# DEFINITION, PROPERTY, RULE AND THEOREMS

1.1 Vectors in Euclidean Spaces

## PROPERTIES OF VECTOR ALGEBRA IN $\mathbb{R}^n$

$\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$, $r$ and $s$ be any scalars in $\mathbb{R}$

| | | | |
|---|---|---|---|
| $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ | associative law | $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$ | distributive law |
| $\vec{v}+\vec{w} = \vec{w}+\vec{v}$ | commutative law | $(r+s)\vec{v} = r\vec{v}+s\vec{v}$ | distributive law |
| $\vec{0}+\vec{v} = \vec{v}$ | $\vec{0}$ as additive identity | $r(s\vec{v}) = (rs)\vec{v}$ | associative law |
| $\vec{v}+(-\vec{v}) = \vec{0}$ | $-\vec{v}$ as additive inverse of $\vec{v}$ | $1\vec{v} = \vec{v}$ | preservation of scale |

## LINEAR COMBINATION

Given vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ in $\mathbb{R}^n$ and scalars $r_1, r_2, \cdots, r_k$ in $\mathbb{R}$, the vector

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$$

is a linear combination of the vectors $v_1, v_2, \cdots, v_k$ with scalar coefficients $r_1, r_2, \cdots, r_k$

### SPECIAL VECTORS

$e_r = [0, 0, \cdots, 0, \overset{r^{th}\text{ element}}{1}, 0, \cdots 0]$ $r^{th}$ standard basis vector $\|\vec{v}\| = 1$ unit vector

$\vec{0} = [0, 0, 0, \cdots 0]$ zero vector

## SPAN OF $V_1, V_2, \cdots V_k$

Let $v_1, v_2, \cdots v_k$ be vectors in $\mathbb{R}^n$

The span of these vectors is the set of all linear combination of them and is denoted by $sp(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k)$ or $\{r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k \mid r_1, r_2, \cdots r_k \in \mathbb{R}\}$

### NUMBERS

$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ set of natural numbers

$\mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$ set of integers

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ set of rational numbers

$\overline{\mathbb{Q}} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$ set of irrationals

$\mathbb{R}$ is the set of all real numbers

1.2 The Norm and the Dot product

## MAGNITUDE OF A VECTOR / NORM

$\|\vec{v}\|$ of $\vec{v} = [v_1, v_2, \cdots, v_n] = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ is called the magnitude or norm

## PROPERTIES OF THE NORM IN $\mathbb{R}^n$

For all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and for all scalars $r$, we have

$\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$ positivity

$\|r\vec{v}\| = |r|\|\vec{v}\|$ homogeneity

$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ triangle inequality

## DOT PRODUCT / INNER PRODUCT / SCALAR PRODUCT

Let $\vec{v} = [v_1, v_2, \cdots, v_n]$ and $\vec{w} = [w_1, w_2, \cdots, w_n]$ be vectors in $\mathbb{R}^n$. The dot product is the $\mathbb{R}$

$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$$

## ANGLE BETWEEN TWO NONZERO VECTORS

$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$ where $-1 \leq \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} \leq 1$

## Properties of the Dot Product in $\mathbb{R}^n$

$\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and $r$ be any scalar in $\mathbb{R}$

$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$ commutative law — $\vec{v}\perp\vec{w} \iff \vec{v}\cdot\vec{u} = 0$

$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ distributive law — $\|\vec{v}-\vec{u}\|$ = magnitude of $\vec{v}-\vec{u}$ = distance

$r(\vec{v}\cdot\vec{w}) = (r\vec{v})\cdot\vec{w} = \vec{v}\cdot(r\vec{w})$ homogeneity — between $\vec{v}, \vec{u}$ $\vec{v}, \vec{u}$ in standard position

$\vec{v}\cdot\vec{v} \geq 0$ and $\vec{v}\cdot\vec{v} = 0 \iff \vec{v} = 0$ positivity — $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$ $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

**Cauchy-Schwartz Inequality**

$\vec{u}, \vec{v} \in \mathbb{R}^n$ $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$

**Triangle Inequality**

$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

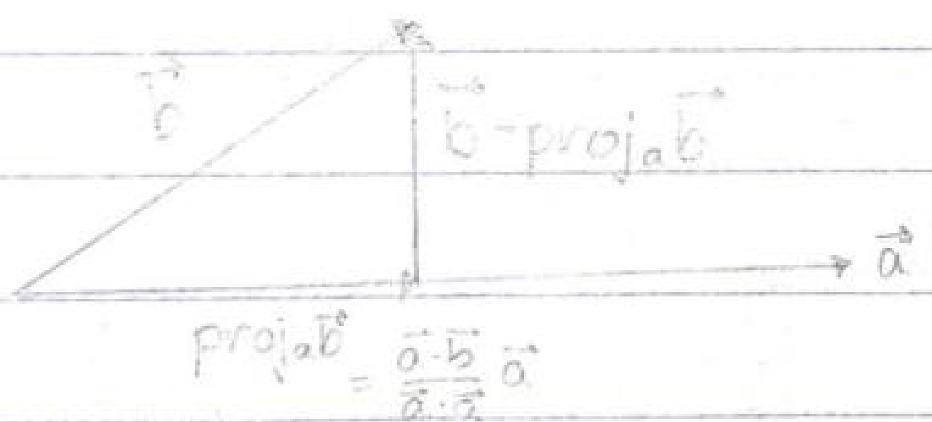
## Projections

A nonzero vector $\vec{b}$ can be "decomposed" into a sum of two other vectors with one parallel to a nonzero vector $\vec{a}$ and one perpendicular to $\vec{a}$.

