ECE570 Lecture 18: Congruence Closure

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Axioms of Equality—I

$=$ is a predicate symbol of arity 2

Reflexivity

$$(\forall x)x = x$$

Symmetry

$$(\forall x)(\forall y)x = y \rightarrow y = x$$

Transitivity

$$(\forall x)(\forall y)(\forall z)(x = y \wedge y = z) \rightarrow x = z$$

Axioms of Equality—II

Function Substitution Schema

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)$$
$$(x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \rightarrow f(x_1,\ldots,x_n) = f(y_1,\ldots,y_n)$$

Predicate Substitution Schema

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)$$
$$(x_1 = y_1 \wedge \cdots \wedge x_n = y_n) \rightarrow (P(x_1,\ldots,x_n) \leftrightarrow P(y_1,\ldots,y_n))$$

Axioms of Group Theory—I

constant symbols: 0
function symbols: i (arity 1), $+$ (arity 2)
predicate symbols: $=$ (arity 2)

0 is Right Identity

$$(\forall x)x + 0 = x$$

$i(x)$ is Right Inverse

$$(\forall x)x + i(x) = 0$$

Associativity of $+$

$$(\forall x)(\forall y)(\forall z)x + (y + z) = (x + y) + z$$

Axioms of Group Theory—II

Substitution Axiom for i

$$(\forall x)(\forall y)x = y \rightarrow i(x) = i(y)$$

Substitution Axiom for $+$

$$(\forall u)(\forall v)(\forall x)(\forall y)(u = x \wedge v = y) \rightarrow u + v = x + y$$

Reflexivity, Symmetry, and Transitivity Axioms

$$(\forall x)x = x$$
$$(\forall x)(\forall y)x = y \rightarrow y = x$$
$$(\forall x)(\forall y)(\forall z)(x = y \wedge y = z) \rightarrow x = z$$

A Theorem from Group Theory

0 is Left Identity

$$(\forall x)0 + x = x$$

An Abbreviated Semantic Tableaux Proof—I

1	$\forall x(x+0=x)$	Axiom
2	$\forall x(x+i(x)=0)$	Axiom
3	$\forall x\forall y\forall z(x+(y+z)=(x+y)+z)$	Axiom
4	$\forall x(x=x)$	Axiom
5	$\forall x\forall y(x=y\rightarrow y=x)$	Axiom
6	$\forall x\forall y\forall z((x=y\wedge y=z)\rightarrow x=z)$	Axiom
7	$\forall x\forall y(x=y\rightarrow i(x)=i(y))$	Axiom
8	$\forall u\forall v\forall x\forall y$ $((u=x\wedge v=y)\rightarrow u+v=x+y)$	Axiom
9	$\neg\forall x(0+x=x)$	Negated Goal
10	$\exists x 0+x\neq x$	from 9

An Abbreviated Semantic Tableaux Proof—II

11	$0+a \neq a$	EI(10,$\{x \mapsto a\}$)
12	$a+0=a$	UI(1,$\{x \mapsto a\}$)
13	$i(a)+0=i(a)$	UI(1,$\{x \mapsto i(a)\}$)
14	$a+i(a)=0$	UI(2,$\{x \mapsto a\}$)
15	$i(a)+i(i(a))=0$	UI(2,$\{x \mapsto i(a)\}$)
16	$a+(i(a)+a)=(a+i(a))+a$	UI(3,$\{x \mapsto a, y \mapsto i(a), z \mapsto a\}$)
17	$i(a)+((a+i(a))+i(i(a)))=$ $(i(a)+(a+i(a)))+i(i(a))$	UI(3,$\{x \mapsto i(a), y \mapsto a+i(a),$ $z \mapsto i(i(a))\}$)
18	$a+(i(a)+i(i(a)))=$ $(a+i(a))+i(i(a))$	UI(3,$\{x \mapsto a, y \mapsto i(a), z \mapsto i(i(a))\}$)
19	$a=0+a \rightarrow 0+a=a$	UI(5,$\{x \mapsto a, y \mapsto 0+a\}$)

