Lecture 3
Linear Systems and Elimination

Yiping Lu
Based on Dr. Ralph Chikhany's Slide

Strang Sections 2.1 – Vectors and Linear Equations and 2.2 – The Idea of Elimination

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

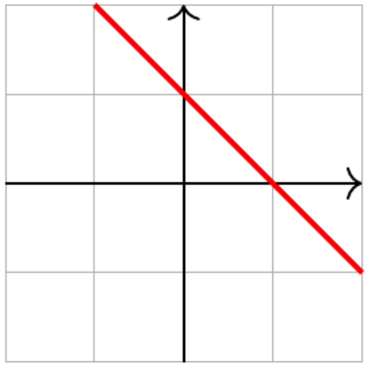
Systems of Equations

Systems of Equations

What does the solution set of a linear equation look like?

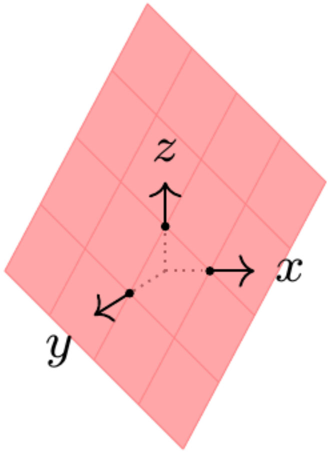
- $x + y = 1$

  $\rightsquigarrow$ a line in the plane: $y = 1 - x$



- $x + y + z = 1$

  $\rightsquigarrow$ a plane in space: $z = 1 - x - y$



- $x + y + z + w = 1$

  $\rightsquigarrow$ a "3-plane" in "4-space"...

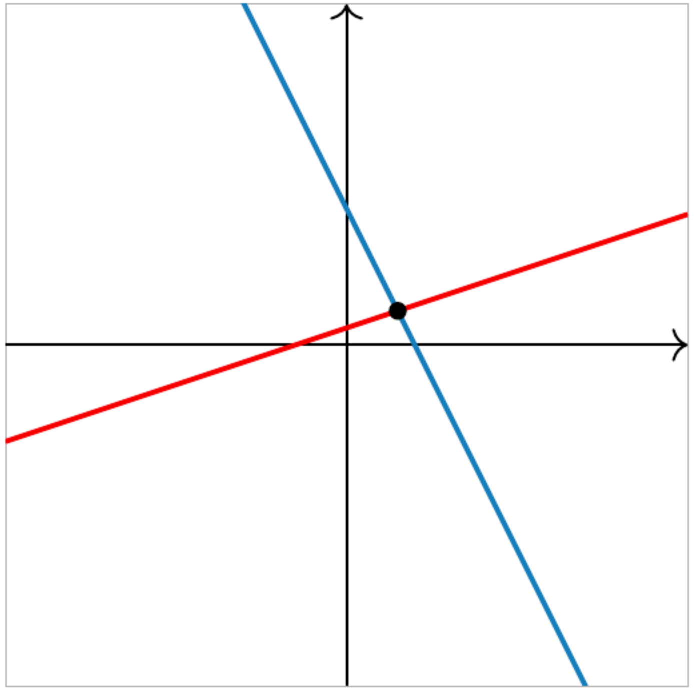
[not pictured here]

Systems of Equations

What does the solution set of a *system* of more than one linear equation look like?

$$\color{red}{x - 3y = -3}$$
$$\color{blue}{2x + y = 8}$$

…is the *intersection* of two lines, which is a *point* in this case.



