NYU

Linear Algebra

Lecture 15

Determinants

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Recap

- Orthogonal $q_i^T q_j = 0 \quad i \neq j$

  Orthonormal $q_i^T q_j = 0 \quad i \neq j \quad \|q_i\| = 1$

- $q_1 \cdots q_n$ orthonormal, which means $Q = [q_1 \cdots q_n]$

  Then $Q^T Q$ is identity

  - projection is as easy as. { Coef is $Q^T b$; projection matrix is QQ^T }

Example. $q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad qq^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

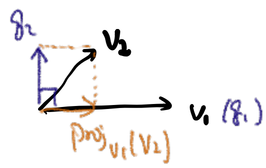
- if Q is a square matrix, all column vectors are orthonormal

  $\Downarrow$

  orthogonal matrix $\quad Q^T Q = QQ^T = I. \quad Q^T = Q^{-1}$

Gram - Schmidt Process $\underbrace{\{v_1 \cdots v_n\}}_{A} \longrightarrow$ orthogonal basis $\{q_1 \cdots q_n\}$

- $q_1 = v_1$
- $q_2 = v_2 - \text{proj}_{\text{span}\{v_1\}}(v_2) = v_2 - \text{proj}_{\text{span}\{q_1\}}(v_2)$



- $q_3 = v_3 - \text{proj}_{\text{span}\{v_1, v_2\}}(v_3)$

  $= v_3 - \text{proj}_{\text{span}\{q_1, q_2\}}(v_3)$ easier to compute $\quad q_1 \cdot q_2$ are orthogonal.

1. : $\exists$ QR decomposition. $\quad A = QR$
2. : It is easier to compute.

QR decomposition.

① using Gram Schmidt ( ), to compute the Q.

② change each column of Q to unit vector (orthogonal $\to$ orthonormal)

③ $R = Q^T A$ $\quad$ because. $A = QR$. which means $Q^T A = \underbrace{Q^T Q}_{I} R = R$

Strang Sections 5.1 – Properties of Determinants

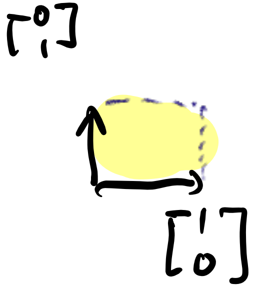
Course notes adapted from N. Hammoud's NYU lecture notes.

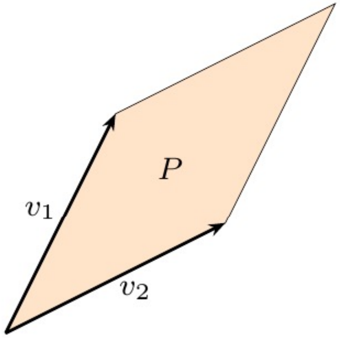
Introduction to Determinants

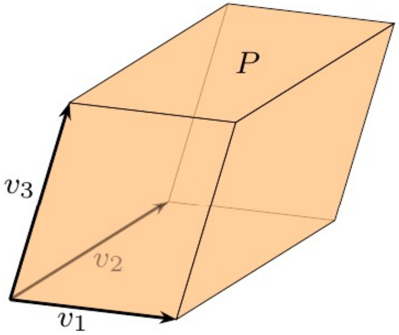
The Idea of Determinants

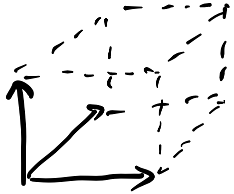
Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.









$A = [\vec{v}_1 \quad \vec{v}_2]$

$x \to Ax$

$A = [V_1, V_2, V_3]$

$x \to Ax$

lies in the same dimension

Determinants, as Volume Change

$\text{Det} = 0 \iff \{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$ are linear dependent.

$\iff A$ is not invertible.

Determinants – 2 × 2 case

We already have a formula in the 2 × 2 case:

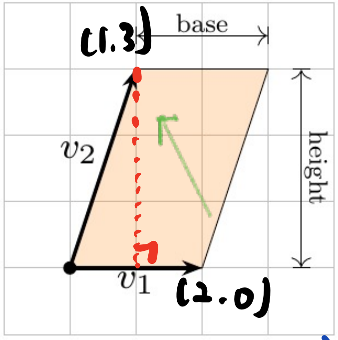
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = 2 \times 3 - 1 \times 0 = 6$

change sign

$v_2 - \text{proj}_{v_1}(v_2)$

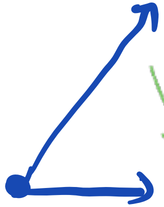
$B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = 1 \times 0 - 2 \times 3 = -6$



A

"signed" volumn.

