ECE570 Lecture 14: Qualitative Physics

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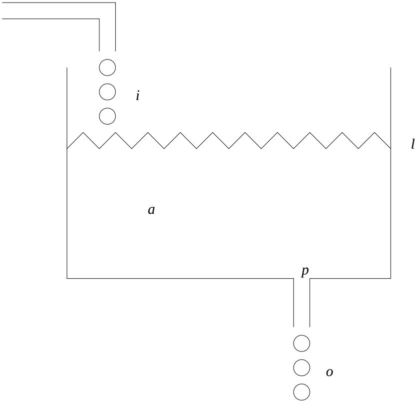
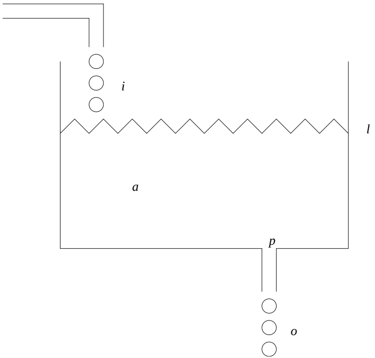
PURDUE UNIVERSITY

A Physical System



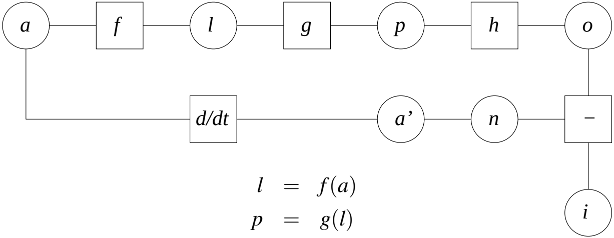
Modelling a Physical System



Differential Equations

$$
\begin{aligned}
l &= f(a)\\
p &= g(l)\\
o &= h(p)\\
n &= i - o\\
n &= \dot{a}
\end{aligned}
$$

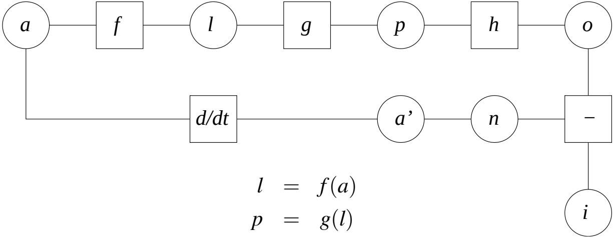
Differential Equations as Constraints



$$
\begin{aligned}
l &= f(a)\\
p &= g(l)\\
o &= h(p)\\
n &= i - o\\
n &= \dot{a}
\end{aligned}
$$

Variables range over functions from reals (time) to reals.
A constraint such as $z = x - y$ means $(\forall t)[z(t) = x(t) - y(t)]$.
$y = \dot{x}$ means that y is the derivative of x.

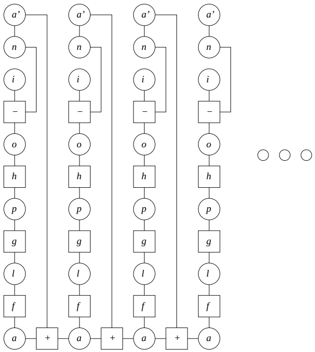
Numerical Solutions as Difference Equations



$$
\begin{aligned}
l &= f(a)\\
p &= g(l)\\
o &= h(p)\\
n &= i - o\\
n &= \dot{a}
\end{aligned}
$$

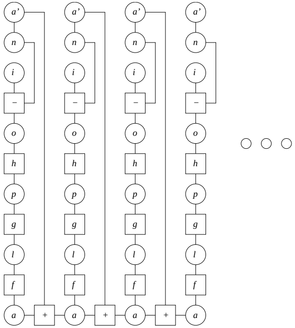
Variables range over (finite) sequences of real numbers.
A constraint such as $z = x - y$ means $(\forall i)[z_i = x_i - y_i]$.
$y = \dot{x}$ means $(\forall i)[x_{i+1} = x_i + y_i]$ (MVT).

Unfolding the CSP



Variables range over real numbers.
A constraint such as $z = x - y$ means $z = x - y$.
No more constraints of the form $y = \dot{x}$.

Need Initial Conditions



$$
\begin{aligned}
a(0) &= 0 \\
(\forall t)i(t) &= c
\end{aligned}
$$

Problems

- Don't know what the functions f, g, and h are.
  Might have only partial information, i.e. that they are monotonically increasing.
- Don't know what the initial condition constant c is.
  Might have only partial information, i.e. that it is positive.

Can we still answer questions like:

- Can the tank fill up?
- Will it fill up?
- Can it overflow?
- Can it empty?
- Can it reach equilibrium?
- Can it oscillate?

