Linear Algebra Cheat Sheet

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1 Vector and Matrix

Vectors

$$\vec{a} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}, \vec{b} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector Addition:$\vec{a} + \vec{b} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{bmatrix}$

Vector Scalar Multiplication:$c\vec{A} = \begin{bmatrix} cA_x \\ cA_y \\ cA_z \end{bmatrix}$

Dot Product: $\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$ (Dot Product is a linear combination, $\vec{A} \cdot \vec{B} = \vec{A}^\top \vec{B}$ in lecture 3)

Length: $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Angle Between Two Vectors: $\theta = \arccos\left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\cdot\|\vec{B}\|}\right)$

Span *by ChatGPT* The "span" of a set of vectors is the set of all possible linear combinations of those vectors. In other words, it's the space formed by stretching, shrinking, or combining the vectors using scalar multiplication and addition. Geometrically, the span of vectors in 2D or 3D space forms a plane or a subspace. In higher-dimensional spaces, it forms a hyperplane or a subspace.

Matrix Examples see `https://2prime.github.io/files/linear/matrixvector.pdf`

Matrix $A \in \mathbb{R}^{m\times n}$ means m rows and n colums matrix.
Matrix A can be multiply with a vector v in $\mathbb{R}^n$

- **Row Representation**

$$A = \begin{bmatrix} r_1^\top \\ r_2^\top \\ \cdots \\ r_m{}^\top \end{bmatrix}, r_i \in \mathbb{R}^n, Av = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ \cdots \\ r_m \cdot v \end{bmatrix} = \begin{bmatrix} r_1^\top v \\ r_2^\top v \\ \cdots \\ r_m^\top v \end{bmatrix}$$

- **Column Representation**

$$A = \left[\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\right], \vec{v}_i \in \mathbb{R}^m, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \; A\vec{x} = \underbrace{x_1}_{\text{scalar}} \underbrace{\vec{v}_1}_{\text{vector}} + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$$

Linear Systems For a linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

We can understand it as

- a linear combiniation problem: if vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$ lies in the span of

$$\underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix}, \cdots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}}_{\text{column vecrtors of matrix } A}$$

- a matrix equation $A\vec{x} = \vec{b}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ (n unknown), $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$ (right hand side of m equations)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Linear depedent and linear independent:

- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linear depedent if there exists $c_1, \cdots, c_n$ which are not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots + c_n\vec{v}_n = \vec{0}$
- $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linear indepedent if only $c_1 = c_2 = \cdots = c_n = 0$ can make $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots + c_n\vec{v}_n = \vec{0}$

Linear system $Ax = b$ (NOT Required! Only need to know what's the inverse and not all matrix have an inverse)

- A don't have an inverse:
  - have solution if and only if $b \in \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ where $\vec{v}_1, \cdots, \vec{v}_n$ are column vectors of matrix A
  - In this case A is not invertible, the linear system have 0 solutions or infinite solutions.
- have a unique solution if and only if matrix A have an inverse matrix A^{-1}. The unique solution is $x = A^{-1}b$. In this case, (*important*) A must be a square matrix and $\vec{v}_1, \cdots, \vec{v}_n$ are linear independent.

2 Matrix Operations

$A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times k}$ then $C = AB \in \mathbb{R}^{m\times k}$.

$$AB = \begin{bmatrix} \vec{r}_1^\top \\ \cdot \\ \cdot \\ \cdot \\ \vec{r}_m^\top \end{bmatrix} \begin{bmatrix} \vec{v}_1, & \cdots, & \vec{v}_k \end{bmatrix} = \begin{bmatrix} \underbrace{\vec{r}_1^\top \vec{v}_1}_{\text{scalar}} & \vec{r}_1^\top \vec{v}_2 & \cdots & \vec{r}_1^\top \vec{v}_k \\ \vec{r}_2^\top \vec{v}_1 & \vec{r}_2^\top \vec{v}_2 & \cdots & \vec{r}_2^\top \vec{v}_k \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \vec{r}_m^\top \vec{v}_1 & \vec{r}_m^\top \vec{v}_2 & \cdots & \vec{r}_m^\top \vec{v}_k \end{bmatrix} \tag{2}$$

- $(AB)C = A(BC)$, $A(B + C) = AB + AC, c(AB) = A(cB)$
- $AB \neq BA$
- $A[\vec{v}_1, \cdots, \vec{v}_k] = [A\vec{v}_1, \cdots, A\vec{v}_k]$
- $\begin{bmatrix} \vec{r}_1^\top \\ \vec{r}_2^\top \\ \cdots \\ \vec{r}_k^\top \end{bmatrix} A = \begin{bmatrix} \vec{r}_1^\top A \\ \vec{r}_2^\top A \\ \cdots \\ \vec{r}_k^\top A \end{bmatrix}$
- $AI_n = A, I_m A = A$
- For a square matrix A, $AA^{-1} = A^{-1}A = I$
- $(AB)^\top = B^\top A^\top$
- *A upper triangular matrix times a upper triangular matrix is a upper triangular matrix. A Lower triangular matrix times a lower triangular matrix is a upper triangular matrix.*

3 Elimination

The Agumentation matrix of linear system $Ax = b$ is $[A|b]$. Opertations you can choose to simplify the system:

- Scalar multiply a row
- Replace row (j) by $** \text{row}(i) + \text{row}(j)$.
- switch two row

First Way Eliminate to **Upper Triangular Form** and then solve the upper triangular form.

