NYU

Linear Algebra

Lecture 2
Spans and Matrices

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Based on Dr. Ralph Chikhany’s Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Spans

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Reminder: Linear Combination

$$w = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

For example, what is the span of $(2, -4)$ and $(1, 1)$?

Span

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$$

Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

Is $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

More Precise Definition

"the set of" "such that"

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

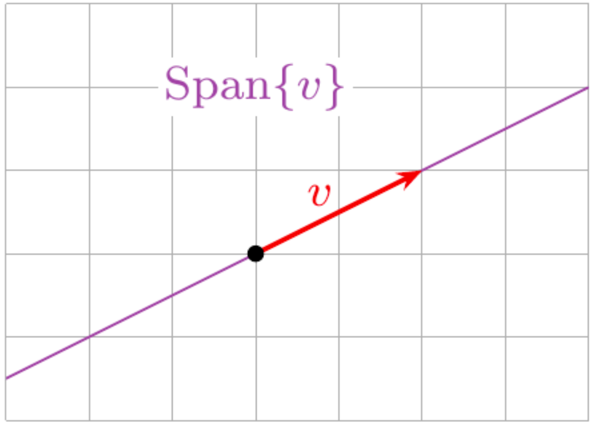
$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1v_1 + x_2v_2 + \cdots + x_pv_p \;\middle|\; x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \right\}.}$$

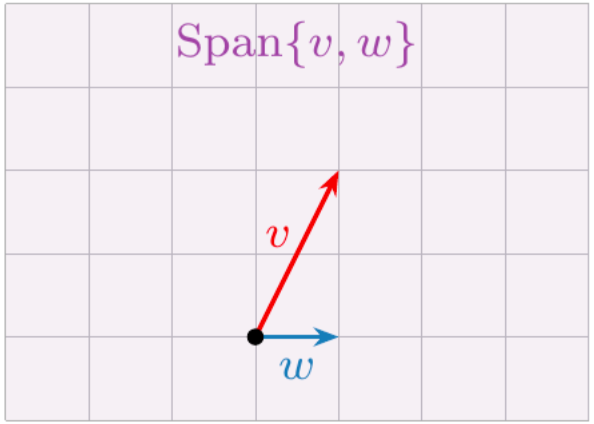
Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

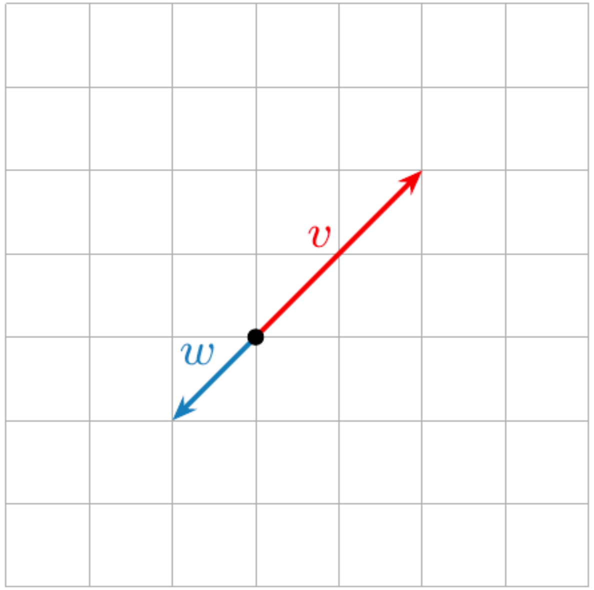
This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Span in $\mathbb{R}^2$

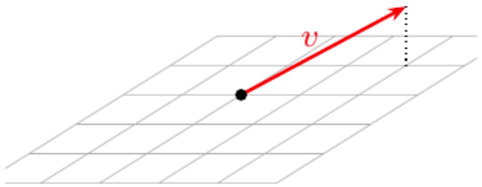
Drawing a picture of $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.