An Abbreviated Semantic Tableaux Proof—III

20	$(i(a)+0)+i(i(a))=0$	SUBST(13,15)
21	$(i(a)+(a+i(a)))+i(i(a))=0$	SUBST(14,20)
22	$i(a)+((a+i(a))+i(i(a)))=0$	SUBST(17,21)
23	$i(a)+(a+(i(a)+i(i(a))))=0$	SUBST(18,22)
24	$i(a)+(a+0)=0$	SUBST(15,23)
25	$i(a)+a=0$	SUBST(12,24)
26	$a+0=(a+i(a))+a$	SUBST(25,16)
27	$a+0=0+a$	SUBST(14,26)
28	$a=0+a$	SUBST(12,27)
29	$a\neq 0+a \vee 0+a=a$	from 19
30	$a\neq 0+a$	from 29 CONTRA(28)
31	$0+a=a$	from 29 CONTRA(11)

A Less-Abbreviated Version of Step 20—I

20	$(i(a)+0)+i(i(a))=0$	SUBST(13,15)
20.1	$i(i(a))=i(i(a))$	UI(4,$\{x\mapsto i(i(a))\}$)
20.2	$(i(a)+0=i(a)\wedge i(i(a))=i(i(a)))\rightarrow$ $(i(a)+0)+i(i(a))=i(a)+i(i(a))$	UI(8,$\{u\mapsto i(a)+0, x\mapsto i(a),$ $v\mapsto i(i(a)), y\mapsto i(i(a))\}$)
20.3	$\neg(i(a)+0=i(a)\wedge i(i(a))=i(i(a)))\vee$ $(i(a)+0)+i(i(a))=i(a)+i(i(a))$	from 20.2
20.4	$((i(a)+0)+i(i(a))=i(a)+i(i(a))\wedge$ $i(a)+i(i(a))=0)\rightarrow$ $(i(a)+0)+i(i(a))=0$	UI(6,$\{x\mapsto(i(a)+0)+i(i(a)),$ $y\mapsto i(a)+i(i(a)), z\mapsto 0\}$)
20.5	$\neg(i(a)+0=i(a)\wedge i(i(a))=i(i(a)))$	from 20.3
20.6	$i(a)+0\neq i(a)\vee i(i(a))\neq i(i(a))$	from 20.5
20.7	$i(a)+0\neq i(a)$	from 20.6 CONTRA(13)

A Less-Abbreviated Version of Step 20—II

20.8	$i(i(a)) \neq i(i(a))$	from 20.6 CONTRA(20.1)
20.9	$(i(a)+0)+i(i(a)) = i(a)+i(i(a))$	from 20.3
20.10	$\neg((i(a)+0)+i(i(a)) = i(a)+i(i(a)) \wedge$ $i(a)+i(i(a)) = 0) \vee$ $(i(a)+0)+i(i(a)) = 0$	from 20.4
20.11	$\neg((i(a)+0)+i(i(a)) = i(a)+i(i(a)) \wedge$ $i(a)+i(i(a)) = 0)$	from 20.10
20.12	$(i(a)+0)+i(i(a)) \neq i(a)+i(i(a)) \vee$ $i(a)+i(i(a)) \neq 0$	from 20.11
20.13	$(i(a)+0)+i(i(a)) \neq i(a)+i(i(a))$	from 20.12 CONTRA(20.9)
20.14	$i(a)+i(i(a)) \neq 0$	from 20.12 CONTRA(15)
20.15	$(i(a)+0)+i(i(a)) = 0$	from 20.10

Congruence Closure Algorithm—I

notation

- Infinite set of colors $c_1, c_2, \ldots$
- C is a *partial* map from tuples $\langle f, c_1, \ldots, c_n \rangle$ to colors where $f \in F_n$ and $c_1, \ldots, c_n$ are colors
- **gensym** creates a new color
- $C[c_1 \mapsto c_2]$ denotes the map with all instances of c_1 replaced with c_2
- $\Gamma[c_1 \mapsto c_2]$ denotes the queue Γ with all instances of c_1 replaced with c_2
- $M[c_1 \mapsto c_2]$ denotes the model M with all instances of c_1 replaced with c_2
- $\textsc{ColorsIn}(\Gamma)$ set of all colors in Γ
- $\textsc{ColorsIn}(M)$ set of all colors in M

