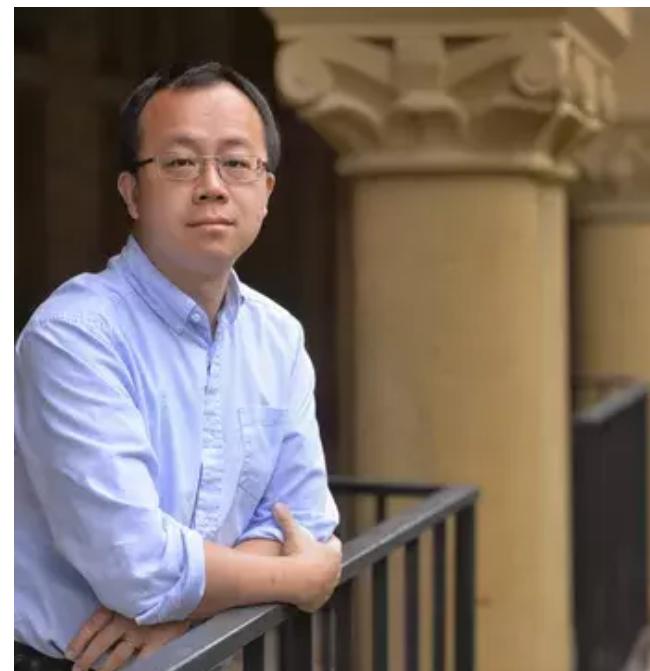


# Two Tales, One Resolution for Physics-Informed Inference-time Scaling

## Debiasing and Precondition

**Yiping Lu**

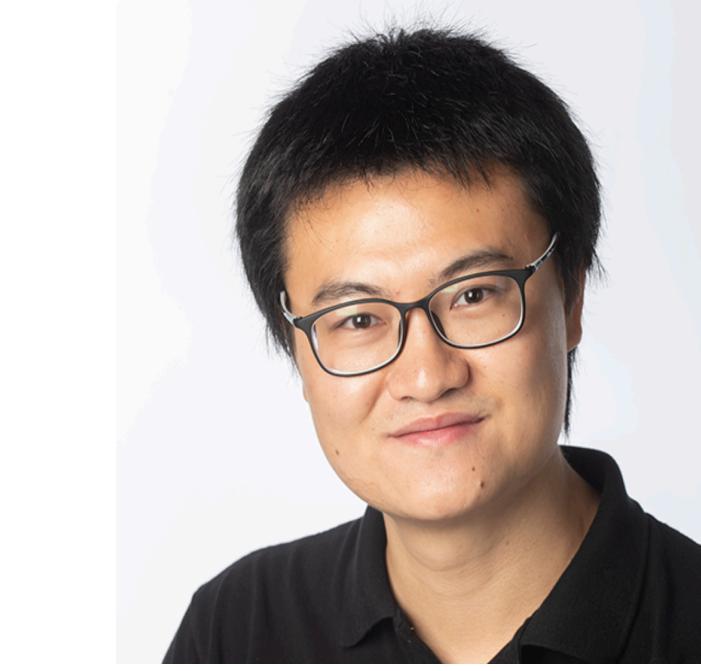
**Northwestern** | McCORMICK SCHOOL OF  
ENGINEERING



Lexing Ying (Stanford)



Jose Blanchet (Stanford)



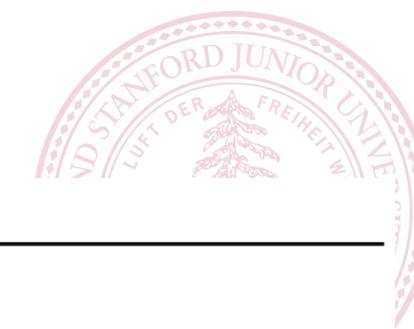
Shihao Yang (Gatech)



Ruihan Xu (UChicago)

# My Journey

- Undergrad: Peking University: 2015-2019
  - Work with Prof. Bin Dong and Prof. Liwei Wang
- Ph.D. Stanford University: 2019-2023
  - Work with Prof. Lexing Ying and Jose E



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## Beyond Finite Layer Neural Networks: Bridging Deep Architectures and Numerical Differential Equations

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Yiping Lu<sup>1</sup> Aoxiao Zhong<sup>2</sup> Quanzheng Li<sup>2,3,4</sup> Bin Dong<sup>5,6,4</sup>

### Abstract

Deep neural networks have become the state-of-the-art models in numerous machine learning tasks. However, general guidance to network architecture design is still missing. In our work, we bridge deep neural network design with numerical differential equations. We show that many effective networks, such as ResNet, PolyNet, FractalNet and RevNet, can be interpreted as different numerical discretizations of differential equations. This finding brings us a brand new perspective on the design of effective deep architectures. We can take advantage of the rich knowledge in numerical analysis to guide us in de-

s while maintaining a similar performance. This can be explained mathematically using the concept of modified equation from numerical analysis. Last but not least, we also establish a connection between stochastic control and noise injection in the training process which helps to improve generalization of the networks. Furthermore, by relating stochastic training strategy with stochastic dynamic system, we can easily apply stochastic training to the networks with the LM-architecture. As an example, we introduced stochastic depth to LM-ResNet and achieve significant improvement over the original LM-ResNet on CIFAR10.

### PDE-NET: LEARNING PDES FROM DATA

Zichao Long\*, Yiping Lu\*

School of Mathematical Sciences  
Peking University, Beijing, China  
[{zlong,luyiping9712}@pku.edu.cn](mailto:{zlong,luyiping9712}@pku.edu.cn)

Xianzhong Ma\*

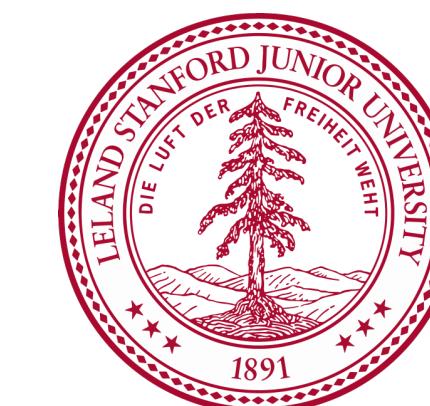
School of Mathematical Sciences, Peking University  
Beijing Computational Science Research Center  
Beijing, China  
[xianzhongma@pku.edu.cn](mailto:xianzhongma@pku.edu.cn)

Bin Dong

Beijing International Center for Mathematical Research, Peking University  
Center for Data Science, Peking University  
Beijing Institute of Big Data Research  
Beijing, China  
[dongbin@math.pku.edu.cn](mailto:dongbin@math.pku.edu.cn)

# My Journey

- Undergrad: Peking University: 2015-2019
  - Work with Prof. Bin Dong and Prof. Liwei Wang
- Ph.D. Stanford University: 2019-2023
  - Work with Prof. Lexing Ying and Jose Blanchet



## Machine Learning For Elliptic PDEs: Fast Rate Generalization Bound, Neural Scaling Law and Minimax Optimality

Yiping Lu\*, Haoxuan Chen<sup>†</sup>, Jianfeng Lu <sup>‡</sup>, Lexing Ying<sup>§</sup>, and Jose Blanchet <sup>¶</sup>

**Abstract.** In this paper, we study the statistical limits of deep learning techniques for solving elliptic partial differential equations (PDEs) from random samples using the Deep Ritz Method (DRM) and Physics-Informed Neural Networks (PINNs). To simplify the problem, we focus on a prototype elliptic PDE: the Schrödinger equation on a hypercube with zero Dirichlet boundary condition, which is applied in quantum-mechanical systems. We establish upper and lower bounds for both methods, which improve upon concurrently developed upper bounds for this problem via a fast rate generalization bound. We discover that the current Deep Ritz Method is sub-optimal and propose a modified version of it. We also prove that PINN and the modified version of DRM can achieve minimax optimal bounds over Sobolev spaces. Empirically, following recent work which has shown that the deep model accuracy will improve with growing training sets according to a power law, we supply computational experiments to show similar behavior of dimension dependent power law for deep PDE solvers.

## Minimax Optimal Kernel Operator Learning via Multilevel Training

Jikai Jin,<sup>\*,1</sup> Yiping Lu,<sup>\*,2</sup> Jose Blanchet,<sup>\*,3</sup> and Lexing Ying<sup>\*,4</sup>

<sup>1</sup>School of Mathematical Sciences, Peking University, Beijing, China

<sup>2</sup>ICME, Stanford University, CA, USA

<sup>3</sup>Management Science & Engineering, Stanford University, CA, USA

<sup>4</sup>Department of Mathematics, Stanford University, Stanford, CA, USA

\*Corresponding author. Email: [jkjin@pku.edu.cn](mailto:jkjin@pku.edu.cn); [yplu@stanford.edu](mailto:yplu@stanford.edu); [jose.blanchet@stanford.edu](mailto:jose.blanchet@stanford.edu); [lexing@stanford.edu](mailto:lexing@stanford.edu)

# Do you trust your theorem?

Big Constant

**Theorem** If you randomly collect ( ) data, then you can achieve ( ) accuracy with your AI!

Relu network is **optimal** for function regression

PINN is **optimal** for differential equation solving

Diffusion Model is **optimal** for density estimation

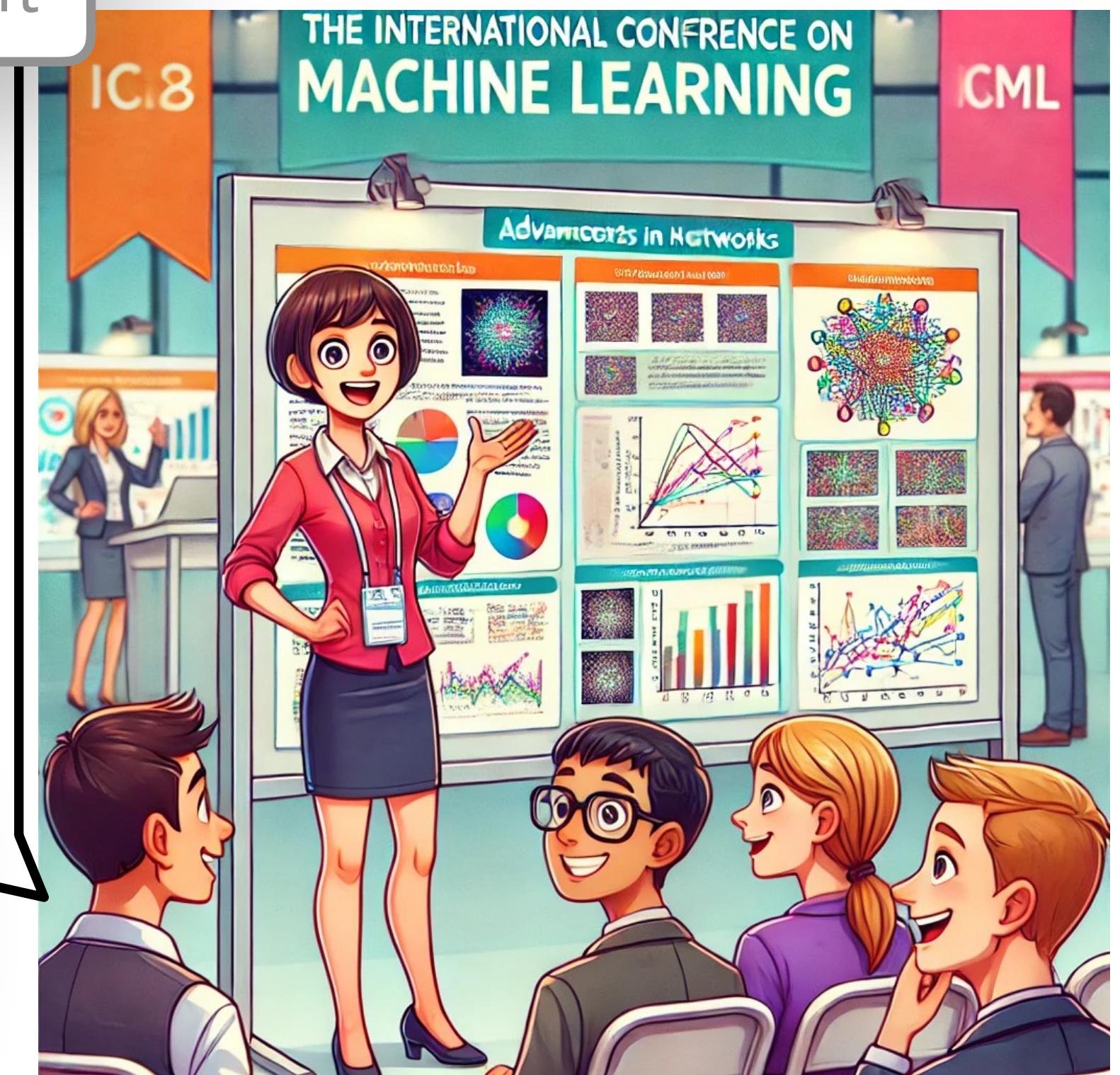
Transformer is **optimal** for in-context learning

...

Global optimization, the structure of target function,  
The smoothness of ReLU, lower bound of  $p(x)$ ...



Does Optimal results really gives me guarantee?



# Consider How you use ChatGPT...

What is the most efficient route from Times Square to Chelsea Market that minimizes walking?

## Option 1: Subway

1. **Walk to a Nearby Subway Entrance:** Head to the Times Sq–42nd Street station.
2. **Board the E Train:** Catch an E train heading downtown (toward World Trade Center).
3. **Ride to 8th Avenue–14th Street Station:** Get off at this station (roughly 4 stops).
4. **Walk to Chelsea Market:** Exit the station and walk east on 14th Street for a few blocks until you reach Chelsea Market at 75 9th Avenue.



double-check  
with a map

Port Authority

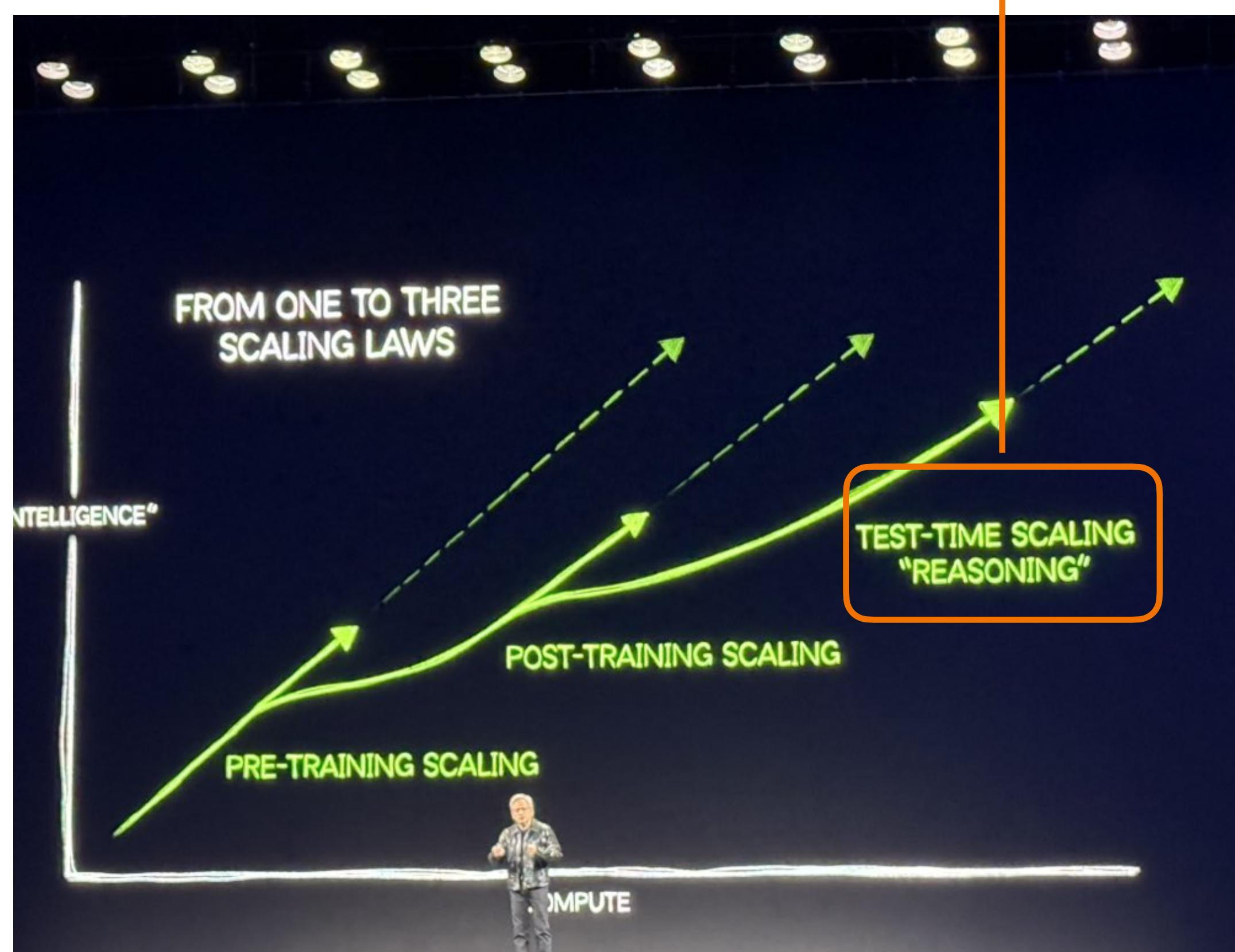
2 stops for A  
3 stops for C/E

1. **At Times Square:** Enter the Times Square–42nd Street station.
2. **Board the 1 Train:** Hop on a downtown 1 train (the red line).
3. **Ride to 14th Street:** Stay on until you reach the 14th Street station.
4. **Exit Appropriately:** Use the exit that leads toward 9th Avenue—this drop-off point is just a short walk from Chelsea Market (located at 75 9th Ave).



# Inference Time Scaling Law

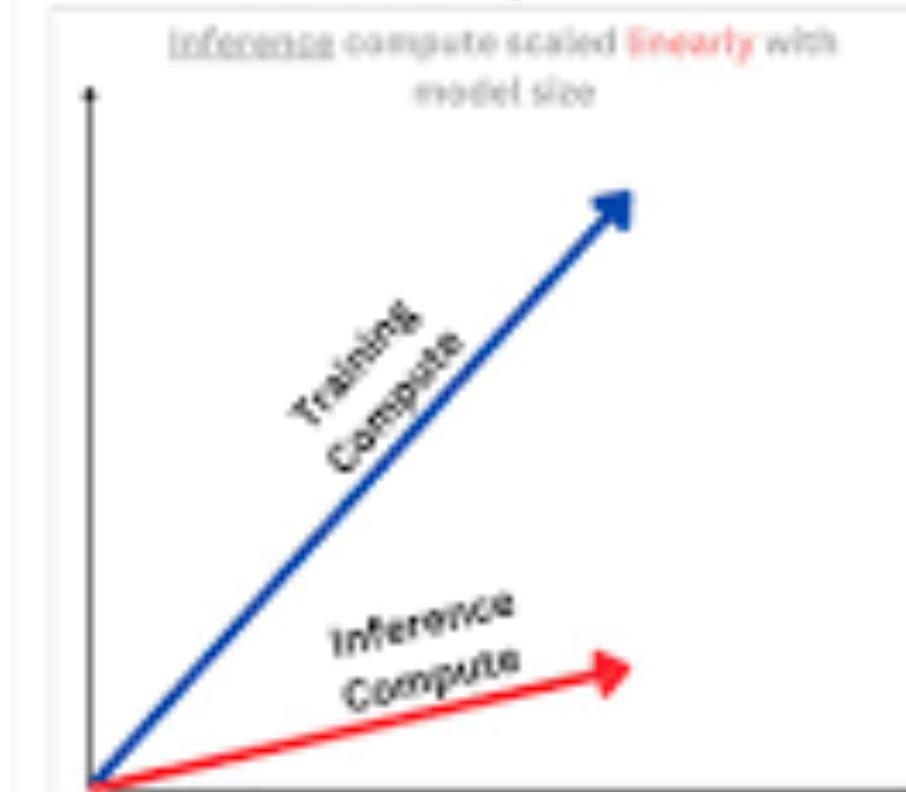
“No training”  
e.g. answer question 10 times



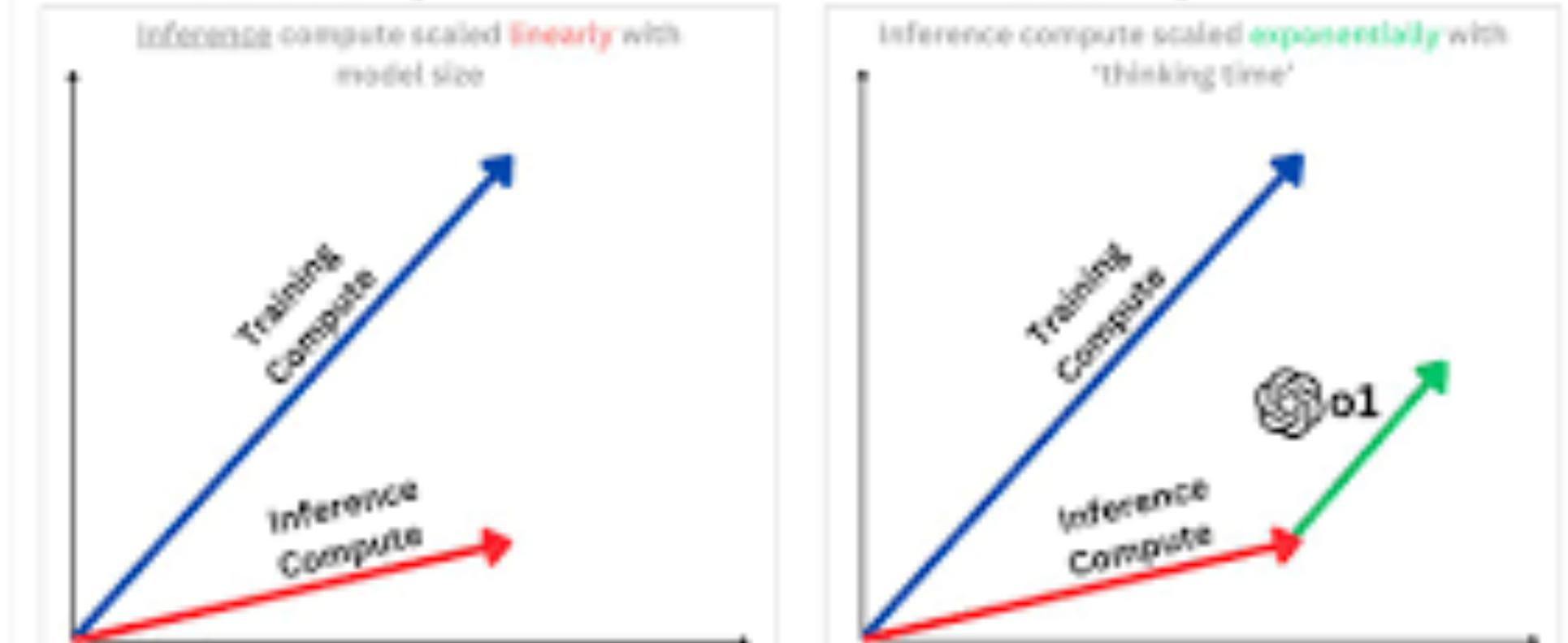
@DrJimFan

New scaling law: why OpenAI's o1 model matters  
OpenAI created a new way to scale - through more compute during generation

Before OpenAI o1



After OpenAI o1



Source: Peter Dzintars (<https://www.linkedin.com/in/peterdzintars/>)

# How can we perform Inference-Time Scaling for Scientific Machine Learning?

With trustworthy guarantee

don't fine-tune/retrain/add a new surrogate model

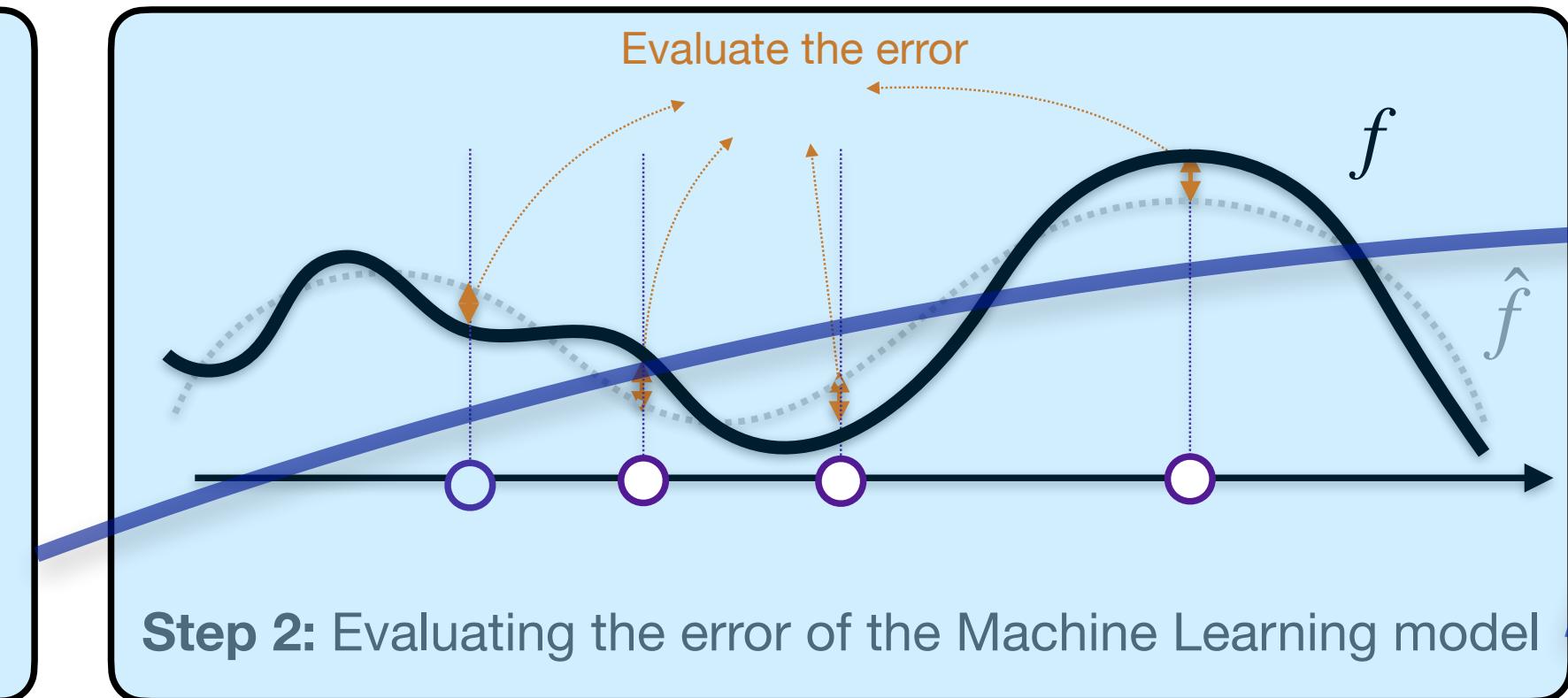
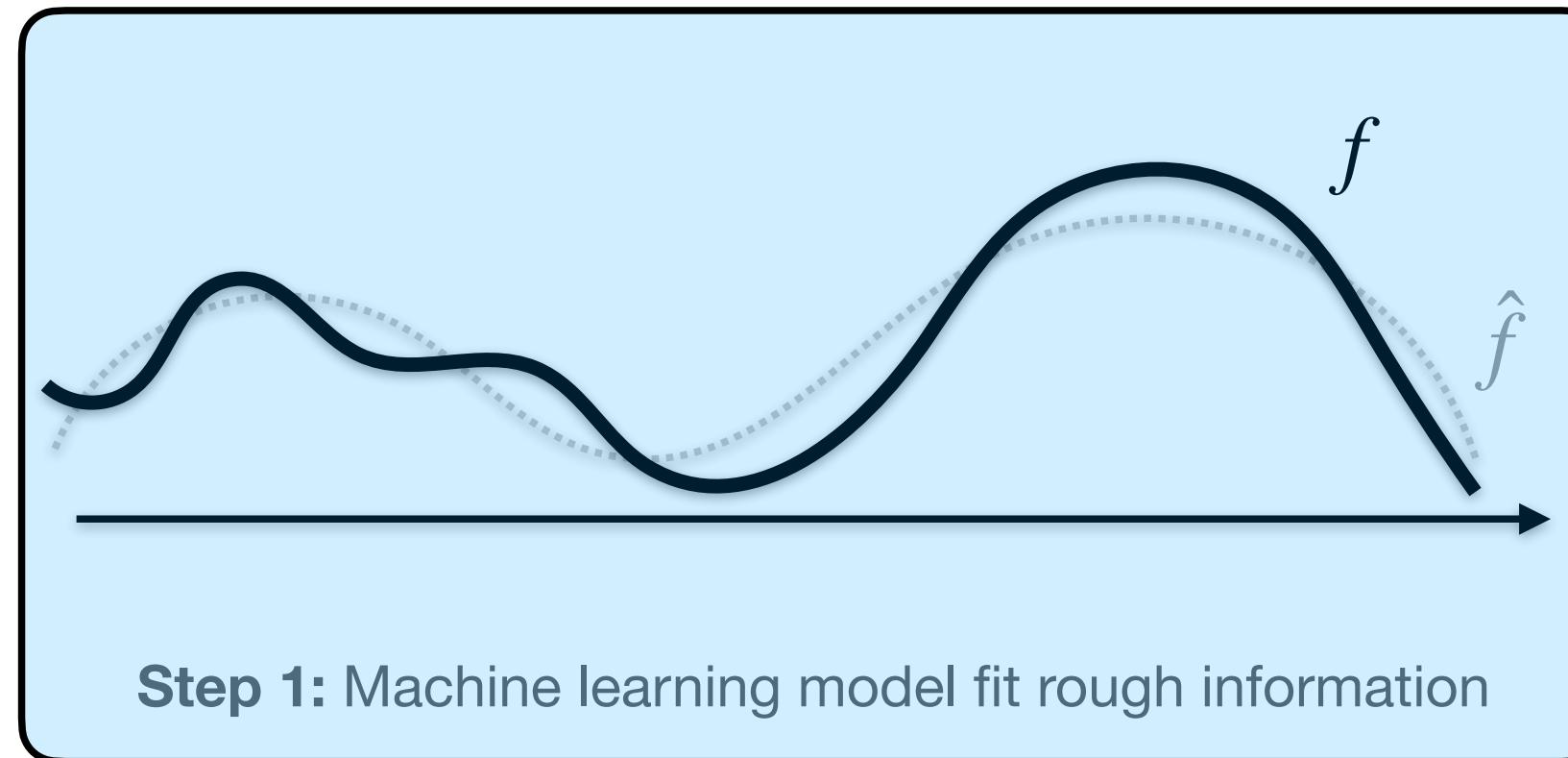
# How can we perform Inference-Time Scaling for Scientific Machine Learning?