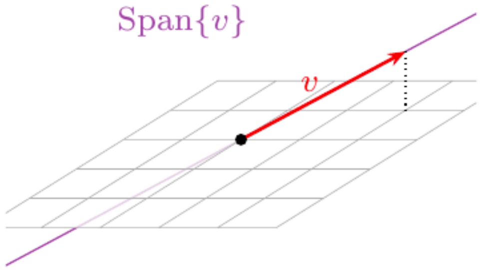




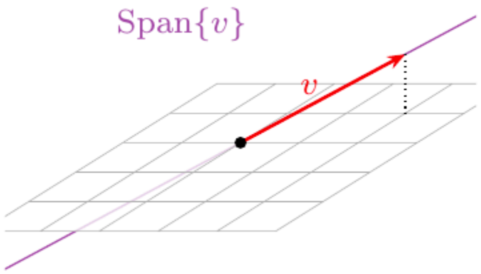
Span in $\mathbb{R}^3$

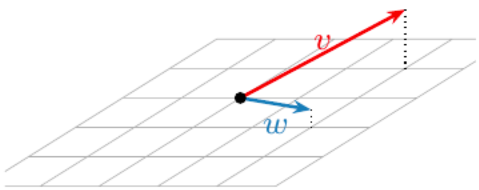


Span in $\mathbb{R}^3$

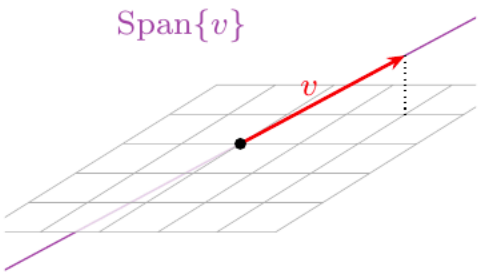


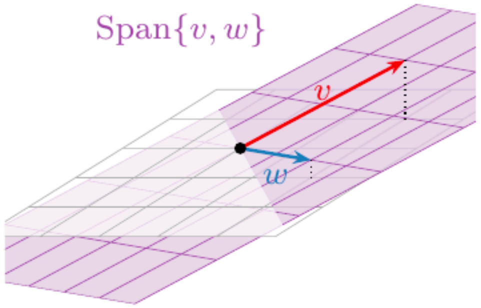
Span in $\mathbb{R}^3$



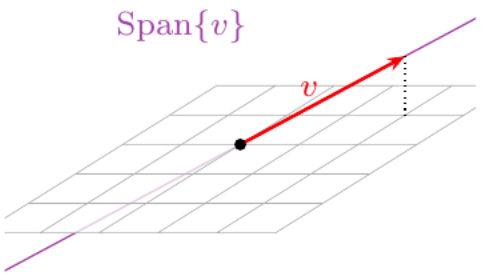


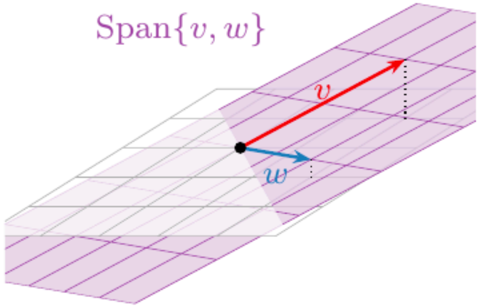
Span in $\mathbb{R}^3$

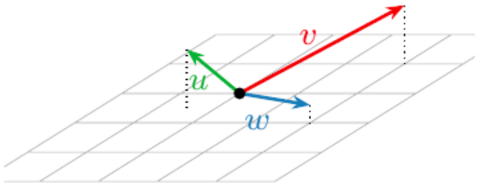




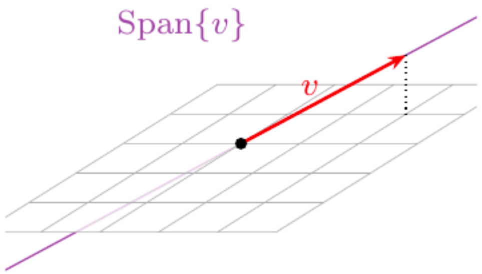
Span in $\mathbb{R}^3$

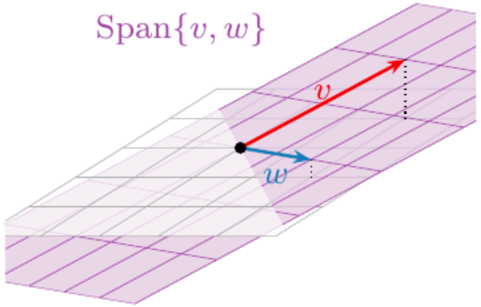


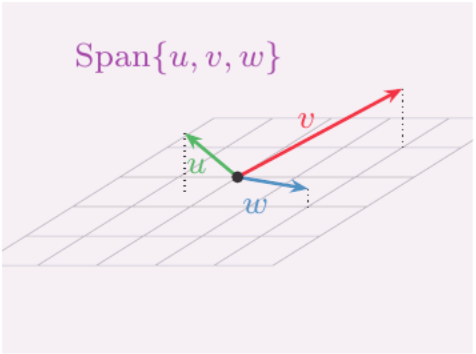




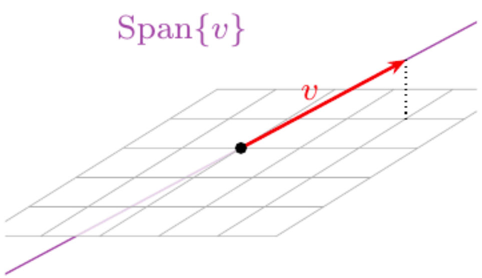
Span in $\mathbb{R}^3$

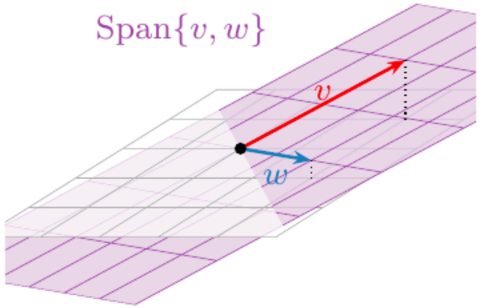


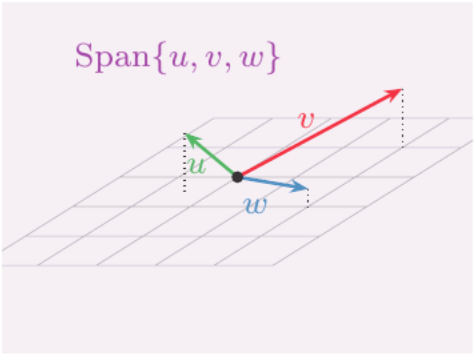


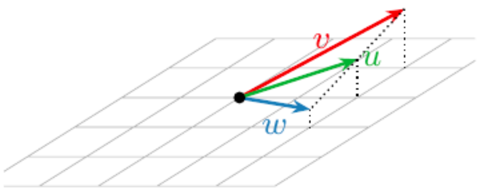


Span in $\mathbb{R}^3$

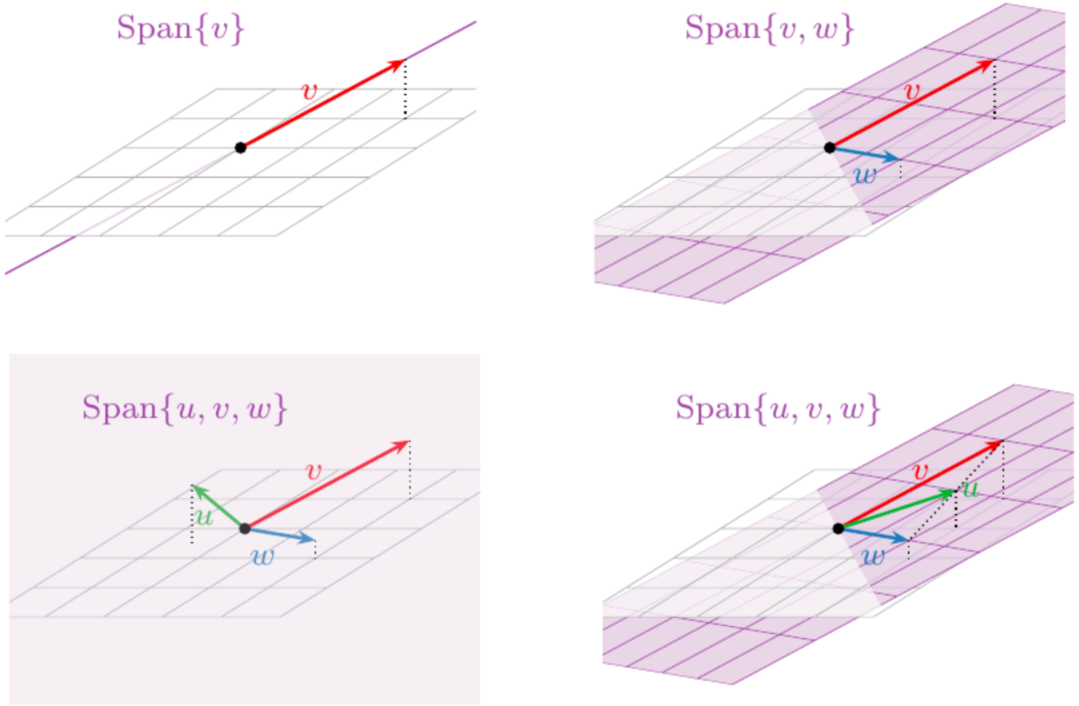








Span in $\mathbb{R}^3$



Strang Section 1.3 - Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Matrices

An $m \times n$ matrix A is a rectangular array of (real) numbers a_{ij} with m rows and n columns, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix is called **square** if it is $n \times n$, i.e., it has the same number of rows and columns.

Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

<u>Column</u> vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear Combination in Matrix Notation

A linear combination of n vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, in $\mathbb{R}^m$ is given by

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

This can be expressed as an $m \times n$ matrix A multiplying a vector $\vec{x} \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}}$$

Linear Combination in Matrix Notation

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute $A\vec{x}$.

Examples

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

The system $Ax = b$

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

The system $Ax = b$: What if x is unknown?

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When A and $\vec{x}$ are given, computing $\vec{b}$ is straight forward. However, the reverse is not always true (or even possible). That is, if A and $\vec{b}$ are given , it is not always possible to find $\vec{x}$.

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$.

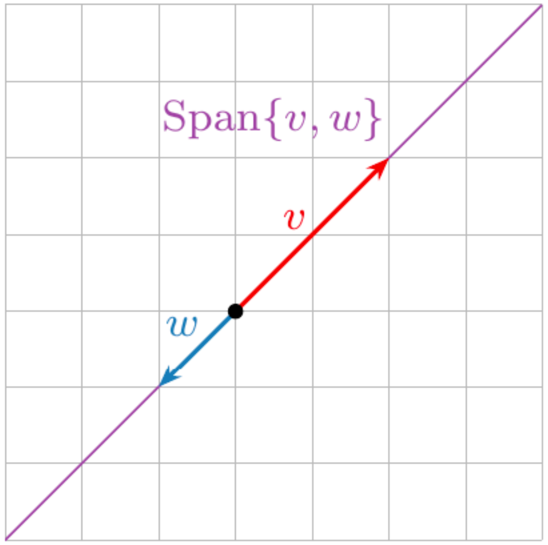
Suppose that b_1, b_2, and b_3 are given, and you want to compute x_1, x_2, and x_3 in terms of the components of $\vec{b}$.

