MATC09

0-order logic

SYMBOLS

Countably many objects in 0-order logic language $\mathcal{L}_P$.

1. (,) brackets
2. $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ connectives
3. $\{A_n \mid n \in \mathbb{N}\} = \mathcal{L}_P^S$ sentence symbols

EXPRESSION

A string (or sequence) of finitely many symbols in $\mathcal{L}_P$.
Length of an expression: number of symbols

WELL-FORMED FORMULAS

An expression in $\mathcal{L}_P$ with following characteristics

1. every sentence symbol is a wff ($\mathcal{L}_P^S \subset \mathcal{W}_P$)
2. if α and β are wff then so are the following:

$$\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$$

$$\mathcal{E}_{\wedge}(\alpha,\beta) = (\alpha\wedge\beta) \qquad \mathcal{E}_{\vee}(\alpha,\beta) = (\alpha\vee\beta)$$

$$\mathcal{E}_{\rightarrow}(\alpha,\beta) = (\alpha\rightarrow\beta) \qquad \mathcal{E}_{\leftrightarrow}(\alpha,\beta) = (\alpha\leftrightarrow\beta)$$

3. nothing else is a wff

The set of wffs is called $\mathcal{W}_P$.

CONSTRUCTION SEQUENCE

Sequences $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$, $n \in \mathbb{N}$, $n \geq 1$ s.t. $\forall i = 1, 2, \cdots, n$ at least one of the following is true:

1. $\alpha_i \in \mathcal{L}_P^S$
2. $\alpha_i = (\neg\alpha_j)$ for some $1 \leq j < i$
3. $\mathcal{E}_{\square}(\alpha_k, \alpha_j) = \alpha_i$ where $\square = \{\wedge, \vee, \rightarrow, \leftrightarrow\}$ for some $1 \leq j, k < i$

Expression α is a wff $\Leftrightarrow$ $\exists$ a construction sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_m \rangle$ $\alpha = \alpha_m$. Length of minimal construction sequence is UNIQUE.

INDUCTION PRINCIPLE FOR wff

Let $S \subseteq W_p$. If

1. $\mathcal{L}_p^s \subseteq S$
2. S is closed under the 5 formula building operations

Then $S = W_p$.

SENTENCE SYMBOL SET

For $\alpha \in W_p$, let $S(\alpha) = \{A \in \mathcal{L}_p^s \mid A \text{ occurs in } \alpha\}$ = sentence symbol of α.

For $A \in \mathcal{L}_p^s$, let $\bar{A} = \{\varphi \in W_p \mid S(\alpha) = \{A\}\}$ = all wff that consists only of A.

TRUTH ASSIGNMENT

Let $S \subseteq \mathcal{L}_p^s$. $S \neq \emptyset$. A truth assignment is a function $v: S \to \{F, T\}$ (Mapping a sentence symbol to boolean value F/T)

If $S = \{A_i\}_{i=1}^n$ is a finite subset of $\mathcal{L}_p^s$ with n elements, then S has 2^n different TAs.

If S is an infinite subset of $\mathcal{L}_p^s$, then S has uncountably many TA.

If $S \neq \emptyset$, $S \subset E \subseteq \mathcal{L}_p^s$, then a TA $w: E \to \{F, T\}$ is also a TA for S.

$\bar{S}$, wffs THAT IS MADE FROM $A \in S$.

Let $S \subseteq \mathcal{L}_p^s$, $S \neq \emptyset$. $\bar{S} = \{\alpha \in W_p \mid \text{If } A \in S \text{ then } A \text{ occurs in } \alpha\}$

$\bar{S}$ is all elements in W_p (and only those) that can be "built" up from the elements of S. Always countably infinite.

$\bar{v}$

Let $S \subseteq \mathcal{L}_p^s$, let $v: S \to \{F, T\}$. Extend $\bar{v}: \bar{S} \to \{F, T\}$ by:

1. $\bar{v}(A) = v(A)$, $\forall A \in S$
2. Let $\alpha, \beta \in \bar{S}$ and $\bar{v}(\alpha)$; $\bar{v}(\beta)$ be defined.

   We could evaluate $\bar{v}((\neg\alpha))$, $\bar{v}((\neg\beta))$, $\bar{v}((\alpha \wedge \beta))$, $\bar{v}((\alpha \vee \beta))$, $\bar{v}((\alpha \to \beta))$ and $\bar{v}((\alpha \leftrightarrow \beta))$

FUNDAMENTAL TRUTH TABLE

Let $v: S \to \{F, T\}$. $S \subseteq \mathcal{L}_p^s$, $S \neq \emptyset$. $\alpha, \beta \in W_p$. $dom(v) = S$.
Let $\bar{v}(\alpha)$, $\bar{v}(\beta)$ be defined.

$\bar{v}(\alpha)$	$\bar{v}(\beta)$	$\bar{v}((\neg\alpha))$	$\bar{v}((\alpha\wedge\beta))$	$\bar{v}((\alpha\vee\beta))$	$\bar{v}((\alpha\to\beta))$	$\bar{v}((\alpha\leftrightarrow\beta))$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

UNIQUE EXTENSION THM

Let $S \subseteq \mathcal{L}_p^s$, $S \neq \emptyset$ and let $v: S \to \{F, T\}$ be a TA. There exists a unique function $\bar{v}: \bar{S} \to \{F, T\}$ that extends v.

