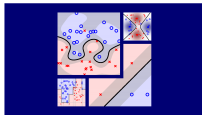


Machine Learning Techniques (機器學習技法)



Lecture 12: Neural Network

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Roadmap

- 1 Embedding Numerous Features: Kernel Models
- 2 Combining Predictive Features: Aggregation Models

Lecture 11: Gradient Boosted Decision Tree

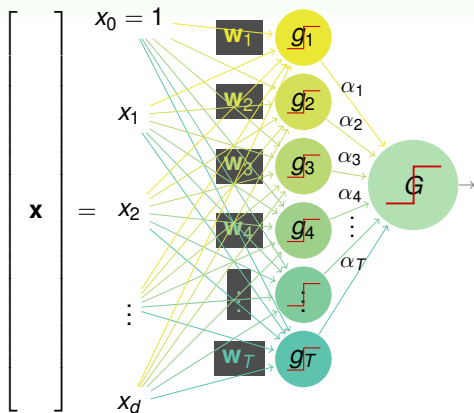
aggregating trees from **functional gradient** and **steepest descent** subject to **any error measure**

- 3 Distilling Implicit Features: Extraction Models

Lecture 12: Neural Network

- Motivation
- Neural Network Hypothesis
- Neural Network Learning
- Optimization and Regularization

Linear Aggregation of Perceptrons: Pictorial View

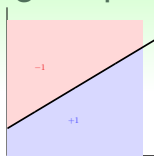
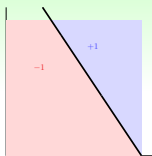
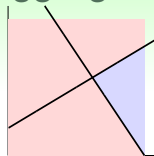
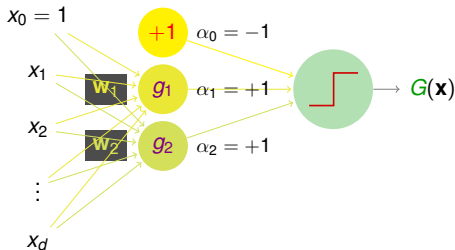


$$G(\mathbf{x}) = \text{sign} \left(\sum_{t=1}^T \alpha_t \underbrace{\text{sign}(\mathbf{w}_t^T \mathbf{x})}_{g_t(\mathbf{x})} \right)$$

- two layers of weights: \mathbf{w}_t and α
- two layers of sign functions: in g_t and in G

what boundary can G implement?

Logic Operations with Aggregation

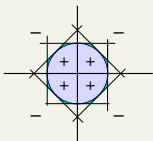

 g_1

 g_2

 $\text{AND}(g_1, g_2)$


$$G(\mathbf{x}) = \text{sign}(-1 + g_1(\mathbf{x}) + g_2(\mathbf{x}))$$

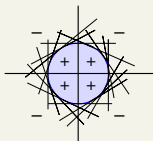
- $g_1(\mathbf{x}) = g_2(\mathbf{x}) = +1$ (TRUE):
 $G(\mathbf{x}) = +1$ (TRUE)
- otherwise:
 $G(\mathbf{x}) = -1$ (FALSE)
- $G \equiv \text{AND}(g_1, g_2)$

OR, NOT can be **similarly implemented**

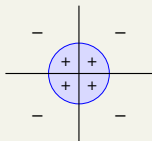
Powerfulness and Limitation



8 perceptrons

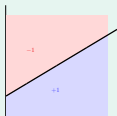
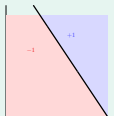
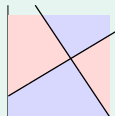


16 perceptrons



target boundary

- 'convex set' hypotheses implemented: $d_{VC} \rightarrow \infty$, **remember? :-)**
- powerfulness: enough perceptrons \approx **smooth boundary**

 g_1  g_2 XOR(g_1, g_2)