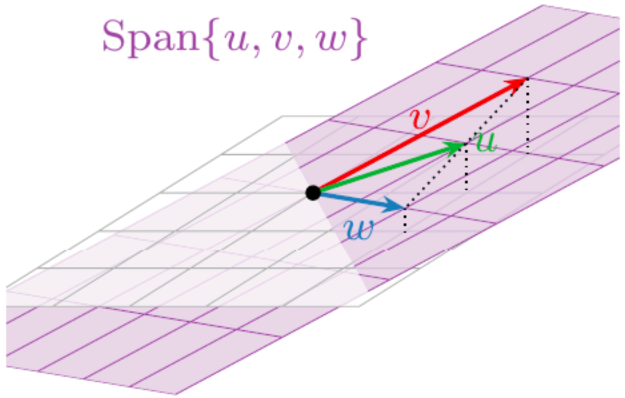
Break + Class Attendance

Linear Dependence and Independence

Linear In/Dependence

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.





This can mean many things. For example, it can mean you're using too many vectors to write your solution set.

Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

We will formalize this idea in the concept of *linear (in)dependence*.

Linear Dependence

Two vectors are said to be linearly dependent if they are multiples of each other, i.e., $\vec{u}$ and $\vec{v}$ are linearly dependent if $\vec{u} = c\vec{v}$ for some constant c.

Three vectors are linearly dependent if they all lie in the same plane, i.e., one of them is a linear combination of the other two. For example, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent if

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

for scalars a, b, and c not all zero.

In general, n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

for scalars $c_1, c_2, \ldots, c_n$ not all zero.

Linear Independence

A set of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is said to be linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

has **only** one solution $c_1 = c_2 = \cdots = c_n = 0$.

Combining Both

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \dots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \dots, x_p$, not all equal to zero, such that

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0.$$

This is called a **linear dependence relation**.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

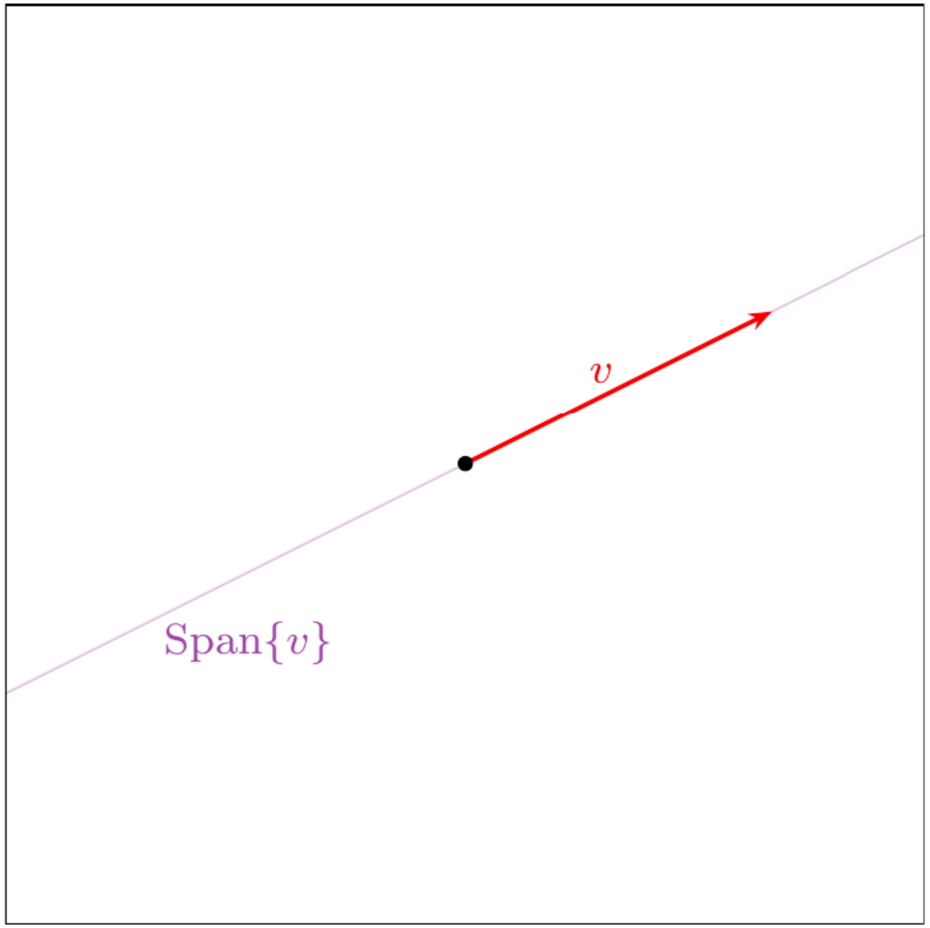
Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

