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Tutorial 5 Surface integrals and divergence theorem

1. Flux integrals $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ (page 450). Evaluate the second type surface integral for the given data. Describe the kind of surface. Show the details of your work.

(1) $\boldsymbol{F} = < -x^2, y^2, 0 >$, S: $\boldsymbol{r} = < u, v, 3u-2v >$, $0 \le u \le \frac{3}{2}, -2 \le v \le 2$.

Solution: Set $x=u$, $y=v$, then the surface S can be represented by $z = 3x-2y$, which is a surface in plane.

$\vec{r}_u = \langle 1,0,3\rangle$. $\vec{r}_v = \langle 0,1,-2\rangle$ ∴ The normal vector $\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{vmatrix} = \langle -3,2,1\rangle$.

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\cdot\vec{N}\,du\,dv = \int_{-2}^{2}\int_0^{\frac{3}{2}} \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv$$

$$= \int_{-2}^{2}\int_0^{\frac{3}{2}} \langle -u^2, v^2, 0\rangle\cdot\langle -3,2,1\rangle\,du\,dv = \int_{-2}^{2}\int_0^{\frac{3}{2}} 3u^2+2v^2\,du\,dv$$

$$= \int_{-2}^{2} \left[u^3+2v^2u\right]_0^{\frac{3}{2}} dv = \int_{-2}^{2} \frac{27}{8}+3v^2\,dv = \left[\frac{27}{8}v+v^3\right]_{-2}^{2} = \frac{59}{2}.$$

(2) $\boldsymbol{F} = < e^y, e^x, 1 >$, S: $x+y+z=1, x\ge 0, y\ge 0, z\ge 0$.

Solution: The surface S is a plane in space.

Set $x=u$, $y=v$, Surface S can be represented by

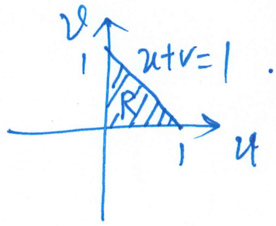
$\vec{r}(u,v) = \langle u, v, 1-u-v\rangle$.

$\therefore \vec{r}_u = \langle 1,0,-1\rangle$, $\vec{r}_v = \langle 0,1,-1\rangle$. $\therefore \vec{N} = \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,1,1\rangle$

$\vec{F}(\vec{r}(u,v)) = \langle e^v, e^u, 1\rangle$.

$\therefore \vec{F}(\vec{r}(u,v))\cdot\vec{N} = e^v+e^u+1$.

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}\,du\,dv.$$



On the uv-plane, $z=0$ ∴ $u+v=1$ ∴ R is: $\begin{cases} 0\le v\le 1-u \\ 0\le u\le 1 \end{cases}$.

$$= \int_0^1\int_0^{1-u} e^u+e^v+1\,dv\,du = \int_0^1 \left[ve^u+e^v+v\right]_0^{1-u} du$$

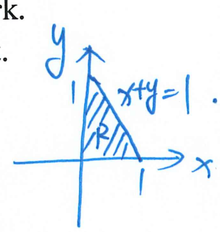
$$= \int_0^1 (1-u)e^u+e^{1-u}+1-u-1\,du = \int_0^1 e^u+e^{1-u}-u-ue^u\,du.$$

$$= \left[e^u\right]_0^1 - \left[e^{1-u}\right]_0^1 - \frac{1}{2}\left[u^2\right]_0^1 - \int_0^1 ue^u\,du = e-1-1+e-\frac{1}{2}-\left\{\left[ue^u\right]_0^1-\left[e^u\right]_0^1\right\}$$

$$= 2e-\frac{5}{2}-(e-e+1) = 2e-\frac{7}{2}.$$

2. Surface integrals $\iint_S G(\boldsymbol{r})dA$ (page 450). Evaluate the first type surface integral for the following data. Indicate the kind of surface. Show the details of your work.

(1) $G = \cos x + \sin x$, S is the portion of $x + y + z = 1$ in the first octant.



Solution: $S: \ z = 1 - x - y, \quad 0 \le x \le 1-y, \ 0 \le y \le 1$

Let $f(x,y) = 1 - x - y$, $\therefore f_x = -1, \ f_y = -1$.

$\therefore \iint_S G(r)dA = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\, dx dy = \iint_R (\sin x + \cos x)\sqrt{1+1+1}\, dx dy$

$= \int_0^1 \int_0^{1-y} \sqrt{3}\,(\sin x + \cos x)\, dx dy = \sqrt{3}\int_0^1 [\sin x - \cos x]_0^{1-y} dy$

$= \sqrt{3}\int_0^1 \sin(1-y) - \cos(1-y) + 1\, dy = \sqrt{3}\,[\cos(1-y) + \sin(1-y) + y]_0^1$

$= \sqrt{3}\,(1 + 1 - \cos 1 - \sin 1) = \sqrt{3}\,(2 - \cos 1 - \sin 1)$.

