 + g'(y) = x^2+2xy. \\ \frac{\partial f}{\partial y} = x^2+2xy \quad \therefore g'(y) = 0 \quad \therefore g(y) = \text{Constant} \quad \therefore f(x,y) = xy^2 + x^2y + \text{Constant}. \end{cases}$$

$\therefore \int_{(-1,2)}^{(3,1)} (y^2+2xy)dx + (x^2+2xy)dy = f(3,1) - f(-1,2) = 3\cdot1 + 9\cdot1 + \text{Constant} - (-1)\cdot4 - 1\times2$

$= 3+9+4-2 = 14$.

(2) $\int_{(0,0,0)}^{(\pi,\pi,0)} (\cos x + 2yz)dx + (\sin y + 2xz)dy + (z+2xy)dz$.

Solution: $F_1 = \cos x + 2yz$, $F_2 = \sin y + 2xz$, $F_3 = z + 2xy$. curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x+2yz & \sin y+2xz & z+2xy \end{vmatrix} = \langle 2x-2x, 2y-2y, 2z-2z \rangle = 0$.

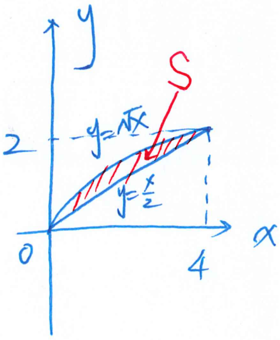
So the line integral is independent of path.

$$\begin{cases} \frac{\partial f}{\partial x} = \cos x + 2yz \quad ① \rightarrow f(x,y,z) = \sin x + 2xyz + g(y,z) \rightarrow \frac{\partial f}{\partial y} = 2xz + \frac{\partial g}{\partial y} = \sin y + 2xz \rightarrow \frac{\partial g}{\partial y} = \sin y \\ \frac{\partial f}{\partial y} = \sin y + 2xz \quad ② \quad \therefore g(y,z) = -\cos y + h(z) \rightarrow f(x,y,z) = \sin x + 2xyz - \cos y + h(z). \\ \frac{\partial f}{\partial z} = z + 2xy \quad ③ \quad \rightarrow \frac{\partial f}{\partial z} = 2xy + h'(z) = z + 2xy \rightarrow h'(z) = z \rightarrow h(z) = \frac{1}{2}z^2 + \text{Constant}. \end{cases}$$

$\therefore f(x,y,z) = \sin x + 2xyz - \cos y + \frac{1}{2}z^2 + \text{Constant} \quad \therefore \int_{(0,0,0)}^{(\pi,\pi,0)} F_1dx + F_2dy + F_3dz = f(\pi,\pi,0) - f(0,0,0) = 2$.

10. Sketch the region S and evaluate the line integral by Green's theorem.

(1) $\oint_C 2xydx + y^2dy$, where C is the closed curve formed by $y = \frac{x}{2}$ and $y = \sqrt{x}$ between (0,0) and (4,2).



Solution: $\oint_C 2xydx + y^2dy = \iint_S \frac{\partial(y^2)}{\partial x} - \frac{\partial(2xy)}{\partial y} dA$

$= \iint_S -2x\,dA = \int_0^2 \int_{y^2}^{2y} -2x\,dx\,dy = -\int_0^2 [x^2]_{y^2}^{2y} dy$

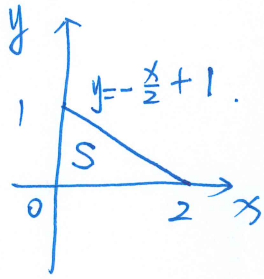
$= -\int_0^2 4y^2 - y^4 dy = \left[\frac{1}{5}y^5 - \frac{4}{3}y^3\right]_0^2 = -\frac{64}{15}$.

or: $= \iint_S -2x\,dA = \int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} -2x\,dy\,dx = \int_0^4 -2x\cdot(\sqrt{x} - \frac{x}{2})dx = \int_0^4 -2x^{\frac{3}{2}} + x^2 dx$.

$= \left[-2\cdot\frac{2}{5}\cdot x^{\frac{5}{2}} + \frac{1}{3}x^3\right]_0^4 = -\frac{4}{5}\cdot 4^{\frac{5}{2}} - \frac{1}{3}\times 4^3 = -\frac{64}{15}$.

(2) $\oint_C xydx + (x+y)dy$, where C is the triangle with vertices (0,0), (2,0) and (0,1).

Solution: The equation of the straight line passing through (0,1) and (2,0) is $y=-\frac{x}{2}+1$.



