

Lecture 5

Matrix Operations and Inverse Matrix

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Matrix Operations

Matrix multiplication

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

*i*th row
*j*th column
ij entry

Question : A is a matrix, write a matrix-vector multiplication as the first column

Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix,

What is Dx ?

Inverse Matrix

- Inverse Matrix $Ax = b \Leftrightarrow x = A^{-1}b$

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

Example

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix} \text{ provided that } d_{ii} \neq 0.$$

Recall

Suppose we are given a system of m equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$


$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form


$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

Two Operations

- Linear combine two rows
- Permutation



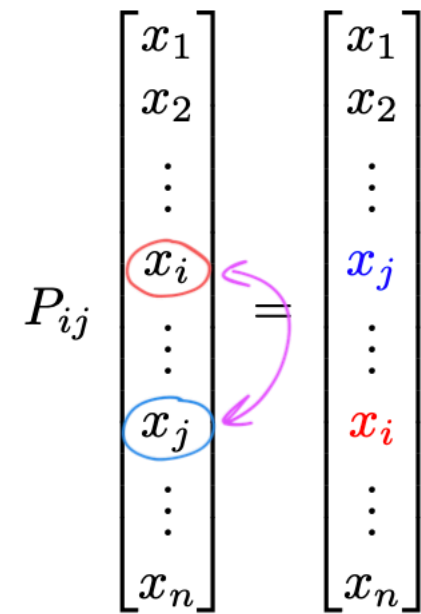
Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I \vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$


Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{P_{31} \vec{x}} \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$



Elimination Matrices

Elimination Matrices

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Col i Col j

Row i

Row j

Elimination Matrices

$$E_{ji} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Col i

Col j

Row i

Row j

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?



Solving Linear Systems

Elimination

Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$R_2 = R_2 - 2R_1$$

$$R_3 = R_3 - 3R_1$$

We want these to be zero.
So we subtract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

$$R_2 \longleftrightarrow R_3$$

$$R_2 = R_2 \div -5$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7 , but that
would produce ugly fractions.

$$R_1 = R_1 - 2R_2$$

Let's swap the last two rows first.

$$R_3 = R_3 + 7R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

We want these to be zero.

Let's make this a 1 first.

$$R_3 = R_3 \div 10$$

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$$R_1 = R_1 + R_3$$

~~~~~>

$$R_2 = R_2 - 2R_3$$

~~~~~>

translates into

~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\begin{array}{rcl} x & = & 1 \\ y & = & -2 \\ z & = & 3 \end{array}$$

Success!

Check:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

substitute solution

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$$1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

$$2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 = 14$$

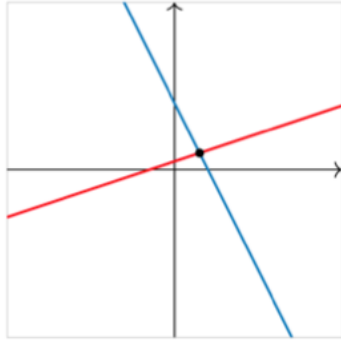
$$3 \cdot 1 + (-2) - 3 = -2$$



# Three Cases

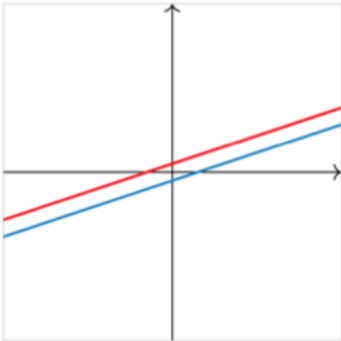
$$x - 3y = -3$$

$$2x + y = 8$$



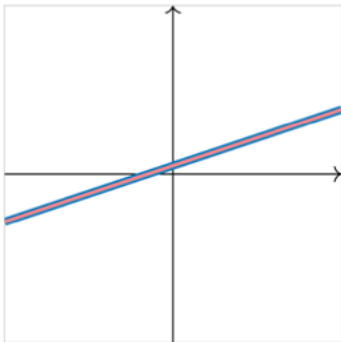
$$x - 3y = -3$$

$$x - 3y = 3$$



$$x - 3y = -3$$

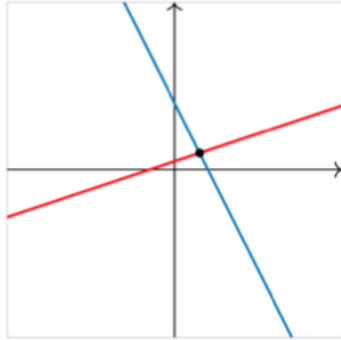
$$2x - 6y = -6$$



# Three Cases

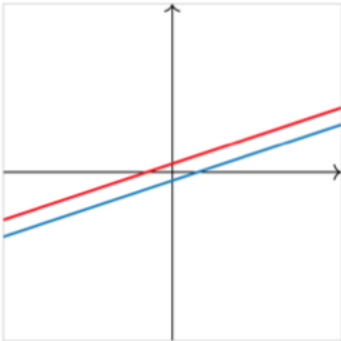
$$x - 3y = -3$$

$$2x + y = 8$$



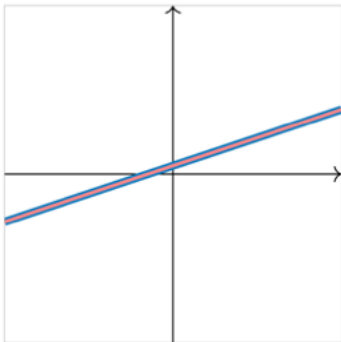
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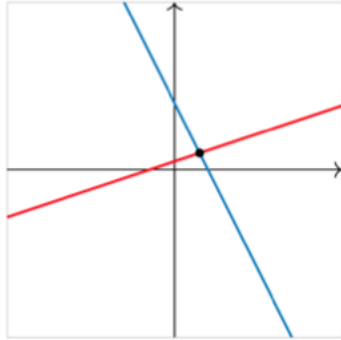