Second Way (Easier to implement) These transformations provides equivalent systems.

Solving the equation equivalent to transform augmented matrix $[A|b]$ to $[I|x]$, we do it column by column

Example:

- Change the first column to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

-2 means replace Row 2 by (-2) * Row 1 + Row 2

-20 means replace Row 3 by (-20) * Row 1 + Row 3

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 4 & 6 & 4 \\ 20 & 26 & 28 & 4 \end{array}\right) -2 \quad -20 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & -2 \\ 0 & 6 & 8 & -56 \end{array}\right) \times \frac{1}{2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 8 & -56 \end{array}\right)$$

- Change the second column to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ by using the second row as **pivot**

-1 means replace Row 1 by (-1) * Row 2 + Row 1

-6 means replace Row 3 by (-6) * Row 2 + Row 3

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 8 & -56 \end{array}\right) \quad -1 \quad -6 \qquad \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -4 & -62 \end{array}\right)$$

- Change the third column to $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ by using the third row as **pivot**.

  do it by yourself

Elimination as Matrix Operation We can write the operations to change equivalent linear system by $[A|b] \rightarrow [E_{ji}A|E_{ji}b]$ and $[P_{ij}A|P_{ij}b]$.

- Elimination matrix E_{ji}: (be carefull about the index)
  - Replace row (j) by $*\times$row(i) + row (j)
  - Identity matrix except $a_{ji} = *$
- Permutation matrix P_{ij}:
  - Swtich Row (i) with Row (j)
  - Identity matrix except $a_{ij} = a_{ji} = 1, a_{ii} = a_{jj} = 0$

Solving Linear System

- The linear system have a single solution: n non-zero pivots
- The linear system have no solution: There exists a line is

$$[0, 0, 0 \cdots, 0|x]$$

  where $x \neq 0$

- The linear system have infinite solutions: **all** rows with $0, 0, 0, \cdots, 0$, the right hand side is also 0 (*i.e.* $[0, 0, \cdots, 0|0]$).

Matrix Inverse

- $AA^{-1} = A^{-1}A = I_n$
- ways to calculate the inverse matrix: change $[A|I]$ to $[I|A^{-1}]$
- $(AB)^{-1} = B^{-1}A^{-1}$

- *The inverse of a upper triangular matrix is a upper triangular matrix.*
- $(A^\top)^{-1} = (A^{-1})^\top$
- **Inverse of a elimination matrix:** change a_{ji} of E_{ji} to $-a_{ji}$
- Inverse of a permutation matrix: its transpose
- $AB = AC$ doesn't mean $B = C$. (A, B, C are both matrices) This is true only when A is invertible

4 LU Decomposition

- **LU decomposition**: $A = LU$, L is a lower traingular matrix, U is a upper traingular matrix
  - LU decomposition can be derived from eliminate a linear system to a upper traingular form.
  - We can represent eliminate process using eliminate matrix E_{ij}: Replace row (i) by $*$*row(i) + row (j)
  -

$$\underbrace{E_{n,n-1}}_{\text{eliminate column } n-1 \text{ using row } n-1} \underbrace{E_{n,n-2}E_{n-1,n-2}}_{\text{eliminate column } n-2 \text{ using row } n-2} \cdots \underbrace{E_{n,1}E_{n-1,1}\cdots E_{2,1}}_{\text{eliminate column 1 using row 1}} A = U \tag{3}$$

  - $L^{-1} = (E_{n,n-1})(E_{n,n-2}E_{n-1,n-2})\cdots(E_{n,1}E_{n-1,1}\cdots E_{2,1})$ is the elimination process and U is the upper traingular form
  - **Example for** 3×3: `https://2prime.github.io/files/linear/LU.pdf`
  - Using LU Decompostion to solve linear system $Ax = b$. We can change the equation to $L\underbrace{Ux}_{y} = b$
    - Solve the lower triangular system $Ly = b$ (one by one from top to bottom)
    - Solve the upper traingular system $Ux = y$ (one by one from bottom to top)
- **LDU decomposition**: $A = LDU$, L is a lower traingular matrix with all 1 diagonal elements, D is a diagonal matrix, U is a upper traingular matrix with all 1 diagonal elements
- **LDL decompostion** If A is symmetric, $A = LDL^\top$

- A is symmetric matrix means $A = A^\top$ ($a_{ij} = a_{ji}$, A is symmetric according to the diagonal)
- A is symmetric then A^{-1} is symmetric
- $A^\top A$ and $A^\top A$ are symmetric square matrix even if A is not square.