Congruence Closure Algorithm—II

$$
\begin{aligned}
\mathcal{C}(f(t_1,\ldots,t_n)) &\triangleq \mathcal{C}(\langle f,\mathcal{C}(t_1),\ldots,\mathcal{C}(t_n)\rangle)\\
\mathcal{C}(p(t_1,\ldots,t_n)) &\triangleq p(\mathcal{C}(t_1),\ldots,\mathcal{C}(t_n))\\
\mathcal{C}(\neg\Phi) &\triangleq \neg\mathcal{C}(\Phi)\\
&\vdots
\end{aligned}
$$

Congruence Closure Algorithm—III

$t_1 = t_2$	$\textsc{Equate}(\mathcal{C}(t_1), \mathcal{C}(t_2))$
Φ atomic	If $\mathcal{C}(\neg\Phi) \in M$ then fail else $M := M \cup \{\mathcal{C}(\Phi)\}$
$\neg\Phi$ atomic	If $\mathcal{C}(\Phi) \in M$ then fail else $M := M \cup \{\mathcal{C}(\neg\Phi)\}$
$\exists x \Phi$	$\textsc{Push}(\Phi[x \mapsto \textbf{gensym}], \Gamma)$
$\forall x \Phi$	Let $T = \textsc{ColorsIn}(\Gamma) \cup \textsc{ColorsIn}(M)$
	If $T = \{\}$ then $T := \{\textbf{gensym}\}$
	For each $t \in T$ do $\textsc{Push}(\Phi[x \mapsto t], \Gamma)$
	$\textsc{Push}(\forall x \Phi, \Gamma)$

Congruence Closure Algorithm—IV

$$\textsc{Equate}(c_1, c_2)$$

1. $C := C[c_1 \mapsto c_2]$
2. $\Gamma := \Gamma[c_1 \mapsto c_2]$
3. $M := M[c_1 \mapsto c_2]$
4. If M contains some Φ and $\neg\Phi$ then fail
5. If C contains $\langle f, c_1, \ldots, c_n \rangle \mapsto d_1$ and $\langle f, c_1, \ldots, c_n \rangle \mapsto d_2$ then $\textsc{Equate}(d_1, d_2)$

A Semantic Tableaux Proof Using Congruence Closure

1	$\forall x(x+0=x)$	Axiom
2	$\forall x(x+i(x)=0)$	Axiom
3	$\forall x\forall y\forall z(x+(y+z)=(x+y)+z)$	Axiom
4	$\neg\forall x(0+x=x)$	Negated Goal
5	$\exists x 0+x\neq x$	from 4
6	$0+a\neq a$	UI(5,$\{x\mapsto a\}$)
7	$a+0=a$	UI(1,$\{x\mapsto a\}$)
8	$i(a)+0=i(a)$	UI(1,$\{x\mapsto i(a)\}$)
9	$a+i(a)=0$	UI(2,$\{x\mapsto a\}$)
10	$i(a)+i(i(a))=0$	UI(2,$\{x\mapsto i(a)\}$)
11	$a+(i(a)+a)=(a+i(a))+a$	UI(3,$\{x\mapsto a, y\mapsto i(a), z\mapsto a\}$)
12	$i(a)+((a+i(a))+i(i(a)))=$ $(i(a)+(a+i(a)))+i(i(a))$	UI(3,$\{x\mapsto i(a), y\mapsto a+i(a),$ $z\mapsto i(i(a))\}$)
13	$a+(i(a)+i(i(a)))=$ $(a+i(a))+i(i(a))$	UI(3,$\{x\mapsto a, y\mapsto i(a), z\mapsto i(i(a))\}$)