MTH201 Name________________ ID______________ Nov. 2017

Tutorial 09 Basic concepts of PDEs and wave equation

1. Verify that the function $u(x,y) = a\ln(x^2+y^2)+b$ satisfies Laplace's equation
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$
And determine a, b so that u satisfies the boundary conditions u=110 on the circle $x^2+y^2=1$ and u=0 on the circle $x^2+y^2=100$.

Because $u(x,y) = a\ln(x^2+y^2)+b$, $\therefore$ $\frac{\partial u}{\partial x} = \frac{2ax}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2} = \frac{2a(x^2+y^2)-4ax^2}{(x^2+y^2)^2} = \frac{2a(y^2-x^2)}{(x^2+y^2)^2}$.

$\frac{\partial u}{\partial y} = \frac{2ay}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2} = \frac{2a(x^2+y^2)-4ay^2}{(x^2+y^2)^2} = \frac{2a(x^2-y^2)}{(x^2+y^2)^2}$

Therefore $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{2a(y^2-x^2)}{(x^2+y^2)^2}+\frac{2a(x^2-y^2)}{(x^2+y^2)^2}=0$.

So $u(x,y)=a\ln(x^2+y^2)+b$ is a solution of Laplace's equation.

The boundary conditions are $\begin{cases} u=110, & \text{on } x^2+y^2=1 \\ u=0, & \text{on } x^2+y^2=100. \end{cases} \Rightarrow \begin{cases} 110 = a\ln 1+b \\ 0 = a\ln 100 + b. \end{cases}$

$\therefore \begin{cases} b=110 \\ 2a\ln 10+b=0 \end{cases} \Rightarrow \begin{cases} a=-\frac{55}{\ln 10} \\ b=110. \end{cases}$ $(\ln 100 = \ln 10^2 = 2\ln 10)$.

Therefore $u(x,y) = -\frac{55}{\ln 10}\ln(x^2+y^2)+110$.

2. Solve the following PDEs using the same method to solve ODEs.
(a) $u_{xx}+16\pi^2 u=0$

Consider the following ODE: $u''+16\pi^2 u=0$

The Characteristic equation is $\lambda^2+16\pi^2=0. \to \lambda^2=-16\pi^2 \to \lambda=\pm 4\pi i$.

Thus, the general solution of the ODE is $u(x) = A\cos 4\pi x + B\sin 4\pi x$, A, B are constants

Since there are no y-derivatives in the given PDE, there are only partial derivatives with respect to x, any functions of variable y is considered as constants, So the solution of the PDE is

$$u(x,y) = A(y)\cos 4\pi x + B(y)\sin 4\pi x.$$

by replacing the constants A and B by $A(y)$ and $B(y)$ respectively.

Here $A(y)$ and $B(y)$ are arbitrary function of y.

(b) $25u_{yy} - 4u = 0$

Consider the following ODE: $25u'' - 4u = 0$.

The characteristic equation of the ODE is

$$25\lambda^2 - 4 = 0 \rightarrow \lambda^2 = \frac{4}{25} \rightarrow \lambda = \pm\frac{2}{5}.$$

Then the solution of the ODE is

$$u = Ae^{\frac{2}{5}y} + Be^{-\frac{2}{5}y},$$ A and B are arbitrary constants.

Therefore, the solution of the given PDE is

$$u(x,y) = A(x)\, e^{\frac{2}{5}y} + B(x)\, e^{-\frac{2}{5}y}.$$

where $A(x)$ and $B(x)$ are arbitrary functions of x.

(c) $u_y + y^2 u = 0$

Consider the following first order homogeneous linear ODE, with $u = u(y)$.

$$u' + y^2 u = 0 \rightarrow u' = -y^2 u \rightarrow \frac{u'}{u} = -y^2.$$

By integrating the both sides, we have.

$$\ln|u| = -\frac{1}{3}y^3 + C_1 \rightarrow |u| = e^{-\frac{1}{3}y^3 + C_1}$$

$\therefore$ $u = Ce^{-\frac{1}{3}y^3}$, where $C = \pm e^{C_1}$ is an arbitrary constant.

Then we obtain the solution of the PDE is

$$u(x,y) = C(x)\, e^{-\frac{1}{3}y^3},$$

where $C(x)$ is an arbitrary function of x.

