MATH-UA 140 - Linear Algebra

Midterm, Spring 2024

Name: ______________________ **NetID:** ______________________

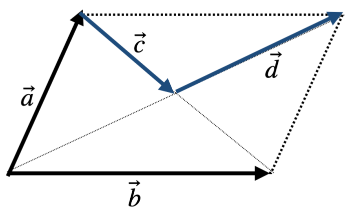
While you wait, please read and check ☑ the following boxes:

- ☐ Unless I have extra time with the Moses Center, the time limit is **75 minutes**.
- ☐ I am taking this exam because I am a student enrolled in Professor Lu's section during this time. If this is not the case, I will leave the room immediately.
- ☐ I wrote my name and NetID (e.g. ab1234) at the top of this page.
- ☐ I will not detach any pages, especially not the scratch pages at the end.
- ☐ Except for multiple choice questions, I will show my work.
- ☐ If I need more space for an exercise, I will make a note and continue on one of the scratch pages.
- ☐ If I am caught in violation of academic integrity, including but not limited to peaking at another student's work, allowing another student to copy from my work, or speaking with another student, or using unauthorized resources, I will be asked to leave the exam and get a zero.

Do not start the exam until you are permitted to.

Exercise I [20 points]

1. Represent $\vec{c}$ and $\vec{d}$ as linear combination of $\vec{a}$ and $\vec{b}$



Solution: $\vec{c} = \frac{1}{2}\vec{a} - \frac{1}{2}\vec{b}$, $\vec{d} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$

10 points

2. Calculate the inverse matrix of $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -5 \end{bmatrix}$

Solution:

Given matrix A and the identity matrix E, we start with:

$$[A|E] = \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 10 & -2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{array}\right] \quad (1)$$

10 points, one time of computational error deduct 2 points.

Exercise II [20 points]

1. Use elimination to put the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ in row echelon form. Show all your steps!

   Solution: Using Gaussian elimination we get:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & -2 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_3+r_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{r_4-2r_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

   so:

$$U = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   5 points

2. Use part (1) to write $A = LU$, where L is lower triangular and U is upper triangular.

   Solution: We can rewrite the row operations in part (1) as multiplications by elimination matrices. The first step is give $E_{21}^{(1)}$, the second by $E_{32}^{(1)}$ and the third by $E_{43}^{(-2)}$. Thus: $E_{43}^{(-2)}E_{32}^{(1)}E_{21}^{(1)}A = U$ By using $(E_{ij}^{(\lambda)})^{-1} = E_{ij}^{(-\lambda)}$, we get: $A = E_{21}^{(-1)}E_{32}^{(-1)}E_{43}^{(2)}U$ hence:

$$L = E_{21}^{(-1)}E_{32}^{(-1)}E_{43}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   5 points: 2points for writing down E_{ij}, 2points for writing down $E_{ij}^{-}1$, 1 point for L.

   If the students's L is right, he can get all the points.

3. Provide a basis of Col(A).

   Solution: span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix} \right\}$ 5 points

4. Provide a basis of Row(A).

   Solution: span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ 5 points

5. Explain why for any 4×4 matrix X, the product AX cannot be invertible.

   (*hint*: What is the rank of matrix A? Use the relation between rank(A) and rank(AX))

   Solution: By part (2), the columns of A are linearly dependent (rank(A) = 3), so they span a vector space of dimension at most $3 < 4$. Since the columns of AX are linear combinations of the columns of A, we conclude that the columns of AX also span vector space of dimension at most $3 < 4$. So AX cannot be invertible, since invertible matrices have full dimensional column space.

Exercise III [20 points]

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

1. If $\mathbf{b} = \begin{pmatrix} \alpha \\ 6 \\ 1 \end{pmatrix}$, for what values of α will $A\mathbf{x} = \mathbf{b}$ have a solution?

   Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \alpha \\ 1 & 2 & 4 & 6 & 6 \\ 0 & 0 & 1 & 2 & \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \alpha \\ 0 & 0 & 1 & 2 & 6-\alpha \\ 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \alpha \\ 0 & 0 & 1 & 2 & 6-\alpha \\ 0 & 0 & 0 & 0 & -5+\alpha \end{array}\right]$$

   Solution:We see that $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \alpha \\ 6-\alpha \\ -5+\alpha \end{pmatrix}$$

   Note that the last row of this system implies that there is a solution only if $\alpha = 5$.

   right logic but wrong calculation deduct 2points in 10 points

2. For the α from the previous question, give the *complete* solution to $A\mathbf{x} = \mathbf{b}$.

   Solution:To find all solutions, we need to find any solution and add $N(A)$ to it. To find a solution of $A\mathbf{x} = \mathbf{b}$ we can use the same elimination as in part (b) and look for a solution of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

   The back-substitution gives $x_3 = 1 - 2x_4$ and $p_1 = 2 - 2x_2 + 2x_4$. (Treat x_2, x_4 as free variable.) Hence, all solutions have the form

$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - 2x_x + 2x_4 \\ x_2 \\ 1 - 2x_4 \\ x_4 \end{pmatrix}$$

   where x_2, x_4 are arbitrary reals.

   right logic but wrong calculation deduct 2points in 10 points

Exercise IV [20 points]

1. What is the value of c that makes matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & t \\ 1 & 9 & t^2 \end{bmatrix}$ non-invertible?