“Physics-informed”

With trustworthy guarantee

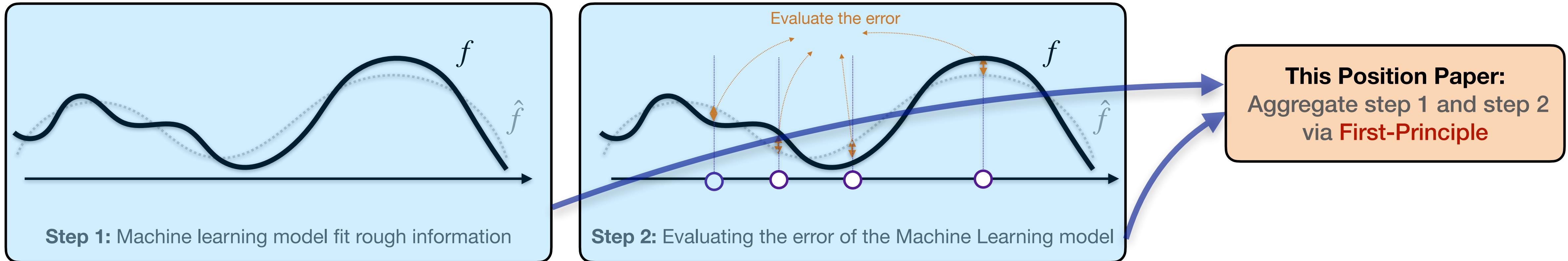
# Tale 1: Debiasing Hybrid Scientific Computing and Machine Learning

# Physics-Informed Inference Time Scaling



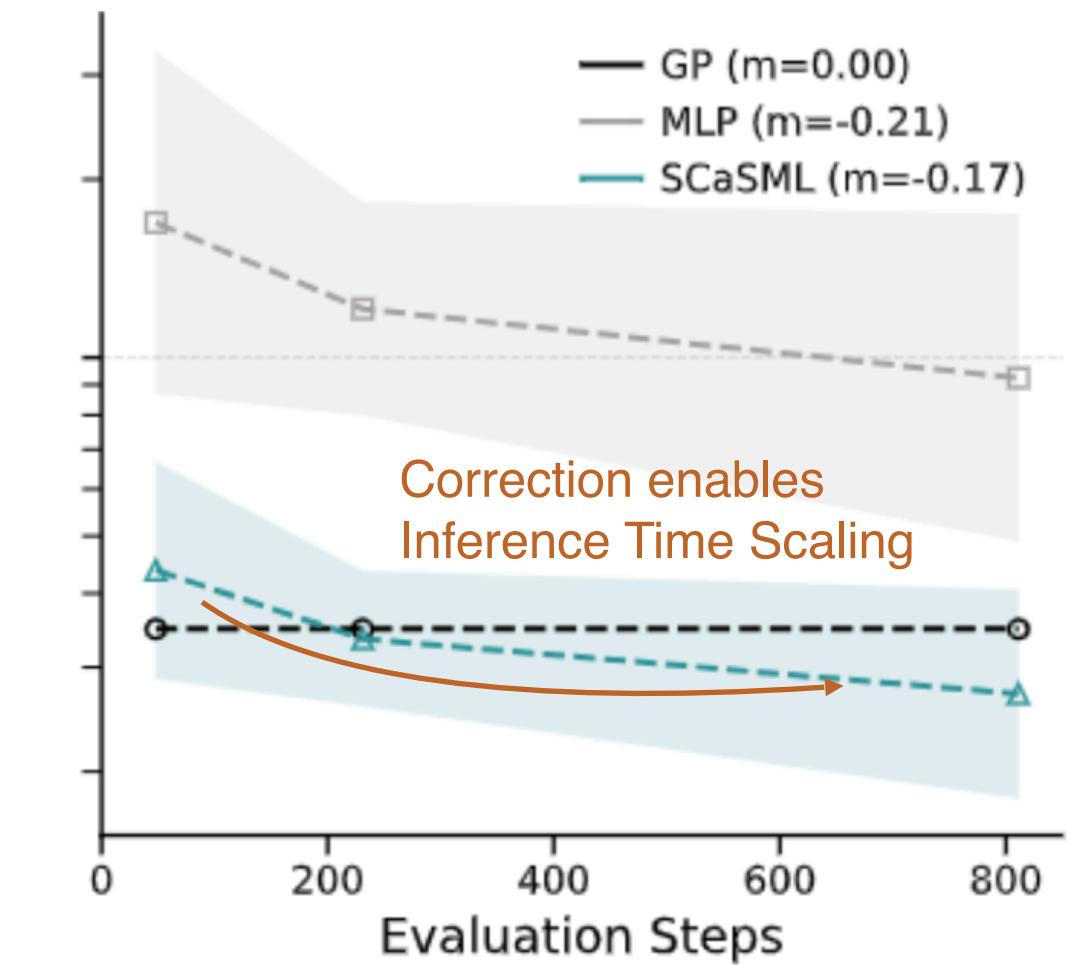
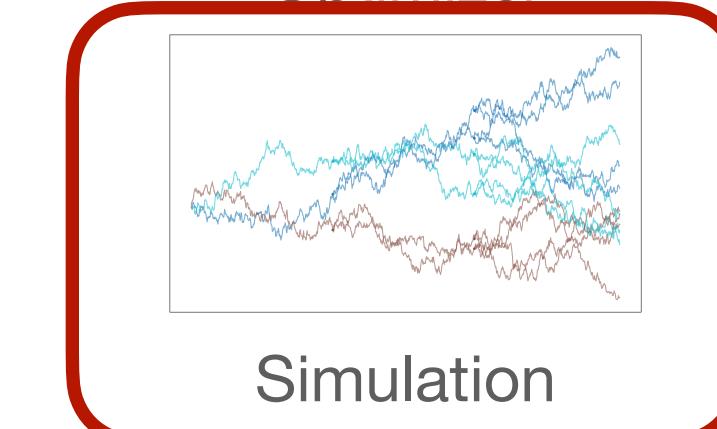
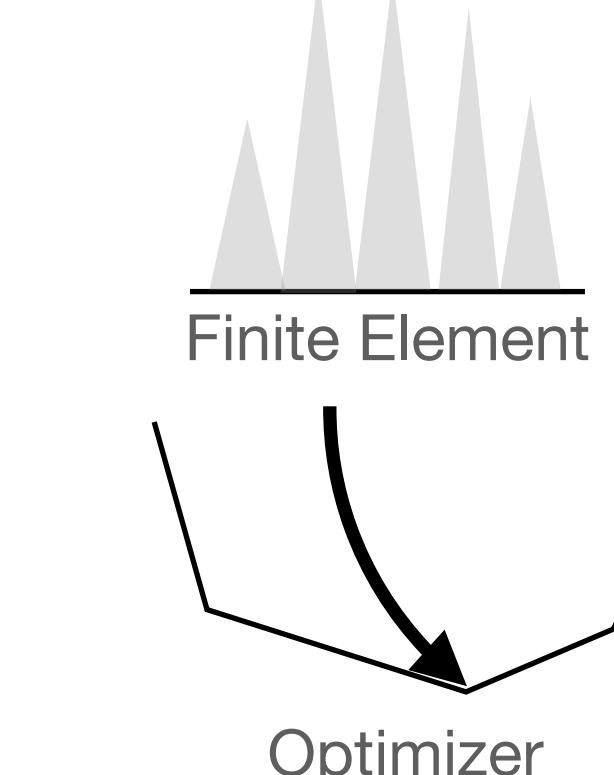
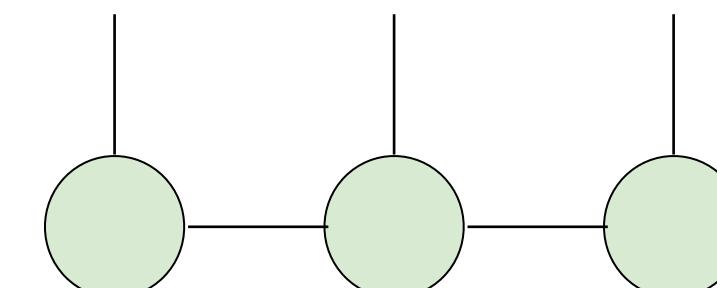
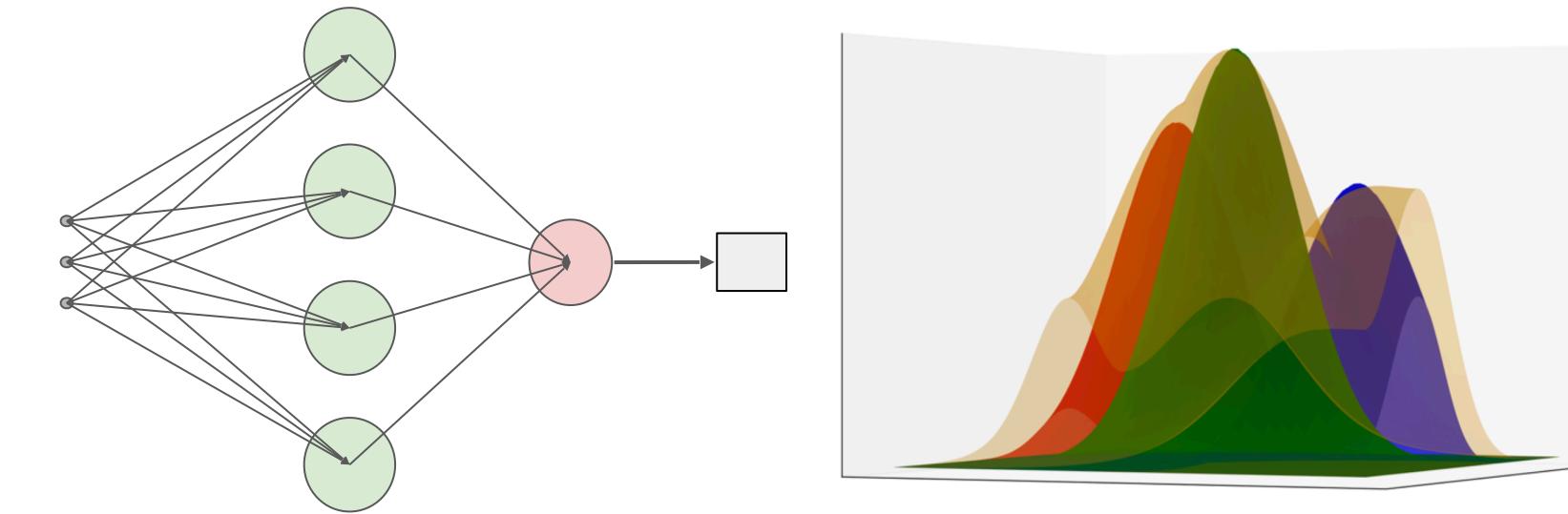
This Position Paper:  
Aggregate step 1 and step 2  
via **First-Principle**

# Physics-Informed Inference Time Scaling



## Step 2. Correct with a Trustworthy Solver

### Step 1. Train a Surrogate (ML) Model



# Our Framework

## Step 1: Sceintific Computing as Machine Learning

$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

### Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

Function fitting

### Example 2

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i)) \quad \text{Solving } \Delta u = f$$

Solving PDE

### Example 3

$$\theta = A, \quad X_i = (x_i, Ax_i)$$

Estimation  $\hat{A}$  via Randomized SVD

# Our Framework

## Step 2: Consider a Downstream Application



**Example 1**

Scientific Machine Learning

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

Downstream application

$$\Phi(\theta) = \int f(x)dx$$

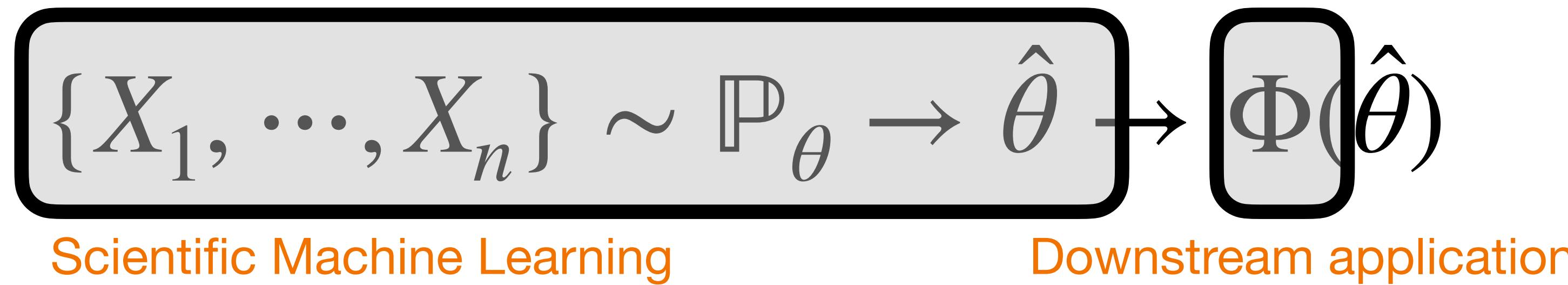
**Example 2**

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i)) \quad \Phi(\theta) = (\Delta^{-1}f)(x)$$

**Example 3**

$$\theta = A, \quad X_i = (x_i, Ax_i) \quad \Phi(\theta) = \text{tr}(A), \text{eigs}(A)$$

# Our Framework



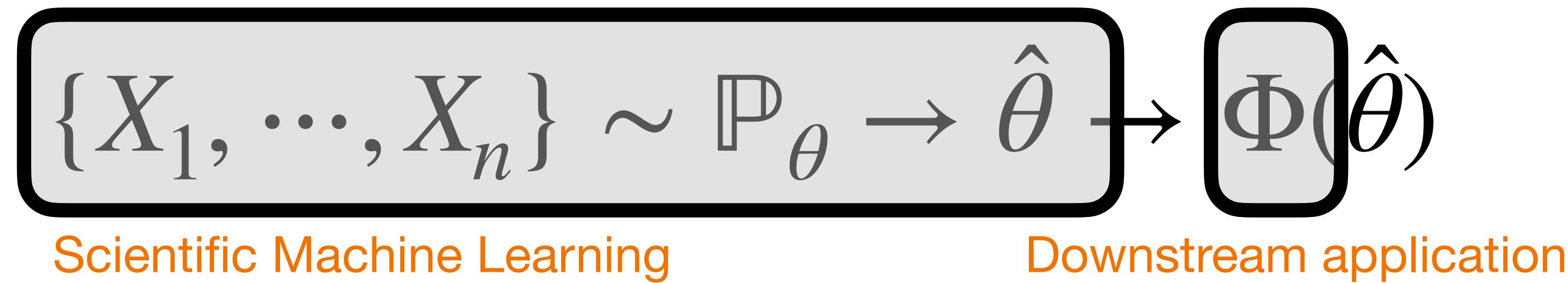
**AIM:** Unbiased prediction even with biased machine learning estimator

AIM: Compute  $\Phi(\hat{\theta}) - \Phi(\theta)$  during Inference time



Using (stochastic) simulation to calibrate the (scientific) machine learning output !

# Our Framework



**AIM:** Unbiased prediction even with biased machine learning estimator

How to estimate  $\Phi(\hat{\theta}) - \Phi(\theta)$ ?

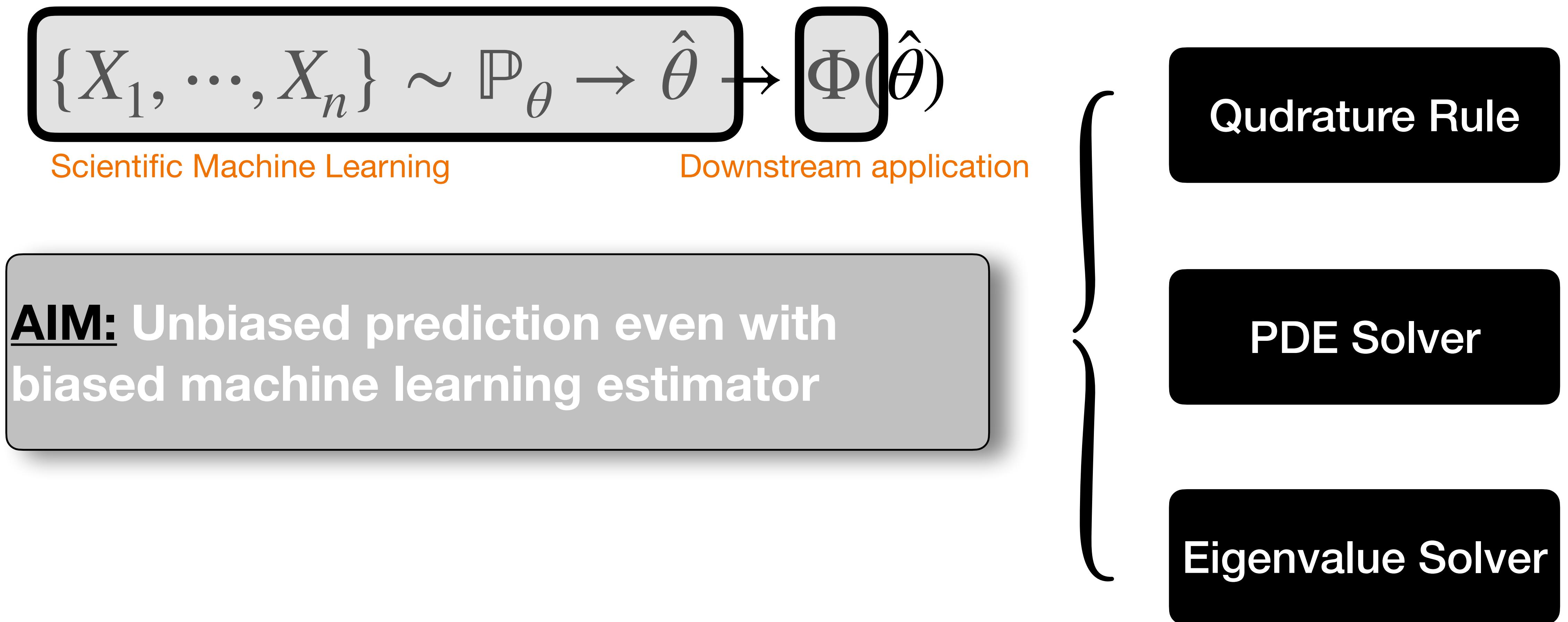


Physics-Informed! (Structure of  $\Phi$ )

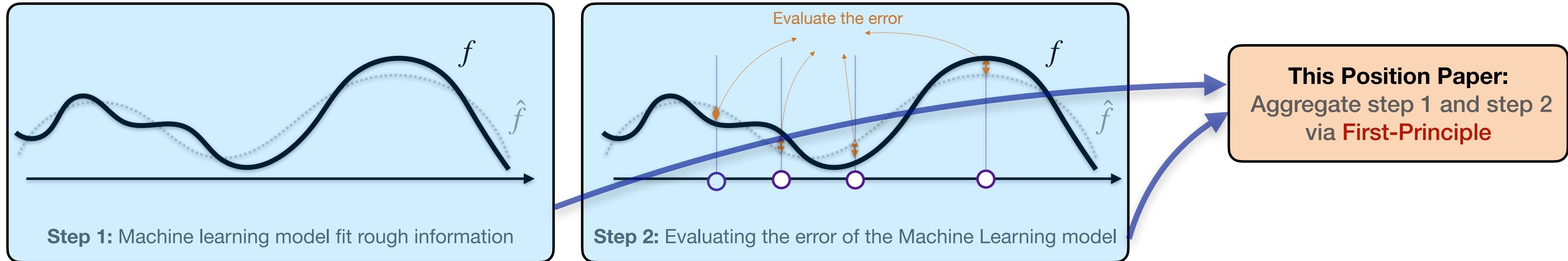
Why it is easier than directly estimate  $\Phi(\theta)$ ?

Variance Reduction

# Our Framework



# Debiasing a Machine Learning Solution



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

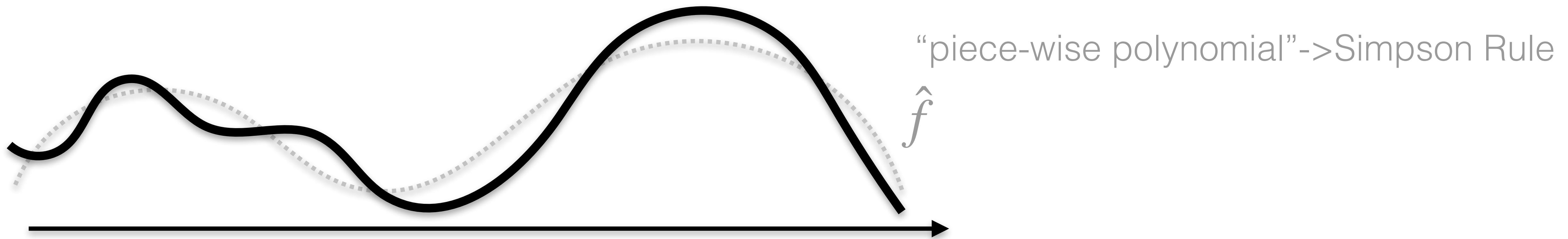
**Example 1**

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

# Debiasing a Machine Learning Solution



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

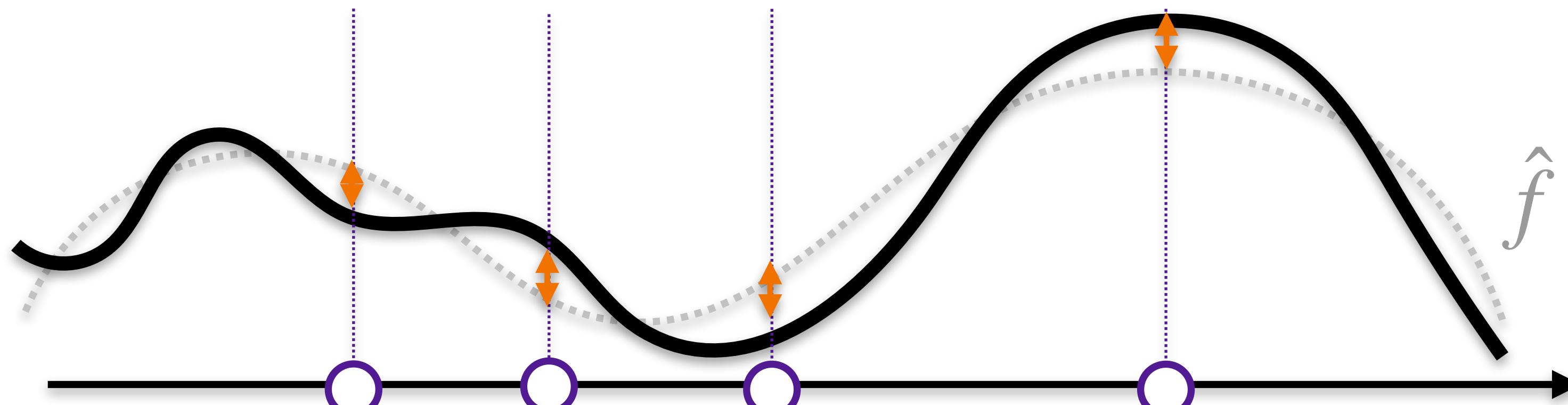
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

# Debiasing a Machine Learning Solution



Our Approach

$$\text{Estimate } \mathbb{E}_P f \approx \mathbb{E}_P \hat{f} + \mathbb{E}_{\hat{P}} f - \hat{f}$$

An estimate to  $\Phi(\hat{\theta}) - \Phi(\theta)$



Scientific Machine Learning

Downstream application

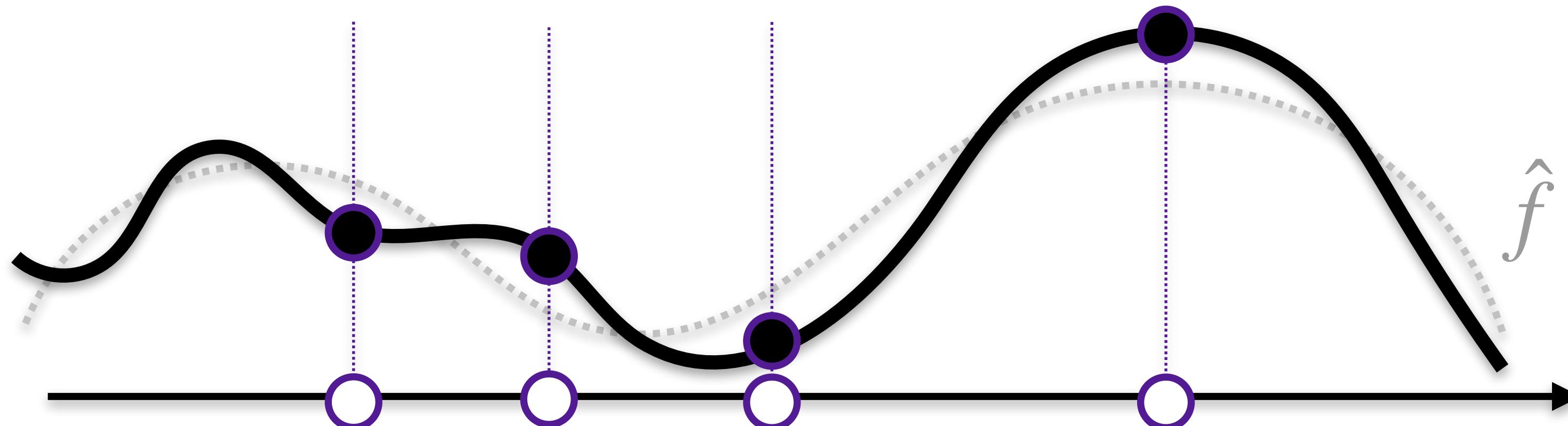
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

# Debiasing a Machine Learning Solution



Monte Carlo?

Estimate  $\mathbb{E}_P f \approx \mathbb{E}_{\hat{P}} \hat{f}$



Scientific Machine Learning

Downstream application

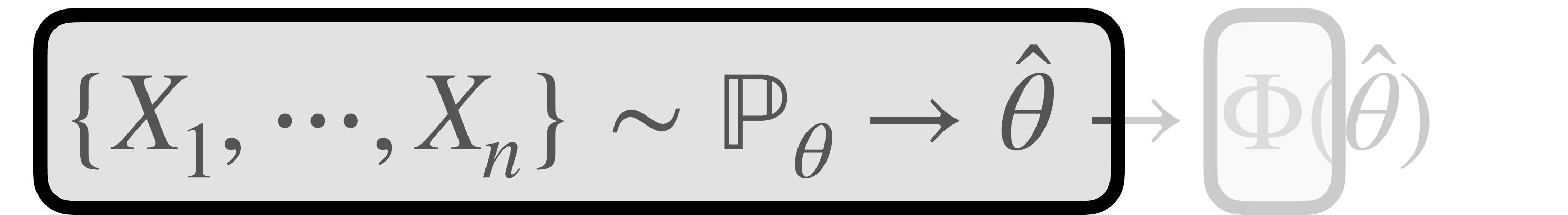
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

# The 101 Example



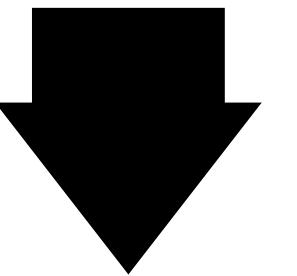
Scientific Machine Learning

Downstream application

## Example

$$\theta = f, \underbrace{X_i}_{=} (x_i, f(x_i))$$

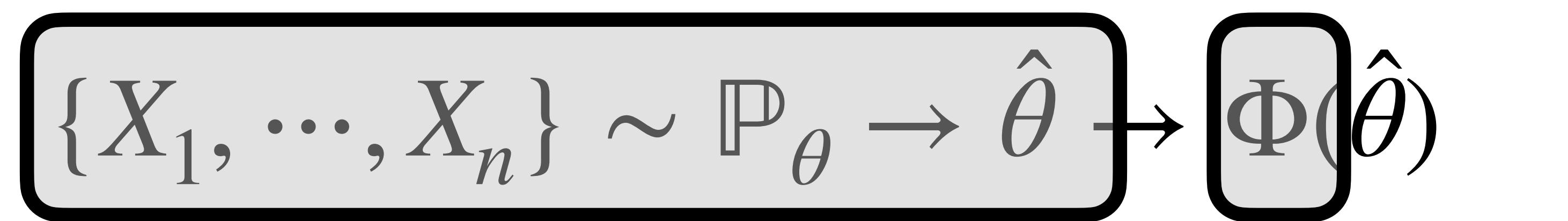
$$\Phi(\theta) = \int (f(x))dx$$



**Machine Learning:**  $\hat{\theta} = \hat{f}$

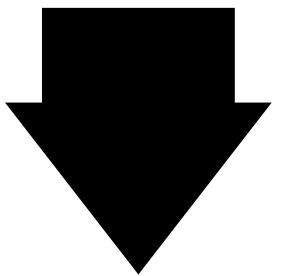
# The 101 Example

**Example**



$$\theta = f, \quad \underbrace{X_i}_{= (x_i, f(x_i))}$$

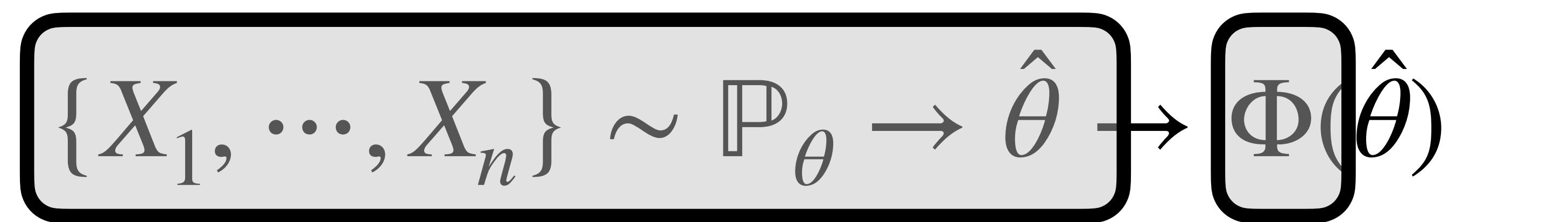
$$\Phi(\theta) = \int (f(x))dx$$



**Machine Learning:**  $\hat{\theta} = \hat{f} \longrightarrow \Phi(\hat{\theta}) = \int f(x)dx$

# The 101 Example

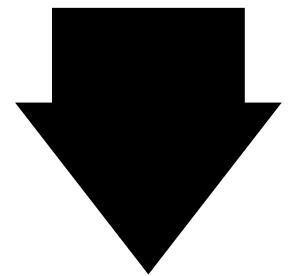
## Example



$$\theta = f, \underbrace{X_i}_{=} (x_i, f(x_i))$$

$$\Phi(\theta) = \int (f(x))dx$$

||



Machine Learning:  $\hat{\theta} = \hat{f}$



$$\Phi(\hat{\theta}) = \int f(x)dx$$

+

$$\Phi(\theta) - \Phi(\hat{\theta}) = \int (f(x) - \hat{f}(x))dx$$

# Debiasing a Machine Learning Solution



Regression-adjusted Control Variates

Doubly Robust Estimator

Multi-fidelity monte carlo

...

- Investigated the **optimality** of the SCaSML Framework

- Jose Blanchet, Haoxuan Chen, Yiping Lu, Lexing Ying. When can Regression-Adjusted Control Variates Help? Rare Events, Sobolev Embedding and Minimax Optimality Neurips 2023

- Extend to **nonlinear** functional estimation using iterative methods

Later



Scientific Machine Learning

Downstream application

**Example 1**

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

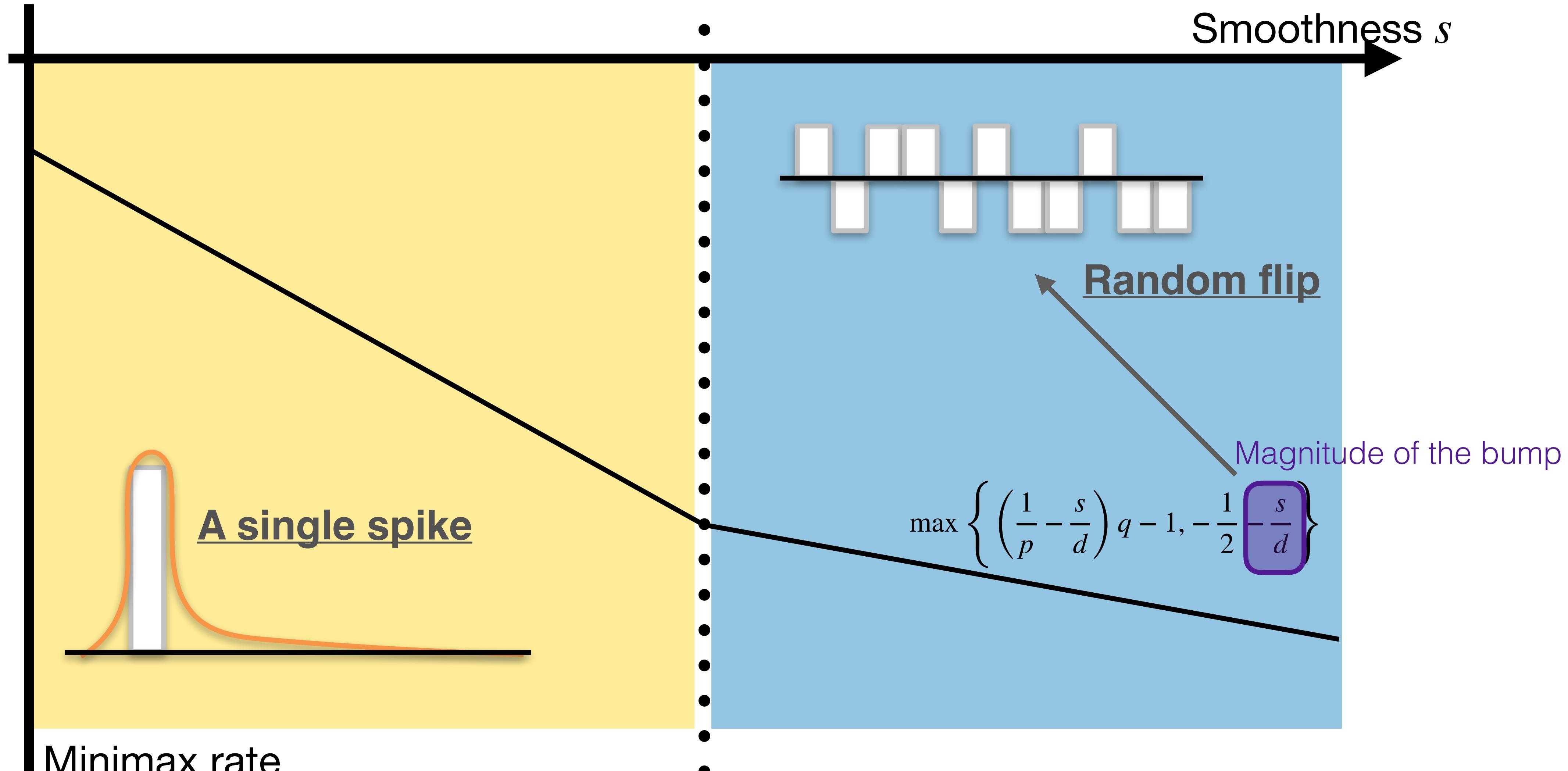
Temperature, overall velocity...

