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# Efficient and robust recurrence relations for the Zernike circle polynomials and their derivatives in Cartesian coordinates

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**Abstract:** For some time it has been known and recommended that the calculation of Zernike polynomials to radial orders higher than 8 to 10 should be performed using recurrence relations rather than explicit expressions due increasingly large cancellation errors. This paper presents a set of simple recurrence relations that can be used for the unit-normalized Zernike polynomials in polar coordinates and easily adapted to Cartesian coordinates as well. The recurrence relations are also well suited for the calculation of the Cartesian derivatives of the Zernike polynomials. The recurrence relations are easily extended to arbitrarily high orders. Assessments of the precision achievable with standard 64-bit floating point arithmetic show that Zernike polynomials up to radial order 30 can be calculated over the unit disc with errors not exceeding 5E-14, and up to radial order 50 with errors not exceeding 1.2E-13. Comparison with the Zernike capability in OpticStudio (Zemax) shows that the recurrence relations are superior in performance (both speed and precision) over the existing algorithm implemented in the software. General pseudo-code for the calculation of Zernike polynomials and their derivatives is also presented.



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## 1. Introduction

The Zernike circle polynomials are widely used in optical design and analysis [1–15], opto-mechanical design and analysis [16–19], and other disciplines such as ophthalmology [20–23], image analysis and pattern recognition [24–28], and anatomy [29]. Due to their mathematical properties, Zernike polynomials can be used to represent optical surfaces such as free-form surfaces, perturbations to ideal optical surfaces due to fabrication errors or environmental effects, aberrations in optical wavefronts, or patterns and structures in captured images. As part of these types of analysis for calculating fitting coefficients (or Zernike moments) [30–34], it is important to calculate Zernike polynomials up to high orders at selected sample points in the unit disc, and for certain applications such as ray tracing through optical surfaces with Zernike polynomial definitions it is also important to calculate the Cartesian derivatives of the Zernike polynomials [9, 35–38].

Numerous workers have pointed out that the explicit expressions for the Zernike polynomials [1-2, 39] are inefficient and suffer from large cancellation errors for higher-order polynomials and have devised various schemes using recurrence relations for their efficient calculation [9, 13, 25, 38, 40–47]. These recurrence relations are all recurrence relations for the radial component of the Zernike polynomials, and the relations vary in the number of terms involved and the complexity of the coefficients involved which may complicate the implementation of such schemes for the calculation of Zernike polynomials.

Papakostas *et al*. [48] studied the stability of recursive algorithms used to calculate Zernike moments of a digitized image, and Singh and Walla [49] also considered the numerical stability of three recursive algorithms for calculating the radial Zernike polynomials.

The stability of factorials larger than 21 was the concern of Camacho-Bello *et al*. [50] who developed a high-precision and fast algorithm for calculating Jacobi-Fourier moments of digitized images that can also be applied to calculating Zernike moments. Janssen and Dirksen [51] used the connection between the radial Zernike polynomials and Chebyshev polynomials to calculate Zernike polynomials of arbitrary radial order using a discrete Fourier (cosine) transform.

In this paper, we have selected one simple recurrence relation with coefficients that do not depend on radial or azimuthal orders and which contains no singularities. In Sec. 2 we introduce the nomenclature selected for this paper, and in Sec. 3 we introduce the recurrence relation for the radial polynomials and for the angular components. In Sec. 4 we present an alternate representation for this recurrence relation, and in Sec. 5 we use these recurrence relations to derive a set of recurrence relations for the Zernike polynomials in Cartesian coordinates that do not involve polar coordinates at all. In Sec. 6 we derive a set of recurrence relations for the Cartesian derivatives of the Zernike polynomials that are closely related to the Cartesian recurrence relations in Sec. 5. In Sec. 7 we present analysis of the accuracy to be expected when using these recurrence relations to calculate Zernike polynomials to very high radial orders in standard 64-bit floating point precision. In Sec. 8 we compare the presented recurrence relations with algorithms in current commercial software for optical design and analysis and it is shown that the recurrence relations are far superior in performance for both execution time and arithmetic precision over the existing software. Details on the implementation of these recurrence relations are also given in two appendices in the form of explicit expressions and general pseudo-code.

Details in works on Zernike polynomials are often obfuscated by the different notations and conventions used in literature. The differences are in the way the polynomials are normalized, how the azimuthal index is defined, and how the 2-index numbering scheme gets mapped to a single index. In this paper, we use unit-normalized Zernike polynomials arranged according to the azimuthal scheme set forth by Rimmer and Wyant [39], and we refer to discussions by Schwiegerling [52], Doyle, Genberg, and Michels [17] and others [9, 53, 54] for details on normalizations and numbering schemes for Zernike polynomials.

## 2. Zernike polynomials – definitions

The Zernike polynomials is a set of polynomials defined on the unit disc $\mathcal{C}$

$$x^2 + y^2 \le 1 \qquad . \tag{1}$$

However, the Zernike polynomials are most often defined using polar coordinates $(r,\theta)$

$$x = r\cos\theta \,, y = r\sin\theta \tag{2}$$

because the polynomials are separable in polar coordinates. Thus, the polynomials can be defined in a double-index form for integers $n \ge 0$ and $0 \le m \le n$ [5, 39]

$$U_{nm}(r,\theta) = \begin{cases} R_n^{|n-2m|}(r)\sin(n-2m)\theta \,, for\, n-2m > 0 \\ R_n^{|n-2m|}(r)\cos(n-2m)\theta \,, for\, n-2m \le 0 \end{cases} \qquad , \tag{3}$$

where the radial polynomials $R_n^{\mu}(r)$ are defined as

$$R_n^{\mu}(r) = \sum_{k=0}^{\frac{n-\mu}{2}} (-1)^k \frac{(n-k)!}{k!\left(\frac{n-\mu}{2}-k\right)!\left(\frac{n+\mu}{2}-k\right)!} r^{n-2k} \,, n \ge 0 \,, 0 \le \mu \le n \,, n-\mu\, even \tag{4}$$

The polynomials defined in (3) are orthogonal over the unit disc $\mathcal{C}$ and are unit-normalized in the sense that for each polynomial $U_{nm}$

$$\max_{\mathcal{C}} |U_{nm}| = 1 \qquad . \tag{5}$$

For many applications it is more convenient to consider the $\sigma$-normalized polynomials (normalized to unit standard deviation)

$$Z_{nm}(r,\theta) = N_{nm} U_{nm}(r,\theta) \qquad , \tag{6}$$

where

$$N_{nm} = \sqrt{\left(2-\delta_{n,(2m)}\right)(n+1)} \tag{7}$$

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is a normalization factor and where $\delta_{ij}$ is the usual Kronecker delta. Thus for a linear combination of Zernike polynomials

$$W(r,\theta)=\sum_{n=0}^{n_{max}}\sum_{m=0}^{n}a_{nm}U_{nm}(r,\theta) \tag{8}$$

we have the following expression for the variance of $W$ over the unit disc $\mathcal{C}$, due to the orthogonality of the Zernike polynomials,

$$\operatorname*{var}_{\mathcal{C}} W=\sum_{n=0}^{n_{max}}\sum_{m=0}^{n}a_{nm}{}^{2}N_{nm}{}^{2} \quad . \tag{9}$$

## 3. Recurrence relations for radial $R$ -polynomials

The explicit expression in Eq. (4) for the radial polynomials $R_n^{\mu}(r)$ contains terms with alternating signs and for large radial orders $n$ the polynomials may suffer from relatively large errors due to cancellation of several large terms [42]. For this reason, many recursive relations have been developed that allow the radial polynomials of higher radial orders to be calculated recursively from radial polynomials of lower radial orders [9, 38, 40, 41, 43, 44]. Many of these recurrence relations differ in number of polynomials involved and in complexity of the coefficients. Here, we shall focus on a recurrence relation given by Shakibaei and Paramesran [44] who compared it favorably for speed with earlier relations given by Kintner [10] and Prata and Rusch [40]. Noll [9] showed the relationship between the radial polynomials and the ordinary Bessel functions

$$\begin{aligned} R_n^{\mu}(r) &= 2\pi(-1)^{(n-\mu)/2}\int_0^{\infty}J_{n+1}(2\pi k)J_{\mu}(2\pi rk)\,dk \\ &= (-1)^{(n-\mu)/2}\int_0^{\infty}J_{n+1}(x)J_{\mu}(rx)\,dx \quad , \end{aligned} \tag{10}$$

where $J_{n+1}$ and $J_{\mu}$ are ordinary Bessel functions of the first kind. Many recurrence relations for the radial polynomials can be derived using recurrence relations for the ordinary Bessel functions [55]. In particular, using the recurrence relation [55]