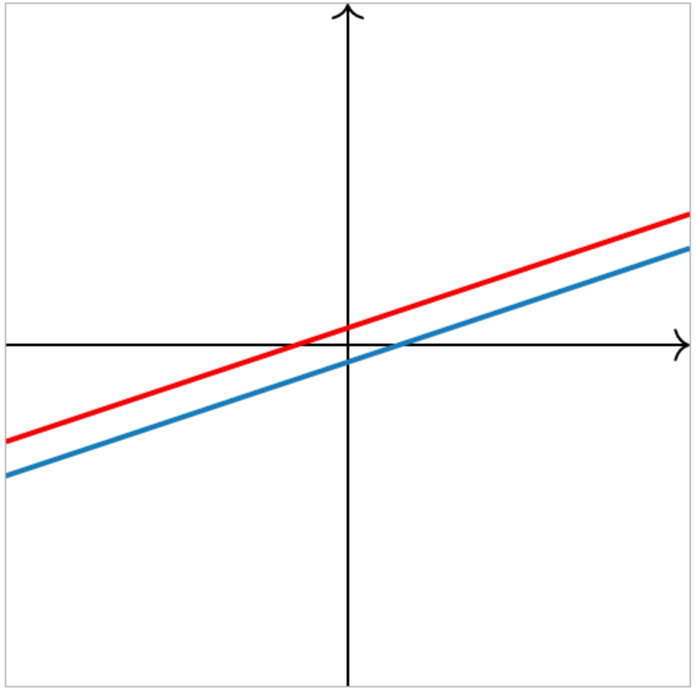
In general it's an intersection of lines, planes, etc.

Systems of Equations

In what other ways can two lines intersect?

$$\color{red}{x - 3y = -3}$$
$$\color{blue}{x - 3y = 3}$$

has no solution: the lines are *parallel*.



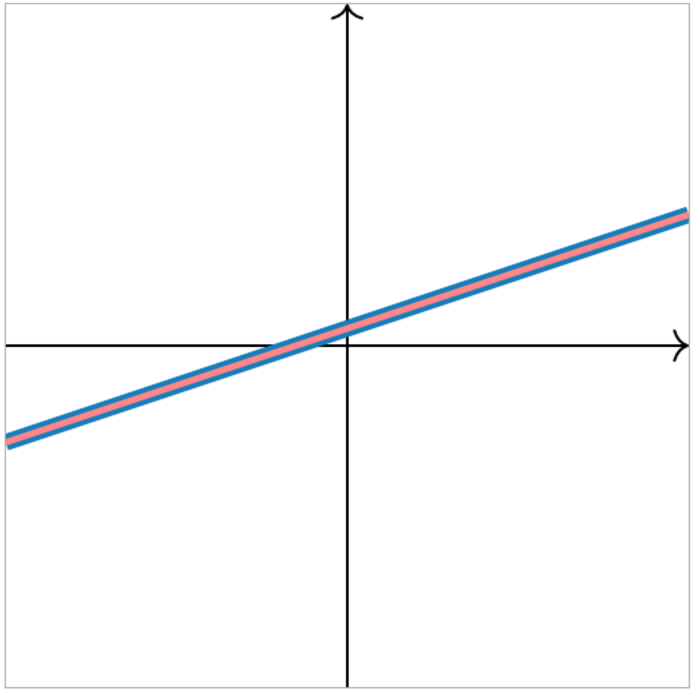
A system of equations with no solutions is called **inconsistent**.

Systems of Equations

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$2x - 6y = -6$$

has infinitely many solutions: they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for a good portion of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that make *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set.

Systems of Equations

Consider the following system of two equations in two unknowns

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 + 2x_2 &= 11 \end{aligned}$$

This system could be expressed in matrix notation as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Row picture: $(1, -2) \cdot (x_1, x_2) = 1 \implies x_1 - 2x_2 = 1$

$(3, 2) \cdot (x_1, x_2) = 11 \implies 3x_1 + 2x_2 = 11$

Systems of Equations – 2D – Row vs. Column Picture

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Column picture: $$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

If we have three equations with three unknowns, it is still possible to draw a picture of what a solution looks like. Each of the three equations represents a plane in 3D, and their intersection gives the solution of the system. As soon as you go above 3D, visualization becomes impossible.

Consider the following system of three equations in three unknowns

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Row picture:

$$(1, 2, 3) \cdot (x_1, x_2, x_3) = 6 \implies x_1 + 2x_2 + 3x_3 = 6$$

$$(2, 5, 2) \cdot (x_1, x_2, x_3) = 4 \implies 2x_1 + 5x_2 + 2x_3 = 4$$

$$(6, -3, 1) \cdot (x_1, x_2, x_3) = 2 \implies 6x_1 - 3x_2 + x_3 = 2$$

Systems of Equations – 3D – Row vs. Column Picture

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Column picture: $x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

Elimination

Systems of Equations

Example: Solve the system
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

What strategies do you know?

