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Language as an intellectual tool: From hieroglyphics to APL

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Language as an intellectual tool: From hieroglyphics to APL

by D. B. McIntyre

We learn elementary mathematics before understanding the source of its symbols and procedures, which therefore appear, incorrectly, to have been decreed ready-made. Language and reason are intimately related, and the embodiment of an idea in a symbol may be essential to its comprehension. APL unifies algebra into a single consistent notation; it allows us to exploit the powerful concepts of functions and operators; and it helps us to escape from the tyranny of scalars by giving us the tools to think in terms of arrays, or multiple quantity, as J. J. Sylvester so eloquently urged us to do a century ago. APL has an intellectual consistency that is a source of satisfaction and pleasure. This paper traces the history of symbols from hieroglyphics to APL.

The APL language, a language with symbols and not words, is one of the intellectual triumphs of our time. Its modern incarnation began with Iverson notation,^{1,2} but its roots go far back into the past.

In the beginning

Perhaps the earliest record of what came to be APL was carved on a sculptured mace of granite about 3100 BC, before the invention of papyrus. Of course you cannot read it, unless as is the case with contemporary APL, you know the meaning of the symbols.

We shroud in mystery whatever we do not understand. In crystal optics we speak of "extra-ordinary" rays, though there is, of course, nothing extra-ordinary about them. Negative numbers were called absurd or fictitious. Even after Leonardo of Pisa (known as Fibonacci), in the year 1202, had taught us to recognize debt as a negative asset, it took another 400 years before the number scale was represented geometrically. Intellectual progress is slow, and an additional 250 years passed before Sylvester showed how absurd it was to style as imaginary the quantities represented by the symbols i , j , k of "complex" numbers and quaternions.

I remind you of the words of Whitehead: "Mathematics is often considered to be a difficult and mysterious science, because of the numerous symbols which it employs. Of course, nothing is more incomprehensible than symbolism we do not understand."³

The inscription illustrated in Figure 1 is a record of the triumph of Menes, founder of the first dynasty

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Figure 1 An example of early hieroglyphics

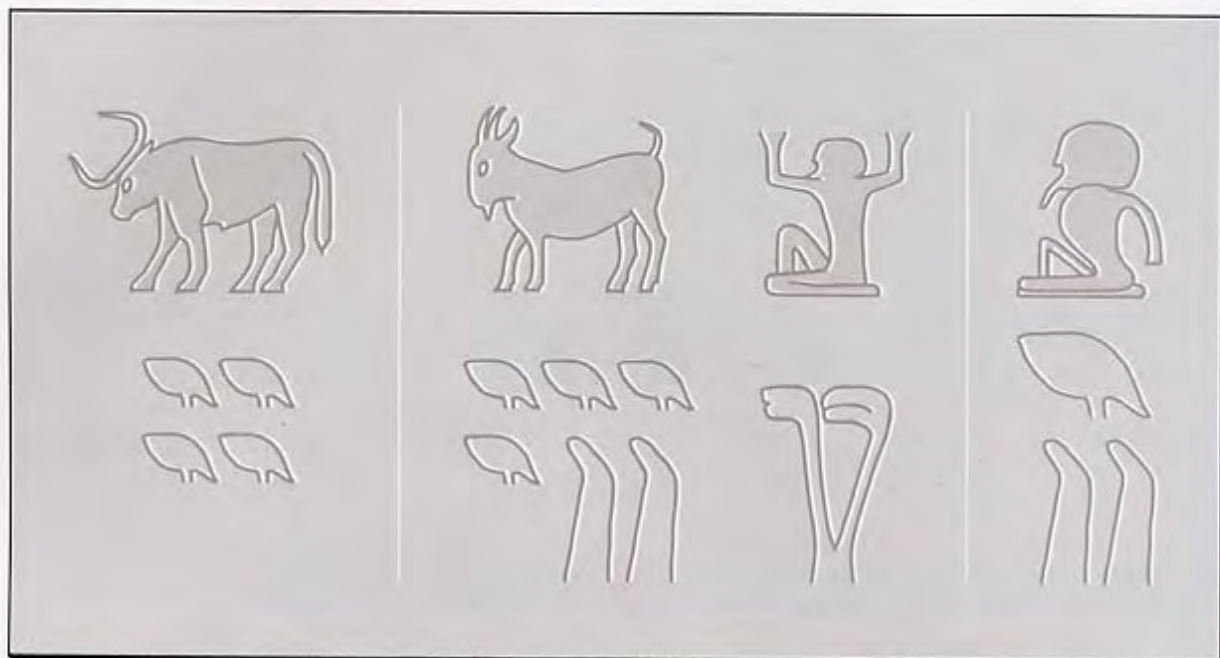
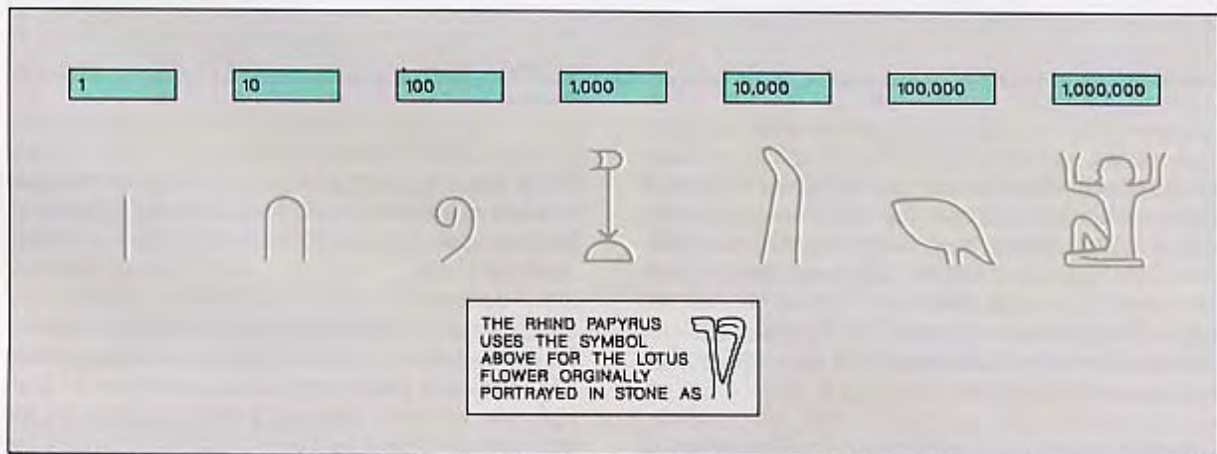


Figure 2 The key to hieroglyphic numbers

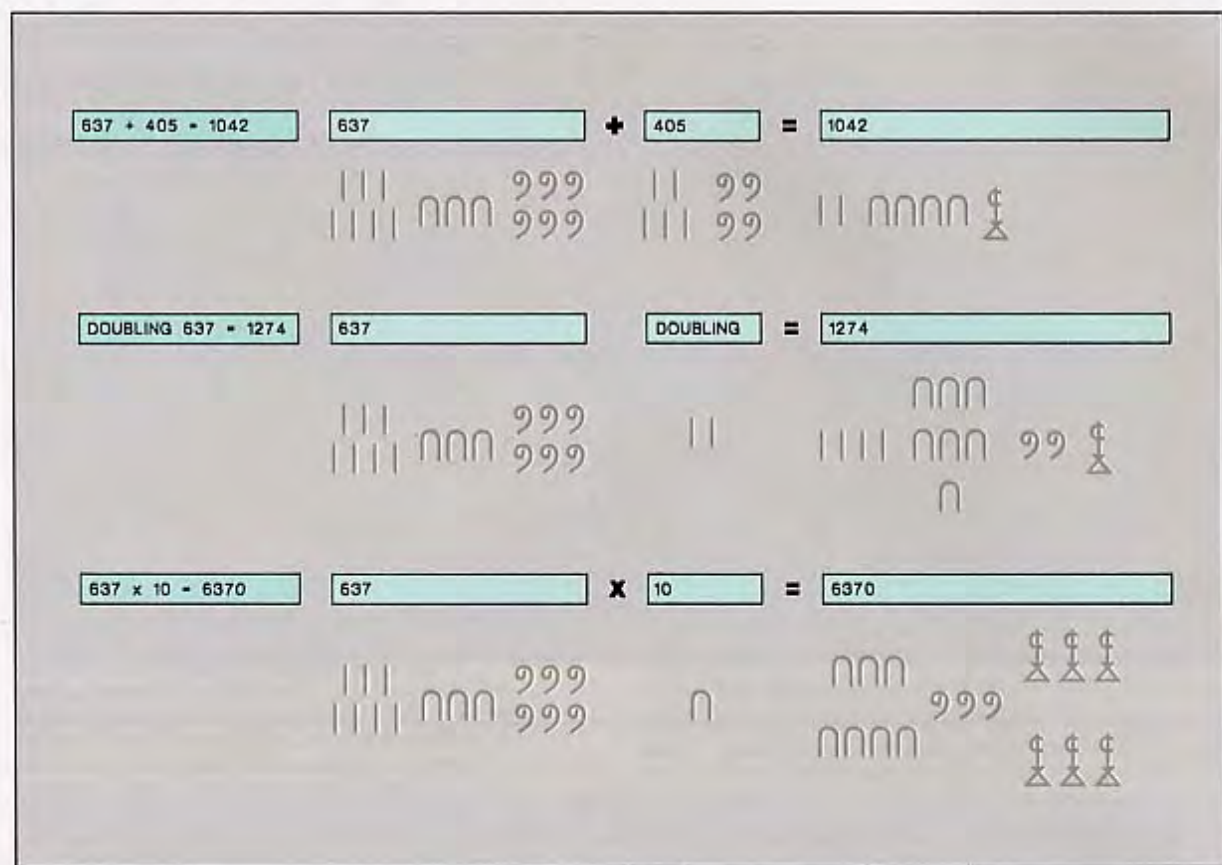


of historical pharaohs, who united the two kingdoms of Egypt. With Figure 2 as our key, we read that he captured 400 000 oxen, 1 422 000 goats, and 120 000 prisoners.

Although the variables are named, the example lacks the equivalent of APL's assignment arrow. A

hundred is represented in hieroglyphics by a picture of the coiled rope used by Egyptian surveyors, or "rope-stretchers," whose descendants today use the "chain" as a unit of measurement. We should remember that Eratosthenes, the director of the great library in Alexandria, was the first to measure the earth's circumference, thus initiating the science of

Figure 3 Examples of Egyptian methods of arithmetic



geophysics. Lotus flowers and tadpoles represent large numbers, and one can only hold up one's hands in amazement at so large a number as a million. The base is, of course, 10. Poor though 10 is as a base,⁴ it was and remains popular because we have 10 fingers to count on. The Egyptian system, like the Roman, did not use place notation, and so had no need for zero.

