Sept 2017 MTH201 Name________________ID________ S.Guo

Tutorial 2 Path dependence

1. (Page 425) Show that the form under the integral sign is exact in the plane or in space and evaluate the integral. Wow the details of your work.

(1) $\int_{(\pi/2,\pi)}^{(\pi,0)}\left(\frac{1}{2}\cos\frac{x}{2}\cos 2y dx-2\sin\frac{x}{2}\sin 2y dy\right)$ $\vec{F}=\langle F_1, F_2\rangle=\langle \frac{1}{2}\cos\frac{x}{2}\cos 2y, -2\sin\frac{x}{2}\sin 2y\rangle$.

We want to find a $f(x,y)$ such that $\vec{F}=\text{grad} f$, so $\begin{cases}\frac{\partial f}{\partial x}=F_1=\frac{1}{2}\cos\frac{x}{2}\cos 2y & ① \\ \frac{\partial f}{\partial y}=F_2=-2\sin\frac{x}{2}\sin 2y. & ②\end{cases}$

From ①, we have: $f(x,y)=\sin\frac{x}{2}\cos 2y+g(y)$

So $\frac{\partial f}{\partial y}=-2\sin\frac{x}{2}\cos 2y+g'(y)$. Then by ②, we know $g'(y)=0$ $\therefore g(y)=$ constant.

Let the constant be 0, then $f(x,y)=\sin\frac{x}{2}\cos 2y$ is a potential of $\vec{F}$.

Therefore $\vec{F}(r)\cdot d\vec{r}$ is exact, and then the line integral is path independent.

$\therefore \int_{(\frac{\pi}{2},\pi)}^{(\pi,0)}\vec{F}\cdot d\vec{r}=f(\pi,0)-f(\frac{\pi}{2},\pi)=\sin\frac{\pi}{2}\cos 0-\sin\frac{\pi}{4}\cos 2\pi=1-\frac{\sqrt{2}}{2}$.

(2) $\int_{(4,0)}^{(6,1)} e^{4y}(2\,x dx+4\,x^2 dy)$

$\vec{F}=\langle F_1, F_2\rangle=\langle 2xe^{4y}, 4x^2e^{4y}\rangle$. Find $f(x,y)$ such that $\begin{cases}\frac{\partial f}{\partial x}=2xe^{4y} & ① \\ \frac{\partial f}{\partial y}=4x^2e^{4y} & ②\end{cases}$

From ①, $f(x,y)=x^2e^{4y}+g(y)\rightarrow\frac{\partial f}{\partial y}=4x^2e^{4y}+g'(y)$.

By ②, $g'(y)=0$, $\therefore g(y)=$ constant. $\therefore f(x,y)=x^2e^{4y}$. (Take the constant be 0).

So the form under the integral sign is exact.

So the line integral is path independent, and

$\int_{(4,0)}^{(6,1)} e^{4y}(2xdx+4x^2dy)=f(6,1)-f(4,0)=36e^4-16e^0=36e^4-16$.

(3) $\int_{(0,0,\pi)}^{(2,1/2,\pi/2)} e^{xy}(y\sin z dx+x\sin z dy+\cos z dz)$

Find $f(x,y,z)$ such that $\begin{cases}\frac{\partial f}{\partial x}=e^{xy}y\sin z & ① \\ \frac{\partial f}{\partial y}=e^{xy}x\sin z & ② \\ \frac{\partial f}{\partial z}=e^{xy}\cos z & ③\end{cases}$

From ①, $f(x,y,z)=e^{xy}\sin z+g(y,z)\longrightarrow\frac{\partial f}{\partial y}=xe^{xy}\sin z+\frac{\partial g}{\partial y}$, by ②, $\frac{\partial g}{\partial y}=0$.

$\therefore g(y,z)=C_1+h(z)$, C_1 is an arbitrary constant.