5 Vector Space

- Vector space V means if $v_1, \cdots, v_n \in V$ then all linear combinition of them $c_1 v_1 + \cdots + c_n v_n \in V$ also lies in V. (This should holds for all $c_1, \cdots, c_n$)

  (checking VS1-VS8 only needed for problem set)

- $\{x|Ax = 0\}$ (Null space) is a vector space, $\{x|Ax = b\}$ ($b \neq 0$) is not a vector space

- a vector space is a line/plane/...(generalization of line and plane) that go through the origin

  $\{x|\text{condition}\}$ means all the possible x satisfies the condition

- $\text{Col}(A) := \{Ax|x \in \mathbb{R}^n\} = \text{span}(\vec{v}_1, \cdots, \vec{v}_n)\Big(:= \{x|c_1\vec{v}_1 + \cdots + c_n\vec{v}_n\}\Big)$ (:= means definition) is a vector space

  $\{Ax|x \in \mathbb{R}^n\}$ means $\{b|b = Ax, x \in \mathbb{R}^n\}$, *i.e.* all the possible b that can be represented as Ax

Complete Solutions = Special Solution + element in the null space
Row Echelon Form

- All nonzero rows are above any rows of all zeros.
- The leading entry of each nonzero row (pivot) occurs to the right of the pivot of the previous row.

Additional properties

- free **columns** $\in$ span {pivot columns}
- pivot **columns** are linear indepedent.

Rank

- Rank r is number of pivots means the true size of the matrix
- $r \leq m, r \leq n$
- Number of free Varibale: $n - r$
- **full row rank**:
  - $r = m$ means column space is $\mathbb{R}^m$
  - all row columns are linear independent

  - a linear system have at least one solution.

- **full column rank**:
  - $r = n$ means no free variable, which means null space is $\{\vec{0}\}$. So linear system have at most one solution
  - column vectors are linear dependent (because they are all pivot columns)

- $m < n$ means $r < n$ (because $r \leq m$), $n < m$ means $r < n$ (because $r \leq n$)

The four possibilities for linear equations depend on the rank r:

$r = m$	and	$r = n$	*Square and invertible*	$Ax = b$	has 1 solution
$r = m$	and	$r < n$	*Short and wide*	$Ax = b$	has ∞ solutions
$r < m$	and	$r = n$	*Tall and thin*	$Ax = b$	has 0 or 1 solution
$r < m$	and	$r < n$	*Not full rank*	$Ax = b$	has 0 or ∞ solutions

Basis: A basis of a vector space is a set of vectors in the subspace that are linearly independent and span the entire vector space

Dimension: The dimension of a subspace is the number of vectors in every basis for the subspace

Four Subspaces

- elimination will not change the row space but will change the column space
- elimination will not change the null space
- $\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \dim(\text{Col}(A^\top)) = \dim(\text{Row}(A^\top)) = \text{rank}(A)$
- $\dim(\text{Nul}(A)) = n - \text{rank}(A), \dim(\text{Nul}(A^\top)) = m - \text{rank}(A)$
- $\text{rank}(A) = \text{rank}(A^\top)$
- $\text{rank}(AB) \leq \text{rank}(A), \text{rank}(AB) \leq \text{rank}(B)$

How to calculate the four subspaces

- **Row Space**: Reduce to REF and pivot rows are the basis
- **Column Space**: $\text{Col}(A) = \text{Row}(A^\top)$
- **Null space** Reduce to REF and find the free variable. Use pivot rows to solve the pivot variables.

6 Important properties

- A upper triangular matrix times a upper triangular matrix is a upper triangular matrix. The inverse of a upper triangular matrix is a upper triangular matrix
- $\{x|Ax = b\}$ is a vector space when and only when $b = \vec{0}$.
- $A^\top A$ and $A^\top A$ are symmetric square matrix even if A is not square.
- rank $=$ number of variable - number of free variables
- Full row rank means the linear system at least have one solution. (and why?)
- Full column rank means the linear system at most have one solution. (and why?)
- $r < n$ (rank $<$ number of variables) then the linear system have 0 or infinite solutions. (and why?)
- $\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A^\top)) = \dim(\mathrm{Row}(A^\top)) = \mathrm{rank}(A)$
- $\dim(\mathrm{Nul}(A)) = n - \mathrm{rank}(A), \dim(\mathrm{Nul}(A^\top)) = m - \mathrm{rank}(A)$
- A is invertible equals to A is a square matrix and A is full column/row rank.
- $\mathrm{rank}(A) = \mathrm{rank}(A^\top)$
- $\mathrm{rank}(AB) \leq \mathrm{rank}(A), \mathrm{rank}(AB) \leq \mathrm{rank}(B)$