Egyptian methods of arithmetic are illustrated in Figure 3, reading the symbols from right to left, i.e., the more significant figures are to the right. The three examples represent: adding 637 and 405; doubling 637; and multiplying 637 by 10. The system has been derided as clumsy, but for more than a thousand years no nation was able to improve on the Egyptian notation and methods.⁵ Again, Figure 2 is the key to understanding the notation in Figure 3. This system makes addition, subtraction, dou-

bling, and multiplying by 10 easy. We, on the other hand, must memorize 55 combinations in order to add, and we must learn another table in order to multiply.

Most of us probably imagine that children always learned addition and multiplication tables, but in 1542 *Recorde* had to explain at length how to multiply two numbers between 5 and 10. Consider the implication of Samuel Pepys's entry in his diary for July 4, 1662: "Comes Mr. Cooper of whom I intend to learn mathematiques, and do begin with him today. After an hour's being with him at arithmetique, my first attempt being to learn the multiplication table." Five days later he records: "Up by four o'clock, and at my multiplicacion-table [sic] hard, which is all the trouble I meet withal in my arithmetique." Now Pepys was a 30-year-old graduate of Cambridge, an able man of business, soon

Figure 4 First appearance of symbols in print

Plus, Minus	+ -	Johann Widman	1489
Equals	=	Robert Recorde	1557
Times	×	William Oughtred	1631
Greater than, Less than	> <	Thomas Harriot	1631
Exponentiation	A^{111}	James Hume	1636
Greater than or equal to, Less than or equal to	$\geq \leq$	John Wallis	1655
Divide	÷	Johann H. Rahn	1659
Summation	Σ	Leonard Euler	1755
Factorial	!	Christian Kramp	1808
Absolute Value	A	K. Weierstrass	1841
Membership	∈	Giuseppe Peano	1889
Logical Not	~	Giuseppe Peano	1893
Inclusive Or	∨	Whitehead and Russell	1909
And	∧	Alfred Tarski	1933

to become a Fellow of the Royal Society (as president of the society, Pepys gave his imprimatur to Newton's "Principia"). As Secretary of the Navy he became one of the nation's leading financiers.

How seldom do we look back in maturity at what we learned by rote as children, and that is why I like the title (as well as the content) of Klein's *Elementary Mathematics from an Advanced Standpoint*.⁶ We are taught as if the common mathematical symbols came to humankind in antiquity engraved on stone; as if they had no history. The dates when some of these symbols first appeared in print show that our notation evolved over centuries^{7,8} (see Figure 4). The imprints on our bank checks show that in our own time technology has changed some of our familiar symbols.

The acceptance of symbols

The symbol for *plus* is probably an abbreviation for the Latin *et*, and that for *minus* may be "a simple bar used by merchants to separate the indication of the tare, for a long time called 'minus,' from that of the total weight of the merchandise."⁹ De Morgan thought the symbols might be marks on sacks or barrels showing whether they were over or under weight. Recorde, in 1557, first used these signs in an English book, the same one in which he gave us the

equals symbol, which he chose "because noe 2 thynges can be moare equalle." Euler's Σ (sigma) suggests *summation*; epsilon is the first letter of the Greek *esti* (is a), which suggests *membership*; and the symbol for *or* is the first letter of the Latin *vel*.

In his survey of the development of mathematics, Kline pointed out that Leibniz "certainly appreciated the great saving of thought that good symbols make possible. Thus by the end of the seventeenth century, the deliberate use of symbolism—as opposed to incidental or accidental use—and the awareness of the power and generality it confers entered mathematics."¹⁰

Our notation having been at least 500 years in the making, it is no surprise that the story is not yet at an end. What is remarkable is that Iverson is apparently the first to look at the consistency and completeness of the notation as a whole. Function syntax is inconsistent; e.g., *summation* has its argument to the right, *factorial* to the left, and *absolute value* is written on both sides of its argument. *Exponentiation* has no symbol at all; its second argument is merely written as a superscript. Iverson also considered which other functions have sufficient utility to warrant separate graphic symbols. He showed that function names should not be elided, and pointed out the advantage of each symbol rep-

Figure 5 Exponents enclosed in circles (Stevinus, 1585)

$$3x^2 - 5x + 1 \quad 3(2) - 5(1) + 1(0)$$

$$34.735 \quad 34(0) 7(1) 3(2) 5(3)$$

representing related monadic and dyadic functions. Iverson simplified syntax by abandoning function hierarchy (originally imposed for writing polynomials) and making each function take everything to its right as its right argument.

Acceptance of good symbols has, however, never been easy. After introducing the *times* symbol (Saint Andrew's cross) in 1631, Oughtred wrote: "This manner of setting downe Theoremes, whether they be Proportions, or Equations, by Symboles or notes of words, is most excellent, artificiall, and doctrinall [i.e., serving to teach]. Wherefore I earnestly exhort every one, that desireth though but to looke into these noble Sciences Mathematicall, to accustome themselves unto it: and indeede it is easie, being most agreeable to reason, yea even to sence. And out of this working may many singular conseq[ue]ntaries [i.e., conclusions] be drawne: which without this would, it may be, for ever lye hid."¹¹

But 15 years later, still more encouragement was needed: "[My] Treatise being not written in the usuall synthetical manner, nor with verbous expressions, but in the inventive way of Analitice, and with symboles or notes of things instead of words, seemed unto many very hard; though indeed it was but their owne diffidence, being scared by the newness of the delivery; and not any difficulty in the thing it selfe. For this specious [i.e., pleasing to the eye] and symbolically manner, neither racketh the memory with multiplicity of words, nor chargeth the phantasie with comparing and laying things together; but plainly presenteth to the eye the whole course and processe of every operation and argumentation."¹¹

It seems that not much has changed, judging from the experience of Giuseppe Peano (who provided two of APL's symbols). We are told that he "used a great deal of symbolism because he wished to sharpen the reasoning. . . . Peano used this symbolism in his presentation of all of mathematics,

notably in his *Formulario mathematico* (5 vols., 1895–1908). He used it also in his lectures, and his students rebelled. He tried to satisfy them by passing all of them, but that did not work, and he was obliged to resign his professorship at the University of Turin."¹²

Smith, quoting Nesselmann's *Algebra of the Greeks* (1842), says that mathematics evolves through three stages: *rhetorical*, with words and sentences in full; *syncopated*, in which words are condensed by abbreviation; and *symbolic*, in which there are no words at all.^{13,14} Consider the way we write equations. Comparison of 20 examples from 1463 to 1693¹⁵ shows how long it took to pass from words to our present symbolic system. Simon Stevin (Stevinus, 1548–1620), for instance, made great progress by identifying exponents, writing them enclosed in circles (See Figure 5). His books (1585, 1586) were influential in promoting the use of the new methods. (See Reference 16.)

The superscript method of denoting a to the power b (that is, a^b) was used by Hume in 1636, though his use of Roman numerals for the exponent shows he thought only of integer powers. The form we use now was first used by Descartes in 1637. John Wallis, a distinguished predecessor of Sylvester's as Savilian Professor of Geometry in Oxford, was one of the first to write equations in the form we use today, though even he often wrote $aaaa$ for a^4 . Until the end of the eighteenth century it was, indeed, common practice to write aa for a^2 . Wallis, who gave us our symbols for *greater-than-or-equal-to* (\geq) and *less-than-or-equal-to* (\leq) and our symbol for infinity (∞), found a meaning for negative exponents (1655, 1657), but Newton was the first to permit the exponent to be positive, negative, integer, or fractional (1676).

Euler, in 1777, introduced the symbol i (impossible or imaginary) for $\sqrt{-1}$, and by 1837 Sir William Rowan Hamilton had so adopted the geometrical interpretation of complex numbers (Wessel, Gauss, Argand) that it could be said that exponentiation had been extended to the case of a negative number with a fractional exponent. Cayley further extended the scope of exponentiation by raising matrices to positive integer powers and to the power -1 , which he called the "inverse or reciprocal" matrix.^{17,18} Today's APL handles all these cases directly.

To indicate that a word was abbreviated, the practice used to be to put a stroke (solidus) through the last letter. This accounts for the lines still seen in

symbols for the British pound (£, Latin *libra*), the dollar (\$, an abbreviation of *pesos*), the cent (¢), and the sign R (for the Latin *recipe*, or the imperative "take") displayed by pharmacists.^{19,20} Cardan used R for "root" (Latin *radix*) in 1539, and we still talk of "extracting" (pulling out) the root. Although Euler believed the square root symbol ($\sqrt{}$) to be the deformed letter *r* (abbreviating *radix*), Cajori doubts this, suggesting its origin might be a dot.²¹

We are taught that it is a simple step from exponents to logarithms, and few developments have been more important. Laplace recognized our immense debt to Napier in his well-known remark about logarithms, that, by halving the labor, they had doubled the life of the astronomer and mathematician; but we seldom think of the primitive state of the conceptual tools available in 1614, or recognize Napier's genius. In his day, algebra differed little from arithmetic, and the notation we take for granted was almost nonexistent. Napier's discovery came three years before he invented the decimal point, and less than 60 years after Recorde introduced the equals sign and first used the signs + and - in an English book. Just how Napier succeeded in calculating his table of logarithms is well described by Gittleman.²²

In a volume commemorating the 300th anniversary of Napier's *Description of the Marvellous Canon of Logarithms*, Glaisher well expressed the power of good notation: "Nothing in the history of mathematics is to me so surprising or impressive as the power it has gained by its notation or language. . . . By his invention [of logarithms] Napier introduced a new function into mathematics. . . . When mathematical notation has reached a point where the product of n x s was replaced by x^n , and the extension of the law $x^m \cdot x^n = x^{m+n}$ has suggested $x^{1/2} \cdot x^{1/2} = x$, so that $x^{1/2}$ could be taken to denote the square root of x , then the fractional exponents would follow as a matter of course, and the tabulation of x in the equation $10^x = y$ for integral values of y might naturally suggest itself as a means of performing multiplication by addition. But in Napier's time, when there was practically no notation, his discovery or invention was accomplished by mind alone without any aid from symbols."²³ (See also Reference 24.)