the ~~ot~~ orthogonal projection of $\vec{b}$ on $\vec{a}$ $\text{proj}_a b = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$ is the vector parallel to $\vec{a}$ (vector; dot products)

the vector component of $\vec{b}$ orthogonal to $\vec{a}$

~~$\vec{a}$ — P~~ $\vec{b} - \text{proj}_a\vec{b}$ is the vector perpendicular to $\vec{a}$




shortest distance between $\vec{a}$ and $\vec{b}$ $= \|\vec{b} - \text{proj}_a\vec{b}\|$

$\vec{b}\cdot\vec{a}$ = length of $\text{proj}_a b$ × length of $\vec{a}$ – $\vec{b}\cdot\vec{a}$ = length of $\text{proj}_b\vec{a}$ × length of $\vec{b}$

If both $\text{proj}_a\vec{b}$ and $\vec{b}$ are in the same direction, dot product is positive; if in opposite direction dot product is negative

1.3 Matrices and their Algebra

## Matrix

A matrix is an ordered rectangular array of numbers, usually enclosed in square brackets. An $m\times n$ matrix has $m$ rows and $n$ columns.

$M_{m,n}(\mathbb{R})$ set of all $m\times n$ matrices with $\mathbb{R}$ $A = [a_{ij}]$ $a_{ij}$ is the entry in $i$th row and $j$th column of $A$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ an $m\times n$ matrix $A$

**Column Vector**

An $m\times 1$ matrix in $\mathbb{R}^m$ $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$

**Row Vector**

A $1\times n$ matrix in $\mathbb{R}^n$ $[a_1\ a_2 \cdots a_n]$

**Square Matrix**

An $n\times n$ matrix

**Diagonal Matrix**

A square matrix with $a_{ij} = 0\ \forall i\neq j$ $\begin{bmatrix} 1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&0 \end{bmatrix}$

**Identity Matrix**

A diagonal matrix with $a_{ij} = \begin{cases} 0 & \forall i\neq j \\ 1 & \forall i=j \end{cases}$

**Zero Matrix**

A $m\times n$ matrix with all entries equal to zero

**Upper Triangular Matrix**

A square matrix with $a_{ij} = 0$ if $i > j$

**Lower Triangular Matrix**

A square matrix with $a_{ij} = 0$ if $i < j$

## Matrix Manipulation

$A = [a_{ij}] \quad B = [b_{ij}] \quad k \in \mathbb{R}$

$A+B = (A+B)_{ij} = a_{ij} + b_{ij}$

$A-B = (A-B)_{ij} = a_{ij} - b_{ij}$

$kA = (kA)_{ij} = ka_{ij}$

$AB = (AB)_{ij} = (\text{ith row of } A) \cdot (\text{jth column of } B) = \sum_{l=1}^{n} A_{il} \cdot B_{lj}$

## Transpose of a Matrix

The matrix B is the transpose of the matrix A, written as $B = A^T$, if each entry $b_{ij}$ in B is the same as the entry $a_{ji}$ in A, and conversely.

$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & -7 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ 5 & -7 \end{bmatrix}$

## Symmetric Matrix

$A^T = A$

## Skew Symmetric Matrix

$A^T = -A$

$a \cdot b = a^T b = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n]$

## Properties of Matrix Algebra

| | |
|---|---|
| $A+B = B+A$ | commutative law |
| $(A+B)+C = A+(B+C)$ | associative law |
| $A+0 = 0+A = A$ | identity |
| $r(A+B) = rA+rB$ | left distributive law |
| $(r+s)A = rA+sA$ | right distributive law |
| $(rs)A = r(sA)$ | associative law |
| $(rA)B = A(rB) = r(AB)$ | scalars pull through |
| $A(BC) = (AB)C$ | associative law |
| $IA = A$ and $BI = B$ | identity |
| $A(B+C) = AB+AC$ | left distributive |
| $(A+B)C = AC+BC$ | right distributive |
| $A+AB = A(I+B)$ | |

## Properties of the Transpose Operation

| | |
|---|---|
| $(A^T)^T = A$ | transpose of the transpose |
| $(A+B)^T = A^T + B^T$ | transpose of a sum |
| $(AB)^T = B^T A^T$ | transpose of a product |
| $(rA)^T = rA^T$ | transpose of scalar multiplication |

$AB = \begin{bmatrix} | & | & | & & | \\ Ab_1 & Ab_2 & Ab_3 & \cdots & Ab_n \\ | & | & | & & | \end{bmatrix}$

## Trace

$A \in M_{n,n}(\mathbb{R})$

trace of A is defined as $tr(A) = \sum_{i=1}^{n} a_{ii}$

## Properties of the Trace of a Matrix

| | |
|---|---|
| $tr(A^T) = tr(A)$ | $tr(A+B) = tr(A) + tr(B)$ |
| $tr(rA) = r\,tr(A)$ | $tr(AB) = tr(BA)$ |

1.4 Solving systems of linear equations

## A LINEAR SYSTEM

A set of m equations with n unknowns

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$ can be written as $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

$A \quad \vec{x} = \vec{b}$

and

$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ so $\vec{b}$ is a linear combination of the column vectors of A with coefficients $x_1, x_2, \cdots, x_n$

$A\vec{x} = \vec{b}$ has a solution $\iff \vec{b}$ is in the span of the columns of A

## AUGMENTED MATRIX / PARTITIONED MATRIX

$[A|b]$ where A is called the coefficient matrix of the system and

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

## ELEMENTARY ROW OPERATIONS

We may solve the linear system using the following operations

1. Row interchange ($R_i \leftrightarrow R_j$) Interchange the ith and jth vectors in a matrix
2. Row scaling ($R_i \to sR_i$) Multiply the ith row vector ~~ano~~ in a matrix by a nonzero scalar s.
3. Row addition ($R_i \to R_i + sR_j$) Add to the ith row vector ~~the~~ of a matrix s times the jth row vector

## ROW EQUIVALENT MATRICES

If a matrix B can be obtained from A by performing a sequence of elementary row operations, then A is row equivalent to B. $A \sim B$

If $[A|\vec{b}]$ and $[H|\vec{c}]$ are row equivalent matrices then $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{c}$ have the same solution set.