3. Boundary value problem

(a) $\begin{cases} \dfrac{\partial u}{\partial x} - 3\dfrac{\partial u}{\partial y} = 0 & ① \\ u(x,0) = 3e^{5x} & ② \end{cases}$

(1). Assume $u(x,y) = X(x)Y(y)$ and substitute into ①, we have

$$X'Y - 3XY' = 0 \Leftrightarrow \frac{X'}{X} = 3\frac{Y'}{Y} = c. \quad c \text{ is a constant.}$$

$$\Leftrightarrow \begin{cases} X' - cX = 0 \\ Y' - \frac{c}{3}Y = 0 \end{cases} \Rightarrow \begin{cases} X = Ae^{cx} & (\lambda = c) \\ Y = Be^{\frac{c}{3}y}. & (\lambda = \frac{c}{3}). \end{cases}$$ A, B and C are arbitrary constants.

Thus $u(x,y) = XY = ABe^{c(x+\frac{y}{3})} = ke^{c(x+\frac{y}{3})}$. is a solution of ①.

(2). Find k and c such that $u(x,y)$ satisfies the boundary conditions ②.

We have $u(x,0) = ke^{cx} = 3e^{5x} \Rightarrow k=3,\ c=5.$

In conclusion, the solution of the PDE satisfying the boundary condition is

$$u(x,y) = 3e^{5(x+\frac{y}{3})}.$$

(b) $\begin{cases} 2u_x + u_y = 0 & ① \\ u(0,y) = 5e^{-7y} & ② \end{cases}$

(1). Suppose $u(x,y) = X(x)Y(y)$. and substitute into ①, we get

$$2X'Y + XY' = 0 \Leftrightarrow \frac{2X'}{X} = -\frac{Y'}{Y} = c. \quad c \text{ is a constant.}$$

$$\Leftrightarrow \begin{cases} \frac{2X'}{X} = c \\ -\frac{Y'}{Y} = c \end{cases} \Leftrightarrow \begin{cases} X' - \frac{c}{2}X = 0 \\ Y' + cY = 0 \end{cases} \Rightarrow \begin{cases} X = Ae^{\frac{c}{2}x} & (\lambda = \frac{c}{2}) \\ Y = Be^{-cy} & (\lambda = -c), \end{cases}$$

where A, B, C are arbitrary constants. Thus

$$u(x,y) = XY = ABe^{c(\frac{x}{2}-y)} = ke^{c(\frac{x}{2}-y)}$$

is a solution of PDE ①.

(2). Find k and c such that $u(x,y)$ satisfies the boundary condition ②

i.e. $u(0,y) = ke^{-cy} = 5e^{-7y}$ $\therefore k=5.\ c=7.$

Therefore, the solution of the PDE is

$$u(x,y) = 5e^{7(\frac{x}{2}-y)}.$$

(c) $\begin{cases} 5\dfrac{\partial u}{\partial x} = 6\dfrac{\partial u}{\partial y} & ① \\ u(0,y) = 10e^{2y} + 2e^{y} & ② \end{cases}$

(1) Assume $u(x,y) = X(x)Y(y)$, and substitute into ①, we have

$$5X'Y = 6XY' \Leftrightarrow \frac{5X'}{X} = \frac{6Y'}{Y} = C.$$

$$\Leftrightarrow \begin{cases} X' - \frac{C}{5}X = 0 \\ Y' - \frac{C}{6}Y = 0 \end{cases} \Rightarrow \begin{cases} X = Ae^{\frac{C}{5}x}, & (\lambda = \frac{C}{5}) \\ Y = Be^{\frac{C}{6}y} & (\lambda = \frac{C}{6}). \end{cases}$$

Thus, $u(x,y) = X(x)Y(y) = ABe^{C(\frac{x}{5}+\frac{y}{6})} = ke^{C(\frac{x}{5}+\frac{y}{6})}$

is a solution of ①, where $k=AB$ and C are arbitrary constants.

(2). Find the constants k and C such that $u(x,y)$ satisfies the condition ②.