# **Lower Bound**

# Hardest Examples



SCaML

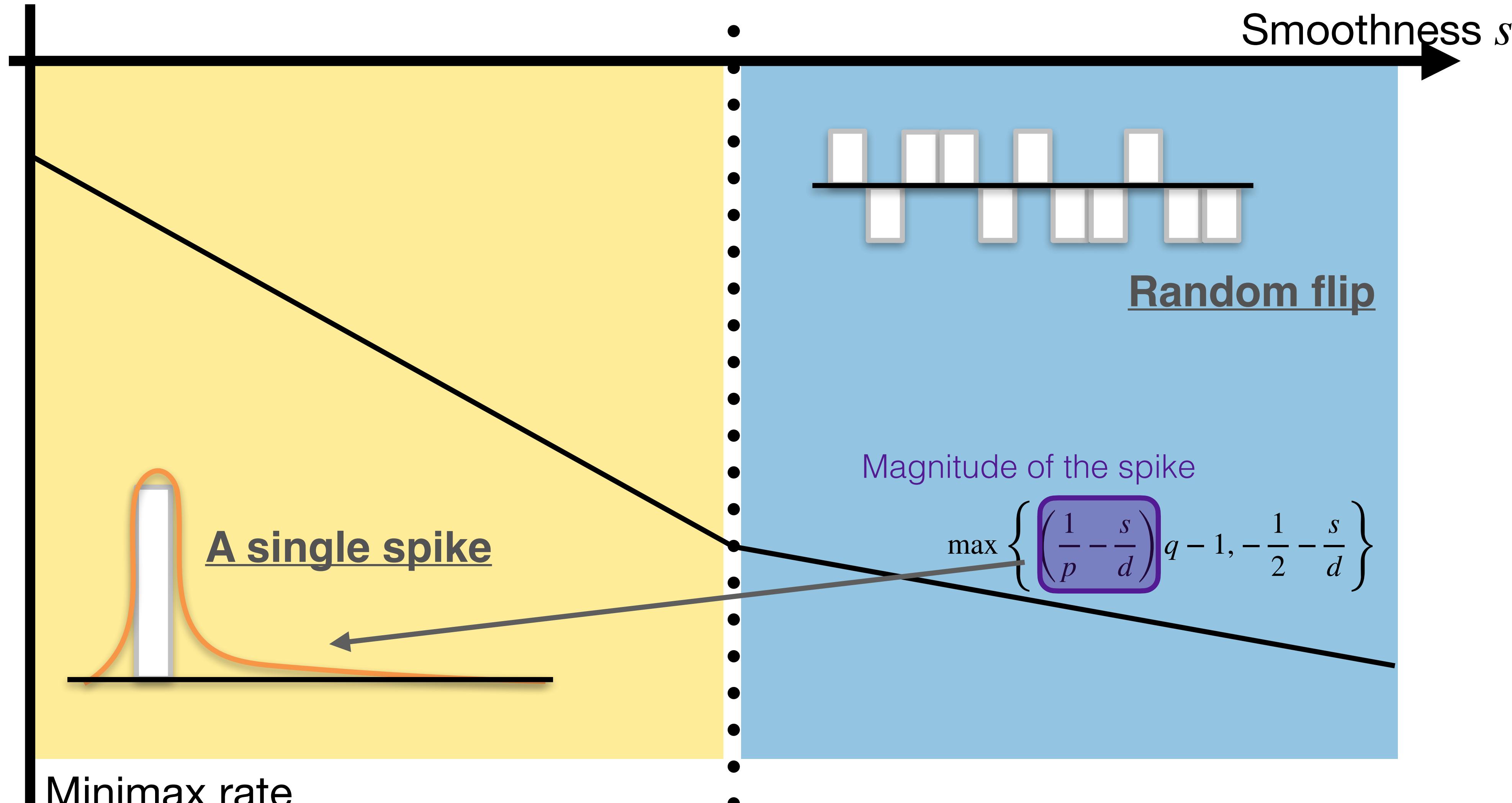


Minimax rate

# Hardest Examples

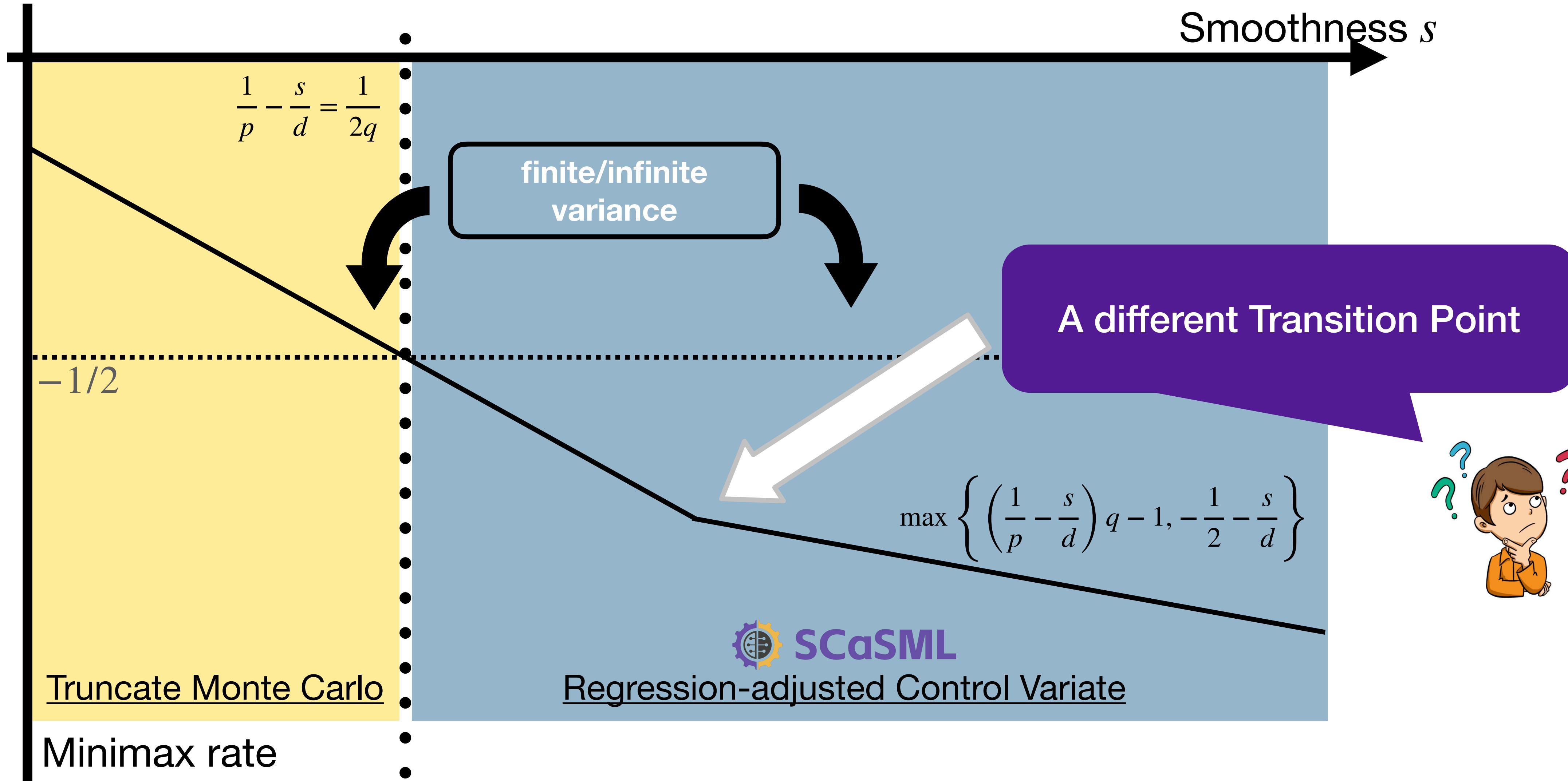


SCaML

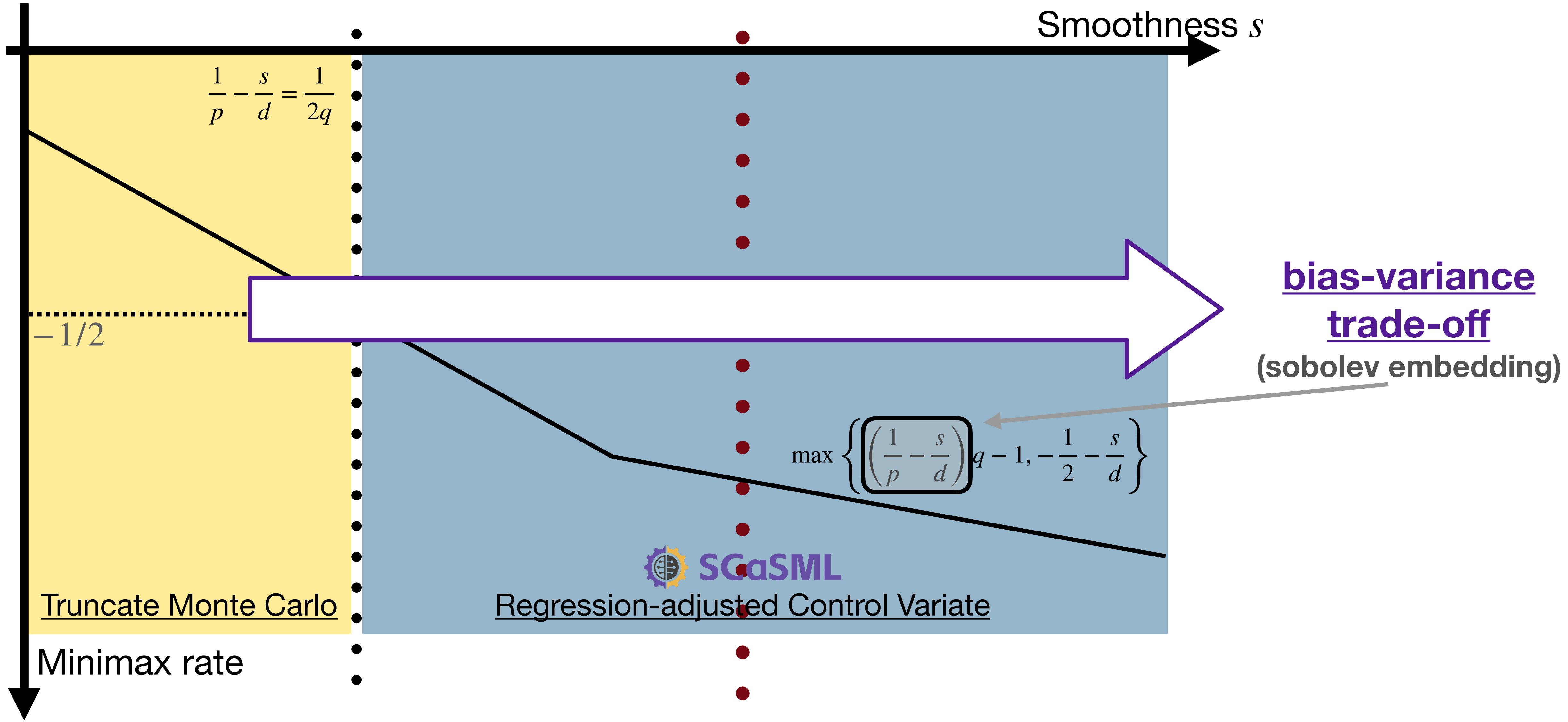


# **Optimal Algorithms**

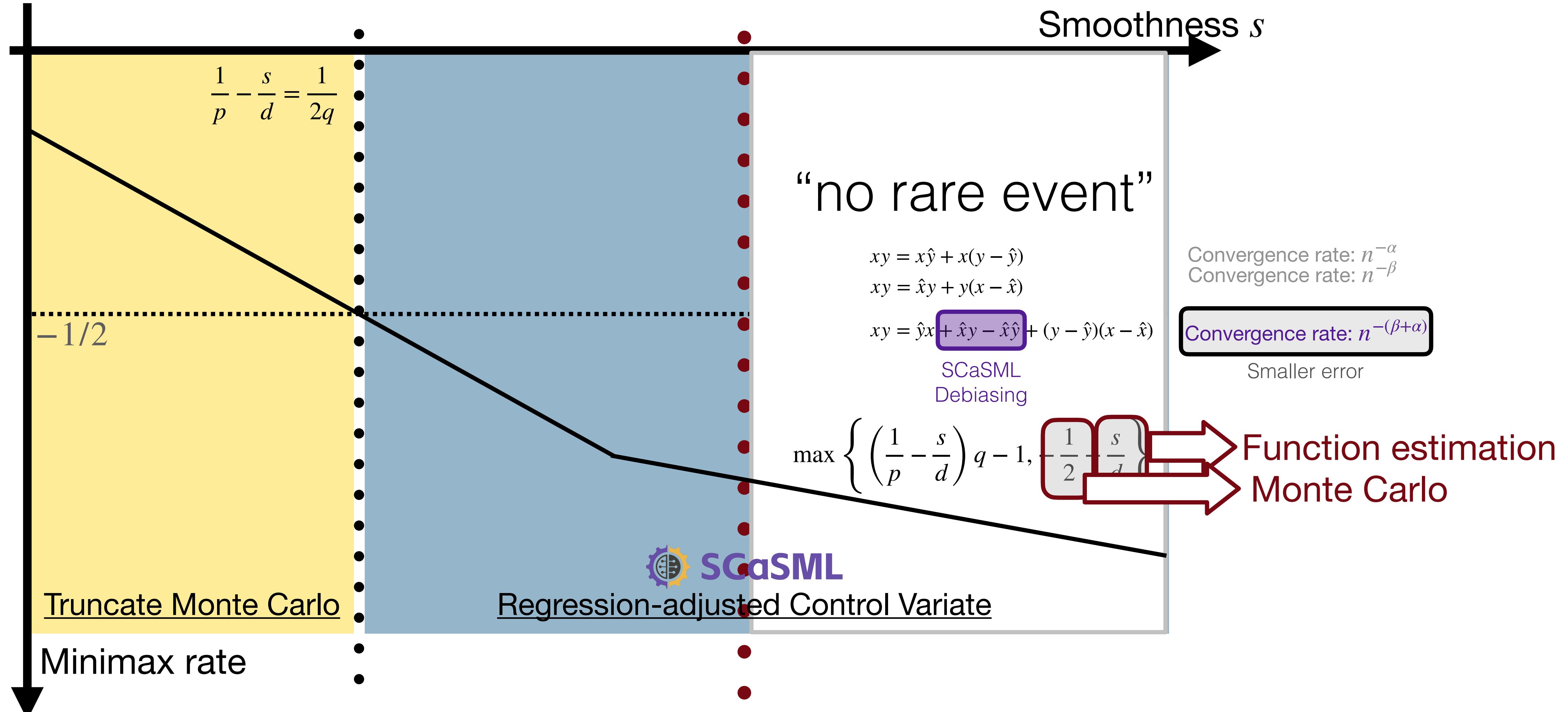
# Easiest Understanding



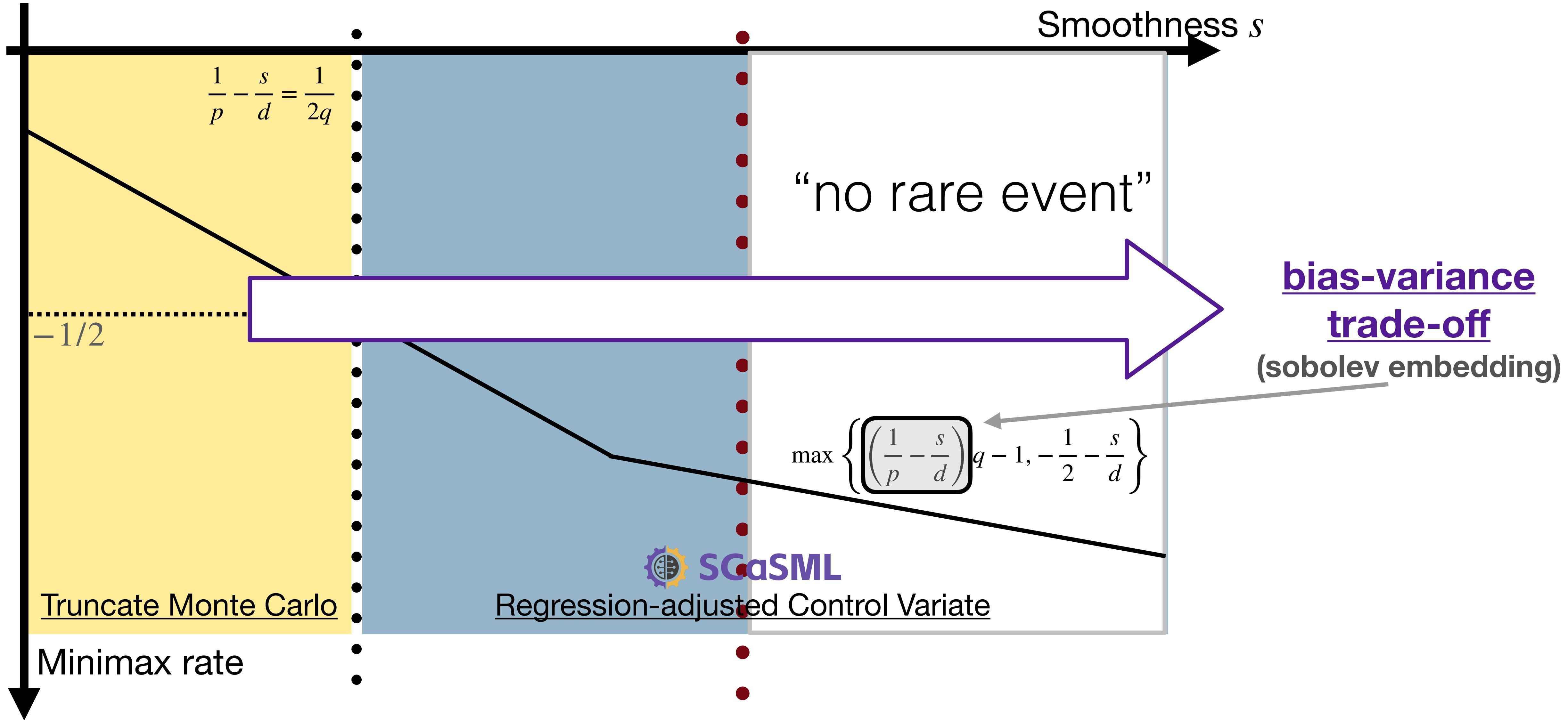
# Easiest Understanding



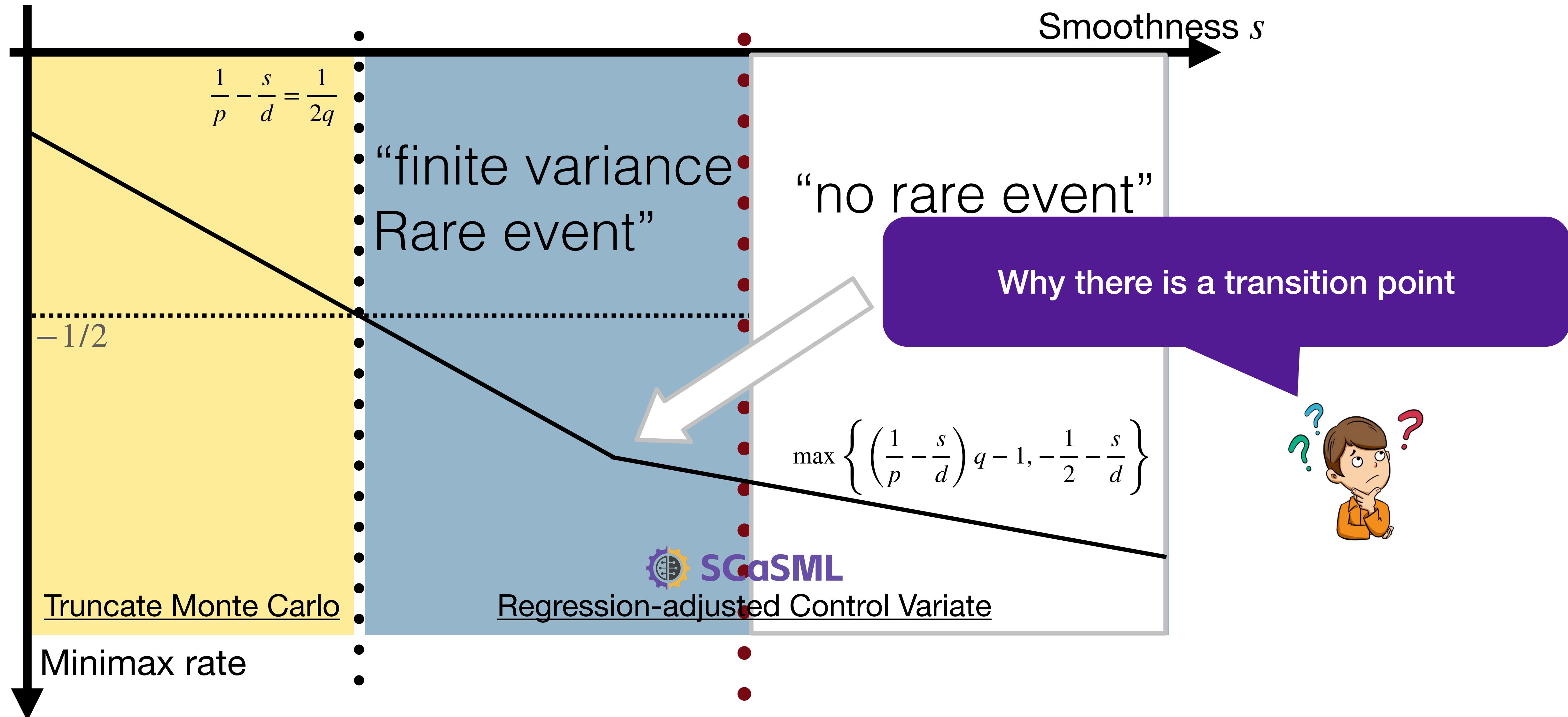
# Easiest Understanding



# Easiest Understanding



# I Why...



# Analysis of Error propagation



 **SCaML** estimate of  $\mathbb{E}_P f^q, f \in W^{s,p}$

**Step 1**

Using half of the data to estimate  $\hat{f}$

**Step 2**

$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$



How does step 2 variance depend on estimation error?

**Hardness = The variance of the debasing step**

# Analysis of Error propagation



**SCaML** estimate of  $\mathbb{E}_P f^q, f \in W^{s,p}$



**Step 1**

Using half of the data to estimate  $\hat{f}$

**Step 2**

$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$

Low order term

$$f^{q-1}(f - \hat{f}) + (f - \hat{f})^q$$

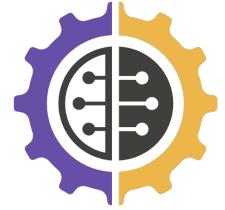
How does step2 variance  
depend on estimation error?

“influence function” (gradient)

Error propagation

# Analysis of Error propagation



 **SCaML** estimate of  $\mathbb{E}_P f^q, f \in W^{s,p}$

**Step 1**

Using half of the data to estimate  $\hat{f}$

**Step 2**

$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$

Low order term

$$f^{q-1}(f - \hat{f}) + (f - \hat{f})^q$$

“influence function” (gradient)

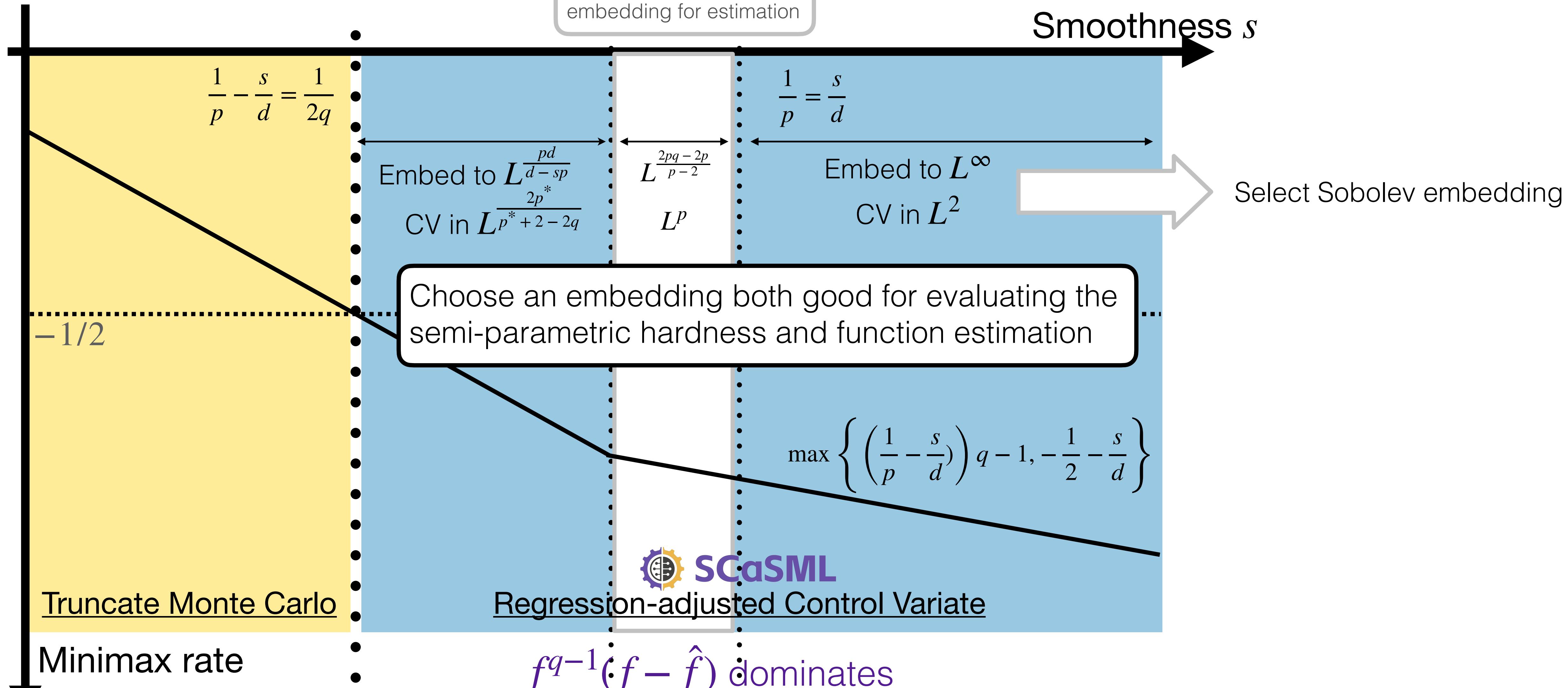
Error p

Embed  $f^{q-1}$  and  $f - \hat{f}$  into “dual” space

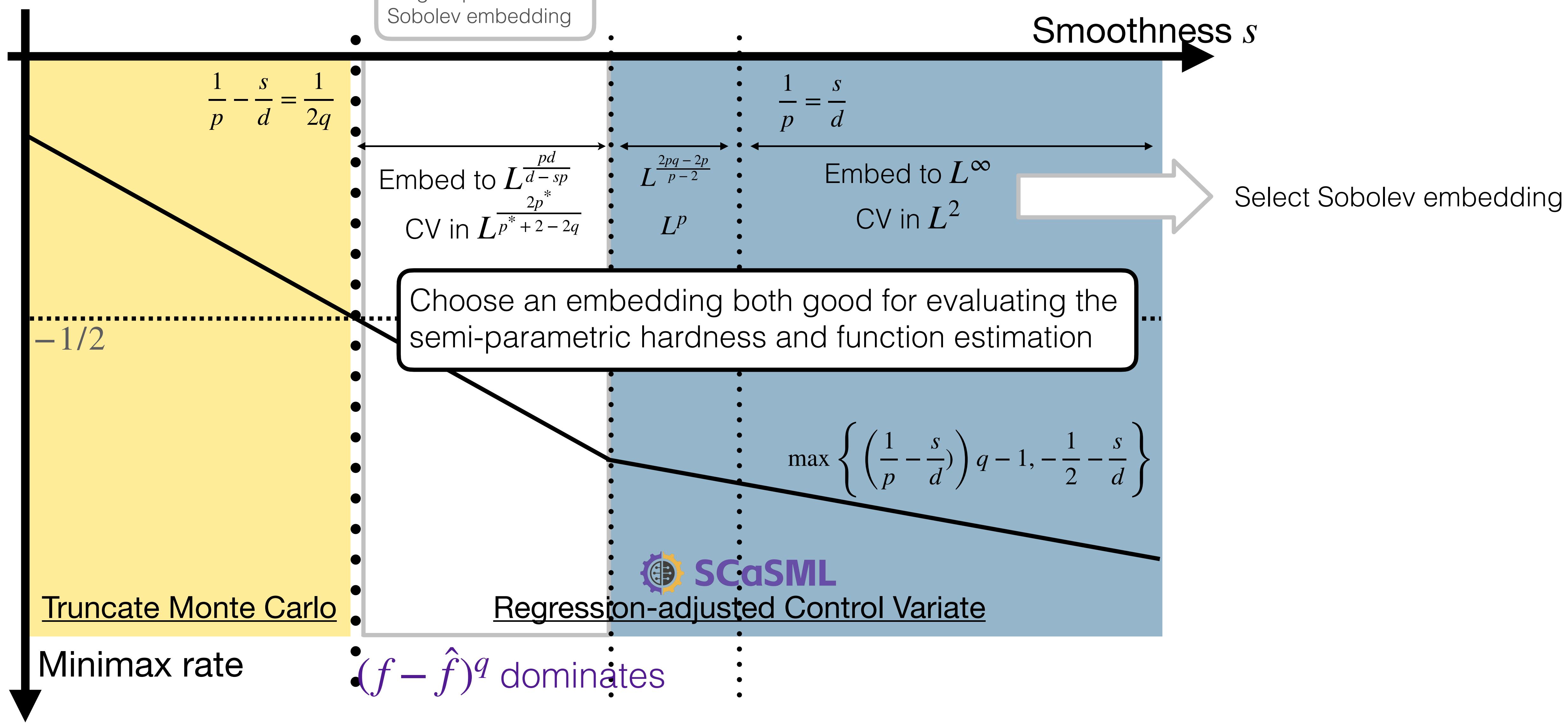
How to select the Sobolev embedding?