"We who live in an age when algebraical notation has been extensively developed can realise only by an effort how slow and difficult was any step in mathematics until its own language had begun to

arise, and how great was the mental power shown in Napier's conception and its realisation. . . . In our days when the rules of computation are precise, and when the construction of instruments has reached a high state of efficiency, the processes of multiplication and other arithmetical operations can be performed by machines designed for the purpose. These apparatuses which save mental strain and time are effective aids to calculation, and they may be regarded as the modern successors to Napier's rods."²³

APL and functional programming

APL's concise notation helps us grasp the intellectual content of an algorithm without the distraction of extraneous and irrelevant matters prescribed by a machine. APL is a succinct and admirably consistent language that not only uses verbs (functions) to act on nouns (data arrays), but uses adverbs and conjunctions (operators) to derive new verbs, and permits definition of new verbs, adverbs, and conjunctions. It has the subtlety and suggestiveness which, as Bertrand Russell said, makes a good notation "seem almost like a live teacher,"²⁵ and, to quote Pledge, "Suggestiveness is the essential service of symbolism."²⁶

With APL, the goal of *functional programming* (Backus, 1978) can be achieved. The word *function* (derived from *functio*, meaning a performance or execution) was used at the end of the 17th century by mathematicians writing in Latin. Leibniz, who gave us many terms such as *constant*, *variable*, and *parameter*, used "function" in our sense in 1673. Euler used the symbol f for a function in 1734, and in 1754 used the notation $f(a,n)$ for a function of the variables a and n , i.e., to state that the result depends upon the current values of a and n . Iverson does better than this; in 1976 his method of *direct definition*²⁷ of functions shows formally exactly how the result is derived from the arguments, and Euler's parentheses are not needed.

The relationship between ordinary APL and direct definition is illustrated by the following examples:

In ordinary APL:

```

      VZ← A PLUS B
[1]   Z← A + B
[2]   V
      3 PLUS 4
7

```



```

      VZ← F N
[1]  ⍺(N=0)/'→0, 0ρZ+1'
[2]  Z←N× F N-1
[3]  ∇

```

24 F 4

In direct definition:

```

      PLUS: α + ω
      3 PLUS 4
7
      F: ω × F ω-1 : ω=0 : 1
F
      F 4
24

```

The left and right arguments are denoted α and ω . The recursive definition of the factorial should be read: "The factorial of ω is ω times the factorial of $\omega-1$ unless ω equals zero, in which case the factorial of ω is 1."

To illustrate the advantage of Iverson's method, consider the problem of cluster analysis. Each entity, described by n variables, can be considered a point in n -dimensional space, and we are required to compute the distance between each point and all the others. If n is 2, the data are given in a matrix of two columns. We then represent each entity as a point, with coordinates x and y , plotting the points on a scatter diagram. The theorem of Pythagoras lets us determine the distance between any two points, and the results complete a square matrix. This *similarity matrix* gives the *closeness* of each entity to every other one based on all measured properties. The matrix is symmetric with zeros on the diagonal. In APL the algorithm automatically extends to higher dimensions.

Hellerman used this as an example of APL notation, in a book that (in both of its editions) is a landmark in the history of APL.²⁸ His solution is as follows:

```

      VZ←DISTANCE P;N;I;J
[1]  N←(ρP)[0]
[2]  D←(N,N)ρ0
[3]  J←0
[4]  L0:I←0
[5]  L1:D[I;J]←(P[I;]-P[J;])
      +.×(P[I;]-P[J;])
[6]  →(N>I+I+1)/L1
[7]  →(N>J+J+1)/L0
      ∇

```

Direct definition allows this to be expressed in a single line:

```
DSQ: (0 2 1 2⊞ω∘.-ω)+.×2
```

If this line seems strange or unduly terse to someone new to APL, I would point out that if we already know how to add, subtract, and square numbers, there are only three APL functions to learn: inner and outer products and dyadic transposition. I remember Adin Falkoff saying that good notation cannot make an inherently recondite concept easy, but it can remove unnecessary impediments by expressing the concept in as simple a manner as possible: Einstein's $E = m \times c^2$ is a simple statement of a relationship that probably can be fully understood by very few.

For a further illustration consider eight statistical functions, first in standard APL notation:

```

      VZ←MEAN X
[1]  Z←(+/X)÷0⊥ρX
[2]  ∇

      VZ←DEV X
[1]  Z←X-(MEAN X)∘.+ (0⊥ρX)ρ0
[2]  ∇

      VZ←SS X
[1]  Z←(DEV X)+.×2
[2]  ∇

      VZ←VAR X
[1]  Z←(SS X)÷-11+0⊥ρX
[2]  ∇

      VZ←SD X
[1]  Z←(VAR X)*0.5
[2]  ∇

      VZ←SP X;M
[1]  Z←M+.×⊞M←DEV X
[2]  ∇

      VZ←COV X
[1]  Z←(SP X)÷-11+0⊥ρX
[2]  ∇

      VZ←COR X;S
[1]  Z←(COV X)÷S∘.×S+SD X
[2]  ∇

```

They define the means, deviations from the means, sums of squares of the deviations, variances, standard deviations, sums of cross products, covariances, and correlation coefficients.

The functions form a pedagogic sequence in the sense that to understand any one of them you must first understand those that precede it. Each function can be directly defined in a single line, and each takes the original data as its argument.

Next, in direct definition:

```
MEAN:(+/w)÷0⊥pw
DEV:w-(MEANw)°.(0⊥pw)p0
SS:(DEVw)+.×2
VAR:(SSw)÷~1+0⊥pw
SD:(VARw)×0.5
SP:M+.×QW+DEVw
COV:(SPw)÷~1+0⊥pw
COR:(COVw)÷S°×S÷SDw
```

Using Iverson's new dialect J,²⁹⁻³¹ the same functions can be defined even more succinctly, and without parentheses. Not only are no variables assigned, no explicit reference is made to the arguments. This is *tacit definition*, or pure functional programming (Backus, 1978), which leads to efficient execution and invites parallel processing. (Version 3.3 of Iverson's J is used for the examples that follow.)³²

```
mean=.,/w#
dev=.-mean
ss=.,/w*:dev
var=.ss%<:@#
sd=.%:@var
sp=.,/ .~|:dev
cov=.sp%<:@#
cor=.cov%*/~@sd
```

The sequence of functions starts with the mean and ends with the correlation coefficient. Is this structured programming? Is it top down or bottom up? Such questions seem to vanish in a sequence that is almost self-documenting.

The style of programming brings to mind the words of Babbage: "The almost mechanical nature of many of the operations of Algebra, which certainly contributes greatly to its power, has been strangely

misunderstood by some who have even regarded it as a defect. When a difficulty is divided into a number of separate ones, each individual will in all probability be more easily solved than that from which they spring. In many cases several of these secondary ones are well known, and methods of overcoming them have already been contrived: it is not merely useless to re-consider each of these, but it would obviously distract the attention from those which are new: something very similar to this occurs in Geometry; every proposition that has been previously taught is considered as a known truth, and whenever it occurs in the course of an investigation, instead of repeating it, or even for a moment thinking on its demonstration, it is referred to as a known datum. It is this power of separating the difficulties of a question which gives peculiar force to analytical investigations, and by which the most complicated expressions are reduced to laws and comparative simplicity."³³

Revisiting our roots

Being aware of the long history of functions in mathematics, and having seen examples written in current APL, we can now use APL to illuminate our roots, which reach back to Egyptian hieroglyphics. The word *algorithm*, according to the Oxford English Dictionary, is an erroneous refashioning of *algorism*, a word derived from "al-Khowarazmi, the native of Khowarazm, surname of the Arab mathematician who flourished early in the 9th Century, and through the translation of whose work on Algebra, the Arabic numbers became generally known in Europe." In its original form it was used by Chaucer, and the Oxford dictionary cites the use of *algorithm* in 1774.^{34,35} I found it first used by Sylvester³⁶ in one of the earliest papers to speak of *matrices* (compare References 27 and 37 for APL treatment of polygons and polyhedra).

The earliest known book of algorithms is the Rhind Papyrus, based on work written 2000-1800 BC and copied by Ahmes the scribe in 1650 BC.³⁸⁻⁴⁰ It is a textbook on solving practical problems. Consider a simple example, shown in Figure 6 and using Figure 2, again, as the key; to multiply 12 by 12 begin by writing down 12, and by successive doublings obtain 1, 2, 4, and 8 times 12. Check the rows 4× and 8× (on the papyrus the check marks are red) and add them to get the required result. The symbol preceding the answer is a rolled-up scroll (*quod erat demonstrandum*), which in fancy we may take as the ancestor of our *equals* and APL's *assignment* symbols.⁴¹

Figure 6 Rhind Papyrus problem 32; multiplying 12 by 12



16, then subtract $6\times$ and add $1\times$. One may ask why the products used by System/360 are $1\times$, $2\times$, $3\times$, and $6\times$ instead of the $1\times$, $2\times$, $4\times$, $8\times$ used by the Egyptians. When I raised this question in a lecture in New York in 1982, John Macpherson (who was the first to implement binary coded decimal on an IBM computer) gave me the explanation in engineering terms.

However unfamiliar its symbols may be to us, the hieroglyphic message is inherently simple. So it is with the symbols of APL, all of which stand for well-known or easily understood operations. Many today, as Oughtred found 350 years ago, are "scared by the newness of the delivery; and not by any difficulty in the thing itself"!

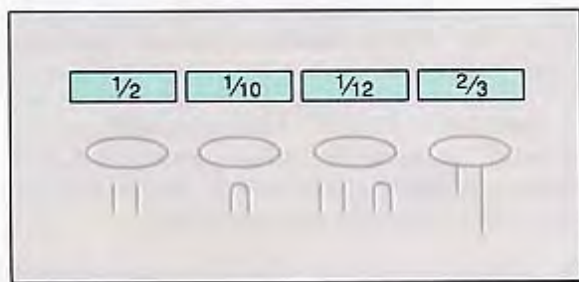
The ancient Egyptians used mathematics for practical purposes, such as paying wages and collecting taxes. Consider the instructive example of salary distribution at the Temple of Illahun—not paid in salt (as the word "salary" implies) but in jugs of beer and loaves of bread. Division, of course, often produces fractions, and the hieroglyphic way to represent fractions can be seen in Figure 7.

All fractions were represented as unit fractions, i.e., with a numerator of 1. Even $2/3$, which seems like an exception, was represented as the unit fraction $1/1.5$. The eye-like symbol is perhaps the earliest of all APL function symbols. It is the reciprocal, or *monadic divide*, which in APL has become an eye closed into a slit, with dots above and below (\div).

