Analysis and Design of Algorithms

Chapter 7: Transform and Conquer

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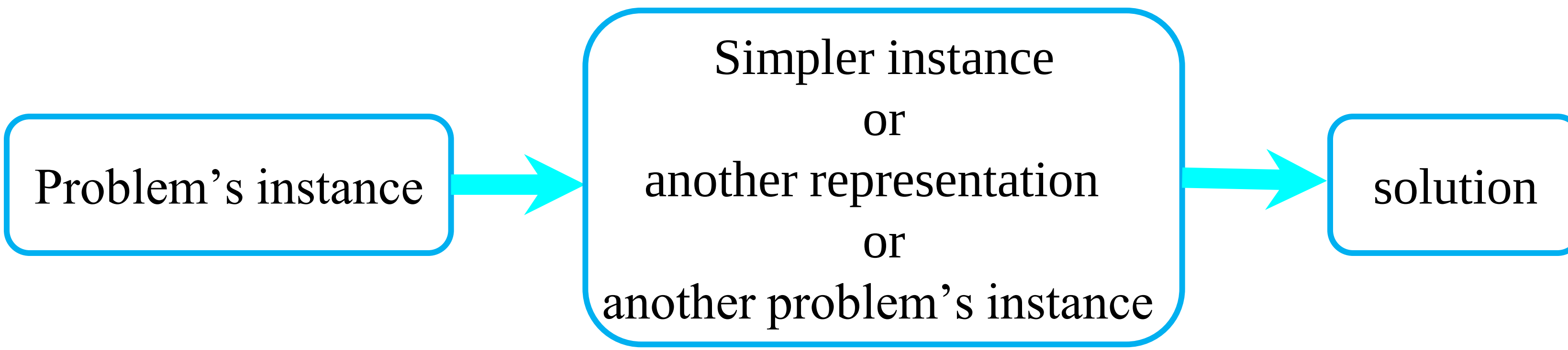
Transform and Conquer

- ***Three variations of Transform and Conquer tech.***

  This group of techniques solves a problem based on a transformation.

  - *Two stage:*



Transform and Conquer

Three variations of Transform and Conquer tech.

Differ by what we transform a given instance to:

- *instance simplification:*

  to a simpler/more convenient instance of the same problem

- *representation change:*

  to a different representation of the same instance

- *problem reduction:*

  to a different problem for which an algorithm is already available

Presorting - Instance simplification

- *why interested in sorting?*

many questions about a list are easier to answer if the list is sorted.

- *benefit from sorting?*

☆ *the benefits of a sorted list should more than compensate for the time spent on sorting*

☆ generally comparison-based sorting alg. ***worst case, at least nlogn***

- ***Selection Sort*** $\Theta(n^2)$
- ***Bubble Sort*** $\Theta(n^2)$
- ***Insertion Sort*** $C_{worst}(n) = \frac{(n-1)n}{2}$ $C_{best}(n) = n-1$ $C_{avg}(n) \approx \frac{n^2}{4}$
- ***Mergesort*** $\Theta(n \log n)$
- ***Quicksort*** $C_w(n) = \Theta(n^2)$ $C_b(n) = \Theta(n \log n)$ $C_{avg}(n) = O(n \log n)$

Presorting

- **Presorting --- Instance simplification**
  - *searching*
  - *computing the median (selection problem)*
  - *checking if all elements are distinct (element uniqueness)*

Presorting

- ***Element Uniqueness with presorting***
  - *Element Uniqueness problem --a brute-force method*
    - *compare all pairs of the array's elements (see Chapt 2)*
    - *until either two equal elements found or no more pairs left*

$$C_{worst}(n) = \frac{n(n-1)}{2} \in \Theta(n^2)$$

ALGORITHM *UniqueElements*$(A[0..n-1])$
  //Determines whether all the elements in a given array are distinct
  //Input: An array $A[0..n-1]$
  //Output: Returns "true" if all the elements in A are distinct
  //   and "false" otherwise
  for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
      if $A[i] = A[j]$ **return false**
  return true

Presorting

- ***Element Uniqueness with presorting***
  - *Element Uniqueness problem -Presorting-based method*
    - *Stage 1: sort by efficient sorting algorithm (e.g. mergesort)*
    - *Stage 2: scan array to check pairs of adjacent elements*
  - *Efficiency Analysis*
    - *sum of*
    - *time spent on sorting : at least nlogn comparisons –determine the overall efficiency*
    - *time spent on checking consecutive elements: no more than n-1 comparisons*
    - ***use a good sorting alg.***

$$C(n)=C_{sort}(n)+ C_{scan}(n)= \Theta(n\log n) + \Theta(n) = \Theta(n\log n)$$

Presorting

Computing a mode

Mode: a value that occurs most often in a given list of numbers

Eg. For {5, 1, 5, 7, 6, 5, 7} mode is 5

Brute-force method

- ***Idea:***

- Scan the list, compute the frequency of all its distinct values

- find the value with the largest frequency

Presorting

Computing a mode

- *Brute-force method (‘cont)*

  - ***implementation:***

- *Store the values already encountered, along with their frequencies, in an auxiliary list (the values in this auxiliary list are all distinct)*