$\therefore \oint_C xydx+(x+y)dy = \iint_S \frac{\partial(x+y)}{\partial x} - \frac{\partial(xy)}{\partial y}dxdy = \iint_S 1-x\,dxdy.$

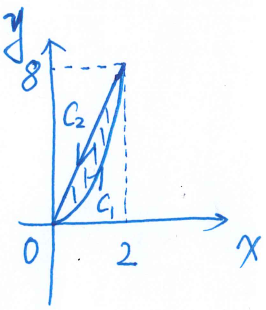
$= \int_0^2\int_0^{-\frac{x}{2}+1} 1-x\,dydx = \int_0^2 (1-x)(1-\frac{x}{2})dx = \int_0^2 1-\frac{3}{2}x+\frac{x^2}{2}dx$

$=[x-\frac{3}{4}x^2+\frac{1}{6}x^3]_0^2 = 2-3+\frac{4}{3}=\frac{1}{3}.$

or. $\iint_S 1-xdxdy = \int_0^1\int_0^{2-2y}1-x\,dxdy = \int_0^1[x-\frac{1}{2}x^2]_0^{2-2y}dy=\int_0^1 2-2y-\frac{1}{2}(2-2y)^2dy=\frac{1}{3}.$

11. Find the area of the region S by using $A(s) = \frac{1}{2}\oint_C xdy - ydx$.

(1) S is bounded by the curves $y=4x$ and $y = 2x^2$.



$A(s) = \frac{1}{2}\oint_C xdy - ydx$, where $C=C_1+C_2$.

$C_1: y=2x^2$, $dy=4xdx$, $0\le x\le 2$, $C_2: y=4x$. $dy=4dx$, $x: 2\to 0$.

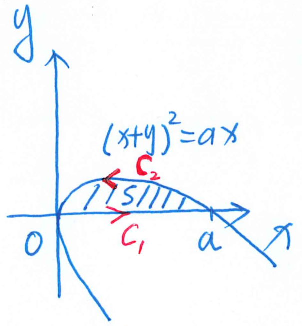
$\therefore A(s)=\frac{1}{2}\left[\int_0^2 x\cdot 4xdx - 2x^2dx + \int_2^0 x\cdot 4dx - 4xdx\right]$

$=\frac{1}{2}\int_0^2 2x^2dx = [\frac{1}{3}x^3]_0^2 = \frac{8}{3}.$

or by the Green's theorem: $A(s)=\frac{1}{2}\iint_S \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}dA = \iint_S dxdy = \int_0^2\int_{2x^2}^{4x} dydx$

$= \int_0^2 4x-2x^2dx = [2x^2 - \frac{2}{3}x^3]_0^2 = 2\times 4 - \frac{2}{3}\times 8 = \frac{8}{3}.$

(2) S is bounded by $(x+y)^2 = ax$ and x-axis. $\boxed{a>0}$



S is bounded by C_1 and C_2

$C_1: y=0$, $\therefore dy=0$, $0\le x\le a$.

$C_2: y=\sqrt{ax}-x$, $dy=(\frac{\sqrt{a}}{2\sqrt{x}}-1)dx$ $x=a\to 0$.

$\therefore A(s)=\frac{1}{2}\oint_C xdy-ydx = \frac{1}{2}\left[\int_{C_1} xdy-ydx+\int_{C_2} xdy-ydx\right]$

$=\frac{1}{2}\left\{\int_0^a x\cdot 0-0dx+\int_a^0 x\cdot(\frac{\sqrt{a}}{2\sqrt{x}}-1)dx - (\sqrt{ax}-x)dx\right\}.$

$=\frac{1}{2}\int_a^0 -\frac{1}{2}\sqrt{ax}\,dx = \frac{\sqrt{a}}{4}\int_0^a \sqrt{x}\,dx = \frac{\sqrt{a}}{4}\cdot\frac{2}{3}[x^{\frac{3}{2}}]_0^a$

$= \frac{\sqrt{a}}{4}\cdot\frac{2}{3}\cdot a\sqrt{a} = \frac{1}{6}a^2.$

or: $A(s)=\frac{1}{2}\oint_C xdy-ydx$

$=\frac{1}{2}\iint_S \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}dxdy$, (Green's Thm

$=\iint_S dxdy = \int_0^a\int_0^{\sqrt{ax}-x} dydx$

$=\int_0^a \sqrt{ax}-xdx=[\sqrt{a}\cdot\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2]_0^a$

$=\frac{2}{3}a^2-\frac{1}{2}a^2 = \frac{a^2}{6}.$

12. Evaluate the surface integral $\iint_S g(x,y,z)dA$, where $g(x,y,z)=x$ and S: $x+y+2z=4$, $0 \le x \le 1, 0 \le y \le 1$.

