NYU

Linear Algebra

Lecture 4

Matrix Operations

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Based on Dr. Ralph Chikhany's Slide

Strang Sections 2.3 – Elimination Using Matrices and 2.4 – Rules for Matrix Operations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Permutation Matrices

Recall

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\xrightarrow{\text{in augmented form}} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{P_{31}\,\vec{x}} \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

An elimination matrix is an $n \times n$ matrix which takes the $n \times n$ identity matrix and changes one of the zeros in the lower triangular or the upper triangular part of the identity matrix to some nonzero entry.

Elimination Matrices

$$I = \begin{array}{cc}
\begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\
\begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\
0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\
\vdots & & \ddots & & & & & \\
0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\
\vdots & & & & \ddots & & & \\
0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\
\vdots & & & & & & \ddots & \\
0 & 0 & \dots & 0 & \dots & 0 & \dots & 1
\end{bmatrix} &
\begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array}
\end{array}$$

Elimination Matrices

$$E_{ji} = \begin{array}{cc} \begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array} \end{array}$$

Elimination Matrices

$$E_{ji} = \begin{array}{c} \begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \color{red}{\star} & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array}$$

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\color{red}{\star \cdot x_i}) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

When this matrix acts on a vector in $\mathbb{R}^n$, it adds $\star$ copies of the first row to the third row.

Elimination Matrices

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$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{31}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \star & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

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Elimination Matrices

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{E_{32}\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \star & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + (\star \cdot x_2) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

Matrix Operations

Recall

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

Matrix Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size.*

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A+B &= B+A & \qquad (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB & (c+d)A &= cA+dA \\ (cd)A &= c(dA) & A+0 &= A \end{aligned}$$

Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & & \\ b_{l1} & \dots & b_{lm} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & \dots & c_{1k} \\ c_{21} & \dots & c_{2k} \\ \vdots & & \\ c_{n1} & \dots & c_{nk} \end{bmatrix}$$

Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

Matrix Multiplication

A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Matrix Multiplication

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Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

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On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

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It follows that

$$\boxed{AB = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}}$$

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array} \end{array} \cdot \begin{array}{c} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} \\ j\text{th column} \end{array} = \begin{array}{c} \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix} \\ ij \text{ entry} \end{array}$$

Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

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This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

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Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Matrix-Matrix and Matrix-Vector

Matrix vector multiplication is a Matrix Matrix multiplication

$$A[\vec{v}_1, \cdots, \vec{v}_k] = [A\vec{v}_1, \cdots, A\vec{v}_k]$$

$$\begin{bmatrix} \vec{r}_1^\top \\ \vec{r}_2^\top \\ \cdots \\ \vec{r}_k^\top \end{bmatrix} A = \begin{bmatrix} \vec{r}_1^\top A \\ \vec{r}_2^\top A \\ \cdots \\ \vec{r}_k^\top A \end{bmatrix}$$

Matrix Multiplication

Let $A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 3 & -1 & 0 & -3 \\ 2 & -2 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$.

Compute AB and BA (if possible).

Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$\begin{aligned} A(BC) &= (AB)C & \qquad A(B+C) &= (AB+AC) \\ (B+C)A &= BA+CA & c(AB) &= (cA)B \\ c(AB) &= A(cB) & I_nA &= A \\ AI_m &= A & & \end{aligned}$$

Most of these are easy to verify.

Matrix Multiplication

Warnings!

- AB is usually not equal to BA.

  In fact, AB may be defined when BA is not.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

- $AB = 0$ does not imply $A = 0$ or $B = 0$.