$v_1 = (2,0)$



$v_2 = (1,3)$

B

Determinants – 3 × 3 case

Here's the formula:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$a_{1f(1)}\, a_{2f(2)}\, a_{3f(3)}$$

$f(1), f(2), f(3)$ is different order of $1, 2, 3$

"permutations"

$f(1)$	$f(2)$	$f(3)$	
1	2	3	+
2	3	1	+
3	1	2	+
3	2	1	−
1	3	2	−
2	1	3	−

$\pm\, a_{1f(1)}\, a_{2f(2)} \cdots a_{nf(n)}$

- $f(1) \cdots f(n)$ is the permutation of $1, \cdots n$
- (not required)
  If you start from $1, 2, \cdots, n$
  each step you can swift two variable
  ① If you switch odd times it's −
  ② - - - - even times it's +

ex. 1. 2. 3
→ 2. 1. 3 (2⟷1)
→ 3 2 1 (1⟷3)
→ 1. 3. 2 (2⟷3)

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Determinants – n × n case

We can think of the determinant as a function of the entries of a matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{aligned} &a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms. So the determinant of a 10×10 matrix has 3,628,800 terms!

When mathematicians encounter a function whose formula is too difficult to write down, we try to *characterize* it in terms of its properties.

Determinants – Definition

Definition

The **determinant** is a function

$$\det\colon \mathbb{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

with the following **defining properties**:

1. $\det(I_n) = 1$
2. If we do a row replacement, the determinant does not change.
3. If we swap two rows of a matrix, the determinant scales by -1.
4. If we scale a row of a matrix by k, the determinant scales by k.

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Why would we think of these properties? This is how volumes work!

1. The volume of the unit cube is 1.
2. Volumes don't change under a shear.
3. Volume of a mirror image is negative of the volume?
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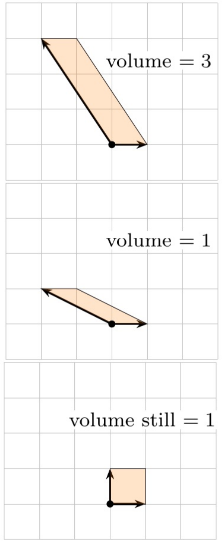
$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

Scale: $R_2 = \frac{1}{3}R_2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$

Row replacement: $R_1 = R_1 + 2R_2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$

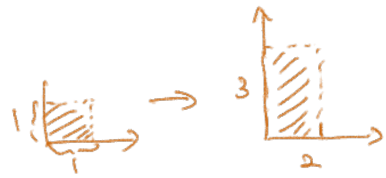


Properties of Determinants

det = Volumn change of linear Transform.

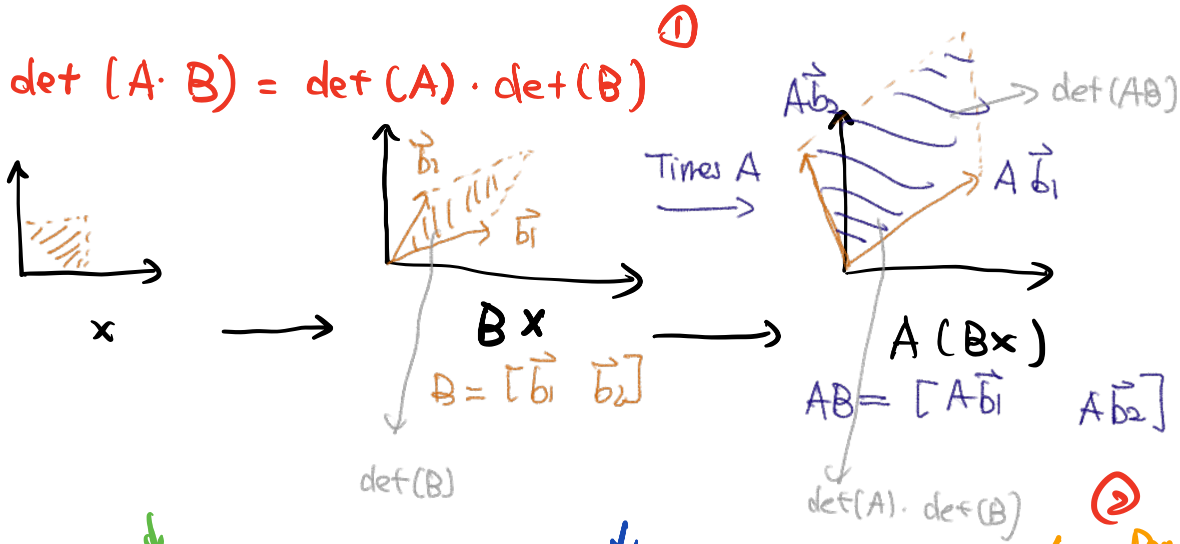
- $\det(I_n) = 1$ — $x \to x$ nothing changes
- $\det(Q) = \pm 1$ — Q is orthogonal rotation
- $\det\left(\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}\right) = a_1 a_2 \cdots a_n$

