Linear Algebra Cheat Sheet
Inverse Matrix

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1 Properties of Inverse Matrix

- $AA^{-1} = I$, $A^{-1}A = I$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$

Question 1 If A and M have inverse matrix A^{-1} and M^{-1} and

X, Y, Z are matrix!

- $AX = B$ $\quad A^{-1}(AX) = A^{-1}B \Rightarrow X = IX = A^{-1}B$
- $YM = C$ $\quad (YM)M^{-1} = CM^{-1} \Rightarrow Y = YI = CM^{-1}$
- $AZM^{\top} = D$ $\quad M^T$ is invertible. $(M^T)^{-1} = (M^{-1})^T$

what is X, Y, Z?

$$A^{-1}(AZM^T)(M^T)^{-1} = A^{-1}D(M^T)^{-1} \Rightarrow Z = I\cdot Z\cdot I$$
$$= A^{-1}\cdot D\cdot (M^T)^{-1} = A^{-1}D(M^{-1})^T$$

2 Elimination

Elimination as Matrix Operation We can write the operations to change equivalent linear system by $[A|b] \to [E_{ij}A|E_{ij}b]$ and $[P_{ij}A|P_{ij}b]$.

- Elimination matrix E_{ij}:
  - Replace row (i) by $*$*row(i) + row (j) $\leftarrow$ $[E_{ij}A | E_{ij}b]$
  - Identity matrix except $a_{ij} = *$

$$\begin{bmatrix} 1 & & & \\ & \ddots \; a_{ij}^{*} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

- Permutation matrix P_{ij}:
  - Swtich Row (i) with Row (j) $\leftarrow$ $[P_{ij}A | P_{ij}b]$
  - Identity matrix except $a_{ij} = a_{ji} = 1, a_{ii} = a_{jj} = 0$

ex. $\overset{3}{E_{32}}\overset{2}{E_{31}}\overset{1}{E_{21}}A$

1. Operate E_{21} first
2. Then operate E_{31}
3. Final operate E_{32}

Question 2 What is the matrix after the following operations

- Change Row 2 of A to Row 2 + 2* Row 1 $\quad E_{21}$ ① $\quad A \to E_{21}A$
- Switch Row 3 and Row 4 of the new matrix $\quad P_{34}$ ② $\to P_{34}(E_{21}A)$
- Change Row 4 of the new matrix to Row 4 + 2* Row 2 $\quad E_{42}$ ③ $\to E_{42}(P_{34}(E_{21}A))$

3 Inverse Matrix

- The inverse of a matrix exists if and only if the matrix is a square matrix and all column vectors are linear independent.
- The inverse of a matrix exists if and only if elimination produced n non-zero pivots.

Questions (answer is in the slide) Can you describe how the upper triangular form and their pivots look like for the following three cases

- The linear system have a single solution
- The linear system have no solution
- The linear system have infinite solutions

Questions Please ensure you know the answer of the following questions

- How to calculate the inverse of a matrix?
- What is the inverse of the elimination matrix? What is the inverse of the permutation matrix?

① Single Solution. n non-zero pivot

$$\left[\begin{array}{cccc|c} * & x & \cdots & x & x \\ & * & & x & x \\ & & \ddots & & \vdots \\ & & & * & x \end{array}\right]$$

*: non-zero

x: any number

② No Solution

pivot is zero → $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & \triangle \end{array}\right]$ $\triangle$: non-zero value $0x_1 + \cdots + 0x_n = \triangle$, not possible!!

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & -1 \\ 0 & \cdots & 0 & 0 \end{array}\right]$$

↑ no solution!!

③ Infinite Solution.

1. have a row $0 \cdots 0 \mid 0$
2. all the row, whose left part is all zero $0 \cdots 0$ the right part is also 0!!!

LU Decomposition!

$$A = L \cdot U$$

lower Traingular (L) ; upper Traingular (U)

LDU Decomposition

$$A = L\,D\,U$$

diagnal Matrix (D)

upper Traingular but all 1 on the diag (U)

Lower Traingular but all 1 on the diag (L)

if A is symmetric

$$A = L \cdot D \cdot L^T$$

(LDL decomposition)

1. LDU, LDL are Unique!!
2. LU is not unique.
   $A = L_1 U_1 = L_2 U_2$
   Then $L_1 = L_2 \cdot D$ (!) (D: diag)

1. LDU Decomposition is Unique!!!

Let's assume $A = L_1 U_1 = L_2 \cdot U_2$.

$$L_2^{-1} L_1 \cancel{U_1 U_1^{-1}} = \cancel{L_2^{-1}} L_2 U_2 U_1^{-1}$$

$$\Rightarrow \boxed{L_2^{-1} \cdot L_1} = \boxed{U_2 \cdot U_1^{-1}}$$

L_2^{-1}: L.T. ; $L_2^{-1} \cdot L_1$: L.T. ; U_2: U.T. ; U_1^{-1}: U.T. ; $U_2 \cdot U_1^{-1}$: U.T.

① Inverse of Lower Traingular is L.T.
② L.T. × L.T. → L.T.

$\Rightarrow$ $L_2^{-1} \cdot L_1$ or $U_2 \cdot U_1^{-1}$ are both L.T. and U.T. means. They are diag!!!

$$L_2^{-1} \cdot L_1 = D = U_2 \cdot U_1^{-1}$$

(D: diagnal)

$$L_1 = L_2 \cdot D \qquad L_2 (L_2^{-1} L_1) = L_2 \cdot D$$

$$U_2 = D \cdot U_1 \qquad D \cdot U_1 = (U_2 \cdot U_1^{-1}) U_1$$

if diag of L_2 and L_1 are 1. the my D is identity

hint!! $L_1 = L_2 \cdot D$

$\Rightarrow (L_1)_{11} = (L_2)_{11} \cdot d_{11} \Rightarrow d_{11} = 1 \Rightarrow D$ is identity!!