At first, find $u_1(x,y) = k_1 e^{C_1(\frac{x}{5}+\frac{y}{6})}$ satisfying $u_1(0,y) = 10e^{2y}$,

i.e. $k_1 e^{\frac{C_1}{6}y} = 10e^{2y}$ $\therefore$ $k_1 = 10$, $C_1 = 12 \Rightarrow u_1 = 10e^{12(\frac{x}{5}+\frac{y}{6})}$.

Secondly, find $u_2(x,y) = k_2 e^{C_2(\frac{x}{5}+\frac{y}{6})}$ satisfying $u_2(0,y) = 2e^{y}$,

i.e. $k_2 e^{\frac{C_2}{6}y} = 2e^{y}$ $\therefore$ $k_2 = 2$, $C_2 = 6 \Rightarrow u_2 = 2e^{6(\frac{x}{5}+\frac{y}{6})}$.

Therefore, by the fundamental theorem on superposition,

$$u = u_1 + u_2 = 10e^{12(\frac{x}{5}+\frac{y}{6})} + 2e^{6(\frac{x}{5}+\frac{y}{6})}.$$

is the solution of the given PDE satisfying condition ②.

4. Wave equation, solution by separating variables.

Consider the following one-dimensional wave equation problem:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < 1, \qquad t \geq 0$$

with the initial boundary conditions:

$$y(0,t) = y(1,t) = 0, \text{for } t \geq 0;$$
$$y_t(x,0) = 0 \text{ and } y(x,0) = f(x).$$

Using separating variables, find the solution of the problem for the following given initial deflections *f(x)* and k. [Hint: let $y(x,t) = X(x)T(t)$] and $\frac{X''}{X} = \frac{T''}{T} = -\lambda$, where $\lambda = \lambda_n = n^2\pi^2$, for n=1,2,⋯.

(a) $f(x) = k\sin 3\pi x$

Suppose $y(x,t) = X(x)T(t) \Rightarrow \frac{\partial y}{\partial x} = X'T,\ \frac{\partial^2 y}{\partial x^2} = X''T.$

$\frac{\partial y}{\partial t} = XT',\ \frac{\partial^2 y}{\partial t^2} = XT''.$

$\therefore X''T = XT'' \Rightarrow \frac{X''}{X} = \frac{T''}{T} = -\lambda$, where λ is a constant.

Then the wave equation becomes $\begin{cases} \frac{X''}{X} = -\lambda \\ \frac{T''}{T} = -\lambda \end{cases} \Rightarrow \begin{cases} X'' + \lambda X = 0 & ① \\ T'' + \lambda T = 0 & ② \end{cases}$.

(1) Solving ①: Using the hint $\lambda = n^2\pi^2$ then: $X'' + n^2\pi^2 X = 0$.

Hence $X(x) = A_n \cos n\pi x + B_n \sin n\pi x$. where A_n and B_n are arbitrary constants. $n=1,2,\cdots$.

$y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$. (Because $T(t) \neq 0$, otherwise $y(x,t) = X(x)T(t) \equiv 0$).

So we get $X(0) = 0$, $n=1,2,\cdots$. $\Rightarrow A_n \cos 0 + B_n \sin 0 = 0 \Rightarrow A_n = 0$.

$\therefore X(x) = B_n \sin n\pi x$.

$y(1,t) = 0 = X(1)T(t) = 0 \Rightarrow X(1) = 0$. $\therefore X(1) = B_n \sin n\pi = 0 \Rightarrow B_n$ could be any constants.

Let $B_n = 1$. then $X(x) = \sin n\pi x$. is a solution of ① satisfying the boundary conditions.

(2). Solving ②: similarly, ② $\Rightarrow T'' + n^2\pi^2 T = 0$ gives

$T(t) = C_n \cos n\pi t + D_n \sin n\pi t$, C_n, D_n are arbitrary constants. $n=1,2,\cdots$.

$\therefore y(x,t) = X(x)T(t) = \sin n\pi x[C_n \cos n\pi t + D_n \sin n\pi t] = y_n(x,t)$.