(2) $G = x + y + z, z = x + 2y, 0 \le x \le \pi, 0 \le y \le x$.

Solution: Let $f(x,y) = x + 2y$, $\therefore f_x = 1, \ f_y = 2$.

$G(x,y,f(x,y)) = \cancel{x} + y + \cancel{x+2y} = 2x + 3y$

$\therefore \iint_S G(\vec{r})d\vec{A} = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\, dx dy = \int_0^{\pi}\int_0^{x} 2x + 3y\sqrt{1+4+1}\, dy\, dx$

$= \sqrt{6}\int_0^{\pi} [2xy + \frac{3}{2}y^2]_0^x dx = \sqrt{6}\int_0^{\pi} 2x^2 + \frac{3}{2}x^2 dx$

$= \sqrt{6}\cdot\frac{7}{2}\int_0^{\pi} x^2 dx = \frac{7\sqrt{6}}{2}\ \frac{1}{3}[x^3]_0^{\pi} = \frac{7\sqrt{6}}{6}\pi^3$.

3. Evaluate the integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ where $\boldsymbol{F} =< 0,0,z >$, and S is the oriented surface parametrized by $\boldsymbol{r}(u,v) =< u\cos v, u\sin v, v >, 0 \le u \le 1, 0 \le v \le 2\pi$.

Solution: $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\cdot\vec{N}\,du dv$. (*)

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\rangle$

$\vec{F}(\vec{r}) = \langle 0,0,z\rangle = \langle 0,0,v\rangle$. $\therefore \vec{F}\cdot\vec{N} = \langle 0,0,v\rangle\cdot\langle \sin v, -\cos v, u\rangle = uv$.

$\therefore (*) = \int_0^1 \int_0^{2\pi} uv\, du\, dv = \int_0^1 v[\frac{1}{2}u^2]_0^{2\pi} dv = \int_0^1 v\cdot 2\pi^2 dv$

$= 2\pi^2\cdot[\frac{1}{2}v^2]_0^1 = \pi^2$.

4. Evaluate the surface integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ where $\boldsymbol{F} =< x, y, 2z >$, and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit square $[0,1]\times[0,1]$ with the downward orientation.

Solution: Let $x=u$, $y=v$. then $S: \vec{r}(u,v)=\langle u, v, 4-u^2-v^2\rangle$, $0\le u\le 1$, $0\le v\le 1$.

$\therefore \vec{N}=\vec{r}_u\times\vec{r}_v=\langle 1,0,-2u\rangle\times\langle 0,1,-2v\rangle=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -2u\\ 0 & 1 & -2v\end{vmatrix}=\langle 2u, 2v, 1\rangle.$

Because the orientation of the surface is downward, we take $\vec{N}=\langle -2u,-2v,-1\rangle$.

$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA=\iint_R \vec{F}\cdot\vec{N}\,du\,dv=\int_0^1\int_0^1\langle u, v, 2(4-u^2-v^2)\rangle\cdot\langle -2u,-2v,-1\rangle\,du\,dv$

$=\int_0^1\int_0^1 -2u^2-2v^2-8+2u^2+2v^2\,du\,dv=-8\int_0^1\int_0^1 du\,dv.$

$=-8\times(\text{area of unit square})=-8.$

5. Evaluate the surface integral $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ by the divergence theorem. Show the details.

(1) $\boldsymbol{F} =< x^2,\ 0,\ z^2 >$, S is the surface of the box $|x|\le 1, |y|\le 3, 0\le z\le 2$.

Solution: $\operatorname{div}\vec{F}=\frac{\partial(x^2)}{\partial x}+\frac{\partial(0)}{\partial y}+\frac{\partial(z^2)}{\partial z}=2x+2z.$

By the divergence theorem: $\iint_S \vec{F}\cdot\vec{n}\,dA=\iiint_T \operatorname{div}\vec{F}\,dV=\iiint_T 2x+2z\,dV.$

$=\int_{-1}^1\int_{-3}^3\int_0^2 2x+2z\,dz\,dy\,dx=\int_{-1}^1\int_{-3}^3[2xz+z^2]_0^2\,dy\,dx$

$=\int_{-1}^1\int_{-3}^3 4x+4\,dy\,dx=\int_{-1}^1(4x+4)[y]_{-3}^3\,dx=24\int_{-1}^1 x+1\,dx$

$=24[\frac{1}{2}x^2+x]_{-1}^1=24(\frac{1}{2}+1-\frac{1}{2}+1)=48.$

(2) $\boldsymbol{F} =< \sin y, \cos x, \cos z >$, S is the surface of $x^2+y^2\le 4,\ |z|\le 2$ (a cylinder and two disks).

