Lecture 10

Independence, Basis and Dimension

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Strang Sections 3.4 – Independence, Basis and Dimension

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed) and N. Hammoud's NYU lecture notes.

Row Space of a Matrix

Theorem

The row space of an $m \times n$ matrix A is the span of the nonzero rows in REF(A).

For example, if $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, then REF($A$)$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \implies \text{Row } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Another way to compute the row space of A is by finding the column space of A^T, since the columns of A^T are equal to the rows of A.

$$\boxed{\text{Row } A = \text{Col } A^T = \text{span}\left\{\text{linearly independent columns of } A^T\right\}}$$

Example

Describe the column space and the row space of $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$.

Basis of a Vector Space

What is a Basis?

A basis β for a vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, such that

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, and

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V.

Examples

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- standard basis for $\mathbb{R}^2$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

- standard basis for $\mathbb{R}^3$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\vdots$

- standard basis for $\mathbb{R}^n$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

Examples

A basis β for a vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, such that

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, and

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V.

- another basis for $\mathbb{R}^2$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$
- the pivot columns form a basis for $\operatorname{Col} A$
- the nullspace solutions form a basis for $\operatorname{Nul} A$

Vector as a Linear Combination of Basis Vectors

Theorem: If $\vec{v} \in V$, then there is a unique way to write $\vec{v}$ as a linear combination of the basis vectors of V.

Example

Find bases for the column and row spaces of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$.

Examples

Find a basis for $\mathbb{M}_{2\times 2}$, the vector space of all 2×2 matrices.

Find a basis for the vector space of all 3×3 diagonal matrices.

Dimension of a Vector Space

Meaning of Dimension

The dimension of a vector space V is the number of vectors in a basis β for V.

- $\dim(\mathbb{R}^n) = n$

- For an $m \times n$ matrix A with $\operatorname{rank}(A) = r$.
  - $\dim(\operatorname{Col} A) = r$
  - $\dim(\operatorname{Row} A) = r$
  - $\dim(\operatorname{Nul} A) = n - r$

Theorem and Proof Outline

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are basis for a vector space V, then $m = n$.

Theorem and Full Proof (Optional Reading)

If $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{w}_1, \dots, \vec{w}_n$ are basis for a vector space V, then $m = n$.

Suppose $n > m$

Then $\beta_v = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis. Then for $\vec{w}_1 \in V$ we have $\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{m1}\vec{v}_m$. Similarly, for $\vec{w}_2 \in V$ we have $\vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{m2}\vec{v}_m$ In general, any $\vec{\omega}_i$ can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

$$\underbrace{[\vec{w}_1 \vec{w}_2 \dots \vec{w}_n]}_{W} = \underbrace{[\vec{v}_1 \vec{v}_2 \dots \vec{v}_m]}_{V} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Notice that the matrix (call it A) is short and wide since we assumed $n > m$. Thus $A\vec{x} = \vec{0}$ has a nonzero solution.

$$A\vec{x} = \vec{0} \Rightarrow VA\vec{x} = \vec{0} \Rightarrow W\vec{x} = \vec{0}$$

The columns of W are not linearly independent , they can't form a basis (contradiction with the initial assumption)

Suppose $m > n$.

Repeat the same steps and eventually we have:

$$\underbrace{[\vec{v}_1 \vec{v}_2 \dots \vec{v}_n]}_{V} = \underbrace{[\vec{w}_1 \vec{w}_2 \dots \vec{w}_m]}_{W} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{m2} & \cdots & b_{nm} \end{bmatrix}$$

Notice that the matrix (call it B) is short and wide since we assumed $m > n$. Thus $B\vec{x} = \vec{0}$ has a nonzero solution.

$$B\vec{x} = \vec{0} \Rightarrow WB\vec{x} = \vec{0} \Rightarrow V\vec{x} = \vec{0}$$

The columns of V are not linearly independent , they can't form a basis (contradiction with the initial assumption)

Conclusion: The only way to avoid these contradictions is to have $m = n$.

Example

Find a basis and the dimension of Col A and Nul A, where

$$A = \begin{bmatrix} 1 & -3 & -6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Example

Find a basis and the dimension of Col A and Nul A, where

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Find a basis and the dimension of Col A and Nul A, where

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