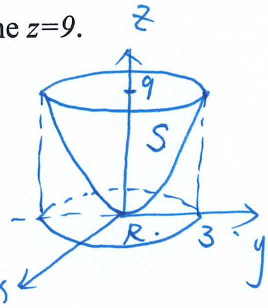
$\iint_S g(x,y,z)\, dA = \iint_R g(x,y,z)\sqrt{z_x^2+z_y^2+1}\, dxdy$, where R is the projection of S in xy-plane

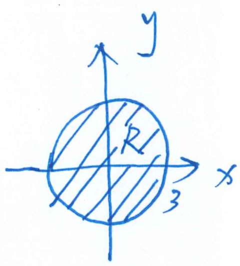
$= \int_0^1\int_0^1 x\cdot\sqrt{z_x^2+z_y^2+1}\, dx\,dy$. $\because x+y+2z=4 \therefore z = 2-\frac{1}{2}x-\frac{1}{2}y$.

$= \int_0^1\int_0^1 x\sqrt{\frac{1}{4}+\frac{1}{4}+1}\, dx\,dy = \int_0^1\int_0^1 \frac{\sqrt{6}}{2}x\, dx\,dy$

$= \int_0^1 \frac{\sqrt{6}}{2}\left[\frac{1}{2}x^2\right]_0^1 dy = \frac{\sqrt{6}}{4}\cdot\int_0^1 dy = \frac{\sqrt{6}}{4}$.

13. Find the area of the surface $z = x^2 + y^2$ below the plane $z=9$.





The surface area is given by: $\iint_R \sqrt{f_x^2+f_y^2+1}\, dxdy$.

Here $f(x,y) = x^2+y^2$ $\therefore \frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$.

$\therefore$ Area $= \iint_R \sqrt{4x^2+4y^2+1}\, dxdy$

The shape of R is a circular disk, which reminds us to use polar coordinates.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ and $dxdy = r\,dr\,d\theta$.

So Area $= \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta$. Let $u = 4r^2+1$, then $du = 8r\,dr$, when $r=0$, $u=1$; $r=3$, $u=37$

$= \int_0^{2\pi}\int_1^{37}\sqrt{u}\,\frac{1}{8}du\,d\theta = \int_0^{2\pi}\frac{1}{8}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{37}d\theta = \int_0^{2\pi}\frac{1}{12}(37^{\frac{3}{2}}-1)\,d\theta = \frac{\pi}{6}(37^{\frac{3}{2}}-1)$.

14. Evaluate the surface integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dA$.

(1) $\boldsymbol{F} = \langle -x^2, y^2, 0\rangle$, S: $\boldsymbol{r}(u,v) = \langle u, v, 3u-2v\rangle$, $0 \le u \le \frac{3}{2}$, $-2 \le v \le 2$.

Solution: $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\cdot\vec{N}\,du\,dv$, where $\vec{N}$ is the normal vector of S.

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle 1,0,3\rangle \times \langle 0,1,-2\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} = \langle -3, 2, 1\rangle$.

$\vec{F}(\vec{r}(u,v)) = \langle -x^2, y^2, 0\rangle = \langle -u^2, v^2, 0\rangle$.

$\therefore \vec{F}(\vec{r}(u,v))\cdot\vec{N} = \langle -u^2, v^2, 0\rangle\cdot\langle -3,2,1\rangle = 3u^2+2v^2$.

$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \int_{-2}^{2}\int_0^{\frac{3}{2}} 3u^2+2v^2\,du\,dv = \int_{-2}^{2}\left[u^3+2v^2u\right]_0^{\frac{3}{2}}dv = \int_{-2}^{2}\frac{27}{8}+3v^2\,dv$.

$= \left[\frac{27}{8}v+v^3\right]_{-2}^{2} = \frac{27}{8}\times 2+2^3-\frac{27}{8}\times(-2)-(-2)^3 = \frac{27}{4}+8+\frac{27}{4}-8 = \frac{27}{2}+16 = \frac{59}{2}$.

(2) $\boldsymbol{F} =< \tan(xy), x, y >, S: y^2+z^2=1, 2 \le x \le 5, y \ge 0, z \ge 0.$

S can be represented by the parametric equation: $\vec{r}(u,v) = \langle u, \cos v, \sin v \rangle$, $2 \le u \le 5$, $0 \le v \le \frac{\pi}{2}$.