# Three Cases

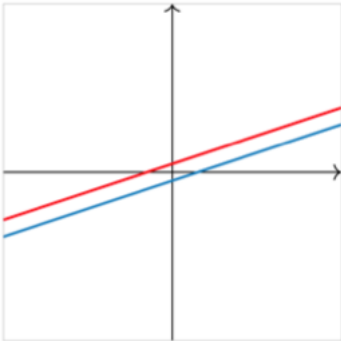
$$x - 3y = -3$$

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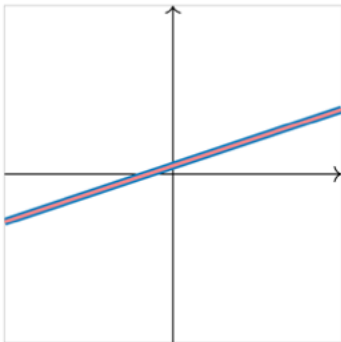
$$x - 3y = -3$$

$$x - 3y = 3$$



$$x - 3y = -3$$

$$2x - 6y = -6$$



# The idea of Inverse Matrices

Consider the following system:

$$\begin{aligned}2x_1 + 4x_2 - 2x_3 &= 2 \\4x_1 + 9x_2 - 3x_3 &= 8 \\-2x_1 - 3x_2 + 7x_3 &= 10\end{aligned}$$

Our goal is to find  $x_1$ ,  $x_2$ , and  $x_3$ . In matrix form, this system is:

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\vec{b}}$$

**idea**  $\longrightarrow$   $\vec{x} = A^{-1} \vec{b}$

# The idea of Inverse Matrices

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ \underbrace{-2 & -3 & 7}_A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$\vec{x} = \vec{b}$

$$\Leftrightarrow \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$A \quad \vec{x} = I \quad \vec{b}$

idea  $\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$I \quad \vec{x} = A^{-1} \quad \vec{b}$

# The idea of Inverse Matrices

$$\left[ \begin{array}{c|c} A & I \end{array} \right]$$

$\Downarrow$  **elimination**

$$\left[ \begin{array}{c|c} I & A^{-1} \end{array} \right]$$

## Example

**Example:** Find the inverse of  $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$ .

# Example

# Example

# Example



# Note on Infinite Solutions

$$\left[ \begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$



# Block Matrix\*

# Block Matrices

4 by 6 matrix  
2 by 2 blocks

$$A = \left[ \begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

**Block multiplication** If the cuts between columns of  $A$  match the cuts between rows of  $B$ , then block multiplication of  $AB$  is allowed:

$$\begin{matrix} m_1 \\ m_2 \end{matrix} \begin{bmatrix} \overset{n_1}{A_{11}} & \overset{n_2}{A_{12}} \\ \overset{n_1}{A_{21}} & \overset{n_2}{A_{22}} \end{bmatrix} \begin{bmatrix} \overset{k_1}{B_{11}} & \cdots \\ B_{21} & \cdots \end{bmatrix} \overset{n_1}{=} \begin{bmatrix} \overset{n_1}{\overset{k_1}{m_1} A_{11} B_{11} + \overset{n_2}{\overset{k_1}{m_2} A_{12} B_{21}}} & \cdots \\ \overset{n_1}{\overset{k_1}{m_1} A_{21} B_{11} + \overset{n_2}{\overset{k_1}{m_2} A_{22} B_{21}}} & \cdots \end{bmatrix}. \quad (1)$$

**Example 3 (Important special case)** Let the blocks of  $A$  be its  $n$  columns. Let the blocks of  $B$  be its  $n$  rows. Then block multiplication  $AB$  adds up *columns times rows*:

$$\begin{array}{l} \text{Columns} \\ \text{times} \\ \text{rows} \end{array} \quad \left[ \begin{array}{c|c|c} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{array} \right] \left[ \begin{array}{c|c|c} - & b_1 & - \\ & \vdots & \\ - & b_n & - \end{array} \right] = \left[ a_1 b_1 + \cdots + a_n b_n \right]. \quad (2)$$

$$AB = \left( \begin{array}{c} -r_1- \\ \vdots \\ -r_m- \end{array} \right) \left( \begin{array}{c|c|c} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{array} \right) = \left( \begin{array}{cccc} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{array} \right)$$

# Elimination by Block

**One at a time**

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}. \quad E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

**Block  
elimination**

$$\left[ \begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array} \right] \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[ \begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array} \right].$$