SATISFIES

A TA v satisfies a wff $\varphi \in W_p \Leftrightarrow \bar{v}(\varphi) = T$. (where $dom(v) \subseteq \mathcal{L}_p^s$) ($S(\varphi) \subseteq dom(v)$)

SATISFIABLE

A wff $\alpha \in W_p$ is satisfiable $\Leftrightarrow \exists$ a TA w s.t. $\bar{w}(\alpha) = T$
$\Sigma \subseteq W_p$ is satisfiable $\Leftrightarrow \exists$ a TA w s.t. $\bar{w}(\sigma) = T$ $\forall \sigma \in \Sigma$ (there is a TA that satisfies α)

TAUTOLOGY

An element $\tau \in W_p$ is a tautology $\Leftrightarrow \forall$ TA v st. $S(\tau) \subseteq dom(v)$ we have that $\bar{v}(\tau) = T$

CONTRADICTION

A wff $k \in W_p$ is a contradiction $\Leftrightarrow \forall$ TA w st. $S(k) \subseteq dom(w)$ we have that $\bar{w}(k) = F$.

TAUTOLOGICALLY IMPLIES $\Sigma \models \tau$

Let $\Sigma \subseteq W_p$, $\tau \in W_p$, w say that Σ tautologically implies τ $\Leftrightarrow \forall$ TA v st. $S(\Sigma) \cup S(\tau) \subseteq dom(v)$. $\bar{v}(\tau) = T$ if $\bar{v}(\sigma) = T$ $\forall \sigma \in \Sigma$

TAUTOLOGICALLY EQUIVALENT

Let $\alpha, \beta \in W_p$. α and β are tautologically equivalent $\Leftrightarrow \alpha \models \beta$ and $\beta \models \alpha$ ($\alpha \models\!\dashv \beta$)

$\tau \in W_p$ is a tautology $\Leftrightarrow \emptyset \models \tau$ (vacuousness)

if $\Omega \subseteq W_p$ and Ω not satisfiable then $\Omega \models \varphi \ \forall \varphi \in W_p$ (F → t is T)

BOOLEAN FUNCTION

Let $k \in \mathbb{N}^+$, a k-place boolean function is a function
$f: \{F,T\}^k \rightarrow \{F,T\}$ $\quad (\{F,T\}^k = \{F,T\} \times \{F,T\} \times \cdots \times \{F,T\})$

Thus, $\forall \vec{x} = \langle x_1, x_2, \cdots, x_k \rangle \in \{F,T\}^k$, $f(\vec{x}) = f(x_1, x_2, \cdots, x_k) \in \{F,T\}$

There are 2^{2^k} such functions $\forall k \in \mathbb{N}^+$. (k symbols $\Rightarrow$ 2^k TAs $\Rightarrow$ (T/F) for each of the 2^k to make 1 BF $\Rightarrow 2^{2^k}$

$W_p \rightarrow \mathcal{B}$ (boolean function)

Let $\mathcal{B}^k = \{k\text{-place boolean functions}\}$, $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}^k$.

Let $\alpha \in W_p$, let $S(\alpha) = \{A_1, A_2, \cdots, A_n\}$. The boolean function $\mathcal{B}_\alpha^n$ induced by α as $\mathcal{B}_\alpha^n : \{F,T\}^n \rightarrow \{F,T\}$ where $\mathcal{B}_\alpha^n(x_1, x_2, \cdots, x_n) = \bar{v}(\alpha)$ where $v: S(\alpha) \rightarrow \{F,T\}$ and $v(A_i) = x_i$, $1 \leq i \leq n$

THM 15B

Let $G: \{F,T\}^n \rightarrow \{F,T\}$ be an n-place boolean function, $n \in \mathbb{N}^+$.
There is some $\alpha \in W_p$ s.t. $\mathcal{B}_\alpha^n = G$

$\mathcal{B}_\alpha^n = G \Leftrightarrow \mathcal{B}_\alpha^n(\vec{x}) = G(\vec{x}) \quad \forall \vec{x} \in \{F,T\}^n$

DISJUNCTIVE NORMAL FORM (DNF)

$\{\neg A \mid A \in \mathcal{L}_p^s\}$

Let $\alpha \in W_p$. α has the ~~DF~~ DNF iff $\alpha = \bigvee(\bigwedge \beta_i)$ where $\beta_i \in (\mathcal{L}_p^s \cup \neg \mathcal{L}_p^s)$

(easiest way to obtain a wff from given a boolean function)

ORDERING

For $\{F,T\}$, let $F < T$. Then we can say the following:

1. $\mathcal{B}^n = \{f \mid f: \{F,T\}^n \rightarrow \{F,T\}\}$, Let $G, H \in \mathcal{B}^n$. $G(\vec{x}) < H(\vec{x}) \Leftrightarrow G(\vec{x}) = F$ and $H(\vec{x}) = T$
2. $G < H \Leftrightarrow G(\vec{x}) < H(\vec{x}) \quad \forall \vec{x} \in \{F,T\}^n$