$\therefore \vec{N} = \vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle \times \langle 0, -\sin v, \cos v \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -\sin v & \cos v \end{vmatrix} = \langle 0, -\cos v, -\sin v \rangle.$

$\therefore \vec{F}(\vec{r}(u,v)) \cdot \vec{N} = \langle \tan(u\cos v), u, \cos v \rangle \cdot \langle 0, -\cos v, -\sin v \rangle = -u\cos v - \cos v \sin v.$

$\therefore \iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R -u\cos v - \cos v \sin v \, du\, dv = \int_0^{\frac{\pi}{2}} \int_2^5 -u\cos v - \cos v \sin v \, du\, dv$

$= \int_0^{\frac{\pi}{2}} -\cos v \cdot \left[\frac{u^2}{2}\right]_2^5 - \cos v \sin v \cdot [u]_2^5 \, dv = -\int_0^{\frac{\pi}{2}} \frac{21}{2}\cos v + \frac{3}{2}\sin 2v \, dv = -\frac{21}{2}[\sin v]_0^{\frac{\pi}{2}} + \frac{3}{4}[\cos 2v]_0^{\frac{\pi}{2}}$

$= -\frac{21}{2} + \frac{3}{4}(-1-1) = -12.$

15. Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA$ by the divergence theorem.

(1) $\boldsymbol{F} =< x^2, 0, z^2 >$, S is the surface of the box $|x| \le 1, |y| \le 3, 0 \le z \le 2.$

$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \iiint_T \frac{\partial(x^2)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(z^2)}{\partial z}\, dV = \iiint_T 2x+2z\, dV.$

$= \int_{-1}^{1} \int_{-3}^{3} \int_0^2 2x+2z \, dz\, dy\, dx = \int_{-1}^{1} \int_{-3}^{3} \left[2xz + z^2\right]_0^2 dy\, dx$

$= \int_{-1}^{1} \int_{-3}^{3} 4x+4 \, dy\, dx = 24 \int_{-1}^{1} x+1\, dx = 24\left[\frac{1}{2}x^2 + x\right]_{-1}^{1}$

$= 24\left(\frac{1}{2}+1-\frac{1}{2}+1\right) = 48.$

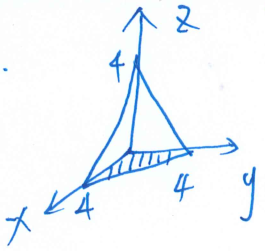
(2) $\boldsymbol{F} =< x, 2y+z, z+x^2 >$, S is the surface of $1 \le x^2+y^2+z^2 \le 4.$

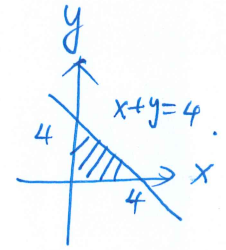
$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \iiint_T 1+2+1\, dV = 4 \iiint_T dV = 4 \times$ Volumn of T.

$= 4 \cdot \left(\frac{4}{3}\pi \cdot 2^3 - \frac{4}{3}\pi \cdot 1^3\right) = \frac{112\pi}{3}.$

(3) $\boldsymbol{F} =< x^2, y^2, z^2 >$, S is the surface of the solid bounded by $x+y+z=4, x=0, y=0, z=0.$

$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \iiint_T 2x+2y+2z\, dV.$





T is the tetrahedron bounded by four triangles.

$= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} 2x+2y+2z\, dz\, dy\, dx$

$= \int_0^4 \int_0^{4-x} \left[(2x+2y)z + z^2\right]_0^{4-x-y} dy\, dx = \int_0^4 \int_0^{4-x} (2x+2y)(4-x-y) + (4-x-y)^2 dx$

$= \int_0^4 \int_0^{4-x} \cancel{8x+8y} - 4xy - 2x^2 - 2y^2 + 16 - \cancel{8x} - 8y + 2xy + x^2 + y^2 dy\, dx.$

$= \int_0^4 \int_0^{4-x} -x^2-y^2-2xy\, dy\, dx = -\int_0^4 \left[x^2y - \frac{1}{3}y^3 + xy^2\right]_0^{4-x} dx = -\int_0^4 x^2(4-x) - \frac{1}{3}(4-x)^3 + x(4-x)^2 dx$

$= 64.$

16. Verify Stokes's theorem for $\boldsymbol{F} = < y, -x, yz >$ if S is the paraboloid $z = x^2 + y^2$ with the circle $x^2 + y^2 = 1, z = 1$ as its boundary. Here we take the normal vector of the surface as upward.

