Linear Algebra - Problem Set 6 - Solutions

Exercise I (30 + 10 = 40 points)

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

1. Find the singular value decomposition, $A = U\Sigma V^T$.

   First, we find the eigenvalues of either A^TA, or AA^T. Since A^TA is 3×3 and AA^T is 2×2, it is easier to compute the eigenvalues of AA^T, but both procedures are correct. We proceed as follows:

$$AA^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

   The eigenvalues of AA^T can be computed by setting

$$\det(AA^T - \lambda I) = 0 \implies (5-\lambda)^2 - 16 = 0 \implies \lambda_1 = 9, \lambda_2 = 1.$$

   Note that the nonzero eigenvalues of A^TA are equal to the nonzero eigenvalues of AA^T, hence A^TA has eigenvalues $\lambda_1 = 9, \lambda_2 = 1$. But since A^TA is 3×3, it must have a third eigenvalue $\lambda_3 = 0$.

   Next, we find the eigenvectors of A^TA, which will form the matrix A.

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

   - $\lambda_1 = 9$: Solve $[\,A^TA - 9I\,]\vec{v}_1 = \vec{0}$

$$\left[\begin{array}{ccc|c} -8 & 0 & 2 & 0 \\ 0 & -8 & 2 & 0 \\ 2 & 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -\frac{1}{2} & 0 \\ -8 & 0 & 2 & 0 \\ 0 & -8 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 8 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

     Then, an eigenvector $\vec{v}_1$ associated with $\lambda_1 = 9$ is given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}.$$

   - $\lambda_2 = 1$: Solve $[\,A^TA - I\,]\vec{v}_2 = \vec{0}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

     Then, an eigenvector $\vec{v}_2$ associated with $\lambda_2 = 1$ is given by

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

- $\lambda_3 = 0$: Solve $[\,A^TA - 0I\,]\vec{v}_3 = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 2 & 8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, an eigenvector $\vec{v}_3$ associated with $\lambda_3 = 0$ is given by

$$\vec{v}_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Then, the matrix V is given by

$$V = \begin{bmatrix} 1/\sqrt{18} & -1/\sqrt{2} & -2/3 \\ 1/\sqrt{18} & 1/\sqrt{2} & -2/3 \\ 4/\sqrt{18} & 0 & 1/3 \end{bmatrix}.$$

The matrix of singular values, Σ is given by

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

To compute the matrix U containing the left singular vectors, we find

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\implies A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} & 1/\sqrt{18} & 4/\sqrt{18} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

2. Deduce an orthonormal basis for each of the four fundamental subspaces of A.

$$\beta_{\text{Row}A} = \left\{ \begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

$$\beta_{\text{Nul}A} = \left\{ \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\}$$

$$\beta_{\text{Col}A} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

$$\beta_{\text{Nul}A^T} = \emptyset$$

Exercise II ($5 \times 10 = 50$ points)

Are the following transformations linear? If so, provide a complete proof. If not, explain.

1. $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3^2)$.

   T is not a linear transformation. To see this, let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, then

   $$T(\vec{u}+\vec{v}) = \begin{bmatrix} (u_1+v_1) - (u_2+v_2) \\ (u_2+v_2) + (u_3+v_3)^2 \end{bmatrix} \text{ but } T(\vec{u})+T(\vec{v}) = \begin{bmatrix} u_1 - u_2 \\ u_2 + u_3^2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ v_2 + v_3^2 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 + v_1 - v_2 \\ u_2 + u_3^2 + v_2 + v_3^2 \end{bmatrix} \neq T(\vec{u}+\vec{v})$$

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(x_1, x_2, x_3) = (x_1 + x_3, \, -2x_2)$.

   Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, and let $c \in \mathbb{R}$, then:

   (i) $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1+u_3+v_3 \\ -2u_2 - 2v_2 \end{bmatrix} = T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} u_1+u_3 \\ -2u_2 \end{bmatrix} + \begin{bmatrix} v_1+v_3 \\ -2v_2 \end{bmatrix} = \begin{bmatrix} u_1+u_3+v_1+v_3 \\ -2u_2 - 2v_2 \end{bmatrix}$

   (ii) $T(c\vec{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 + cu_3 \\ -2cu_2 \end{bmatrix} = c\,T(\vec{u}) = c\,T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = c\begin{bmatrix} u_1+u_3 \\ -2u_2 \end{bmatrix} = \begin{bmatrix} cu_1 + cu_3 \\ -2cu_2 \end{bmatrix}$

   Therefore, by (i) and (ii), T is a linear transformation.

3. $T : \mathbb{R}^3 \to \mathbb{R}^4$, where $T(x_1, x_2, x_3) = (2x_1 - x_3, \, x_2 + x_3, \, 0, \, x_1 - 3x_2)$.

   Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, and let $c \in \mathbb{R}$. T is linear since:

   (i) $T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} 2(u_1+v_1) - (u_3+v_3) \\ (u_2+v_2) + (u_3+v_3) \\ 0 \\ (u_1+v_1) - 3(u_2+v_2) \end{bmatrix} = \begin{bmatrix} 2u_1 + 2v_1 - u_3 - v_3 \\ u_2 + v_2 + u_3 + v_3 \\ 0 \\ u_1 + v_1 - 3u_2 - 3v_2 \end{bmatrix}$

   $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 2u_1 - u_3 \\ u_2 + u_3 \\ 0 \\ u_1 - 3u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 - v_3 \\ v_2 + v_3 \\ 0 \\ v_1 - 3v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 - u_3 + 2v_1 - v_3 \\ u_2 + u_3 + v_2 + v_3 \\ 0 \\ u_1 - 3u_2 + v_1 - 3v_2 \end{bmatrix} = T(\vec{u}+\vec{v})$

   (ii) $T(c\vec{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} 2cu_1 - cu_3 \\ cu_2 + cu_3 \\ 0 \\ cu_1 - 3cu_2 \end{bmatrix} = c\,T(\vec{u}) = c\,T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = c\begin{bmatrix} 2u_1 - u_3 \\ u_2 + u_3 \\ 0 \\ u_1 - 3u_2 \end{bmatrix} = \begin{bmatrix} 2cu_1 - cu_3 \\ cu_2 + cu_3 \\ 0 \\ cu_1 - 3cu_2 \end{bmatrix}$

4. $T : \mathbb{P}_2 \to \mathbb{P}_3$, where $T[p(x)] = xp(x) + (x-1)^2 p''(x)$.

Let $p(x), q(x) \in \mathbb{P}_2$, and let $c \in \mathbb{R}$, then T is linear since:

(i) $T[p(x)+q(x)] = x[p(x)+q(x)]+(x-1)^2[p''(x)+q''(x)] = xp(x)+xq(x)+(x-1)^2p''(x)+(x-1)^2q''(x)$

$T[p(x)] + T[q(x)] = xp(x) + (x-1)^2p''(x) + xq(x) + (x-1)^2q''(x) = T[p(x) + q(x)]$

(ii) $T[cp(x)] = cxp(x) + c(x-1)^2p''(x) = cT[p(x)]$.

5. $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3, 1 + x_3)$

T is not a linear transformation. To see this, let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, then

$$T(c\vec{v}) = \begin{bmatrix} cv_1 - cv_2 \\ cv_2 + cv_3 \\ 1 + cv_3 \end{bmatrix} \text{ but } cT(\vec{v}) = \begin{bmatrix} cv_1 - cv_2 \\ cv_2 + cv_3 \\ c + cv_3 \end{bmatrix} \neq T(c\vec{v})$$

Exercise III (10 points)

Find the change of basis matrix that changes β'-coordinates into β-coordinates (both are bases for $\mathbb{R}^2$)

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \qquad \beta' = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

Change of basis from β' to β:

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{aligned}$$

Thus, the change of basis matrix from β' to β is

$$\begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix}$$