An Important Result

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

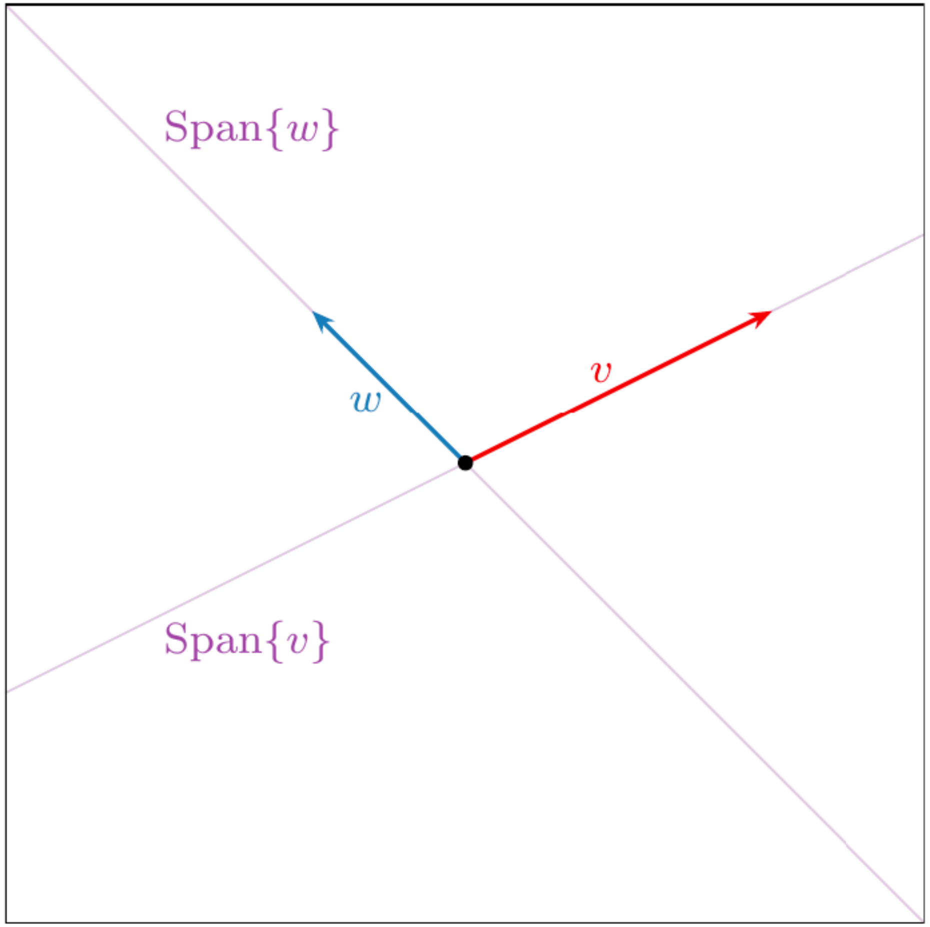
Linear In/Dependence – Visuals in $\mathbb{R}^2$



In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Linear In/Dependence – Visuals in $\mathbb{R}^2$

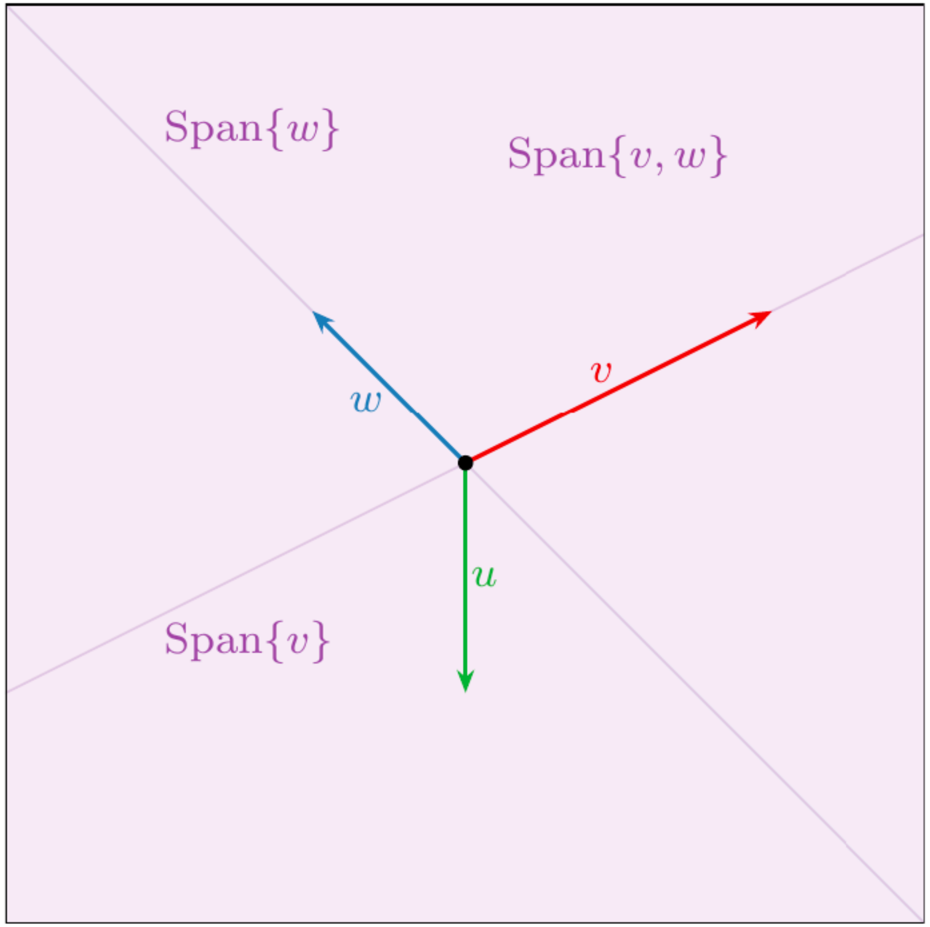


In this picture

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear In/Dependence – Visuals in $\mathbb{R}^2$