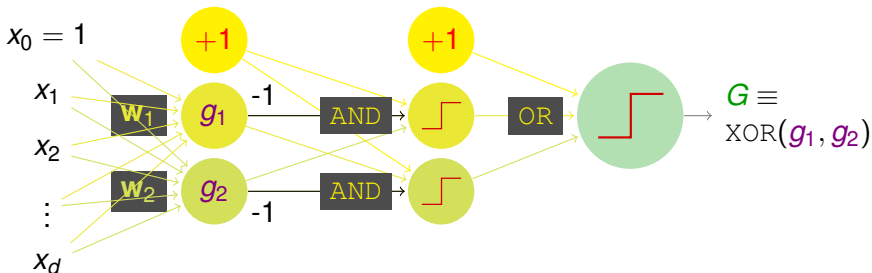
- limitation: XOR **not 'linear separable'** under $\phi(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}))$

how to implement XOR(g_1, g_2)?

Multi-Layer Perceptrons: Basic Neural Network

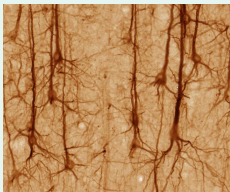
- non-separable data: can use more **transform**
- how about **one more layer of AND transform**?

$$\text{XOR}(g_1, g_2) = \text{OR}(\text{AND}(-g_1, g_2), \text{AND}(g_1, -g_2))$$



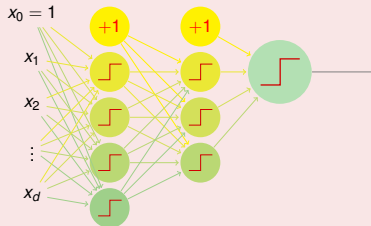
perceptron (simple)
 \Rightarrow aggregation of perceptrons (powerful)
 \Rightarrow **multi-layer perceptrons (more powerful)**

Connection to Biological Neurons



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neural network: **bio-inspired** model

Fun Time

Let $g_0(\mathbf{x}) = +1$. Which of the following $(\alpha_0, \alpha_1, \alpha_2)$ allows

$G(\mathbf{x}) = \text{sign} \left(\sum_{t=0}^2 \alpha_t g_t(\mathbf{x}) \right)$ to implement $\text{OR}(g_1, g_2)$?

- 1 $(-3, +1, +1)$
- 2 $(-1, +1, +1)$
- 3 $(+1, +1, +1)$
- 4 $(+3, +1, +1)$

Fun Time

Let $g_0(\mathbf{x}) = +1$. Which of the following $(\alpha_0, \alpha_1, \alpha_2)$ allows

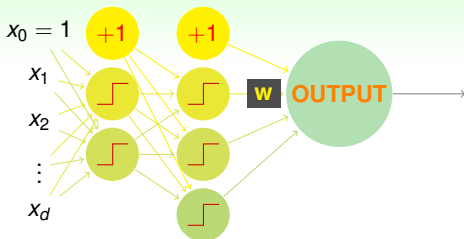
$G(\mathbf{x}) = \text{sign} \left(\sum_{t=0}^2 \alpha_t g_t(\mathbf{x}) \right)$ to implement $\text{OR}(g_1, g_2)$?

- ① $(-3, +1, +1)$
- ② $(-1, +1, +1)$
- ③ $(+1, +1, +1)$
- ④ $(+3, +1, +1)$

Reference Answer: ③

You can easily verify with all four possibilities of $(g_1(\mathbf{x}), g_2(\mathbf{x}))$.

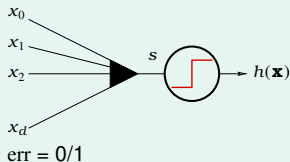
Neural Network Hypothesis: Output



- **OUTPUT**: simply a **linear model** with $\mathbf{s} = \mathbf{w}^T \phi^{(2)}(\phi^{(1)}(\mathbf{x}))$
- any linear model can be used—**remember? :-)**