$$J_{n-1}(x)-J_{n+1}(x)=2J_n'(x) \quad , \tag{11}$$

we now have, for $n\geq 2$, $n-2\geq\mu\geq 1$ and $n-\mu$ even,

$$
\begin{aligned}
&R_n^{\mu}(r)+R_{n-2}^{\mu}(r)\\
&=(-1)^{(n-\mu)/2}\int_0^{\infty}J_{n+1}(x)J_{\mu}(rx)\,dx+(-1)^{(n-2-\mu)/2}\int_0^{\infty}J_{n-1}(x)J_{\mu}(rx)\,dx\\
&=(-1)^{(n-\mu)/2}\int_0^{\infty}(J_{n+1}(x)-J_{n-1}(x))J_{\mu}(rx)\,dx\\
&=-(-1)^{(n-\mu)/2}\int_0^{\infty}2J_n'(x)J_{\mu}(rx)\,dx\\
&=-2(-1)^{(n-\mu)/2}\left[J_n(x)J_{\mu}(rx)\right]_0^{\infty}+2(-1)^{(n-\mu)/2}\int_0^{\infty}J_n(x)rJ_{\mu}'(rx)\,dx\\
&=(-1)^{(n-\mu)/2}\int_0^{\infty}J_n(x)r\left(J_{\mu-1}(rx)-J_{\mu+1}(rx)\right)dx\\
&=(-1)^{(n-\mu)/2}\int_0^{\infty}J_n(x)rJ_{\mu-1}(rx)\,dx-(-1)^{(n-\mu)/2}\int_0^{\infty}J_n(x)rJ_{\mu+1}(rx)\,dx\\
&=(-1)^{\frac{n-1-(\mu-1)}{2}}\int_0^{\infty}J_n(x)rJ_{\mu-1}(rx)\,dx+(-1)^{\frac{n-1-(\mu+1)}{2}}\int_0^{\infty}J_n(x)rJ_{\mu+1}(rx)\,dx\\
&=rR_{n-1}^{\mu-1}(r)+rR_{n-1}^{\mu+1}(r)\quad .
\end{aligned}
\tag{12}
$$

For $n\geq 2$, $\mu=0$ and $n$ even, we have

$$
\begin{aligned}
&R_n^{0}(r)+R_{n-2}^{0}(r)\\
&=(-1)^{n/2}\int_0^{\infty}J_{n+1}(x)J_0(rx)\,dx+(-1)^{(n-2)/2}\int_0^{\infty}J_{n-1}(x)J_0(rx)\,dx\\
&=(-1)^{n/2}\int_0^{\infty}(J_{n+1}(x)-J_{n-1}(x))J_0(rx)\,dx\\
&=-(-1)^{n/2}\int_0^{\infty}2J_n'(x)J_0(rx)\,dx\\
&=-2(-1)^{n/2}\left[J_n(x)J_0(rx)\right]_0^{\infty}+2(-1)^{n/2}\int_0^{\infty}J_n(x)rJ_0'(rx)\,dx\\
&=-2(-1)^{n/2}\int_0^{\infty}J_n(x)rJ_1(rx)\,dx\\
&=2(-1)^{(n-1-1)/2}\int_0^{\infty}J_n(x)rJ_1(rx)\,dx\\
&=2rR_{n-1}^{1}(r)\quad .
\end{aligned}
\tag{13}
$$

Since, for $\mu=n$, we have, see [56],

$$
R_{n-2}^{n}(r)=\int_0^{\infty}J_{n-1}(x)J_n(rx)dx=\begin{cases}0,\text{for } 0\leq r<1\\ \dfrac{1}{2},\text{for } r=1\end{cases}\quad , \tag{14}
$$

by setting $R_n^\mu(r)=0$ identically for $\mu>n$, Eqs. (12) and (13) can then be combined, for $n\geq 2$, $n\geq\mu\geq 0$ and $n-\mu$ even, into the recurrence relation

$$R_n^\mu(r)=rR_{n-1}^{|\mu-1|}(r)+rR_{n-1}^{\mu+1}(r)-R_{n-2}^\mu(r) \tag{15}$$

which Shakibaei and Paramesran derived using a recurrence relation for Chebyshev polynomials [44]. With the starting polynomials

$$R_0^0(r)=1 \quad , \quad R_1^1(r)=r \quad , \tag{16}$$

all radial polynomials of radial orders $n\geq 2$ can be calculated recursively from Eq. (15). Shakibaei and Paramesran [44] gave tables with the amount of calculations needed to calculate all radial polynomials up to a certain radial order $n$ and they showed the computational flow involved to calculate the radial polynomials.

In order to calculate the full set of Zernike polynomials $U_{nm}$ in Eq. (3) we must also calculate the trigonometric functions $\sin m\theta$ and $\cos m\theta$ for $m=1,\ldots,n$ and multiply these with the corresponding radial polynomials. The calculation of the trigonometric functions can be done efficiently recursively using the addition formulas for the trigonometric functions or de Moivre's formula in matrix form

$$\begin{pmatrix}\cos m\theta\\ \sin m\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos(m-1)\theta\\ \sin(m-1)\theta\end{pmatrix} \quad . \tag{17}$$

Hence, to calculate all Zernike polynomials $U_{nm}$ up to a certain radial order $n$ at a point $(x,y)$ in the unit disc, we first calculate $r$ and $(\cos\theta,\sin\theta)$ using Eq. (2):

$$r=\sqrt{x^2+y^2} \quad , \quad \cos\theta=\frac{x}{r} \quad , \quad \sin\theta=\frac{y}{r} \tag{18}$$

and then apply Eqs. (15) and (17) recursively with Eq. (16) providing the starting values and where

$$\mu=n-2m \quad . \tag{19}$$

We can reformulate the recurrence relation Eq. (15) as

$$R_n^\mu(r)=r\left(R_{n-1}^{\mu-1}(r)+R_{n-1}^{\mu+1}(r)\right)-R_{n-2}^\mu(r) \tag{20}$$

with the following exceptions:

$$R_n^n(r)=rR_{n-1}^{n-1}(r) \quad ,for\ \mu=n \quad , \tag{21}$$

$$R_n^0(r)=2rR_{n-1}^1(r)-R_{n-2}^0(r) \quad ,for\ \mu=0 \quad . \tag{22}$$

## 4. Recurrence relations for radial $A$ -polynomials

From the explicit expression for the radial polynomials in Eq. (4) it follows that every term in the polynomial $R_n^\mu(r)$ contains the factor $r^\mu$ and since each such radial polynomial ends up being multiplied with $\sin\mu\theta$ or $\cos\mu\theta$, it is of interest to consider the polynomials $A_n^\mu(r)$ defined by

$$R_n^\mu(r)=r^\mu A_n^\mu(r) \quad . \tag{23}$$

With this definition, the equivalent of Eq. (3) then becomes

$$U_{nm}(r,\theta)=\begin{cases} A_n^{|n-2m|}(r)\,r^{|n-2m|}\sin(n-2m)\theta \text{ , for } n-2m>0 \\ A_n^{|n-2m|}(r)\,r^{|n-2m|}\cos(n-2m)\theta \text{ , for } n-2m\le 0 \end{cases} , \quad (24)$$

and the counterpart to Eq. (17) is then

$$\begin{pmatrix} r^m\cos m\theta \\ r^m\sin m\theta \end{pmatrix}=\begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} r^{m-1}\cos(m-1)\theta \\ r^{m-1}\sin(m-1)\theta \end{pmatrix} \\ =\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} r^{m-1}\cos(m-1)\theta \\ r^{m-1}\sin(m-1)\theta \end{pmatrix} . \quad (25)$$

We notice that to use the recurrence relation Eq. (25) it is not necessary to calculate $\cos\theta$ and $\sin\theta$ since the relation only involves $x=r\cos\theta$ and $y=r\sin\theta$.

If we insert Eq. (23) into the recurrence relation Eqs. (20)-(22), we obtain the recurrence relations for these $A$-polynomials

$$A_n^{\mu}(r)=A_{n-1}^{\mu-1}(r)+r^2A_{n-1}^{\mu+1}(r)-A_{n-2}^{\mu}(r) \quad (26)$$

with the following exceptions:

$$A_n^{n}(r)=A_{n-1}^{n-1}(r) \text{ , for } \mu=n , \quad (27)$$

$$A_n^{0}(r)=2r^2A_{n-1}^{1}(r)-A_{n-2}^{0}(r) \text{ , for } \mu=0 . \quad (28)$$

For the $A$-polynomials we have the starting polynomials

$$A_0^0(r)=1 , \quad A_1^1(r)=1 . \quad (29)$$

The recurrence relations for the $A$-polynomials appear to be slightly simpler than the corresponding recurrence relations for the $R$-polynomials in Sec. 3.