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & t \\ 1 & 9 & t^2 \end{pmatrix} \xrightarrow{R_2-R_1, R_3-R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & t-1 \\ 0 & 8 & t^2-1 \end{pmatrix} \xrightarrow{R_3-4R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & t=1 \\ 0 & 0 & t^2-1-4(t-1) \end{pmatrix}$$

Matrix A is non-invertible means $t^2-1-4(t-1)=t^2-4t+3=(t-1)(t-3)\neq 0$. So $t\neq 1$ and $t\neq 3$.

If student did calculation error in elimination, but all logic below is right. deduct 4 points.

If student have logical error in doing the rank. deduct 4 points.

If student have calculation error in calculating a. deduct 2 points.

For parts 2 and 3, circle the right answer. No justification needed.

2. $\{\vec{x}\in\mathbb{R}^3 | \vec{x}\cdot\vec{z}\leq 1, \vec{z}\in\mathbb{R}^3\}$ is a vector space whose dimension is 2.

A. True B. False

Solution: False 2.5pt

3. $Y=XA$ and A is an invertible matrix, then $\text{rank}(Y)=\text{rank}(X)$.

A. True B. False

Yes, because $Y=XA$ so $\text{rank}(Y)\leq\text{rank}(X)$. For A is invertible matrix, so $X=YA^{-1}$ which tells us $\text{rank}(X)\leq\text{rank}(Y)$. The only possiblity is $\text{rank}(Y)=\text{rank}(X)$ 2.5pt

4.If C is any 4 by 7 matrix of rank $r = 4$, find the column space of C. Explain clearly why $Cx = b$ always has infinitely many solutions.

Solution: The rank of the matrix is equal to the dimension of the column space. Thus the dimension of the column space is 4. Therefore, the column space spans all of R4. A basis of the column space is $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]$ and $[0, 0, 0, 1]$. Hence, for any vector b there exists a solution. In addition, the dimension of the nullspace is 3. Therefore, there are infinitely many solutions.

full row rank 2.5pt, Nul(A) is 3 dimensional (or dim > 0 is enough) 2.5pt

Exercise V [10 points]

Show that the set of all polynomials of degree less than 3 (any polynomial that can be written in the form $ax^2 + bx + c$) forms a vector space. What is the dimension of the subspace? Provide a basis of the subspace.

Yes! It's a vector space. This is because for any tow polynomials of degree less than 3

$$a_1x^2 + b_1x + c_1, a_2x^2 + b_2x + c_2,$$

their linear combination

$$d_1(a_1x^2 + b_1x + c_1) + d_2(a_2x^2 + b_2x + c_2) = (d_1a_1 + d_2a_2)x^2 + (d_1b_1 + d_2b_2)x + d_1c_1 + d_2c_2$$

is also a polynomial of degree less than 3. It's dimension is 3. The basis is

$$1, x, x^2$$

Subspace proof 6pts. using $0 \in V$ $cx \in V$ $x_1 + x_2 \in V$ is also right.
$\dim(V) = 3$ 1 pt
The rank of student's basis:
rank =1 1 pt
rank =2 2 pt
rank =3 3 pt

Exercise VI [10 points]

Find all values of a such that

- $\text{rank}(A) = 1$
- $\text{rank}(A) = 2$
- $\text{rank}(A) = 3$

where

$$\begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

Solution:Using the elimination method, we obtain:

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \xrightarrow[R_3 - aR_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} = B$$

Let us consider two cases. (2.5pt)
Case 1: $a = -1$. Then the matrix B is equal to

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, B (and hence A) has rank 1. (2.5pt)
Case 2: $a \neq -1$. Then we divide the second and the third rows of B by $4a+4$ and $-2-2a$ respectively:

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \xrightarrow[R_3/(-2-a)]{R_2/(4a+4)} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{a^2-1}{-2-2a} \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = C.$$

Let us again consider two cases.
Case 2a: $a = 2$. Then $\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ has rank 2. (2.5pt)
Case 2b: $a \neq 2$. Then

$$\begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{a-2}{2} \end{bmatrix} \xrightarrow{R_3/\left(\frac{a-2}{2}\right)} \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

The last matrix has rank 3. (2.5pt)
If student did calculation error in elimination, but all logic below is right. deduct 4 points.
If student have logical error in doing the rank. deduct 4 points.
If student have calculation error in calculating a. deduct 2 points.

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