Solution: $\operatorname{div}\vec{F}=0+0-\sin z=-\sin z.$

$\therefore \iint_S \vec{F}\cdot\vec{n}\,dA=\iiint_T \operatorname{div}\vec{F}\,dV=\iiint_T -\sin z\,dV$, where T is $x^2+y^2\le 4$, $|z|\le 2$.

So T can be represented by:
$T: \langle r,\theta,z\rangle: 0\le r\le 2, 0\le\theta\le 2\pi, -2\le z\le 2$

$=\int_{-2}^2\int_0^{2\pi}\int_0^2 -\sin z\,r\,dr\,d\theta\,dz$

$=\int_{-2}^2\int_0^{2\pi}-\frac{1}{2}\sin z\,[r^2]_0^2\,d\theta\,dz$

$=-\frac{1}{2}\int_{-2}^2\int_0^{2\pi}4\sin z\,d\theta\,dz=-2\int_{-2}^2 2\pi\sin z\,dz=-4\pi\int_{-2}^2\sin z\,dz$

$=-4\pi[-\cos z]_{-2}^2=4\pi(-\cos 2+\cos 2)=0.$

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6. Evaluate the integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA$ directly or, if possible, by the divergence theorem. Show the details.

(1) $\boldsymbol{F} =< ax, by, cz >$, S is the sphere $x^2 + y^2 + z^2 = 36$.

Solution: Because S is a closed surface, so we can use the divergence theorem.

$\operatorname{div} \vec{F} = a+b+c$.

$\therefore \iint_S \vec{F}\cdot\vec{n}\, dA = \iiint_T \operatorname{div}\vec{F}\, dV = \iiint_T (a+b+c)\, dV$.

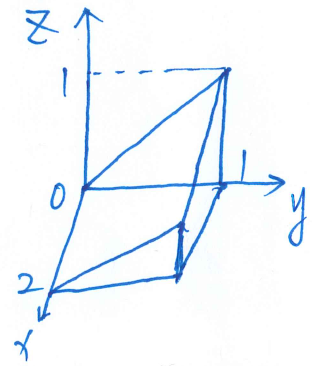
$= (a+b+c) \iiint_T dV = (a+b+c) \times$ (Volumn of the sphere)

$= (a+b+c) \cdot \frac{4}{3}\pi 6^3$

$= 288\pi (a+b+c)$.

(2) $\boldsymbol{F} =< y + z, 20y, 2z^3 >$, S is the surface of $0 \le x \le 2, 0 \le y \le 1, 0 \le z \le y$.

Solution: Since S is a closed surface, the divergence theorem is applicable. $\operatorname{div}\vec{F} = 0 + 20 + 6z^2 = 20 + 6z^2$.



$\therefore \iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_T \operatorname{div}\vec{F}\, dV = \int_0^2\int_0^1\int_0^y 20 + 6z^2\, dz\, dy\, dx$

$= \int_0^2\int_0^1 [20z + 2z^3]_0^y\, dy\, dx = \int_0^2\int_0^1 20y + 2y^3\, dy\, dx$

$= \int_0^2 [10y^2 + \frac{1}{2}y^4]_0^1\, dx = \int_0^2 (10 + \frac{1}{2})\, dx$

$= \frac{21}{2}[x]_0^2 = \frac{21}{2} \times 2 = 21$.

(3) $\boldsymbol{F} =< y^2, x^2, z^2 >$, S is $\boldsymbol{r} =< u, u^2, v >, 0 \le u \le 2, -2 \le v \le 2$.

Solution: S is $y = x^2$, $0 \le x \le 2$, $-2 \le z \le 2$, So this is not a closed surface, the divergence theorem is not applicable, So we evaluate the integral directly.

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle 1, 2u, 0\rangle \times \langle 0, 0, 1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u, -1, 0\rangle$.

$\because \vec{F}(\vec{r}) = \langle u^4, u^2, v^2\rangle$. $\therefore \vec{F}(\vec{r})\cdot\vec{N} = \langle u^4, u^2, v^2\rangle \cdot \langle 2u, -1, 0\rangle = 2u^5 - u^2$.

$\therefore \iint_S \vec{F}\cdot\vec{n}\, dA = \int_0^2\int_{-2}^2 2u^5 - u^2\, dv\, du = \int_0^2 (2u^5 - u^2)\cdot 4\, du$.

$= 4[\frac{1}{3}u^6 - \frac{1}{3}u^3]_0^2 = 4(\frac{1}{3}\times 2^6 - \frac{1}{3}\times 2^3) = \frac{4}{3}\times 56 = \frac{224}{3}$.