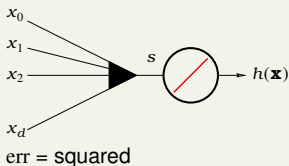
linear classification

$$h(\mathbf{x}) = \text{sign}(s)$$



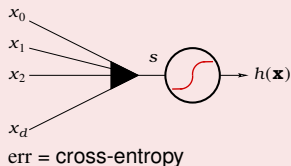
linear regression

$$h(\mathbf{x}) = s$$



logistic regression

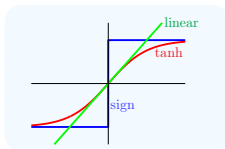
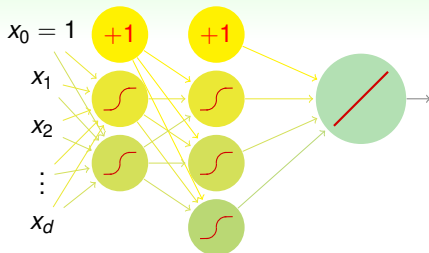
$$h(\mathbf{x}) = \theta(s)$$



will discuss **'regression' with squared error**

Neural Network Hypothesis: Transformation

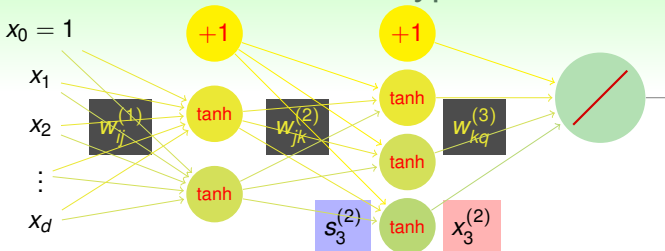
- \lceil : **transformation** function of score (signal) s
- **any transformation?**
 - \diagup : whole network linear & thus **less useful**
 - \lceil : discrete & thus **hard to optimize** for \mathbf{w}
- popular choice of **transformation**: $\frown = \tanh(s)$
 - 'analog' approximation of \lceil : **easier to optimize**
 - somewhat **closer to biological** neuron
 - **not that new! :-)**



$$\begin{aligned}\tanh(s) &= \frac{\exp(s) - \exp(-s)}{\exp(s) + \exp(-s)} \\ &= 2\theta(2s) - 1\end{aligned}$$

will discuss with **tanh** as **transformation function**

Neural Network Hypothesis



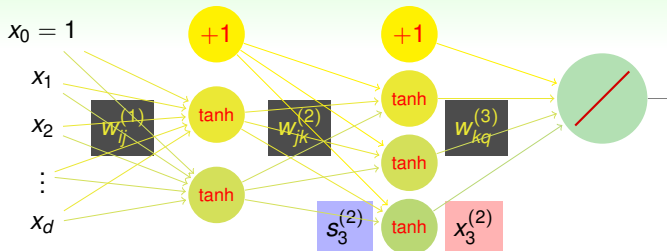
$d^{(0)} - d^{(1)} - d^{(2)} - \dots - d^{(L)}$ Neural Network (NNet)

$$w_{ij}^{(\ell)} : \begin{cases} 1 \leq \ell \leq L & \text{layers} \\ 0 \leq i \leq d^{(\ell-1)} & \text{inputs} \\ 1 \leq j \leq d^{(\ell)} & \text{outputs} \end{cases}, \text{ score } s_j^{(\ell)} = \sum_{i=0}^{d^{(\ell-1)}} w_{ij}^{(\ell)} x_i^{(\ell-1)},$$

$$\text{transformed } x_j^{(\ell)} = \begin{cases} \tanh(s_j^{(\ell)}) & \text{if } \ell < L \\ s_j^{(\ell)} & \text{if } \ell = L \end{cases}$$

apply \mathbf{x} as input layer $\mathbf{x}^{(0)}$, go through hidden layers to get $\mathbf{x}^{(\ell)}$, predict at output layer $x_1^{(L)}$

Physical Interpretation



- each layer: **transformation** to be **learned** from data

- $\phi^{(\ell)}(\mathbf{x}) = \tanh \left(\begin{bmatrix} \sum_{i=0}^{d^{(\ell-1)}} w_{i1}^{(\ell)} x_i^{(\ell-1)} \\ \vdots \end{bmatrix} \right)$

—whether \mathbf{x} ‘matches’ weight vectors in pattern

NNet: **pattern extraction** with
layers of **connection weights**

Fun Time

How many weights $\{w_{ij}^{(\ell)}\}$ are there in a 3-5-1 NNet?