In this picture

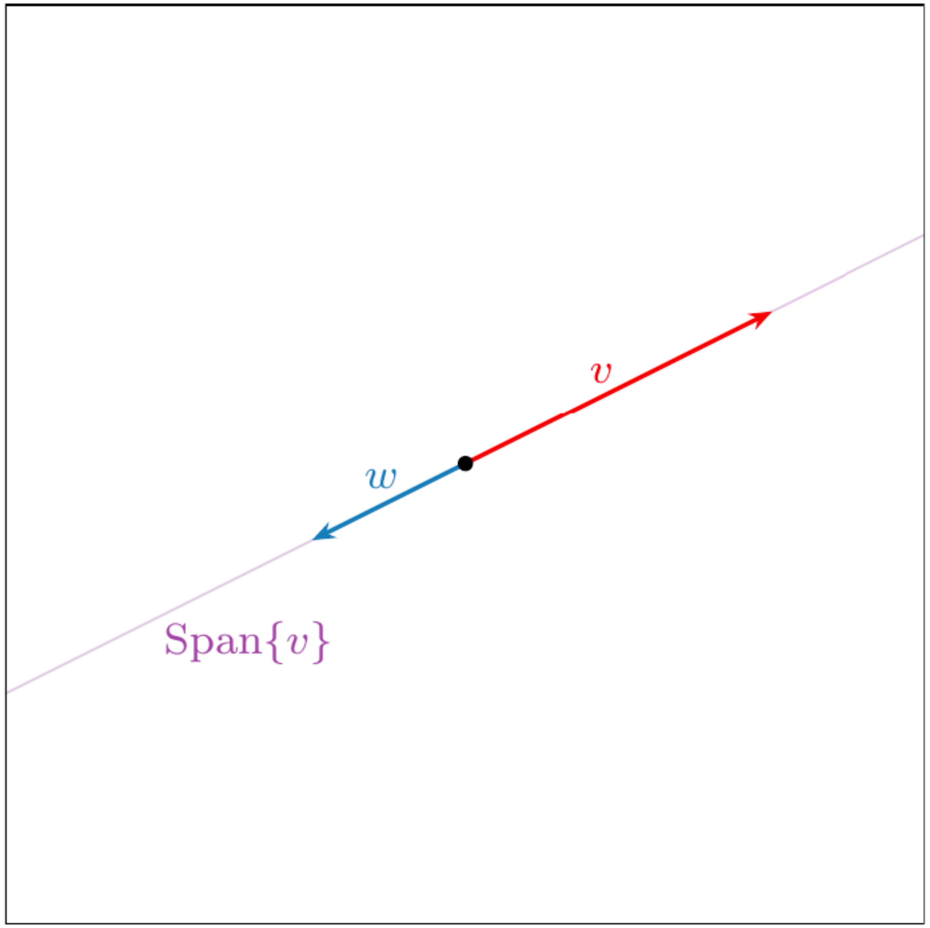
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly dependent: u is in Span$\{v, w\}$.

Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.

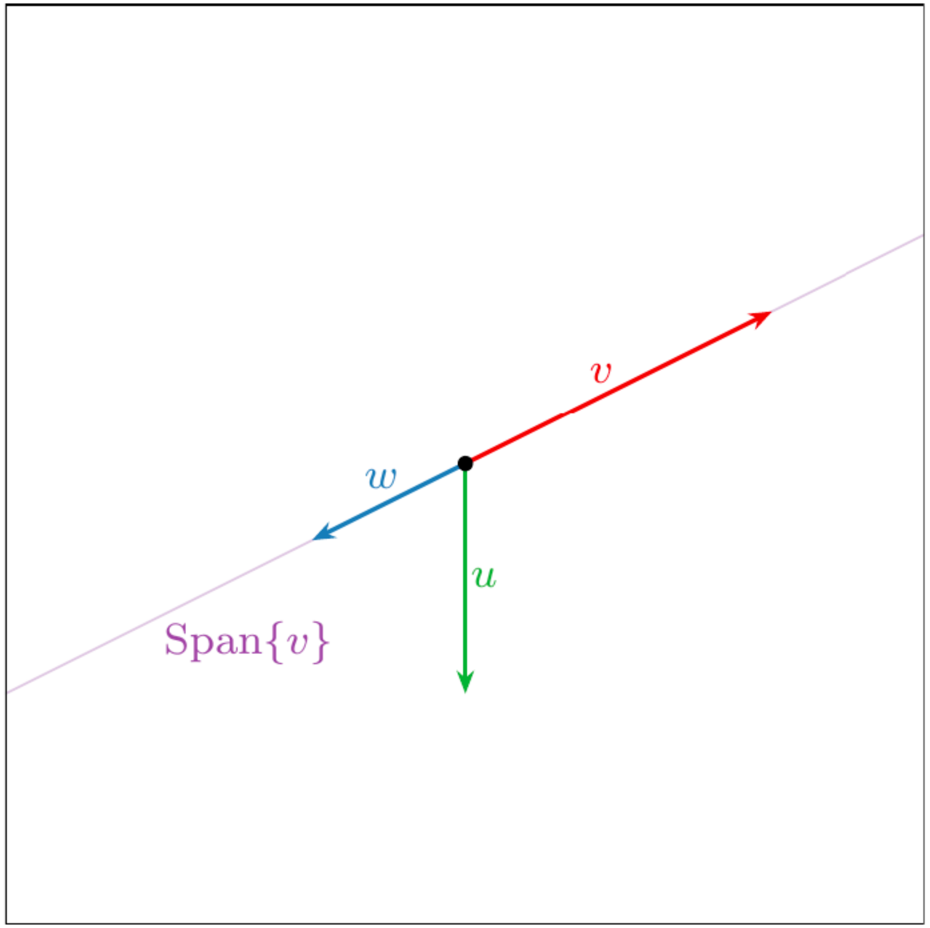
Linear In/Dependence – Visuals in $\mathbb{R}^2$