Let $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin n\pi x[C_n \cos n\pi t + D_n \sin n\pi t]$ ③

From the initial condition: $y_t(x,0)=0$, so

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\sin n\pi x\left[-n\pi C_n\sin n\pi t+n\pi D_n\cos n\pi t\right]\Big|_{t=0}=0.$$

$$=\sum_{n=1}^{\infty}\sin n\pi x\left[0+n\pi D_n\right]=0\Rightarrow\sum_{n=1}^{\infty}n\pi D_n\sin n\pi x=0\Rightarrow D_n=0.$$

Therefore $y(x,t)=\sum_{n=1}^{\infty}C_n\sin(n\pi x)\cos(n\pi t)$,

where C_n are unknown to find by using $y(x,0)=f(x)$.

If $f(x)=k\sin 3\pi x$, then

$$y(x,0)=\sum_{n=1}^{\infty}C_n\sin n\pi x=k\sin 3\pi x.$$

$$\therefore\begin{cases}C_n=0 & \text{if } n\neq 3\\ C_3\sin 3\pi x=k\sin 3\pi x\end{cases}\Rightarrow\begin{cases}C_3=k\\ C_n=0, & \text{if } n\neq 3.\end{cases}$$

In conclusion, the solution of the PDE satisfying the conditions are

$$y(x,t)=k\sin(3\pi x)\cos(3\pi t).$$

(b) $f(x) = kx(1-x)$

The method for solving the PDE is the same as in (1). The only difference is that we need to determine C_n by using the condition $y(x,0) = f(x)$, i.e.

$\sum_{n=1}^{\infty} C_n \sin(n\pi x) = kx(1-x) = f(x)$.

Therefore C_n is the coefficients of the Fourier series of $f(x)$. In the Fourier series we have $\frac{n\pi}{L} = n\pi$, so $L=1$. So we should extend $f(x)$ as an odd periodic function with period $=2L=2$: $F(x) = f(x)$, $0 \le x \le 1$. $F(-x) = -F(x)$, $F(x+2) = F(x)$

$$\therefore C_n = \int_{-1}^{1} F(x) \sin n\pi x \, dx$$

$$= 2\int_0^1 F(x) \sin n\pi x \, dx = 2\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\int_0^1 kx(1-x) \sin n\pi x \, dx. = 2k\int_0^1 x(1-x)\sin n\pi x \, dx.$$

Let $u = x(1-x) = x - x^2$. $v' = \sin n\pi x$.

$u' = 1-2x$, $\quad v = -\frac{1}{n\pi}\cos n\pi x$

$$= \underbrace{2k\cdot\left[(x-x^2)\frac{-1}{n\pi}\cos n\pi x\right]_0^1}_{=0} - 2k\int_0^1 -\frac{1}{n\pi}\cos n\pi x \cdot (1-2x)\,dx.$$

$= 0 + \frac{2k}{n\pi}\int_0^1 (1-2x)\cos n\pi x \, dx$. Let $u = 1-2x$, $v' = \cos n\pi x$

$u' = -2 \quad v = \frac{1}{n\pi}\sin n\pi x$.

$$= \frac{2k}{n\pi}\left\{\left[(1-2x)\frac{1}{n\pi}\cdot\sin n\pi x\right]_0^1 - \int_0^1 \frac{1}{n\pi}\sin n\pi x\cdot(-2)\,dx\right\}$$

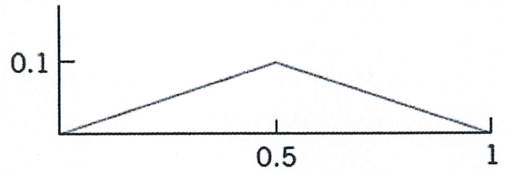
$$= \frac{2k}{n\pi}\left\{0 - 0 + 2\,\frac{1}{n\pi}\int_0^1 \sin n\pi x\,dx\right\} = \frac{4k}{n^2\pi^2}\cdot\frac{-1}{n\pi}\left[\cos n\pi x\right]_0^1 = \frac{-4k}{n^3\pi^3}(\cos n\pi - 1).$$

$$= \frac{4k}{n^3\pi^3}(1-\cos n\pi).$$

Therefore the solution $y(x,t)$ is

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin n\pi x \cos n\pi t$$

$$= \sum_{n=1}^{\infty} \frac{4k}{n^3\pi^3}(1-\cos n\pi)\sin n\pi x \cdot \cos n\pi t.$$



(c) . .