# Selecting the Sobolev Embedding



# Selecting the Sobolev Embedding



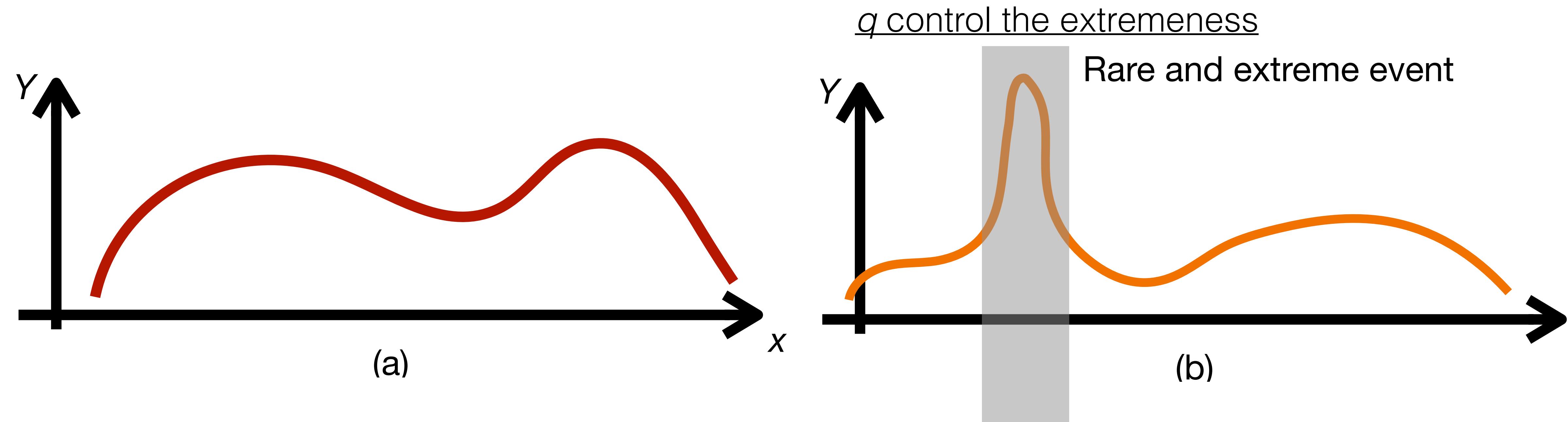
# | Take Home Message on the Theory



SCaML

- a) Statistical optimal regression is the optimal control variate
- b) It helps only if there isn't a hard to simulate (infinite variance)

Rare and extreme event



# When can Regression-Adjusted Control Variates Help? Rare Events, Sobolev Embedding and Minimax Optimality

---

**Jose Blanchet**

Department of MS&E and ICME  
Stanford University  
Stanford, CA 94305  
[jose.blanchet@stanford.edu](mailto:jose.blanchet@stanford.edu)

**Haoxuan Chen**

ICME  
Stanford University  
Stanford, CA 94305  
[haoxuanc@stanford.edu](mailto:haoxuanc@stanford.edu)

**Yiping Lu**

Courant Institute of Mathematical Sciences  
New York University  
New York, NY 10012  
[yiping.lu@nyu.edu](mailto:yiping.lu@nyu.edu)

**Lexing Ying**

Department of Mathematics and ICME  
Stanford University  
Stanford, CA 94305  
[lexing@stanford.edu](mailto:lexing@stanford.edu)

# PDE Solver

# The PDE Example

Let's consider  $\Delta u = f$



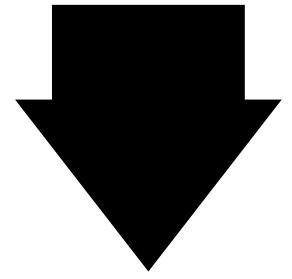
$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

$$\theta = u, \quad \underbrace{X_i}_{=} = (x_i, f(x_i))$$

$$\Phi(\theta) = u(x), \text{ or } \int (u(x)) dx$$



FEM/PINN/DGM/Tensor/Sparse Grid/...:

$$\hat{\theta} = \hat{u}$$

# The PDE Example

Let's consider  $\Delta u = f$



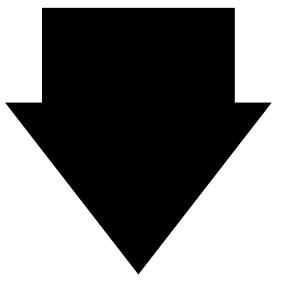
$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

$$\theta = u, \quad \underbrace{X_i}_{=} (x_i, f(x_i))$$

$$\Phi(\theta) = u(x), \text{ or } \int (u(x))dx$$



What is  $\Phi(\theta) - \Phi(\hat{\theta}) = u(x) - \hat{u}(x)$  ?

FEM/PINN/DGM/Tensor/Sparse Grid/...:

$$\hat{\theta} = \hat{u} \quad \longrightarrow \quad \Phi(\hat{\theta}) = \hat{u}(x)$$

# The PDE Example

Let's consider  $\Delta u = f$



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

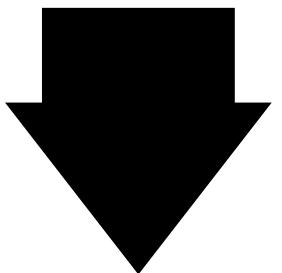
Scientific Machine Learning

Downstream application

$$\Delta u = f$$

$$\theta = u, \quad \underbrace{X_i}_{(x_i, f(x_i))}$$

$$\Phi(\theta) = u(x), \text{ or } \int (u(x))dx$$

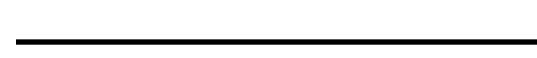


What is  $\Phi(\theta) - \Phi(\hat{\theta}) = u(x) - \hat{u}(x)$  ?

$$\Delta \hat{u} = \hat{f}$$

FEM/PINN/DGM/Tensor/Sparse Grid/...:

$$\hat{\theta} = \hat{u}$$



$$\Phi(\hat{\theta}) = \hat{u}(x)$$

||

$$\Delta(u - \hat{u}) = f - \hat{f}$$



$$(u - \hat{u})(x) = \mathbb{E} \int (f - \hat{f})(X_t) dt$$

# Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation



Can you do simulation  
for nonlinear equation?



**Δ is linear!**

# Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation

$$\frac{\partial \hat{U}}{\partial t}(x, t) + \boxed{\Delta \hat{U}(x, t)} + f(\hat{U}(x, t)) = g(x, t)$$

*g(x, t) is the error made by NN*

# Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation

$$\frac{\partial \hat{U}}{\partial t}(x, t) + \boxed{\Delta \hat{U}(x, t)} + f(\hat{U}(x, t)) = g(x, t)$$

*g(x, t) is the error made by NN*

Subtract two equations

$$\frac{\partial(U - \hat{U})}{\partial t}(x, t) + \boxed{\Delta(U - \hat{U})(x, t)} + \underbrace{f(t, \hat{U}(x, t) + U(x, t) - \hat{U}(x, t)) - f(t, \hat{U}(x, t))}_{G(t, (U - \hat{U})(x, t))} = g(x, t).$$

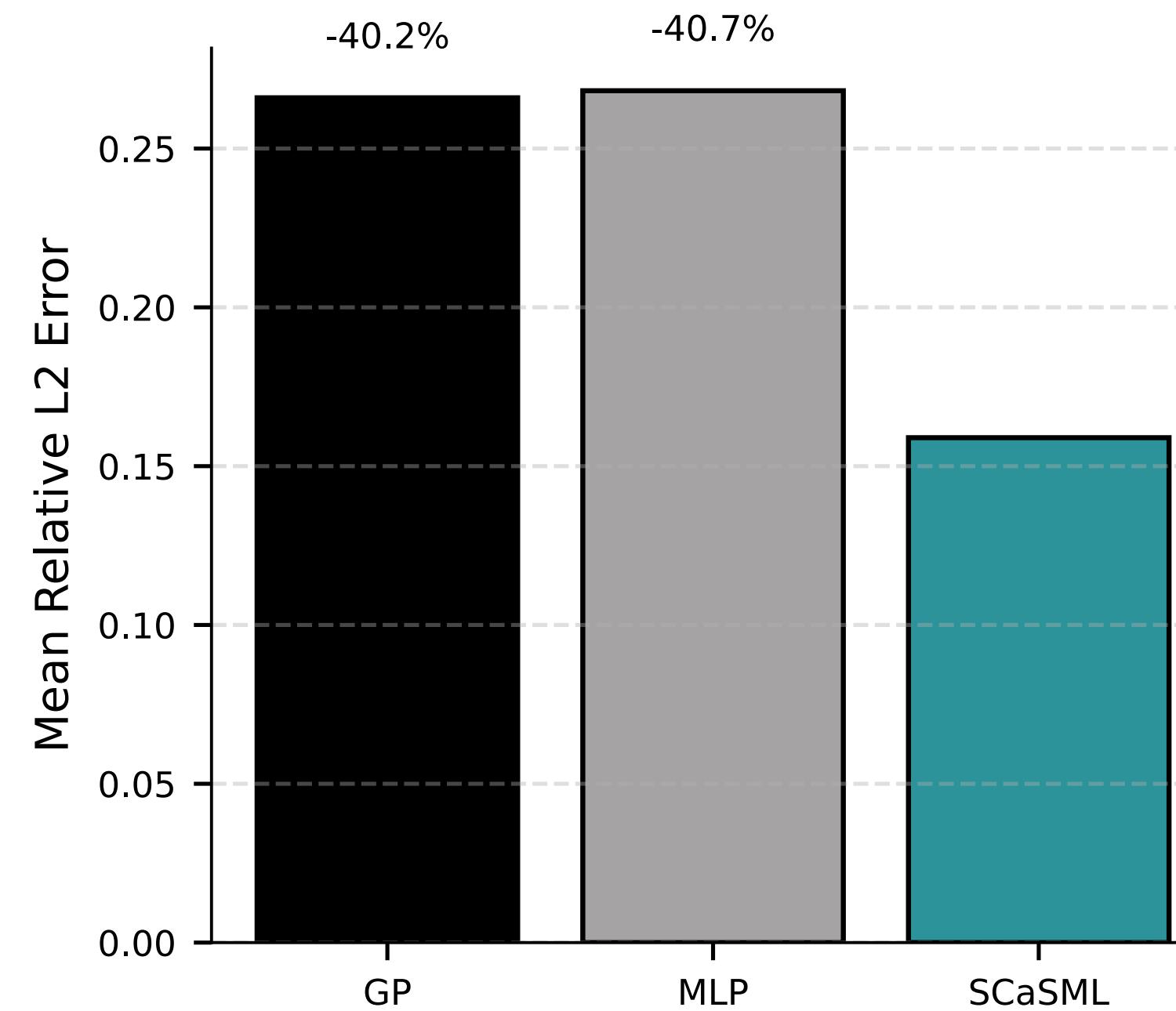
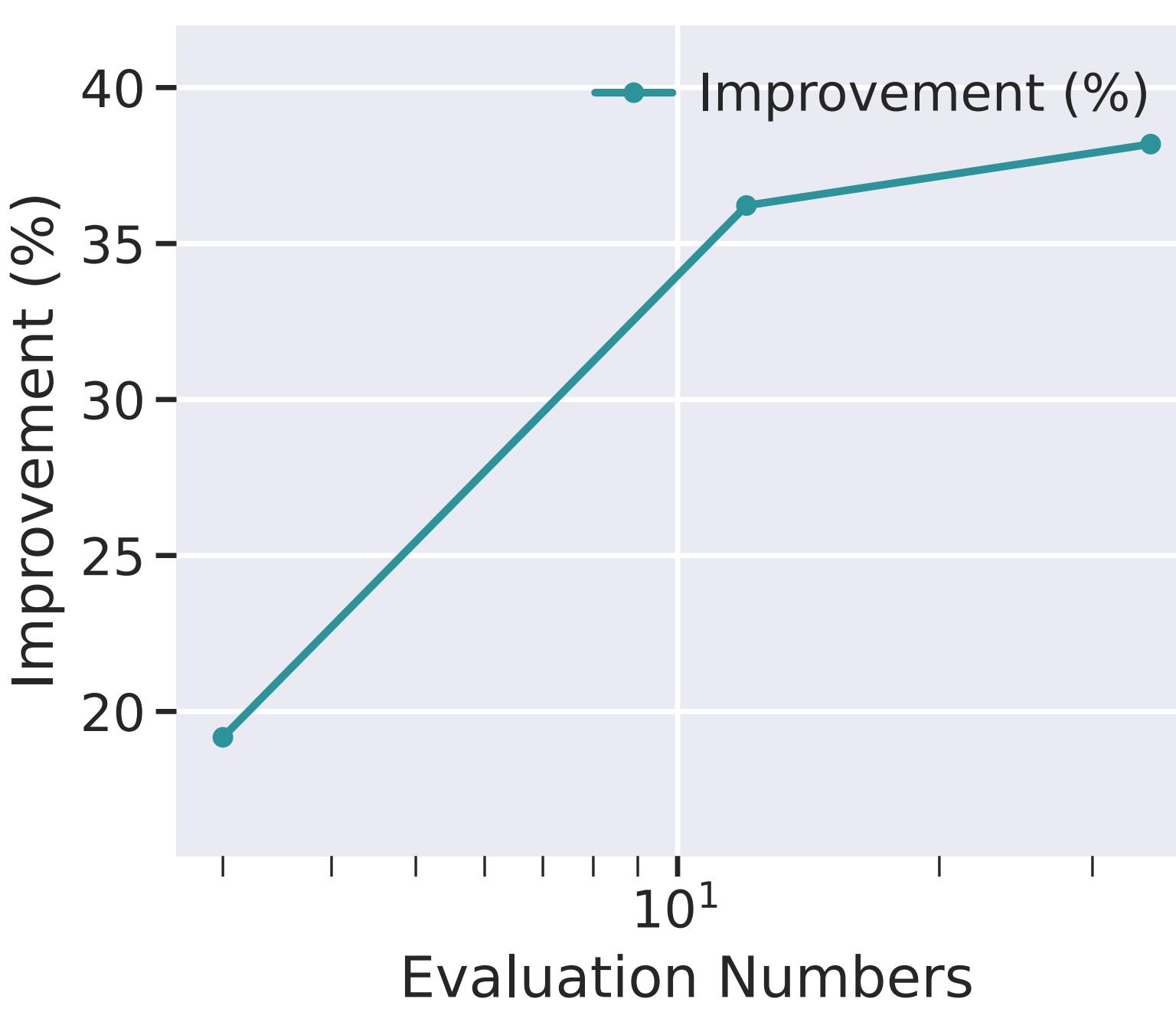
# Numerical Results

		Time (s)			Relative $L^2$ Error			$L^\infty$ Error			$L^1$ Error		
		SR	MLP	SCaSML	SR	MLP	SCaSML	SR	MLP	SCaSML	SR	MLP	SCaSML
<b>LCD</b>	10d	2.64	11.24	23.75	5.24E-02	2.27E-01	<b>2.73E-02</b>	2.50E-01	9.06E-01	<b>1.61E-01</b>	3.43E-02	1.67E-01	<b>1.78E-02</b>
	20d	1.14	7.35	17.59	9.09E-02	2.35E-01	<b>4.73E-02</b>	4.52E-01	1.35E+00	<b>3.28E-01</b>	9.47E-02	2.37E-01	<b>4.52E-02</b>
	30d	1.39	7.52	25.33	2.30E-01	2.38E-01	<b>1.84E-01</b>	4.73E+00	1.59E+00	<b>1.49E+00</b>	<b>1.75E-01</b>	2.84E-01	1.91E-01
	60d	1.13	7.76	35.58	3.07E-01	2.39E-01	<b>1.32E-01</b>	3.23E+00	2.05E+00	<b>1.55E+00</b>	5.24E-01	4.07E-01	<b>2.06E-01</b>
<b>VB-PINN</b>	20d	1.15	7.05	13.82	1.17E-02	8.36E-02	<b>3.97E-03</b>	3.16E-02	2.96E-01	<b>2.16E-02</b>	5.37E-03	3.39E-02	<b>1.29E-03</b>
	40d	1.18	7.49	16.48	3.99E-02	1.04E-01	<b>2.85E-02</b>	8.16E-02	3.57E-01	<b>7.16E-02</b>	1.97E-02	4.36E-02	<b>1.21E-02</b>
	60d	1.19	7.57	19.83	3.97E-02	1.17E-01	<b>2.90E-02</b>	8.10E-02	3.93E-01	<b>7.10E-02</b>	1.95E-02	4.82E-02	<b>1.24E-02</b>
	80d	1.32	7.48	21.99	6.78E-02	1.19E-01	<b>5.68E-02</b>	1.89E-01	3.35E-01	<b>1.79E-01</b>	3.24E-02	4.73E-02	<b>2.49E-02</b>
<b>VB-GP</b>	20d	1.97	10.66	65.46	1.47E-01	8.32E-02	<b>5.52E-02</b>	3.54E-01	<b>2.22E-01</b>	2.54E-01	7.01E-02	3.50E-02	<b>1.91E-02</b>
	40d	1.68	10.14	49.38	1.81E-01	1.05E-01	<b>7.95E-02</b>	4.01E-01	3.47E-01	<b>3.01E-01</b>	9.19E-02	4.25E-02	<b>3.43E-02</b>
	60d	1.01	7.25	35.14	2.40E-01	2.57E-01	<b>1.28E-01</b>	3.84E-01	9.50E-01	<b>7.10E-02</b>	1.27E-01	9.99E-02	<b>6.11E-02</b>
	80d	1.00	7.00	38.26	2.66E-01	3.02E-01	<b>1.52E-01</b>	3.62E-01	1.91E+00	<b>2.62E-01</b>	1.45E-01	1.09E-01	<b>7.59E-02</b>
<b>LQG</b>	100d	1.54	8.67	26.95	7.96E-02	5.63E+00	<b>5.51E-02</b>	7.78E-01	1.26E+01	<b>6.78E-01</b>	1.40E-01	1.21E+01	<b>8.68E-02</b>
	120d	1.25	8.17	27.46	9.37E-02	5.50E+00	<b>6.64E-02</b>	9.02E-01	1.27E+01	<b>8.02E-01</b>	1.73E-01	1.22E+01	<b>1.05E-01</b>
	140d	1.80	8.27	29.72	9.79E-02	5.37E+00	<b>6.78E-02</b>	1.00E+00	1.27E+01	<b>9.00E-01</b>	1.91E-01	1.23E+01	<b>1.11E-01</b>
	160d	1.74	9.07	32.08	1.11E-01	5.27E+00	<b>9.92E-02</b>	1.38E+00	1.28E+01	<b>1.28E+00</b>	2.15E-01	1.23E+01	<b>1.79E-01</b>
<b>DR</b>	100d	1.62	7.75	60.86	9.52E-03	8.99E-02	<b>8.87E-03</b>	7.51E-02	6.37E-01	<b>6.51E-02</b>	1.13E-02	9.74E-02	<b>1.11E-02</b>
	120d	1.26	7.28	65.66	1.11E-02	9.13E-02	<b>9.90E-03</b>	7.10E-02	5.74E-01	<b>6.10E-02</b>	1.40E-02	9.97E-02	<b>1.23E-02</b>
	140d	2.38	7.82	76.90	3.17E-02	8.97E-02	<b>2.94E-02</b>	1.79E-01	8.56E-01	<b>1.69E-01</b>	3.96E-02	9.77E-02	<b>3.67E-02</b>
	160d	1.75	7.42	82.40	3.46E-02	9.00E-02	<b>3.23E-02</b>	2.08E-01	8.02E-01	<b>1.98E-01</b>	4.32E-02	9.75E-02	<b>4.02E-02</b>

# Inference-Time Scaling

$$\frac{\partial}{\partial t} u + \left[ \sigma^2 u - \frac{1}{d} - \frac{\bar{\sigma}^2}{2} \right] (\nabla \cdot u) + \frac{\bar{\sigma}^2}{2} \Delta u = 0$$

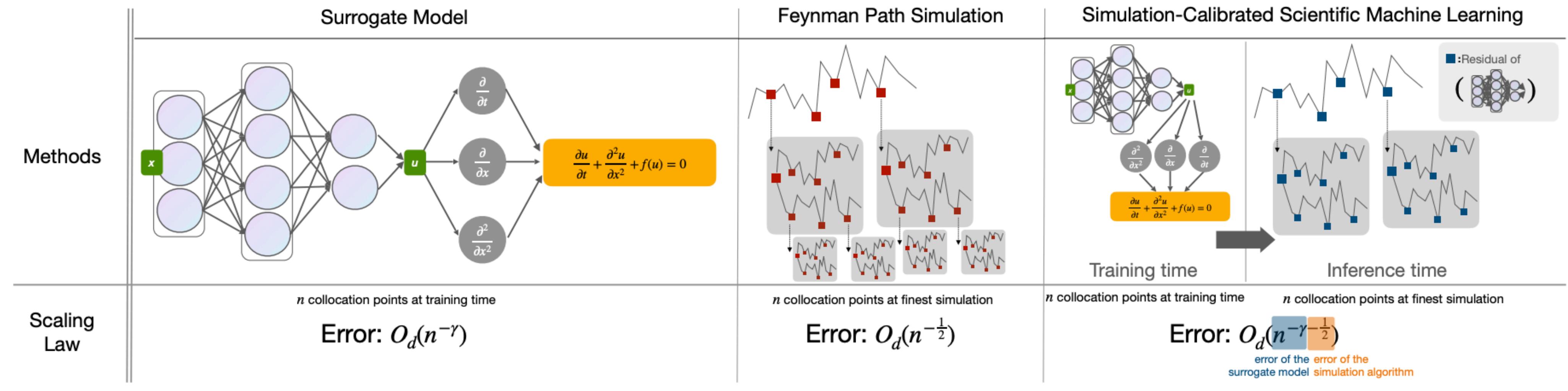
have closed-form solution  $g(x) = \frac{\exp(T + \sum_i x_i)}{1 + \exp(T + \sum_i x_i)}$



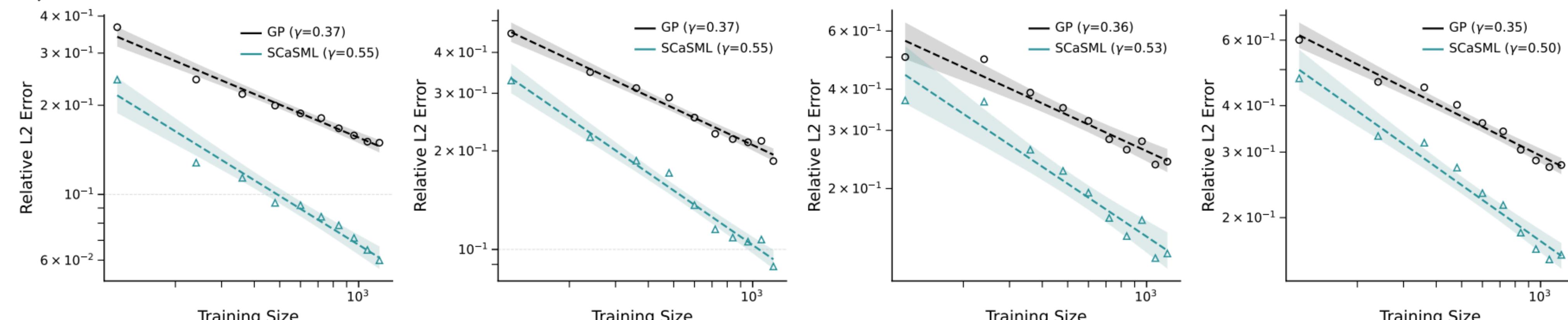
Method	Convergence Rate
PINN	$O(n^{-s/d})$
MLP	$O(n^{-1/4})$
SCaSML	$O(n^{-1/4-s/d})$

# Better Scaling Law

a)



b)

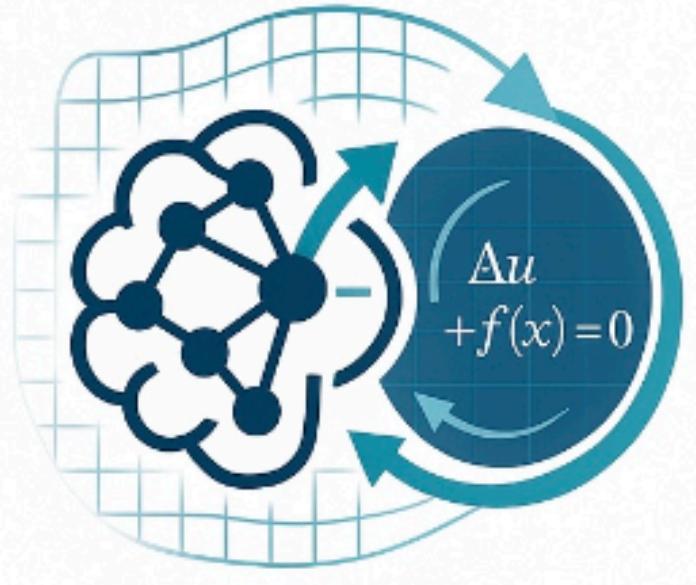


(a)  $d = 20$

(b)  $d = 40$

(c)  $d = 60$

(d)  $d = 80$



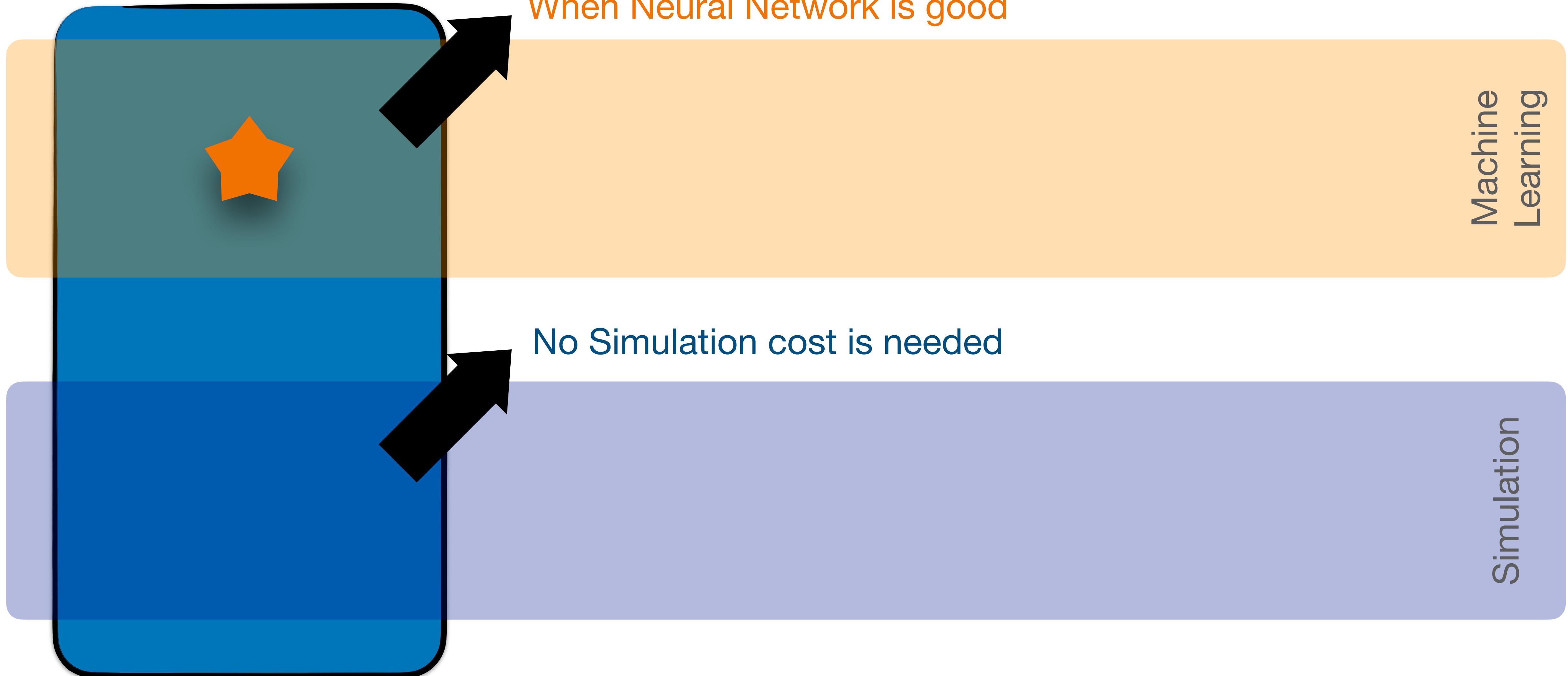
# Physics-Informed Inference Time Scaling via Simulation-Calibrated Scientific Machine Learning

Zexi Fan<sup>1</sup>, Yan Sun<sup>2</sup>, Shihao Yang<sup>3</sup>, Yiping Lu<sup>\*4</sup>

<sup>1</sup> Peking University <sup>2</sup> Visa Inc. <sup>3</sup> Georgia Institute of Technology <sup>4</sup> Northwestern University  
[fanzexi\\_francis@stu.pku.edu.cn](mailto:fanzexi_francis@stu.pku.edu.cn), [yansun414@gmail.com](mailto:yansun414@gmail.com),  
[shihao.yang@isye.gatech.edu](mailto:shihao.yang@isye.gatech.edu), [yiping.lu@northwestern.edu](mailto:yiping.lu@northwestern.edu)

[https://2prime.github.io/files/scasml\\_techreport.pdf](https://2prime.github.io/files/scasml_techreport.pdf)