THM 15A

Let $S = \{A_i\}_{i=1}^n \subseteq \mathcal{L}_p^s$. Let $\alpha, \beta \in W_p$ s.t. $S(\alpha), S(\beta) \subseteq S$. Then:

1. $\alpha \models \beta \Leftrightarrow \mathcal{B}_\alpha \leq \mathcal{B}_\beta$
2. $\alpha \models\!\!\dashv \beta \Leftrightarrow \mathcal{B}_\alpha = \mathcal{B}_\beta$
3. $\models \alpha \Leftrightarrow \mathcal{B}_\alpha = T$ (tautology)

$\models\!\!\dashv$ is a equivalence relation on $\bar{\bar{S}} = \{\varphi \in W_p \mid S(\varphi) = S\}$

COMPLETE

Let C be a non-empty set of connectives (any of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ or other generalized connectives")

C is complete $\Leftrightarrow \forall \varphi \in W_p\ \exists \alpha \in W_p$ s.t.

1. $\varphi \models\!\!\!\dashv \alpha$
2. The connectives in α ($C(\alpha)$) are in C, i.e. $C(\alpha) \subseteq C$

e.g. $C = \{\neg, \vee, \wedge\}$ is a complete set of connectives.

FINITELY SATISFIABLE

Let $\Sigma \subseteq W_p$. Σ is finitely satisfiable $\Leftrightarrow$ every subset of Σ is satisfiable.

Σ is fs $\Leftrightarrow \forall \Sigma_0 \subseteq \Sigma$, Σ_0 finite, $\exists$ TA v s.t. $\bar{v}(\sigma) = T$, $\forall \sigma \in \Sigma_0$

if $|\Sigma| = n$, let $m_i = |S(\sigma_i)|$, $1 \le i \le n$, $\sigma_i \in \Sigma = \{\sigma_i\}_{i=1}^{n}$, $|S(\Sigma)| \le \sum_{i=1}^{n} m_i = M$

$\Rightarrow$ we need a truth table with $\le 2^M$ lines to determine whether or not Σ is satisfiable.

- if Σ is finite and Σ is fs, then Σ is satisfiable
- if $\Sigma \subseteq \mathcal{L}_p^s$ then Σ is fs
- if Σ is satisfiable (TA v s.t. $\bar{v}(\sigma) = T\ \forall \sigma \in \Sigma$), then Σ is fs (same v)
- if Σ contains a contradiction or $\exists \gamma \in W_p$ s.t. $\{\gamma, \neg\gamma\} \subseteq \Sigma$, then Σ is not fs and not satisfiable.

THE COMPACTNESS THM

Let $\Sigma \subseteq W_p$. Σ is satisfiable $\Leftrightarrow$ Σ is finitely satisfiable.

THM

Let $\Sigma \subseteq W_p$ be fs. Then $\forall \alpha \in W_p$, $\Sigma; \alpha$ or $\Sigma; \neg\alpha$ is fs.

THM

Let $\Delta \subseteq W_p$ such that

1. Δ is fs
2. $\forall \alpha \in W_p$, $\alpha \in \Delta$ or $\neg\alpha \in \Delta$

Then $\exists$ a TA $v: \mathcal{L}_p^s \rightarrow \{F, T\}$ s.t. $v(A) = T \Leftrightarrow A \in \Delta$

$\Rightarrow \forall \varphi \in W_p$, $\bar{v}(\varphi) = T \Leftrightarrow \varphi \in \Delta$

FINITE TAUTOLOGICAL IMPLICATION (FTI)

Let $\Sigma \subseteq W_p$ and $\alpha \in W_p$. If $\Sigma \models \alpha$ then $\exists$ a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \alpha$.

DEDUCTIONS $\Sigma \vdash \alpha$

Let $\Sigma \subseteq W_p$ (can be empty). A deduction is a finite sequence $\langle \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ such that $\alpha_i \in W_p$ for $i = 0, 1, \cdots, n$ and ~~so~~ α_i's satisfy $\geq$ one of the following:

1. $\alpha_i \in T_p = \{\text{tautologies of } W_p\}$
2. $\alpha_i \in \Sigma$ (think of Σ as "hypotheses")
3. α_i can be created by MODUS PONENS (MP) $(i \geq 2)$
   $\exists j, k \in \{0, 1, 2, \cdots i-1\}$ s.t. $\alpha_j = \alpha_k \to \alpha_i$

If $\alpha_n = \alpha$, then this is a deduction of α from Σ.

e.g. Horizontal Deduction $\Sigma = \{\varphi, \varphi \to \psi\}$ Vertical Deduction

$\langle \varphi \quad \varphi \to \psi \quad \psi \rangle$

1	φ	$(\in \Sigma)$
2	$\varphi \to \psi$	$(\in \Sigma)$
3	ψ	(MP; 1,2) ∎

DEDUCTION THM

Let $\Gamma \subseteq W_p$, $\alpha, \beta \in W_p$. $\Gamma; \beta \models \alpha \iff \Gamma \models \beta \to \alpha$.