- *On each iteration, the ith element of the original list is compared with the values already encountered by traversing this an auxiliary list*

- *If a matching value is found, its frequency is incremented;*

- *otherwise, the current element is added to the auxiliary list with frequency of* 1

Presorting

Computing a mode

Brute-force method (‘cont)

- ***Worst case analysis***

- when a list with no equal elements,

- i th element is compared with i-1 elements of the auxiliary list

number of comparisons in creating the frequency auxiliary list

$$C(n) = \sum_{i=1}^{n}(i-1) = \frac{(n-1)n}{2} \in \theta(n^2)$$

number of comparisons to find the largest frequency in the auxiliary list n-1

Presorting

Computing a mode

Computing a mode with presorting

- ***Idea :***

  - sort the input firstly, then all equal values will be adjacent

  - find the longest run of the adjacent equal values in the sorted array

- ***Efficiency analysis***

  sum of

  *- time spent on sorting : at least **nlogn** comparisons –determine the overall efficiency*

  *- time spent on checking longest run of the adjacent : **linear***

- ***Conclusion**: using a good sorting algorithm*

Presorting

- **Searching problem** *Search for a given K in A*[*0..n*-1]
  - *Brute-force method*
    - ***sequential search :*** *(see Chapt2)*

$$T_{avg}(n) = \frac{p(n+1)}{2} + n(1-p) \qquad T_{\text{worst}}(n)=n \qquad T_{\text{best}}(n)=1$$

    - ***Binary Search (see Chapt5)***

$$C_w(n) = \lfloor \log_2 n \rfloor + 1 = \lceil \log_2 (n+1) \rceil = \Theta(\log n)$$

$$C_b(n) = 1$$

$$C_{avg}(n) = \frac{1}{n}\sum_{i=1}^{k} i2^{i-1} \approx \log(n+1) - 1$$

Presorting

- ***Searching problem***
  - *Searching with presorting*

    sum of $C(n)=C_{sort}(n)+ C_{search}(n)= \Theta(n\log n) + \Theta(\log n) = \Theta(n\log n)$

    -*time spent on sorting : at least* ***nlogn*** *comparisons –determine the overall efficiency*

    -*time spent on binary search:*

    $C_w(n) = \lfloor \log_2 n \rfloor +1 = \Theta(\log n)$; , $C_{avg}(n) = \Theta(\log n)$;

  - *if to search in the same list more than once, the time spent on sorting might be justified*

Gaussian Elimination - *Instance simplification*

Gaussian Elimination 高斯消去法

Problem: Given: a system of n linear equations in n unknowns with an arbitrary coefficient matrix.

Idea:

*Stage1: **Elementary operations**: Transform to an **equivalent** system of n linear equations in n unknowns with an **upper triangular** coefficient matrix.*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \longrightarrow \begin{aligned} a_{1,1}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

Gaussian Elimination - Instance simplification

Gaussian Elimination

Stage2: *Solve the latter by* ***backward substitutions*** *starting with the last equation and moving up to the first one.*

Specifically:

① *find the value of* x_n *from the last equation immediately*

② *Substitute this value into the next to last equation to get* x_{n-1}

③ *And so on, until we substitute the known values of the last n-1 variables into the first equation, to find the value of* x_1

Gaussian Elimination - Instance simplification

Gaussian Elimination *What's Elementary operations*

*To change from a system with an arbitrary coefficient matrix to an **equivalent** system with an upper triangular coefficient matrix by*

- *exchanging two equations of the system*
- *replacing an equation with its nonzero multiple*
- *replacing an equation with a sum or difference of this equation and some multiple of the former*

Specifically

① *we use a_{11} as a pivot to make all x_1 coefficients zeros in the equations below the first one*

② *replace the second equation with the difference between it and **the first equation multiplied by a_{21}/a_{11}** to get an equation with zero coefficient for x_1*

③ *doing the same for the third, fourth, and finally nth equation – with the multiples a_{31}/a_{11}, a_{41}/a_{11}, …. a_{n1}/a_{11} of the first equation*

Gaussian Elimination - Instance simplification

Gaussian Elimination

e.g. Solve

$$2x_1 - x_2 + x_3 = 1$$
$$4x_1 + x_2 - x_3 = 5$$
$$x_1 + x_2 + x_3 = 0$$

Gaussian elimination

2	-1	1	1		2	-1	1	1	
4	1	-1	5	row2 – (4/2)*row1	0	3	-3	3	
1	1	1	0	row3 – (1/2)*row1	0	3/2	1/2	- 1/2	row3–(1/2)*row2

2	-1	1	1
0	3	-3	3
0	0	2	-2

Backward substitution

$x_3 = (-2) / 2 = -1$

$x_2 = (3- (-3)\, x_3) / 3 = 0$

$x_1 = (1 – x_3 –(-1)\, x_2)/2 = 1$

Gaussian Elimination - Instance simplification

Gaussian Elimination

- *Two considerations*

1. *if* $A[i, i] = 0$

→ *exchange the ith row with some row below it with a nonzero coefficient in the ith column*

