EEE319 Optimisation
Lecture 7 Nonlinear Programming (1)

Prof. Xinheng Wang

xinheng.wang@xjtlu.edu.cn

Office: EE512

Nonlinear programming

- Previous lectures
  - Linear programming

- Nonlinear programming
  - Single variable nonlinear programming
  - Line search
    - Bisection
    - Golden section
    - Newton method
    - Secant method

Definition of Nonlinear Programming (NLP)

- Nonlinear programming (NLP): If the objective function is nonlinear and/or the feasible region is determined by nonlinear constraints, this is called a nonlinear programming optimisation.
- In other words, NLP is an optimisation problem when either objective function or constraints have a nonlinear function.
- Linearity vs nonlinearity: Linearity is data can be expressed with a straight line. Nonlinearity is the opposite.

Fundamental Approach to NLP

- Searching is a general approach to find the optimal solutions.
  - Line search
    - Bisection
    - Golden section
    - Newton method
    - Secant

Fundamental Approach to NLP

- Searching is normally operated on an iterative algorithm
- Recall iterative algorithm:

$$x_{i+1} = x_i + \alpha_i d_i$$

where α_i is the step length, which is a positive scalar, d_i is the direction.

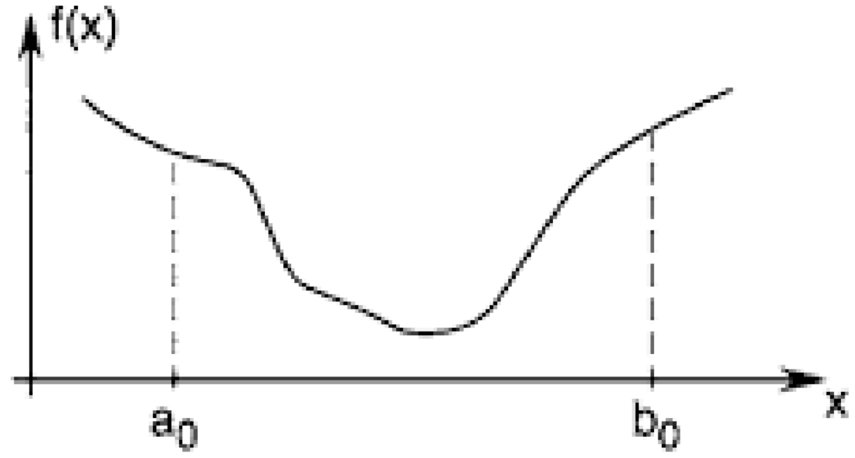
- Procedure:
    1. Start from an initial candidate solution, x_0 ;
    2. Generate a sequence of candidate solutions, $x_1, x_2, x_3, \ldots$
    3. Stop when a certain condition is met; and return the candidate solution.

Line Search

- Line search method seeks the minimum of a defined nonlinear function by selecting a reasonable direction, d_i, that, when computed iteratively with a reasonable step size, will provide a function value closer to the absolute minimum of the function.

One Variable Line Search Methods

- Starting from one variable objective function, with a *unimodal* mode
- Unimodal: only one minimum at interval $[a_0, b_0]$



One Variable Line Search Methods –Bisection

- Working with mid-point of a_0, b_0 and derivative
- Mid-point:

$$x_0 = \frac{1}{2}(a_0 + b_0)$$

- Conditions:
  - If $f'(x_0) = 0$, x_0 is the minimizer, the minimum value is $f(x_0)$;
  - If $f'(x_0) > 0$, then narrow to $[a_0, x_0]$
  - If $f'(x_0) < 0$, then narrow to $[x_0, b_0]$

- Every evaluation of f' reduces the interval by half.

One Variable Line Search Methods –Bisection

- Min $f(x) = x^4 - 14x^3 + 60x^2 - 70x$

at interval [0,2] with required accuracy of $\epsilon = 0.3$, which is how far from local minimizer.