SOUNDNESS THM (0-ORDER)

Let $\Sigma \subseteq W_p$. Let $\alpha \in W_p$. If $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$

COMPLETENESS THM (0-ORDER)

Let $\Sigma \subseteq W_p$. Let $\alpha \in W_p$. If $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$

INCONSISTENT

Let $\Gamma \subseteq W_p$. Γ is inconsistent $\iff \exists \varphi \in W_p$ s.t. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$

If Γ is not inconsistent, then Γ is consistent

THM

Γ is consistent $\iff \Gamma$ is satisfiable.

$\nexists \varphi \in W_p$ s.t. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi \iff \exists$ TA v s.t. $\bar{v}(\sigma) = T \ \forall \sigma \in \Gamma$.

no contradictions

1st order logic

There are infinitely many 1st order languages. Let $\mathcal{L}$ be any 1st ~~od~~ order language.

LOGICAL SYMBOLS (countably many, fixed)

1. (,) brackets
2. $\to$, $\neg$ connectives (this is complete; for readability $\wedge, \vee, \leftrightarrow$ will be used as well, even though not in $\mathcal{L}$)
3. v_k, $k \in \mathbb{N}^+$ variables
4. $=$ equality (optional) just a 2-place predicate with same meaning $\forall \mathcal{L}$

PARAMETER SYMBOLS (≥ 1, finitely or infinitely many, varies from $\mathcal{L}_a$ to $\mathcal{L}_b$)

1. $\forall$ universal quantifier
2. $\mathcal{P}$, $n \in \mathbb{N}^+$ set of n-place predicates (≥ 1 (not ≥ 1 for each n)) (relationship amongst objects)
3. $\mathcal{C}$, set of constant symbols (labels object)
4. $\mathcal{F}$, $n \in \mathbb{N}^+$ set of n-place function (mapping of n-variables)

EXPRESSION

Let $\mathcal{L}$ be a 1st order language. An expression is a finite sequence of symbols from $\mathcal{L}$.

TERM

An expression of $\mathcal{L}$ characterized by the following:

1. variables are terms V
2. constants are terms
3. if f is a k-place function symbol and t_i is a term, $1 \leq i \leq k$, then $f t_1 t_2 \cdots t_k$ is a term
4. nothing else is a term $T = \{\text{terms}\}$

ATOMIC FORMULA

An expression of the form $P t_1 t_2 \cdots t_k$, where P is a k-place predicate symbol and t_i, $1 \leq i \leq k$ are terms.

$A = \{\text{atomic formula}\}$, these are the minimal expressions for which we will assign "truthness"/"falseness"

Does not contain ( ), connectives or quantifiers

WELL-FORMED FORMULA

A well-formed formula (formula) of $\mathcal{L}$ is an expression characterized as follows:

1. All atomic formulas are formulas
2. if α is a formula, then $(\neg\alpha)$ is a formula
3. if α, β are a formulas, then $(\alpha \rightarrow \beta)$ is a formula
4. if α is a formula and x is a variable, then $\forall x \alpha$ is a formula
5. Nothing else is a formula.

$W = \{\text{formula}\}$

OCCURS IN

Let $x \in V$ and $\alpha \in W$. x occurs in $\alpha \Leftrightarrow x$ is one of the symbols in the expression that α is. $x \in V(\alpha) = \{\text{variables in } \alpha\}$

FREE

Let $x \in V$. x has a free occurence iff x is not quantified (by $\forall$ or $\exists$)

BOUND

Let $x \in V$. x has a bound location iff x is quantified (by $\forall$ or $\exists$)

STRUCTURE

Let $\mathcal{L}$ be a 1^{st} order language. Let PS (parameter set) $= \{\forall, C, P, F\}$

A structure $\mathcal{U}$ for $\mathcal{L}$ is a "function defined" on PS as follows:

1. $\forall \mapsto |\mathcal{U}|$ (non empty set, universe (or domain) of $\mathcal{U}$)
2. $c \mapsto c^{\mathcal{U}}$, $c^{\mathcal{U}} \in |\mathcal{U}|$ (giving mean to symbol within language)
3. $P \mapsto P^{\mathcal{U}}$, $P^{\mathcal{U}} \subseteq |\mathcal{U}|^n$ ($P^{\mathcal{U}}$ is set of n tuples, $|\mathcal{U}|^n = \{\langle u_1, u_2, \cdots, u_n \rangle \mid u_i \in |\mathcal{U}|\}$
4. $f \mapsto f^{\mathcal{U}}$, $f^{\mathcal{U}} := |\mathcal{U}|^n \mapsto |\mathcal{U}|$

$\mathcal{U} = (|\mathcal{U}|, c^{\mathcal{U}}, p^{\mathcal{U}}, f^{\mathcal{U}})$

SUBSTITUTION / ASSIGNMENT FUNCTION

Let $\mathcal{L}$ be a language and $\mathfrak{U}$ be a structure for $\mathcal{L}$. Let $V = \{\text{variables for } \mathcal{L}\} = \{v_i\}_{i=1}^{\infty}$. An assignment function is a function $s: V \to |\mathfrak{U}|$. $\forall x \in V$, $s(x) \in |\mathfrak{U}|$

- $|\mathfrak{U}| \neq \emptyset \Rightarrow s(x) \in |\mathfrak{U}| \ \forall x \in V$
- $|\mathfrak{U}| = \{a\} \Rightarrow s(x) = a \ \forall x \in V$
- $|\mathfrak{U}|$ has >1 element, then $\mathfrak{U}$ has uncountably many assignment functions.