$\therefore f(x,y,z)=e^{xy}\sin z+C_1+h(z)\rightarrow\frac{\partial f}{\partial z}=e^{xy}\cos z+h'(z)$, by ③, $h'(z)=0$.

$\therefore h(z)=C_2$, C_2 is an arbitrary constant. $\therefore f(x,y,z)=e^{xy}\sin z$. (Take C_1, C_2 be 0).

is a potential of $\vec{F}=\langle e^{xy}y\sin z, e^{xy}x\sin z, e^{xy}\cos z\rangle$.

$\therefore \int_{(0,0,\pi)}^{(2,\frac{1}{2},\frac{\pi}{2})}\vec{F}\cdot d\vec{r}=f(2,\frac{1}{2},\frac{\pi}{2})-f(0,0,\pi)=e\sin\frac{\pi}{2}-e^0\sin\pi=e$.

(4) $\int_{(0,0,0)}^{(1,1,0)} e^{x^2+y^2+z^2}(xdx+ydy+zdz)$ $\vec{F}=\langle xe^{x^2+y^2+z^2}, ye^{x^2+y^2+z^2}, ze^{x^2+y^2+z^2}\rangle$.

Find $f(x,y,z)$ such that $\vec{F}=\nabla f$, therefore:
$$\begin{cases} \frac{\partial f}{\partial x} = xe^{x^2+y^2+z^2} & ① \\ \frac{\partial f}{\partial y} = ye^{x^2+y^2+z^2} & ② \\ \frac{\partial f}{\partial z} = ze^{x^2+y^2+z^2}. & ③ \end{cases}$$

By ①: $f(x,y,z)=\frac{1}{2}e^{x^2+y^2+z^2}+g(y,z)$ ∴ $\frac{\partial f}{\partial y}=ye^{x^2+y^2+z^2}+\frac{\partial g}{\partial y}$.

By ②: $\frac{\partial g}{\partial y}=0$ ∴ $g(y,z)=C_1+h(z)$, ∴ $f(x,y,z)=\frac{1}{2}e^{x^2+y^2+z^2}+C_1+h(z)$

∴ $\frac{\partial f}{\partial z}=ze^{x^2+y^2+z^2}+h'(z)$ ∴ By ③, $h'(z)=0$ ∴ $h(z)=C_2$, where C_1, C_2 are arbitrary constants.

∴ $f(x,y,z)=\frac{1}{2}e^{x^2+y^2+z^2}$ (Take C_1, C_2 be 0) ∴ $\vec{F}\cdot d\vec{r}$ is exact. ∴ $\int_{(0,0,0)}^{(1,1,0)}\vec{F}\cdot d\vec{r}=f(1,1,0)-f(0,0,0)$
$=\frac{1}{2}e^2-\frac{1}{2}$.

(5) $\int_{(0,2,3)}^{(1,1,1)}(yz\sinh(xz)\,dx+\cosh(xz)\,dy+xy\sinh(xz)\,dz)$

$F_1=yz\sinh xz$, $F_2=\cosh xz$, $F_3=xy\sinh xz$, We want to find $f(x,y,z)$, such that $\vec{F}=\text{grad} f$.

$$∴\begin{cases} \frac{\partial f}{\partial x}=yz\sinh xz & ① \\ \frac{\partial f}{\partial y}=\cosh xz & ② \\ \frac{\partial f}{\partial z}=xy\sinh xz. & ③ \end{cases}$$

From ① $f(x,y,z)=y\cosh xz+g(y,z)$ ∴ $\frac{\partial f}{\partial y}=\cosh xz+\frac{\partial g}{\partial y}$, by ② $\frac{\partial g}{\partial y}=0$,