General Case

Suppose we are given a system of m equations in n unknowns:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
$$

our goal is to find $x_1, \ldots, x_n$.

The Process of Elimination

$$\boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1} \quad \text{first pivot}$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\boxed{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2}$$

this or any other equation with nonzero coefficient in front of ! % can also be chosen as **first pivot**

$$\vdots$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$\boxed{a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j}$$

this or any other equation with
nonzero coefficient in front of ! %
can also be chosen as **first pivot**

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

To do that, we choose one equation which has a nonzero coefficient multiplying x_1 and use that to eliminate x_1 from all the remaining equations. This equation is referred to as the *first pivot*, and it could be any of the m equations in our system, e.g, the j^{th} equation $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$.

The Process of Elimination

$$a_{j1} x_1 + a_{j2} x_2 + \cdots + a_{jn} x_n = b_j$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇤$$

$$\vdots$$

$$\bullet\, x_2 + \cdots + \bullet\, x_n = ⇧$$

the system after choosing the !'(equation as first pivot and using it to eliminate $% from the remaining equations

The Process of Elimination

$$
\begin{aligned}
a_{j\,1}\,x_1 + a_{j\,2}\,x_2 + \cdots + a_{j\,n}\,x_n &= b_j \\
\boxed{\bullet\; x_2 + \cdots + \bullet\; x_n = ⇤} \\
&\vdots \\
\bullet\; x_2 + \cdots + \bullet\; x_n &= \Uparrow
\end{aligned}
$$

choose **second pivot** with nonzero !) coefficient, and use it to eliminate !) from all remaining equations except the first pivot

Once we have eliminated x_1 from all equations except the first pivot, we move the pivot to the top, and leave it unaltered, then we choose another pivot from the remaining $m - 1$ equations, which has a nonzero coefficient multiplying x_2. We use this *second pivot* to eliminate x_2 from the $m - 2$ equations, i.e., all equations except the pivot equations (first and second).

The Process of Elimination

$$\begin{array}{rl}
a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \cdots + a_{jn}x_n = b_j & \text{first pivot} \\
\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\, x_n = \text{⇤} & \text{second pivot} \\
\boxed{\bullet\, x_3 + \cdots + \bullet\, x_n = \circ} & \text{third pivot} \\
\vdots & \\
\bullet\, x_3 + \cdots + \bullet\, x_n = \text{⌦} &
\end{array}$$

Once that is done, we move the second pivot and place it right under the first, and we leave it unaltered. We proceed by selecting a third pivot, which we use to eliminate x_3 from the remaining $m - 3$ equations.

Systems in Upper Triangular Form

$$
\begin{array}{rcl}
a_{j\,1}\,x_1 + a_{j\,2}\,x_2 + a_{j\,3}\,x_3 + \cdots + a_{j\;n}\,x_n & = & b_j \\
\bullet\, x_2 + \bullet\, x_3 + \cdots + \bullet\;\, x_n & = & |\!\leftarrow \\
\bullet\, x_3 + \cdots + \bullet\;\, x_n & = & \circ \\
\ddots & & \\
\bullet\;\, x_n & = & ?
\end{array}
$$

We continue with this procedure, until the system is upper triangular. Once that is achieved, we can use the last equation to solve for x_n and then back-solve for all the remaining unknowns.

Example

Example: Solve the system
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

Example

Example: Solve the system
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 + 5x_2 + 2x_3 &= 4 \\ 6x_1 - 3x_2 + x_3 &= 2 \end{aligned}$$

This process is known as the Gauss-Jordan elimination method.
We can go even further to make the work more practical.

Gauss-Jordan

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

General Case

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\xrightarrow{\text{in augmented form}} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy: fiddle with it so we only have ones and zeros.

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ \boxed{2} & -3 & 2 & 14 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & -2 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} \checkmark$$

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Another Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Note

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

The linear equations of row-equivalent matrices have the *same solution set.*

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.