EXTENDED ASSIGNMENT I

$\bar{s}: T \to |\mathfrak{U}|$ such that

1. $\forall x \in V$, $\bar{s}(x) = s(x)$
2. $\forall$ constant symbols c, $\bar{s}(c) = c^{\mathfrak{U}}$
3. If f is an n-place function symbol and $t_i \in T$, $1 \leq i \leq n$, then $\bar{s}(f(t_1, t_2, \cdots, t_n)) = f(\bar{s}(t_1), \bar{s}(t_2), \cdots, \bar{s}(t_n))$

IDEA OF φ IS TRUE IN $\mathfrak{U}$

II. Let $A = \{\text{atomic formulas}\}$, let $\varphi \in A$ ($\varphi := P(t_1 t_2 \cdots t_n)$ or $\varphi := t_1 = t_2$)

1. if $t_1, t_2 \in T$ then $\mathfrak{U} \models (t_1 = t_2)[s] \Leftrightarrow \bar{s}(t_1) = \bar{s}(t_2)$
2. if P is an n-place predicate symbol and $t_i \in T$, $1 \leq i \leq n$, then $\mathfrak{U} \models P(t_1, t_2, \cdots, t_n)[s] \Leftrightarrow \langle \bar{s}(t_1), \bar{s}(t_2), \cdots, \bar{s}(t_n) \rangle \in P^{\mathfrak{U}}$

III. Let $\alpha, \beta \in W$ and assume we have defined $\mathfrak{U} \models \alpha[s]$ and $\mathfrak{U} \models \beta[s]$.

1. if $\alpha, \beta \in A$, then follow definitions above.
2. $\mathfrak{U} \models (\neg\alpha)[s] \Leftrightarrow \mathfrak{U} \not\models \alpha[s]$ $\quad (\neg(\mathfrak{U} \models \alpha[s]))$
3. $\mathfrak{U} \models (\alpha \to \beta)[s] \Leftrightarrow (\mathfrak{U} \models \alpha[s]) \to (\mathfrak{U} \models \beta[s])$
4. $\mathfrak{U} \models (\alpha \wedge \beta)[s] \Leftrightarrow (\mathfrak{U} \models \alpha[s]) \wedge (\mathfrak{U} \models \beta[s])$
5. $\mathfrak{U} \models (\alpha \vee \beta)[s] \Leftrightarrow \mathfrak{U} \models \alpha[s] \vee \mathfrak{U} \models \beta[s]$
6. $\mathfrak{U} \models (\alpha \leftrightarrow \beta)[s] \Leftrightarrow \mathfrak{U} \models \alpha[s] \leftrightarrow \mathfrak{U} \models \beta[s]$

   assignment of variable x replaced with $d \in |\mathfrak{U}|$

7. Let $x \in V$, $\mathfrak{U} \models \forall x\, \alpha[s] \Leftrightarrow \forall d \in |\mathfrak{U}|$, $\mathfrak{U} \models \alpha[s(x|d)]$
8. Let $x \in V$, $\mathfrak{U} \models \exists x\, \alpha[s] \Leftrightarrow \exists d \in |\mathfrak{U}|$, $\mathfrak{U} \models \cancel{\varphi}\alpha[s(x|d)]$

$S(x|d)$

$S(x|d): V \to |\mathfrak{U}|$ and $s(x|d)(y) = \begin{cases} s(y) & y \neq x \\ d & y = x \end{cases}$

An assignment function s altered by setting the value of x to d and the rest unchanged.

SATISFIABLE

$\varphi \in W$ is satisfiable $\Leftrightarrow$ $\exists$ a structure $\mathfrak{A}$ for the language and an assignment w such that $\mathfrak{A} \models \varphi\,[w]$ is true.

CONTRADICTION

$k \in W$ is a contradiction $\Leftrightarrow$ k is not satisfiable.

i.e. $\forall$ structure $\mathfrak{B}$ and $\forall$ assignment s, $\mathfrak{B} \not\models k\,[s]$.

VALID FORMULA

$\alpha \in W$ is valid $\Leftrightarrow$ $\forall$ structure $\mathfrak{U}$ and $\forall$ assignment s, $\mathfrak{U} \models \alpha\,[s]$.

LOGICAL IMPLICATION

Let $\Gamma \subseteq W$ and $\varphi \in W$. We write $\Gamma \models \varphi$ $\Leftrightarrow$ whenever $\mathfrak{U}$ is a structure and s is an assignment such that $\mathfrak{U} \models \gamma\,[s]$, $\forall \gamma \in \Gamma$, then $\mathfrak{U} \models \varphi\,[s]$ also.