## 5. Recurrence relations in Cartesian coordinates for Zernike $U$-polynomials

We will now derive a set of recurrence relations for the Zernike polynomials $U_{nm}(x,y)$ directly in Cartesian coordinates $(x,y)$. For this, we use the recurrence relations for the radial $R$-polynomials from Sec. 3 along with the addition formulas for the trigonometric functions. For $n-2m>0$ we have in general from Eq. (3), with $\mu=n-2m$,

$$\begin{aligned} U_{nm}&=R_n^{\mu}(r)\sin\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)\sin\mu\theta+rR_{n-1}^{\mu+1}(r)\sin\mu\theta-R_{n-2}^{\mu}(r)\sin\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)\sin((\mu-1)\theta+\theta)+rR_{n-1}^{\mu+1}(r)\sin((\mu+1)\theta-\theta)-R_{n-2}^{\mu}(r)\sin\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)(\sin(\mu-1)\theta\cos\theta+\cos(\mu-1)\theta\sin\theta)+rR_{n-1}^{\mu+1}(r)(\sin(\mu+1)\theta\cos\theta-\cos(\mu+1)\theta\sin\theta)-R_{n-2}^{\mu}(r)\sin\mu\theta \\ &=xU_{n-1,m}+yU_{n-1,n-1-m}+xU_{n-1,m-1}-yU_{n-1,n-m}-U_{n-2,m-1} \end{aligned} \quad (30)$$

Similarly, for $n-2m\le 0$ we have

$$\begin{aligned} U_{nm}&=R_n^{\mu}(r)\cos\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)\cos\mu\theta+rR_{n-1}^{\mu+1}(r)\cos\mu\theta-R_{n-2}^{\mu}(r)\cos\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)\cos((\mu-1)\theta+\theta)+rR_{n-1}^{\mu+1}(r)\cos((\mu+1)\theta-\theta)-R_{n-2}^{\mu}(r)\cos\mu\theta \\ &=rR_{n-1}^{\mu-1}(r)(\cos(\mu-1)\theta\cos\theta-\sin(\mu-1)\theta\sin\theta)+rR_{n-1}^{\mu+1}(r)(\cos(\mu+1)\theta\cos\theta+\sin(\mu+1)\theta\sin\theta)-R_{n-2}^{\mu}(r)\cos\mu\theta \\ &=xU_{n-1,m}+yU_{n-1,n-1-m}+xU_{n-1,m-1}-yU_{n-1,n-m}-U_{n-2,m-1} \end{aligned} \quad (31)$$

Hence, in Cartesian coordinates we have the recurrence relation

$$U_{n,m} = xU_{n-1,m} + yU_{n-1,n-1-m} + xU_{n-1,m-1} - yU_{n-1,n-m} - U_{n-2,m-1} \tag{32}$$

with the following exceptions

$$U_{n,0} = xU_{n-1,0} + yU_{n-1,n-1} \text{ , } for\, m = 0 \tag{33}$$

$$U_{n,n} = xU_{n-1,n-1} - yU_{n-1,0} \text{ , } for\, m = n \tag{34}$$

$$U_{n,m} = yU_{n-1,n-1-m} + xU_{n-1,m-1} - yU_{n-1,n-m} - U_{n-2,m-1}, for\, n\, odd\, and\, m = \frac{n-1}{2} \tag{35}$$

$$U_{n,m} = xU_{n-1,m} + yU_{n-1,n-1-m} + xU_{n-1,m-1} - U_{n-2,m-1}, for\, n\, odd\, and\, m = \frac{n-1}{2} + 1 \tag{36}$$

$$U_{n,m} = 2xU_{n-1,m} + 2yU_{n-1,m-1} - U_{n-2,m-1} \text{ , } for\, n\, even\, and\, m = \frac{n}{2} \tag{37}$$

The starting polynomials are

$$U_{00} = 1 \text{ , } U_{10} = y \text{ , } U_{11} = x \quad . \tag{38}$$

## 6. Recurrence relations for Cartesian derivatives of Zernike polynomials

Since the Cartesian derivatives $\frac{\partial U_{nm}}{\partial x}$ and $\frac{\partial U_{nm}}{\partial y}$ are functions of $(x, y)$ and in fact are polynomials in $x$ and $y$, they can be expressed as finite linear combinations of Zernike polynomials up to radial order $n-1$, see, e.g., Noll [9] or Zhao and Burge [35]. In this Section, we shall provide a set of recurrence relations for the derivatives closely related to the Cartesian recurrence relations given in Sec. 5. Janssen [38] considered the complex-valued functions

$$Z_n^{\mu}(x, y) = R_n^{\mu}(\cos\mu\theta + i\sin\mu\theta) \tag{39}$$

and derived the following expressions for the Cartesian derivatives

$$\frac{\partial Z_n^{\mu}}{\partial x} = \frac{\partial Z_{n-2}^{\mu}}{\partial x} + n\left(Z_{n-1}^{\mu-1} + Z_{n-1}^{\mu+1}\right) \tag{40}$$

$$\frac{\partial Z_n^{\mu}}{\partial y} = \frac{\partial Z_{n-2}^{\mu}}{\partial y} + in\left(Z_{n-1}^{\mu-1} - Z_{n-1}^{\mu+1}\right) \quad . \tag{41}$$

When making use of the translation

$$Z_n^{\mu} = U_{n,\frac{n+\mu}{2}} + iU_{n,\frac{n-\mu}{2}} \tag{42}$$

inserted into Eq. (40), we get, by separating the real and imaginary parts and using Eq. (19),

$$\frac{\partial U_{n,n-m}}{\partial x} = nU_{n-1,n-m-1} + nU_{n-1,n-m} + \frac{\partial U_{n-2,n-m-1}}{\partial x} \tag{43}$$

and

$$\frac{\partial U_{n,m}}{\partial x} = nU_{n-1,m} + nU_{n-1,m-1} + \frac{\partial U_{n-2,m-1}}{\partial x} \quad . \tag{44}$$

We see, by replacing $n-m$ by $m$ that Eq. (43) is equivalent to Eq. (44).

Similarly, when inserting Eq. (42) into Eq. (41), we get

$$\frac{\partial U_{n,m}}{\partial y} = nU_{n-1,n-m-1} - nU_{n-1,n-m} + \frac{\partial U_{n-2,m-1}}{\partial y} \quad . \tag{45}$$

Equations (44) and (45) are the general recursive relations for the Cartesian derivatives of the Zernike polynomials. They have exceptions similar to the exceptions for the Zernike polynomials in Sec. 5:

$$\left.\begin{aligned} \frac{\partial U_{n,0}}{\partial x} &= nU_{n-1,0} \\ \frac{\partial U_{n,0}}{\partial y} &= nU_{n-1,n-1} \end{aligned}\right\} for\, m = 0 \quad , \tag{46}$$

$$\left.\begin{aligned} \frac{\partial U_{n,n}}{\partial x} &= nU_{n-1,n-1} \\ \frac{\partial U_{n,n}}{\partial y} &= -nU_{n-1,0} \end{aligned}\right\} for\, m = n \quad , \tag{47}$$

$$\left.\begin{aligned} \frac{\partial U_{n,m}}{\partial x} &= nU_{n-1,m-1} + \frac{\partial U_{n-2,m-1}}{\partial x} \\ \frac{\partial U_{n,m}}{\partial y} &= nU_{n-1,n-m-1} - nU_{n-1,n-m} + \frac{\partial U_{n-2,m-1}}{\partial y} \end{aligned}\right\} for\, n\, odd\, and\, m = \frac{n-1}{2} \quad , \tag{48}$$

$$\left.\begin{aligned} \frac{\partial U_{n,m}}{\partial x} &= nU_{n-1,m} + nU_{n-1,m-1} + \frac{\partial U_{n-2,m-1}}{\partial x} \\ \frac{\partial U_{n,m}}{\partial y} &= nU_{n-1,n-m-1} + \frac{\partial U_{n-2,m-1}}{\partial y} \end{aligned}\right\} for\, n\, odd\, and\, m = \frac{n-1}{2} + 1 \, , \tag{49}$$

$$\left.\begin{aligned} \frac{\partial U_{n,m}}{\partial x} &= 2nU_{n-1,m} + \frac{\partial U_{n-2,m-1}}{\partial x} \\ \frac{\partial U_{n,m}}{\partial y} &= 2nU_{n-1,n-m-1} + \frac{\partial U_{n-2,m-1}}{\partial y} \end{aligned}\right\} for\, n\, even\, and\, m = \frac{n}{2} \quad . \tag{50}$$

The starting expressions for the Cartesian derivatives are

$$\frac{\partial U_{00}}{\partial x} = \frac{\partial U_{00}}{\partial y} = 0 \, , \, \frac{\partial U_{10}}{\partial x} = \frac{\partial U_{11}}{\partial y} = 0 \, , \, \frac{\partial U_{11}}{\partial x} = \frac{\partial U_{10}}{\partial y} = 1 \quad . \tag{51}$$

The recursive relations for the derivatives at a given radial order $n$ at a point $(x, y)$ only involve the Zernike polynomials and derivatives at lower radial orders at that point. The derivatives can be calculated after the polynomials have been calculated, or they can be calculated concurrently with the polynomials. In Appendix A we provide the recursive formulas for both the polynomials and their derivatives for sufficiently high radial orders that the recursive patterns with their exceptions should be clear. In Appendix B we provide pseudo-code for the efficient calculation of both the Zernike polynomials and their Cartesian derivatives.