2. *if* $A[i, i]$ is so small that consequently the scaling factor $A[j, i]/A[i, i]$ so large that new $A[j, k]$ might distorted by a round-off error caused by a subtraction of two numbers of greatly different magnitudes

→ *look for a row in the largest absolute value of the coefficient in the ith column, exchange it with the ith row* --- *partial pivoting*

Gaussian Elimination - Instance simplification

Gaussian Elimination

Efficiency: $\Theta(n^3) + \Theta(n^2) = \Theta(n^3)$

*stage*1: *Elementary operations*

ALGORITHM *GaussElimination* ($A[1\ldots n, 1\ldots n]$, $b[1\ldots n]$)

for $i \leftarrow 1$ **to** n-1 **do**

  for $j \leftarrow i$+1 **to** n **do**

    temp $\leftarrow A[j, i] / A[i, i]$

    for $k \leftarrow i$ **to** n+1 **do**

      $A[j, k] \leftarrow A[j, k] - A[i, k] * temp$

A[i,i] ?= 0

basic operation: multiplication

*stage*2: *Backward substitution*

for $j \leftarrow n$ **downto** 1 **do**

  $t \leftarrow 0$

  for $k \leftarrow j$ +1 **to** n **do**

    $t \leftarrow t + A[j, k] * x[k]$

  $x[j] \leftarrow (A[j, n+1] - t) / A[j, j]$

Gaussian Elimination - Instance simplification

Gaussian Elimination

Efficiency analysis

- *stage1: Elementary operations*

$$C(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{n+1} 1 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (n+1-i+1) = \sum_{i=1}^{n-1} (n+2-i)(n-(i+1)+1)$$

$$= \sum_{i=1}^{n-1} (n+2-i)(n-i) = (n+1)(n-1) + n(n-2) + \ldots + 3 \cdot 1 = \sum_{j=1}^{n-1} (j+2) \cdot j$$

$$= \sum_{j=1}^{n-1} j^2 + \sum_{j=1}^{n-1} 2j = \frac{(n-1)n(2n-1)}{6} + 2\frac{(n-1)n}{2} = \frac{(n-1)n(2n+5)}{6} \in \theta(n^3)$$

- *stage2: Backward substitution* $\Theta(n^2)$

Gaussian Elimination - Instance simplification

- Some discussions

☆ *Gaussian Elimination either yields* ***an exact solution*** *to a system of linear equations when the system has a unique solution*

☆ *or discovers that no such solution exists, in this case, the system will have either* ***no solutions or infinitely many of them***

☆ *the principal difficulty lies in preventing the accumulation of* ***round-off error***

Gaussian Elimination

Applications of Gaussian Elimination

- *LU decomposition*
- *Computing a matrix inverse*
- *Computing a determinant*

Heaps and Heapsort

Heaps

- Heap is suitable for implementing priority queues

  maintaining a set S of elements, each with an associated value called a key/priority. It supports the following operations:

  - Finding an item with the highest priority
  - Deleting an item with the highest priority
  - Adding a new item to the multiset

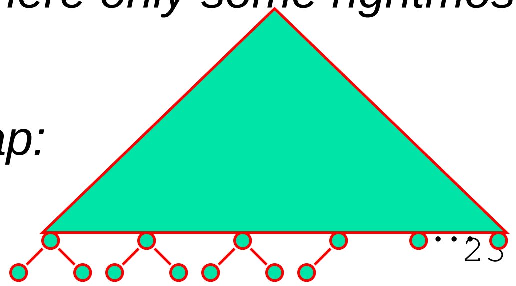
Notion of the Heap

- A binary tree with keys assigned to its nodes, one key per node

- Shape requirement: the binary tree is essentially complete, i.e. all its levels are full except possibly the last level, where only some rightmost leaves may missing

- Parental dominance requirement: for max-heap:

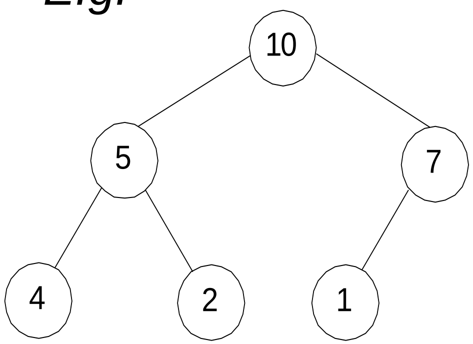
  key at each node $\geq$ keys at its children