If a loaf of bread is divided into 10 parts, and you are to get 1 share, your portion is $1/10$; if you are to get 2 shares your portion is $1/5$; and if you are to get 5 shares your portion is $1/2$. From these simple fractions, other shares can be computed by combination.⁴² For example, 3 shares are the same as $1 + 2$ shares, i.e., $1/5 + 1/10$; 4 shares are the same as $2 + 2$ shares, i.e., $1/5 + 1/5$, which, by consulting a table of values of $2/n$, is set down as $1/3 + 1/15$.

Sylvester became interested in the unit fractions of the Egyptians when reading "the chapter in Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions. It was their curious custom to resolve every fraction into a sum of simple fractions according to a certain traditional method, not leading, I need hardly say, except in a few of the simplest cases, to the expansion under the special form to which I have the name of a fractional sorites."⁴³

Figure 7 Hieroglyphic method of showing fractions



IBM's System/360* and its descendants use this ancient method to multiply integers. Microcode for fixed-point multiplication builds the $1\times$, $2\times$, $3\times$, and $6\times$ products of the multiplicand in local storage. Then, just as the scribes did nearly 4000 years ago, it combines the products corresponding to the multiplier. If the multiplier is 8 or more, a shift of 4 is first made (corresponding to multiplication by 16), and then products are subtracted rather than added; e.g., to multiply by 11, first shift to multiply by

Sylvester's algorithm is expressed in APL with tolerance as left argument:

```

      ∇Z←T F X
[1]  ⍺(X≤T)/'→0,0ρZ+10'
[2]  Z←Z, T F X→Z+1÷X
[3]  ∇

```

Sylvester's example is:

```

      1E-16 F 335+336
2 3 7 48

```

In direct definition, this leads to a useful paradigm for writing recursive functions in APL:

```

      F:Z,αFω→Z+1÷ω : ω≤α 10
      (○1) = +/÷1E-16 F ○1
1

```

Roger Hui (in a personal communication) translated this into the *purely functional* form in J, using @. for *agenda*:

```

      f=. 1.@8: ' (>.@ @ 1 , [f ]->.&.%@) @.<:
      1e_16 f 335%336
2 3 7 48

```

The initial result of the function must be the identity element for the primary function, which for catenation is an empty array of the appropriate shape—in the case of Sylvester's algorithm this is an empty vector.

An example using recursion

A good way to introduce recursion is by one of the oldest of all algorithms: the calculation of pi by approximating inscribed (and circumscribed) polygons.⁴⁴ The symbol pi (π) was chosen by William Jones (1706) because pi is the length of the perimeter of a circle of unit diameter. An inscribed hexagon has 6 sides each of length 0.5, which gives 3 as the first approximation.⁴⁵

Doubling the sides of the hexagon gives a better approximation, and further doublings give still closer values. The secret is, therefore, to compute the length of a new chord from the length of an old one, which is not difficult to do once the theorem of Pythagoras is known. CH gives the new chord as a function of the old one.

$\nabla Z \leftarrow CH\ X$

```

[1]  Z←(0.5×1-(1-X*2)*0.5)*0.5
[2]  ∇

```

For a circle of unit diameter, the first approximation is given by the perimeter of a hexagon whose sides are each equal to the radius, i.e., the approximation to pi is 3.

After 8 doublings (8 applications of CH), pi is given by:

```

      6×(2×2×2×2×2×2×2)×CH
      CH CH CH CH CH CH CH CH 0.5
3.14159

```

We have a notation (exponentiation) that allows us to abbreviate this to:

```

      6×(2^8)×CH CH CH CH CH
      CH CH CH 0.5
3.14159

```

With APL we can use recursion to effect successive applications of the function CH:

```

      ∇Z←N C X
[1]  ⍺(N=0)/'→0, 0ρZ+ X'
[2]  Z←(N-1) C CH X
[3]  ∇

```

```

      ∇Z←PI N
[1]  Z←6×(2*N)× N C 0.5
[2]  ∇

```

In direct definition these functions can be given more concisely:

```

      CH:(.5×1-(1-ω*2)*.5)*.5
      C:(α-1)C CHω: α=0 : ω

```

```

      PI:6×(2*ω)× ω C 0.5

```

```

      PI 8
3.14159

```

Because Iverson's J includes primitives for square root (%:), halve (-:), and square (*:), and a conjunction (dyadic operator) for raising a function to a power (^:), we have the following formulation:

```

      ch=. '%: -: 1- %: 1- *: y.' : ''
      6*(2^8)*ch ch ch ch ch ch ch ch 0.5
3.14159
      6*(2^8)*(ch^:8) 0.5
3.14159

```


Though the ancient Egyptians used *heap* as a general term for an unknown quantity,^{46,47} Diophantus, a Greek mathematician in Alexandria about 300 AD, was probably the original inventor of an algebra using letters for unknown quantities.⁴⁸ Diophantus used the Greek capital letter *delta* (not for his own name!) for the word *power* (“*dynamis*”; compare “*dynamo*,” “*dynamic*,” and “*dynamite*”), which is therefore one of the oldest terms in mathematics.⁴⁹ Today we use a conjunction to raise a function to a power. The syntax brings out the parallelism between raising a number to a power and applying a function an equal number of times. The algorithm fails when the number of doublings is further increased.⁴⁹

Hindu-Arabic numerals and zero

Hindu-Arabic numerals were introduced to the western world by Leonardo of Pisa (Fibonacci) in 1202 with these words: “*Novem figure Indorum he sunt 9 8 7 6 5 4 3 2 1. Cum his itaque nouem figuris, et cum hoc signo 0, quod arabic cephirum appellatur, scribitur quilibet numeros.*” [The nine numerals of the Indians are these: 9 8 7 6 5 4 3 2 1. With them and with this sign 0, which in Arabic is called cipher, any desired number can be written.⁵⁰] (Slightly different in Reference 51.)

It was, however, far easier for most people to add and subtract with Roman numerals (or with Egyptian hieroglyphics for that matter), and this was sufficient for their needs. They also believed that, with the new system, accounts could be more easily falsified—for instance by changing zero into 6 or 9. Adoption of the new symbols was therefore very slow. The oldest known Hindu-Arabic numerals on a gravestone are dated 1371, and their earliest use on coins outside Italy was in 1424. They were not used on an English coin until 1551.⁵² Even today Roman numerals are used for royalty. Clocks not powered by digital technology still commonly display old-style symbols on their dials.

As long as calculations were performed on *counters* or *boards* (see the etymology of *bank* and *bankrupt*) there was no need for a symbol to show an *empty* column. Menninger has some excellent sentences on the subject: “Zero is something that must be there to show that nothing is there, [for] only the abstract place-notation needs zero. Zero first liberated the digits from the counting board.”⁵³

Surely one of the most remarkable inscriptions in Europe⁵⁴ is: $I \cdot V^e \cdot V$. It records the date 1505 in

symbols which, though Roman, are used with a positional significance unknown in Rome. The scribe “had heard about the new place-value system and now tried to find it in the Roman numerals. Since the meaning of the zero was still not clear to him, $I V 0 V = 1505$; at the critical point he yielded and retreated into the ‘named’ place-value notation.”⁵⁵ He solved his problem by inserting a superscript letter *e* to identify the hundreds column (compare Sylvester’s *locative symbols*). It is exciting to catch the conversion from the old way to the new as it was happening!

If it took so long for Hindu-Arabic numerals to make their way in the western world, we can hardly expect APL to be universally adopted in 25 years. But we can find encouragement in Menninger’s words: “These ten symbols which today all peoples use to record numbers, symbolize the world-wide victory of an idea. There are few things on earth that are universal, and the universal customs which man has successfully established are fewer still. But this one boast he can make: the new Indian numerals are universal.”⁵⁶

One of the satisfactions in working with APL comes from its consistency and completeness, exemplified by its recognition of *identity elements*, i.e., arguments that, used with a dyadic function, give a result identical to the other argument. If at each iteration in a FORTRAN loop, we accumulate by adding to a variable named SUM, why must we set SUM to zero before entering the loop? The reason is that zero is the identity element for addition, as 1 is of multiplication. APL, being rich in scalar dyadic functions, needs more kinds of identity elements than other languages do.

Although the computation of pi by inscribed polygons is recursive, we did not accumulate intermediate results, but proceeded at once to the next approximation. On the other hand, Sylvester’s algorithm for Egyptian unit-fractions constructs a vector, and the starting point must therefore be an empty vector.

We can calculate interest payments on a declining balance by following the same recursive paradigm.

Ordinary APL:

```

      ∇Z← A where W
[1]   Z←A
[2]   ∇

```



```

VZZ←N IB W;Z;B;R;I
[1]  ⍺(0>A+N-1)/'→0, 0pZZ←0 3p0'
[2]  ZZ←Z,[0]A IB W[0 1],1+Z+B,R,I
      where B←W[2]-R←W[1]-I←×/W[0 2]
[3]  ∇

```

Direct definition:

where: $\alpha:0:\omega$

```

IB:Z,[0]A IBω[0 1],1+Z+B,R,I
      where B←ω[2]-R←ω[1]-I←×/ω[0 2]:
            0>A←α-1:0 3p0

```

where B = current balance; R = amount going to reduce principal; I = amount going to pay interest.

If the principal is \$20,000, the interest is 10 percent, and the monthly payment is \$1,000, the function IB computes a table for 12 months (numbers are rounded):

```

12 IB 10 1000 20000÷1200 1 1
19167 833 167
18326 840 160
17479 847 153
16625 854 146
15763 861 139
14895 869 131
14019 876 124
13136 883 117
12245 891 109
11347 898 102
10442 905 95
9529 913 87

```

In J's pure functional form, define *i*, *r*, and *b* as three forks:

```

i=. 2&| * |.
r=. 1&| - i
b=. 2&| - r
ib=.(b,r,i)@].
<:@[ ib (2&|.b)@]]'
(0 3&$ @ 0:) @.(-0:)

```

To understand the structure of this function, condense it as follows:

```

ib=.(f@]. <:@[ ib g@]]'h
@.(-0:)

```

Read it thus:

To the result of function *f* of the right argument, concatenate the item (row) resulting from the function *ib* with a decremented left argument, and a right argument computed by function *g* from the previous right argument. Function *h* gives the identity

element for catenation of rows to a table with 3 columns. Terminate when the left argument is zero.

Because calculation of interest payments on a declining balance builds a table, we must start with 0 rows and 3 columns. Zero, then, is not enough; any language is incomplete if it fails to include different kinds of emptiness.