- 1 9
- 2 15
- 3 20
- 4 26

Fun Time

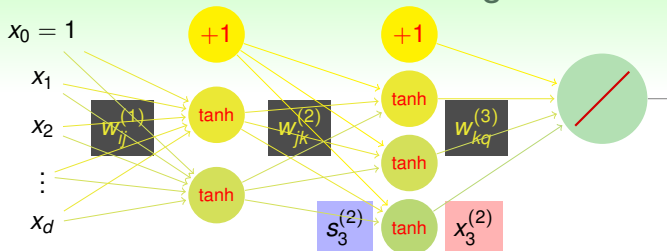
How many weights $\{w_{ij}^{(\ell)}\}$ are there in a 3-5-1 NNet?

- ① 9
- ② 15
- ③ 20
- ④ 26

Reference Answer: ④

There are $(3 + 1) \times 5$ weights in $w_{ij}^{(1)}$, and $(5 + 1) \times 1$ weights in $w_{jk}^{(2)}$.

How to Learn the Weights?



- goal: learning all $\{w_{ij}^{(\ell)}\}$ to **minimize** $E_{\text{in}}(\{w_{ij}^{(\ell)}\})$
- one hidden layer: simply **aggregation of perceptrons**
—**gradient boosting** to determine hidden neuron one by one
- multiple hidden layers? **not easy**
- let $e_n = (y_n - \text{NNet}(\mathbf{x}_n))^2$:
can apply **(stochastic) GD** after computing $\frac{\partial e_n}{\partial w_{ij}^{(\ell)}}$!

next: efficient computation of $\frac{\partial e_n}{\partial w_{ij}^{(\ell)}}$

Computing $\frac{\partial e_n}{\partial w_{i1}^{(L)}}$ (Output Layer)

$$e_n = (y_n - \text{NNet}(\mathbf{x}_n))^2 = (y_n - s_1^{(L)})^2 = \left(y_n - \sum_{i=0}^{d^{(L-1)}} w_{i1}^{(L)} x_i^{(L-1)} \right)^2$$

specially (output layer)
 $(0 \leq i \leq d^{(L-1)})$

$$\begin{aligned} & \frac{\partial e_n}{\partial w_{i1}^{(L)}} \\ &= \frac{\partial e_n}{\partial s_1^{(L)}} \cdot \frac{\partial s_1^{(L)}}{\partial w_{i1}^{(L)}} \\ &= -2 \left(y_n - s_1^{(L)} \right) \cdot \left(x_i^{(L-1)} \right) \end{aligned}$$

generally $(1 \leq \ell < L)$
 $(0 \leq i \leq d^{(\ell-1)}; 1 \leq j \leq d^{(\ell)})$

$$\begin{aligned} & \frac{\partial e_n}{\partial w_{ij}^{(\ell)}} \\ &= \frac{\partial e_n}{\partial s_j^{(\ell)}} \cdot \frac{\partial s_j^{(\ell)}}{\partial w_{ij}^{(\ell)}} \\ &= \delta_j^{(\ell)} \cdot \left(x_i^{(\ell-1)} \right) \end{aligned}$$

$$\delta_1^{(L)} = -2 \left(y_n - s_1^{(L)} \right), \text{ how about others?}$$

Computing $\delta_j^{(\ell)} = \frac{\partial e_n}{\partial s_j^{(\ell)}}$

$$s_j^{(\ell)} \xrightarrow{\tanh} x_j^{(\ell)} \xrightarrow{w_{jk}^{(\ell+1)}} \begin{bmatrix} s_1^{(\ell+1)} \\ \vdots \\ s_k^{(\ell+1)} \\ \vdots \end{bmatrix} \Rightarrow \dots \Rightarrow e_n$$