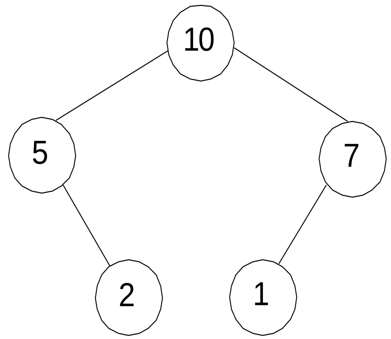
Heaps and Heapsort

Heaps

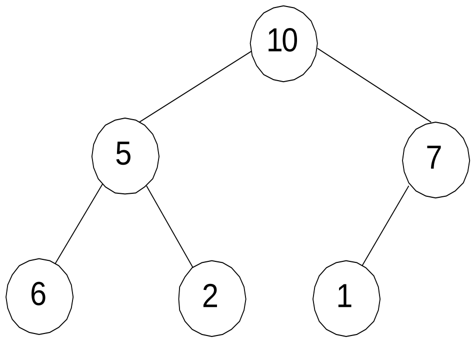
E.g.



a heap



not a heap



not a heap

- ☆ *Heap's elements are ordered top down (a sequence of values along any path down from its root is decreasing or non-increasing if equal keys are allowed)*
- ☆ *but they are not ordered left to right*

Heaps and Heapsort

Heaps

Properties of Heaps

- *There exits exactly one essentially <u>complete binary tree</u> with n nodes, its height is* $\lfloor \log_2 n \rfloor$
  - *<u>Height of a node</u>: the number of edges on the longest simple downward path from the node to a leaf.*
  - *<u>Height of a tree</u>: the height of its root.*
  - *<u>level of a node</u>: A node's level + its height = h, the tree's height.*
- *The root of a heap always has the largest key (for a max-heap)*
- *A node of a heap considered with all its descendants is also a heap (The subtree rooted at any node of a heap is also a heap)*
- *<u>Max-heap</u> property and <u>min-heap</u> property*
  - *Max-heap: for every node other than root, A[PARENT(i)] >= A(i)*
  - *Min-heap: for every node other than root, A[PARENT(i)] <= A(i)*

Heaps and Heapsort

Heaps

Properties of Heaps

- *it is more efficient to implement a heap as an array, by storing the heap's elements in top-down left-to-right order*

- Parental nodes are represented in the first $\lfloor n/2 \rfloor$ locations of the array

- *Leaf keys occupy the last $\lceil n/2 \rceil$ locations*
- *Relationships between indexes of parents and children.*

PARENT(i)

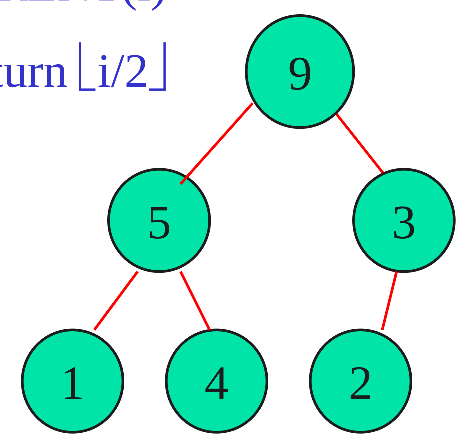
return $\lfloor i/2 \rfloor$

LEFT(i)

return 2i

RIGHT(i)

return 2i+1



1	2	3	4	5	6
9	5	3	1	4	2

Heaps and Heapsort

Heaps Construction

How to construct a heap with the given list of keys?

Bottom-up Heap construction

- *Build an essentially complete binary tree by inserting n keys in the given order.*
- *Heapify the tree*
  - *Starting with the last (rightmost) parental node, heapify/fix the subtree rooted at it; if the parental dominance condition does not hold for the key at this node:*
    - *exchange its key K with the key of its larger child*
    - *Heapify/fix the subtree rooted at the K's new position*
    - *until the parental dominance requirement for K is satisfied*
  - *Proceed to do the same for the node's immediate predecessor.*
  - *Stops after this is done for the tree's root.*

Heaps and Heapsort

Heaps Construction

Bottom-up Heap construction (A Recursive version)

ALGORITHM *HeapBottomUp*(H[1..n])
//Constructs a heap from the elements
//of a given array by the bottom-up algorithm
//Input: An array H[1..n] of orderable items
//Output: A heap H[1..n]
for $i \leftarrow \lfloor n/2 \rfloor$ downto 1 do
    MaxHeapify(H, i)

Given a heap of n nodes, what's the index of the last parent? $\lfloor n/2 \rfloor$