# Our Aim Today : A Marriage



# Our Aim Today : A Marriage

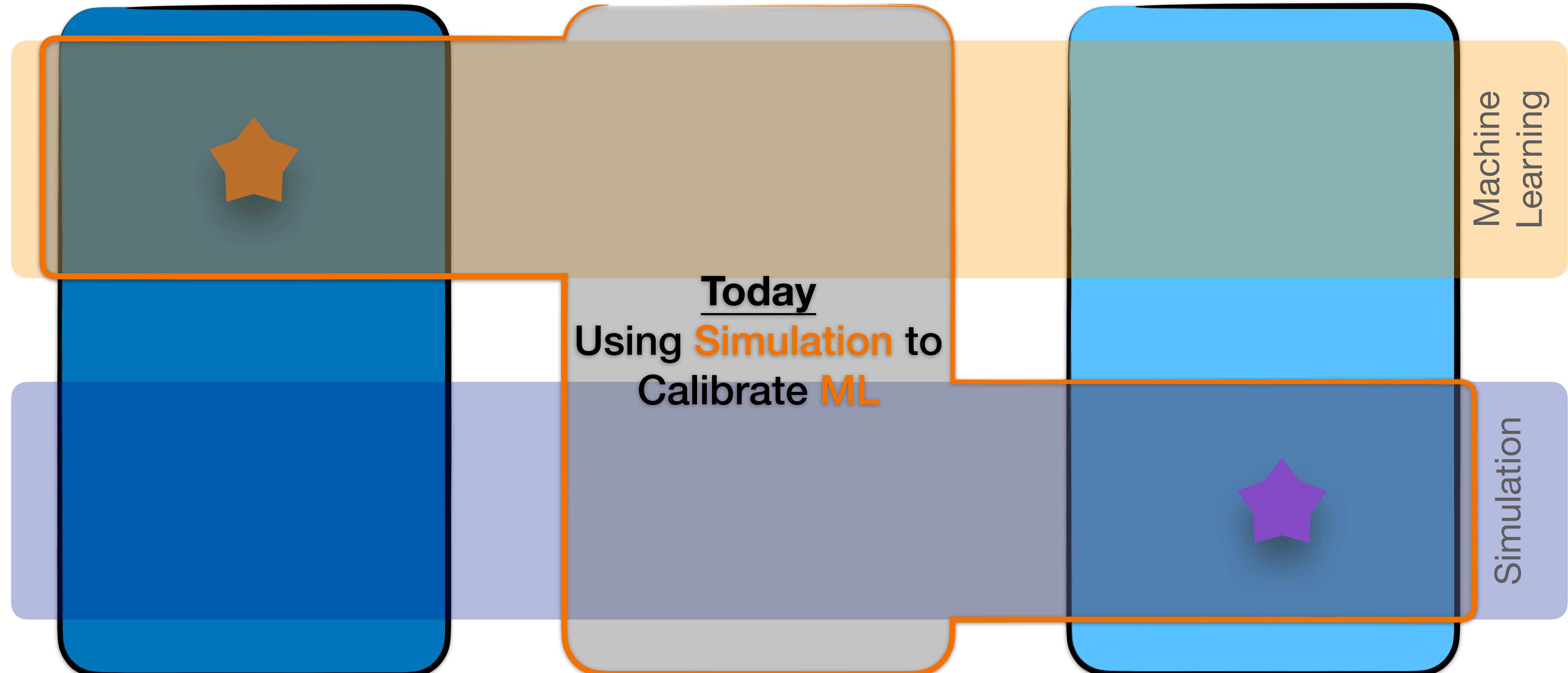
When Neural Network is bad

Provide pure Simulation solution

Machine  
Learning

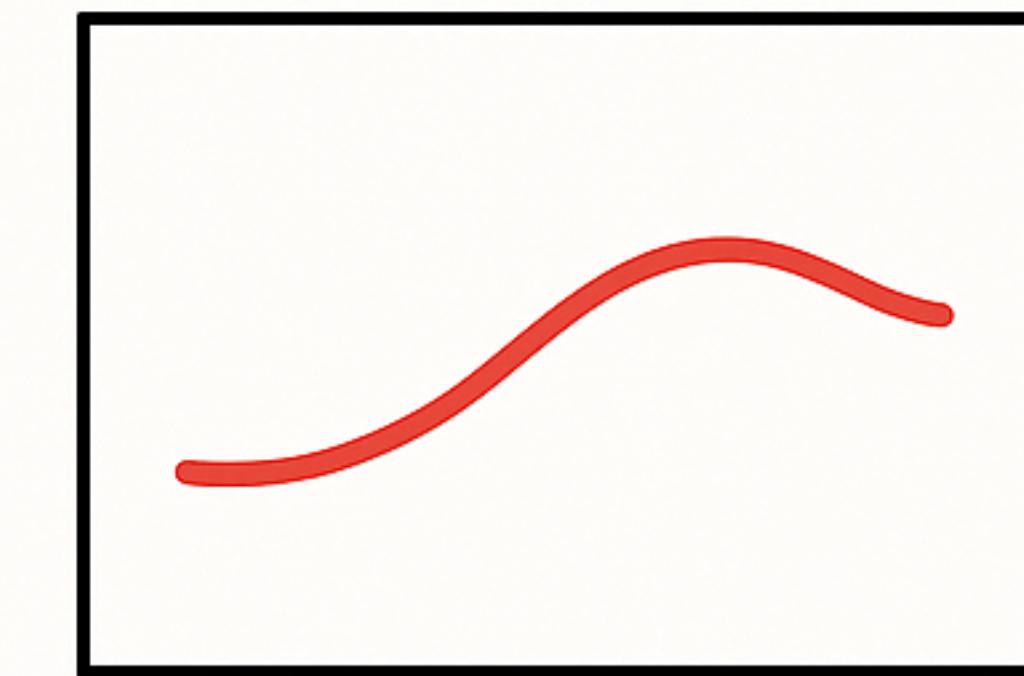
Simulation

# Our AIM Today: A Marriage



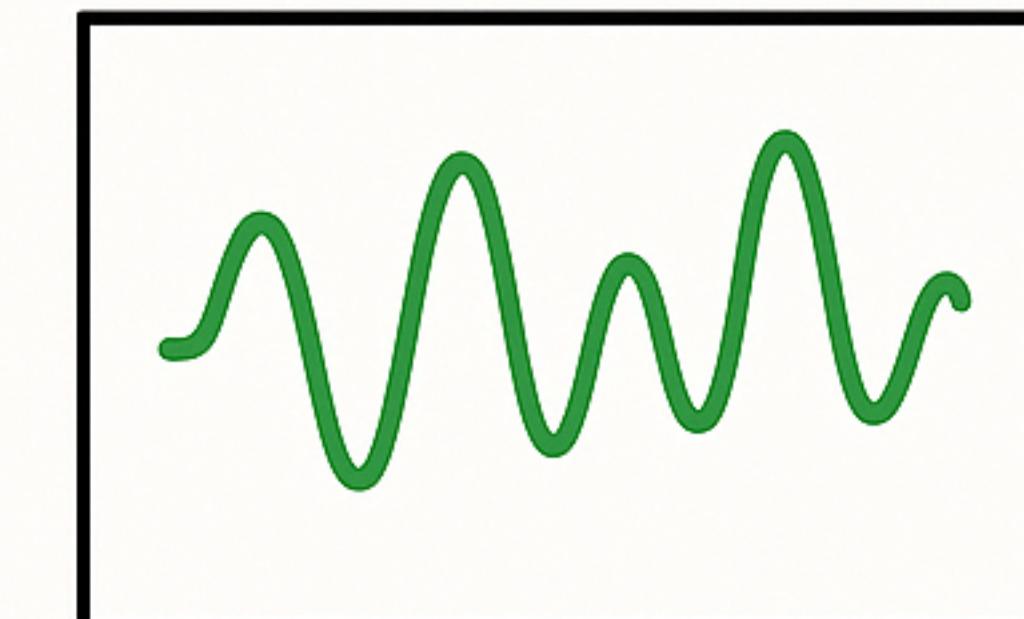
# A multiscale view

Capture via surrogate model



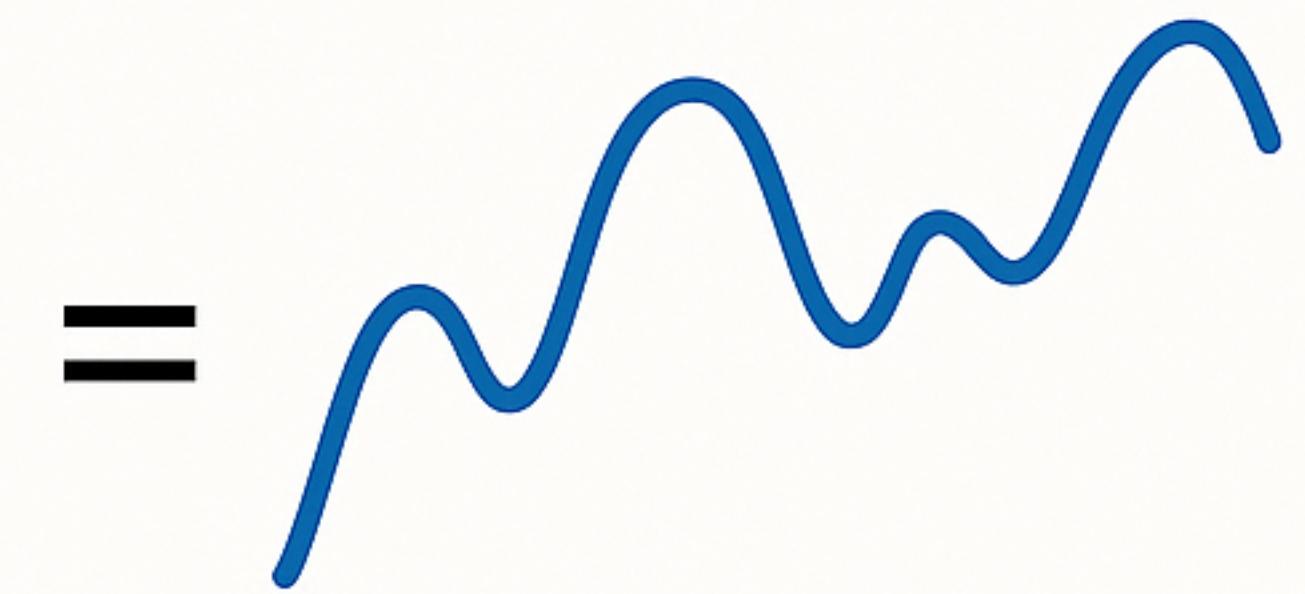
Coarse Scale

+



Fine Scale

True  
Function



# A Numerical Linear Algebra Example

$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

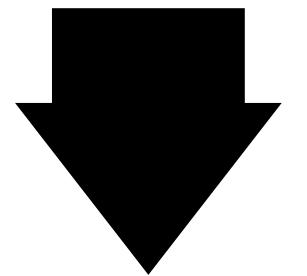
Scientific Machine Learning

Downstream application

## Example

$$\theta = A, \quad \underbrace{X_i}_{(Randomized) \text{ Subspace methods}} = (x_i, Ax_i)$$

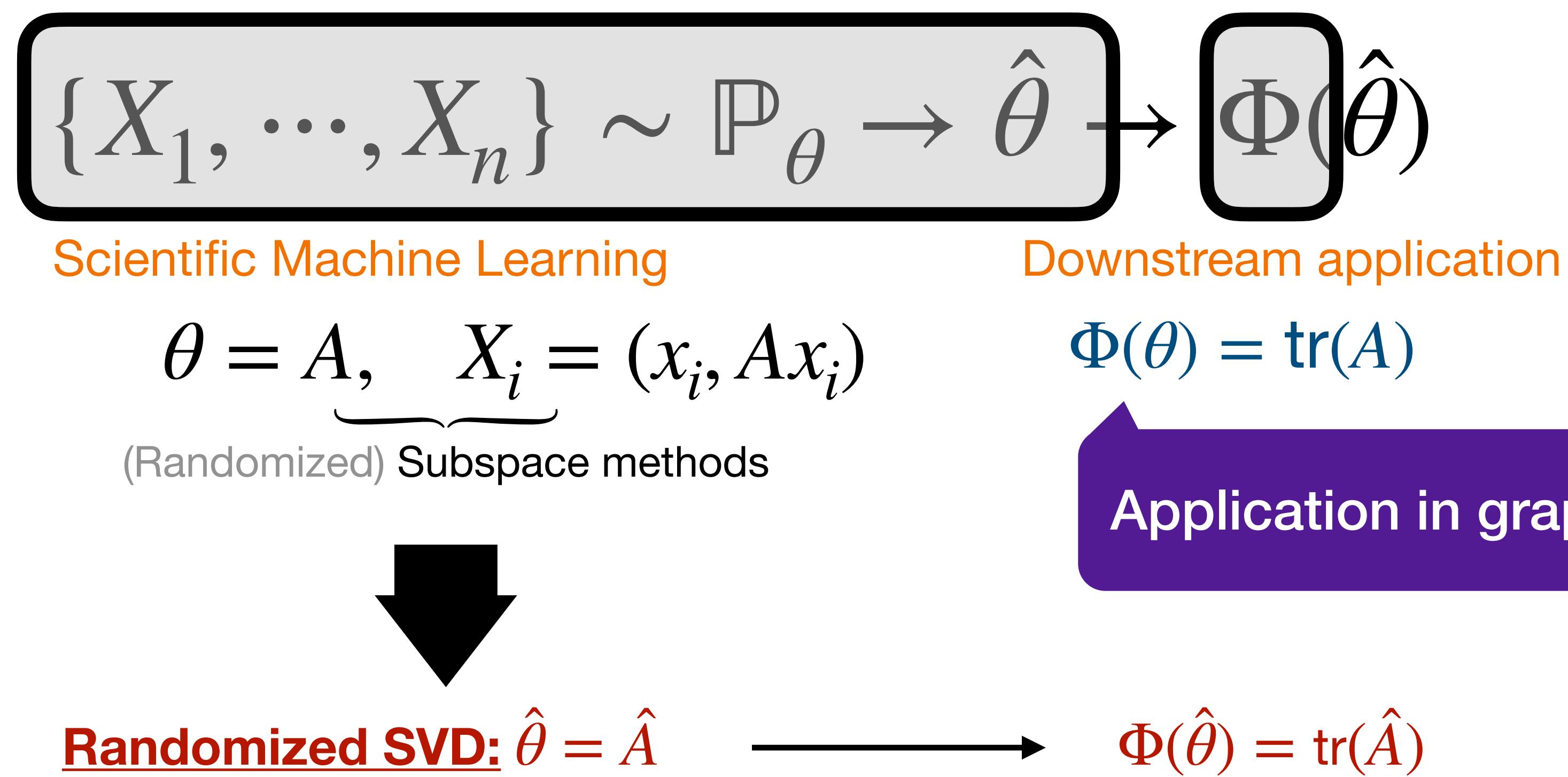
$$\Phi(\theta) = \text{tr}(A)$$



Randomized SVD:  $\hat{\theta} = \hat{A}$

# A Numerical Linear Algebra Example

## Example



# A Numerical Linear Algebra Example

**Example**

$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

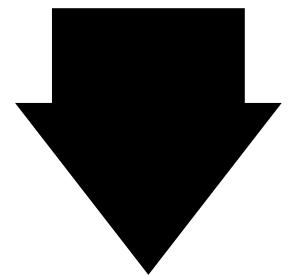
Scientific Machine Learning

Downstream application

$$\theta = A, \quad \underbrace{X_i}_{(Randomized) \text{ Subspace methods}} = (x_i, Ax_i)$$

$$\Phi(\theta) = \text{tr}(A)$$

||



Randomized SVD:  $\hat{\theta} = \hat{A}$   $\longrightarrow \Phi(\hat{\theta}) = \text{tr}(\hat{A})$

+

$$\Phi(\theta) - \Phi(\hat{\theta}) = \text{tr}(A - \hat{A})$$

Estimate  $\text{tr}(A - \hat{A})$  via Hutchinson's estimator

# More Examples...



Scientific Machine Learning

Downstream application

**Example 1**

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

**Example 2**

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \theta(x)$$

**Example 3**

$$\theta = A, \quad X_i = (x_i, Ax_i)$$

Estimation  $\hat{A}$  via Randomized SVD

$$\Phi(\theta) = \text{tr}(A)$$

Estimate  $\text{tr}(A - \hat{A})$  via Hutchinson's estimator

**Example 4**

Siegel J W, Xu J. Sharp bounds on the approximation rates, metric entropy, and n-widths of shallow neural networks.

Foundations of Computational Mathematics, 2024, 24(2): 481-537.

# More Examples... (Uncertainty Quantification)

$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

## Example 5

$$\theta = \theta, \quad X_i \sim P_\theta$$

Quantile regression

Confidence Interval of  
Point Estimation

Conformal Prediction

Romano Y, Patterson E, Candes E. Conformalized quantile regression. Neurips 2019.

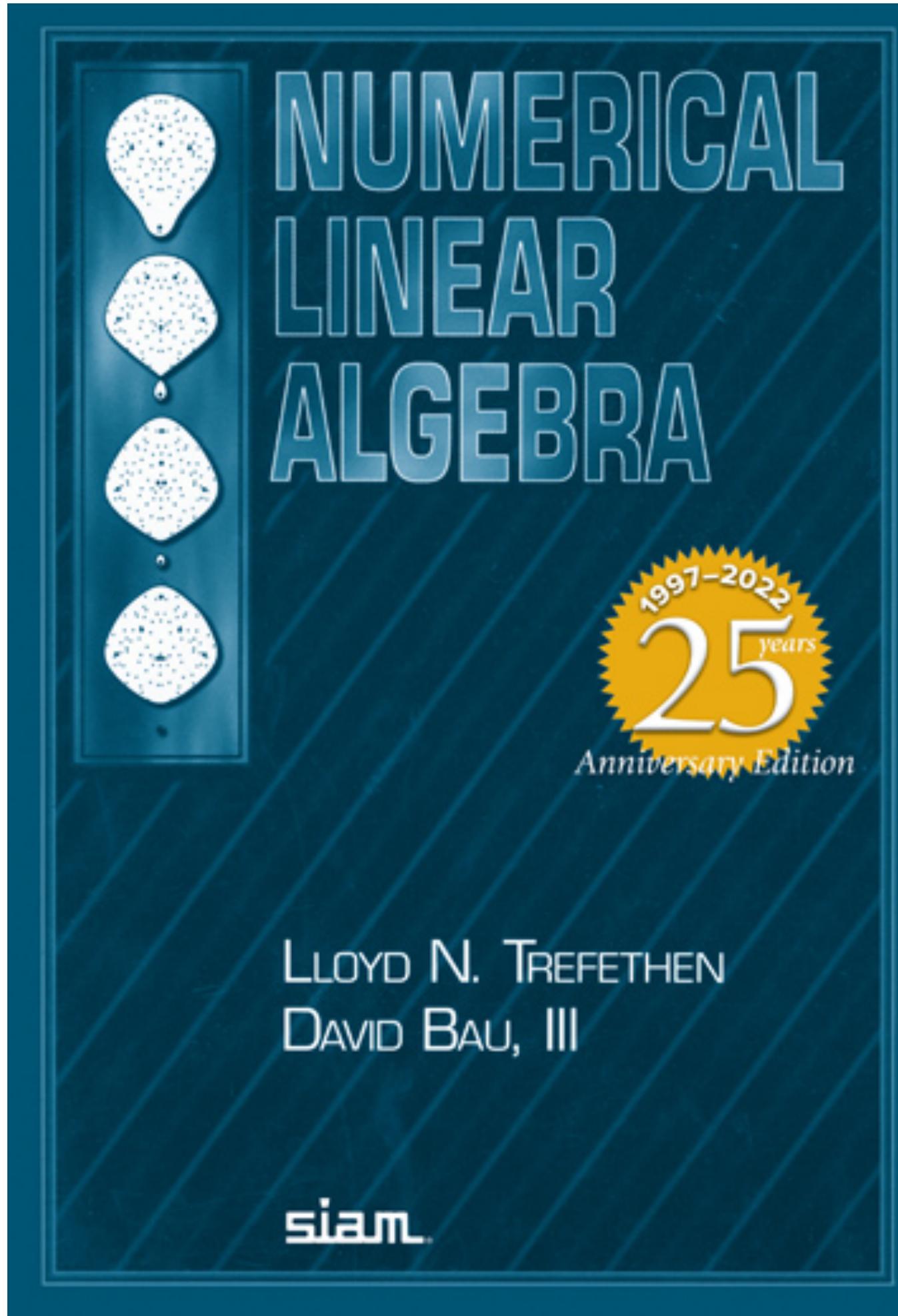
Influence Function

Bootstrap

Liu K, Blanchet J, Ying L, et al. Orthogonal bootstrap: efficient simulation of input uncertainty. ICML 2024.

# Tale 2: Pre-condition with a surprising connection with debiasing

# Tale 2: Preconditioning



*"In ending this book with the subject of preconditioners, we find ourselves at the philosophical center of the scientific computing of the future."*

— L. N. Trefethen and D. Bau III, Numerical Linear Algebra [TB22]



Nothing will be more central to computational science in the next century than **the art of transforming a problem that appears intractable into another whose solution can be approximated rapidly**.

# What is precondition

- Solving  $Ax = b$  is equivalent to solving  $B^{-1}Ax = B^{-1}b$   
hardness depend on  $\kappa(A)$       hardness depend on  $\kappa(B^{-1}A)$   


Become easier when  $B \approx A$

# A New Way to Implement Precondition

- Debiasing is a way of solving  $Ax = b$ 
  - Using an approximate solver  $Bx_1 = b$

# A New Way to Implement Precondition

- Debiasing is a way of solving  $Ax = b$ 
  - Using an approximate solver  $Bx_1 = b$
  - $x - x_1$  satisfies the equation  $A(x - x_1) = b - Ax_1$
  - Using the approximate solver to approximate  $x - x_1$  via  $Bx_2 = b - Ax_1$

Easy to solve for  $b - Ax_1$  is small

# A New Way to Implement Precondition

- Debiasing is a way of solving  $Ax = b$ 
  - Using an approximate solver  $Bx_1 = b$

Iterative Refinement Algorithm

- $x - \sum_{i=1}^t x_i$  satisfies the equation  $A(x - \sum_{i=1}^t x_i) = b - A \sum_{i=1}^t x_i$
- Using the approximate solver to approximate  $x - \sum_{i=1}^t x_i$  via  $Bx_{i+1} = b - A \sum_{i=1}^t x_i$

# A New Way to Implement Precondition

- Debiasing is a way of solving  $Ax = b$

- Using an approximate solver  $Bx_1 = b$

Iterative Refinement Algorithm

.  $x - \sum_{i=1}^t x_i$  satisfies the equation  $A(x - \sum_{i=1}^t x_i) = b - A \sum_{i=1}^t x_i$

. Using the approximate solver to approximate  $x - \sum_{i=1}^t x_i$  via  $Bx_{i+1} = b - A \sum_{i=1}^t x_i$

$$x_{i+1} = (I - B^{-1}A)x_i + B^{-1}b$$

Preconditioned Jacobi Iteration

# This Talk: A New Way to Implement Precondition Via Debiasing

- **Step 1:** Aim to solve (potentially nonlinear) equation  $A(u) = b$

use Machine Learning

- **Step 2:** Build an approximate solver  $A(\hat{u}) \approx b$

Unreliable approximate  
solver as preconditioner

- Via machine learning/sketching/finite element....

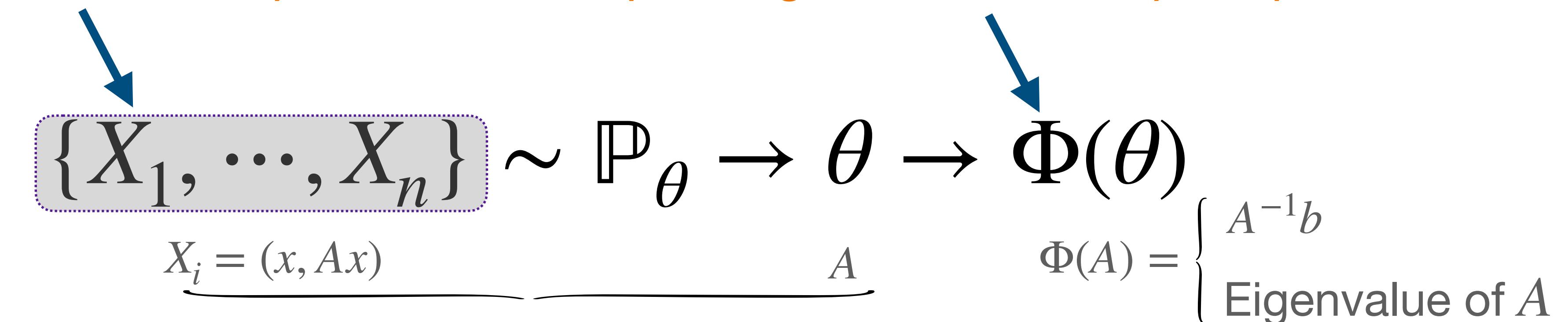
- **Step 3:** Solve  $u - \hat{u}$



AIM: Debiasing a Learned Solution = Using Learned Solution as preconditioner!

# Randomized NLA as Machine Learning

**AIM:** using matrix-vector multiplication to compute eigenvalue/least square problem



“Randomized Numerical Linear Algebra”/Sketching



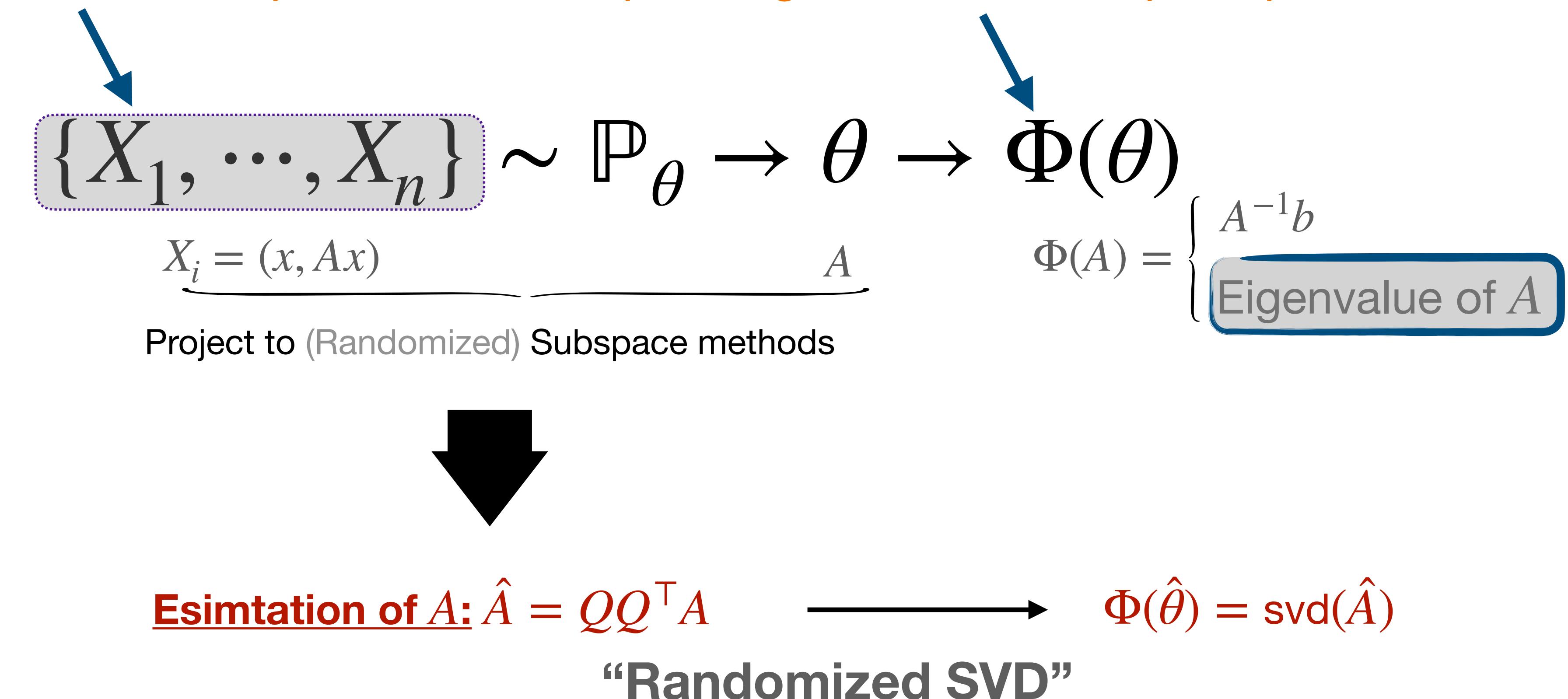
“Sketch-and-Solve”

*It seems easier to train a bi-directional LSTM with attention than to compute the SVD of a large matrix. –Chris Re*

NeurIPS 2017 Test-of-Time Award, Rahimi and Recht  
(Rahimi and Recht, 2017).

