Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Spring 2020 Term Project

Date Due: 12:00 PM (noon), Friday, the 30th of April 2020

JOINT INSTITUTE
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General Information

The goal of this term project is to help you apply your new-found knowledge of Green's functions and differential equatuons in extended tasks that are beyond the scope of ordinary assignments. You are free to choose any one of the 10 projects described below. Some place more emphasis on pure mathematics, others on physics and one is focused on programming. If you have a suggestion for some other project that you are interested in doing instead, please feel free to let me know and we can discuss it.

It is strongly recommended that you do not leave the entire project to the last minute but rather commence work on individual parts as soon as you are able to do so.

Project Report

The term project will be submitted **electronically only** as a typed report of **no more than 20 pages** in length. Handwritten submission will not be accepted! It is recommended that you use a professional type-setting program (such as LaTeX) for your report. Unless you are able to ensure a unified font size and style for formulas and text in Microsoft Word, use of Word is *not recommended*.

Your report should have the appearance, style and contents of a professional report. It should be comprehensible without reference to this document and should be comprehensible by any other student in this course. It is strongly suggested that all members of the project team proof-read the report before submission. **The report should not look like the solution to an assignment.** Do not structure the section titles as "Answer to Question i)" or similarly.

Grading Policy

This term project accounts for 50% of the course grade; it will be scored based on

- **Form (10 points):** Does the report contain essential elements, such as a cover page (with title, date, list of authors), a synopsis (abstract giving the main conclusions of the project), table of contents, clear section headings, introduction, clear division into sections and appendices with informative titles and bibliography (if applicable)? Are the pages numbered? Are the text and formulas composed in a unified font? Are all figures (graphs and images) clearly labeled with identifiable source?
- **Language (10 points):** Is the style of english appropriate for a technical report? Do not treat the project as an assignment and simply number your results like part-exercises. Your text should be a single, coherent whole. The text should be a pleasant read for anyone wanting to find out about the subject matter.

  Errors in grammar and orthography (use a spell-checker!) will be penalized. Make sure that the report is interesting to read. Avoid simply repeating sentences by cut-and-paste.
- **Content (30 points):** Are the mathematical and statistical methods and deductions clearly exhibited and easy to follow? Are the conclusions well-supported by the mathematical analysis? It is important to not just copy calculations from elsewhere, but to fully make them your own, adding details and comments where necessary.

On Plagiarism

Study JI's Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text. Follow the rules regarding the correct attribution of sources that you have learned in your English course (e.g., Vy100, Vy200). You are responsible for the correct attribution of all sources in all parts of the project essay, i.e., any plagiarism will be considered a violation of the Honor Code.

The following list includes some specific examples of plagiarism:

- Use of any passage of three words or longer from another source without proper attribution. Use of any phrase of three words or more must be enclosed in quotation marks ("example, example, example"). This excludes set phrases (e.g., "and so on", "it follows that") and very precise technical terminology (e.g., "without loss of generality", "reject the null hypothesis") that cannot be paraphrased,
- Use of material from an uncredited source, making very minor changes (like word order or verb tense) to avoid the three-word rule.
- Inclusion of facts, data, ideas or theories originally thought of by someone else, without giving that person (organization, etc.) credit.
- Paraphrasing of ideas or theories without crediting the original thinker.
- Use of images, computer code and other tools and media without appropriate credit to their creator and in accordance with relevant copyright laws.

Important: Please read this

You should write the report out as a single coherent paper, citing all sources. **Do not** treat this like as assignment and write it in a style like "Exercise 1: Solution:" etc. Rather, write it as a harmonious report detailing your investigations into this subject. Throughout your text, you should use the notation that we have developed in our course and adapt the notation/nomenclature as appropriate (e.g., x' becomes ξ, a Green function on an infinite domain with no boundary conditions is referred to as a fundamental solution, etc.)

1 The Helmholtz Equation from Periodic Heating

The basis for this project is the section of the *Library of Green's Functions* on the Helmholtz equation arrived at from the heat equation with steady-periodic heating in $(1 + 1)$ dimensions [1].

Using Cole's article [1] and the references given therein, do the following:

i) Explain what "steady-periodic heating" is and derive the relevant Helmholtz equation. Explain in detail what is meant by "the solution is interpreted as the steady-periodic response at a single frequency ω."

ii) Derive the solution formula [1, (5)] using the methods we have studied in our class.

iii) Derive the Green's functions given for the case "Plate, 1-D, Steady-Periodic GF" with X11, X12 and X13.

iv) Derive the Green's functions for the case "Semi-infinite body, 1-D, Steady-Periodic GF" with X10, X20 and X30. The case X00 ("Infinite Body, 1-D, Steady-Periodic GF") has been treated in class - have a look at the solution given there and comment.

v) Construct some examples to illustrate the solution formulas.

vi) Elaborate on the above according by your own volition.