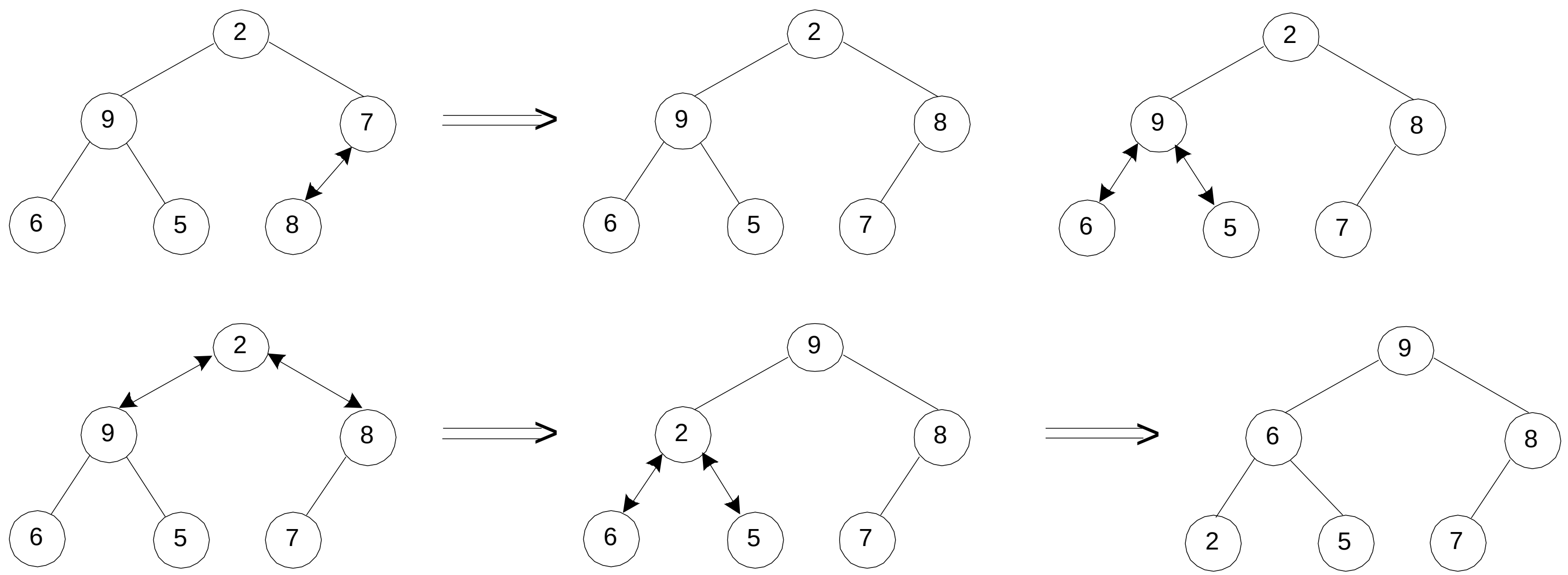
ALGORITHM *MaxHeapify*(H, i)
$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
if $l <= n$ and $H[l] > H[i]$
  then *largest* $\leftarrow l$
  else *largest* $\leftarrow i$
if $r <= n$ and $H[r] > H[largest]$
  then *largest* $\leftarrow r$
if *largest* $\neq i$
  then exchange $H[i] \leftrightarrow H[largest]$
    MaxHeapify(H, *largest*)

Heaps and Heapsort

Heaps Construction

Bottom-up Heap construction(‘cont)

- *Example 1: Construct a heap for the list 2, 9, 7, 6, 5, 8*



- *Example 2: 4 1 3 2 16 9 10 14 8 7 → 16 14 10 8 7 9 3 2 4 1*

Heaps and Heapsort

Heaps Construction

Worst-Case Efficiency for Bottom-up

- *assume n* =2^k-1, *so the heap is full, the maximum number of nodes occurs on each level*
- *Worst case: each key on level i will travel to the leaf level h*
  - *height of the tree* $h = \lfloor \log_2 n \rfloor$
  - *moving to the level down needs two comparisons*
    - *one to find the larger child*
    - *one to determine whether the exchange is required*
  - *number of key comparisons for a key on level i: 2(h-i)*

$$C_{worst}(n) = \sum_{i=0}^{h-1} \sum_{\text{nodes at level } i} 2(h-i) = \sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1))$$

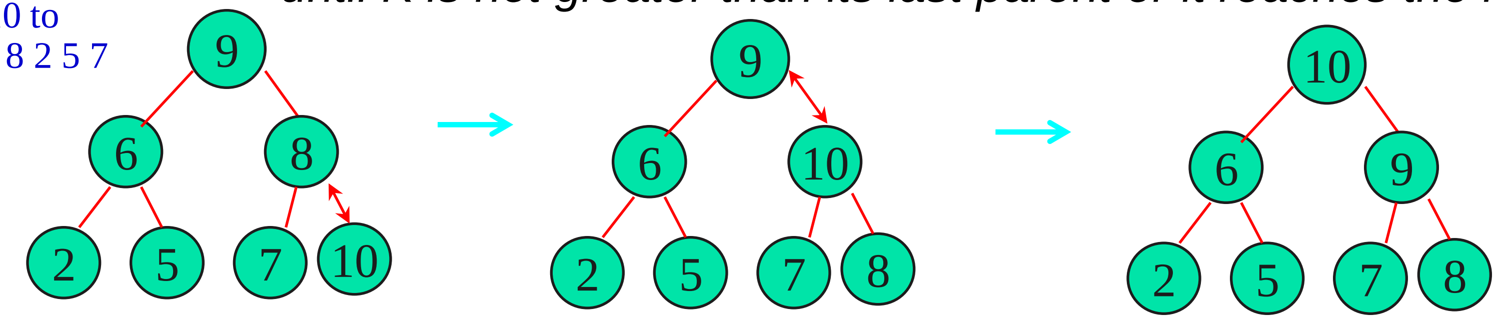
Heaps and Heapsort

Heaps Construction

Top-down Heap Construction

- *Successive insertions of new key into a previously constructed heap*
- *Insertion of a new key K*
    - *Insert the new node with key K at the last position in heap, i.e. after the last leaf of the existing heap*
    - *sift K up to its appropriate position*
        - *Compare with its parent, and exchange them if it violates the parental dominance condition.*
        - *Continue comparing the element with its new parent,*
        - *until K is not greater than its last parent or it reaches the root*