We mention here that recurrence relations similar to Eqs. (44) and (45) with the exceptions Eqs. (46)-(50) were derived by Stephenson [37] for the Cartesian derivatives of the $\sigma$ -normalized Zernike polynomials.

It is also noticed that the recurrence relations presented here can be implemented in code that calculates the Zernike polynomials at lower orders using explicit expressions and then feed these into the recurrence relations for calculation for the higher radial orders.

## 7. Estimations of accuracy of recurrence relations for Zernike polynomials

In order to assess the accuracy of the recurrence relations for Zernike polynomials presented in Sections 3, 4, and 5, we implemented these in compiled software using both 64-bit and 128-bit floating point precision. Generally, 64-bit floating point precision (double precision) is considered to provide full 15-17 decimal digits of precision, whereas 128-bit floating precision (quadruple precision) is considered to provide 34 decimal digits of precision. When estimating errors in the double precision algorithms, we used the difference between the result calculated in 64-bit precision and that calculated in 128-bit precision. The unit-normalized Zernike polynomials considered here have values over the unit disc that are always in the interval $[-1,1]$.

We first estimated the errors in the algorithm using $R$ -polynomials in Sec. 3. Here, we calculated the radial polynomials up to radial order 30 in the interval $[0,1]$ using a very fine subdivision. In addition, we calculated the angular factors in Eq. (17) up to azimuthal orders 30 in the interval $[0,2\pi]$ using a very fine subdivision of 200000 points. For the angular factors, we found that the maximum error over the interval $[0,2\pi]$ in $\cos m\theta$ or $\sin m\theta$ using Eq. (17) in 64-bit precision could be estimated very accurately as

$$\varepsilon_{ang} = 1.281 \cdot 10^{-15} m \qquad . \qquad (52)$$

We then estimated the error in calculating all the Zernike polynomials up to a radial order $n$ by combining the maximum error in the radial polynomials with the maximum error in the angular factors, either conservatively by just adding the errors (assuming worst-case combination), or more empirically by rss-ing the errors together.

This analysis was repeated for the algorithm using $A$ -polynomials in Sec. 4. Here, each calculated $A$ -polynomial was multiplied with the corresponding factor $r^{\mu}$, see Eq. (23), to recover the corresponding $R$ -polynomial, and the estimated errors were determined.

To validate these error estimation methods, separating the radial and angular variables, we also calculated all the Zernike polynomials up to a radial order $n$ over the full unit disc Eq. (1) using first the $A$ -polynomial method from Sec. 4, and secondly the recurrence relations in Sec. 5. The results of all these analyses are combined in Fig. 1. From this we see, that the $A$ -polynomials provide better accuracy than the $R$ -polynomials, and that the rss-estimate of the errors from the separated radial and angular variables provides a very good estimate when compared with the rigorous calculation over the full unit disc. Finally, it is also seen that the Cartesian algorithm in Sec. 5 gives the best accuracy of the three methods examined here.

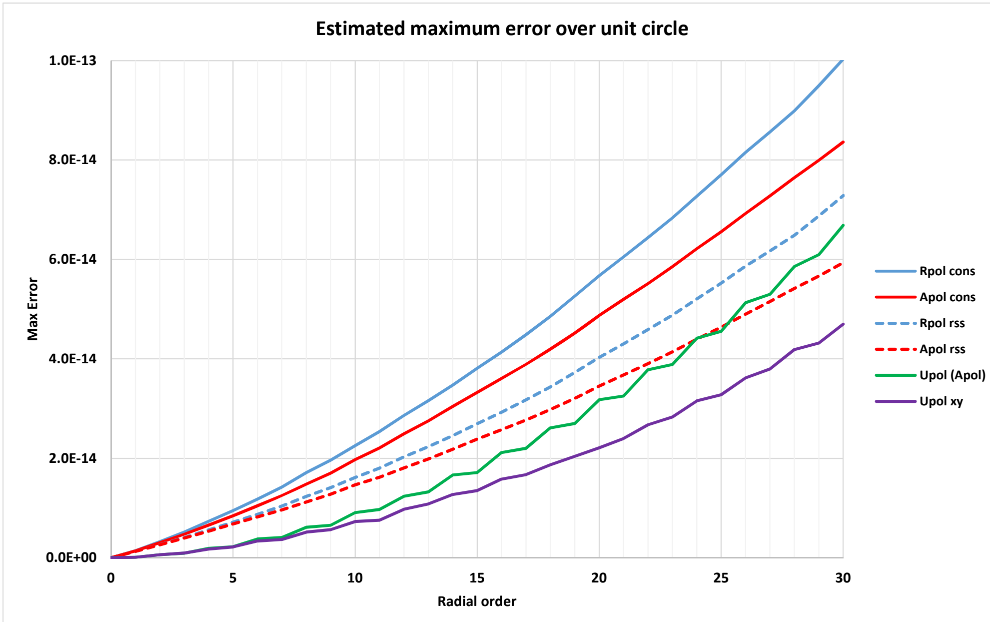


Fig. 1. Estimated maximum error in calculating Zernike polynomials over the unit disc. Solid blue: conservative estimate using $R$ -polynomials. Solid red: conservative estimate using $A$ -polynomials. Dashed blue: rss-estimate using $R$ -polynomials. Dashed red: rss-estimate using $A$ -polynomials. Green: using $A$ -polynomials (Sec. 4). Purple: using Cartesian recurrence relations (Sec. 5).

Hence, up to radial order 20, this method will calculate the Zernike polynomials over the unit disc in 64-bit precision with an error not exceeding $2 \cdot 10^{-14}$, and up to radial order 30, the error is not exceeding $5 \cdot 10^{-14}$.

This analysis only considered the maximum error over the unit disc among all Zernike polynomials with the same radial order. We also conducted a more detailed analysis, capturing the maximum error over the unit disc for each Zernike polynomial $U_{nm}$ up to a certain maximum radial order $n$ using the Cartesian recurrence relations of Sec. 5. The results for radial orders up to 30 are shown in Fig. 2, and for radial orders up to order 50 in Fig. 3. From these figures it is seen that the largest errors are encountered in Zernike polynomials with azimuthal indices $m$ close to $n/2$. These are the polynomials that have the highest number of predecessor polynomials in the Cartesian recursive scheme for the Zernike polynomials, hence they will generally suffer from having the largest accumulative errors. Up to radial order 30 using 64-bit floating point arithmetic, errors in calculating any Zernike polynomial anywhere in the unit disc do not exceed $5 \cdot 10^{-14}$, and for radial orders up to 50, the errors do not exceed $1.2 \cdot 10^{-13}$.

Whereas the use of Zernike polynomials with radial orders 8 to 12 or smaller may be sufficient in for much analysis in optical design and fabrication, papers in image analysis [24–28] may indicate a need for calculating Zernike polynomials with radial orders as high as 40. Zernike polynomials with radial orders as high as 20 to 30 can also be useful in modeling environmental and mount effects on optical surface deformations [19].