References

[1] Cole, K. D. Helmholtz equation: steady periodic. In: Green's function library. `http://www.greensfunction.unl.edu/glibcontent/stharm/stharm.html`, August 10, 2004. [Online; accessed April 5, 2020].

2 The $(1 + 1)D$ Heat Equation on the Semi-Infinite Line

The basis for this project is the section of the *Library of Green's Functions* on the Green's function for the heat equation on the semi-infinite line [1].

i) Derive the Green's functions given for the cases X10, X20 and X30 using whichever method seems most appropriate.

ii) Attempt to derive the Green's function for the cases X40 - this is not a case we treated in class.

iii) Construct some examples to illustrate the solution formulas.

iv) Elaborate on the above according by your own volition.

References

[1] Cole, K. D. Semi-infinite body, transient 1-D. In: Green's function library. `http://www.greensfunction.unl.edu/glibcontent/node4.html`, December 31, 2002. [Online; accessed April 5, 2020].

3 The $(1+1)D$ Heat Equation on a Bounded Interval I

The basis for this project is the section of the *Library of Green's Functions* on the Green's function for the heat equation on a bounded interval [1].

i) Derive the two Green's function expansions for the case X11 using whichever methods seem most appropriate (method of images, partial eigenfunction expansion).

ii) Explain why the two expansions converge well for small vs. large $t-\tau$.

iii) Attempt to show directly that the two expansions yield the same solution formula (this may be difficult and may involve some complex analysis or other techniques).

References

[1] Cole, K. D. Plate, transient 1-D. In: Green's function library. `http://www.greensfunction.unl.edu/glibcontent/node5.html`, December 31, 2002. [Online; accessed April 5, 2020].

4 The $(1+1)D$ Heat Equation on a Bounded Interval II

The basis for this project is the section of the *Library of Green's Functions* on the Green's function for the heat equation on a bounded interval [1].

i) Derive the two Green's function expansions for the case X21 using whichever methods seem most appropriate (method of images, partial eigenfunction expansion).

ii) Explain why the two expansions converge well for small vs. large $t-\tau$.

iii) Attempt to show directly that the two expansions yield the same solution formula (this may be difficult and may involve some complex analysis or other techniques).

References

[1] Cole, K. D. Plate, transient 1-D. In: Green's function library. `http://www.greensfunction.unl.edu/glibcontent/node5.html`, December 31, 2002. [Online; accessed April 5, 2020].

5 The $(1+1)D$ Heat Equation on a Bounded Interval III

The basis for this project is the section of the *Library of Green's Functions* on the Green's function for the heat equation on a bounded interval [1].

i) Derive the two Green's function expansions for the case X22 using whichever methods seem most appropriate (method of images, partial eigenfunction expansion).

ii) Explain why the two expansions converge well for small vs. large $t-\tau$.

iii) Attempt to show directly that the two expansions yield the same solution formula (this may be difficult and may involve some complex analysis or other techniques).

References

[1] Cole, K. D. Plate, transient 1-D. In: Green's function library. `http://www.greensfunction.unl.edu/glibcontent/node5.html`, December 31, 2002. [Online; accessed April 5, 2020].

6 The $2D$ Laplace Equation on an Infinite Strip

The basis for this project is the model calculation to be found in Assignments 12-13 at `http://umji.sjtu.edu.cn/~horst/greenfunctions/pde.html`.

i) Derive the Green's function eigenfunction expansions for the Laplace equation on the infinite strip $S = \mathbb{R} \times (0, a)$, $a > 0$, in $\mathbb{R}^2$ with Neumann boundary conditions, i.e., the problem

$$-\Delta u = F, \qquad x \in S, \qquad \frac{\partial u}{\partial x_2}\Big|_{x_2=0} = h(x_1), \qquad \frac{\partial u}{\partial x_2}\Big|_{x_2=a} = f(x_1), \qquad x_1 \in \mathbb{R}.$$

Use both the x_1 and the x_2 eigenfunctions.

ii) Derive the explicit solution formula.

iii) Attempt to show directly that the two expansions yield the same solution formula (this may be difficult and may involve some complex analysis or other techniques).