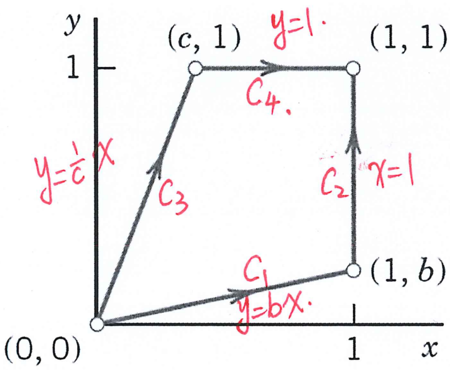
∴ $g(y,z)=C_1+h(z)$ ∴ $f(x,y,z)=y\cosh xz+C_1+h(z)$

∴ $\frac{\partial f}{\partial z}=xy\sinh xz+h'(z)$. By ③ $h'(z)=0$ ∴ $h(z)=C_2$. (C_1, C_2 are constants).

∴ $f(x,y,z)=y\cosh xz$. (we take C_1 and C_2 be 0) ∴ $\vec{F}\cdot d\vec{r}$ is exact.

∴ $\int_{(0,2,3)}^{(1,1,1)}\vec{F}\cdot d\vec{r}$ is path independent $=f(1,1,1)-f(0,2,3)=\cosh 1-2$.

2. (a) Show that $I=\int_C(x^2ydx+2xy^2dy)$ is path dependent in the xy-plane.



$F_1=x^2y$, $F_2=2xy^2$.

∴ $\frac{\partial F_1}{\partial y}=x^2$, $\frac{\partial F_2}{\partial x}=2y^2$

∴ $\frac{\partial F_1}{\partial y}\neq\frac{\partial F_2}{\partial x}$ so I is line dependent.

Or. We calculate $\text{curl}\,\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2xy^2 & 0\end{vmatrix}=\langle 0,0,2y^2-x^2\rangle\neq 0$

So I is not path independent, i.e. I is path dependent.

(b) Integrate from (0,0) along the straight-line segment to $(1,b)$, $0 \le b \le 1$, and then vertically up to (1,1); see the figure below. For which b is I maximum? What is its maximum value?

Let C_1 and C_2 be the following pathes respectively: $\begin{cases} C_1: \vec{r}_1 = \langle t, bt \rangle & 0 \le t \le 1 \\ C_2: \vec{r}_2 = \langle 1, t \rangle. & b \le t \le 1. \end{cases}$

Then $I_1 = \int_{C_1} x^2y\,dx + 2xy^2dy = \int_0^1 \langle x^2y, 2xy^2 \rangle \cdot \vec{r}_1{}'(t)\,dt = \int_0^1 \langle t^2bt, 2tb^2t^2 \rangle \cdot \langle 1, b \rangle dt$

$= \int_0^1 bt^3 + 2b^3t^3dt = (b+2b^3)\left[\frac{t^3}{4}\right]_0^1 = \frac{b+2b^3}{4}.$

or: $I_1 = \int_0^1 x^2 \cdot (bx)dx + 2x(bx)^2 b\,dx$. (replace y by bx, so $dy = bdx$)

$= \int_0^1 bx^3 + 2bx^3dx = \frac{b+2b^3}{4}.$

$I_2 = \int_{C_2} x^2y\,dx + 2xy^2dy = \int_{C_2} \langle x^2y, 2xy^2 \rangle \cdot \vec{r}_2{}'(t)\,dt = \int_b^1 \langle t, 2t^2 \rangle \cdot \langle 0, 1 \rangle dt = \int_b^1 2t^2dt = \frac{2}{3}(1-b^3)$

or. $I_2 = \int_b^1 1^2 \cdot y\,dx + 2 \cdot 1 \cdot y^2dy$, but $x=1$ $\therefore dx = 0$ $\therefore I_2 = \int_b^1 2y^2dy = \frac{2}{3}(1-b^3)$.

$\therefore I = I_1 + I_2 = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \frac{b+2b^3}{4} + \frac{2}{3}(1-b^3) = -\frac{1}{6}b^3 + \frac{1}{4}b + \frac{2}{3}$, is a function of b.

