Recap and Fundamental Theorem of Linear Algebra.

$Col(A) \subseteq \mathbb{R}^m$. span {all n Column Vectors}
$Row(A) \subseteq \mathbb{R}^n$ span {all m row Vectors}
$Nul(A) \subseteq \mathbb{R}^n$ $\{x \mid$ all solution of $Ax=0\}$
$Nul(A^T) \subseteq \mathbb{R}^m$ $\{y \mid$ all Solution of $A^Ty=0\}$ $= y^TA=0$
left Null space "left Nul space"
$(A^Ty)^T = y^TA$

same size
same size

REF $\begin{cases} \dim[Col(A)] = r \\ \dim[Row(A)] = r \end{cases}$
#Free Variable
$\dim(Nul(A)) = n-r$ add given
$\dim(Nul(A^T)) = m-r$ add give m

$A \in \mathbb{R}^{m\times n}$. $rank(A) = r$.
m rows and n columns
$Ax=b$ have m Eq n Variable
$A^T \in \mathbb{R}^{n\times m}$. $A^Ty=b$ have n Eq and m Variable.

Geometry Meaning of
$$\dim[Row(A)] + \dim(Nul(A)) = n$$

Let's write down A using row Representation

$$Ax = 0 \iff Ax = \begin{bmatrix} \vec{r_1}^T \\ \vec{r_2}^T \\ \vdots \\ \vec{r_m}^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r_1}\cdot\vec{x} \\ \vec{r_2}\cdot\vec{x} \\ \vdots \\ \vec{r_m}\cdot\vec{x} \end{bmatrix} = 0$$

$\vec{x} \in \mathbb{R}^n$

$\vec{r_1} \cdots \vec{r_m}$ all all row vectors. $\vec{r_i} \in \mathbb{R}^n$!!

The Solution $\vec{x}$ should orthogonal to all row vector

$\vec{r_1}\cdot\vec{x} = 0$ $\vec{r_1} \perp \vec{x}$
$\vec{r_2}\cdot\vec{x} = 0$ $\vec{r_2} \perp \vec{x}$
$\vdots$
$\vec{r_m}\cdot\vec{x} = 0$ $\vec{r_m} \perp \vec{x}$

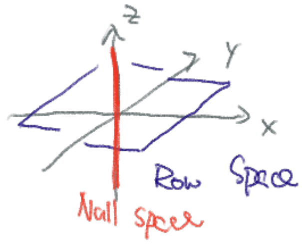
$$Nul(A) = (Row(A))^\perp$$

$Nul(A) = \{x \mid Ax=0\} = \{x \mid$ all vectors that is orthogonal to the row vectors$\}$
Row Space

Example if $Row(A), Nul(A) \subseteq \mathbb{R}^3$ $n=3$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ← x-axis ← y-axis all solution of $Ax=0$ is $x = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \middle| x_3 \in \mathbb{R} \right\}$

$\begin{cases} Row(A) = x\text{-}y \text{ plane} \\ Nul(A) = z\text{-axis} \end{cases}$



Similarily. $$Nul(A^T) = (Col(A))^\perp$$

Linear Algebra

Midterm Review Question

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Exercise Consider the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Elimination!

- Write down $A = LU$ where L is an lower traingular matrix and U is a REF. Same as $L\ U \to$ REF
- Calculate the four fundamental subspaces

REF $= U$

$$A \to \underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{E_{31}A} R_3 - R_1 \to \underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{E_{32}(E_{31}A) = E_{32}E_{31}A} R_3 - R_2$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$$

be careful about the order!

$$E_{32}E_{31}A = U \qquad A = (E_{32}E_{31})^{-1}U = E_{31}^{-1}E_{32}^{-1}A$$

Third.

↑

→ Eliminate Second

first Eliminate first col

$$E_{31}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \qquad E_{32}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$L = E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 1 & 1 & 1 \end{bmatrix}$$

!! You should follow My order of Elimination Method. if you want to copy it.

$$L = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & 1 & 1 \end{bmatrix}, \quad U = \begin{array}{c} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\quad x_2 \ \ x_3 \quad\quad x_5 \end{array} \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

Make sure to check whether $LU = A$

1. $\text{Row}(A) = \text{Row}(U) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

Copy the pivot rows !!!