$$\begin{aligned} \delta_j^{(\ell)} = \frac{\partial e_n}{\partial s_j^{(\ell)}} &= \sum_{k=1}^{d^{(\ell+1)}} \frac{\partial e_n}{\partial s_k^{(\ell+1)}} \frac{\partial s_k^{(\ell+1)}}{\partial x_j^{(\ell)}} \frac{\partial x_j^{(\ell)}}{\partial s_j^{(\ell)}} \\ &= \sum_k \left(\delta_k^{(\ell+1)} \right) \left(w_{jk}^{(\ell+1)} \right) \left(\tanh' \left(s_j^{(\ell)} \right) \right) \end{aligned}$$

$\delta_j^{(\ell)}$ can be computed **backwards** from $\delta_k^{(\ell+1)}$

Backpropagation (Backprop) Algorithm

Backprop on NNet

initialize all weights $w_{ij}^{(\ell)}$

for $t = 0, 1, \dots, T$

- ① stochastic: randomly pick $n \in \{1, 2, \dots, N\}$
- ② forward: compute all $x_i^{(\ell)}$ with $\mathbf{x}^{(0)} = \mathbf{x}_n$
- ③ backward: compute all $\delta_j^{(\ell)}$ subject to $\mathbf{x}^{(0)} = \mathbf{x}_n$
- ④ gradient descent: $w_{ij}^{(\ell)} \leftarrow w_{ij}^{(\ell)} - \eta x_i^{(\ell-1)} \delta_j^{(\ell)}$

return $g_{\text{NNET}}(\mathbf{x}) = \left(\dots \tanh \left(\sum_j w_{jk}^{(2)} \cdot \tanh \left(\sum_i w_{ij}^{(1)} x_i \right) \right) \right)$

sometimes ① to ③ is (parallelly) done many times and $\text{average}(x_i^{(\ell-1)} \delta_j^{(\ell)})$ taken for update in ④, called **mini-batch**

basic NNet algorithm: backprop to compute the gradient **efficiently**

Fun Time

According to $\frac{\partial e_n}{\partial w_{i1}^{(L)}} = -2 \left(y_n - s_1^{(L)} \right) \cdot \left(x_i^{(L-1)} \right)$ when would $\frac{\partial e_n}{\partial w_{i1}^{(L)}} = 0$?

- ① $y_n = s_1^{(L)}$
- ② $x_i^{(L-1)} = 0$
- ③ $s_i^{(L-1)} = 0$
- ④ all of the above

Fun Time

According to $\frac{\partial e_n}{\partial w_{i1}^{(L)}} = -2 \left(y_n - s_1^{(L)} \right) \cdot \left(x_i^{(L-1)} \right)$ when would $\frac{\partial e_n}{\partial w_{i1}^{(L)}} = 0$?

- ① $y_n = s_1^{(L)}$
- ② $x_i^{(L-1)} = 0$
- ③ $s_i^{(L-1)} = 0$
- ④ all of the above

Reference Answer: ④

Note that $x_i^{(L-1)} = \tanh(s_i^{(L-1)}) = 0$ if and only if $s_i^{(L-1)} = 0$.

Neural Network Optimization

$$E_{\text{in}}(\mathbf{w}) = \frac{1}{N} \sum_{n=1}^N \text{err} \left(\left(\cdots \tanh \left(\sum_j w_{jk}^{(2)} \cdot \tanh \left(\sum_i w_{ij}^{(1)} x_{n,i} \right) \right) \right), y_n \right)$$

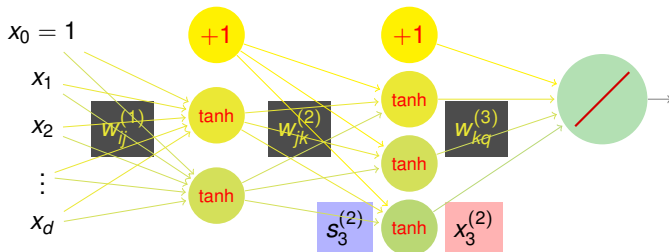
- generally **non-convex** when multiple hidden layers
 - not easy to reach **global minimum**
 - GD/SGD with **backprop** only gives **local minimum**
- different initial $w_{ij}^{(\ell)} \Rightarrow$ different **local minimum**
 - somewhat '**sensitive**' to initial weights
 - large weights** \Rightarrow **saturate** (small gradient)
 - advice: try **some random** & **small** ones