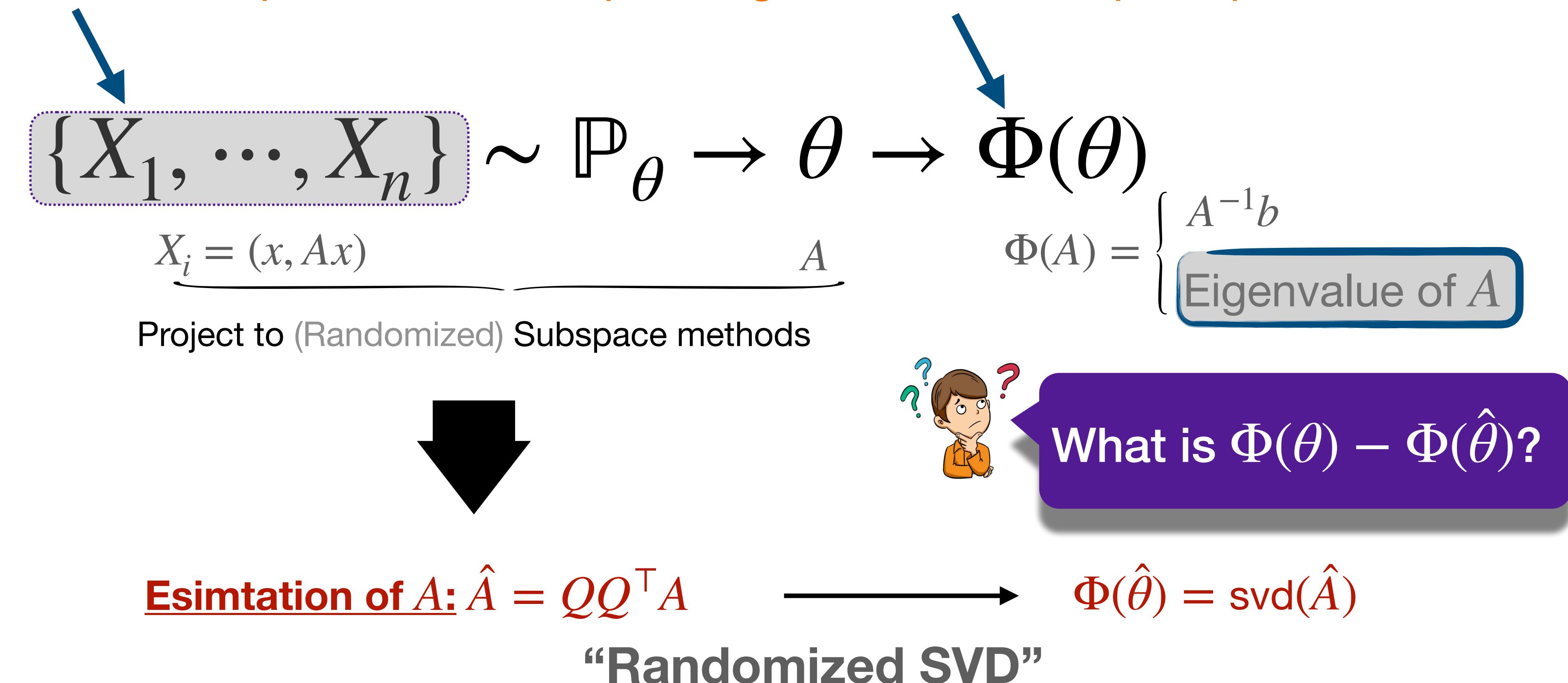
# Randomized NLA as Machine Learning

**AIM:** using matrix-vector multiplication to compute eigenvalue/least square problem



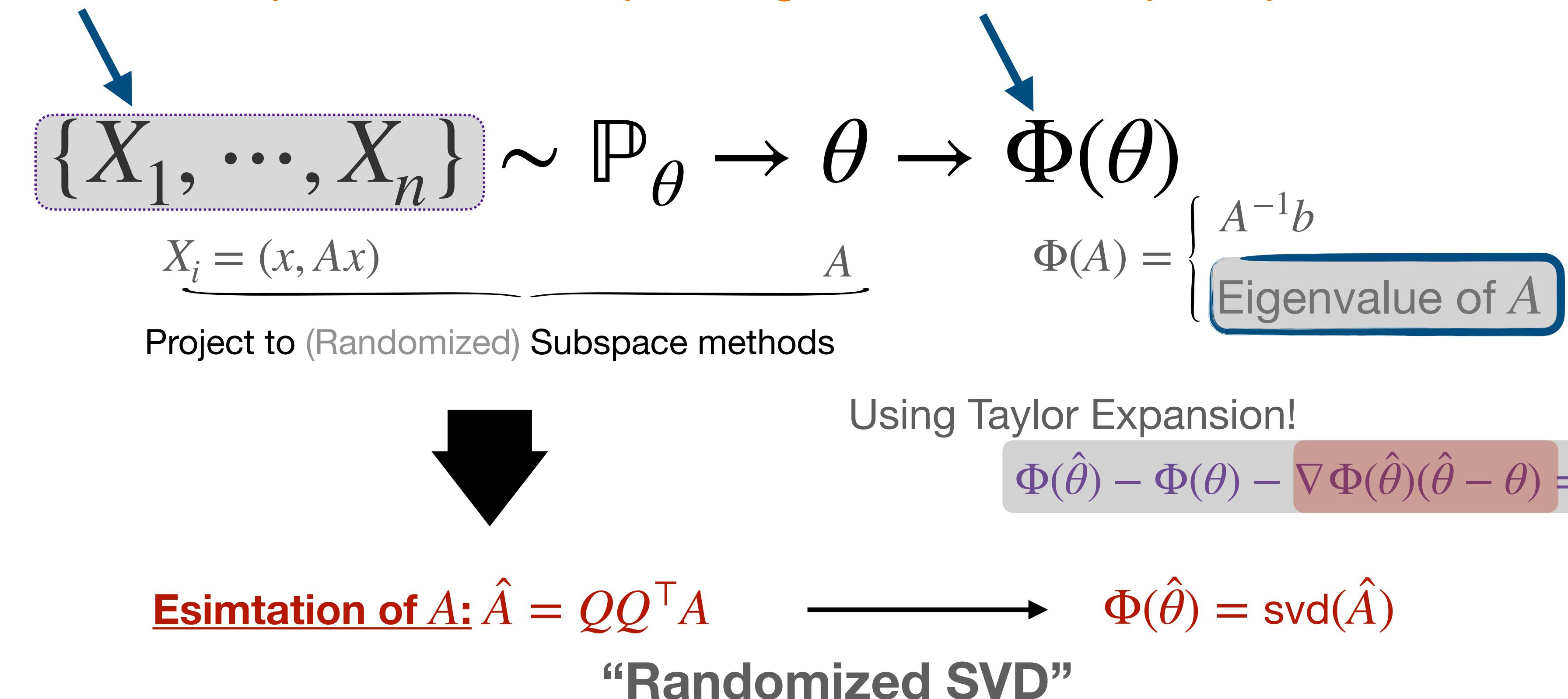
# Randomized NLA as Machine Learning

**AIM:** using matrix-vector multiplication to compute eigenvalue/least square problem



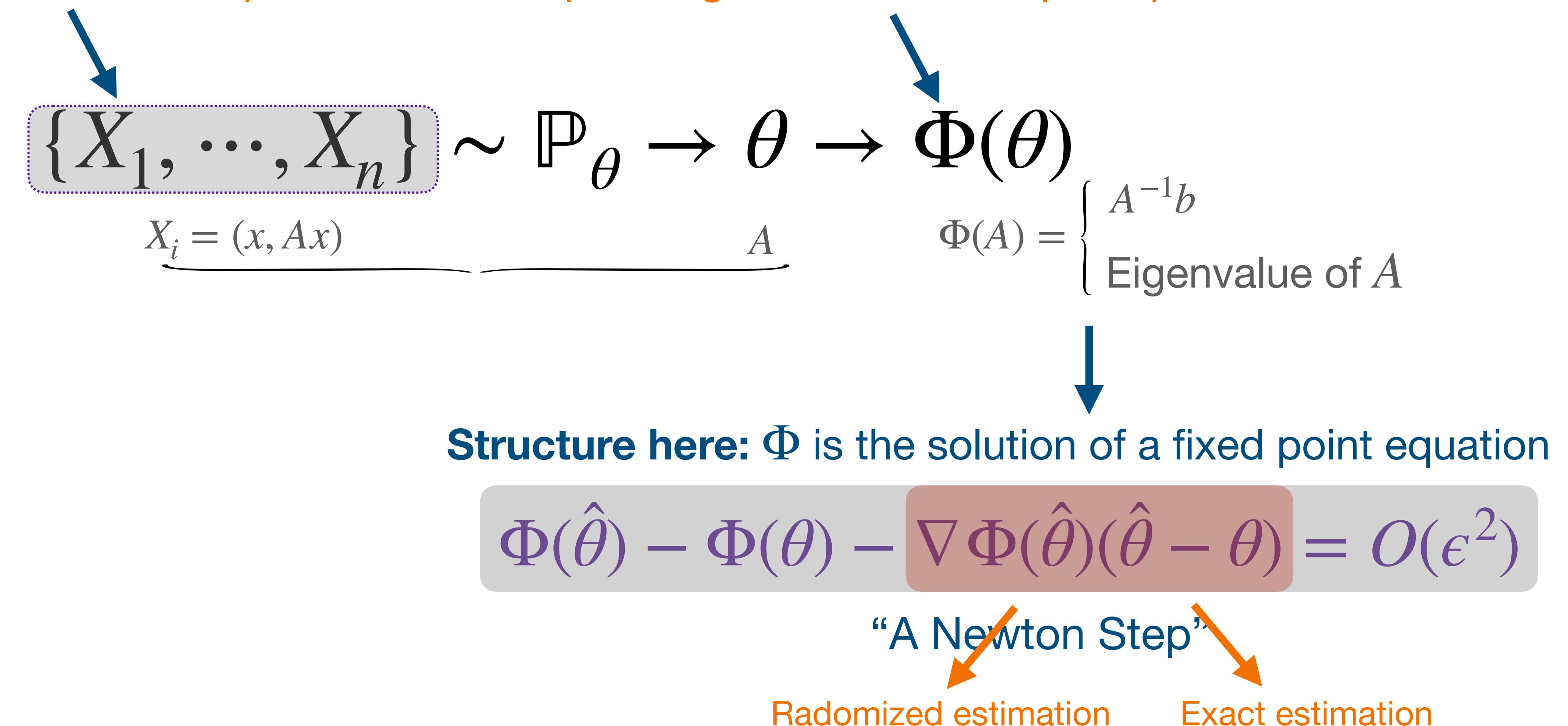
# Randomized NLA as Machine Learning

**AIM:** using matrix-vector multiplication to compute eigenvalue/least square problem



# Randomized NLA as Machine Learning

**AIM:** using matrix-vector multiplication to compute eigenvalue/least square problem



(In)exact Sub-sample Newton Method/Sketch-and-Precondition

# Relationship with Inverse Power Methods

(Approximate) Inverse Power Method	Our Method
$X_{n+1} = (\lambda I - A)^\dagger X_n$	$X_{n+1} = \boxed{(\lambda I - \hat{A})^\dagger}_{\nabla \Phi(\hat{\theta})} \boxed{(A - \hat{A})X_n}_{(\theta - \hat{\theta})}$

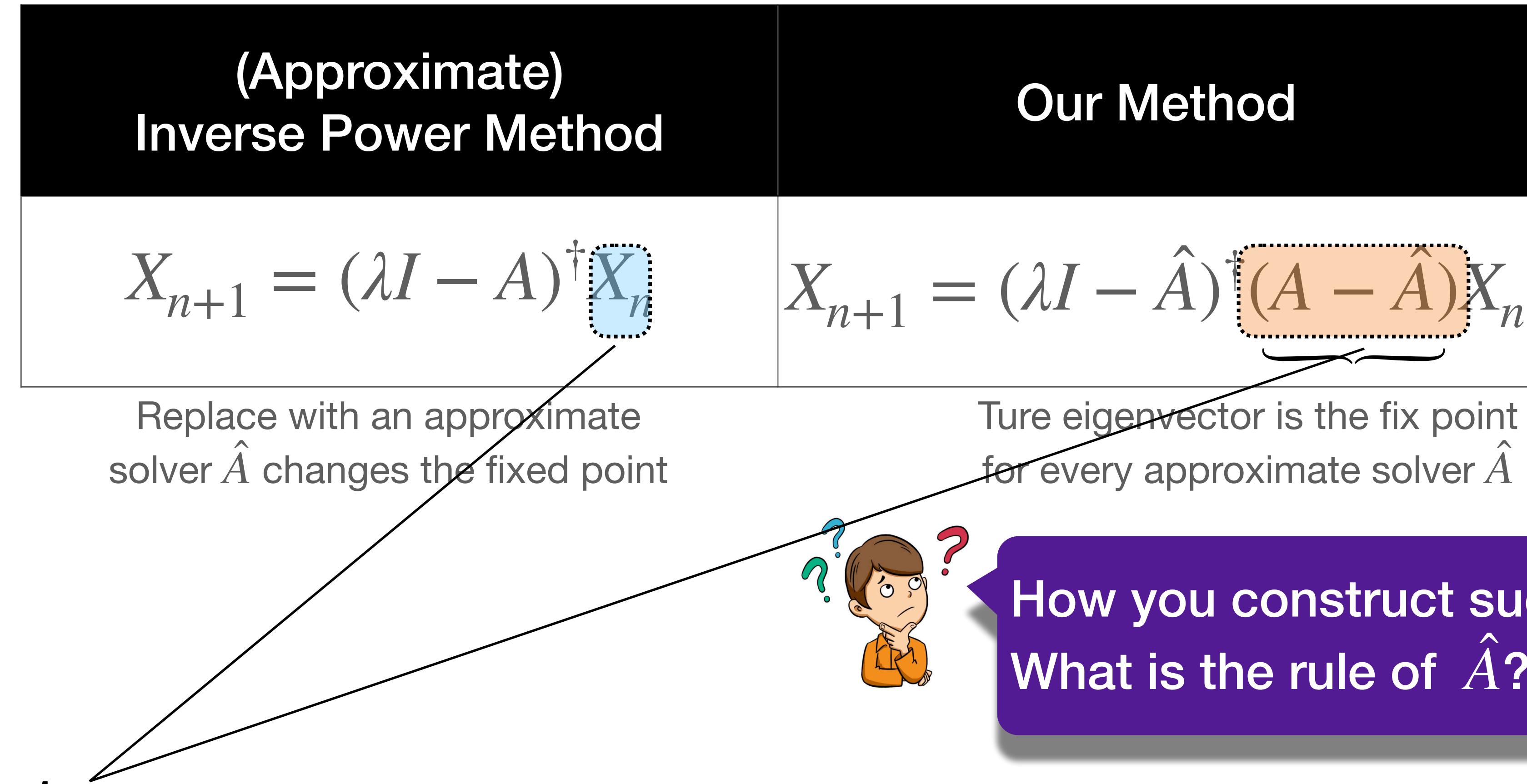
# Relationship with Inverse Power Methods

(Approximate) Inverse Power Method	Our Method
$X_{n+1} = (\lambda I - A)^\dagger X_n$	$X_{n+1} = (\lambda I - \hat{A})^\dagger \underbrace{(A - \hat{A})}_{\text{True eigenvector is the fix point}} X_n$

Replace with an approximate  
solver  $\hat{A}$  changes the fixed point

True eigenvector is the fix point  
for every approximate solver  $\hat{A}$

# Relationship with Inverse Power Methods



## Take Home Message 1:

Power the Residual but not Power the vector

# Why better than Directly DMD

## “Sketch-and-Solve” VS “Sketch-and-Precondition”

	Sketch-and-Solve	Sketch-and-Precondition
Least Square		Sketch-and-precondition, Sketch-and-project, Iterataive Sketching, ....
Low rank Approx	Idea 1: plug in a SVD Solver: Random SVD Idea 2: plug in a inverse power method	 <b><u>Our Work!</u></b>

Use sketched matrix  $\hat{A}$  as  
an approximation to  $A$

Use sketched matrix  $\hat{A}$  as  
an precondition to the probelm



Sorry... but I can't see the  
relationship....

# Why better than Directly DMD

## “Sketch-and-Solve” VS “Sketch-and-Precondition”

	<b>Sketch-and-Solve</b>	<b>Sketch-and-Precondition</b>
<b>Least Square</b>		Sketch-and-precondition, Sketch-and-project, Iterataive Sketching, ....
<b>Low rank Approx</b>	Idea 1: plug in a SVD Solver: Random SVD Idea 2: plug in a inverse power method	 <b><u>Our Work!</u></b>

Use sketched matrix  $\hat{A}$  as  
an approximation to  $A$

Use sketched matrix  $\hat{A}$  as  
an precondition to the probelm



**Idea:** using (approximate) Newton method to solve the Lagrange from  

$$\min u^\top A u - \lambda(x^\top x - 1)$$

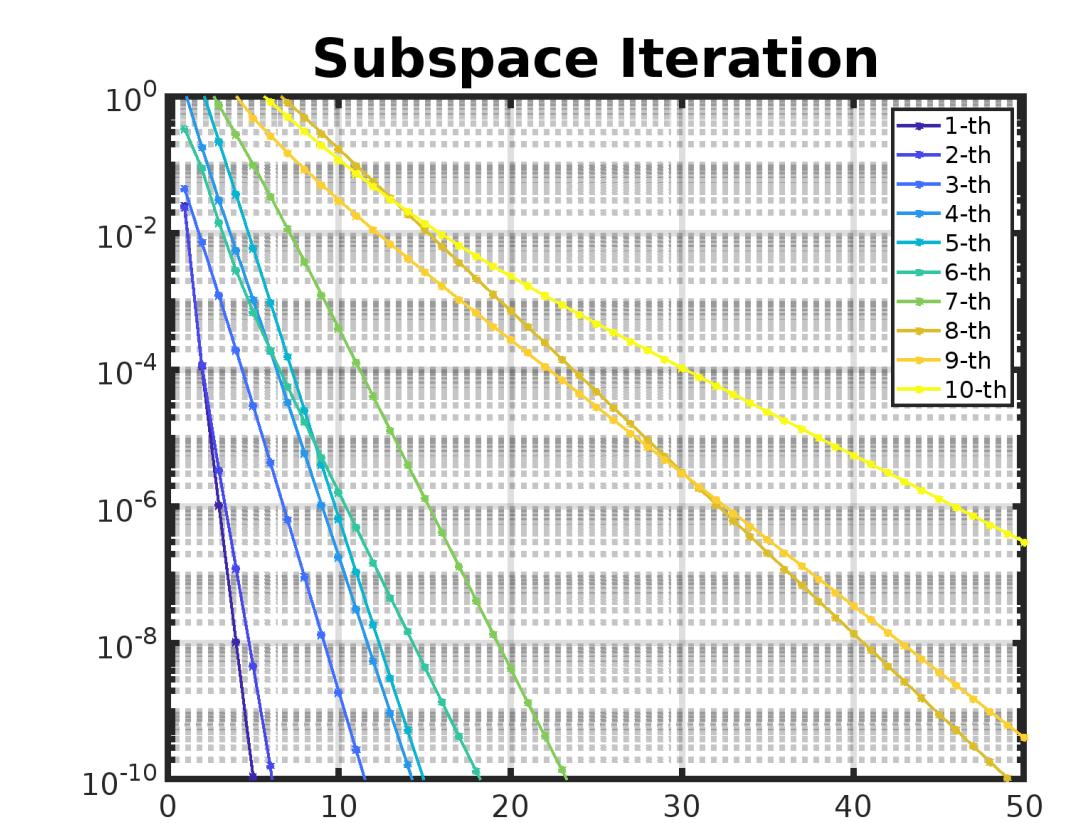
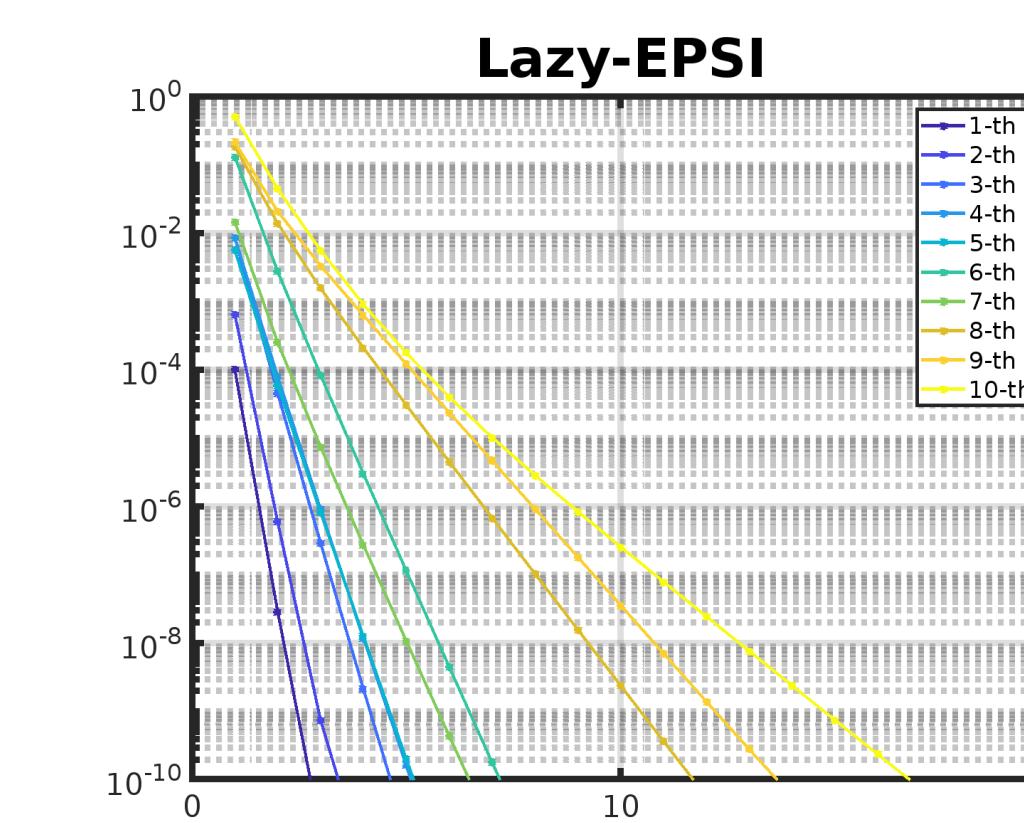
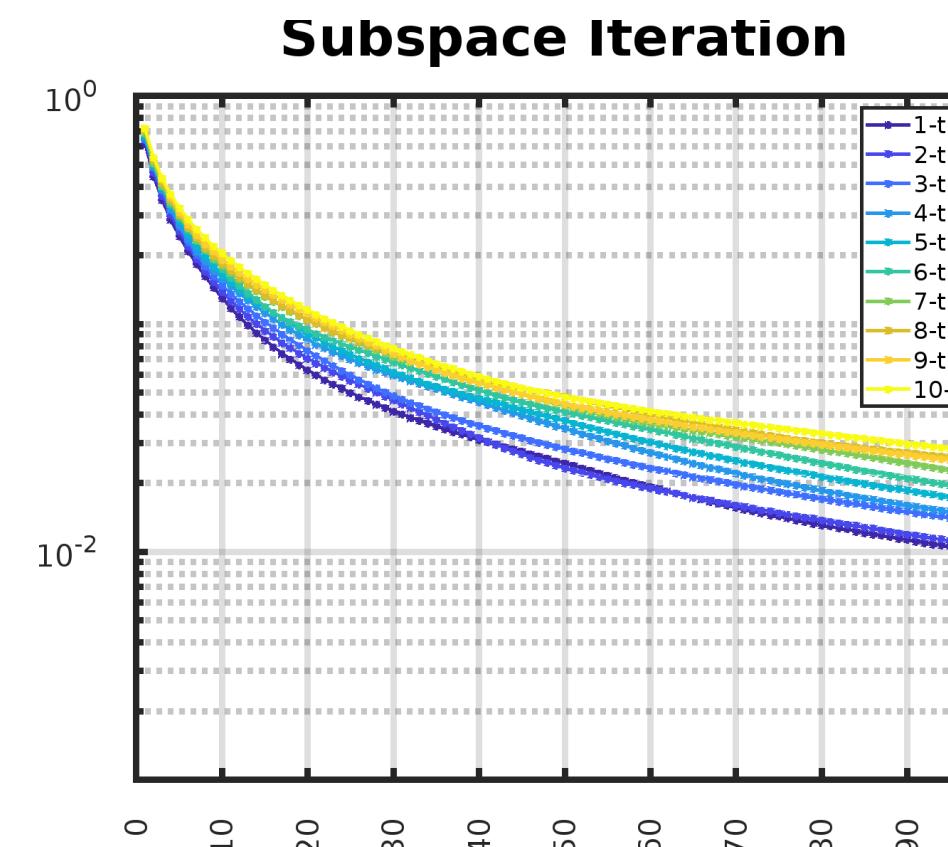
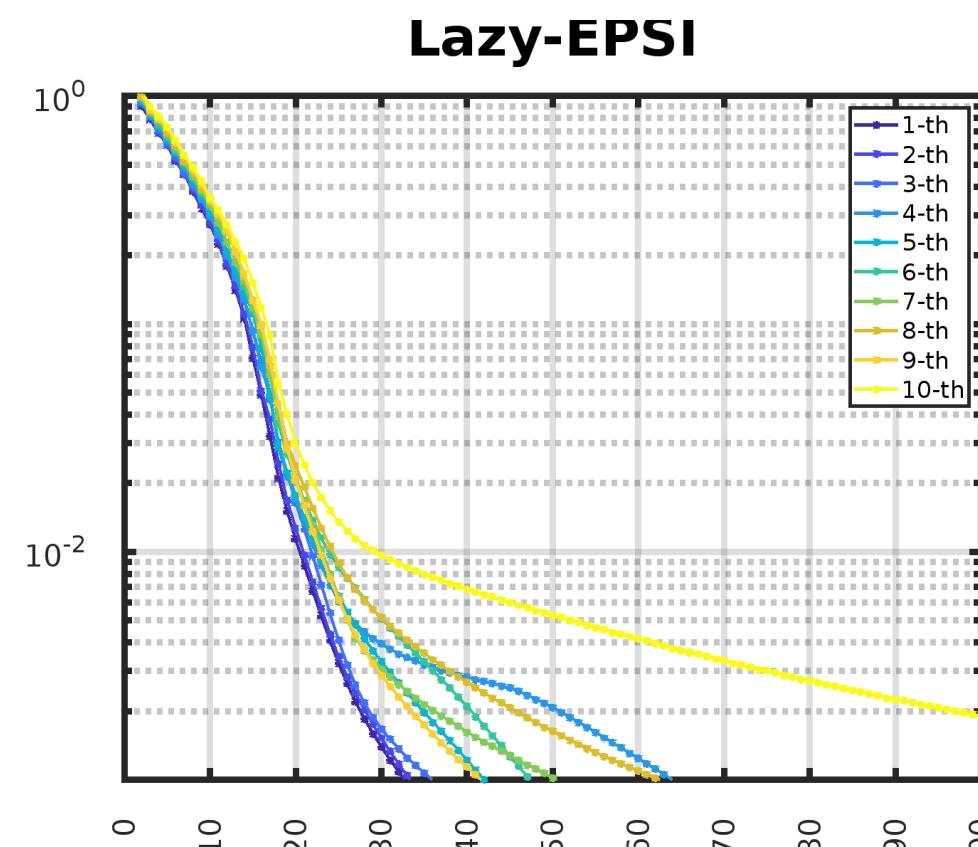
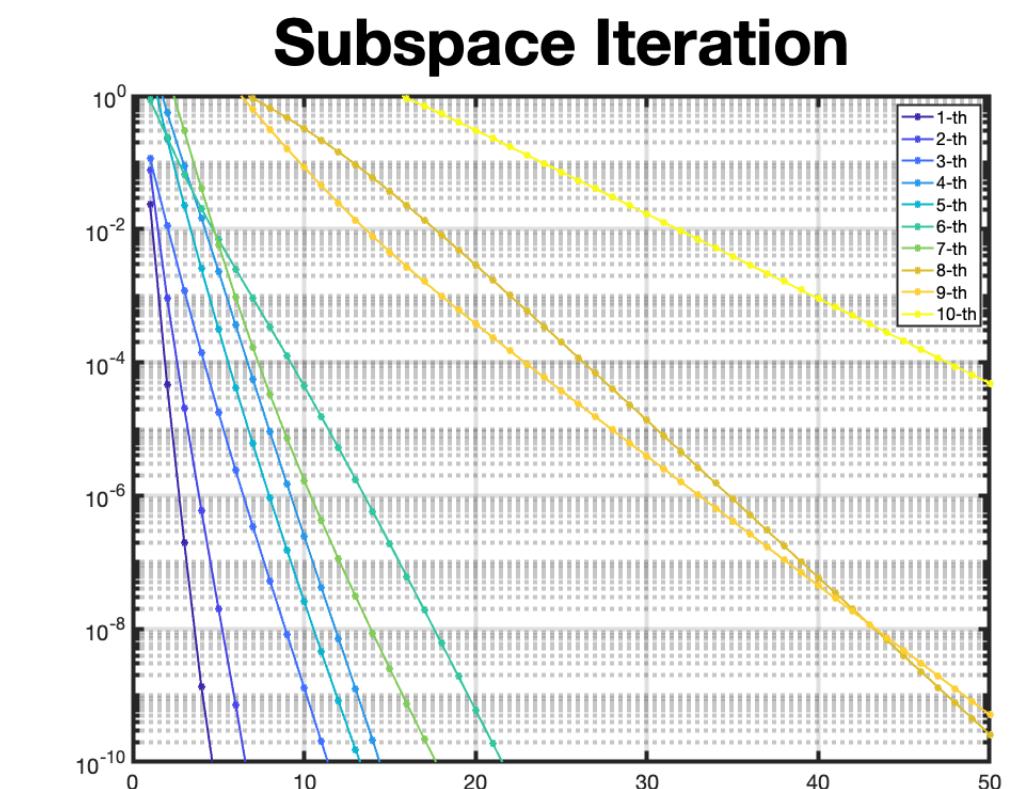
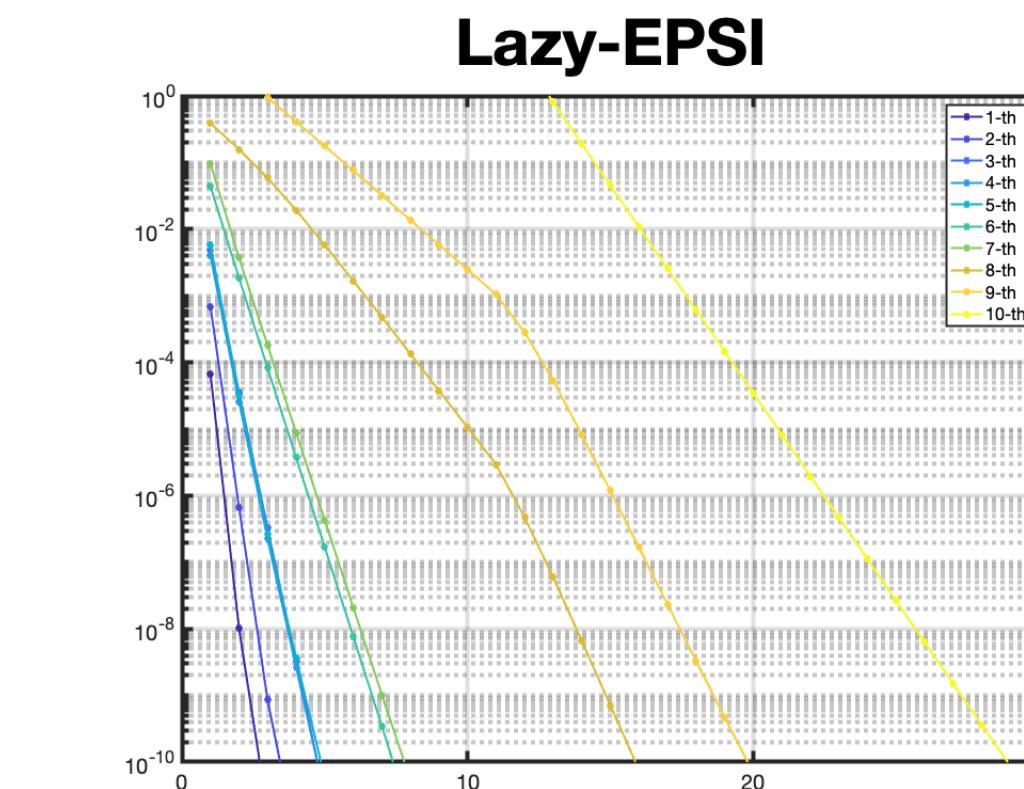
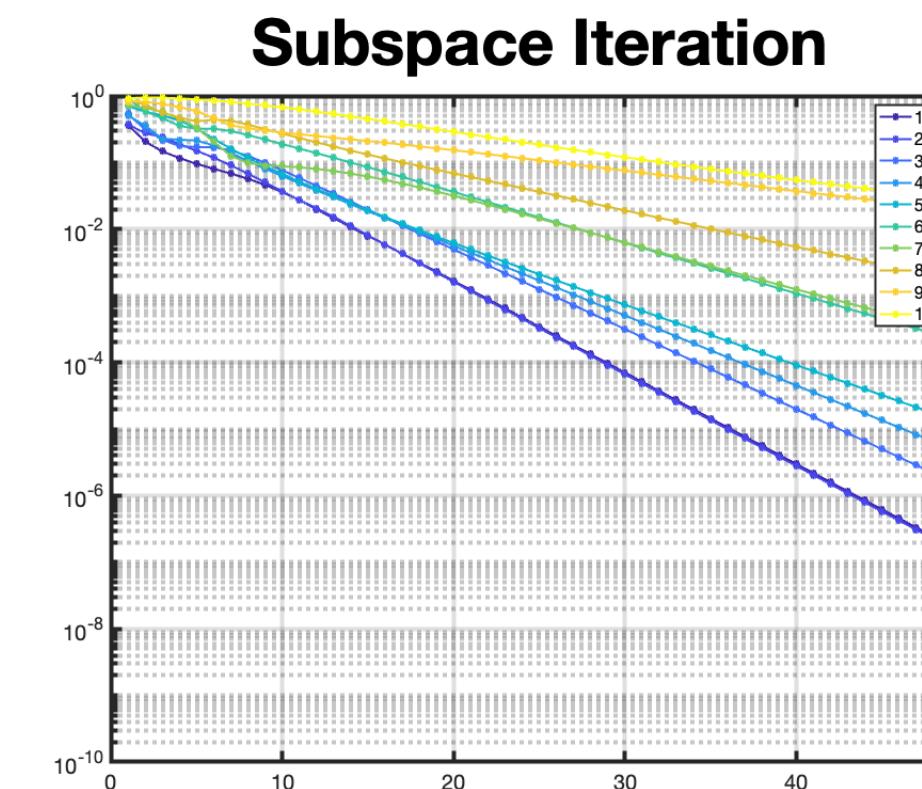
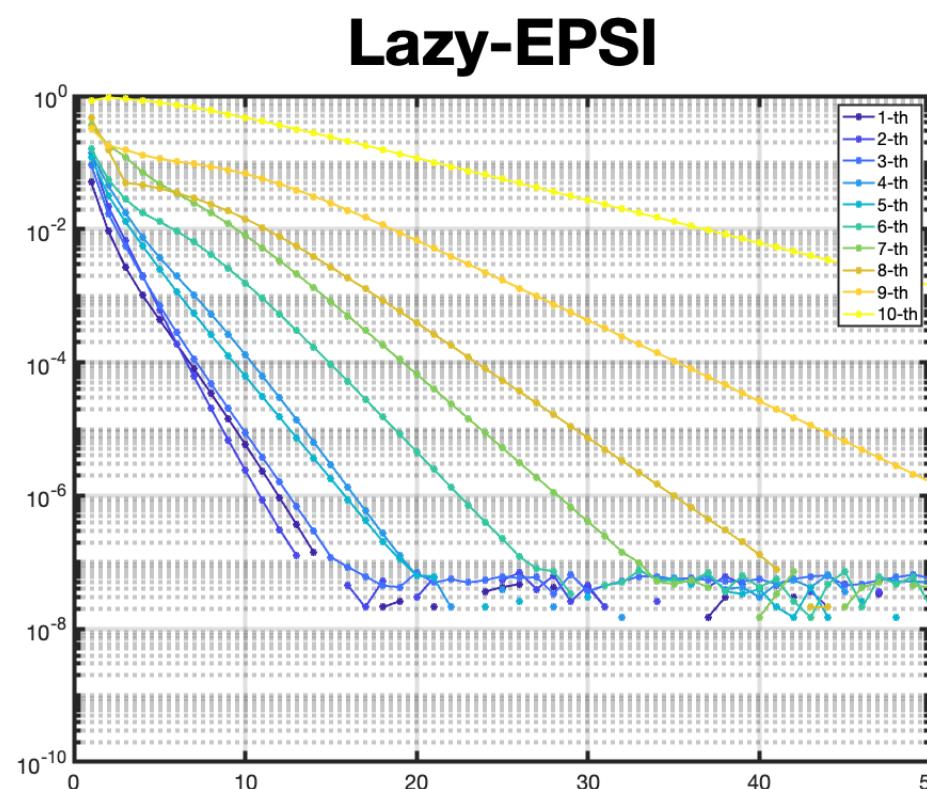
Thus Our convergence is  $u$  linear-quadratic