- Solution by bisection method
- First of all, to calculate the first-order derivative

$$f' = 4x^3 - 42x^2 + 120x^1 - 70$$

One Variable Line Search Methods –Bisection

- Solution by bisection method
- First of all, to calculate the first-order derivative

$$f' = 4x^3 - 42x^2 + 120x^1 - 70$$

- Iteration 0: $a_0 = 0, b_0 = 2; x_0 = \frac{1}{2}(a_0 + b_0)$=1; $f' = 12 > 0.$
- Iteration 1: $a_1 = 0, b_1 = 1; x_1 = \frac{1}{2}(a_1 + b_1)$=0.5; $f' = -20 < 0,$ $\lfloor x_1 - x_0 \rfloor = 0.5$.
- Iteration 2: $a_2 = 0.50, b_2 = 1; x_2 = \frac{1}{2}(a_2 + b_2)$=0.75; $f' = -1.9375 < 0,$ $\lfloor x_2 - x_1 \rfloor = 0.25 < \epsilon,$ stops.
- $f(x^*) = 0.75^4 - 14 * 0.75^3 + 60 * 0.75^2 - 70 * 0.75 \approx -24.34$

One Variable Line Search Methods – Golden Section

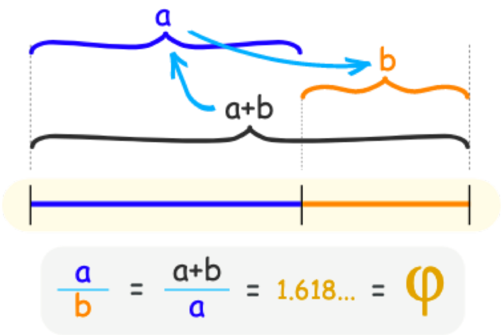
- Bisection is to separate the interval into 2 sections in the middle. How about three sections? Golden Section is such an algorithm to separate an interval into 3 sections by following the famous Golden Ratio rule.
- Golden Ratio
  - A line with two sections *a* and *b*,

the long section divided by the short section

is equal to

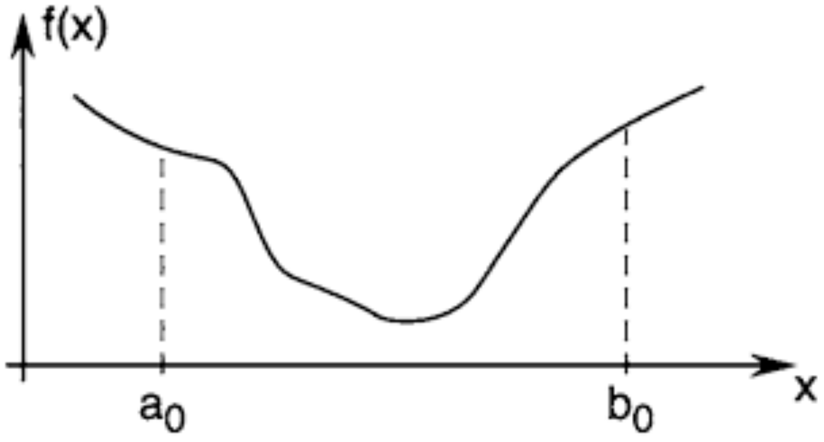
the whole length divided by the long section.

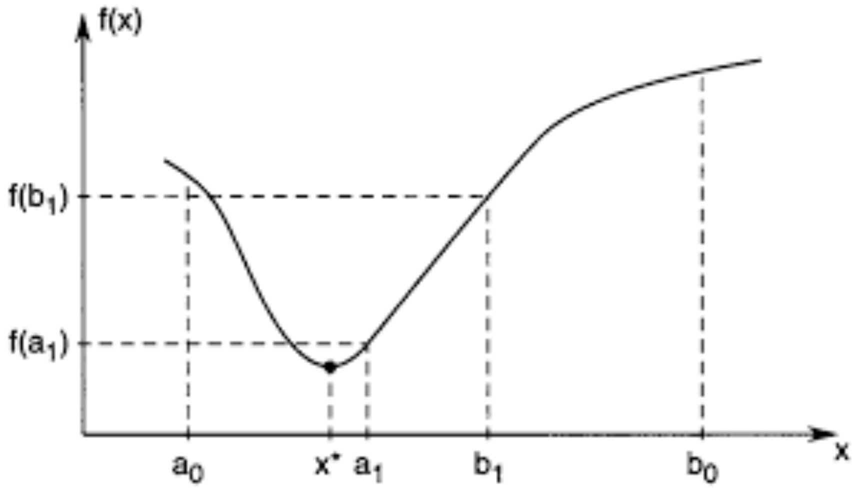
$$\frac{a}{b} = \frac{a+b}{a} = 1.618 = \varphi$$