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Linear In/Dependence – Visuals in $\mathbb{R}^2$



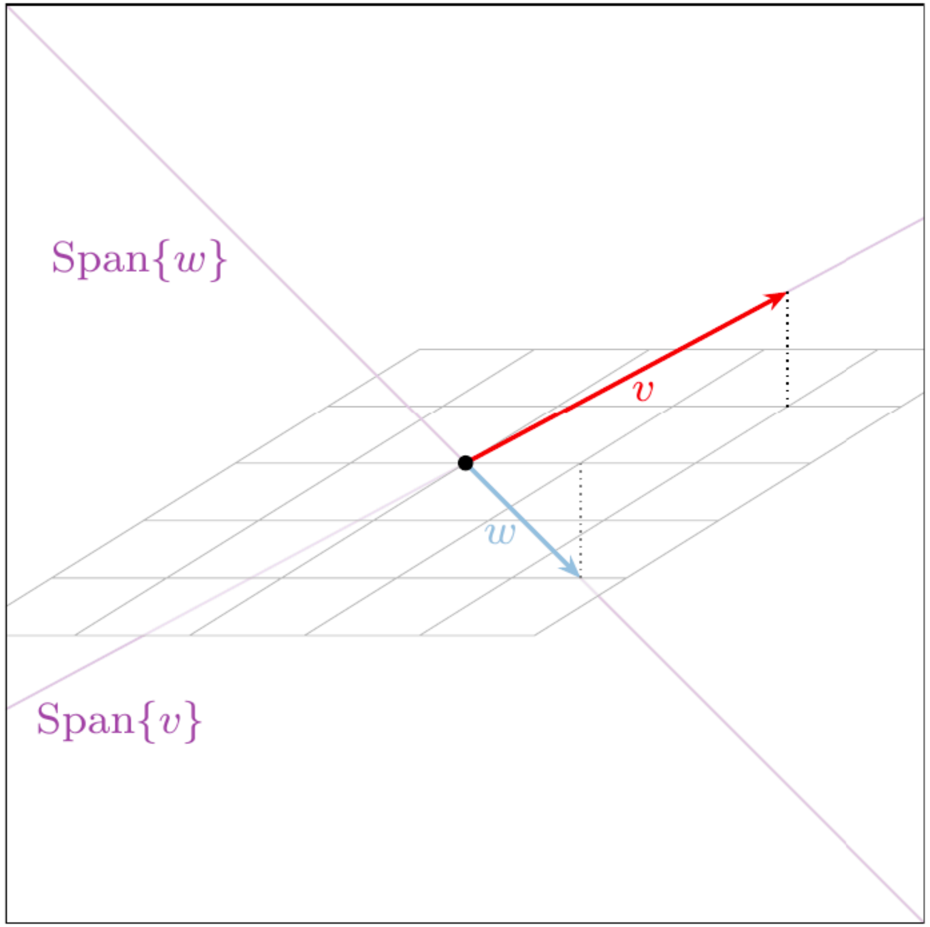
Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

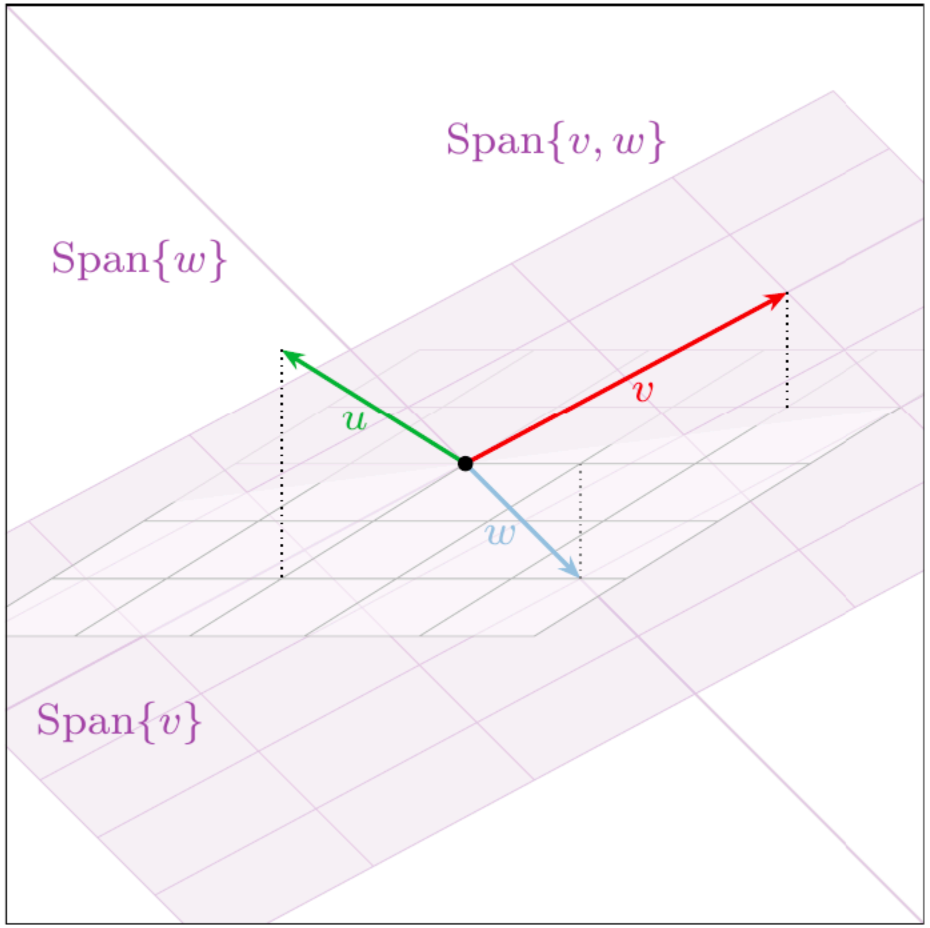
Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Linear In/Dependence – Visuals in $\mathbb{R}^3$