## ROW-ECHELON FORM

A matrix is in row-echelon form if

1. All rows containing only zeros appear below rows of nonzero rows
2. The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any row above it.

## PIVOT

The first nonzero entry in a row

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Reducing a Matrix A to Row-Echelon Form

1. Move all the zero rows to the bottom
2. Identify the first pivot and use row addition to change all entries below that pivot zero
3. Repeat step two until $A \sim H$ in ~~RR~~REF (use row scaling and row interchange if it makes the process easier)

SOLUTIONS OF $A\vec{x}=\vec{b}$

Let $A\vec{x}=\vec{b}$ be a linear system, let $[A|\vec{b}] \sim [H|\vec{c}]$ where H is in REF

$A\vec{x}=\vec{b}$ is inconsistent $\iff$ $[H|\vec{c}]$ has a row with all zeros to the left of the partition and a nonzero entry to the right of the partition.

$A\vec{x}=\vec{b}$ is consistent and every column of H contains a pivot $\iff$ unique solution

$A\vec{x}=\vec{b}$ is consistent and some columns has no pivot $\iff$ infinitely many solutions with free variables as pivot-free columns in H.

REDUCED ROW ECHELON FORM

The matrix is in REF

Every pivot equals to 1

Each pivot is the only nonzero entry in its column

Reducing a matrix to RREF

1. Reduce it to REF
2. Idenify a row with nonzero entry ~~on top~~ above a pivot and perform row addition on that row with the row with the pivot
3. Repeat step 2 ~~until all~~ RREF
4. Use Row scaling to make all the pivots to 1.

GAUSS REDUCTION WITH BACK SUBSTITUTION METHOD

The method of solving $A\vec{x}=\vec{b}$ by reducing $[A|\vec{b}]$ to a row echelon form and then using back substitition

GAUSS-JORDAN METHOD

The method of solving the system $A\vec{x}=\vec{b}$ by reducing $[A|\vec{b}]$ to a reduced row echelon form

ELEMENTARY MATRIX

Any matrix that can be obtained by performing one elementary row operation on the identity matrix

Doing one row operation on A $\iff$ ~~mult~~ multiplying an elementary matrix to the left of A

1.5 Inverses of square matrices

INVERTIBLE MATRIX

An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $C$ (the inverse of $A = A^{-1}$) such that $AC = CA = I$. If $A$ is not invertible, it is singular

UNIQUENESS OF AN INVERSE

$A$ be $n \times n$

If $C$ and $D$ are matrices such that $AC = DA = I$, then $C = D$. In particular, if $AC = CA = I$, then $C$ is the unique matrix with this property

INVERSE OF PRODUCTS

Let $A$ and $B$ be invertible $n \times n$ matrices

$AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

CONDITION FOR $A\vec{x} = \vec{b}$ TO BE SOLVABLE FOR ALL $\vec{b}$

Let $A$ be $n \times n$

The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of $\vec{b} \in \mathbb{R}^n$ $\Leftrightarrow$ $A$ is row equivalent to the $n \times n$ identity matrix

COMMUTATIVITY PROPERTY

$A$ and $C$ be $n \times n$

$AC = I \Leftrightarrow CA = I$

FINDING $A^{-1}$, if it exists

1. Form the augmented matrix $[A|I]$
2. Apply Gauss-Jordan method to find RREF of $[A|I]$
3. If RREF of $[A|I] = [I|C]$ then $A$ is invertible and $A^{-1} = C$
4. If RREF of $[A|I]$ does not have the identity matrix on the left of partition, then $A$ is not invertible

$C$ is just the product of elementary matrices, thus if $A^{-1}$ exists, then $A^{-1}$ can be written as a product of elementary matrices

$A^{-1}$ can also be found with $\mathrm{Adj}(A)$ and $\det(A)$

Cayley Hamilton ~~theor~~ theorem

INVERTIBILITY OF TRANSPOSE

An $n \times n$ matrix $A$ is invertible $\Leftrightarrow$ $A^T$ is invertible

1.6 Homogeneous Systems Subspaces and Bases

THE SOLUTION SET OF A HOMOGENEOUS SYSTEM

A linear system $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b} = \vec{0}$. A homogeneous linear system $A\vec{x} = \vec{0}$ is always consistent ($\vec{x} = \vec{0}$ is the trivial solution)

If $A \sim H$ and $H$ has a pivot in every column, then $\vec{0}$ is the only solution

If $m < n$ then it is impossible for $H$ to have pivots in every column, $A\vec{x} = \vec{0}$ has infinitely many sols.