One Variable Line Search Methods – Golden Section

- As illustrated in the figures below, given an internal $[a_0, b_0]$ with a unimodal objective function, find a point that is no more than ε away from the local minimizer x^*.





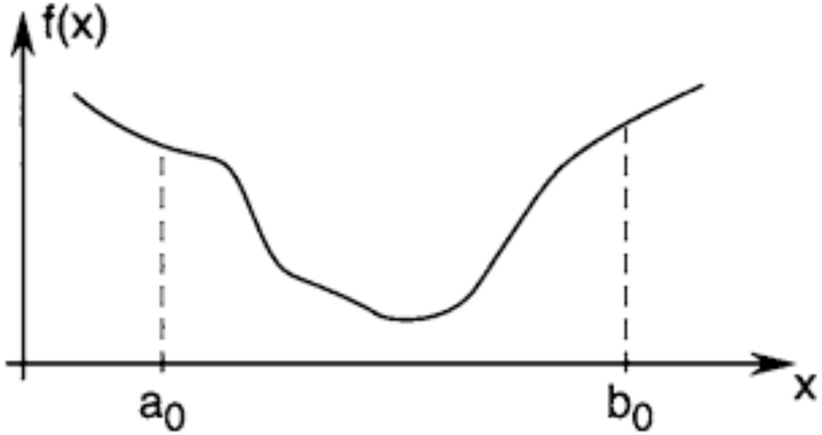
One Variable Line Search Methods – Golden Section

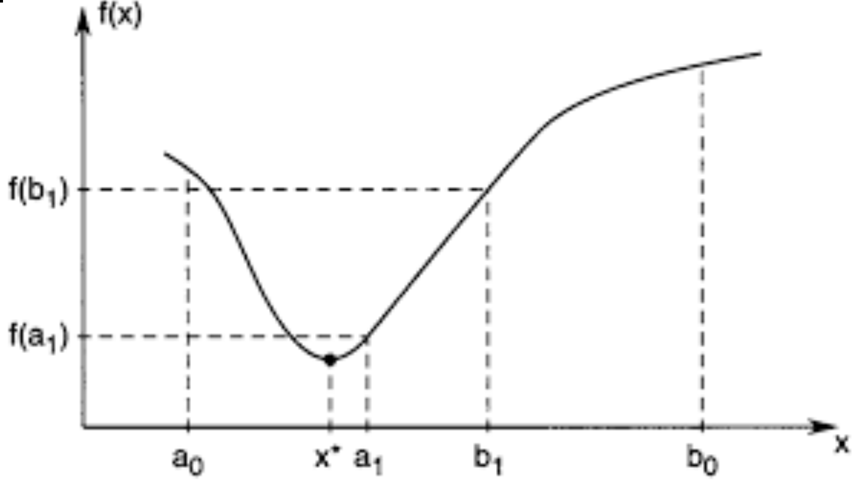
- Steps to do it
- Evaluate $a_1, b_1 \in [a_0, b_0]$, where $\in a_1 < b_1$