To find the maximum value of $I(b)$, Let $I'(b) = 0$ we get $-\frac{1}{2}b^2 + \frac{1}{4} = 0$ $\therefore b = \pm\frac{\sqrt{2}}{2}$, As $b>0$, So $b = \frac{\sqrt{2}}{2}$.

Since $I''(b) = -b$, so $I''(\frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{2} < 0$. So when $b = \frac{\sqrt{2}}{2}$, I takes its maximum value, which is $I(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}+8}{12}$.

(c) Integrate I from (0,0) along the straight-line segment to $(c,1)$, $0 \le c \le 1$, and then horizontally to (1,1). For c=1, do you get the same value as for b=1 in (b)? For which c is I maximum? What is its maximum value?

C_3: $y = \frac{1}{c}x$, $0 \le x \le c$, or $\vec{r}_3(t) = \langle t, \frac{1}{c}t \rangle$. $0 \le t \le c$.

C_4: $y = 1$. $c \le x \le 1$, or $\vec{r}_4(t) = \langle t, 1 \rangle$, $c \le t \le 1$.

$\therefore I_3 = \int_{C_3} \vec{F} d\vec{r} = \int_0^c \langle x^2y, 2xy^2 \rangle \cdot \vec{r}_3{}'(t)\,dt = \int_0^c \langle \frac{1}{c}t^3, \frac{2}{c^2}t^3 \rangle \cdot \langle 1, \frac{1}{c} \rangle dt = \int_0^c \frac{1}{c}t^3 + \frac{2}{c^3}t^3dt$
$= \frac{1}{4}(c^3+2c)$

or $I_3 = \int_0^c x^2 \cdot \frac{1}{c}x\,dx + 2x \cdot (\frac{1}{c}x)^2 \cdot \frac{1}{c}dx = \int_0^c \frac{1}{c}x^3 + \frac{2}{c^3}x^3dx = \frac{1}{4}(c^3+2c)$.

$I_4 = \int_{C_4} \vec{F} \cdot d\vec{r} = \int_c^1 \langle x^2y, 2xy^2 \rangle \cdot \vec{r}_4{}'(t)\,dt = \int_c^1 \langle t^2, 2t \rangle \cdot \langle 1, 0 \rangle dt = \int_c^1 t^2dt = \frac{1}{3}(1-c^3)$.

or $I_4 = \int_c^1 x^2dx + 2x\,dy$, but $y=1$, so $dy=0$ $\therefore I_4 = \int_c^1 x^2dx = \frac{1}{3}(1-c^3)$.

$\therefore I = I_3 + I_4 = \frac{1}{4}(c^3+2c) + \frac{1}{3}(1-c^3) = -\frac{1}{12}c^3 + \frac{1}{2}c + \frac{1}{3}$, which is a function of c.

To find the maximum value of $I(c)$, Let $I'(c) = 0$ we get $-\frac{1}{4}c^2 + \frac{1}{2} = 0$ $\therefore c = \pm\sqrt{2}$.

but $0 \le c \le 1$, So the maximum value of $I(c)$ does not happen at $c = \pm\sqrt{2}$.

As $0 \le c \le 1$, $I'(c) = -\frac{1}{4}c^2 + \frac{1}{2} > 0$, so $I(c)$ is increasing in $[0,1]$.

Therefore, when $c=1$, $I(c)$ attains its maximum $I(1) = \frac{3}{4}$.

3. Check, and if independent, integrate from (0,0,0) to (a,b,c).

(1) $2e^{x^2}(x\cos 2y\,dx - \sin 2y\,dy)$

$F_1 = 2xe^{x^2}\cos 2y$, $F_2 = -2e^{x^2}\sin 2y$. $\therefore \frac{\partial F_1}{\partial y} = -4xe^{x^2}\sin 2y$. $\frac{\partial F_2}{\partial x} = -4xe^{x^2}\sin 2y$

$\because \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ so the line integral is path independent.