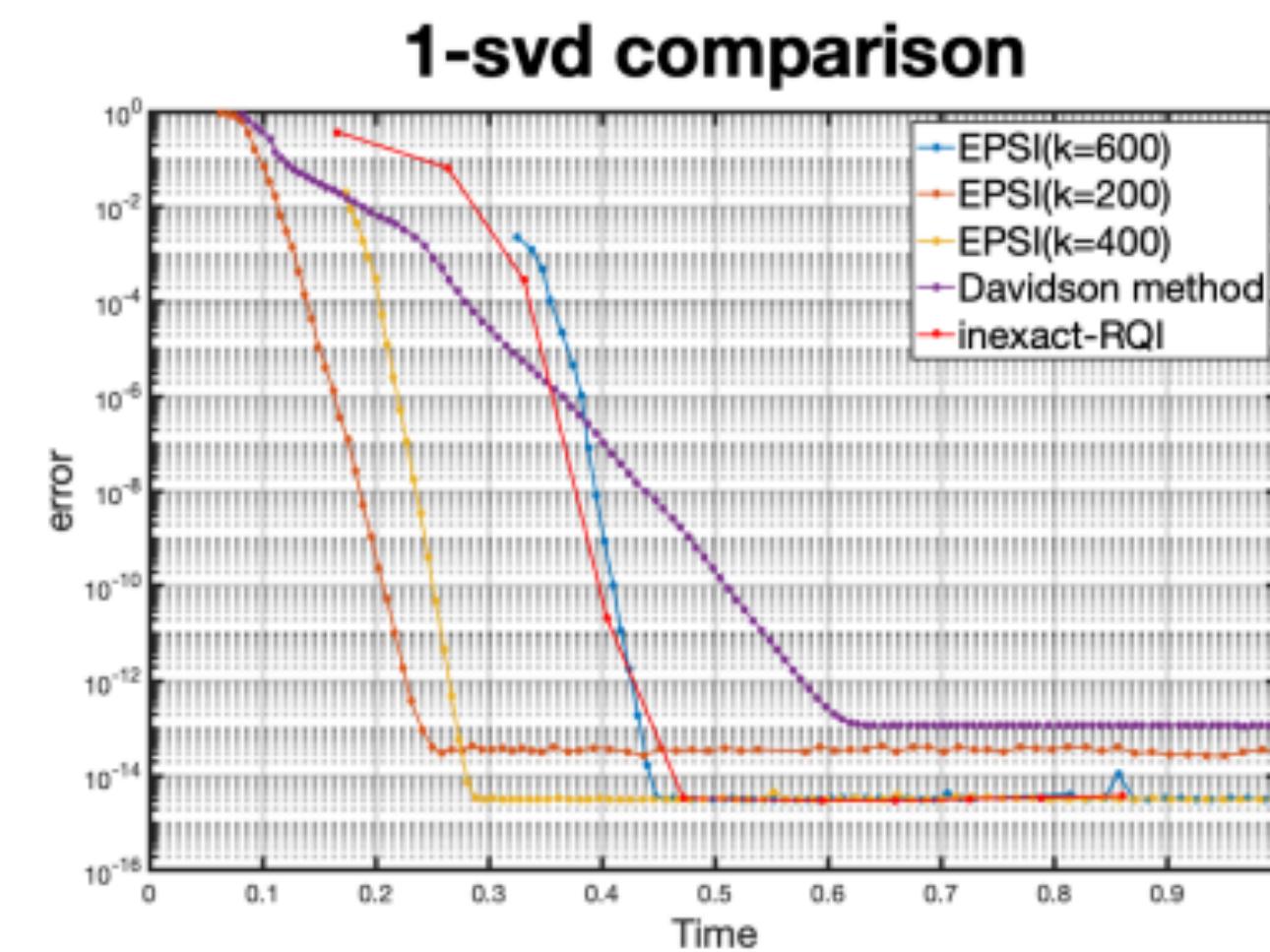
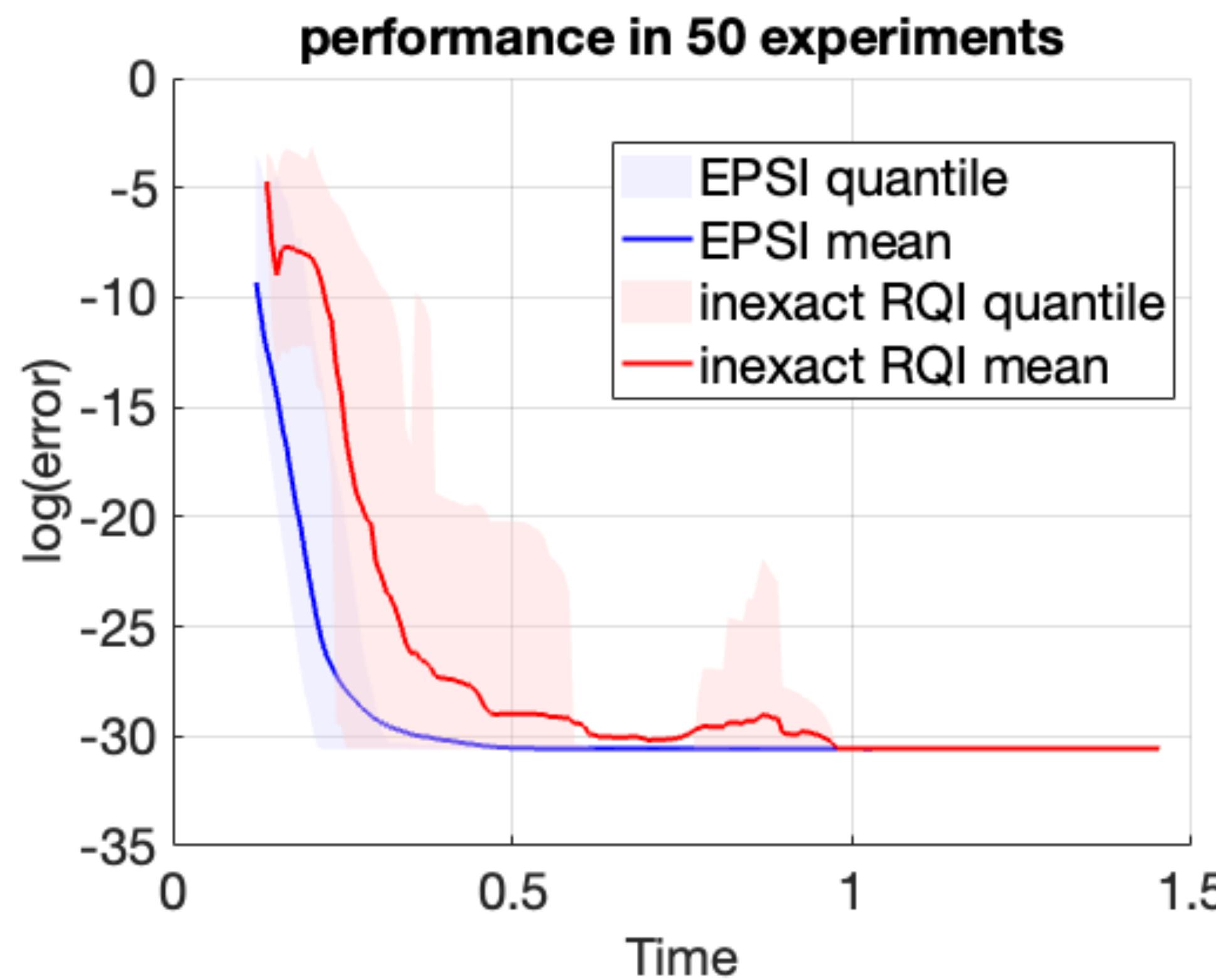
Contraction coefficient improves when sketching quality increases



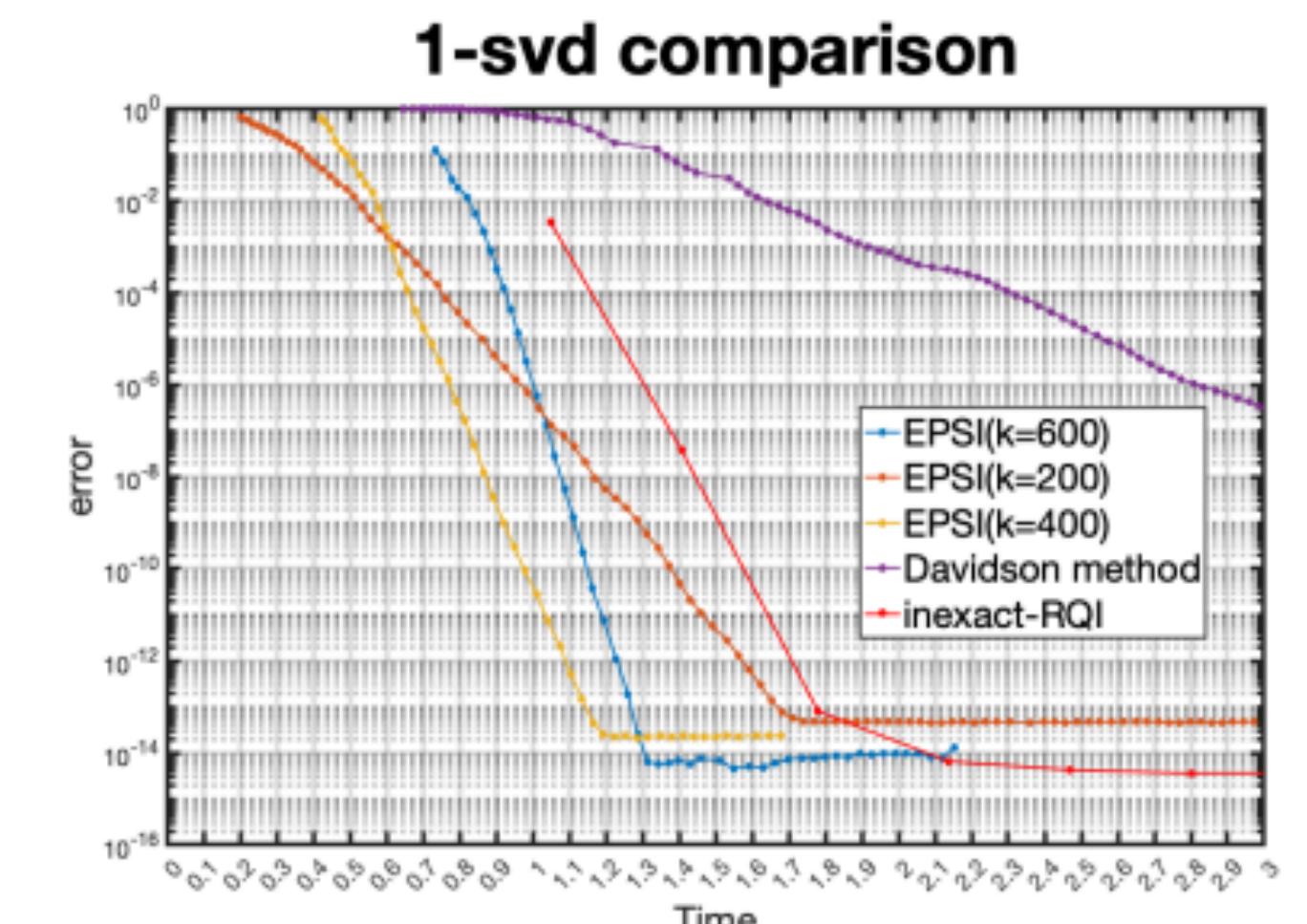
# Eigenvalue Computation



# Running Time



(b)  $n = 2000, \kappa = 10^{-6}$



(c)  $n = 4000, \kappa = 10^{-6}$

# What is a Sketch-and-Precondition Derivation for Low-Rank Approximation? Inverse Power Error or Inverse Power Estimation?

Ruihan Xu \*

Yiping Lu †

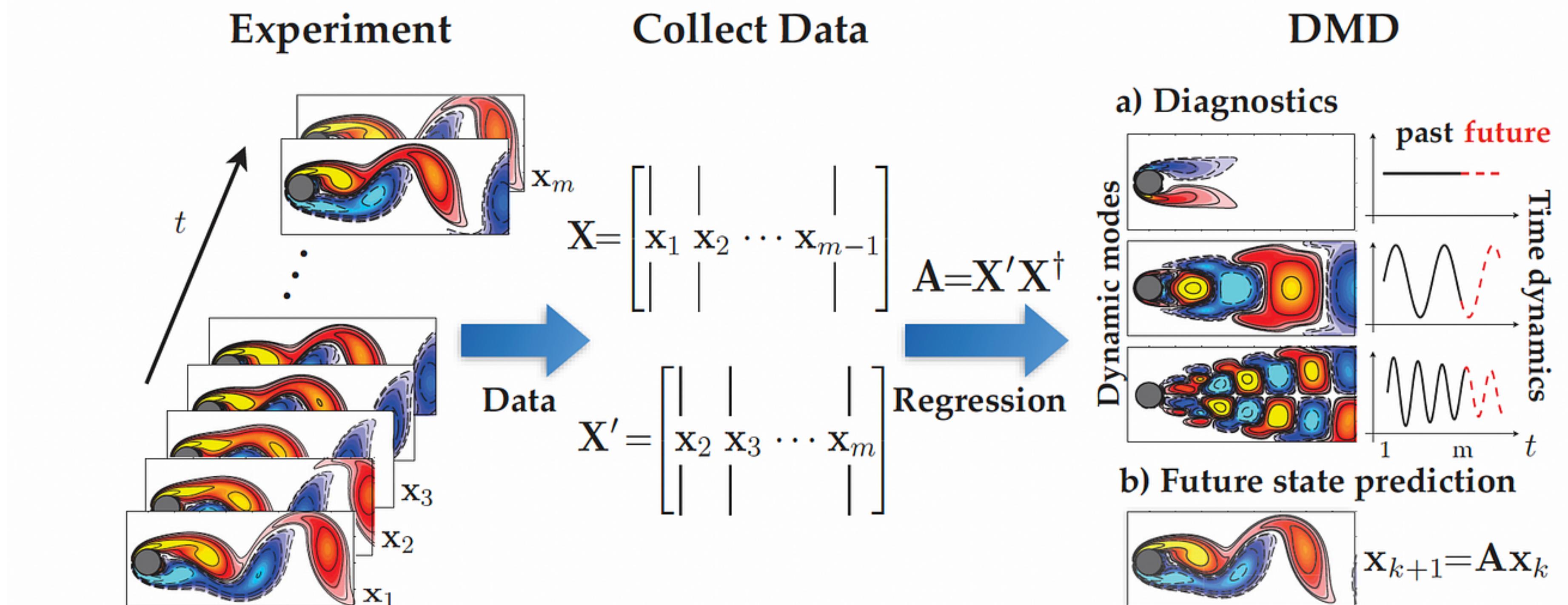
## Abstract

Randomized sketching accelerates large-scale numerical linear algebra by reducing computational complexity. While the traditional sketch-and-solve approach reduces the problem size directly through sketching, the sketch-and-precondition method leverages sketching to construct a computational friendly preconditioner. This preconditioner improves the convergence speed of iterative solvers applied to the original problem, maintaining accuracy in the full space. Furthermore, the convergence rate of the solver improves at least linearly with the sketch size. Despite its potential, developing a sketch-and-precondition framework for randomized algorithms in low-rank matrix approximation remains an open challenge. We introduce the *Error-Powered Sketched Inverse Iteration* (EPSI) Method via run sketched Newton iteration for the Lagrange form as a sketch-and-precondition variant for randomized low-rank approximation. Our method achieves theoretical guarantees, including a convergence rate that improves at least linearly with the sketch size.

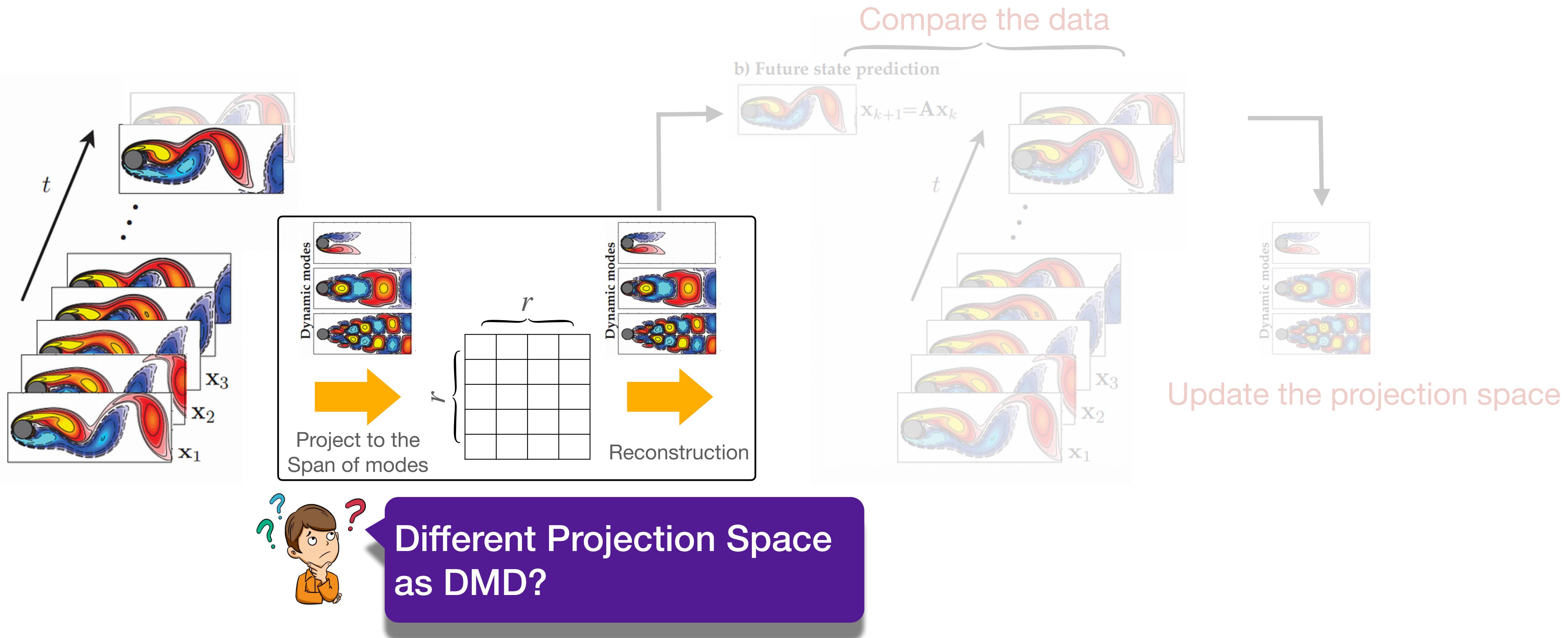
# Another Supersing Fact...

## Iteration lies in the Krylov Subspace

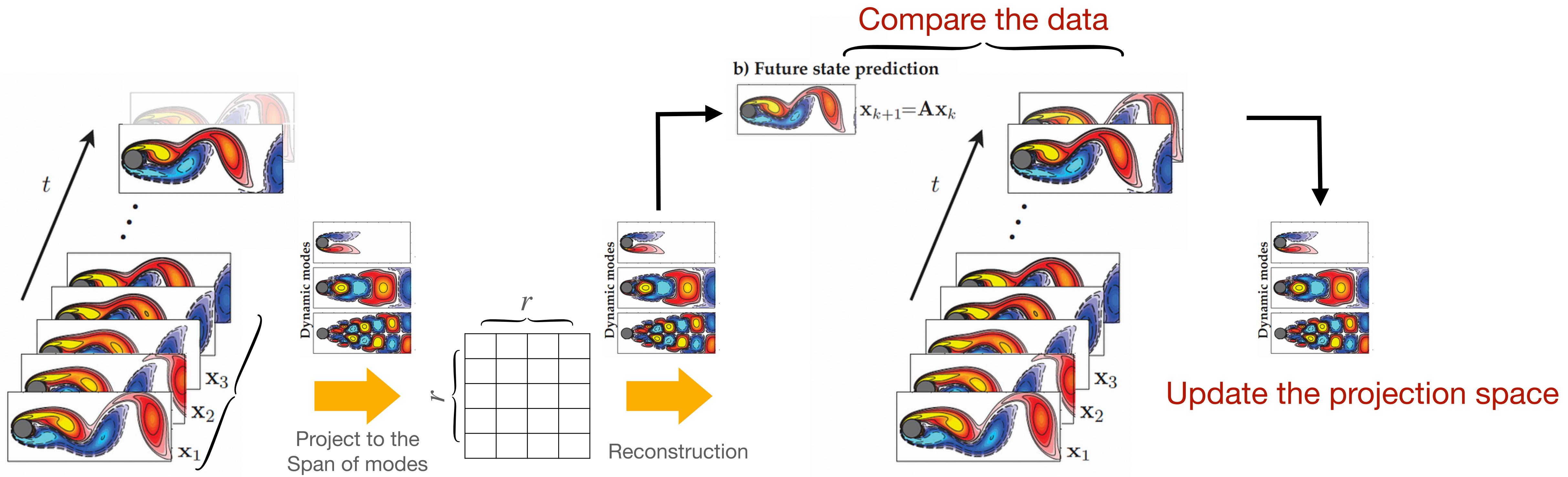
- enable dynamic mode decomposition
- Online fast update
- Much better than DMD



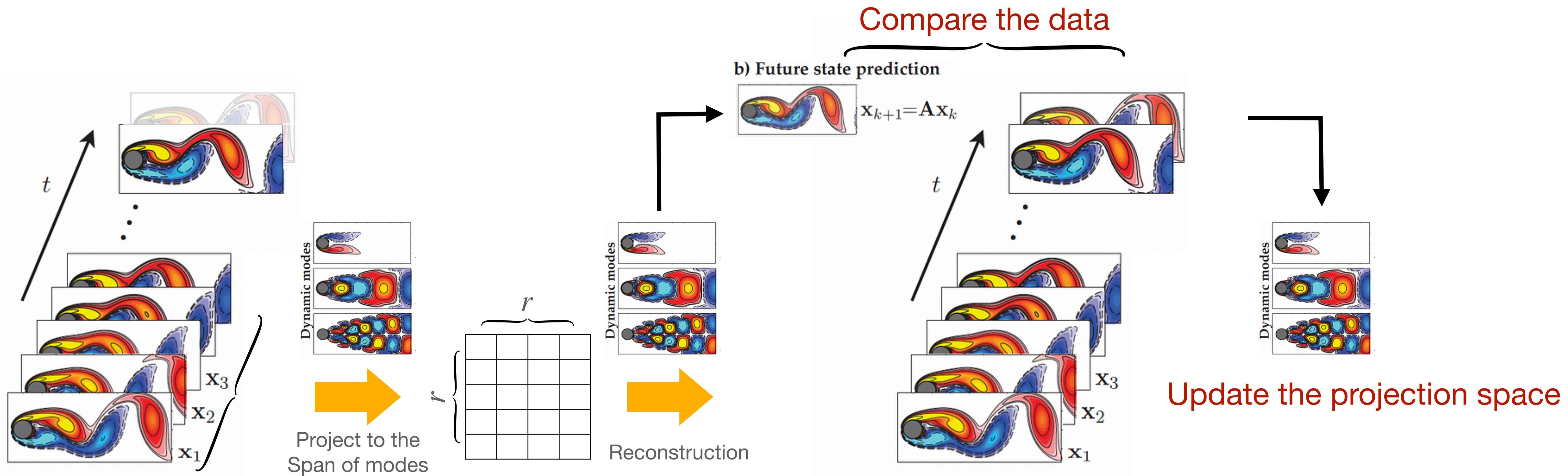
# DMD with First-Order Feedback



# DMD with First-Order Feedback

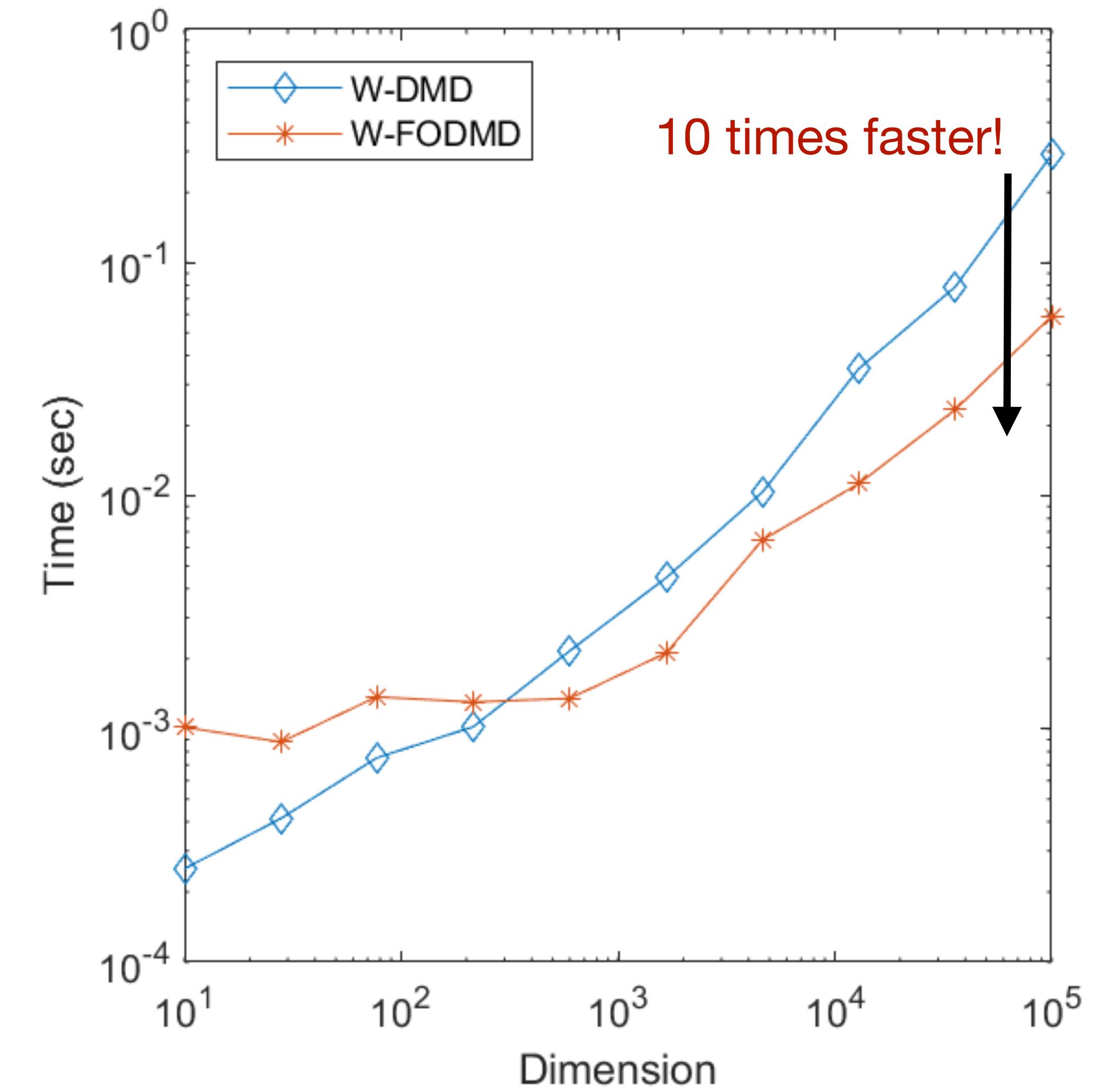
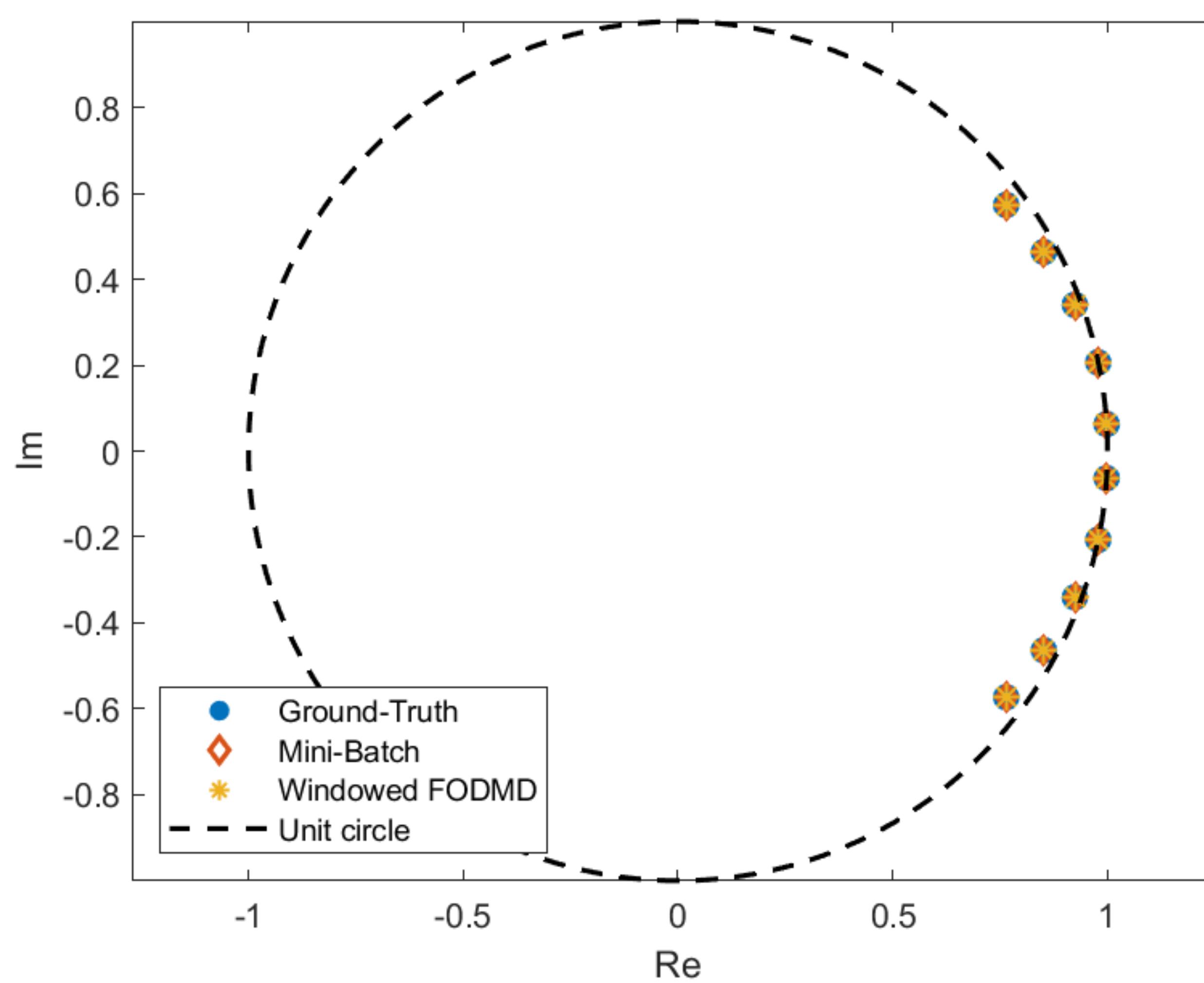


# DMD with First-Order Feedback

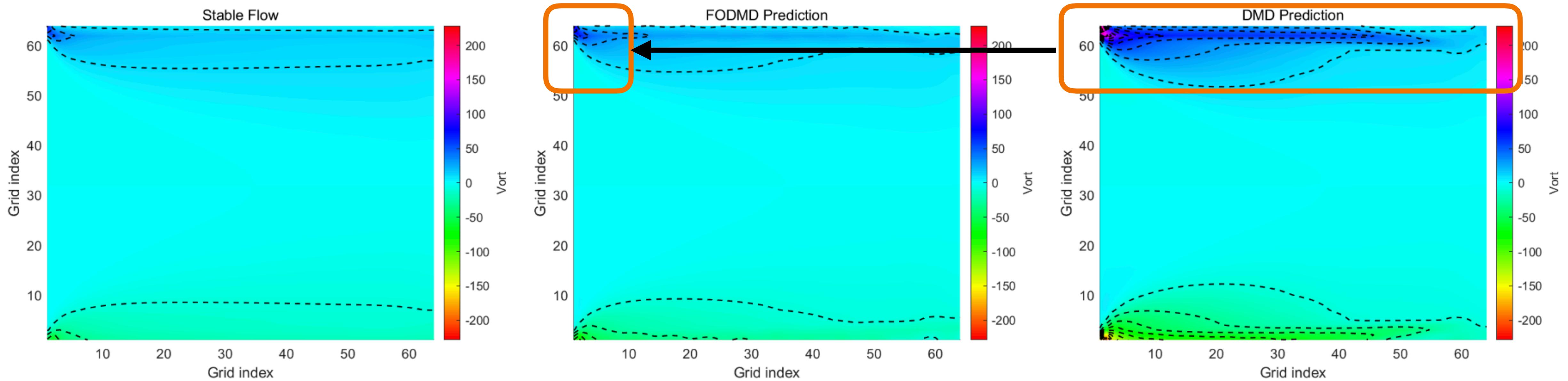


No matrix inverse, No SVD computation  
Only a  $n \times r$  QR decomposition  
(Everything has a closed-form solution)

# Faster than Recomputation!



# Prediction of Tube Flow



# One more thing...

Iterative  
debiasing

Easier for numerical stability computation  
Algorithms can do **online** computation



Newton  
Methods

Easier for convergence analysis



## Randomized Iterative Solver as Iterative Refinement A Simple Fix Towards Backward Stability

Ruihan Xu  
University of Chicago

Yiping Lu  
Northwestern University

### Abstract

Iterative sketching and sketch-and-precondition are well-established randomized algorithms for solving large-scale over-determined linear least-squares problems. In this paper, we introduce a new perspective that interprets Iterative Sketching and Sketching-and-Precondition as forms of Iterative Refinement. We also examine the numerical stability of two distinct refinement strategies: iterative refinement and recursive refinement, which progressively improve

tive tools for developing approximate matrix factorizations. These methods are remarkable for their simplicity and efficiency, often producing surprisingly accurate results.

