Tutorial 8

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Give the first three non-zero terms of the Fourier series.

1. $f(x) = \begin{cases} x, & \text{if } -\pi < x \le 0 \\ \pi - x, & \text{if } 0 < x \le \pi \end{cases}$.

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \pi - x\,dx = \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$

$= \frac{1}{2\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = -\frac{\pi}{4} + \frac{1}{2\pi}\cdot\frac{\pi^2}{2} = -\frac{\pi}{4} + \frac{\pi}{4} = 0.$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} x\cos nx\,dx + \int_0^{\pi}(\pi - x)\cos nx\,dx\right) = \frac{1}{\pi}(I_1 + I_2).$

$I_1 = \int_{-\pi}^{0} x\cos nx\,dx$. Let $u = x$, $v' = \cos nx$ $\therefore$ $u' = 1$, $v = \frac{1}{n}\sin nx$.

$= \left[x\cdot\frac{1}{n}\sin nx\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{1}{n}\sin nx\,dx = 0 + \left[\frac{1}{n^2}\cos nx\right]_{-\pi}^{0} = \frac{1}{n^2}(1 - \cos n\pi).$

$I_2 = \int_0^{\pi}(\pi - x)\cos nx\,dx$, Let $u = \pi - x$, $v' = \cos nx$, $\therefore$ $u' = -1$, $v = \frac{1}{n}\sin nx$.

$= \left[(\pi - x)\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\cdot(-1)\,dx = -\frac{1}{n^2}\left[\cos nx\right]_0^{\pi} = \frac{1}{n^2}(1 - \cos n\pi).$

$\therefore$ $a_n = \frac{1}{\pi}\cdot(I_1 + I_2) = \frac{1}{\pi}\cdot\frac{2}{n^2}(1 - \cos n\pi) = \frac{2}{\pi n^2}(1 - \cos n\pi).$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\cdot\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{\pi}(\mathrm{II}_1 + \mathrm{II}_2).$

$\mathrm{II}_1 = \int_{-\pi}^{0} x\sin nx\,dx$. Let $u = x$, $v' = \sin nx$. $\therefore$ $u' = 1$, $v = -\frac{1}{n}\cos nx$.

$= \left[x\cdot\frac{-1}{n}\cos nx\right]_{-\pi}^{0} - \int_{-\pi}^{0} -\frac{1}{n}\cos nx\,dx = 0 - (-\pi)\frac{-1}{n}\cos(n\pi) + \frac{1}{n^2}\left[\sin nx\right]_{-\pi}^{0}$

$= -\frac{\pi}{n}\cos n\pi.$

$\mathrm{II}_2 = \int_0^{\pi}(\pi - x)\sin nx\,dx$. Let $u = \pi - x$, $v' = \sin nx$ $\therefore$ $u' = -1$, $v = -\frac{1}{n}\cos nx$.

$= \left[(\pi - x)\frac{-1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nx\cdot(-1)\,dx = 0 - \pi\cdot\frac{-1}{n}\cos 0 - \int_0^{\pi}\frac{1}{n}\cos nx\,dx$

$= \frac{\pi}{n} - \frac{1}{n^2}\left[\sin nx\right]_0^{\pi} = \frac{\pi}{n}.$ $\therefore$ $b_n = \frac{1}{\pi}(\mathrm{II}_1 + \mathrm{II}_2) = \frac{1}{\pi}\left(\frac{\pi}{n} - \frac{\pi}{n}\cos n\pi\right) = \frac{1}{n}(1 - \cos n\pi).$

$\therefore$ The Fourier Series of $f(x)$ is

$f(x) = \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}(1 - \cos n\pi)\cos nx + \frac{1}{n}(1 - \cos n\pi)\sin nx\right].$

2. $f(x) = x^2 \ (0 < x \le 2\pi)$.

