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NEWYORK UNIVERSITY
Linear Algebra Final Review

Subject: MATH-UA 140 Linear Algebra Year: 2024(Sem 2)
Name of Examiners: . Time allow: 1001

Instruction to Candidate : (only on page 1)
(1) This paper contains ________ questions.
(2) Candidates must answer ________ questions.

Question No 1

Diagonalize A and compute $V\mathbf{A}kV^{-1}$ to prove this formula for A^k:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ has } A^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

and what is the meaning of $\lim_{k\to\infty} \frac{1}{3^k} A^k$.

Solution:

The eigenvalues of A are 3 and 1, and the corresponding eigenvectors are $v_1 = (-1, 1)$, $v_2 = (1, 1)$. Therefore, A can be diagonalized as $A = VAV^{-1}$, where $V = [v_1, v_2]$, $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $A^k = V\Lambda^k V^{-1} = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$.

$\lim_{k\to\infty} \frac{1}{3^k} A^k = \lim_{k\to\infty} \frac{1}{2}\begin{pmatrix} \frac{1}{3^k}+1 & \frac{1}{3^k}-1 \\ \frac{1}{3^k}-1 & \frac{1}{3^k}+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (the largest eigen vector)

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Question No 2

For u is a unit vector prove that $Q = I - 2uu^\top$ is an symmetric orthogonal matrix. Prove $\|Qx\| = \|x\|$.

Solution:

Q is symmetric because $uu^\top$ is symmetric.
Then $QQ^\top = QQ = Q^2 = (I - 2uu^\top)^2 = I - 2uu^\top - 2uu^\top + 4u \underbrace{u^\top u}_{u^\top u = \|u\|^2 = 1} u^\top = I - 4uu^\top + 4uu^\top = I$

For all orthogonal matrix Q, we have

$$\|Qx\|^2 = (Qx)^\top (Qx) = x^\top Q^\top Q x = x^\top x = \|x\|^2$$

for $Q^\top Q = I$

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Question No 3

What are the four fundamental subspaces of $M = I - P$ in terms of the column space of P.

Solution

For a projection matrix P: (projection matrix is always symmetric)

- $x \in \text{col}(P) = \text{row}(P)$: Px=x
- $x \in \text{Nul}(P) = \text{LeftNul}(P)$: Px=0

For matrix $I - P$

- $x \in \text{col}(P) = \text{row}(P)$: (I-P)x=x-Px=x-x=0
- $x \in \text{Nul}(P) = \text{LeftNul}(P)$: (I-P)x=x-Px=x-0=x

is a also a projection matrix.
Left Null space = Right Null space = Colume space of P.
Column space = Row space = orthogonal complement of the colume space of P.

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Question No 4

P is a Projection Matrix, prove P is symmetric and $P^2 = P$. What is the eigenvalue of Projection matrix P. Prove that $I - 2P$ is an orthogonal matrix

Solution

$P = A(A^TA)^{-1}A^T$ then

- $P^2 = A\underbrace{(A^TA)^{-1}A^TA}_{=I}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$
- $P^T = (A(A^TA)^{-1}A^T)^T = A^T(A^TA)^{-T}A = A(A^TA)^{-1}A^T$ (For A^TA symmetric)

Eigenvalue is $1, 0$ (for $P^2 = P$ so eigenvalues should satisfies $\lambda^2 = \lambda$)

Since P is a projection matrix, we have $P = P^T$. To show that Q is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$QQ^T = (I - 2P)(I - 2P)^T$$

$$= (I - 2P)(I^T - 2P^T)$$

$$= (I - 2P)(I - 2P) \quad \text{(since } I \text{ and } P \text{ are symmetric)}$$

$$= I - 4P + 4P^2$$

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

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Question No 5

If $A^2 = -A$, what is the possible value of $\det(A)$.

Solution

$A^2 = -A$ means $\det(A^2) = \det(-A)$ however

- $\det(A^2) = \det(A)^2$
- $\det(-A) = (-1)^n \det(A) = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$

Thus

$$\det(A)^2 = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$$

which means

$$\det(A) = \begin{cases} 0, -1 & \text{if } n \text{ is odd} \\ 0, 1 & \text{if } n \text{ is even} \end{cases}$$

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Question No 5

Suppose an $m \times n$ matrix A has rank r. What are the ranks of

(a) A^T?

(b) AA^T?

(c) $AA^T + \lambda I$ $(\lambda > 0)$?

(d) $A^T AA^T$?

Solution

Answer 1

(A) r

(B) we showed in class it's r (page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

(C) it's a positive definite matrix with all eigenvalues lareger than λ, think why.

(D) r (similar page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

Answer 2 Using SVD

(A) $\text{rank}(A^T) = \dim(\text{row}(A^T)) = \dim(\text{col}(A)) = \text{rank}(A) = r$.

(B) Let $A = U\Sigma V^T$ be a full SVD. Then,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T.$$

Thus, $U\Sigma^2 U^T$ is a SVD of AA^T. If Σ has r positive singular values then so will Σ^2. Therefore, the rank of AA^T is r.

(C) Since $I_m = UU^T$, the equation above yields $AA^T + \lambda I = U\Sigma^2 U^T + \lambda I = U(\Sigma^2 + \lambda I_m)U^T$. Since $\Sigma^2 + \lambda I = \text{diag}(\sigma_1^2 + \lambda, \ldots, \sigma_r^2 + \lambda, \ldots, \lambda)$, the rank is m.

(D) $A^T AA^T = (U\Sigma V^T)^T (U\Sigma V^T)(U\Sigma V^T)^T = V\Sigma^T U^T U\Sigma V^T U^T U\Sigma V^T = V\Sigma^T \Sigma\Sigma^T V^T = V\Sigma^3 V^T$. Σ^3 has r positive singular values as like Σ. Therefore, the rank is r.

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Question No 6

The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

For a matrix A, the eigenspace with eigenvalue λ is the kernel of the matrix $A - \lambda I$. Here we have $\lambda = 1$, so we subtract I from each of the matrices above:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and find the dimensions of the kernels.
The ranks of these matrices are 0, 2, 2, 1 respectively, so by the rank-nullity theorem the dimensions of the kernels are 3, 1, 1, 2.
Answer: 3, 1, 1, 2.

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Question No 7

For $A \in \mathbb{R}^n$ has singular value $\sigma_1, \cdots, \sigma_n$ prove

- $\mathrm{tr}(A^\top A) = \sigma_1^2 + \cdots + \sigma_n^2$
- $\mathrm{tr}((A^\top A + \lambda I)^{-1} A^\top A) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$

Solution

Using SVD $A = U\Sigma V^\top$ Then we have

- $A^\top A = V\Sigma^\top \Sigma V^\top$ so $\mathrm{tr}(A^\top A) = \mathrm{tr}(\Sigma^\top \Sigma) = \sigma_1^2 + \cdots + \sigma_n^2$
- $(A^\top A + \lambda I) = V(\Sigma^\top \Sigma + \lambda I)V^\top$, $(A^\top A + \lambda I)^{-1} = V(\Sigma^\top \Sigma + \lambda I)^{-1} V^\top$
- $(A^\top A + \lambda I)^{-1} A^\top A = V(\Sigma^\top \Sigma + \lambda I)^{-1} \Sigma^\top \Sigma V^\top = V \begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix} V^\top$
- $$\mathrm{trace}((A^\top A+\lambda I)^{-1} A^\top A) = \mathrm{trace}\left(\begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix}\right) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$$

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