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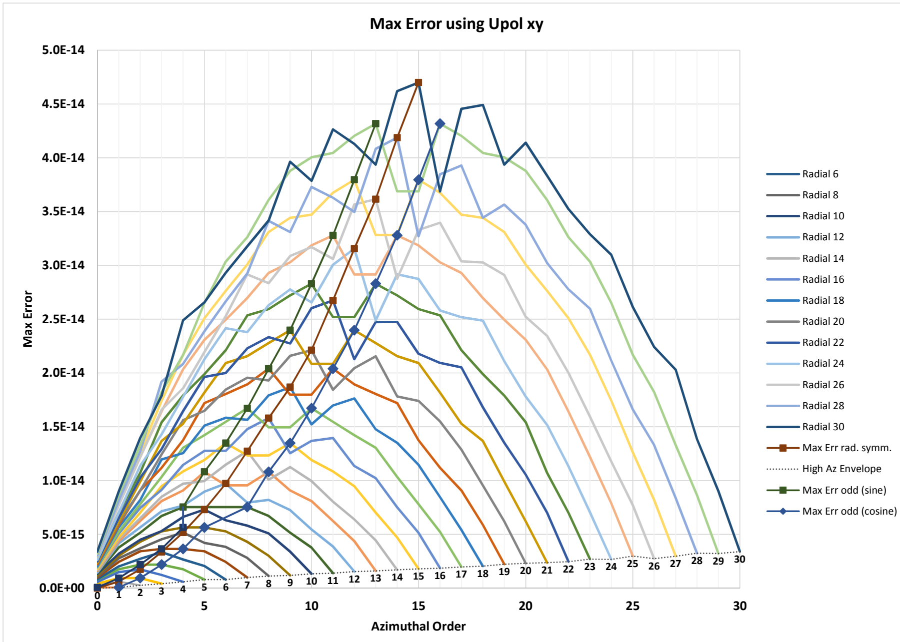


Fig. 2. Maximum error over the unit disc for individual Zernike polynomials with radial orders up to 30. The separate curves show the trends for the even-ordered rotationally symmetric polynomials (rad. symm.), and for the odd-ordered sine and cosine polynomials where the maximum errors are the largest.

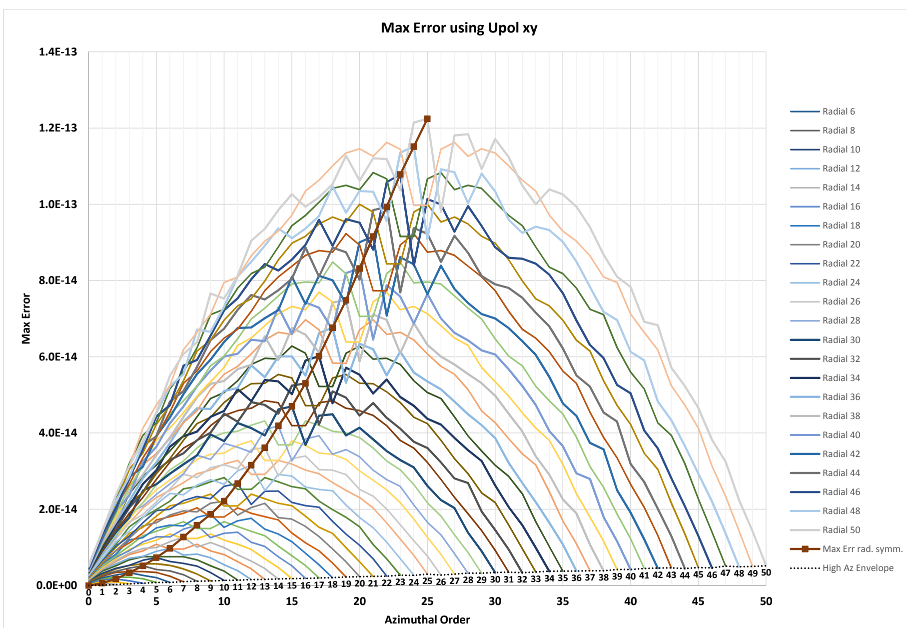


Fig. 3. Maximum error over the unit disc for individual Zernike polynomials with radial orders up to 50. The separate curve shows the trends for the even-ordered rotationally symmetric polynomials (rad. symm.).

## 8. Comparison of performance with OpticStudio

In this section we report results of comparing the performance in both accuracy and speed of the recursive relations for Zernike polynomials with the performance of Zernike polynomials provided by the commercial software OpticStudio by Zemax [57]. For this analysis version 17.5 of OpticStudio was available. OpticStudio provides support for Zernike polynomials up to radial order 20. The Zernike polynomials in OpticStudio are $\sigma$-normalized following a linear numbering scheme set forth by Noll [9] in which Zernike polynomials with a cosine-term are always given an even number [17, 52]. When comparing Zernike polynomial values computed by OpticStudio with those calculated using the recurrence relations presented here, the normalization constants of Eq. (7) and the difference in numbering schemes were taken into account.

We first calculated all Zernike polynomials through radial order 20 (231 terms) at selected points in the unit disc by assigning unit coefficients to a Standard Zernike Sag surface, one polynomial at a time, and requesting the surface sag written out with a sufficient number of digits. These values were then compared with 128-bit calculated values using recurrence relations. Here, we considered 3 points in the unit disc, $(0.663,-0.396)$, $(0.5,0.5)$, and $(-0.873,0.485)$. The distances of these points from the center of the unit disc are 0.7714, 0.7071, and 0.9987, respectively. The errors at each of these points for all Zernike polynomials from radial orders 9 through 20 are shown in Fig. 4.

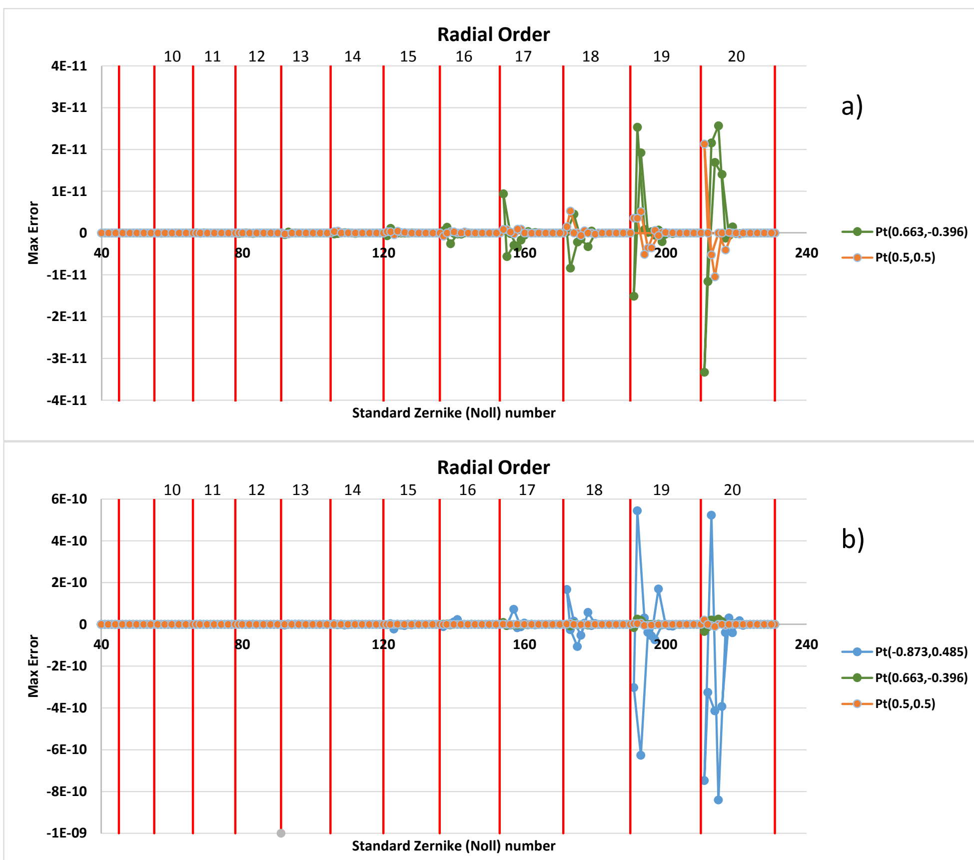


Fig. 4. Errors in individual Zernike Standard polynomials evaluated by OpticStudio at selected points in the unit disc. a) at the points (0.663,-0.396), (0.5,0.5). b) at the two points in a) plus

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the point (−0.873,0.485). The values are for $\sigma$ -normalized polynomials, using the linear numbering scheme by Noll [9].

The analysis clearly shows that for radial orders from 15 and up to 20, some Zernike polynomial values in OpticStudio may suffer a loss in precision of 3 to 5 decimal digits.