## Structure Of The Solution Set Of $A\vec{x}=\vec{0}$

Let $\vec{a}, \vec{b}$ be solutions to $A\vec{x}=\vec{0}$

$s_1\vec{a}+s_2\vec{b}$ is ~~abs~~ also a solution to $A\vec{x}=\vec{0}$, for any $s_1, s_2 \in \mathbb{R}$

## Subspace

A subspace of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ having the following properties

W is nonempty

if $\vec{u}, \vec{v} \in W$, then $\vec{u}+\vec{v} \in W$ closure under vector addition

if $\vec{u} \in W$ and $r \in \mathbb{R}$, then $r\vec{u} \in W$ closure under scalar multiplication

A span is a subspace

## Nullspace

A be $m \times n$ matrix

null(A) is the set of all solutions to $A\vec{x}=\vec{0}$

~~ant~~ $\text{null}(A) = \mathcal{N} = \{\vec{x} \in \mathbb{R} \mid A\vec{x}=\vec{0}\}$

## Rowspace

A be $m \times n$ matrix

row(A) is the span of row vectors of A row(A) $\subseteq \mathbb{R}^n$

## Column Space

A be $m \times n$ matrix

col(A) is the span of column vectors of A

## Structure Of The Solution Set Of $A\vec{x}=\vec{b}$

$A\vec{x}=\vec{b}$ be a linear system of equations with solution $\vec{p}$

if $\vec{h}$ is in nullspace of A, then $\vec{p}+\vec{h}$ is also a solution to $A\vec{x}=\vec{b}$

if $\vec{q}$ is any solution to $A\vec{x}=\vec{b}$, then $\vec{q}=\vec{p}+\vec{h}_1$ for some $\vec{h}_1$ in the nullspace of A

## Linearly Independent And Linearly Dependent

~~if $\vec{0}$~~ the vectors $v_1, v_2, \cdots v_k$ are linearly independent $\Leftrightarrow$ $r_1\vec{v}_1+r_2\vec{v}_2+\cdots+r_k\vec{v}_k=\vec{0}$

only when ~~where~~ $r_1, r_2 \cdots r_k = 0$

if $r_1\vec{v}+r_2\vec{v}_2+\cdots+r_k\vec{v}=\vec{0}$ has more than one solution then the vectors are linearly dependent.

## Basis

A set of vectors that are linearly independent and spans the given subspace.

i.e. Every vector in subspace W can be uniquely expressed as a linear combination of the vectors in the basis.

## Case With Fewer Equations Than Unknowns

If $A\vec{x}=\vec{b}$ is consistent, then it has infinitely many solutions

CASE WITH MORE EQUATIONS THAN UNKNOWNS

A be $m \times n$ , $m > n$ The following are equivalent

Each consistent system $A\vec{x} = \vec{b}$ has a unique solution

The RREF of A consists of the $n \times n$ identity matrix on top followed by $m-n$ rows of zeros

The column vectors of A form a basis for the column space of A.

2.1 TO FIND A BASIS FOR $W = sp(\vec{w_1}, \vec{w_2}, \cdots, \vec{w_k})$ where $w_i \in \mathbb{R}^n$ FOR $i = 1, 2, \cdots, k$

Independence and dimension

1. Construct the matrix $A = \begin{bmatrix} | & | & | \\ \vec{w_1} & \vec{w_2} \cdots & \vec{w_k} \\ | & | & | \end{bmatrix}$
2. Row reduce A to get H in a REF or in a RREF so $A \sim H$
3. A basis for W consists only of all the columns of A, corresponding to the columns of H that contain pivots.

RELATIVE SIZES OF SPANNING AND INDEPENDENT SETS

W be a subspace of $\mathbb{R}^n$ , $\vec{w_1}, \vec{w_2}, \vec{w_3}, \vec{w_k}$ be vectors in W that span W

$\vec{v_1}, \vec{v_2}, \vec{v_3}, \cdots \vec{v_m}$ be vectors that are linearly independent in W

$k \geq m$ can be proved by contrapositive

INVARIANCE OF DIMENSION

Any two bases of a subspace W of $\mathbb{R}^n$ contain the same number of vectors

DIMENSION OF A SUBSPACE

Let W be a subspace of $\mathbb{R}^n$. The number of elements in a basis for W is called the dimension of W. $\dim(W)$

EXISTENCE AND DETERMINATION OF BASES

Every nonzero subspace W of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Every linearly independent set of vectors in $\mathbb{R}^n$ can be enlarged, if necessary to become a basis for $\mathbb{R}^n$

If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = k$, then

every independent set of k vectors in W is a basis for W, and

every set of k vectors that spans W, is a basis for W.

2.2 FINDING BASES FOR SPACES ASSOCIATED WITH A MATRIX A be $m \times n$ REF H.

The rank of a Matrix

The nonzero rows form a basis for R (row space)

A basis for C consists of all the columns of A, corresponding to the columns of H containing pivots

For a basis of N(nullspace), use H and back substitution to solve $H\vec{x} = \vec{0}$ and find solution set

$\dim(C) = \dim(R) =$ ~~null~~ number of columns of H with pivots

$\dim(N) =$ number of pivot-free columns of H.