The identity element for matrix multiplication is the appropriately named *identity matrix*, first recognized by Cayley: "A matrix is not altered by its composition, either as first or second component matrix, with the matrix unity."¹⁷ In the following example, the recursive function *MP* raises a matrix (left argument) to an integer power (right argument), and consequently requires the identity matrix of the same shape as the matrix argument.

Ordinary APL:

```

VZ←M MP N;I
[1]  ⍺(N=0)/'→0, 0pZ←I∘.=I+11+pM'
[2]  Z←M+.×M MP N-1
[3]  ∇

```

More succinctly in direct definition:

```

MP:α+.×αMPω-1: ω=0 :I∘.=I+11+pα

```

```

M←3 3p19
M+.×M+.×M
180 234 288
558 720 882
936 1206 1476

```

```

M MP 3
180 234 288
558 720 882
936 1206 1476

```

Zero seems to behave like the queen in chess; for is it not the most powerful piece on the board? Any number multiplied by zero is reduced to zero. But *emptiness* is more powerful still, because any number, including zero, is reduced to emptiness when multiplied by an empty vector. Emptiness is not, however, to be confused with *nothing*, which is the result of executing an empty vector. You cannot multiply a number by nothing—a *value error* results if you try. Shakespeare made the fool touch something profound in saying to the king without a throne: "Now thou art an O without a figure. I am better than thou art now; I am a Fool, thou art nothing."⁵⁷

Unlike the play on words in Lewis Carroll's *Through the Looking Glass*, the distinctions between *zero*, *emptiness*, and *nothing* are not only useful but essential. The recursive APL functions already given include in a single line, *zero*, an *empty vector*, and (when the end condition obtains) *nothing*.

Logic

Because logic deals with two states, true and false, the mathematics of 0 and 1 is said to be logical. Propositions, or statements that may be judged true or false, are logical statements, and computers are logical machines because they manipulate binary digits (bits). The mathematics of logic began with Boole,⁵⁸⁻⁶¹ just at the time Sylvester introduced the term *matrix*. Jevons considered Boole's work to be, perhaps, "one of the most marvellous and admirable pieces of reasoning ever put together."⁶² Bertrand Russell thought highly of Boole's work, going so far as to claim that "Pure mathematics was discovered by Boole in a work which he called 'The Land of Thought.'"⁶³

"Let us conceive, then," wrote Boole, "of an Algebra in which the symbols x , y , z , etc. admit indifferently of the values of 0 and 1, and of these values alone."⁶⁴ Today we call a vector consisting of 1s and 0s a logical or Boolean vector, and Iverson notation, from its outset, used Boolean vectors to select from arrays, whether or not they were logical.¹ Where Boole used $x(s)$ to stand for the selection of all the x s from subset s , Iverson used u/s in APL (or $u\#s$ in J), which is compression if u is Boolean and replication if it is not.

Because a computer's memory and registers can be described as arrays of 1s and 0s, we now recognize that Boole laid the foundation for the design and description of modern computers—which are logical machines. But to most of his contemporaries his work seemed of little significance. The obituary notice in *The Athenaeum* (December 17, 1864) dryly reported that "The Professor's principal works were 'An Investigation into the Laws of Thought,' and 'Differential Equations,' books which sought a very limited audience, and we believe found it."

The Oxford English Dictionary cites the use of *Boolian algebra* [sic] in 1895 and 1902, but however we spell it, the usage is questionable. As Sylvester emphasized, there is only one universal algebra, which must, of course, include logic: "I have also a great

repugnance to being made to speak of Algebras in the plural; I would as lief acknowledge a plurality of Gods as of Algebras."⁶⁵ I am sure he would have approved of APL, which incorporates logical functions so that they can be used together with arithmetic functions in a single expression. For example, from Iverson:⁶⁶

"A theorem is a proposition which is claimed to be universally true, i.e., to have the value 1 when applied to any element in the universe of discourse. For example, the proposition

$$((0=2|X) \wedge (0=3|X)) \leq 0=6|X$$

is a theorem which may be verbalized in a variety of ways:

" X is divisible by 2 and X is divisible by 3 implies that X is divisible by 6.

"Any number divisible by both 2 and 3 is also divisible by 6.

"If X is divisible by both 2 and 3 then X is divisible by 6.

"Divisibility by 2 and 3 implies divisibility by 6."

According to John Venn (whose name is well known in connection with the diagrams that so effectively illustrate the meanings of *and*, *or*, and *not*), Jevons "was certainly the first to popularize the new conceptions of symbolic logic." The boldness, originality, and beauty of Boole's system fascinated him, and Jevons's book⁶⁷ was largely founded on Boole. Jevons, unlike Boole, emphasized the importance of the *inclusive or* and his symbol $(\cdot | \cdot)$ survives (though without the dots) in PL/I and in countless IBM technical manuals.

In 1865, Jevons completed construction of his reasoning machine, or logical abacus, adapted to show the workings of Boole's logic in a half mechanical manner, a full account of which was published by the Royal Society in 1870.⁶⁸ Mechanical devices had been designed by Napier, Pascal, Thomas of Colmar, and in Jevons's own time by Babbage, Stanhope,⁶⁹ and Smee,^{70,71} but Jevons claimed that until the work of Boole, logic had remained substantially as molded by Aristotle 2200 years ago. De Morgan, whose *Formal Logic*⁷² was published, by coincidence, on the same day as Boole's book,⁵⁸ pointed to the connection between two revealing facts: "logic

is the only science which has made no progress since the revival of letters; logic is the only science which has produced no growth of symbols." In my view APL is in the best tradition of Boole, De Morgan, Jevons, and Venn.⁷³

One of the most striking features in Iverson's *A Programming Language* is his demonstration that "the generalized matrix product and the selection operations together provide an elegant formulation in several established areas of mathematics. A few examples will be chosen from two such areas, symbolic logic and matrix algebra."⁷⁴ Iverson proceeded to show how his notation leads to a natural extension of De Morgan's laws.⁷⁵

De Morgan's law:

$$A \wedge B \leftrightarrow \sim(\sim A) \vee \sim B$$

Iverson's extensions:

$$\wedge/U \leftrightarrow \sim\vee/\sim U$$

$$\neq/U \leftrightarrow \sim=/\sim U$$

In ordinary APL:

$$U \leftarrow ? \ 5 \ 4 \ 3 \ \rho \ 2$$

$$V \leftarrow ? \ 3 \ 6 \ 7 \ \rho \ 2$$

$$\wedge/, (U \neq, \wedge V) = \sim (\sim U) =. \vee (\sim V)$$

1

In J, the latest form of Iverson's notation, his 1962 example is executed as follows:

$$u = .?5 \ 4 \ 3 \ \$2$$

$$v = .?3 \ 6 \ 7 \ \$2$$

$$(u \sim: / . * . v) \sim: - . (-, u) = / . + . (-, v)$$

1

where:

$\sim:$ is NOT EQUAL; $*$ is AND; $\sim:$ is MATCH;
 $-.$ is NOT; and $+$ is OR.

In algebra a leading negative can be removed by changing the signs of all quantities in the expression that follows; in APL a leading NOT (\sim) can be removed by interchanging the pairs AND and OR, EQUALS and NOT-EQUALS, etc. In the following example both functions *F* and *G* remove redundant blanks from a string.

Ordinary APL:

$$\forall Z \leftarrow F \ S; U$$

$$[1] \quad Z \leftarrow (\sim U \wedge 1 \phi U \leftarrow S = ' \ ') / S$$

$$[2] \quad \nabla$$

$$\forall Z \leftarrow G \ S; U$$

$$[1] \quad Z \leftarrow (U \vee 1 \phi U \leftarrow S \neq ' \ ') / S$$

$$[2] \quad \nabla$$

Direct definition:

$$F: (\sim U \wedge 1 \phi U \leftarrow \omega = ' \ ') / \omega$$

$$G: (U \vee 1 \phi U \leftarrow \omega \neq ' \ ') / \omega$$

APL continues to grow in power, and Iverson's final example,⁷⁶ written but not executable as $+. /$ in APL, can be executed in J as follows.

Given:

$$A = . \ 1 \ 3 \ 2 \ 0. \ 2 \ 1 \ 0 \ 1. : 4 \ 0 \ 0 \ 2$$

$$B = . \ 4 \ 1. \ 0 \ 3. \ 0 \ 2. : 2 \ 0$$

$$f = . \sim: \&0$$

$$h = . \ + / \ @ \ \# \ 0$$

Then:

$$(f \ A) \ + \ / \ . \ h \ B$$

$$4 \ 6$$

$$6 \ 4$$

$$6 \ 1$$

Iverson's generalized matrix product found immediate application in his formal description of indexed addressing on the IBM 7090 computer,⁷⁷ which in one line made clear what takes half a page of text in the *Principles of Operation* manual for that machine. There are, of course, many similar examples in Reference 78.

Arrays and locative symbols

APL is often referred to as the *array processing language*, and its power does to a great extent come from its ability to work with arrays directly, a feature of increasing importance as vector processors and parallel computing become available. When we specify a place by giving its latitude and longitude, or define a point on a scatter diagram by giving its X and Y coordinates, we intend that two numbers should be taken together to identify one object. This is the first step in thinking in terms of what Sylvester called *multiple quantity*.

Stevinus was the first to show how forces combine in the manner we know as the parallelogram of

forces.⁷⁹⁻⁸¹ The discovery is so important that Newton⁸² stated it as Corollary I immediately after his Laws of Motion. Authors of modern textbooks often suggest that the rule for vector addition is quite arbitrary by saying that the sum of two vectors is defined to be a third vector whose components are given by the sum of the corresponding components of the given vectors. Such a statement disguises the fact that in the real world we observe that forces combine in this manner.

Many first encounter the word *vector* in Kepler's so-called Second Law of Planetary Motion: the radius vector sweeps out equal areas in equal times. Kepler's prodigious calculations are even more remarkable when we remember how few mathematical symbols were available—logarithms, and even the decimal point had not yet been invented.

Once Kepler had found a mathematical relationship that held throughout space, he looked for a deeper reason. Introducing the *Newtonian* concept of force into science, he claimed that a magnetic force (*anima motrix*) emanated from the sun and carried the planets in their orbits.⁸³

Vector is the Latin word for a carrier, and it is used in medicine today in this sense. *Vector meus* is "my horse," and *vehicle, wagon, way, and convection* are from the same root. It was therefore an appropriate word for whatever it is that carries the planets in their orbits round the sun. I looked in vain for it in Kepler,⁸⁴ but Small⁸⁵ gives *radii vectores*. Harris, in 1704, defines *vector* to be "A line supposed to be drawn from any Planet moving round a Centre, or the Focus of an Ellipse, to that Centre or Focus, is by some writers of the New Astronomy, called the Vector; because 'tis that line by which the Planet seems to be carried round its Centre."⁸⁶

A vector in two dimensions can be represented by a complex number (and vice versa). Wessel, a Norwegian surveyor, was the first to realize this, but his work, though published in 1799, was unrecognized until 1897. A modern geometric treatment of the addition and multiplication of complex numbers was given by Argand in 1806, but these ideas received little attention until Gauss took up the topic in 1831.