To better assess the performance of the recursive relations algorithm compared with the current implementation of Zernike polynomials in OpticStudio, we wrote a Dynamic Link Library (DLL) module for a user-defined surface sag using the recursive algorithm with the $\mathcal{A}$ –polynomials of Sec. 4. A DLL is a module of compiled code that gets linked with the main software (OpticStudio) at run-time with very little compute overhead when compared with features that are parts of the standard software. We set up an optical model with a Zernike Standard Sag (type SZERNSAG) on one surface and the user-defined DLL surface on an adjacent surface, each with unit normalization radius. The coefficients were generated in ZPL macros so that the complete surface sag description would be

$$f(x,y)=\sum_{n=0}^{20}\sum_{m=0}^{n}a_{nm}U_{nm}(x,y) \tag{53}$$

where the coefficients were generated using an arbitrary, but deterministic, algorithm

$$a_{nm}=\sin\left(100\frac{m-\frac{n}{2}+0.1}{n+1}\right) \quad . \tag{54}$$

For the Zernike Standard Sag surface, that uses the $\sigma$ -normalized Zernike polynomials, the coefficients in Eq. (54) were multiplied with the corresponding normalization coefficients in Eq. (7). We then used the ZPL macro language to calculate the difference between the “native” Zernike Standard Sag and the user-defined surface using recursively calculated Zernike polynomials. The results are presented in Fig. 5.

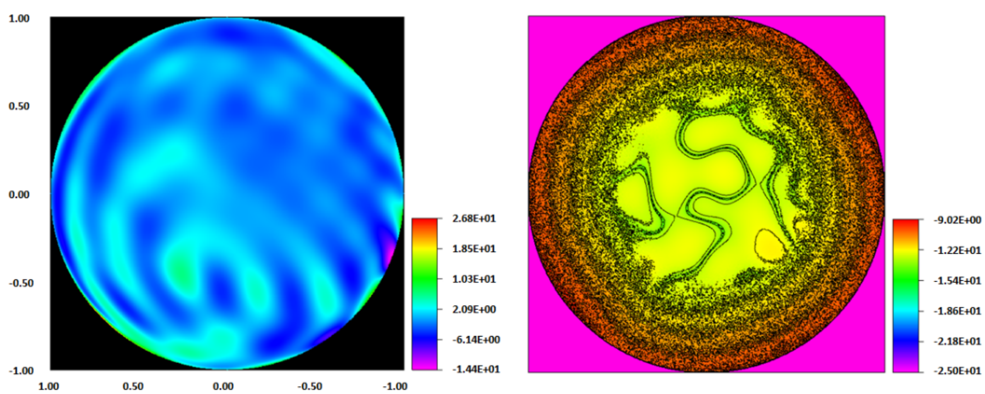


Fig. 5. Surface profile and sag errors for Zernike surface in OpticStudio. Left: the surface profile with values between −14.4 and + 26.8. Right: 10-based logarithm of the absolute value of the difference between the Standard Surface sag and the user-defined surface sag. The log-values are in the range −25 to −9. The differences themselves were between −9.62E-10 and + 7.47E-10. The grid-size for both plots is 501x501 data points.

It is seen that the larger errors are located near the edge of the unit disc, and that 4 to 5 decimal digits may get lost in the algorithm currently used in OpticStudio.

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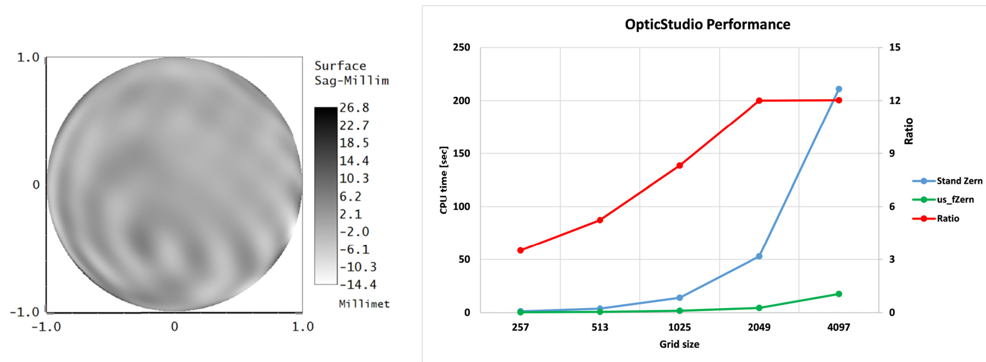


Fig. 6. Comparison of performance in speed between OpticStudio Zernike Standard Sag surface and user-defined surface using Zernike recurrence relations. Left: The surface profile generated in OpticStudio (same as Fig. 5, left). Right: CPU-times for generating 2D grids of surface sag data for different sizes of the square grid-side. The ratio between the CPU times for the Zernike Standard Sag surface and the user-defined surface is also shown.

Finally, we used the user-defined surface DLL to compare execution speeds between the algorithm used in the Zernike Standard Sag surface and the recursive relations algorithm used in the user-defined surface. We used the Surface Sag tool in the Analyze section to generate a grey-scale plot of the surface. We set it up in a ZPL macro that could set the grid-size and choose between the Zernike Standard Sag surface and the user-defined surface and accurately record the time for calculating and displaying the surface plot. Hence, the overhead would be identical for the two surface types, both run on the same hardware. The results of this analysis are shown in Fig. 6 for different grid-sizes. For the larger grid-sizes where the bulk of the CPU time was spent generating Zernike polynomial values, it is seen that a gain of a factor of upward of 10 in speed can be expected if the calculation of Zernike polynomials is implemented using recurrence relations.

Calculations involving Zernike polynomials of radial orders less than about 10 (the first 66 polynomials) are not seriously affected in precision when using existing algorithms. However, from these results we see that the OpticStudio software would benefit in both precision and speed if recursive relations were implemented for the calculation of Zernike polynomials.

## 9. Summary and conclusions

We have presented a set of recursive relations for calculating Zernike polynomials and their Cartesian derivatives. For the Zernike polynomials, we presented two sets of recursive relations for the radial polynomials ( $R$ -polynomials and $A$ -polynomials) operating in polar coordinates, and one set operating directly in Cartesian coordinates. These are useful when all Zernike polynomials up to a certain radial order must be evaluated at a specified point in the unit disc. The Cartesian derivatives of the Zernike polynomials at the given point can be calculated concurrently with the Zernike polynomials or after the Zernike polynomials have been calculated.

We presented detailed assessments of the precision achievable with these recurrence relations in Sec. 7, and we concluded that the algorithm working directly in Cartesian coordinates is superior in accuracy over the algorithms using the radial polynomials and polar coordinates. In Sec. 8 we compared the performance (precision and speed) of these recurrence relations with existing algorithms available in commercial software widely used in industry, concluding that the commercial software can benefit significantly in both accuracy and speed by implementing a recursive relations algorithm for calculating the Zernike polynomials.

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## Appendix A – Recursive formulas in Cartesian coordinates

In this Appendix, we provide explicit details for the recursive relations for the Zernike polynomials and their Cartesian derivatives in Cartesian coordinates as presented in Secs. 5 and 6 (see Fig. 7). We provide these in tabular form giving expressions for the polynomials $U_{nm}$ as well as the derivatives $\partial_x U_{nm}$ (meaning $\partial U_{nm} / \partial x$) and $\partial_y U_{nm}$ (meaning $\partial U_{nm} / \partial y$) for radial orders 0 through 6 and 9 to 10.