φ is valid $\Leftrightarrow$ $\phi \models \varphi$ $(\models \varphi)$

MODEL

Let $\varphi \in W$. A model of φ is a structure $\mathfrak{B}$ such that $\mathfrak{B} \models \varphi\,[s]$ $\forall$ assignments s. ~~(Mod~~

SENTENCE

$\sigma \in W$ is a sentence $\Leftrightarrow$ $\forall x \in V(\sigma)$, x is bounded.

i.e. σ has no free variables.

THM 22A

Let $\mathcal{L}$ be a language, $\mathfrak{U}$ be a structure for $\mathcal{L}$, and let $\varphi \in W$. Let $s_1, s_2: V \to |\mathfrak{U}|$ be assignment functions such that $s_1(x) = s_2(x)$ whenever $x \in V(\varphi)$ and x is free. Then $\mathfrak{U} \models \varphi\,[s_1] \Leftrightarrow \mathfrak{U} \models \varphi\,[s_2]$

COROLLARY 22B

Let σ be a sentence and let $\mathfrak{U}$ be a structure for $\mathcal{L}$. Then exactly one of the following is true:

1. $\mathfrak{U} \models \sigma[s]$ $\forall$ assignment function $s: V \to |\mathfrak{U}|$
2. $\mathfrak{U} \not\models \sigma[w]$ $\forall$ assignment function $w: V \to |\mathfrak{U}|$

DEFINABILITY IN A STRUCTURE

Let $\mathcal{L}$ be a language and $\mathfrak{U}$ is a structure. Let $R \subseteq |\mathfrak{U}|^k$, $k \in \mathbb{N}$, $k \geq 1$ (R is called a "k-ary" relation) (for $k=2$, R is called a relation). R is definable (in $|\mathfrak{U}|$) iff

$\exists \varphi \in \mathcal{W}$ s.t.

1. if $x \in V(\varphi)$ is free, then $x \in \{v_i\}_{i=1}^{k}$ ($k \in \mathbb{N}, k \geq 1$)
   (i.e. free variables in φ are labelled as $v_1, v_2, \cdots, v_k \in V$)
2. $R = \{\langle a_1, a_2, \cdots, a_k \rangle \mid a_i \in |\mathfrak{U}|, 1 \leq i \leq k$ and $\mathfrak{U} \models \varphi [\![a_1, a_2, \cdots, a_k]\!] \}$

$[\![a_1, a_2, \cdots, a_k]\!]$

Let $s: V \to |\mathfrak{U}|$ be an assignment function. $[\![a_1, a_2, \cdots, a_k]\!]$ is a notation used to mean $s(v_i) = a_i$, $1 \leq i \leq k$ where $v_i \in V(\varphi)$ for some $\varphi \in \mathcal{W}$.

HOMOMORPHISM

Let $\mathcal{L}$ be a language and $\mathfrak{U}$ and $\mathfrak{B}$ be two structures of $\mathcal{L}$. A homomorphism of $\mathfrak{U}$ into $\mathfrak{B}$ is a function $h: |\mathfrak{U}| \to |\mathfrak{B}|$ with these 3 properties

1. $\forall$ c constants in $\mathcal{L}$, $h(c^{\mathfrak{U}}) = c^{\mathfrak{B}}$ ($c^{\mathfrak{U}} \in |\mathfrak{U}|$, $c^{\mathfrak{B}} \in |\mathfrak{B}|$)
2. $\forall$ n-place function symbol f in $\mathcal{L}$,
   $$h(f^{\mathfrak{U}}(a_1, a_2, \dots, a_n)) = f^{\mathfrak{B}}(h(a_1), h(a_2), \dots, h(a_n))$$
   where $a_i \in |\mathfrak{U}|$, $h(a_i) \in |\mathfrak{B}|$
3. $\forall$ n-place predicate symbol P of $\mathcal{L}$, $\forall \langle a_1, a_2, \cdots, a_n \rangle \in |\mathfrak{U}|^n$, we have that $\langle a_1, a_2, \cdots, a_n \rangle \in P^{\mathfrak{U}} \Leftrightarrow$
   $$\langle h(a_1), h(a_2), \cdots, h(a_n) \rangle \in P^{\mathfrak{B}}$$

ISOMORPHISM

$h: |\mathfrak{U}| \to |\mathfrak{B}|$ is an isomorphism iff h is homomorphism and injective (1-1)

INJECTIVE (one to one)

$\forall u_1, u_2 \in |\mathfrak{U}|$ if $u_1 \neq u_2$ then $h(u_1) \neq h(u_2)$.

i.e if $h(u_1) = h(u_2)$ then $u_1 = u_2$

SURJECTIVE (onto)

$\forall b \in |\mathfrak{B}|, \exists u \in |\mathfrak{U}|$ s.t. $h(u) = b$

ISOMORPHIC

Structures $\mathfrak{U}$ and $\mathfrak{B}$ are isomorphic ($\mathfrak{U} \cong \mathfrak{B}$)

$\Leftrightarrow \exists h: |\mathfrak{U}| \to |\mathfrak{B}|$ that is an onto isomorphism.

