Rank and Solubility.

- first of all. for Matrix $A \in \mathbb{R}^{m \times n}$

rank must satisfy . rank $\leq m$, rank $\leq n$.

- <u>Case 1</u> if rank $= n$. full Column rank (1. a Solution)

$\Rightarrow \dim(\mathrm{Nul}(A)) = 0$ which means $\mathrm{Nul}(A) = \{\vec{0}\}$

Recall (all) complete Solution is $\vec{x} = \vec{x}_{spec} + \underline{\vec{x}_{nul}}$ Nul (A) is $\{\vec{0}\}$, if exist special solution, you can only have 1 solution

but you may also don't have a special Solution. $\Rightarrow$ 0 Solution or 1 Solution

example $\begin{cases} x = 1 \\ 2x = 1 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- <u>Case 2</u> if rank $< n$. $\Rightarrow \dim(\mathrm{Nul}(A)) = n - \text{rank} > 0$ (0, a Solution)

$\vec{x}_1 \cdots \vec{x}_{n-r}$ is basis of Nul (A)

$\Rightarrow \mathrm{Nul}(A) = \mathrm{span}\{\vec{x}_1 \cdots \vec{x}_{n-r}\}$

Recall (all) complete Solution is $\vec{x} = \vec{x}_{spec} + \vec{x}_{nul}$

$a_1 \cdots a_{n-r} \in \mathbb{R}$ $\Rightarrow \infty$ Solution

① if exist a x_{spec} (special Solution), then $\vec{x}_{spec} + a_1\vec{x}_1 + \cdots + a_{n-r}\vec{x}_{n-r}$ are all Solution

② There don't exist x_{spec}. example $\begin{cases} x + 0y = 1 \\ 2x + 0y = 1 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- <u>Case 3</u> if rank $= m$. Full Row Rank.

$$\left.\begin{aligned} \dim(\mathrm{Col}(A)) &= m \\ \mathrm{Col}(A) &\subseteq \mathbb{R}^m \end{aligned}\right\} \Rightarrow \mathrm{Col}(A) = \mathbb{R}^m$$

(in $\mathbb{R}^2$, 2-(linear independent) vectors can span whole $\mathbb{R}^2$

Similarly, in $\mathbb{R}^m$, m-(linear independent) vectors can span whole $\mathbb{R}^m$)

$\Rightarrow$ For any $b \in \mathbb{R}^m$. $b \in \mathrm{Col}(A)$ So $Ax = b$ must have a solution (1. ∞ Solution)

Example.

1. rank = m = n,

Full row rank ⇒ at least one solution
Full Column rank ⇒ at most one solution
⇒ must have 1 solution

⇒ Matrix is invertible.

2. $A \in \mathbb{R}^{4 \times 5}$ rank(A) = 4

A full row rank ⇒ $Ax=b$ at least one solution,
rank = 4 < n = 5 ⇒ $Ax=b$ have 0 or ∞ solution
⇒ $Ax=b$ must have ∞ solution.