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\cdot\left[\frac{1}{3}x^3\right]_0^{2\pi} = \frac{1}{2\pi}\left(\frac{8}{3}\pi^3\right) = \frac{4}{3}\pi^2.$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$, Let $u = x^2, v' = \cos nx \therefore u' = 2x, v = \frac{1}{n}\sin nx.$

$= \frac{1}{\pi}\left\{\left[x^2\cdot\frac{1}{n}\sin nx\right]_0^{2\pi} - \int_0^{2\pi}\cdot\frac{1}{n}\sin nx\cdot(2x)dx\right\} = \frac{1}{\pi}\left\{0 - \frac{2}{n}\int_0^{2\pi} x\sin nx\, dx\right\}.$

$= \frac{2}{n\pi}\cdot\int_0^{2\pi} x\sin nx\, dx$, Let $u = x$, $v' = \sin nx$, $\therefore u' = 1$, $v = -\frac{1}{n}\cos nx$.

$= \frac{2}{n\pi}\left\{\left[x\frac{-1}{n}\cos nx\right]_0^{2\pi} - \int_0^{2\pi} -\frac{1}{n}\cos nx\cdot dx\right\}$

$= \frac{2}{n\pi}\left\{2\pi\cdot\frac{-1}{n}\cos 2n\pi - 0 + \int_0^{2\pi}\frac{1}{n}\cos nx\, dx\right\}$

$= \frac{4}{n^2}\cdot + \frac{1}{n^2}\cdot\left[\sin nx\right]_0^{2\pi} = \frac{4}{n^2}.$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\, dx$, Let $u = x^2, v' = \sin nx$.

$\therefore u' = 2x, V = -\frac{1}{n}\cos nx.$

$= \frac{1}{\pi}\cdot\left\{\left[x^2\cdot\frac{-1}{n}\cos nx\right]_0^{2\pi} - \int_0^{2\pi} -\frac{1}{n}\cos nx\cdot 2x\, dx\right\}$

$= \frac{1}{\pi}\left\{4\pi^2\cdot\frac{-1}{n}\cos 2n\pi - 0 + \frac{2}{n}\int_0^{2\pi} x\cos nx\, dx\right\}$

$= \frac{-4\pi}{n} + \frac{2}{n\pi}\cdot\int_0^{2\pi} x\cos nx\, dx$, Let $u = x, v' = \cos nx$, $\therefore u' = 1, V = \frac{1}{n}\sin nx$.

$= -\frac{4\pi}{n} + \frac{2}{n\pi}\cdot\left\{\left[x\cdot\frac{1}{n}\sin nx\right]_0^{2\pi} - \int_0^{2\pi}\cdot\frac{1}{n}\sin nx\cdot dx\right\}$

$= -\frac{4\pi}{n} + \frac{2}{n\pi}\left\{0 - \frac{1}{n}\cdot\int_0^{2\pi}\sin nx\, dx\right\}$

$= -\frac{4\pi}{n} - \frac{2}{n^2\pi}\left[\frac{-1}{n}\cos nx\right]_0^{2\pi} = -\frac{4\pi}{n}\cdot + \frac{2}{n^3\pi}[0-0] = -\frac{4\pi}{n}.$

Therefore the Fourier Series of $f(x)$ is

$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx + \frac{-4\pi}{n}\sin nx\right]$

3. $f(x) = x^2\ (-\pi < x \le \pi)$.

Because $f(x)$ is an even function and the given interval $[-\pi, \pi]$ is symmetric about the origin, so $b_n = 0$, $n = 1, 2, \cdots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\cdot\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{1}{3}(\pi^3+\pi^3) = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$, Let $u = x^2$, $v' = \cos nx$.