(1) If $f(a_1) < f(b_1)$, then $x^* \in [a_0, b_1]$

(2) If $f(a_1) > f(b_1)$, then $x^* \in [a_1, b_0]$

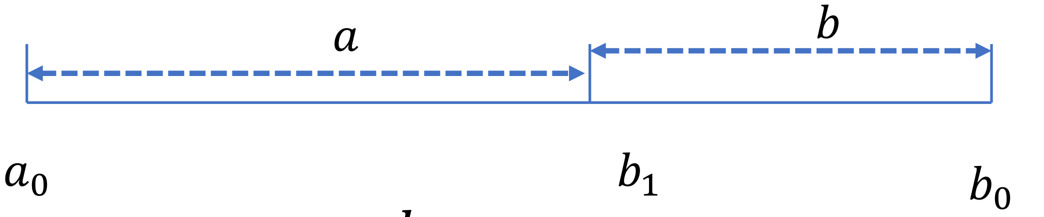
(3) If $f(a_1) = f(b_1)$, then $x^* \in [a_1, b_1]$ This will rarely occur.



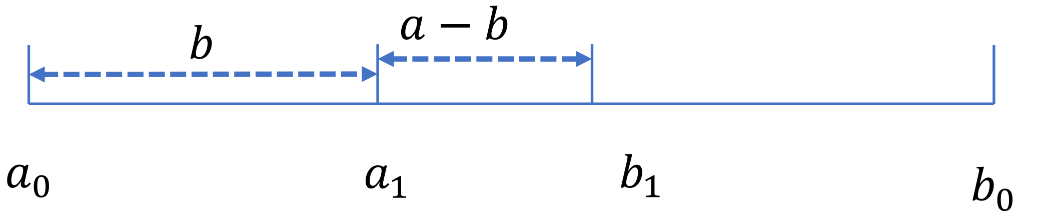


One Variable Line Search Methods – Golden Section

- How to determine the two points a_1 and b_1?
- First point at b_1 is $\frac{a}{b} = \frac{a+b}{a}$, which is the golden ratio.



Second point is at a_1 is $\frac{a}{b} = \frac{b}{a-b}$, which is also the golden ratio.



One Variable Line Search Methods –Newton Method

- The underlying is Taylor series:
- Recall Taylor series: A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. It is expressed as:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where *a* is a real or complex number.

- This can be written in a sigma notation

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

where $f^{(i)}(a)$ denotes the *i*th derivative of evaluated at the point *a*, where *i*! denotes the factorial of *n*.

One Variable Line Search Methods –Newton Method

- Derivation from Taylor series to Newton Method
- Taylor series: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$
- Looking at first two terms and setting $a = x_0$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- Setting $f(x)$=0

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

$$x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Newton Method is by iterating this equation: $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
- High-order derivative could be obtained as well.

One Variable Line Search Methods –Newton Method

- Newton Method by an example
- Find the root of $2x^2 - x - 2 = 0$ between 1 and 2.
- $f(x) = 2x^2 - x - 2; f'(x) = 4x - 1$
- $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
- $x_0 = 1.6; x_1 = x_0 - \frac{2{x_0}^2 - x_0 - 2}{4x_0 - 1} = 1.6 - \frac{5.12 - 1.6 - 2}{6.4 - 1} \approx 1.32$
- $x_2 = x_1 - \frac{2{x_1}^2 - x_1 - 2}{4x_1 - 1} = 1.32 - \frac{3.48 - 1.32 - 2}{5.28 - 1} \approx 1.2826 \approx 1.28$
- $x_3 = x_2 - \frac{2{x_2}^2 - x_2 - 2}{4x_2 - 1} = 1.28 - \frac{5.12 - 1.6 - 2}{5.12 - 1} \approx 1.280776 \approx 1.28$

One Variable Line Search Methods –Secant Method

- In some practical applications, it is impossible to obtain the derivative, an approximation for f' is used in Newton Method. We obtain

$$f'(x_i) = \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

It is called Secant method (the poor man's Newton method).

THANK YOU

VISIT US

WWW.XJTLU.EDU.CN

FOLLOW US

@XJTLU

Xi'an Jiaotong-Liverpool University
西交利物浦大學