AUTOMORPHISM

Let $g: |\mathfrak{U}| \to |\mathfrak{U}|$. g is an automorphism if g is onto.

ELEMENTARILY EQUIVALENT $\mathfrak{U} \equiv \mathfrak{B}$

Structures $\mathfrak{U}$ and $\mathfrak{B}$ are elementarily equivalent $\Leftrightarrow$

$\forall$ sentence σ, $\models_{\mathfrak{U}} \sigma \Leftrightarrow \models_{\mathfrak{B}} \sigma$ ($\mathfrak{U} \models \sigma \Leftrightarrow \mathfrak{B} \models \sigma$)

ISOMORPHIC $\Rightarrow$ ELEMETARILY EQ

$\mathfrak{U} \cong \mathfrak{B} \Rightarrow \mathfrak{U} \equiv \mathfrak{B}$

AUTOMORPHISM AND DEFINABILITY

If $g: |\mathfrak{U}| \to |\mathfrak{U}|$ is an automorphism, then a k-ary relation R is definable $\Leftrightarrow$ $\langle a_1, a_2, \cdots, a_k \rangle \in R \leftrightarrow \langle g(a_1), g(a_2), \cdots, g(a_k) \rangle \in R$

TRUTH / SATISFACTION ISOMORPHISM THM (TSI THM)

Let $\mathfrak{U}$ and $\mathfrak{B}$ be structures, $\mathfrak{U} \cong \mathfrak{B}$. Let $h: |\mathfrak{U}| \to |\mathfrak{B}|$ be an onto isomorphism. Let $s: V \to |\mathfrak{U}|$ be an assignment function.

$T \to |\mathfrak{U}|$ $\quad$ $T \to |\mathfrak{B}|$

$$\begin{tikzcd} |\mathfrak{U}| \arrow[r, "h"] & |\mathfrak{B}| \\ V \arrow[u, "s"] \arrow[ur, "h\circ s"'] & \end{tikzcd}$$

1. if $t \in T$, then $h \circ \bar{s}(t) = \overline{h \circ s}(t)$
2. $\forall \alpha \in W$, $\mathfrak{U} \models \alpha[s] \Leftrightarrow \mathfrak{B} \models \alpha[h \circ s]$

ELEMENTARY CLASS

A class K of structures for a language $\mathcal{L}$ is called an elementary class (EC) $\Leftrightarrow$ $\exists$ a sentence τ for $\mathcal{L}$ s.t. $K = \text{Mod}\,\tau$

CLASS OF MODELS

Let σ be a sentence for a language $\mathcal{L}$. $\text{Mod}\,\sigma$ = class of all models of σ. Thus, $\mathfrak{U} \in \text{Mod}\,\sigma \Leftrightarrow \mathfrak{U} \models \sigma$.

EC_Δ

A class of structures K is EC_Δ (elementary class in the wider sense) $\Leftrightarrow$ $K = \text{mod}\,\Sigma$ for some set of sentences Σ of $\mathcal{L}$.

DEDUCTION (FORMAL PROOFS)

Let $\Gamma \subseteq W$ (Γ can be empty) and $\varphi \in W$. A deduction or formal proof of φ from Γ is a finite sequence $\langle \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \rangle$ such that $\alpha_k \in W$, $1 \le k \le n$ and

1. $\alpha_k \in \Gamma \cup \Lambda$, $k = 1, 2, \cdots, n$ Γ = hypothesis , Λ = logical axioms
2. $\exists\, i, j \in \{1, 2, \cdots, k-1\}$ s.t. $\alpha_j = \alpha_i \to \alpha_k$ $(k \ge 3)$ MP
3. $\alpha_n = \varphi$

When such a deduction exists, we write as $\Gamma \vdash \varphi$.
φ is a theorem of Γ and φ is deducible from Γ.

GENERALIZATION

If $\psi \in W$ and $x_i \in V$, $1 \le i \le n$ then $\beta = \forall x_1 \forall x_2 \cdots \forall x_n \psi$ is called a generalization of ψ.

THE 6 AXIOM TYPES

Ax 1

Tautologies $T \subset$ Valid Formulas $V \subset W$

Tautologies (1st order logic) - Let $\gamma \in W_p$ be a tautology in 0-order logic. Let $S(\gamma) = \{A_i\}_{i=1}^{n}$. For each $i = 1, \cdots, n$, let $\alpha_i \in W$. Replace A_i with α_i to obtain $\tau \in W$. τ is a tautology of W.
note: $\lambda \in \Lambda$ belong to Ax 1 $\Rightarrow$ λ is a generalization of τ ($\forall x_1 \forall x_2 \cdots \tau$)

Ax 2

Let $\alpha \in W$, $x \in V$, $t \in T$

$$\forall x \alpha \rightarrow \alpha_t^x$$

where the term t is substitutable for variable x in α (term t does not contain a variable that is bounded in α)