$\therefore u' = 2x$, $v = \frac{1}{n}\sin nx$

$$= \frac{1}{\pi}\cdot\left\{\left[\frac{1}{n}x^2\sin nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{1}{n}\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left[0 - \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,dx\right]$$, Let $u = x$, $v' = \sin nx$

$\therefore u' = 1$, $v = -\frac{1}{n}\cos nx$.

$$= -\frac{2}{n\pi}\left\{\left[x\cdot\frac{-1}{n}\cos nx\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{n}\cos nx\,dx\right\}$$

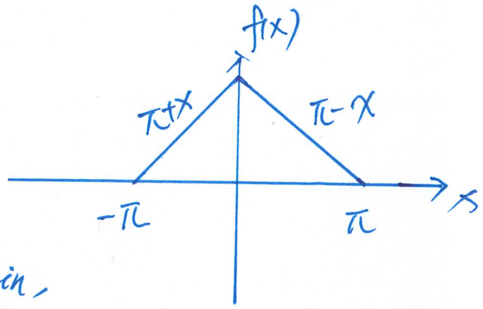
$$= -\frac{2}{n\pi}\cdot\left\{\pi\cdot\frac{-1}{n}\cos n\pi - (-\pi)\cdot\frac{-1}{n}\cos n\pi + \frac{1}{n}\cdot\left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi}\right\}$$

$$= -\frac{2}{n\pi}\cdot\left\{\frac{-\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi\right] = \frac{2}{n\pi}\cdot\frac{2\pi}{n}\cos n\pi = \frac{4}{n^2}\cos n\pi.$$

Or, we can calculate a_n by $a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$.

Therefore, the Fourier Series of $f(x)$ is

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos n\pi\cdot\cos nx\right).$$



4. $f(x) = \begin{cases} x+\pi, & \text{if } -\pi < x \le 0 \\ \pi - x, & \text{if } \quad 0 < x \le \pi \end{cases}$.

Because $f(x)$ is an even function, and the given interval $(-\pi, \pi)$ is symmetric about the origin.

So $b_n = 0$, $n = 1, 2, 3, \ldots$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} \pi + x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \pi - x\,dx.$$

$$= \frac{1}{2\pi}\cdot\left[\pi x + \frac{1}{2}x^2\right]_{-\pi}^{0} + \frac{1}{2\pi}\cdot\left[\pi x - \frac{1}{2}x^2\right]_{0}^{\pi}.$$

$$= \frac{1}{2\pi}\left[0 - \pi\cdot(-\pi) + \frac{\pi^2}{2}\right] + \frac{1}{2\pi}\cdot\left[\pi^2 - \frac{\pi^2}{2} - 0\right] = -\frac{1}{2\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) + \frac{1}{2\pi}\cdot\frac{\pi^2}{2}.$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\cdot\int_{0}^{\pi}(\pi - x)\cos nx\,dx.$$ Let $u = \pi - x$, $v' = \cos nx$

$u' = -1$, $v = \frac{1}{n}\sin nx$.

$$= \frac{2}{\pi}\cdot\left\{\underbrace{\left[(\pi - x)\frac{1}{n}\sin nx\right]_{0}^{\pi}}_{=0} - \int_{0}^{\pi}\frac{1}{n}\sin nx\cdot(-1)\,dx\right\}$$

$$= \frac{2}{n\pi}\cdot\int_{0}^{\pi}\sin nx\,dx = \frac{2}{n\pi}\cdot\left[\frac{1}{n}\cos nx\right]_{0}^{\pi}$$

$$= \frac{-2}{n^2\pi}\left[\cos n\pi - 1\right] = \frac{2}{n^2\pi}(1 - \cos n\pi).$$

Therefore, the Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(1 - \cos n\pi)\cdot\cos nx.$$

5. $f(x) = x, (-\pi < x \leq \pi)$.

Because $f(x)$ is and odd function and $(-\pi, \pi)$ is symmetric about the origin. $a_n = 0$, $n = 0, 1, 2, \cdots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx.$$ Let $u = x$, $v' = \sin nx$,

then $u' = 1$, $v = -\frac{1}{n}\cos nx$.

$$= \frac{2}{\pi}\cdot\left\{\left[x\cdot\frac{-1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\cdot\left\{\pi\cdot\frac{-1}{n}\cos n\pi - 0 + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}$$

$$= -\frac{2}{n}\cos n\pi + \frac{2}{n\pi}\cdot\left[\frac{1}{n}\sin nx\right]_0^{\pi}$$

$$= -\frac{2}{n}\cos n\pi.$$

Therefore, the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}\cos n\pi\cdot\sin nx.$$