Stokes's theorem is : $\oint_C \vec{F} \cdot \overrightarrow{r'(t)}\, dt = \iint_S \text{curl}\vec{F} \cdot \vec{n}\, ds$

To evaluate the left hand side, we may des write the bondary c by the parametric equations :

$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle$. $0 \le t \le 2\pi$ $\therefore$ $\vec{F}(\vec{r}(t)) = \langle y, -x, yz \rangle = \langle \sin t, -\cos t, \sin t \rangle$.

$\therefore \oint_C \vec{F} \cdot \overrightarrow{r'(t)}\, dt = \int_0^{2\pi} \vec{F}(\vec{r}) \cdot \vec{r}'(t) dt = \int_0^{2\pi} \langle \sin t, -\cos t, \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$.

$= \int_0^{2\pi} -\sin^2 t - \cos^2 t\, dt = -2\pi$.

On the other hand, to calculate $\iint_S \text{curl}\vec{F} \cdot \vec{n}\, dA$, we first obtain

$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & yz \end{vmatrix} = \langle z, 0, -2 \rangle$. Take $f(x,y,z) = x^2 + y^2 - z = 0$, then $\nabla f = \langle 2x, 2y, -1 \rangle$, As the normal vector is upward, we take $\vec{N} = -\nabla f = \langle -2x, -2y, 1 \rangle$

$\therefore \iint_S \text{curl}\vec{F} \cdot \vec{n}\, ds = \iint_R \text{curl}\vec{F} \cdot \vec{N}\, dxdy$

$= \iint_R \langle z, 0, -2 \rangle \cdot \langle -2x, -2y, 1 \rangle dxdy = -2 \iint_R zx + 1\, dxdy$

$= -2 \iint_R x(x^2+y^2) + 1\, dxdy = -2 \int_0^{2\pi} \int_0^1 (r^3 \cos\theta + 1)\, r dr d\theta$.

$= -2 \int_0^{2\pi} \int_0^1 r^4 \cos\theta + r\, dr d\theta = -2 \int_0^{2\pi} \left[\frac{1}{5} r \cos\theta + \frac{1}{2} r\right]_0^1 d\theta$.

$= -2 \int_0^{2\pi} \frac{1}{5}\cos\theta + \frac{1}{2}\, d\theta = -2 \cdot \left[\frac{1}{5}\sin\theta + \frac{1}{2}\theta\right]_0^{2\pi} = -2\pi$.

Therefore $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\vec{F} \cdot \vec{n}\, ds$ is verified.

17. Use Stokes's theorem to calculate $\iint_S (curl\boldsymbol{F}) \cdot \boldsymbol{n} dA$, where $\boldsymbol{F} = < xz^2, x^3, \cos(xz) >$; S is the part of the ellipsoid $x^2 + y^2 + 3z^2 = 1$ above the xy-plane and $\boldsymbol{n}$ is the upward normal vector.

The boundary of is C : $x^2 + y^2 = 1$, $z = 0$ (in the xy-plane).

So $\iint_S (\text{curl}\vec{F}) \cdot \vec{n}\, dA = \oint_C \vec{F} \cdot \vec{r}'(t)\, dt$, where $\vec{r}(t)$ is the parametric equation of C.

So $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $\therefore \vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle$. $0 \le t \le 2\pi$.

$\therefore \oint_C \vec{F} \cdot \vec{r}'(t) dt = \int_0^{2\pi} \langle xz^2, x^3, \cos(xz) \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$

$= \int_0^{2\pi} \langle \cos t \cdot 0, \cos^3 t, \cos(0) \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt = \int_0^{2\pi} \cos^4 t\, dt$ (*).

$\cos^4 t = (\cos^2 t)^2 = \left(\frac{\cos 2t + 1}{2}\right)^2 = \frac{1}{4}(\cos^2 2t + 2\cos 2t + 1) = \frac{1}{4}\left(\frac{\cos 4t + 1}{2} + 2\cos 2t + 1\right)$

$\therefore (*) = \frac{1}{4} \int_0^{2\pi} \frac{\cos 4t + 1}{2} + 2\cos 2t + 1\, dt = \frac{1}{4}\left\{\frac{1}{2}\int_0^{2\pi} \cos 4t + 1\, dt + \int_0^{2\pi} 2\cos 2t + 1\, dt\right\}$.

$= \frac{1}{4}\left\{\frac{1}{2}\left[\frac{1}{4}\sin 4t + t\right]_0^{2\pi} + \left[\sin 2t\right]_0^{2\pi} + \left[t\right]_0^{2\pi}\right\} = \frac{1}{4}\left\{\frac{1}{2}(2\pi - 0) + 0 + 2\pi\right\} = \frac{3\pi}{4}$.