2. $\text{Nul}(A) = \text{Nul}(U)$

x_2, x_3, x_5 are free

Solve x_1, x_4

First solve the Eq

- $x_1 = -3x_2 - 5x_3 - 7x_5$
- $x_4 = -2x_5$

Then write down in vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_2 - 5x_3 - 7x_5 \\ x_2 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix}$$

free

$$= x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Exercise 1. all the possible rank of

$$\begin{bmatrix}1&1&a\\3&3&a\\a&a&a\end{bmatrix}$$

← by your self

when a varies.

2. all the possible rank of

$$\begin{bmatrix}1&3&a\\3&1&a\\a&a&a\end{bmatrix}$$

when a varies.

Elimination

$$\begin{pmatrix}1&3&a\\3&1&a\\a&a&a\end{pmatrix} \xrightarrow[R3 \leftarrow R2 - aR1]{R2 \leftarrow R2 - 3R1} \begin{pmatrix}1&3&a\\0&-8&-2a\\0&-2a&a-a^2\end{pmatrix}$$

← don't divide by zero

check if this will be 0 here & ≠ 0

$$\xrightarrow{R3 \leftarrow R3 - \frac{a}{4}R1} \begin{pmatrix}1&1&a\\0&-8&-2a\\0&0&a-\frac{a^2}{2}\end{pmatrix}$$

$a=0$, or $a=1/2$

if $a-\frac{a^2}{2}=0$, then rank $=2$

otherwise rank $\neq 3$

$a\neq 0$, and $a\neq 1/2$

☆ if you set

$$\begin{pmatrix}1&1&a\\&a&-2a\\&&a-\frac{a^2}{2}\end{pmatrix}$$

then $a=0 \Rightarrow$ rank $=1$

$a=1/2 \Rightarrow$ rank $=2$.

otherwise $\Rightarrow$ rank $=3$

Exercise 1. What is all the possible rank of

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

when a, b, c, d varies.

2. When is A invertible?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \to \quad E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \to \quad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \to \quad E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

asking number of 0

in a, b-a, c-b, d-c

calculating Col $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \text{span}\{\cdots\}$

Exercise For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$, where the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$? For those b, write down the complete solution.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 3 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right) \xrightarrow{R2 \leftarrow R2 - 2R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 \end{array}\right)$$

$$\xrightarrow{R3 \leftarrow R3 - R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - (b_2 - 2b_1) \end{array}\right)$$

$= b_3 + 2b_1 - b_2$

$! \; b_3 + 2b_1 - b_2 = 0 \Rightarrow b_3 = b_2 - 2b_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_2 is free

Solve x_1, x_3

$x_1 = -x_2 - b_2 + 3b_1$

$x_3 = b_2 - 2b_1$

Exercise Calculate the inverse matrix of $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$?

Use elmination start from $[M|I]$ to $[I|M^{-1}]$

$$
\begin{aligned}
[M|I] = \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 2 & | & 0 & 1 & 0 \\ 1 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R3 \;\; \leftarrow R3 - R1]{R2 \;\; \leftarrow R2 - R1} \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 2 & 3 & | & -1 & 0 & 1 \end{pmatrix} \\
\text{Use R1 to ellimate the column 1 in R2 and R3} \\
\xrightarrow[R3 \;\; \leftarrow R3 - 2 \cdot R2]{R1 \;\; \leftarrow R1 - 1 \cdot R2} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
\text{Use R2 to ellimate the column 2 in R1 and R3} \\
\xrightarrow[R2 \;\; \leftarrow R2 - 1 \cdot R3]{R1 \;\; \leftarrow R1 - 0 \cdot R3} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 3 & -1 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
\text{Use R3 to ellimate the column 3 in R1 and R2}
\end{aligned} \tag{1}
$$

Thus

$$
M^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}
$$

Check if MM^{-1} is idenity! equal to check

- $(1,1,1) \cdot (2,-2,1) = 1$,$(1,2,2) \cdot (2,-2,1) = 0$, $(1,3,4) \cdot (2,-2,1) = 0$
- $(1,1,1) \cdot (-1,3,-2) = 0$,$(1,2,2) \cdot (-1,3,-2) = 1$, $(1,3,4) \cdot (-1,3,-2) = 0$
- $(1,1,1) \cdot (0,-1,1) = 0$,$(1,2,2) \cdot (0,-1,1) = 0$, $(1,3,4) \cdot (0,-1,1) = 1$

Harder Question:

$Nul(A^T) = m - r$ needs more information!

What is all the possible value?

