Lecture 5
Inverse Matrices

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Recap

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} \begin{array}{cc} k_1 & \end{array} & \\ \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array} = \begin{array}{cc} & \begin{array}{c} n_1 \; k_1 \qquad n_2 \; k_1 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11}B_{11}{}^{n_1} + A_{12}B_{21}{}^{n_2} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix} \end{array}. \tag{1}$$

Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Idea of Inverse Matrices

The idea of Inverse Matrices

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

The idea of Inverse Matrices

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

More on the Transpose of a Matrix

Recall

The transpose of an $m \times n$ matrix A is denoted by A^T, and it has entries $a_{ij}^T = a_{ji}$. That is, the columns of A^T are the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \implies A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Properties of the Transpose

sum: $(A+B)^T = A^T + B^T$

product: $(AB)^T = B^T A^T$

inverse: $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

Revisiting the Dot Product

We can redefine the dot product $\vec{u} \cdot \vec{v}$, where

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \; \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n,$$

as a matrix product $\vec{u}^T \vec{v}$.

$$\vec{u}^T \, \vec{v} = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{n \times 1} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{1 \times 1}$$

Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix $(A+B)$ is not necessarily invertible.

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$$ provided that $d_{ii} \neq 0$.

Inverse of an Elimination Matrix

Consider the elimination matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which adds c copies of the first row to the third row. Then,

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \implies P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \implies P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Inverse of upper triangular matrix

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{c c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{c c} \begin{array}{cc} k_1 & \end{array} & \\ \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array} = \begin{array}{c c} \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} \overset{n_1}{A_{11}}\overset{k_1}{B_{11}}{}^{n_1} + \overset{n_2}{A_{12}}\overset{k_1}{B_{21}}{}^{n_2} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix} \end{array}. \tag{1}$$

Goal

Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\textbf{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

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If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

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d

If $a_{11} = 0$, but $a_{21} \neq 0$, we have to permute first. If both a_{11} and a_{21} are zero, then the matrix is already upper triangular.

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

Computing U – 3×3 case

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

If $b \neq 0$, then we make it second pivot and use it to eliminate d.

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{d}{b} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

$$a_{11} \text{ pivot} \longrightarrow E_{21}A \rightarrow E_{31}E_{21}A \rightarrow E_{41}E_{31}E_{21}A \rightarrow \underbrace{E_{n1}\dots E_{41}E_{31}E_{21}A}_{B}$$

$$b \text{ pivot} \longrightarrow E_{32}B \rightarrow E_{42}E_{32}B \rightarrow E_{52}E_{42}E_{32}B \rightarrow \underbrace{E_{n2}\dots E_{52}E_{42}E_{32}B}_{C}$$

$$e \text{ pivot} \longrightarrow E_{43}C \rightarrow E_{53}E_{43}C \rightarrow E_{63}E_{53}E_{43}C \rightarrow E_{n3}\dots E_{63}E_{53}E_{43}C$$

$$\vdots$$

note that we're assuming we can find a pivot without having to use permutations

Computing L

2×2 case: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $U = E_{21}A$.

$$\implies A = \underbrace{E_{21}^{-1}}_{L} U$$

3×3 case: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $U = E_{32}E_{31}E_{21}A$.

$$\implies A = (E_{32}E_{31}E_{21})^{-1}U$$

$$= \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L} U$$

Questions?