2.3 Linear transformation of Euclidean spaces

**RANK**

The dimension of the column space = The dimension of row space is called the rank of a matrix A. rank(A)

**NULLITY**

The dimension of the nullspace of A is called the nullity of A. ~~m~~ nullity(A)

**RANK EQUATION**

A be $m \times n$ matrix row equivalent to H H in REF or RREF

$$\text{rank}(A) + \text{nullity}(A) = n \leftarrow \# \text{columns}$$

**INVERTIBILITY CRITERION**

An $n \times n$ matrix A is invertible $\Leftrightarrow$ rank(A) = n

**LINEAR TRANSFORMATION**

A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it ~~to~~ satisfies

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ Preservation of addition

$T(r\vec{u}) = r\,T(\vec{u})$ Preservation of scalar multiplication

**IF $T: \mathbb{R}^n \to \mathbb{R}^m$ IS A LINEAR TRANSFORMATION**

$\mathbb{R}^n$ = domain of T $\qquad \mathbb{R}^m$ = codomain of T

$W \subset \mathbb{R}^n$, the image of W under T is $T[W] = \{T(\vec{w}) \mid \vec{w} \in W\}$ smaller set of range

the range of T is $T[\mathbb{R}^n] = \{T(\vec{v}) \mid \vec{v} \in \mathbb{R}^n\}$

$W' \subset \mathbb{R}^m$, the reverse image of W' under T is $T^{-1}[W'] = \{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) \in W'\}$

**KERNEL OF T**

The set $T^{-1}[\{\vec{0}'\}] = \{\vec{v} \in \mathbb{R}^n \mid T(\vec{v}) = \vec{0}'\}$ (where $\vec{0}'$ is the zero vector in $\mathbb{R}^m$)

i.e. the null space

**BASES AND LINEAR TRANSFORMATIONS**

$T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

if $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ then $T(r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_k\vec{v}_k)$

$$= r_1 T(\vec{v}_1) + r_2 T(\vec{v}_2) + \dots + r_k T(\vec{v}_k)$$

$T(\vec{0}) = \vec{0}'$ where $\vec{0} \in \mathbb{R}^n$, $\vec{0}' \in \mathbb{R}^m$

if $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, for any $\vec{v} \in \mathbb{R}^n$, $T(\vec{v})$ can be expressed uniquely as $r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \dots + r_3 T(\vec{b}_n)$

if $\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_k)\}$ is linearly independent then $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is also linearly independent

## FINDING A GENERAL RULE FOR LINEAR TRANSFORMATION

1. Express the general vector ~~ant~~ ~~in~~ as a linear combination of the given vectors. $\vec{v} = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots r_k\vec{v}_k$
2. Find scalars $r_1, r_2, \ldots, r_k$ by solving the linear system of equations with $\vec{v}_i$ as the column vectors of augmented matrix A on the left and entries in $\vec{v}$ on the right
3. Rewrite $\vec{v}$ ~~as~~ as a the linear combination with solutions of $r_i$ and Perform a linear transformation to the equation $T(\vec{v}) = r_1 T(\vec{v}_1) + \cdots r_k T(\vec{v}_k)$

## STANDARD MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

Let $T:\mathbb{R}^n \to T:\mathbb{R}^m$ be a linear transformation, A be $m \times n$ matrix

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix} \quad T(x) = A\vec{x} \;\; \forall \vec{x} \in \mathbb{R}^n.$$

A is the standard matrix representation of T

### PROPERTIES

~~rak~~ rank (T) $\Leftrightarrow$ rank (A) nullity (T) = Ker (T) $\Leftrightarrow$ nullity (A)

T is invertible if $m=n$ and A is invertible. In that case, the standard matrix representation of $T^{-1}$ is $A^{-1}$.

### RANK EQUATION

rank (T) + nullity (T) = dim (domain of T)

### PRESERVATION OF SUBSPACES

$T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

if W is a subspace of $\mathbb{R}^n$, then T[W] is a subspace of $\mathbb{R}^m$

if W' is a subspace of $\mathbb{R}^m$, then T[W'] is a subspace of $\mathbb{R}^n$

## TYPES OF LINEAR TRANSFORMATION

$T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then T is

Every column has pivot

one-to-one $\to$ $T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$ OR $\vec{u} \neq \vec{v} \Rightarrow T(\vec{u}) \neq T(\vec{v})$

[ ] $\to$ Ker(T) = Null (A) = $\{\vec{0}\}$ where $\vec{0}$ is the zero vector in $\mathbb{R}^n$

Every row has pivot

onto $\to$ $T[\mathbb{R}^n] = \mathbb{R}^m$ OR $\forall \vec{v}' \in \mathbb{R}^m \; \exists \; \vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{v}'$ OR range of $T = \mathbb{R}^m$

[ ] $\to$ $T[\mathbb{R}^n]$ = column space (A) = $\mathbb{R}^m$ OR rank (A) = m = dim (col (A))

Every row and column has pivot

isomorphism $\to$ both one-to-one and onto

[ ] $\to$ nullspace (A) = $\{\vec{0}\}$ and rank (A) = m , m = n

4.1
Areas, Volumes and Cross Products

## DETERMINANTS

$2\times 2 \quad \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad det = a_1b_2 - a_2b_1$

$3\times 3 \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \cancel{b}a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

## CROSS PRODUCT

$\vec{a} = [a_1, a_2, a_3] \qquad \vec{b} = [b_1, b_2, b_3] \in \mathbb{R}^3$

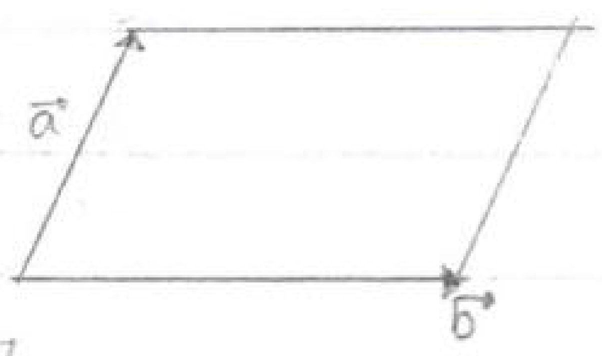
$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = [a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_1]$$