Ex: add 10 to
heap: 9 6 8 2 5 7



Heaps and Heapsort

Heaps Construction

Efficiency for Top-down

- *height of a heap with n node: h=* $\lfloor \log_2 n \rfloor$
- *Inserting one new element to a heap with* n-1 *nodes requires no more comparisons than the heap's height*
- *time efficiency for Top-down insertion is* $O(\log_2 n)$

Heaps and Heapsort

Heaps Construction

Root Deletion

- *swap the root with the last leaf K*
- *Decrease the heap's size by* 1
- *Heapify the smaller tree by sifting K down the tree, in exactly the same way in Bottom-up Heap construction*
    - *verify the parental dominance for K,*
    - *if it holds, we done.*
    - *if not, swap K with the larger of its children*
    - *and repeat this operation until parental dominance holds for K in its new position.*

Heaps and Heapsort

Heaps Construction

Efficiency for Root Deletion

- *It can't make key comparison more than twice the heap's height*
- *Efficiency:* $2h \in \Theta(\log n)$

Ex: 9 8 6 2 5 1

Heaps and Heapsort

Heapsort

Analysis of Heapsort

- *Bottom-up heap construction* $O(n)$
- *Root deletion, Repeat n-1 times until heap contains just one node*

$$C_2(n) \le 2\lfloor \log_2(n-1) \rfloor + 2\lfloor \log_2(n-2) \rfloor + \ldots + 2\lfloor \log_2 1 \rfloor \le 2\sum_{i=1}^{n-1} \log_2 i$$

$$\le 2\sum_{i=1}^{n-1} \log_2(n-1) = 2(n-1)\log_2(n-1) \le 2n\log_2 n \in O(n\log n)$$

- *Analysis shows that $C_1(n)+C_2(n)=$**$\Theta(n\log n)$**, in both the worst and average cases, the same class as mergesort*
- *But not require extra storage ---implemented with arrays*
- *Experiments show that heapsort runs more slowly than quicksort but competitive with mergesort*

Horner's Rule- Representation change

Horner's Rule For Polynomial Evaluation 霍纳法则

- Problem

  Polynomial Evaluation: *Compute the value of a polynomial*

  $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \qquad (1)$$

  at a specific point x --- *fast Fourier Transform, FFT*

- Two brute-force algorithms

```
p ← 0
for i ← n down to 0 do
    power ← 1
    for j ← 1 to i do
        power ← power * x
    p ← p + a_i * power
return p
```

```
p ← a_0;  power ← 1
for i ← 1 to n do
    power ← power * x
    p ← p + a_i * power
return p
```

Horner's Rule- Representation change

Horner's Rule For Polynomial Evaluation

- *Horner's Rule --Representation change*
  - *Obtained from* (1), *successively taking x as a common factor in the remaining polynomials of diminishing degrees*

$$p(x) = (\dots (a_n x + a_{n-1}) x + \dots) \ x + a_0 \qquad (2)$$

算法 Horner($P[0..n]$,x)

//用霍纳法则求一个多项式在一个给定点的值

//输入：一个 n 次多项式的系数数组 $P[0..n]$(从低到高存储)，以及一个数字 x

//输出：多项式在 x 点的值

$p \leftarrow p[n]$

for $i \leftarrow n-1$ **downto** 0 **do**

$\quad p \leftarrow x * p + p[i]$

return p

Horner's Rule- Representation change

Horner's Rule For Polynomial Evaluation

Horner's Rule --Representation change

- *Obtained from* (1), *successively taking x as a common factor in the remaining polynomials of diminishing degrees*