If complex numbers can represent points in a plane, it is natural to try to create hypercomplex numbers to represent points in three-dimensional space. Sir William Rowan Hamilton finally succeeded in doing this in 1843.⁸⁷

In a long paper on "algebraic couples" written in 1837 Hamilton said: "In the THEORY OF SINGLE NUMBERS, the symbol $\sqrt{-1}$ is 'absurd' [it is an impossible root, or an imaginary number]; but in the THEORY OF COUPLES, the same symbol $\sqrt{-1}$ is 'significant' [i.e., it denotes a possible root, or a real couple]." What did he mean? I found the answer more clearly in Hamilton's own words than in modern textbooks.

Knowing that if you double a force you double the vector that represents it, Hamilton looked on 2 *times* as the operator that doubles; it is a special case of what he called a *tensor*, an operator that stretches (not to be confused with the modern use of the word). In the same way -1 *times* is a reversor. Moreover if $\sqrt{2}$ *times* is applied twice it doubles; and if $\sqrt{-1}$ *times* is applied twice it reverses. Consequently *i times* (where *i* is $\sqrt{-1}$) is a versor, or operator that rotates a vector without changing its length; it is taken as producing a counter-clockwise rotation of 90 degrees. Application of $-2i$ *times* would then be the composition of a rotation, a stretch, and a reversal. It is to Hamilton that we owe our terms *scalar* and *vector* (1846).

It seemed plausible that if couplets represent vectors in two dimensions, triplets would represent vectors in three dimensions, but after years of unsuccessful attempts, Hamilton realized, in a flash of genius, that a consistent algebra of triplets is impossible. Four terms (quaternions) are needed, shown in the example below:

complex: $a + bi$
 $i^2 = -1$

quaternion: $a + bi + cj + dk$
 $i^2 = j^2 = k^2 = ijk = -1$
 $ij = -ji$

Quaternions are of interest to the pure mathematician because they do not obey the laws of ordinary arithmetic: multiplication of quaternions is associative but not commutative.

Hermann Grassmann (a German schoolmaster) worked on vector systems at about the same time as Hamilton, and it was Grassmann who, in 1862, gave us *inner* and *outer products*, analogous to the scalar and vector parts of Hamilton's multiplication of quaternions.⁸⁸⁻⁹⁰

All of Arthur Cayley's early papers were on, or used, determinants, and both he and Sylvester pub-

lished on the rotation of a solid body. These are all topics that led naturally to the algebra of matrices. A matrix can, as we know, be looked upon as an array of multidimensional vectors, and so it is interesting that in 1843, the year Hamilton discovered quaternions, Cayley published on "the Geometry of (n) dimensions." Work on matrices was almost bound to follow.

Cayley was much influenced by Hamilton and visited Hamilton in Dublin. Cayley wrote his first paper on quaternions in 1845 at the age of 24, and considered the quaternion theory to be "a generalization of the analysis which occurs in ordinary Algebra." Later the same year he wrote on "The octuple system of imaginaries," showing that consistent arithmetics exist for couples, quadruples (but not triplets), and eight-fold hypercomplex numbers. Two years later he demonstrated that "in the octuple system of imaginary quantities neither the commutative nor the distributive law holds."

In 1848 Cayley showed that the combined effect of two rotations could be represented as the product of two quaternions, and shortly afterwards Sylvester (in the year he introduced the term *matrix*) pointed out that any number of rotations can be represented by a single rotation about one axis. As we would now say: each rotation can be represented by a matrix, and the product of these matrices is a matrix completely describing the combined rotation, whose axis is an eigenvector of this matrix, and the angle of rotation can be found from the corresponding eigenvalue. By 1855 Cayley used matrix product (calling it the *composition* of matrices), and in his memoir of 1858 he wrote: "It will be seen that matrices comport themselves as single quantities; they may be added, multiplied, or compounded together, etc.: the law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix . . ." ¹⁷ In this memoir he uses Sylvester's latent roots (eigenvalues), but without naming them.

Sylvester's paper, written in 1882, begins thus: "Professor Sylvester referred to the general question of representing the product of sums of two, four, or eight squares under the form of a like sum, and mentioned that Professor Cayley had been the first

Figure 8 Sylvester's locative symbols

$$\theta + 8h + 8t + 2u$$

is 1882

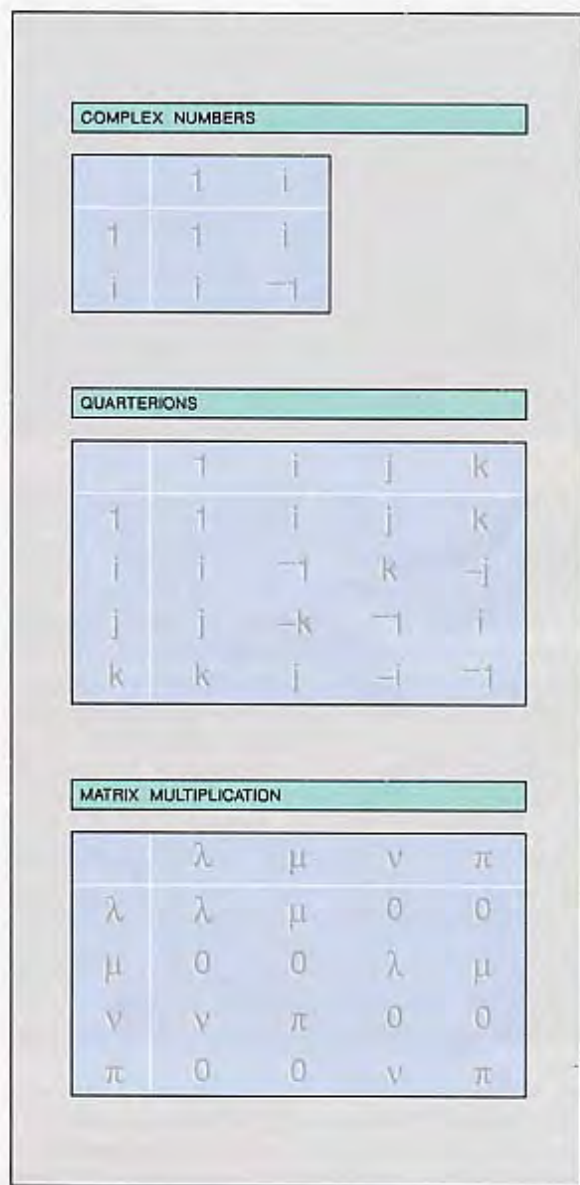
$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ v & \pi \end{pmatrix}$$

$$\rho = a\lambda + b\mu + cv + d\pi$$

to demonstrate, by an exhaustive investigation, the impossibility of extending the law applicable to 2, 4, and 8 to the case of 16 squares. The new kind of so-called imaginaries referred to by Professor Cayley are, as far as Mr. Sylvester is aware, the first example of the introduction into Analysis of *locative symbols* not subject to the strict law of association, and he considers the law regulating the connexion of the two products represented by a succession of three such symbols, most interesting, inasmuch as such products are either identical, or if not identical, of the same absolute value, but with contrary signs: most persons, before this example had been brought forward, would have felt inclined to doubt the possibility of locative symbols ('vulgo' imaginary quantities) whose multiplication table should give results inconsistent with the common associative law, being capable of forming the groundwork of any real accession to algebraical science " ⁹¹

His footnote is illuminating (compare also Reference 92): "Using θ , h , t , u to denote thousands, hundreds, tens, units, the year of grace in which we live may be represented by $\theta + 8h + 8t + 2u$ [—] θ , h , t , u , being locative symbols which it would be absurd to style 'imaginary quantities'; but they are as much entitled to that name as the i , j , k , or any like set of symbols—the only essential difference being that one set of symbols is limited, the other unlimited in number—and accordingly the law of combination of the one set is given by a finite and of the other an infinite 'multiplication table' . . . The 'locatives' indicate out of what 'basket,' so to say, the 'quantities' appearing in an analytical expres-

Figure 9 Sylvester's multiplication tables



sion are to be selected—the multiplication table determines the basket into which their product is to be thrown. . . . The whole analytical side of the theory of quaternions merges into a particular case of the general theory of *Multiple Algebra*. As far as the present writer is aware, Professor Cayley in his Memoir on Matrices (1858), was the first to recognize the parallelism between quaternions and matrices . . . ”⁹¹

Sylvester's locative symbols and multiplication tables for complex numbers, quaternions, and matrix multiplication are given in Figures 8 and 9 (from References 18, 91, 93, 94). By this method of representation Sylvester states in 1884: “a matrix is robbed as it were of its areal dimensions and represented as a linear sum.” Sylvester's 2 by 2 matrices I , L , M , and N are given in Figure 10, where the matrices, “construed as complex quantities, are a linear transformation of the ordinary quaternion system 1, i , j , k .” As he said: “Every matrix of the second order may be regarded as representing a quaternion, and vice versa.”

Sylvester's matrix identities given in Figure 10 can be demonstrated very concisely in Iverson's J, which supports complex numbers. The inner product is given by p , and $square$ computes the product of a matrix with itself; i is $\sqrt{-1}$. One line suffices to express the identities. The $match$ function is —:

```
i=.%:_1
p=.+/. *
square=.p~

I=. 1 0,: 0 1
L=. (i,0),:0, -i
M=. 0 _1,: 1 0
N=. (0,-i),: -i,0

(<-I)-:&.> (square &.> L;M;N),
<L p M p N
```

1	1	1	1
---	---	---	---

These matrices, derived by Sylvester (see also References 71, 95) as an exercise in pure mathematics, are intimately connected to the Pauli spin matrices, which have central significance in relativistic quantum theory; they are also close to the *spinor transformation*,⁹⁶ to *basis quaternions*, and the *basis elements* of the 16-dimensional Clifford numbers,⁹⁷ whose algebraic properties can easily be demonstrated in APL.⁹⁸⁻¹⁰⁰ The three Pauli matrices (σ_1 , σ_2 , and σ_3) describing the spin of an electron, together with all permutations of Pauli's identities, can be stated formally and executed. These are shown below in J with the numbers in square brackets from Pauli.