Recursive formulas for unnormalized Zernike polynomials and their derivatives

Zernike polynomials

| | | | | | | |
|---|---|---|---|---|---|---|
| $U_{00}$ | = | 1 | | | | |
| $U_{10}$ | = | | $+yU_{00}$ | | | |
| $U_{11}$ | = | | | $+xU_{00}$ | | |
| $U_{20}$ | = | $+xU_{10}$ | $+yU_{11}$ | | | |
| $U_{21}$ | = | $+2xU_{11}$ | $+2yU_{10}$ | | | $-U_{00}$ |
| $U_{22}$ | = | | | $+xU_{11}$ | $-yU_{10}$ | |
| $U_{30}$ | = | $+xU_{20}$ | $+yU_{22}$ | | | |
| $U_{31}$ | = | | $+yU_{21}$ | $+xU_{20}$ | $-yU_{22}$ | $-U_{10}$ |
| $U_{32}$ | = | $+xU_{22}$ | $+yU_{20}$ | $+xU_{21}$ | | $-U_{11}$ |
| $U_{33}$ | = | | | $+xU_{22}$ | $-yU_{20}$ | |
| $U_{40}$ | = | $+xU_{30}$ | $+yU_{33}$ | | | |
| $U_{41}$ | = | $+xU_{31}$ | $+yU_{32}$ | $+xU_{30}$ | $-yU_{33}$ | $-U_{20}$ |
| $U_{42}$ | = | $+2xU_{32}$ | $+2yU_{31}$ | | | $-U_{21}$ |
| $U_{43}$ | = | $+xU_{33}$ | $+yU_{30}$ | $+xU_{32}$ | $-yU_{31}$ | $-U_{22}$ |
| $U_{44}$ | = | | | $+xU_{33}$ | $-yU_{30}$ | |
| $U_{50}$ | = | $+xU_{40}$ | $+yU_{44}$ | | | |
| $U_{51}$ | = | $+xU_{41}$ | $+yU_{43}$ | $+xU_{40}$ | $-yU_{44}$ | $-U_{30}$ |
| $U_{52}$ | = | | $+yU_{42}$ | $+xU_{41}$ | $-yU_{43}$ | $-U_{31}$ |
| $U_{53}$ | = | $+xU_{43}$ | $+yU_{41}$ | $+xU_{42}$ | | $-U_{32}$ |
| $U_{54}$ | = | $+xU_{44}$ | $+yU_{40}$ | $+xU_{43}$ | $-yU_{41}$ | $-U_{33}$ |
| $U_{55}$ | = | | | $+xU_{44}$ | $-yU_{40}$ | |
| $U_{60}$ | = | $+xU_{50}$ | $+yU_{55}$ | | | |
| $U_{61}$ | = | $+xU_{51}$ | $+yU_{54}$ | $+xU_{50}$ | $-yU_{55}$ | $-U_{40}$ |
| $U_{62}$ | = | $+xU_{52}$ | $+yU_{53}$ | $+xU_{51}$ | $-yU_{54}$ | $-U_{41}$ |
| $U_{63}$ | = | $+2xU_{53}$ | $+2yU_{52}$ | | | $-U_{42}$ |
| $U_{64}$ | = | $+xU_{54}$ | $+yU_{51}$ | $+xU_{53}$ | $-yU_{52}$ | $-U_{43}$ |
| $U_{65}$ | = | $+xU_{55}$ | $+yU_{50}$ | $+xU_{54}$ | $-yU_{51}$ | $-U_{44}$ |
| $U_{66}$ | = | | | $+xU_{55}$ | $-yU_{50}$ | |
| … | | | | | | |
| $U_{90}$ | = | $+xU_{80}$ | $+yU_{88}$ | | | |
| $U_{91}$ | = | $+xU_{81}$ | $+yU_{87}$ | $+xU_{80}$ | $-yU_{88}$ | $-U_{70}$ |
| $U_{92}$ | = | $+xU_{82}$ | $+yU_{86}$ | $+xU_{81}$ | $-yU_{87}$ | $-U_{71}$ |
| $U_{93}$ | = | $+xU_{83}$ | $+yU_{85}$ | $+xU_{82}$ | $-yU_{86}$ | $-U_{72}$ |
| $U_{94}$ | = | | $+yU_{84}$ | $+xU_{83}$ | $-yU_{85}$ | $-U_{73}$ |
| $U_{95}$ | = | $+xU_{85}$ | $+yU_{83}$ | $+xU_{84}$ | | $-U_{74}$ |
| $U_{96}$ | = | $+xU_{86}$ | $+yU_{82}$ | $+xU_{85}$ | $-yU_{83}$ | $-U_{75}$ |
| $U_{97}$ | = | $+xU_{87}$ | $+yU_{81}$ | $+xU_{86}$ | $-yU_{82}$ | $-U_{76}$ |
| $U_{98}$ | = | $+xU_{88}$ | $+yU_{80}$ | $+xU_{87}$ | $-yU_{81}$ | $-U_{77}$ |
| $U_{99}$ | = | | | $+xU_{88}$ | $-yU_{80}$ | |
| $U_{1000}$ | = | $+xU_{90}$ | $+yU_{99}$ | | | |
| $U_{1001}$ | = | $+xU_{91}$ | $+yU_{98}$ | $+xU_{90}$ | $-yU_{99}$ | $-U_{80}$ |
| $U_{1002}$ | = | $+xU_{92}$ | $+yU_{97}$ | $+xU_{91}$ | $-yU_{98}$ | $-U_{81}$ |
| $U_{1003}$ | = | $+xU_{93}$ | $+yU_{96}$ | $+xU_{92}$ | $-yU_{97}$ | $-U_{82}$ |
| $U_{1004}$ | = | $+xU_{94}$ | $+yU_{95}$ | $+xU_{93}$ | $-yU_{96}$ | $-U_{83}$ |
| $U_{1005}$ | = | $+2xU_{95}$ | $+2yU_{94}$ | | | $-U_{84}$ |
| $U_{1006}$ | = | $+xU_{96}$ | $+yU_{93}$ | $+xU_{95}$ | $-yU_{94}$ | $-U_{85}$ |
| $U_{1007}$ | = | $+xU_{97}$ | $+yU_{92}$ | $+xU_{96}$ | $-yU_{93}$ | $-U_{86}$ |
| $U_{1008}$ | = | $+xU_{98}$ | $+yU_{91}$ | $+xU_{97}$ | $-yU_{92}$ | $-U_{87}$ |
| $U_{1009}$ | = | $+xU_{99}$ | $+yU_{90}$ | $+xU_{98}$ | $-yU_{91}$ | $-U_{88}$ |
| $U_{1010}$ | = | | | $+xU_{99}$ | $-yU_{90}$ | |

Derivatives in x

| | | | |
|---|---|---|---|
| $\partial_x U_{00}$ = | 0 | | |
| $\partial_x U_{10}$ = | 0 | | |
| $\partial_x U_{11}$ = | | $+U_{00}$ | |
| $\partial_x U_{20}$ = | $+2U_{10}$ | | |
| $\partial_x U_{21}$ = | $+4U_{11}$ | | $+\partial_x U_{00}$ |
| $\partial_x U_{22}$ = | | $+2U_{11}$ | |
| $\partial_x U_{30}$ = | $+3U_{20}$ | | |
| $\partial_x U_{31}$ = | | $+3U_{20}$ | $+\partial_x U_{10}$ |
| $\partial_x U_{32}$ = | $+3U_{22}$ | $+3U_{21}$ | $+\partial_x U_{11}$ |
| $\partial_x U_{33}$ = | | $+3U_{22}$ | |
| $\partial_x U_{40}$ = | $+4U_{30}$ | | |
| $\partial_x U_{41}$ = | $+4U_{31}$ | $+4U_{30}$ | $+\partial_x U_{20}$ |
| $\partial_x U_{42}$ = | $+8U_{32}$ | | $+\partial_x U_{21}$ |
| $\partial_x U_{43}$ = | $+4U_{33}$ | $+4U_{32}$ | $+\partial_x U_{22}$ |
| $\partial_x U_{44}$ = | | $+4U_{33}$ | |
| $\partial_x U_{50}$ = | $+5U_{40}$ | | |
| $\partial_x U_{51}$ = | $+5U_{41}$ | $+5U_{40}$ | $+\partial_x U_{30}$ |
| $\partial_x U_{52}$ = | | $+5U_{41}$ | $+\partial_x U_{31}$ |
| $\partial_x U_{53}$ = | $+5U_{43}$ | $+5U_{42}$ | $+\partial_x U_{32}$ |
| $\partial_x U_{54}$ = | $+5U_{44}$ | $+5U_{43}$ | $+\partial_x U_{33}$ |
| $\partial_x U_{55}$ = | | $+5U_{44}$ | |
| $\partial_x U_{60}$ = | $+6U_{50}$ | | |
| $\partial_x U_{61}$ = | $+6U_{51}$ | $+6U_{50}$ | $+\partial_x U_{40}$ |
| $\partial_x U_{62}$ = | $+6U_{52}$ | $+6U_{51}$ | $+\partial_x U_{41}$ |
| $\partial_x U_{63}$ = | $+12U_{53}$ | | $+\partial_x U_{42}$ |
| $\partial_x U_{64}$ = | $+6U_{54}$ | $+6U_{53}$ | $+\partial_x U_{43}$ |
| $\partial_x U_{65}$ = | $+6U_{55}$ | $+6U_{54}$ | $+\partial_x U_{44}$ |
| $\partial_x U_{66}$ = | | $+6U_{55}$ | |
| … | | | |
| $\partial_x U_{90}$ = | $+9U_{80}$ | | |
| $\partial_x U_{91}$ = | $+9U_{81}$ | $+9U_{80}$ | $+\partial_x U_{70}$ |
| $\partial_x U_{92}$ = | $+9U_{82}$ | $+9U_{81}$ | $+\partial_x U_{71}$ |
| $\partial_x U_{93}$ = | $+9U_{83}$ | $+9U_{82}$ | $+\partial_x U_{72}$ |
| $\partial_x U_{94}$ = | | $+9U_{83}$ | $+\partial_x U_{73}$ |
| $\partial_x U_{95}$ = | $+9U_{85}$ | $+9U_{84}$ | $+\partial_x U_{74}$ |
| $\partial_x U_{96}$ = | $+9U_{86}$ | $+9U_{85}$ | $+\partial_x U_{75}$ |
| $\partial_x U_{97}$ = | $+9U_{87}$ | $+9U_{86}$ | $+\partial_x U_{76}$ |
| $\partial_x U_{98}$ = | $+9U_{88}$ | $+9U_{87}$ | $+\partial_x U_{77}$ |
| $\partial_x U_{99}$ = | | $+9U_{88}$ | |
| $\partial_x U_{1000}$ = | $+10U_{90}$ | | |
| $\partial_x U_{1001}$ = | $+10U_{91}$ | $+10U_{90}$ | $+\partial_x U_{80}$ |
| $\partial_x U_{1002}$ = | $+10U_{92}$ | $+10U_{91}$ | $+\partial_x U_{81}$ |
| $\partial_x U_{1003}$ = | $+10U_{93}$ | $+10U_{92}$ | $+\partial_x U_{82}$ |
| $\partial_x U_{1004}$ = | $+10U_{94}$ | $+10U_{93}$ | $+\partial_x U_{83}$ |
| $\partial_x U_{1005}$ = | $+20U_{95}$ | | $+\partial_x U_{84}$ |
| $\partial_x U_{1006}$ = | $+10U_{96}$ | $+10U_{95}$ | $+\partial_x U_{85}$ |
| $\partial_x U_{1007}$ = | $+10U_{97}$ | $+10U_{96}$ | $+\partial_x U_{86}$ |
| $\partial_x U_{1008}$ = | $+10U_{98}$ | $+10U_{97}$ | $+\partial_x U_{87}$ |
| $\partial_x U_{1009}$ = | $+10U_{99}$ | $+10U_{98}$ | $+\partial_x U_{88}$ |
| $\partial_x U_{1010}$ = | | $+10U_{99}$ | |