$$p(x) = (\ldots (a_n x + a_{n-1})\, x + \ldots)\; x + a_0 \qquad (2)$$

E.g.: $p(x) = 2x^4 - x^3 + 3x^2 + x - 5 = x(2x^3 - x^2 + 3x + 1) - 5 =$

$= x(x(2x^2 - x + 3) + 1) - 5 = x(x(x(2x - 1) + 3) + 1) – 5$

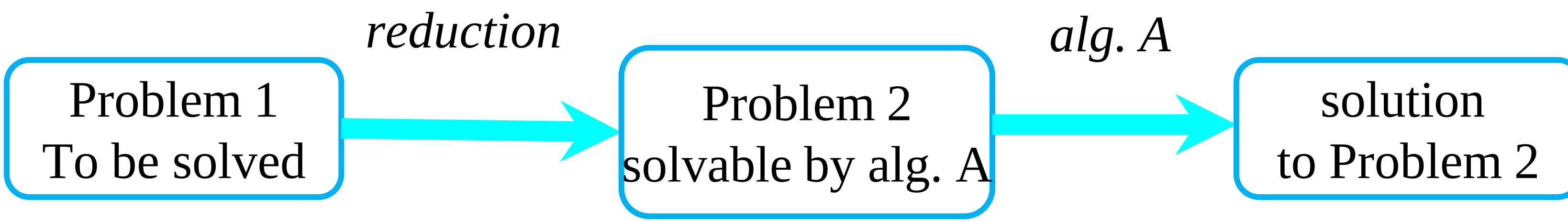
To evaluate p(x) at x=3

coefficients	**2**	**-1**	**3**	**1**	**-5**
x=3	2	3*2+(-1) = 5	3*5+3=18	3*18+1=55	3*55+(-5)=160

Problem Reduction

Problem Reduction

- *To solve a problem, reduce it to another problem that you know how to solve*



two points:

- *finding a problem to which the problem at hand should be reduced*
- *reduction-based algorithm to be more efficient than solving the original problem directly*

Problem Reduction

Problem Reduction

E.g. in analytical geometry, for three arbitrary points in the plane, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$, the determinant is positive if and only if the point p_3 is to the left of the directed line through points $p_1\ p_2$

$$\det\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3$$

i.e. we *reduce* a geometric problem about the relative locations of three points to a problem about the sign of a determinant.

☆ *the entire idea of analytical geometry is based on reducing geometric problems to algebra ones.*

Problem Reduction

Linear programming

Linear programming:

- a problem of optimizing a linear function of several variables subject to constraints in the form of linear equations and linear inequalities.

Maximize(or minimize) $c_1x_1 + \ldots c_nx_n$

Subject to $a_{i1}x_1 + \ldots + a_{in}x_n \le (\text{or} \ge \text{or} =)\ b_i$, for i=1…n

$x_1 \ge 0, \ldots, x_n \ge 0$

Problem Reduction

- ***Linear programming***
  - *Algorithms for Linear programming:*
    - ***simplex method**: worst-case efficiency is to be exponential*
    - *Ellipsoid algorithm: polynomial time.*
    - *Interior-point methods: polynomial time*
    - *Karmarkar's alg.: polynomial worst-case efficiency*
    - *Integer Linear programming: the variables of a Linear programming problem are required to be integers.*
      - *no known polynomial-time alg.*
      - *branch-and-bound method for solving Integer Linear programming*

Problem Reduction

- ***Linear programming***
  - *Investment Problem:*
    - ***Scenario***
      - *A university endowment needs to invest $100million*
      - *Three types of investment:*
        - *Stocks (expected interest: 10%)*
        - *Bonds (expected interest: 7%)*
        - *Cash (expected interest: 3%)*
    - ***Constraints***
      - *The investment in stocks is no more than 1/3 of the money invested in bonds*
      - *At least 25% of the total amount invested in stocks and bonds must be invested in cash*
    - ***Objective**:*
      - *An investment that maximizes the return*

Problem Reduction

- **Linear programming**
  - Investment Problem: (‘cont)
    - ***mathematical model***

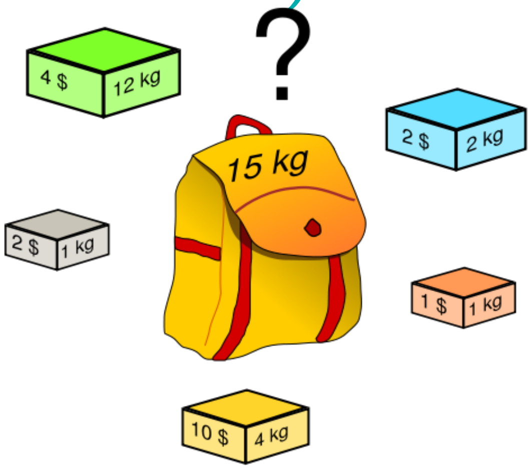
$$\begin{aligned} &\textit{Maximize} && 0.10x + 0.07y + 0.03z \\ &\textit{subject to} && x + y + z = 100 \\ & && x \le (1/3)y \\ & && z \ge 0.25(x + y) \\ & && x \ge 0,\ y \ge 0,\ z \ge 0 \end{aligned}$$

optimal decision making problem ----→ linear programming problem

Problem Reduction

Linear programming

Knapsack Problem (Continuous/Fraction Version):



- ***Scenario***
  - *Given n items:*
    - *weights:* $w_1 \quad w_2 \dots \quad w_n$
    - *values:* $v_1 \quad v_2 \dots \quad v_n$
    - *a knapsack of capacity W*
- ***Constraints***
  - *Any fraction of any item can be put into the knapsack,* x_i
- ***Objective**:*
  - *Find the most valuable subset of the items*

Problem Reduction

- ***Linear programming***
  - *Knapsack Problem (Continuous/Fraction Version): (‘cont)*
    - ***mathematical model***