which is a vector perpendicular to both $\vec{a}$ and $\vec{b}$

## WHAT DETERMINANTS AND CROSS PRODUCTS REPRESENT GEOMETRICALLY

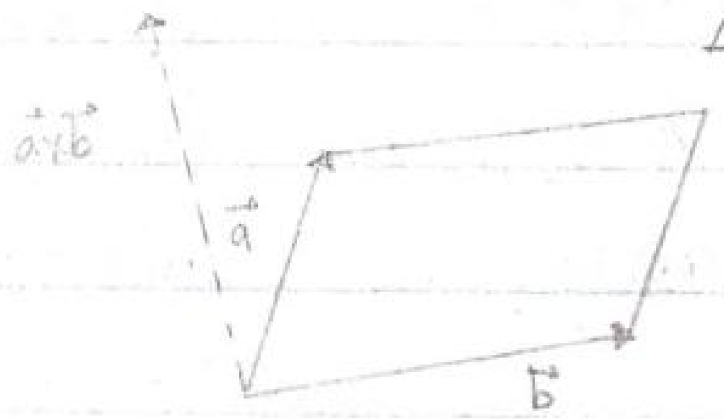
$\vec{a} = [a_1, a_2] \quad \vec{b} = [b_1, b_2] \in \mathbb{R}^2$

Area of parallelogram $= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ (absolute value) in 2D



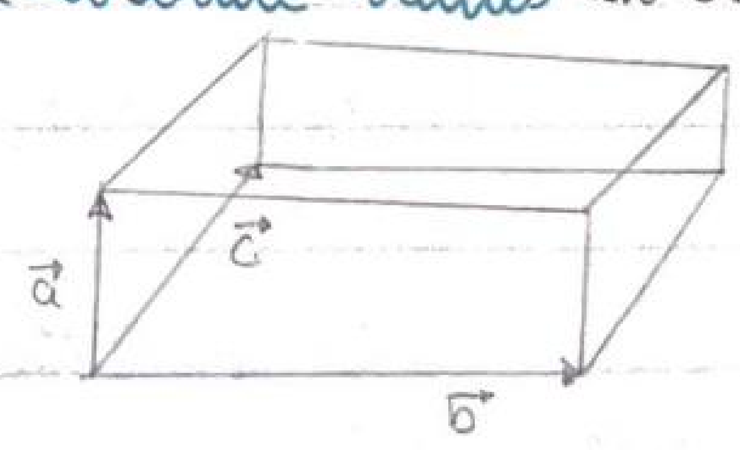

$\vec{a} = [a_1, a_2, a_3] \quad \vec{b} = [b_1, b_2, b_3] \in \mathbb{R}^3$

Area of parallelogram $= \|\vec{a}\times\vec{b}\|$ in 3D




$\vec{a} = [a_1, \cancel{b}a_2, a_3] \quad \vec{b} = [b_1, b_2, b_3] \quad \vec{c} = [c_1, c_2, c_3] \in \mathbb{R}^3$

Volume of parallelepiped $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (absolute value) in 3D




## PROPERTIES OF THE CROSS PRODUCT

$\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$

| | |
|---|---|
| $\vec{b}\times\vec{c} = -(\vec{c}\times\vec{b})$ | Anticommutativity |
| $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$ (usually) | Nonassociativity of cross product |
| $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ | Distributive law |
| $(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c})$ | Distributive law |
| $\vec{b}\cdot(\vec{b}\times\vec{c}) = (\vec{b}\times\vec{c})\cdot\vec{c} = 0$ | Perpendicularity of $\vec{b}\times\vec{c}$ to $\vec{b}$ and $\vec{c}$ |
| $\|\vec{b}\times\vec{c}\|$ = Area of parallelogram | Area property |
| $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = \pm$ volume of parallelepiped | Volume property |
| $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\cancel{b}\,\vec{a}\cdot\vec{b})\vec{c}$ | Formula for computation of $\vec{a}\times(\vec{b}\times\vec{c})$ |

**4.2 The Determinant of a square Matrix**

## MINOR MATRIX

The minor matrix $A_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained by removing the $i$th row and $j$th column of $A$.

## COFACTOR

$a'_{ij} = (-1)^{i+j}\det(A_{ij})$ and the determinant of a matrix can be determined by a cofactor expansion across any row or any column

## PROPERTIES OF DETERMINANTS

$A$ be an $n\times n$ matrix

| | |
|---|---|
| $\det(A) = \det(A^T)$ | transpose property |
| if $A$ is a triangular matrix, then $\det(A) = \prod_{i=1}^{n} a_{ii}$ | |
| $A \overset{R_i \leftrightarrow R_j}{\sim} B$, $(i \neq j)$ $\det(B) = -\det(A)$ | row interchange property |
| if $A$ has 2 equal rows then $\det(A) = 0$ | equal rows property |
| $A \overset{R_i \to sR_i}{\sim} B$ $\det(B) = s\det(A)$ | scalar multiplication property |
| $A \overset{R_i \to R_i + sR_j}{\sim} B$ $\det(B) = \det(A)$ | row addition property |
| $A$ is invertible $\Leftrightarrow \det(A) \neq 0$ | invertibility criterion |
| $\det(AB) = \det(A)\det(B)$ | multiplicative property |
| if $A$ is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$ | |

