Linear Algebra

Midterm Review Question

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Exercise Consider the matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}$$

- Write down $A = LU$ where L is an lower traingular matrix and U is a REF.
- Calculate the four fundamental subspaces

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \xrightarrow{R3 \;\; \leftarrow R3 - R1} \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R3 \;\; \leftarrow R3 - R2} \underbrace{\begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

The elimination matrix we have is $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $E_{32}E_{31}A = U$ (order!)

Thus

$$A = \underbrace{E_{31}^{-1}E_{32}^{-1}}_{L}U$$

and $E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. So

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\text{Col}(A)$: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (first and forth column (pivot) of A)
- $\text{Row}(A)$: $\begin{bmatrix} 1 \\ 3 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ (non-zero rows of REF)
- $\text{Nul}(A) = \text{Nul}(U)$

- $x_1 = -3x_2 - 5x_3 - 7x_5$
- x_2, x_3 is free
- $x_4 = -2x_5$
- x_5 is free

Thus

$$\mathrm{Nul}(A) = \left\{ \begin{bmatrix} -3x_2 & -5x_3 & -7x_5 \\ x_2 & & \\ & x_3 & \\ & & -2x_5 \\ & & x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} | x_2, x_3, x_5 \in \mathbb{R} \right\}$$

so the basis is

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

- $\mathrm{Nul}(A^\top)$:

$$A^\top = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 5 & 0 & 5 \\ 0 & 1 & 1 \\ 7 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(REF)}$$

  - $x_1 = -x_3$
  - $x_2 = -x_3$
  - x_3 is free

  The basis of $\mathrm{Nul}(A^\top)$ is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

What is the dimension of the four fundemental subspaces?

Exercise 1. all the possible rank of

$$\begin{bmatrix} 1 & 1 & a \\ 3 & 3 & a \\ a & a & a \end{bmatrix}$$

when a varies.

$$\begin{pmatrix} 1 & 1 & a \\ 3 & 3 & a \\ a & a & a \end{pmatrix} \xrightarrow[]{\substack{R2 \quad \leftarrow R2 - 3R1 \\ R2 \quad \leftarrow R2 - aR1}} \begin{pmatrix} 1 & 1 & a \\ 0 & 0 & -2a \\ 0 & 0 & a - a^2 \end{pmatrix} \xrightarrow{R3 \quad \leftarrow R3 + \frac{1-a}{2}R1} \begin{pmatrix} 1 & 1 & a \\ 0 & 0 & -2a \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

- $a = 0$,rank=1
- $a \neq 0$, rank=2

2. all the possible rank of

$$\begin{bmatrix} 1 & 3 & a \\ 3 & 1 & a \\ a & a & a \end{bmatrix}$$

when a varies.

$$\begin{pmatrix} 1 & 3 & a \\ 3 & 1 & a \\ a & a & a \end{pmatrix} \xrightarrow[]{\substack{R2 \quad \leftarrow R2 - 3R1 \\ R2 \quad \leftarrow R2 - aR1}} \begin{pmatrix} 1 & 1 & a \\ 0 & -8 & -2a \\ 0 & -2a & a - a^2 \end{pmatrix} \xrightarrow{R3 \quad \leftarrow R3 - \frac{a}{4}R1} \begin{pmatrix} 1 & 1 & a \\ 0 & -8 & -2a \\ 0 & 0 & a - \frac{a^2}{2} \end{pmatrix} \tag{2}$$

- $a - \frac{a^2}{2} = 0$ (which means $a = 0$ or $a = \frac{1}{2}$), rank= 2
- $a - \frac{a^2}{2} \neq 0$ (which means $a \neq 0$ and $a \neq \frac{1}{2}$), rank= 3

Exercise 1. What is all the possible rank of

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

when a, b, c, d varies.

2. When is A invertible?

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \rightarrow \quad E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

1.

- rank=0, $a = b = c = d = 0$
- rank $= k$, k of $a, b-a, c-b, d-c$ is 0

2. $a \neq 0$, $a \neq b$, $b \neq c$, $d \neq c$

Exercise For which $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are there solutions to $Ax = b$, where the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$?

For those b, write down the complete solution.

We first reduce to REF

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R2 \ \leftarrow R2 - 2R1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R3 \ \leftarrow R3 - R2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

- basis of row(A) is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. (First and second row of REF)
- basis of col(A) is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. (First and third column of A)
- Nul(A): solve equation $Ax = 0$ gives solution $x_3 = 0$, x_2 is the free variable, $x_1 = -x_2 - x_3 = -x_2$. So Nul$(A) = \left\{ \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} |x_2 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 3 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right) \xrightarrow{R2 \ \leftarrow R2 - 2R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 \end{array}\right) \xrightarrow{R3 \ \leftarrow R3 - R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right) \tag{4}$$

The solution have solution means $b_3 = b_2 - 2b_1$.