Example. $A = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}$ $x \to Ax$



①

- $\det(A \cdot B) = \det(A) \cdot \det(B)$

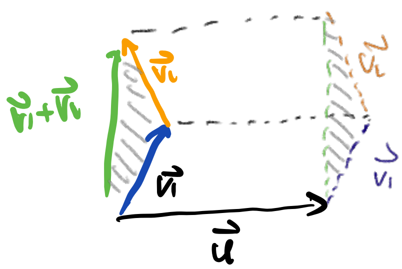


②

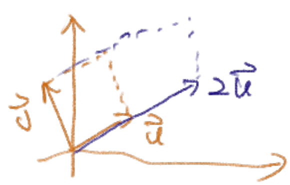
- $\det([\vec{v_1} + \vec{v_2}, \vec{u}]) = \det([\vec{v_1}, \vec{u}]) + \det([\vec{v_2}, \vec{u}])$

Remark. fix n-1 Columns only L.C. 1 Column

(but $\det(A+B) \neq \det(A) + \det(B)$)



Example. $A = [\vec{u}, \vec{v}]$, $B = [2\vec{u}, \vec{v}]$



③

- $\det([c\vec{v_1}, \vec{u}]) = c \cdot \det([\vec{v_1}, \vec{u}])$

(This doesn't mean. $\det[cA] = c \cdot \det[A]$
but it means $\det[cA] = c^n \det(A)$)

④

- $\det([\vec{0}, \vec{v_1}, \cdots, \vec{v_{n-1}}]) = 0$

Properties of Determinants

The determinant of an $n \times n$ matrix A is a number associated with A, and denoted by $\det A$ or $|A|$, with the following properties:

1. The determinant of the $n \times n$ identity matrix is 1. 3
2. The determinant changes sign when two rows are exchanged. (remember)
3. The determinant is a linear function of a fixed row.

- pull out constants: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- break apart sums: $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ← both row and columns are right (see later)

Attention!

$$\det(kA) \neq k \det A$$

$= k^n \det(A)$

$$\det(A+B) \neq \det A + \det B$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$ $\quad |A| = -5$

$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad |B| = 2$

$A + B = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$ $\quad |A+B| = -6$

Properties 1, 2 and 3

1. The determinant of the $n \times n$ identity matrix is 1.
2. The determinant changes sign when two rows are exchanged.
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- pull out constants: $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
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Property 4

4. If A has two equal rows, then $\det A = 0$.
Cloumn

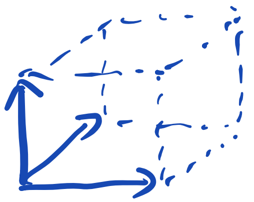
second column × −1

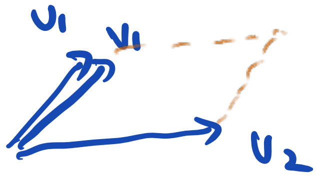
$\det(A) = -\det(B)$

$= 0$

(by Rule ③)

$A = [\vec{v}_1 , \vec{v}_1 , \vec{v}_2] \Rightarrow \det(A) = 0$





map a cube to a plane

$\det(A) = 0$

- proof $B = [\vec{v}_1 , -\vec{v}_1 , \vec{v}_2]$, first prove $\det(B) = 0$

① $\det([\vec{v}_1, -\vec{v}_1, \vec{v}_2]) = -\det([-\vec{v}_1, \vec{v}_1, \vec{v}_2])$ by ⑤ (switched two col)

② $\det([\vec{v}_1, -\vec{v}_1, \vec{v}_2]) = \det([-\vec{v}_1, \vec{v}_1, \vec{v}_2])$

first column × −1, second column × −1 (By Rule ③)

$\det(B) = 0$

Property 5

5. The elementary row operation of adding $l \cdot (\text{row } i)$ to row j leaves the determinant unchanged.

$A = [\vec{v_1}, \vec{v_2}]$. $B = [\vec{v_1} + j\vec{v_2}, \vec{v_2}]$

<u>Thm</u>. $\det(A) = \det(B)$

$$\det(B) = \det([\vec{v_1}, \vec{v_2}]) + \det([j\vec{v_2}, \vec{v_2}])$$

$$= \det([v_1, v_2]) + j\,\underline{\det([v_2, v_2])}^{=0}$$

$$= \det[v_1, v_2] = \det(A)$$

Property 6

6. If A has a row of zeros, then $\det A = 0$.

Property 7

7. If A is triangular, then $\det A$ is the product of diagonal entries.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix}$$

Property 8

8. A is invertible if and only if $\det A \neq 0$.

Property 9

9. $\det(AB) = \det A \cdot \det B$

Corollary – Determinant of the Inverse

Property 10

10. $\det A^T = \det A$

Property 10

10. $\det A^T = \det A$