Given:

```
p=.+/. *
i=.%:_1
```


$$a + bi + cj + dk$$

$$i^2 = j^2 = k^2 = ijk = -1$$

$$a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

I

L

M

N

$$L^2 = M^2 = N^2 = LMN = -I$$

The matrices are:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$s1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$s2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$s3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[33.10]

and the permutations are:

$$z = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \cdot y = s1; s2; s3$$

$$1 \rightarrow 2 \rightarrow 1$$

[33.9]

1 1 1

$$f = p - p^{-1}$$

$$g = \{ (f \ 1 \ 2) \} \rightarrow (2 \ 1) \rightarrow (2 \ 1)$$

$$1 \rightarrow 2 \rightarrow 1$$

[33.11]

1 1 1

$$f = p : - @ p^{-1}$$

$$g = \{ (f \ 2 \ 1) \} \rightarrow (1 \ 2)$$

$$h = g \rightarrow 2 \rightarrow 1 \rightarrow 2$$

$$1 \rightarrow 2 \rightarrow 1$$

[33.12a]

1 1 1

$$f = p + p^{-1}$$

$$g = \{ (f \ 1 \ 2) \}$$

$$(0 \ 0 \ 0) \rightarrow 2 \rightarrow 1 \rightarrow 2$$

[33.12b]

1 1 1

In each of these identities, function f describes the essential relationship; functions g and h make it possible to test all "cyclical permutations of the indices."⁹⁸

After Sylvester returned to England, the principal exponents of the New Algebra in the United States were Benjamin Peirce and J. Willard Gibbs. Sylvester called Peirce's 1870 memoir⁹⁵ "a work which may almost be entitled to take rank as the 'Principia' of the philosophical study of the laws of algebraical operation." Gibbs's address "On Multiple Algebra" to the Section of Mathematics and Astronomy of the American Association for the Advancement of Science is a classic. In it Gibbs wrote the following:

"The multiple quantities corresponding to concrete quantities such as ten apples or three miles are evidently such combinations as ten apples + seven oranges, three miles northwestward + five miles eastward, or six miles in a direction 50 degrees east of north But if we ask what it is in multiple algebra which corresponds to an abstract number like twelve, which is essentially an operator, which changes one mile into twelve miles, and \$1,000 into \$12,000, the most general answer would evidently be: an operator which will work changes as, for example, that of ten apples + seven oranges into fifty apples and 100 oranges, or that of one vector into another. If the operation is distributive, it may not inappropriately be called multiplication, and the result is par excellence the product of the operator and the operand. The sum of operators, quâ operators, is an operator which gives for the product the sum of the products given by the operators to be added. The product of two operators is an operator which is equivalent to the successive operations of the factors."¹⁰¹

Figure 11 Gibbs's example of transformation

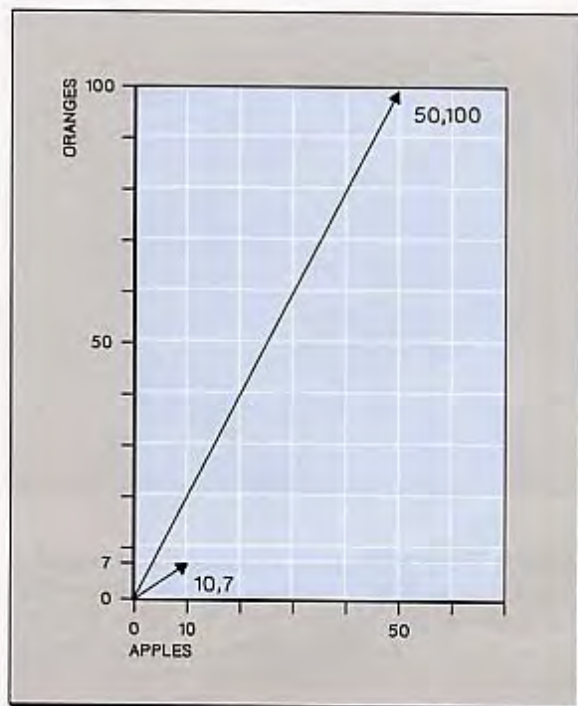


Figure 11 illustrates the problem Gibbs posed and makes the answer obvious. Although Gibbs did not turn to Hamilton, Sylvester, or Cayley for the solution, I betray their influence in Figure 12, where I separate the versor (as a rotation matrix) and the tensor (as a scalar). The example can be worked as follows:

The transformation matrix (with tensor and versor composed):

$$\begin{aligned}
 X &\leftarrow 50 \ 100 \oplus 10 \ ^{-}7, [^{-}0.5] \ 7 \ 10 \\
 N &\leftarrow (1 \ ^{-}1 \times X), [^{-}0.5] \oplus X \\
 N &\leftarrow \times 10 \ 7 \\
 50 \ 100
 \end{aligned}$$

Isolate the tensor and determine the angle of rotation in degrees:

$$\begin{aligned}
 \square &\leftarrow Y \leftarrow (+/- X \times 2) \times 0.5 \\
 9.16 \\
 (180 \div 0.1) \times \ ^{-}2 \ ^{-}1 \circ X &\div Y \\
 28.44 \ 28.44
 \end{aligned}$$

Confirm by composing the tensor and versor, where RFD is Radians From Degrees:

$$\begin{aligned}
 \nabla Z &\leftarrow \text{RFD } X \\
 [1] \quad Z &\leftarrow \circ X \div 180 \\
 [2] \quad \nabla \\
 \nabla Z &\leftarrow F \ X \\
 [1] \quad Z &\leftarrow 2 \ 2p1 \ ^{-}1 \ 1 \ 1 \times 2 \ 1 \ 1 \ 2 \circ \text{RFD } X \\
 [2] \quad \nabla \\
 (9.16 \times F \ 28.44) &\div \times 10 \ 7 \\
 50 \ 100
 \end{aligned}$$

The wondrous tale of multiple quantity

This example, simple though it is, throws light upon the nature of the "new world of thought" to which Sylvester gave the name of "Universal Algebra or the Algebra of multiple quantity" in 1884.

James Joseph Sylvester was born in 1814. In 1837 he completed his studies at Cambridge and published the first of his 342 papers. It was on crystallography. His next two papers were on the motion of fluids and rigid bodies—all topics of importance to my own subject of geology—and all amenable to matrix algebra. Some additional history can be found in Reference 102.

Sylvester,¹⁰³⁻¹⁰⁶ the self-styled mathematical Adam, gave "more names (passed into general circulation) to the creatures of mathematical reason than all the other mathematicians of the age combined" (1888). In 1850, the year he was called to the bar, he introduced the term *matrix* for "a rectangular array of terms, out of which different systems of determinants may be engendered as from the womb of a common parent."^{107,108} Sylvester introduced¹⁰⁹ the Greek letter *lambda* (λ) for the latent roots of a characteristic equation (his terms) in 1852—three-quarters of a century before the term *eigenvalue* was invented; and in 1853 he introduced the *inverse matrix*.¹¹⁰

In 1884, at the age of 70, he published his *Lectures on the Principles of Universal Algebra*, the "apotheosis of algebraical quantity," in the *American Journal of Mathematics*, which he himself founded and edited. His title reminds us that Newton used the term *universal arithmetic* for what we call *algebra*. Emphasizing the importance of matrices as multiple quantity, he speaks of a second birth of algebra, its *avatar* in a new and glorified form.¹¹¹ In the words of this enthusiast, who lived a century before APL was implemented: "A matrix of quadrate

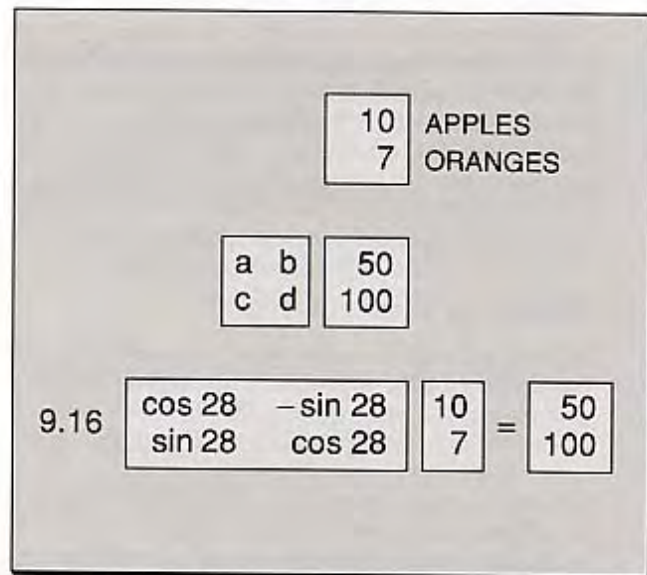
form ... emerges ... in a glorified shape—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. ... The conception of multiple quantity rises upon the field of vision. ... [Matrix] dropped its provisional mantle, its aspect as a mere schema, and stood revealed as bona fide multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity."

"This revolution," he continued, "was effected by a forcible injection into the subject of the concept of addition; that is, by choosing to regard matrices as susceptible to being added to one another; a notion as it seems to me, quite foreign to the idea of substitution, the nidus in which that of multiple quantity was laid, hatched and reared. This step was, as far as I know, first made by Cayley ... in his [immortal] *Memoir on Matrices* [1858], wherein he may be said to have laid the foundation-stone of the science of multiple quantity. That memoir indeed (it seems to me) may in truth be affirmed to have ushered in the reign of Algebra the 2nd; just as Algebra the 1st ... took its rise in Harriot's *Artis Analyticae Praxis*, published in 1631, ... exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University."¹⁸ References 112 to 115 add some additional information about Cayley.

If Sylvester were here today, what pleasure would he find in Iverson's notation, implemented even on our personal computers as an interactive language—this notation that encourages, and as it were expects, us to think in terms of arrays or multiple quantities, manipulating them as entities in the spirit of Sylvester's exhortations! That eloquent mathematician would be even more moved, I am sure, by boxed arrays (arrays of arrays), and array processors, which are APL machines.

A century ago both Sylvester and Gibbs urged us to think in terms of arrays. Most computer languages and what Backus called (perhaps unfairly) the Von Neumann bottleneck, force us, however, to work with scalars. Within the confines of a few pages, I have attempted to trace the development of notation and methods from hieroglyphics to APL. I have tried to show that APL is much more than yet another computer language; that its intellectual importance is great; and that (yet again using Sylvester's words) APL continues "The wondrous tale of Multiple Quantity."