## PROPERTY OF COFACTOR

$A$ be $n\times n$ matrix, $a'_{ji}$ be cofactor of $a_{ji}$

i. $a_{i1}a'_{j1} + a_{i2}a'_{j2} + \cdots + a_{in}a'_{jn} = \begin{cases} \det(A) & i=j \leftarrow \text{expanding across } i^{th} \text{ row} \\ 0 & i\neq j \leftarrow \text{replacing } i^{th} \text{ row by } j^{th} \text{ row} \end{cases}$

ii. $a_{1i}a'_{1j} + a_{2i}a'_{2j} + \cdots + a_{ni}a'_{nj} = \begin{cases} \det(A) & i=j \leftarrow \text{expanding across } \cancel{j^{th} \text{ row}}\ i^{th} \text{ column} \\ 0 & i\neq j \leftarrow \text{replacing } i^{th} \text{ column by } j^{th} \text{ column} \end{cases}$

**4.3 Computation of determinants and Cramer's Rule**

## COMPUTATION OF A DETERMINANT

$A$ be $n\times n$ matrix

1. Reduce $A$ to an echelon form, using only row addition and ~~scalar~~ row interchange
2. If any of the matrices in the reduction contains a row of zeros, then $\det(A) = 0$
3. Otherwise, $\det(A) = (-1)^r \cdot$ (Product of pivots) where $r$ is the number of row interchanges performed.

## ADJOINT

An adjoint of an $n\times n$ matrix $A$ is an $n\times n$ matrix with entry $a_{ij} = A'^{T}_{ij}$

$(\operatorname{adj}(A))A = A(\operatorname{adj}(A)) = \det(A)I$ $\quad \operatorname{adj}(A) = (A')^T$ $\quad A' = [a'_{ij}]^T$

## FINDING $A^{-1}$ USING DET(A) AND ADJ(A)

$A$ is $n \times n$ and $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

for 2x2 matrix $\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1} = \frac{1}{(a_1 b_2 - a_2 b_1)} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}$

if $a_1 b_2 - a_2 b_1 \neq 0$

## CRAMER'S RULE

If $A\vec{x} = \vec{b}$ is a system of $n$ linear equations with $n$ unknowns and $\det(A) \neq 0$, then the unique solution $\vec{x} = [x_1, x_2, \cdots, x_n]$ is of the form

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for } k = 1, 2, \cdots n$$

where $B_k$ is the matrix $A$ with $k^{th}$ column replaced by $\vec{b}$.

5.1 Eigenvalues and Eigenvectors

## EIGENVALUES AND EIGENVECTORS (CHARACTERISTIC VALUE AND CHARACTERISTIC VECTOR) (PROPER VALUE AND PROPER VECTOR)

$A$ be $n \times n$ matrix

A scalar $\lambda$ is an eigenvalue of $A$ if there exists a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. The vector $\vec{v}$ is then an eigenvector corresponding to $\lambda$.

## CHARACTERISTIC POLYNOMIAL

$A$ be $n \times n$ matrix

The characteristic polynomial of $A$ is given by $p(\lambda) = |A - \lambda I|$. If $\lambda$ is an eigenvalue of $A$, then the set $E_\lambda = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\} = \{\vec{0}$ and all eigenvectors of $\lambda\}$ is called the eigenspace of $\lambda$. $E_\lambda = \text{nullspace}(A - \lambda I)$

## COMPUTATION OF EIGENVALUES AND EIGENVECTORS

Eigenvalues of $A$ = solutions to the characteristic polynomial $p(\lambda) = |A - \lambda I| = 0$

~~Eigenve~~ After finding the eigenvalues, substitute each value to the homogeneous system $(A - \lambda I)\vec{x} = \vec{0}$ to find the nontrivial solution i.e. nullspace of $(A - \lambda I)$

## EIGENVALUES AND EIGENVECTORS IN TERMS OF LINEAR TRANSFORMATION

$T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation

A scalar $\lambda$ is an eigenvalue of $T$ if there is a nonzero vector $\vec{v}$ in $\mathbb{R}^n$ such that $T(\vec{v}) = \lambda\vec{v}$. The vector $\vec{v}$ is then an eigenvector of $T$ corresponding to $\lambda$.

## PROPERTIES OF EIGENVALUES AND EIGENVECTORS

$A$ be $n \times n$ matrix, $\lambda$ be an eigenvalue and $\vec{v}$ be an eigenvector

$\lambda^k$ is an eigenvalue of $A^k$ and $\vec{v}$ is an eigenvector of $A^k$ corresponding to $\lambda^k$ where $k \in \mathbb{Z}^+$

$A$ is invertible $\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of $A$

$A$ is invertible $\Rightarrow$ $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$ and $\vec{v}$ is an eigenvector of $A^{-1}$ corresponding to $\frac{1}{\lambda}$.

5.3 Diagonalization

## DIAGONIZABLE MATRIX

An $n \times n$ matrix $A$ is diagonalizable if there exists an invertible $n \times n$ matrix $P$ such that $P^{-1}AP$ is a diagonal matrix

## CRITERION FOR DIAGONALIZATION

An $n \times n$ matrix is diagonalizable $\Leftrightarrow$ the matrix has $n$ linearly independent eigenvectors $\Rightarrow$ $A$ has $n$ distinct eigenvalues $\rightarrow$ $A$ is diagonalizable

## INDEPENDECE OF EIGENVECTORS

$A$ be $n \times n$ matrix, $\lambda_1, \lambda_2\ \lambda_3 \cdots \lambda_k$ be distinct eigenvalues with $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ eigenvectors, $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k$ are linearly independent

## SIMILAR MATRICES

$A, B$ are $n \times n$ matrices such that $B = P^{-1}AP$ for some invertible $n \times n$ matrix $P$

$A$ and $B$ are similar matrices

## PROPERTIES OF SIMILAR MATRICES

$\det(A) = \det(B)$

$A$ is invertible $\Leftrightarrow$ $B$ is invertible

$\text{rank}(A) = \text{rank}(B)$

$\text{nullity}(A) = \text{nullity}(B)$

$\det(A - \lambda I) = \det(B - \lambda I) = p(\lambda)$

## ALGEBRAIC MULTIPLICITY AND GEOMETRIC MULTIPLICITY

The algebraic multiplicity is the "number of times" an eigenvalue is a root of the characteristic ~~polysom~~ polynomial; the multiplicity as a root of the characteristic polynomial of $A$

The geometric multiplicity is the dimension of $E(\lambda_i)$

The geometric multiplicity of an eigenvalue of an $n \times n$ matrix $A$ is less than or equal to its algebraic multiplicity

## INDEPENDENCE OF EIGENVECTORS

A vector cannot be an eigenvector for two distinct eigenvalues of a matrix $A$.