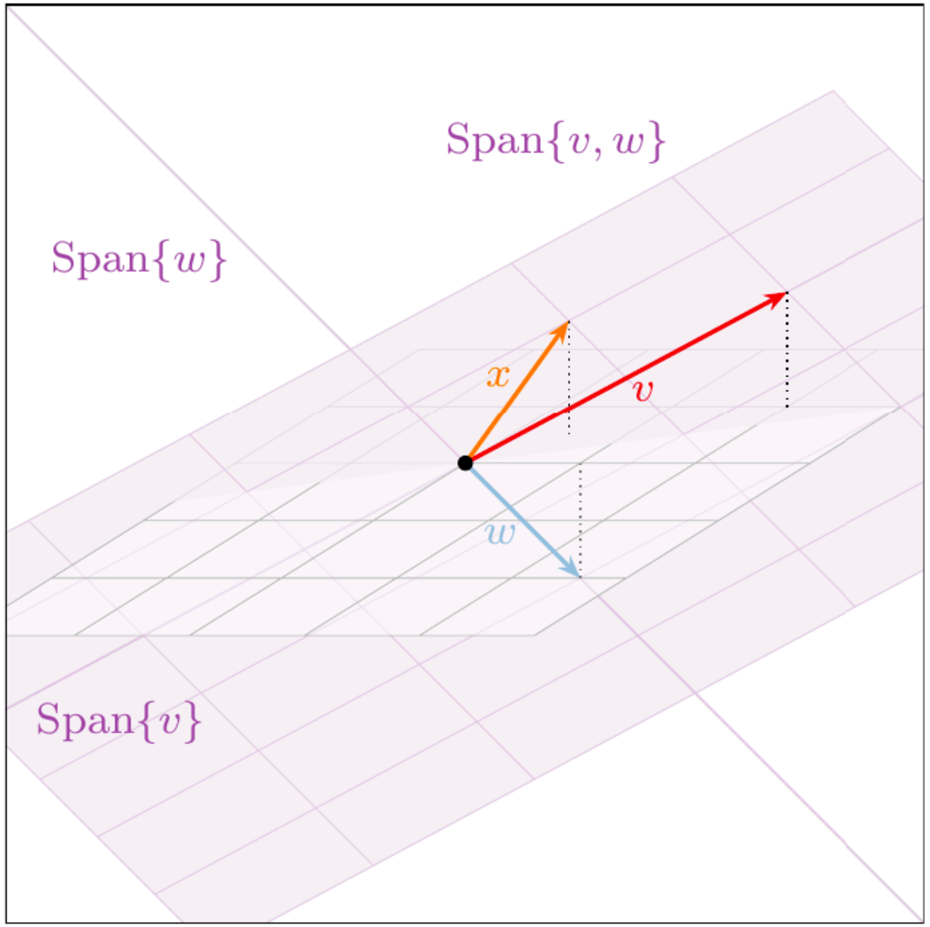


In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.

Linear In/Dependence – Visuals in $\mathbb{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in $\text{Span}\{v, w\}$.

Exercises

If you want more suggestions from the book (solutions easily available), message on the corresponding Campuswire thread

Exercises

Consider the following vectors and matrices

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$$

Compute the following vector-matrix products.

(a) $A\boldsymbol{u}$

(b) $B\boldsymbol{v}$

Exercises

Find A^{-1} by rewriting the following matrix-vector system

$$A\boldsymbol{x} = \boldsymbol{b} \qquad \Longrightarrow \qquad \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Exercises

Write the following system of equations as a matrix-vector system.

$$\textit{Hint:} \text{ Write } \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ so that } A\boldsymbol{x} = \boldsymbol{b}.$$

$$\begin{aligned} 2x + 2y &= 9 \\ -y + z &= 1 \\ x + 6z &= 0 \end{aligned}$$

Exercises

Write the following matrix-vector system as a system of linear equations

$$\begin{bmatrix} 1 & 5 & 2 \\ 3 & 0 & -1 \\ 8 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}$$

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Questions?