$m \geq r$, $\dim(Nul(A^T))$ can be any $\mathbb{Z}^+$

2# Free Variable $\Rightarrow \dim[Nul(A)] = 2$

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\dim(\text{col}(A)) = 3$

$A \in \mathbb{R}^{m \times n}$? what is m? we need more information

what is n? $n = 5$

m is # Eq

n is # Variable

$= \text{rank } A = 3$

$\Rightarrow \dim(Col(A)) = n - \#\text{Free Variable} = 5 - 2 = 3$, $\dim(Row(A)) = \dim(Col(A))$

2. There exist a matrix A whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$ and whose nullspace is spanned by $(1, 2, 3, 6)$

Fix the size of A $n = 4$ $m = 3$

$\dim(Col(A)) = 2$

$\dim(Nul(A)) = 1$

is this possible? No! $\dim(Col(A)) + \dim(Row(A)) = n$ (be careful)

but $2 + 1 \neq 4$!!!

3.

- For a matrix $A \in \mathbb{R}^{4\times5}$, the largest possible rank of A is 5. No $\text{rank}(A) \leq 4$

- For a matrix $A \in \mathbb{R}^{4\times5}$ there are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. No
- For a matrix $A \in \mathbb{R}^{4\times3}$, $\text{rank}(A) = 3$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{5\times4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes
- For a matrix $A \in \mathbb{R}^{4\times5}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. No
- For a matrix $A \in \mathbb{R}^{5\times4}$, $\text{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. Yes

- $Y = AX$ and A is an invertible matrix, then $\text{rank}(Y) = \text{rank}(X)$. Yes

$[Av_1 \cdots Av_n]$ $[v_1 \cdots v_n]$

We know $\text{rank}(AB) \leq \text{rank}(A)$, $\text{rank}(AB) \leq \text{rank}(B)$

Firstly $\text{rank}(AX) \leq \text{rank}(X)$

$Y = AX$ and A is invertible means.

$A^{-1}Y = X$ so $\text{rank}(X) = \text{rank}(A^{-1}Y) \leq \text{rank}(Y)$

$\Rightarrow$ Combine $\text{rank}(Y) = \text{rank}(X)$

Checking linear Subspace.

1. any $V_1, V_2 \in V$ and $c_1, c_2 \in \mathbb{R}$

we have $c_1V_1 + c_2V_2 \in V$

Ex. $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a.b.c.d \in \mathbb{R},\ a+b+c+d=0 \right\}$

$V_1, V_2 \in V$ then $c_1V_1 + c_2V_2 \in V$

is a vector space.

Checking $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $a_1+b_1+c_1+d_1=0$

$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ $a_2+b_2+c_2+d_2=0$

then

$\Rightarrow \begin{pmatrix} x_1a_1+x_2a_2 & x_1b_1+x_2b_2 \\ x_1c_1+x_2c_2 & x_1d_1+x_2d_2 \end{pmatrix}$

satisfies

$(x_1a_1+x_2a_2)+(x_1b_1+x_2b_2)+(x_1c_1+x_2c_2)+(x_1d_1+x_2d_2)=0$

---

a.b.c. are free variables

means once a.b.c are fixed. then d is fixed.

(you can also understand – b.c.d are free. then a fixed.
– a.c.d are free b fixed
– a.b. d are free c fixed)

- Find basis : set one of free variable to 1

↓ all the other to zero

- $a=1$ $b=0$ $c=0$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $a=0$ $b=1$ $c=0$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$
- $a=0$ $b=0$ $c=1$ $\Rightarrow d=-1 \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$

basis

dim = 3!!

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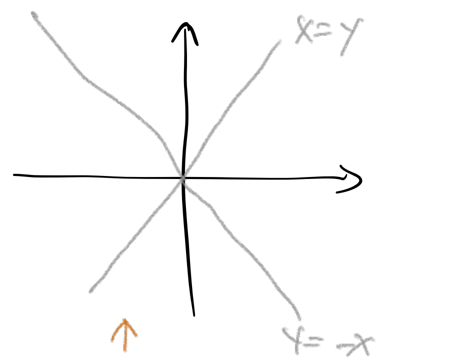
is $V=\{(x,y) \mid x^2-y^2=0\}$ a Vector space

To verify a set is not a vector space, you only to give an example.

$(1,1) \in V$ because $1^2-1^2=0$

$(1,-1) \in V$ $1^2-(-1)^2=0$.

but $(2,0) = (1,1)+(1,-1) \notin V$.



This is how V look like