The basis of all eigenspaces are linearly independent

## CRITERION FOR DIAGONALIZATION

An $n \times n$ matrix is invertible $\Leftrightarrow$ algebraic multiplicity = geometric multiplicity $\forall \lambda$

## DIAGONALIZATION PROCESS

A be $n \times n$ matrix

1. Find the characteristic polynomial $p(\lambda) = det(A - \lambda I_n)$
2. If A is diagonalizable, factorise $p(\lambda)$ to get $p(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_k)^{n_k}$ in $\mathbb{R}$
   where $n_i$ is the algebraic multiplicity of $\lambda_i$. If cannot be factorised, not diagonalizable
3. For each $\lambda_i$, find $E_{\lambda_i}$ = nullspace of $A - \lambda_i I$ $(A - \lambda_i I)\vec{x} = \vec{0}$ and obtain a basis for the eigenspace. The dimension of $E_{\lambda_i}$ = geometric multiplicity of $\lambda_i$.
   - if all $\lambda_i$ has algebraic multiplicity of 1, A is diagonalizable
   - if all $\lambda_i$ has algebraic multiplicity = geometric multiplicity, A is diagonalizable
4. Create the square matrix P with eigenvectors as column vectors of P, then
   $P^{-1}AP = D$ and D is a diagonal matrix with diagonal entries $\lambda_i$.

## COMPUTATION OF $A^k$

A be $n \times n$ and diagonalizable

1. Find $P$, $P^{-1}$ and $D$ using the above process
2. $A^k = PD^kP^{-1}$

## LINEAR RECURRENCE

For recursive functions we can find the $k^{th}$ term by constructing the following system of $A\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix}$ so $A^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$ where $A^k$ can be determined "easily" by diagonalising A and use $A^k = PD^kP^{-1}$ or writing $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ as a linear combination of the eigenvectors of A so $A^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \lambda_1^k \vec{V_1} + \lambda_2^k \vec{V_2}$

## CAYLEY-HAMILTON THEOREM

A be $n \times n$ with characteristic polynomial $p(\lambda) = |A - \lambda I| = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$

$$p(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

if $a_0 \neq 0$ (the determinant of $A \neq 0$)

$$A^{-1} = \frac{-1}{a_0}[A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I]$$

## SYMMETRIC MATRICES

Every symmetric matrix is diagonalizable

## EQUIVALENT STATEMENTS

A be $n \times n$ matrix

1. A is invertible
2. $\det(A) \neq 0$
3. A is row equivalent to $I_n$
4. A can be written as a product of elementary matrices
5. $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$
6. $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$
7. The columns of A are linearly independent
8. The columns of A span $\mathbb{R}^n$
9. A has a left inverse
10. $\text{rank}(A) = n$
11. $\text{nullity}(A) = 0$
12. The rows of A are linearly indepenelent
13. The rows of A span $\mathbb{R}^n$
14. A has a right inverse
15. 0 is not an eigenvalue of A.

9.1 Algebra of Complex Numbers

## COMPLEX NUMBERS

The set of complex numbers $\mathbb{C}$ is the set of all numbers of the form $x+iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$

## FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial equation with coefficients in $\mathbb{C}$ has n solutions in $\mathbb{C}$, where n is the degree of the polynomial and the solutions are counted with their algebraic multiplicity

## MANIPULATION OF $\mathbb{C}$

$(a+bi) \pm (c+di) = (a \pm c) \pm (b \pm d)i$

$r(a+bi) = ra + rbi$

$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

$z = a+bi \Rightarrow \bar{z} = a-bi$

$z_1 z_2 = |z_1||z_2|(\cos(\text{Arg}(z_1)+\text{Arg}(z_2)) + i\sin(\text{Arg}(z_1)+\text{Arg}(z_2)))$

$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos(\text{Arg}(z_1)-\text{Arg}(z_2)) + i\sin(\text{Arg}(z_1)-\text{Arg}(z_2)))$

## PROPERTIES OF CONJUGATION IN $\mathbb{C}$

$\overline{z+w} = \bar{z} + \bar{w}$ $\quad$ $\overline{zw} = \bar{z}\,\bar{w}$ $\quad$ $\bar{\bar{z}} = z$ $\quad$ $Re(z) = \frac{z+\bar{z}}{2}$ $\quad$ $|\bar{z}|^2 = z\bar{z}$

$\overline{z-w} = \bar{z} - \bar{w}$ $\quad$ $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ $\quad$ $z^{-1} = \frac{\bar{z}}{|z|^2}$ $\quad$ $|z_1 + z_2| \leq |z_1| + |z_2|$

## POLAR FORM OF $\mathbb{C}$

$z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

## FINDING ROOTS OF Z

$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \Rightarrow z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$ for $k = 0, 1, 2, \dots, n-1$