Complete solution: Solve the equation by set x_2 as free varible:

- $x_3 = b_2 - 2b_1$,
- x_2 is the free variable,
- $x_1 = -x_2 - x_3 + b_1 = -x_2 - (b_2 - 2b_1) + b_1 = -x_2 - b_2 + 3b_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - b_2 + 3b_1 \\ x_2 \\ b_2 - 2b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} -b_2 + 3b_1 \\ 0 \\ b_2 - 2b_1 \end{bmatrix}}_{\text{find special solution by set free varaible to zero}} + x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{null space}}$$

Exercise Calculate the inverse matrix of $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{pmatrix}$?

Use elmination start from $[M|I]$ to $[I|M^{-1}]$

$$
\begin{aligned}
[M|I] = \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 1 & 2 & 2 & | & 0 & 1 & 0 \\ 1 & 3 & 4 & | & 0 & 0 & 1 \end{pmatrix} & \xrightarrow{\substack{R2 \ \leftarrow R2 - R1 \\ R3 \ \leftarrow R3 - R1}} \begin{pmatrix} 1 & 1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 2 & 3 & | & -1 & 0 & 1 \end{pmatrix} \\
& \text{Use R1 to ellimate the column 1 in R2 and R3} \\
& \xrightarrow{\substack{R1 \ \leftarrow R1 - 1 \cdot R2 \\ R3 \ \leftarrow R3 - 2 \cdot R2}} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 1 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
& \text{Use R2 to ellimate the column 2 in R1 and R3} \\
& \xrightarrow{\substack{R1 \ \leftarrow R1 - 0 \cdot R3 \\ R2 \ \leftarrow R2 - 1 \cdot R3}} \begin{pmatrix} 1 & 0 & 0 & | & 2 & -1 & 0 \\ 0 & 1 & 0 & | & -2 & 3 & -1 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \\
& \text{Use R3 to ellimate the column 3 in R1 and R2}
\end{aligned} \tag{5}
$$

Thus

$$M^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Check if MM^{-1} is idenity! equal to check

- $(1,1,1) \cdot (2,-2,1) = 1$,$(1,2,2) \cdot (2,-2,1) = 0$, $(1,3,4) \cdot (2,-2,1) = 0$
- $(1,1,1) \cdot (-1,3,-2) = 0$,$(1,2,2) \cdot (-1,3,-2) = 1$, $(1,3,4) \cdot (-1,3,-2) = 0$
- $(1,1,1) \cdot (0,-1,1) = 0$,$(1,2,2) \cdot (0,-1,1) = 0$, $(1,3,4) \cdot (0,-1,1) = 1$

1. The complete solution of linear system $Ax = b$ is $\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then $\dim(\mathrm{col}(A)) = 3$

Yes, A have 5 column ($n = 5$). For we have 2 free variables, thus $\dim(\mathrm{Nul}(A)) = 2$ So rank $r = n - \dim(\mathrm{Nul}(A)) = 5 - 2 = 3$

2. There exist a matrix A whose column space is spanned by $(1, 2, 3)$ and $(1, 0, 1)$and whose nullspace is spanned by $(1, 2, 3, 6)$

No. The dimensions of such a matrix must be 3 by 4 ($m = 3$ and $n = 4$). The dimension of the column space is 2, because the given vectors are independent. That means the dimension of the nullspace must be $4 - 2 = 2$. The null space cannot be spanned by 1 vector.

3.

- For a matrix $A \in \mathbb{R}^{4\times 5}$, the largest possible rank of A is 5. No, rank$\leq m$, rank $\leq n$
- For a matrix $A \in \mathbb{R}^{4\times 5}$ there are possibility that linear system $Ax = b$ have one and only have one solution. No, rank ≤ 4, so this can't be a full column rank matrix, $\dim(\mathrm{Nul}(A))$ can't be zero. Futhermore, this linear sytem must have infinite solution. Because this is a full row rank matrix, thus the system at least have one solution.($\mathrm{row}(A) = \mathbb{R}^4$)
- For a matrix $A \in \mathbb{R}^{4\times 5}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have one and only have one solution. No, same as above. $\dim(\mathrm{Nul}(A)) = 1$
- For a matrix $A \in \mathbb{R}^{4\times 3}$, $\mathrm{rank}(A) = 3$. There are possibility that linear system $Ax = b$ have one and only have one solution. Yes, this is a full column rank matrix, so $\mathrm{Nul}(A) = \{\vec{0}\}$
- For a matrix $A \in \mathbb{R}^{4\times 5}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. No. This is a full row rank matrix, so $\dim(\mathrm{row}(A)) = 4$ and $\mathrm{row}(A) \subset \mathbb{R}^4$. This means $\mathrm{row}(A) = \mathbb{R}^4$. Every system $Ax = b$ must have a solution.
- For a matrix $A \in \mathbb{R}^{5\times 4}$, $\mathrm{rank}(A) = 4$. There are possibility that linear system $Ax = b$ have no solution. Yes, this is because this is not a full row rank matrix.
- $Y = AX$ and A is an invertible matrix, then $\mathrm{rank}(Y) = \mathrm{rank}(X)$. Yes,because $Y = AX$ so $\mathrm{rank}(Y) \leq \mathrm{rank}(X)$. For A is invertible matrix, so $X = A^{-1}Y$ which tells us $\mathrm{rank}(X) \leq \mathrm{rank}(Y)$. The only possiblity is $\mathrm{rank}(Y) = \mathrm{rank}(X)$