NNet: **difficult to optimize**,
but **practically works**

VC Dimension of Neural Network Model

roughly, with **tanh-like transfer functions**:

$d_{VC} = O(VD)$ where $V = \#$ of neurons, $D = \#$ of weights



- pros: can **approximate 'anything'** if enough neurons (V large)
- cons: can **overfit** if too many neurons

NNet: watch out for overfitting!

Regularization for Neural Network

basic choice:

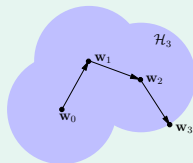
old friend **weight-decay** (L2) regularizer $\Omega(\mathbf{w}) = \sum \left(w_{ij}^{(\ell)} \right)^2$

- **'shrink' weights:**
large weight \rightarrow **large shrink**; **small weight** \rightarrow **small shrink**
- want $w_{ij}^{(\ell)} = 0$ (sparse) to effectively **decrease** d_{vc}
 - **L1** regularizer: $\sum \left| w_{ij}^{(\ell)} \right|$, but **not differentiable**
 - **weight-elimination** ('**scaled**' **L2**) regularizer:
large weight \rightarrow **median shrink**; **small weight** \rightarrow **median shrink**

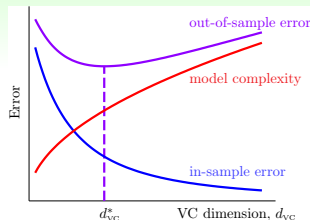
weight-elimination regularizer: $\sum \frac{\left(w_{ij}^{(\ell)} \right)^2}{1 + \left(w_{ij}^{(\ell)} \right)^2}$

Yet Another Regularization: Early Stopping

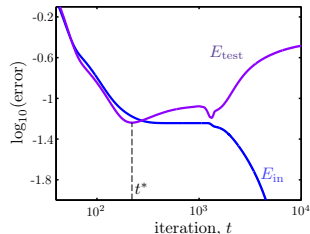
- **GD/SGD (backprop)** visits more weight combinations as t increases



- smaller t effectively decrease d_{VC}
- better 'stop in middle': **early stopping**



(d_{VC}^* in middle, remember? :-))



when to stop? **validation!**

Fun Time

For the weight elimination regularizer $\sum \frac{(w_{ij}^{(\ell)})^2}{1 + (w_{ij}^{(\ell)})^2}$, what is $\frac{\partial \text{regularizer}}{\partial w_{ij}^{(\ell)}}$?

1 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^1$

2 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^2$

3 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^3$

4 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^4$

Fun Time

For the weight elimination regularizer $\sum \frac{(w_{ij}^{(\ell)})^2}{1 + (w_{ij}^{(\ell)})^2}$, what is $\frac{\partial \text{regularizer}}{\partial w_{ij}^{(\ell)}}$?

- 1 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^1$
- 2 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^2$
- 3 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^3$
- 4 $2w_{ij}^{(\ell)} / \left(1 + (w_{ij}^{(\ell)})^2\right)^4$

Reference Answer: (2)

Too much calculus in this class, huh? :-)

Summary

- 1 Embedding Numerous Features: Kernel Models
- 2 Combining Predictive Features: Aggregation Models
- 3 Distilling Implicit Features: Extraction Models

Lecture 12: Neural Network

- Motivation
 - multi-layer for power with biological inspirations**
 - Neural Network Hypothesis
 - layered pattern extraction until linear hypothesis**
 - Neural Network Learning
 - backprop to compute gradient efficiently**
 - Optimization and Regularization
 - tricks on initialization, regularizer, early stopping**
- **next: making neural network 'deeper'**