Derivatives in y

| | | | |
|---|---|---|---|
| $\partial_y U_{00}$ = | 0 | | |
| $\partial_y U_{10}$ = | $+U_{00}$ | | |
| $\partial_y U_{11}$ = | 0 | | |
| $\partial_y U_{20}$ = | $+2U_{11}$ | | |
| $\partial_y U_{21}$ = | $+4U_{10}$ | | $+\partial_y U_{00}$ |
| $\partial_y U_{22}$ = | | $-2U_{10}$ | |
| $\partial_y U_{30}$ = | $+3U_{22}$ | | |
| $\partial_y U_{31}$ = | $+3U_{21}$ | $-3U_{22}$ | $+\partial_y U_{10}$ |
| $\partial_y U_{32}$ = | $+3U_{20}$ | | $+\partial_y U_{11}$ |
| $\partial_y U_{33}$ = | | $-3U_{20}$ | |
| $\partial_y U_{40}$ = | $+4U_{33}$ | | |
| $\partial_y U_{41}$ = | $+4U_{32}$ | $-4U_{33}$ | $+\partial_y U_{20}$ |
| $\partial_y U_{42}$ = | $+8U_{31}$ | | $+\partial_y U_{21}$ |
| $\partial_y U_{43}$ = | $+4U_{30}$ | $-4U_{31}$ | $+\partial_y U_{22}$ |
| $\partial_y U_{44}$ = | | $-4U_{30}$ | |
| $\partial_y U_{50}$ = | $+5U_{44}$ | | |
| $\partial_y U_{51}$ = | $+5U_{43}$ | $-5U_{44}$ | $+\partial_y U_{30}$ |
| $\partial_y U_{52}$ = | $+5U_{42}$ | $-5U_{43}$ | $+\partial_y U_{31}$ |
| $\partial_y U_{53}$ = | $+5U_{41}$ | | $+\partial_y U_{32}$ |
| $\partial_y U_{54}$ = | $+5U_{40}$ | $-5U_{41}$ | $+\partial_y U_{33}$ |
| $\partial_y U_{55}$ = | | $-5U_{40}$ | |
| $\partial_y U_{60}$ = | $+6U_{55}$ | | |
| $\partial_y U_{61}$ = | $+6U_{54}$ | $-6U_{55}$ | $+\partial_y U_{40}$ |
| $\partial_y U_{62}$ = | $+6U_{53}$ | $-6U_{54}$ | $+\partial_y U_{41}$ |
| $\partial_y U_{63}$ = | $+12U_{52}$ | | $+\partial_y U_{42}$ |
| $\partial_y U_{64}$ = | $+6U_{51}$ | $-6U_{52}$ | $+\partial_y U_{43}$ |
| $\partial_y U_{65}$ = | $+6U_{50}$ | $-6U_{51}$ | $+\partial_y U_{44}$ |
| $\partial_y U_{66}$ = | | $-6U_{50}$ | |
| … | | | |
| $\partial_y U_{90}$ = | $+9U_{88}$ | | |
| $\partial_y U_{91}$ = | $+9U_{87}$ | $-9U_{88}$ | $+\partial_y U_{70}$ |
| $\partial_y U_{92}$ = | $+9U_{86}$ | $-9U_{87}$ | $+\partial_y U_{71}$ |
| $\partial_y U_{93}$ = | $+9U_{85}$ | $-9U_{86}$ | $+\partial_y U_{72}$ |
| $\partial_y U_{94}$ = | $+9U_{84}$ | $-9U_{85}$ | $+\partial_y U_{73}$ |
| $\partial_y U_{95}$ = | $+9U_{83}$ | | $+\partial_y U_{74}$ |
| $\partial_y U_{96}$ = | $+9U_{82}$ | $-9U_{83}$ | $+\partial_y U_{75}$ |
| $\partial_y U_{97}$ = | $+9U_{81}$ | $-9U_{82}$ | $+\partial_y U_{76}$ |
| $\partial_y U_{98}$ = | $+9U_{80}$ | $-9U_{81}$ | $+\partial_y U_{77}$ |
| $\partial_y U_{99}$ = | | $-9U_{80}$ | |
| $\partial_y U_{1000}$ = | $+10U_{99}$ | | |
| $\partial_y U_{1001}$ = | $+10U_{98}$ | $-10U_{99}$ | $+\partial_y U_{80}$ |
| $\partial_y U_{1002}$ = | $+10U_{97}$ | $-10U_{98}$ | $+\partial_y U_{81}$ |
| $\partial_y U_{1003}$ = | $+10U_{96}$ | $-10U_{97}$ | $+\partial_y U_{82}$ |
| $\partial_y U_{1004}$ = | $+10U_{95}$ | $-10U_{96}$ | $+\partial_y U_{83}$ |
| $\partial_y U_{1005}$ = | $+20U_{94}$ | | $+\partial_y U_{84}$ |
| $\partial_y U_{1006}$ = | $+10U_{93}$ | $-10U_{94}$ | $+\partial_y U_{85}$ |
| $\partial_y U_{1007}$ = | $+10U_{92}$ | $-10U_{93}$ | $+\partial_y U_{86}$ |
| $\partial_y U_{1008}$ = | $+10U_{91}$ | $-10U_{92}$ | $+\partial_y U_{87}$ |
| $\partial_y U_{1009}$ = | $+10U_{90}$ | $-10U_{91}$ | $+\partial_y U_{88}$ |
| $\partial_y U_{1010}$ = | | $-10U_{90}$ | |

Fig. 7. Recursive formulas for unnormalized Zernike polynomials and their derivatives

## Appendix B – Pseudo code for Zernike polynomials and derivatives

In this Appendix, we provide a reference to pseudo code, as shown in Code File 1 (Ref [58].), in sufficient detail for the calculation of all Zernike polynomials and their Cartesian derivatives at a point $(x, y)$ in the unit disc up to a certain radial order using the recurrence relations presented in Secs. 5 and 6. The code is similar to C-language code without

Optics EXPRESS

terminating semicolons and with some {} brackets omitted. The exception clauses from Secs. 5 and 6 are clearly marked in the code. The statements calculating the Cartesian derivatives are also easily identifiable. If code for just calculating the Zernike polynomials is desired, the statements for the derivatives should be left out. These could instead be implemented in a separate procedure for the calculation of the derivatives that is called after the Zernike polynomials have been calculated.

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