Quantity Spaces

Instead of ranging over real numbers, variables will range over a finite set of *qualitative values*.
Qualitative values are either *landmarks* or *ranges* (open intervals between landmarks).
A set of qualitative values is called a *quantity space*.

a	$\{0, (0, \mathit{full}), \mathit{full}, (\mathit{full}, \infty), \infty\}$
$\dot{a}$	$\{-\infty, (-\infty, 0), 0, (0, \infty), \infty\}$
l	$\{0, (0, \mathit{top}), \mathit{top}, (\mathit{top}, \infty), \infty\}$
p	$\{0, (0, \infty), \infty\}$
i	$\{0, (0, c), c, (c, \infty), \infty\}$

Qualitative Multiplication

$\times$	$-\infty$	$(-\infty, 0)$	0	$(0, \infty)$	∞
$-\infty$					
$(-\infty, 0)$					
0					
$(0, \infty)$					
∞					

Qualitative Addition

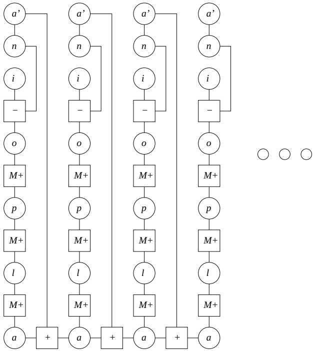
$+$	$-\infty$	$(-\infty, 0)$	0	$(0, \infty)$	∞
$-\infty$					
$(-\infty, 0)$					
0					
$(0, \infty)$					
∞					

Qualitative Monotonic Functions

x	$-\infty$	$(-\infty, 0)$	0	$(0, \infty)$	∞
$M^+(x)$					

x	$-\infty$	$(-\infty, 0)$	0	$(0, \infty)$	∞
$M^-(x)$					

Qualitative CSP



Qualitative Time

Time is also represented with a quantity space.
Landmark qualitative time values are called *instants*.
Range qualitative time values are called *intervals*.
Quantities are indexed by qualitative time: x_t, $\dot{x}_t$, $\ldots$.
Predicates: $\textsc{Instant}(t)$, $\textsc{Interval}(t)$, $\textsc{Landmark}(x_t)$, $\textsc{Range}(x_t)$, $\textsc{Adjacent}(t, t')$, and $\textsc{Adjacent}(x, x')$.
t_s is the *start* time.
t_f is the *finish* time.
Boundary conditions: $\textsc{Interval}(t_s)$ and $\textsc{Interval}(t_f)$.

Condition I—Continuity

Quantities must take on adjacent or equal qualitative values at adjacent qualitative times.

$$(\forall x)(\forall t)(\forall t')\{\textsc{Adjacent}(t,t') \rightarrow [\textsc{Adjacent}(x_t,x_{t'}) \vee (x_t = x_{t'})]\}$$

Condition II—Landmarks

If a quantity takes on a landmark value during an interval then it must take on that same landmark value during the adjacent instants.

$$(\forall x)(\forall t)(\forall t')\left\{\begin{array}{l}[\textsc{Interval}(t) \wedge \textsc{Landmark}(x_t) \wedge \textsc{Adjacent}(t,t')] \\ \rightarrow (x_t = x_{t'})\end{array}\right\}$$

Condition III—Stationarity

If a quantity takes on a landmark value during an interval then its qualitative derivative must be zero during that interval.

$$(\forall x)(\forall t)\left\{\left[\textsc{Interval}(t) \wedge \textsc{Landmark}(x_t)\right] \rightarrow (\dot{x}_t = 0)\right\}$$

Condition IV—Merging Identical Adjacent Intervals

There cannot be an interval followed by an instant followed by an interval where no quantity changes.

$$(\forall t)(\forall t')(\forall t'')\left\{\begin{array}{l}[\textsc{Adjacent}(t,t') \wedge \textsc{Adjacent}(t',t'') \wedge t'' > t \wedge \textsc{Interval}(t)] \\ \rightarrow (\exists x)\,[(x_t \neq x_{t'}) \vee (x_{t'} \neq x_{t''})]\end{array}\right\}$$

Condition V—Qualitative Mean-Value Theorem

The qualitative difference between the qualitative value of a quantity at an adjacent interval and instant must be equal to the qualitative derivative of that quantity during that interval.

$$(\forall t)(\forall t')(\forall t'')\left\{\begin{array}{l}[\textsc{Adjacent}(t,t')\wedge\textsc{Adjacent}(t',t'')\wedge t''>t\wedge\textsc{Interval}(t')]\\ \rightarrow[(x_{t''}=x_{t'}+\dot{x}_{t'})\wedge(x_{t'}=x_t+\dot{x}_{t'})]\end{array}\right\}$$

Condition VI—Termination

The system cannot be quiescent except during the last interval.

$$(\forall t)\left[(t \neq t_f) \rightarrow (\exists \dot{x})(\dot{x} \neq 0)\right]$$