Figure 12 Separation of versor and tensor



The story will, of course, never be completed. We have seen the recent introduction of two hitherto undefined phrases now called *hooks* and *forks*.¹¹⁶ One example of each must suffice here.

$+/\% \# y$ computes the sum over the reciprocals of the tally of y , which is unlikely to be useful, whereas, if we unify the phrase, placing it in parentheses, it becomes a fork $(+/\% \#) y$ equivalent to $(+/y) \% (\#y)$, which computes the mean (or means over the leading axis if the rank exceeds 1).

$(- \text{mean}) y$ is a hook, equivalent to $y - (\text{mean } y)$, which gives the deviations from the mean, a necessary step in computing variance.

It should be noted that when we define the phrase, as for example $\text{mean} = +/\% \#$ the phrase is unified without requiring parentheses. The functions used above for the Pauli identities are examples of forks. The statistical examples above include hook (sums of cross products) and fork (correlation coefficients). The function for interest on a declining balance (*ib*) includes a train of five functions, three of which (*i*, *r*, *b*) are forks, and it ends with an interesting hook.

In a paper published in 1866 we find Sylvester writing on the subject of operators. "The force of the bracket [i.e., parentheses] explains itself. This wonderful symbol has the faculty of extending itself

without ambiguity to every possible development, however new, of mathematical language. It is susceptible only of a metaphysical definition as signifying the exercise, with regard to its content, of that faculty of the human mind whereby a multitude is capable of being regarded as an individual, or a complex as a monad. In a word, it is the symbol of individuality and unification." I am unable to assert that Sylvester foresaw the *phrasal forms* of modern APL 125 years ago, but his words seem remarkably apt in reference to these new developments.

Notation as a tool of thought

In ending I wish to quote from some of our great predecessors who appreciated the power of symbols as an aid to reasoning, or in Ken Iverson's memorable phrase, "notation as a tool of thought."

Lavoisier wrote a memoir in 1787 on the necessity of reforming the nomenclature of chemistry. In it he made this statement: "Languages are intended, not only to express by signs, as is commonly supposed, the ideas and images of the mind; but are also analytical methods, by the means of which, we advance from the known to the unknown, and to a certain degree in the manner of mathematicians. . . . Algebra is the analytical method by excellence [sic]; it has been invented to facilitate the operations of the understanding, and to render reasoning more concise, and to contract into a few lines what would have required whole pages of discussion; in fine, to lead, in a more agreeable and laconic method [*plus commode, plus prompt et plus sûre*], to the solution of the most complicated questions. Even a moment's reflection is sufficient to convince us that algebra is in fact a language: like all other languages it has its representative signs, its method and its grammar, if I may use the expression: thus an analytical method is a language; a language is an analytical method; and these two expressions are, in a certain respect synonymous [sic]."¹¹⁷

In 1821, Babbage, in his thought-provoking paper "On the Influence of Signs in Mathematical Reasoning," said: "The quantity of meaning compressed into small space by algebraic signs is a circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this out-

line by a tedious description, which in some instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience, and presented to the eye a picture perfect in all its parts, disclosing at a glance, not merely the conclusion in which it terminated, but every stage of its progress. At first it appeared probable that this triumph of signs over words would have limits to its extent: a time it might be feared would arrive, when oppressed by the multitude of its productions, the language of signs would sink under the obscurity produced by its own multiplication. . . . Fortunately however such anticipations have proved unfounded.

"Examples of the power of a well-contrived notation to condense into small space a meaning which would—in ordinary language—require several lines, or even pages, can hardly have escaped the notice of most of my readers: in the calculus of functions, this condensation is carried to a far greater extent than in any other branch of analysis, and yet, instead of creating any obscurity, the expressions are far more readily understood than if they were written at length. . . . The power we possess by the aid of symbols of compressing into small compass the several steps of a chain of reasoning, whilst it contributes greatly to abridge the time which our enquiries would otherwise occupy, in difficult cases influences the accuracy of our conclusions: for from the distance which is sometimes interposed between the beginning and the end of a chain of reasoning, although the separate parts are sufficiently clear, the whole is often obscure. . . . The closer the succession between two ideas which the mind compares, provided those ideas are clearly perceived, the more accurate will be the judgement that results."¹¹⁸

"The advantage of selecting in our signs, those which have some resemblance to, or which from some circumstance are associated in the mind with the thing signified has scarcely been stated with sufficient force: the fatigue, from which such an arrangement saves the reader, is very advantageous to the more complete devotion of his attention to the subject examined; and the more complicated the subject, the more numerous the symbols and the less their arrangement is susceptible of symmetry, the more indispensable will such a system be found. This rule is by no means confined to the choice of the letters which represent quantity, but is meant to extend, when it is possible, to cases

where new arbitrary signs are invented to denote operators. . . . The more complicated the enquiries on which we enter, and the more numerous the quantities which it becomes necessary to represent symbolically, the more essentially necessary it will be found to assist the memory by contriving such signs as may immediately recall the thing which they are intended to represent."¹¹⁹

Sylvester, in 1877, said "It is the constant aim of the mathematician to reduce all his expressions to their lowest terms, to retrench every superfluous word and phrase, and to condense the Maximum of meaning into the Minimum of language."¹²⁰

Whitehead, in 1911, claimed that "By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race. . . . By the aid of symbolism we can make transitions in reasoning almost mechanically by the eye, which would otherwise call into play the higher faculties of the brain. It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them."¹²¹

Bertrand Russell said: "The great master of the art of formal reasoning, among men of our own day, is an Italian, Professor Peano, of the University of Turin. He has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole), to strict symbolic form, in which there are no words at all."

In the first paragraph of his book in 1959, Russell wrote: "There is one major division in my philosophical work: in the years 1899-1900 I adopted the philosophy of logical atomism and the technique of Peano in mathematical logic. This was so great a revolution as to make my previous work, except such as purely mathematical, irrelevant to everything I did later. The change in these years was a revolution; subsequent changes have been of the nature of an evolution."¹²²

And finally, Giuseppe Peano himself, in his paper on "The Importance of Symbols in Mathematics" in 1915 wrote: "The oldest symbols, which are also the most used today, are the digits used in arithmetic,

which we learned about 1200 from the Arabs, and they from the Indians, who were using them about the year 400. The first advantage that one sees in the digits is their brevity. . . . Further reflection reveals that these symbols are not just shorthand, i.e., abbreviations of ordinary language, but constitute a new class of ideas. . . . The use of digits not only makes our expressions shorter, but makes arithmetical calculation essentially easier, and hence makes certain tasks possible, and certain results obtainable, which could not otherwise be the case in practice. For example, direct measure assigned to the number Pi, the ratio of the circumference of a circle its diameter, the value 3. . . .

"Archimedes, about 200 B.C., by inscribing and circumscribing polygons about a circle, or rather by calculating a sequence of square roots, using Greek digits, found Pi to within 1/500. The substitution of Indian digits for the Greek allowed Aryabhata, about the year 500, to extend the calculation to 4 decimal places, and allowed the European mathematicians of 1600 to carry the calculation out to 15 and then 32 places, still following Archimedes' model. Further progress, i.e., the calculation of 100 digits in 1700, and the modern calculation of 700, was due to the introduction of series.

"The same thing may be said for the symbols of algebra. . . . Algebraic equations are much shorter than their expression in ordinary language, are simpler, and clearer, and may be used in calculations. This is because algebraic symbols represent ideas and not words. . . . Algebraic symbols are much less numerous than the words they allow us to represent.

"The evolution of algebraic symbolism went like this: first, ordinary language; then, in Euclid, a technical language in which a one-to-one correspondence between ideas and words was established; and then the abbreviation of the words of the technical language, beginning about 1500 and done in various ways by different people, until finally one system of notation, that used by Newton, prevailed over the others.

"The use of algebraic symbols permits schoolchildren easily to solve problems which previously only great minds like Euclid and Diophantus could solve. . . . The symbols of logic too are not abbreviations of words, but represent ideas, and their principal utility is that they make reasoning easier. All those who use logical symbolism attest to this."¹²³

Concluding remarks

A progression of great thinkers has moved the human race towards the adoption, first of an economical and efficient number system containing zero and based on place value, and then of a universal algebra, APL, which operates on arrays or multiple quantities, and is totally devoid of words.

There have also been those who resisted the inevitable progress, who found it difficult to adopt new and improved tools for thought. In our own time we hear appeals to revert from this high intellectual level and use English words, and to submit to the tyranny of scalars, as if Sylvester's eloquence a century ago had fallen on deaf ears.

Unlike its predecessors, APL is an executable notation. APL represents, in a phrase used by Babbage, the "triumph of symbols over words." As so many of our distinguished predecessors predicted, it makes reasoning easier. APL is the result of brilliant insight, careful thought, and hard work through at least 5000 years. Iverson is the latest in a succession that includes Peano, Sylvester, Cayley, De Morgan, Boole, Newton, Leibniz, Napier, Stevinus, Fibonacci, Diophantus, and the unknown Egyptian whose work was copied by Ahmes the scribe.

In 1866 Sylvester proclaimed that: "To attain clearness of conception, the first condition is 'language,' the second 'language,' the third 'language'—Protestant speech—the child and parent of thought."¹²⁴

In reflecting on the significance of APL I have adopted a historical approach. Having done so I find that Sylvester had something to say on that subject also. The occasion was his Presidential Address to the British Association¹²⁵ in 1869 when he said: "the relation of master and pupil is acknowledged as a spiritual and lifelong tie, connecting successive generations of great thinkers with each other in an unbroken chain."

We think in a different way because of APL.

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102. In 1839, at the age of 25, Sylvester was elected a Fellow of the Royal Society. Although, in his own phrase, he was "one of the first holding the faith in which the Founder of Christianity was educated to compete for high honours in the Mathematical Tripos at Cambridge," he could not obtain his B.A. degree until 1872, after all religious tests had been abolished. At different times he was Professor of Physics in London, where he was a colleague of De Morgan's; Professor of Mathematics at the University of Virginia, where he left in haste after successfully defending himself with a sword-cane against the brother of a student whose work he had criticized; and Professor at the Royal Military Academy.
103. The currently popular movement that enjoys "debunking history and toppling eminent Victorians" has not spared Sylvester and Cayley. Hawkins (see Reference 104) says that "the significance of Cayley's memoir on matrices of 1858 has been grossly exaggerated." Sylvester is not even mentioned. Those interested may, however, consult References 105 and 106.
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