Lecture 2

Spans and Matrices

Yiping Lu

Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Strang Section 1.3 - Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Matrices

An $m \times n$ matrix A is a rectangular array of (real) numbers a_{ij} with m rows and n columns, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A matrix is called **square** if it is $n \times n$, i.e., it has the same number of rows and columns.

Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{array}{cc} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} & i\text{th row} \\ j\text{th column} & \end{array}$$

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices

The **zero matrix** (of size $m\times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n\times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$\overset{A}{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \rightsquigarrow \overset{A^T}{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}}$$

flip

<u>Column</u> vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Linear Combination in Matrix Notation

A linear combination of n vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, in $\mathbb{R}^m$ is given by

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

This can be expressed as an $m \times n$ matrix A multiplying a vector $\vec{x} \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}}$$

Pool

Ax lie in the span of the column vectors of matrix A

What is the size of matrix A

For all the vector v in the span of the column vectors of matrix A, we can find a vector x, such that $Ax = v$

Dot product as matrix vector multiplication

$x \cdot y$ is $x^\top y$

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Linear Combination in Matrix Notation

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute $A\vec{x}$.

Dot Product with Rows

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

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Matrix times vector

Matrix times vector $$Ax = \begin{bmatrix} & & \\ u & v & w \\ & & \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = cu + dv + ew. \quad (3)$$

Dot Product View:

Examples

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

The system $Ax = b$

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

The system $Ax = b$: What if x is unknown?

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When A and $\vec{x}$ are given, computing $\vec{b}$ is straight forward. However, the reverse is not always true (or even possible). That is, if A and $\vec{b}$ are given , it is not always possible to find $\vec{x}$.

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$.

Suppose that b_1, b_2, and b_3 are given, and you want to compute x_1, x_2, and x_3 in terms of the components of $\vec{b}$.

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}.$

Suppose that b_1, b_2, and b_3 are given, and you want to compute x_1, x_2, and x_3 in terms of the components of $\vec{b}$.

$$\begin{aligned} x_1 \qquad\qquad &= b_1 \\ -x_1 + x_2 \qquad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \qquad \textbf{Solution} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned}$$

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

Two ways to calculate the matrix vector multiplication

Linear combination

Dot product

Not every matrix have an inverse

Cyclic $$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

$b = (1,3,5)$

Not every matrix have an inverse

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$b = (0,0,0)$

Not every matrix have an inverse

Cyclic
$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}.$$

$b = (0,0,0)$

Questions?