Maximize

$$\sum_{i=1}^{n} v_i x_i$$

subject to

$$\sum_{i=1}^{n} w_i x_i \le W$$

$$0 \le x_i \le 1 \qquad \text{for } i = 1, \ldots, n$$

Problem Reduction

Linear programming

Knapsack Problem (Discrete Version)

- ***Scenario***
  - *Given n items:*
    - *weights:* $w_1 \quad w_2 \ldots \quad w_n$
    - *values*: $v_1 \quad v_2 \ldots \quad v_n$
    - *a knapsack of capacity W*
- ***Constraints***
  - *an item can either be put into the knapsack in its entirely or not be put into the knapsack.*
- ***Objective***:

  Find the most valuable subset of the items

Problem Reduction

Linear programming

Knapsack Problem (Discrete Version) (‘cont)

- ***mathematical model***

Maximize

$$\sum_{i=1}^{n} v_i x_i$$

subject to

$$\sum_{i=1}^{n} w_i x_i \le W$$

$$x_i \in \{0,1\} \qquad \text{for } i = 1,\ldots,n$$

Problem Reduction

Reduction to Graph

- *many problems can be solved by reduction to one of the standard graph problems*
- *state-space graph: vertices of a graph represent possible states of the problem, edges indicate permitted transitions among such states*
- *one of the graph's vertices represents the initial state, another represents a goal state of the problem*
- *puzzles and games*
- *not always a straightforward task*

problem ----→ a path from the initial-state vertex to a goal-state vertex

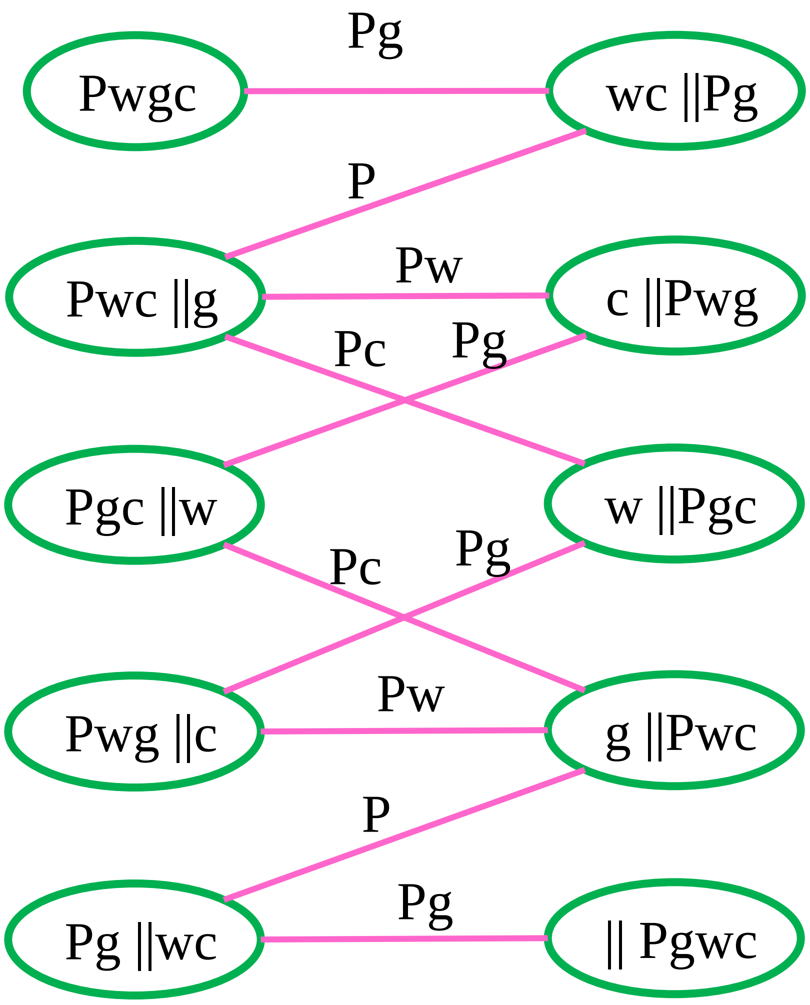
Problem Reduction

Reduction to Graph

- *River-crossing puzzle*
  - ***Problem**: The wolf, goat and bag of cabbage puzzle.*
  - *A peasant must transport a wolf, goat and bag of cabbage from one side of a river to another using a boat*
  - *the boat can only hold one item in addition to the peasant ,*
  - *subject to the constraints that the wolf cannot be left alone with the goat , and the goat cannot be left alone with the cabbage .*

Problem Reduction

- **Reduction to Graph**
  - *River-crossing puzzle*
    - ***state-space graph***



Summary

1. 变治法是一种基于变换思想，把问题变换成一种更容易解决的类型。

2. 变治法的三种类型：实例化简，改变表现，和问题化简

3. 变治法三种类型对应的算法举例

4. 堆的概念，堆排序的思想：在排列好堆中的数组元素后，再从剩余堆中连续删除最大的元素。在最差以及平均情况下，该算法都属于在位的排序算法，时间复杂度Θ(nlogn)
5. 高斯消去法
6. 霍纳法则
7. 线性规划及整数线性规划