In this paper, we consider randomized algorithms to solve the overdetermined linear least-squares problem

$$x = \arg \min_{y \in \mathbb{R}^n} \|b - Ay\| \quad (A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m) \quad (1)$$

# More Examples... (Uncertainty Quantification)



Scientific Machine Learning

Downstream application

## Example 5

$$\theta = \theta, \quad X_i \sim P_\theta$$

Quantile regression

Confidence Interval of Point Estimation

Conformal Prediction

Romano Y, Patterson E, Candes E. Conformalized quantile regression. Neurips 2019.

Influence Function

Bootstrap

Liu K, Blanchet J, Ying L, et al. Orthogonal bootstrap: efficient simulation of input uncertainty. ICML 2024.

LLM

Taylor Expansion

Angelopoulos A N, Bates S, Fannjiang C, et al. Prediction-powered inference. Science, 2023

# More Examples... (Sampling)

$$\boxed{\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})}$$

Scientific Machine Learning

Downstream application

**Example 6**

$$\theta = V, \quad X_i = (x_i, V(x_i), -\nabla V(x_i)) \quad \Phi(\theta) = \int e^{-V(x)} dx$$

Holzmüller D, Bach F. Convergence rates for non-log-concave sampling and log-partition estimation.  
arXiv preprint arXiv:2303.03237, 2023.

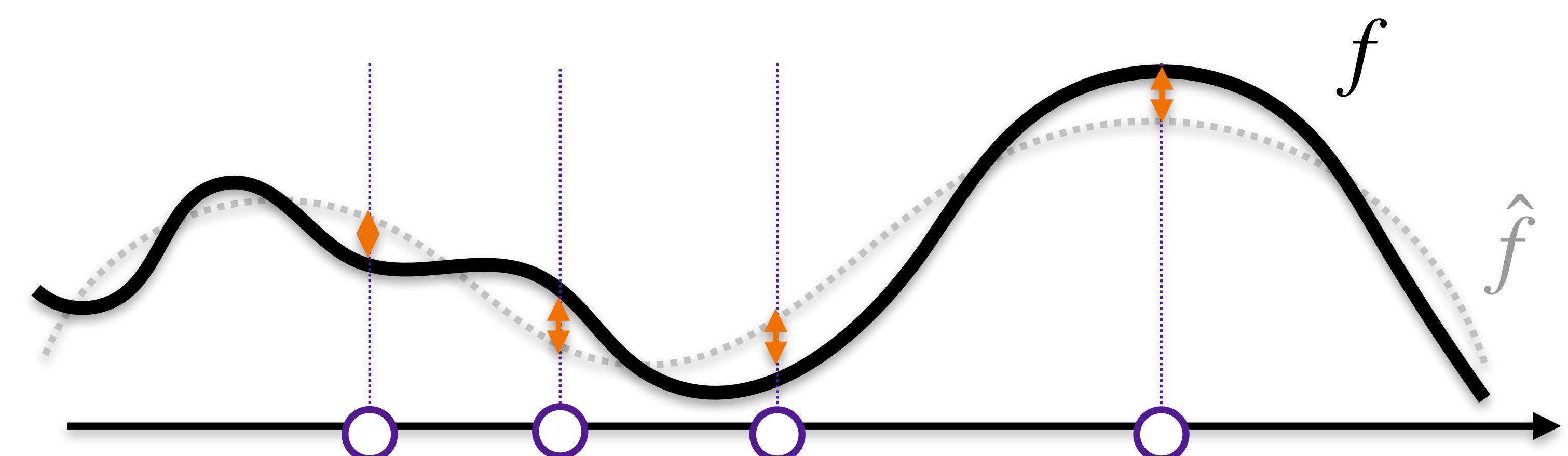
# I What is SCaSML about?



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \theta \rightarrow \Phi(\theta)$$

**Step 1: Using Machine Learning to fit the rough function/environment**

**Step 2: Using validation dataset to know how much mistake machine learning algorithm has made**



**Step 3: Using Simulation algorithm to estimate  $\Phi(\theta) - \Phi(\hat{\theta})$**

*Using ML surrogate during inference time to improve ML solution*

# Research Philosophy

Machine Learning

$$X_1, \dots, X_n \rightarrow \theta$$

Upper Bound

Lower Bound

PDE Solver

Finite Element/PINN/  
Operator Learning

Upper Bound

Lower Bound ?

???

Numerical  
Linear Algebra

Sketch-and-solve

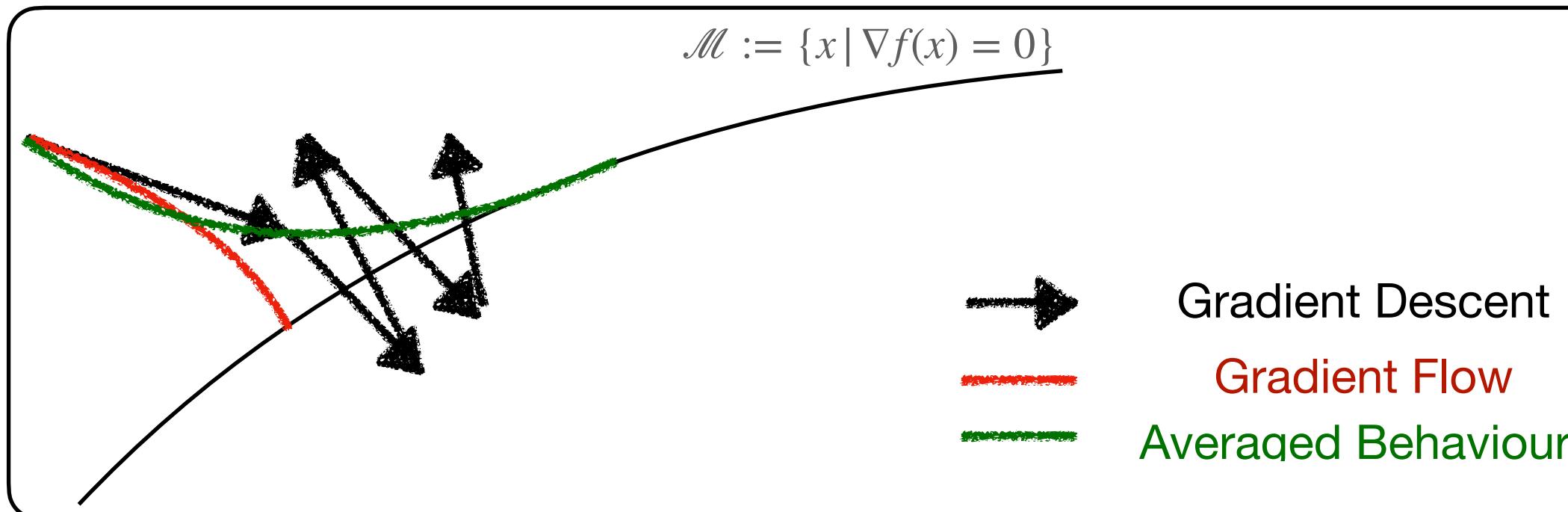
Upper Bound

Lower Bound ?

Sketch-and-precond

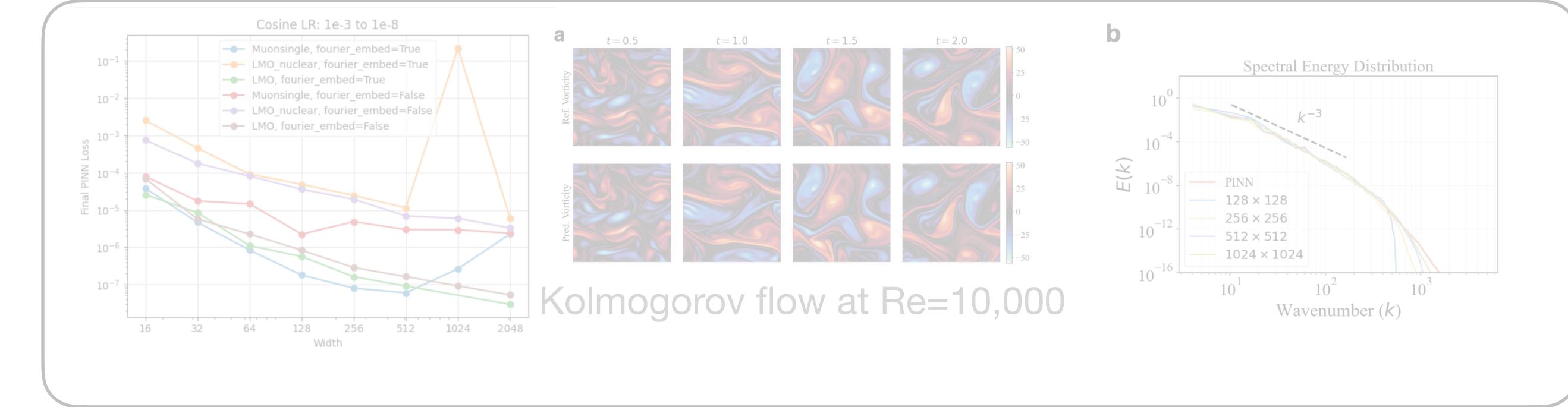
# My Other Research Projects...

## Multiscale analysis of Gradient Descent



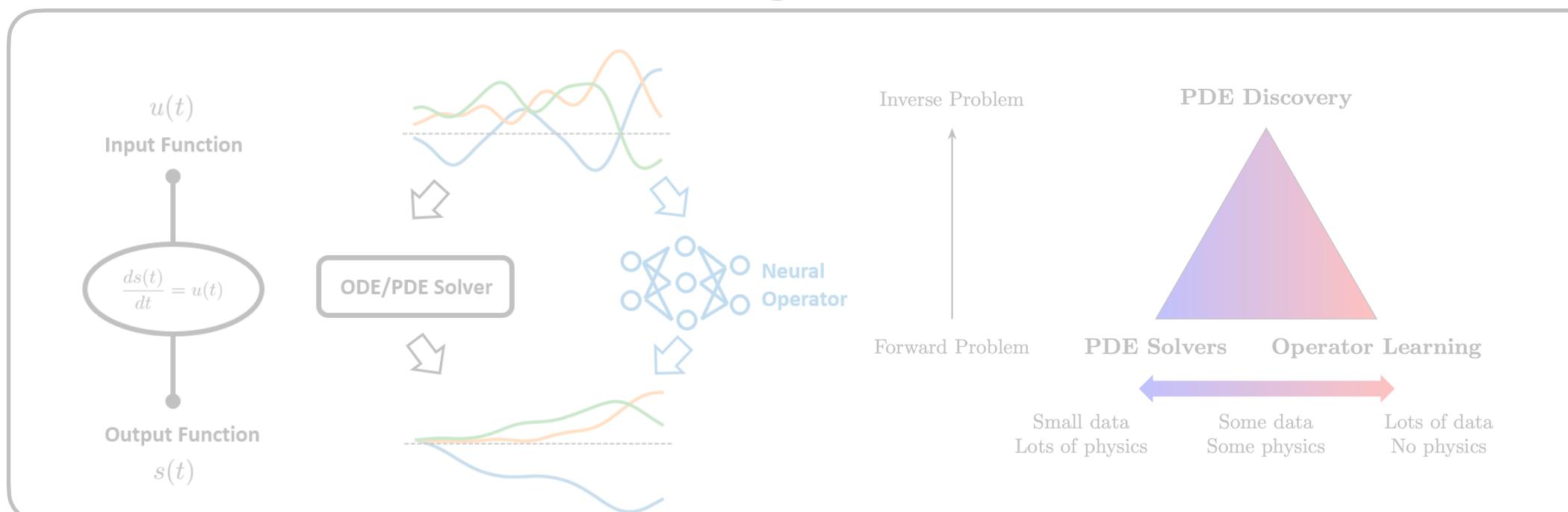
- Gradient flow fails to approximate gradient descent for DL
- Boundary-layer like behaviour for gradient descent
- Implicit regularization via Multiscale analysis

## Neural Solver for PDE

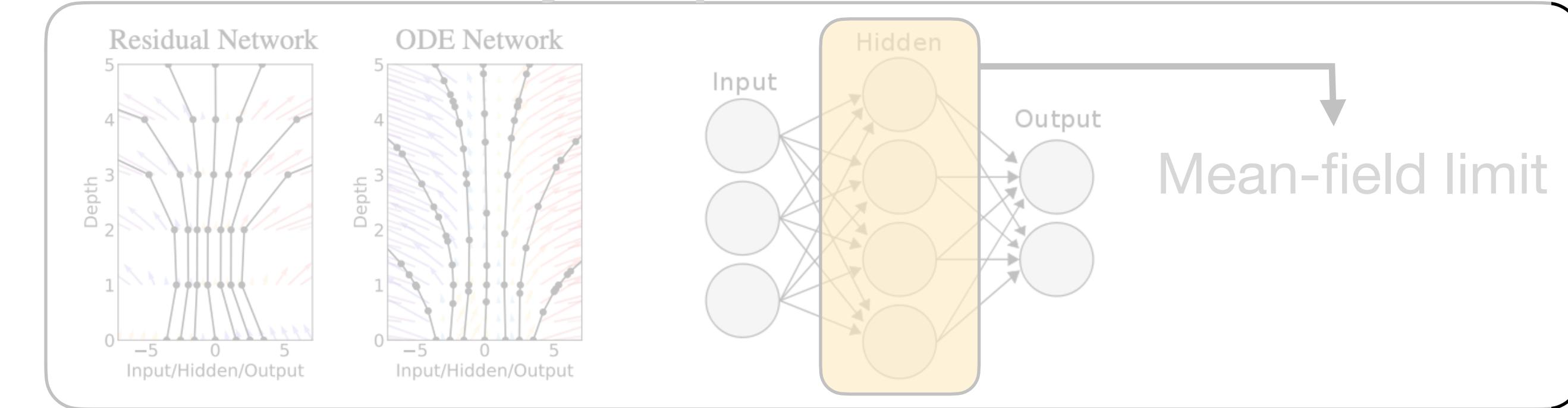


- A new first order optimizer via mirror descent to optimize PINN to machine precision
- The first optimizer works better on larger networks
- Mainly work on high dimensional PDEs!

## Machine Learning in Function Space

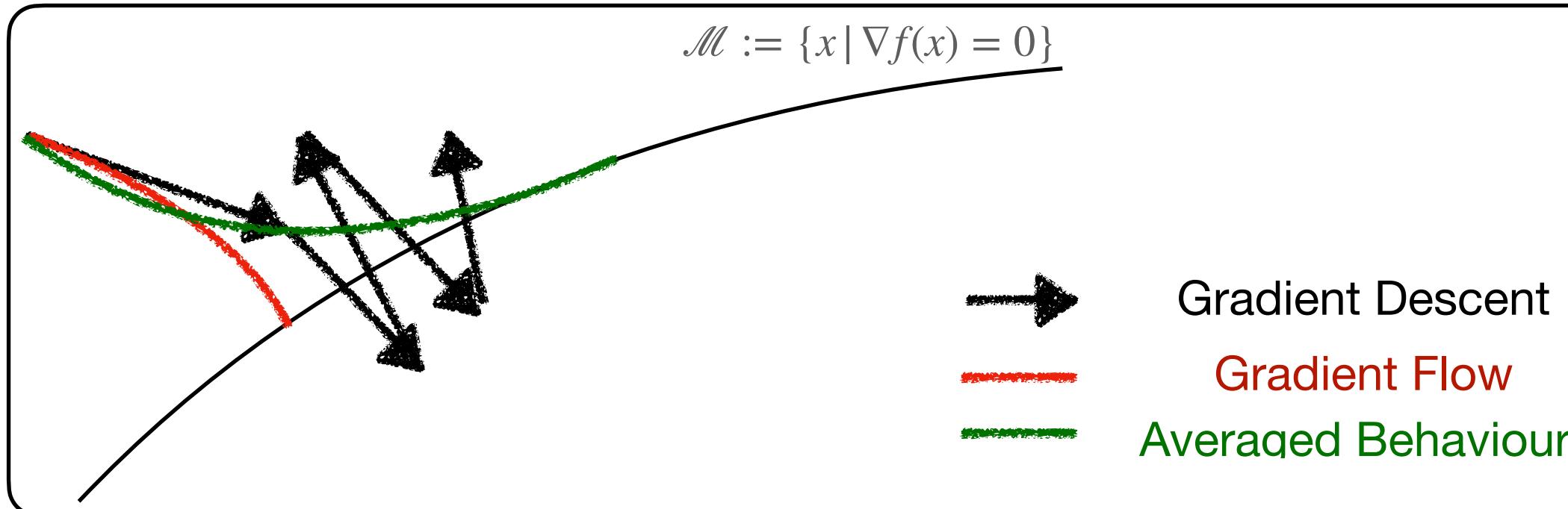


## Infinitely Deep/Wide Networks



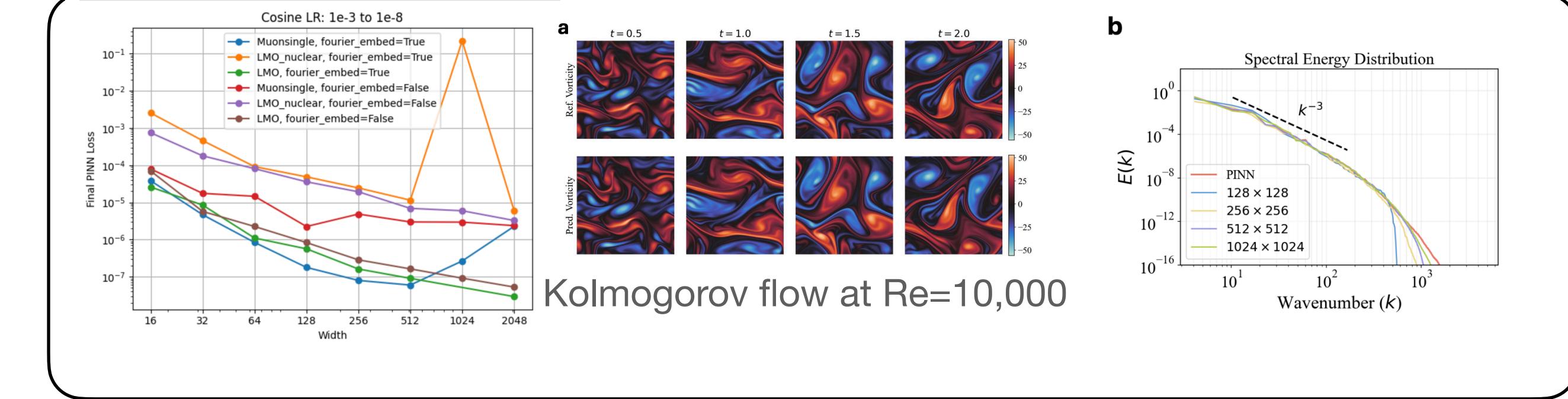
# My Other Research Projects...

## Multiscale analysis of Gradient Descent



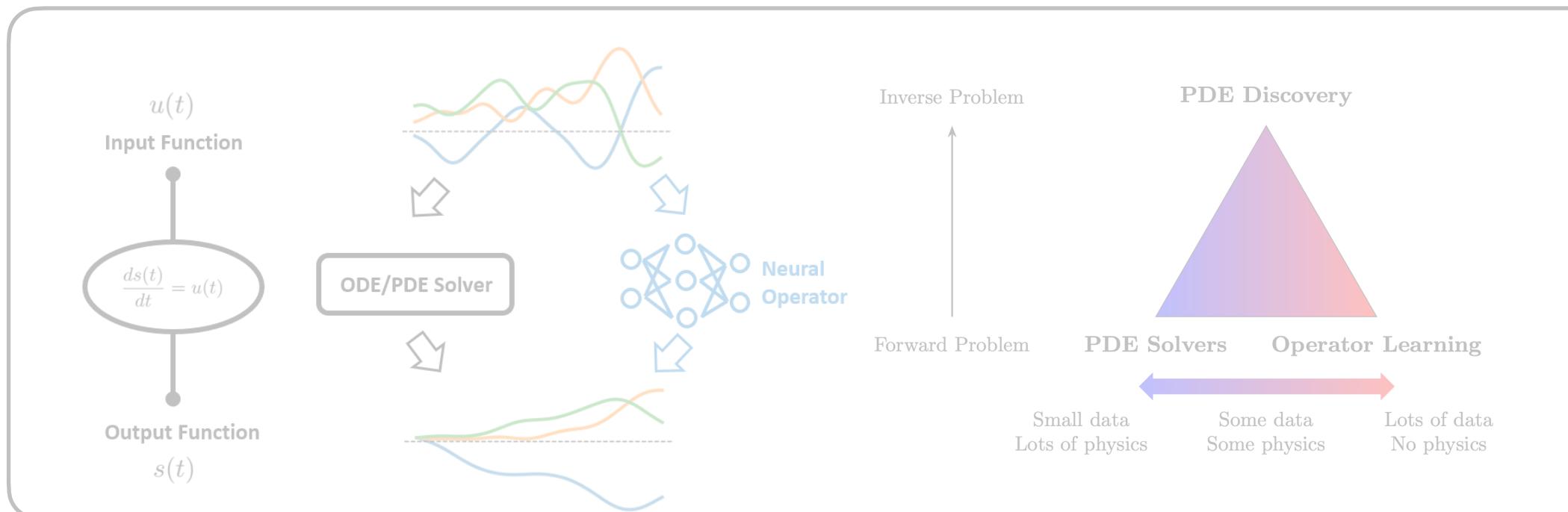
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## Neural Solver for PDE

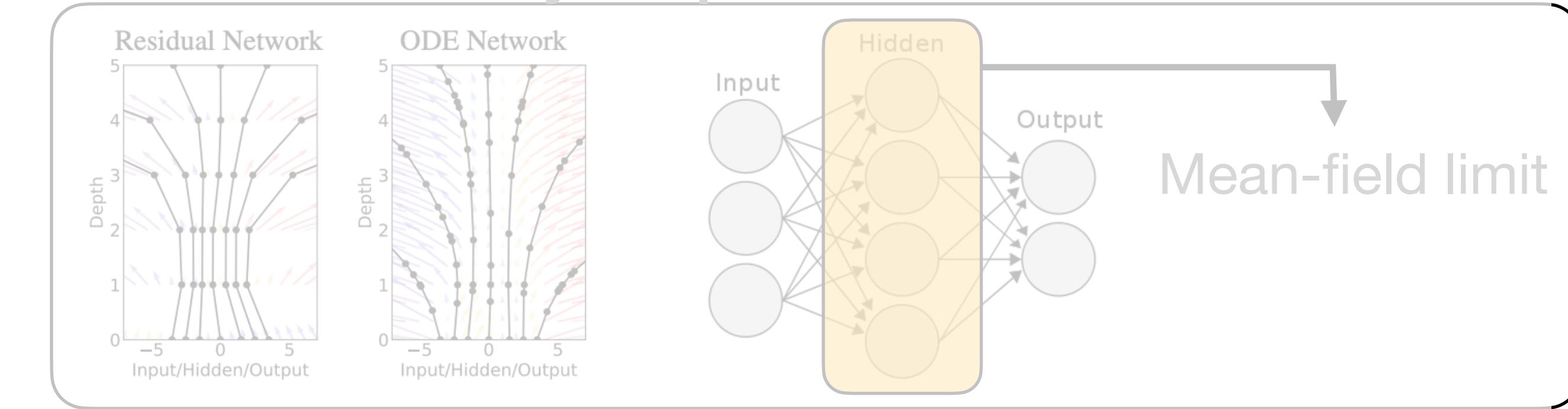


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## Machine Learning in Function Space

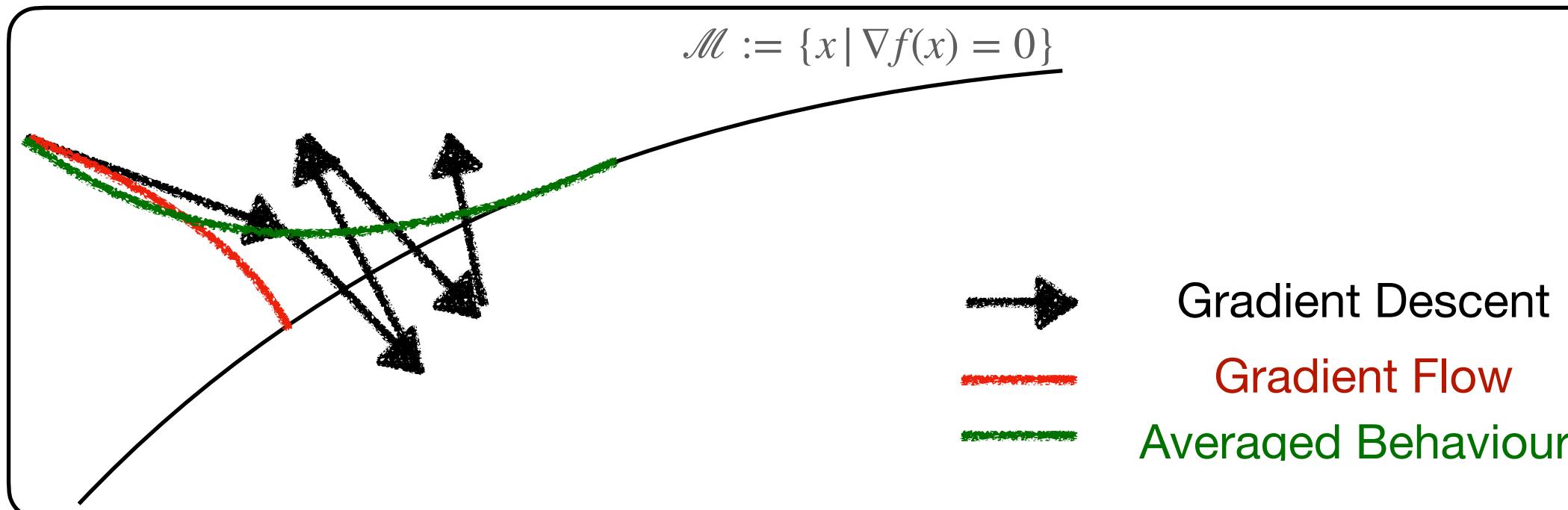


## Infinitely Deep/Wide Networks



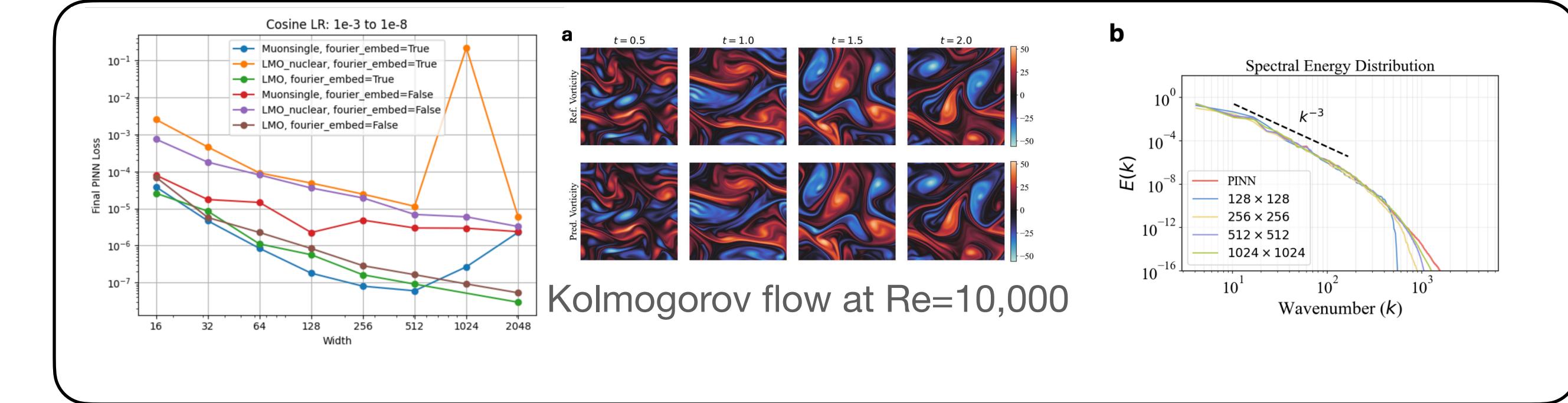
# My Other Research Projects...

## Multiscale analysis of Gradient Descent



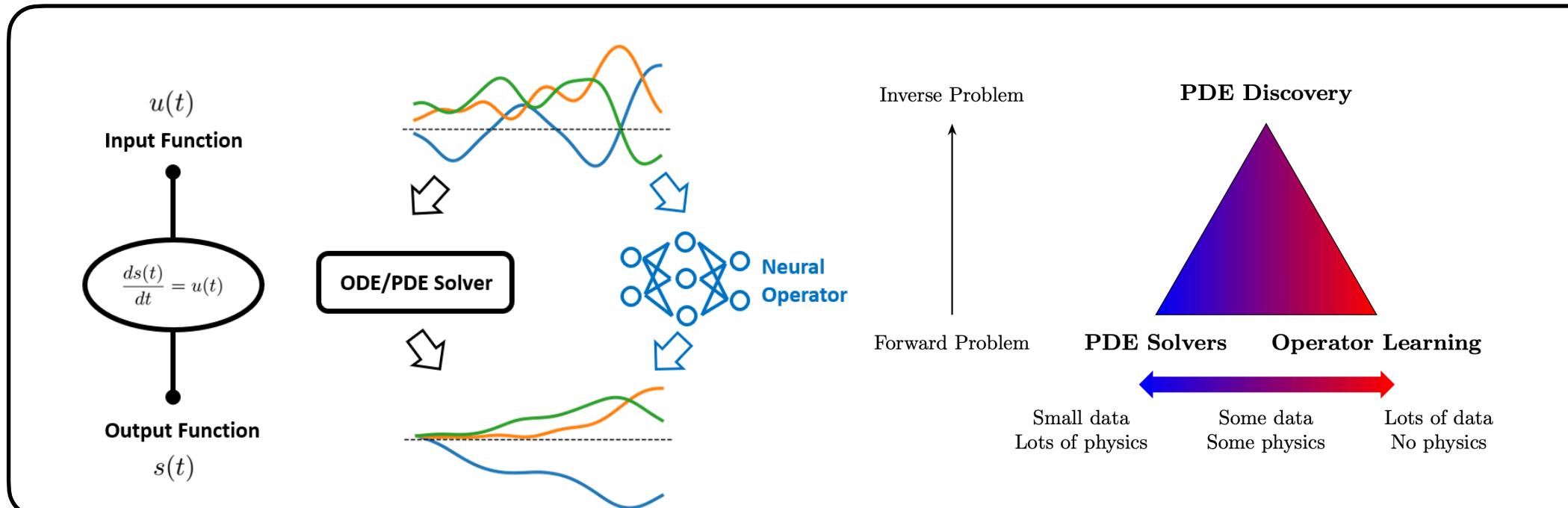
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## Neural Solver for PDE



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## Machine Learning in Function Space



## Infinitely Deep/Wide Networks