Ax 3

Let $\alpha, \beta \in W$, $v \in V$. (Distribution over implication)

$$\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta)$$

Ax 4

$\alpha \rightarrow \forall x \alpha$, where $\alpha \in W$, $x \in V$ and x is not free in α

Ax 5

If the Language contains $\mathcal{L}$, $\quad x = x \quad \forall x \in V$

Ax 6

Let $x, y \in V$, $\alpha \in A$

$$(x = y) \rightarrow (\alpha \rightarrow \alpha')$$

where α' is derived from α by replacing x with y in any occurence of x that is free

I - DEDUCTIONS

Intuitively, an informal deduction (i-deduction) is a vertical presentation that "a deduction exists"

GEN - THE GENERALIZATION THM

Let $\Gamma \subseteq W$, $\varphi \in W$, $x \in V$. Assume x is not free in any $\gamma \in \Gamma$. If $\Gamma \vdash \varphi$ then $\Gamma \vdash \forall x \varphi$

RULE T THEOREM

Let $\Gamma \subseteq W$, $\{\alpha_i\}_{i=1}^n \subseteq W$, $\beta \in W$. If $\Gamma \vdash \alpha_i \quad \forall i \in \{1, 2, \cdots, n\}$ and if $\{\alpha_i\}_{i=1}^n$ tautologically implies β, then $\Gamma \vdash \beta$

i.e. if $\Gamma \vdash \alpha_i$, $i = 1, 2, \cdots, n$ and $\{\alpha_1, \alpha_2, \dots \alpha_n\} \vDash \beta$, then $\Gamma \vdash \beta$

INCONSISTENT

$\Gamma \subseteq \mathcal{W}$ is inconsistent $\Leftrightarrow$ $\exists \beta \in \mathcal{W}$ s.t. $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$.
(in i-deductions)

Γ is consistent $\Leftrightarrow$ Γ is not inconsistent.

Γ is consistent $\Leftrightarrow$ Γ is satisfiable

RAA COROLLARY 24E

Let $\Gamma \subseteq \mathcal{W}$, $\varphi \in \mathcal{W}$. If $\Gamma ; \varphi$ is inconsistent, then $\Gamma \vdash \neg\varphi$.
(usually when we use RAA to show $\Gamma \vdash \alpha$, we assume $\neg\alpha$ and prove $\Gamma ; \neg\alpha$ is inconsistent)

PRIME FORMULAS

$\mathcal{P} = \{\text{atomic formulas } A\} \cup \{\forall x \alpha,\ x \in V \text{ and } \alpha \in \mathcal{W}\}$ is the set of prime formulas where every $\sigma \in \mathcal{W}$ arises from finite applications of $\neg$, $\rightarrow$ upon elements of $\mathcal{P}$.

Imagine elements in ~~R~~ $\mathcal{P}$ as sentence symbols. Then the tautologies $\mathcal{T}$ of 1st order logic are exactly the ~~t~~ elements of $\mathcal{T}_{\mathcal{P}}$ founded on the sentence symbols $\mathcal{P}$.

CONTRAPOSITION COR 24D

$\Gamma ; \varphi \vdash \neg\psi \Leftrightarrow \Gamma ; \psi \vdash \neg\varphi \quad (\varphi \rightarrow \neg\psi \Leftrightarrow \psi \rightarrow \neg\varphi)$

THM 24B

$\Gamma \vdash \varphi \Leftrightarrow (\Gamma \cup \Lambda) \models_0 \varphi$

TAUTOLOGICAL IMPLICATION

Let $\Gamma \subseteq \mathcal{W}$, $\varphi \in \mathcal{W}$. $\Gamma \models_0 \varphi$ (tautological implication) $\Leftrightarrow$ $\forall$ TA $\bar{v} : \Gamma ; \varphi \rightarrow \{F, T\}$
$\bar{v}(\gamma) = T \ \forall \gamma \in \Gamma$, we have that $\bar{v}(\varphi) = T$

where $v : X \rightarrow \{F, T\}$, X is the prime formulas in Γ and φ

$\Gamma \models_0 \varphi \Rightarrow \Gamma \models \varphi$ but not $\Leftarrow$

LEMMA 25A AXIOM VALIDITY THM

Every logical axiom is valid (If $\lambda \in \underbrace{\Lambda}_{\text{set of logical axiom}}$, then $\lambda \in \underbrace{\mathcal{V}}_{\text{set of valid formulas}}$)

SOUNDNESS THM

Let $\Gamma \subseteq W$ and $\varphi \in W$. If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

COMPLETENESS THM (GÖDEL)

Let $\Gamma \subseteq W$ and $\varphi \in W$, If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

COMPACTNESS THM (FINITE TAUTOLOGICAL IMPLICATION)

If $\Gamma \subseteq W$ and $\varphi \in W$ and if $\Gamma \models \varphi$ then $\exists \Gamma_0 \subseteq \Gamma$ such that Γ_0 is finite and $\Gamma_0 \models \varphi$.

DEDUCTION THM DT $\vdash$.

$\Sigma; \alpha \vdash \beta \iff \Sigma \vdash \alpha \to \beta$