References

7 The $2D$ Helmholtz Equation on a Semi-Infinite Strip

The basis for this project is the model calculation to be found in Assignments 12-14 at `http://umji.sjtu.edu.cn/~horst/greenfunctions/pde.html`.

i) Investigate the fundamental solution for the Helmholtz equation in two space dimensions (see, for example, [1, Section 5.2]; this will involve a so-called Hankel function.

ii) Derive two Green's function expansions for the Helmholtz equation on the semi-infinite strip $S = [0, \infty) \times (0, a)$, $a > 0$, in $\mathbb{R}^2$ with Neumann boundary conditions, i.e., the problem

$$-\Delta u = F, \qquad x \in S, \qquad \frac{\partial u}{\partial x_2}\Big|_{x_2=0} = h(x_1), \qquad \frac{\partial u}{\partial x_2}\Big|_{x_2=a} = f(x_1), \qquad x_1 \in \mathbb{R}.$$

Use both an eigenfunction expansion and a series obtained from the method of images.

iii) Derive the explicit solution formula.

iv) Attempt to show directly that the two expansions yield the same solution formula (this may be difficult and may involve some complex analysis or other techniques).

References

[1] Evans, G., Blackledge, J., and Yardley, P. *Analytic Methods for Partial Differential Equations.* Springer Undergraduate Mathematics Series. Springer-Verlag, London, 1999.

8 The Helmholtz Equation and Scattering

This is very much a physical topic, based on the discussion in [1, Chapter 5]. While the book contains exactly what is required to treat this topic, it is full of physical jargon, slightly unusual notation and conventions used in physics - for example, it talks about "incoming" and "outgoing waves", about "free" Green's functions etc. So the main task here is to understand what the authors are saying, to translate the notation and terminology into what we are using in our course and, not least, to fill in all the gaps in the calculations. Some complex analysis (residue theory) may be required.

i) Investigate the fundamental solution for the Helmholtz equation in one, two and three space dimensions. How is the Helmholtz equation related to the wave equation? What are the physical interpretations of these fundamental solutions?

ii) Discuss the use of "incoming" and "outgoing" waves; in what sense are these fundamental solutions?

iii) Does the treatment actually use Green's functions? How is the solution formula obtained and how are boundary conditions dealt with? How is the "reciprocity theorem" obtained?

iv) Describe the application to scattering theory and the Born approximation as well as you can.

References

[1] Evans, G., Blackledge, J., and Yardley, P. *Analytic Methods for Partial Differential Equations.* Springer Undergraduate Mathematics Series. Springer-Verlag, London, 1999.

9 The Helmholtz Equation and Diffraction

This is very much a physical topic, based on the discussion in [1, Chapter 5]. While the book contains exactly what is required to treat this topic, it is full of physical jargon, slightly unusual notation and conventions used in physics - for example, it talks about "incoming" and "outgoing waves", about "free" Green's functions etc. So the main task here is to understand what the authors are saying, to translate the notation and terminology into what we are using in our course and, not least, to fill in all the gaps in the calculations.

i) The basis of this discussion are the Maxwell equations, which yield the scalar wave equation for the potential of an electromagnet wave. Summarize the derivation of this wave equation and explain its physical significance.

ii) Derive the fundamental solution for the Helmholtz equation in three dimensions as used in the text.

iii) Discuss Kirchhoff, Fraunhofer and Fresnel diffraction based on this fundamental solution. Discuss the solution formula used as well as approximations and assumptions that are made.

iv) Apply your results to the scattering of light at a circular aperture. You shoukd obtain a result featuring a Bessel function.

References

[1] Evans, G., Blackledge, J., and Yardley, P. *Analytic Methods for Partial Differential Equations.* Springer Undergraduate Mathematics Series. Springer-Verlag, London, 1999.

10 The Boundary Element Method

This is a topic for those interested in programming. The basis can be found in Ang's book [1]. The relevant chapters are available for free on the author's web page.

i) In class we derived Green's function for the Dirichlet problem for the Laplace equation on the half-disk in the plane,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1,\ y > 0\}.$$

Write down an analytic solution for the problem

$$-\Delta u = 0 \qquad \text{on } \Omega, \qquad\qquad u(x, y)|_{x^2+y^2=1} = 1, \qquad\qquad u(x, 0) = 0 \quad \text{for } -1 < x < 1.$$

ii) Produce a program in Mathematica that implements the boundary element method using the standard fundamental solution for the Laplace operator in $\mathbb{R}^2$ for the above problem. Compare the numerical solution with the analytic solution.

iii) Produce a program in Mathematica that implements the boundary element method using Green's function for the upper half-plane (not the Green's function obtained above) for the above problem. Compare the numerical solution with the analytic solution.

iv) For a domain of your own choosing in $\mathbb{R}^2$ or $\mathbb{R}^3$ implement the BEM and investigate further.

References

[1] Ang, W. T. *A Beginner's Course in Boundary Element Methods.* Universal Publishers, Boca Raton, USA, 2007.