Find $f(x,y)$ such that $\frac{\partial f}{\partial x} = 2xe^{x^2}\cos 2y$, ① $\frac{\partial f}{\partial y} = -2e^{x^2}\sin 2y$ ②.

By ①: $f(x,y) = e^{x^2}\cos 2y + g(y)$ $\therefore \frac{\partial f}{\partial y} = -2e^{x^2}\sin 2y + g'(y)$, by ②. $g'(y) = 0$ $\therefore g(y) =$ Constant.

$\therefore f(x,y) = e^{x}\cos 2y$, here we take the constant be 0.

$\therefore \int_C \vec{F}\cdot d\vec{r} = f(a,b) - f(0,0) = e^{a}\cos 2b - 1$.

(2) $\sinh(xy)(z\,dx - x\,dz)$

$F_1 = z\sinh xy$, $F_2 = 0$, $F_3 = -x\sinh xy$.

$\therefore \frac{\partial F_1}{\partial z} = \sinh xy$, $\frac{\partial F_3}{\partial x} = -\sinh xy - xy\cosh xy$ $\therefore \frac{\partial F_1}{\partial z} \neq \frac{\partial F_3}{\partial x}$ So it is path dependent.

Or, we calculate the curl of $\vec{F}$:

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\sinh xy & 0 & -x\sinh xy \end{vmatrix} = \vec{i}(-x^2\cosh xy - 0) - \vec{j}(-\sinh xy - xy\cosh xy - \sinh xy) + \vec{k}(0 - xy\cosh(xy)) \neq 0.$$

(3) $x^2y\,dx - 4xy^2\,dy + 8z^2x\,dz$

$F_1 = x^2y$, $F_2 = -4xy^2$. $F_3 = 8z^2x$.

$\therefore \frac{\partial F_1}{\partial y} = x^2$. $\frac{\partial F_2}{\partial x} = -4y^2$ $\therefore \frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$ $\therefore$ the integral is path dependent.

Or, we calculate

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -4xy^2 & 8z^2x \end{vmatrix} = \langle 0, -8z^2, -x^2-4y^2\rangle \neq 0$$

So the integral is path dependent.

(4) $e^y dx + (xe^y - e^z)dy - ye^z dz$

$F_1 = e^y$, $F_2 = xe^y - e^z$. $F_3 = -ye^z$. $\therefore \frac{\partial F_1}{\partial y} = e^y$, $\frac{\partial F_2}{\partial x} = e^y$. $\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

$\frac{\partial F_1}{\partial z} = 0$, $\frac{\partial F_3}{\partial x} = 0$ $\therefore \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$.

$\frac{\partial F_2}{\partial z} = -e^z$. $\frac{\partial F_3}{\partial y} = -e^z$ $\therefore \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.

So the integral is path independent.

Now we find $f(x,y,z)$, such that $\begin{cases} \frac{\partial f}{\partial x} = e^y & ① \\ \frac{\partial f}{\partial y} = xe^y - e^z & ② \\ \frac{\partial f}{\partial z} = -ye^z & ③ \end{cases}$

By ①: $f(x,y,z) = xe^y + g(y,z)$.

$\therefore \frac{\partial f}{\partial y} = xe^y + \frac{\partial g}{\partial y} = xe^y - e^z$ by ②.

$\therefore \frac{\partial g}{\partial y} = -e^z$ $\therefore g(y,z) = -ye^z + h(z)$ $\therefore f(x,y,z) = xe^y - ye^z + h(z)$.

$\therefore \frac{\partial f}{\partial z} = -ye^z + h'(z) = -ye^z$ by ③. So $h'(z) = 0$ $\therefore h(z) =$ constant.

$\therefore f(x,y,z) = xe^y - ye^z$ (Take the constant be zero) $\therefore$ The integral $\int_{(0,0,0)}^{(a,b,c)} e^y dx + (xe^y - e^z)dy - ye^z dz = f(a,b,c) - f(0,0,0) = ae^b - be^c$.