The method for solving the PDE is the same as in (b). The only difference is that we need to determine the unknown coefficients C_n by using the condition $y(x,0)=f(x)$, where
$$f(x)=\begin{cases}\frac{1}{5}x, & 0<x\le\frac{1}{2}\\ -\frac{1}{5}x+\frac{1}{5}, & \frac{1}{2}<x\le 1.\end{cases}$$

$y(x,0)=\sum_{n=1}^{\infty} C_n\sin n\pi x$. there is only sine terms in the series, therefore we should extend $f(x)$ as an odd function: $F(-x)=-F(x)$. $F(x+2)=F(x)$.

$$\text{So } C_n=\frac{1}{1}\int_{-1}^{1}F(x)\sin n\pi x\,dx = 2\int_0^1 f(x)\sin n\pi x\,dx.$$
$$=2\int_0^{\frac{1}{2}}\frac{1}{5}x\sin n\pi x\,dx+2\int_{\frac{1}{2}}^{1}\left(\frac{1}{5}-\frac{1}{5}x\right)\sin n\pi x\,dx = 2(I+II).$$

$I=\int_0^{\frac{1}{2}}\frac{1}{5}x\sin n\pi x\,dx=\frac{1}{5}\cdot\int_0^{\frac{1}{2}}x\sin n\pi x\,dx$. Let $u=x$, $v'=\sin n\pi x$, $u'=1$, $v=\frac{-1}{n\pi}\cos n\pi x$.
$$=\frac{1}{5}\left\{\left[x\cdot\frac{-1}{n\pi}\cos n\pi x\right]_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{-1}{n\pi}\cos n\pi x\cdot dx\right\}=\frac{1}{5}\cdot\left\{-\frac{1}{2n\pi}\cos\frac{n\pi}{2}+\frac{1}{n\pi}\int_0^{\frac{1}{2}}\cos n\pi x\,dx\right\}$$
$$=-\frac{1}{10n\pi}\cos\frac{n\pi}{2}+\frac{1}{5n\pi}\cdot\frac{1}{n\pi}\left[\sin n\pi x\right]_0^{\frac{1}{2}}=-\frac{1}{10n\pi}\cos\frac{n\pi}{2}+\frac{1}{5n^2\pi^2}\cdot\sin\frac{n\pi}{2}.$$

$II=\frac{1}{5}\int(1-x)\sin n\pi x\,dx$. Let $u=1-x$, $v'=\sin n\pi x$, then $u'=-1$, $V=\frac{-1}{n\pi}\cos n\pi x$.
$$=\frac{1}{5}\cdot\left\{\left[(1-x)\cdot\frac{-1}{n\pi}\cos n\pi x\right]_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}\frac{-1}{n\pi}\cos n\pi x\,(-1)\,dx\right\}=\frac{1}{5}\cdot\left\{0-\frac{1}{2}\cdot\frac{-1}{n\pi}\cos\frac{n\pi}{2}-\frac{1}{n\pi}\int_{\frac{1}{2}}^{1}\cos n\pi x\,dx\right\}$$
$$=\frac{1}{10n\pi}\cos\frac{n\pi}{2}-\frac{1}{5n\pi}\cdot\frac{1}{n\pi}\left[\sin n\pi x\right]_{\frac{1}{2}}^{1}=\frac{1}{10n\pi}\cos\frac{n\pi}{2}+\frac{1}{5n^2\pi^2}\cdot\sin\frac{n\pi}{2}.$$

$$\therefore C_n=2(I+II)=2\cdot\left\{\frac{-1}{10n\pi}\cos\frac{n\pi}{2}+\frac{1}{5n^2\pi^2}\sin\frac{n\pi}{2}+\frac{1}{10n\pi}\cos\frac{n\pi}{2}+\frac{1}{5n^2\pi^2}\sin\frac{n\pi}{2}\right\}=\frac{4}{5n^2\pi^2}\sin\frac{n\pi}{2}.$$

Therefore the solution is
$$y(x,t)=\sum_{n=1}^{\infty}C_n\sin n\pi x\cos n\pi t=\sum_{n=1}^{\infty}\frac{4}{5n^2\pi^2}\sin\frac{n\pi}{2}\sin n\pi x\cos n\pi t.$$