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NEWYORK UNIVERSITY
Linear Algebra Final Review

Subject: MATH-UA 140 Linear Algebra

Year: 2024(Sem 2)

Name of Examiners: .

Time allow: 1001

Instruction to Candidate : (only on page 1)

(1) This paper contains ________ questions.

(2) Candidates must answer ________ questions.

Question No 1

Diagonalize A and compute $V\mathbf{A}kV^{-1}$ to prove this formula for A^k:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ has } A^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

and what is the meaning of $\lim_{k\to\infty} \frac{1}{3^k} A^k$. $= \left(\frac{A}{3}\right)^k$.

$\lambda_1 = 1 \quad v_1 = (-1, 1)$

$\lambda_2 = 1/3 \quad v_2 = (1, 1)$

Solution:

The eigenvalues of A are 3 and 1, and the corresponding eigenvectors are $v_1 = (-1, 1)$, $v_2 = (1, 1)$. Therefore, A can be diagonalized as $A = VAV^{-1}$, where $V = [v_1, v_2]$, $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $A^k = V\Lambda^k V^{-1} = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$.

$\lim_{k\to\infty} \frac{1}{3^k} A^k = \lim_{k\to\infty} \frac{1}{2}\begin{pmatrix} \frac{1}{3^k}+1 & \frac{1}{3^k}-1 \\ \frac{1}{3^k}-1 & \frac{1}{3^k}+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (the largest eigen vector)

$\frac{1}{2}v_1 \quad -\frac{1}{2}v_1 \rightarrow$ eigenvector of largest eigenvalue

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Householder Matrix

Question No 2

$P = uu^T$ is a projection

For u is a unit vector prove that $Q = I - 2uu^\top$ is an symmetric orthogonal matrix. Prove $\|Qx\| = \|x\|$.

$Q^TQ = QQ^T = I$

Solution:

Q is symmetric because $uu^\top$ is symmetric.

Then $QQ^\top = QQ = Q^2 = (I - 2uu^\top)^2 = I - 2uu^\top - 2uu^\top + 4u \underbrace{u^\top u}_{u^\top u = \|u\|^2 = 1} u^\top = I - 4uu^\top + 4uu^\top = I$

For all orthogonal matrix Q, we have

$$\|Qx\|^2 = (Qx)^\top(Qx) = x^\top Q^\top Qx = x^\top x = \|x\|^2$$

for $Q^\top Q = I$

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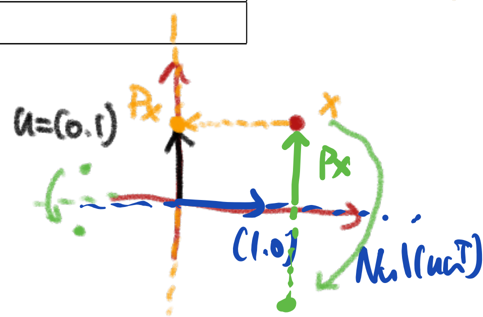
Example $\vec{u} = (0, 1)$

$\vec{u}\,\vec{u}^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \; 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$I - 2\vec{u}\,\vec{u}^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(I - 2uu^T)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
$= \begin{pmatrix} x \\ -y \end{pmatrix}$

col(uu^T) / row(uu^T)



$x - 2Px$

$(I - 2P)x = x - 2Px$

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Question No 3

What are the four fundamental subspaces of $M = I - P$ in terms of the column space of P.

Solution

For a projection matrix P: (projection matrix is always symmetric)

- $x \in \text{col}(P) = \text{row}(P)$: Px=x
- $x \in \text{Nul}(P) = \text{LeftNul}(P)$: Px=0

For matrix $I - P$

- $x \in \text{col}(P) = \text{row}(P)$: (I-P)x=x-Px=x-x=0
- $x \in \text{Nul}(P) = \text{LeftNul}(P)$: (I-P)x=x-Px=x-0=x

is a also a projection matrix.
Left Null space = Right Null space = Colume space of P.
Column space = Row space = orthogonal complement of the colume space of P.

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Question No 4

P is a Projection Matrix, prove P is symmetric and $P^2 = P$. What is the eigenvalue of Projection matrix P. Prove that $I - 2P$ is an orthogonal matrix

Solution

$P = A(A^TA)^{-1}A^T$ then

- $P^2 = A\underbrace{(A^TA)^{-1}A^TA}_{=I}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$
- $P^T = (A(A^TA)^{-1}A^T)^T = A^T(A^TA)^{-T}A = A(A^TA)^{-1}A^T$ (For A^TA symmetric)

Eigenvalue is $1, 0$ (for $P^2 = P$ so eigenvalues should satisfies $\lambda^2 = \lambda$)
Since P is a projection matrix, we have $P = P^T$. To show that Q is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$QQ^T = (I - 2P)(I - 2P)^T$$

.

$$= (I - 2P)(I^T - 2P^T)$$

$$= (I - 2P)(I - 2P) \quad \text{(since } I \text{ and } P \text{ are symmetric)}$$

$$= I - 4P + 4P^2$$

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

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Question No 5

If $A^2 = -A$, what is the possible value of $\det(A)$.

Solution

$A^2 = -A$ means $\det(A^2) = \det(-A)$ however

- $\det(A^2) = \det(A)^2$ $\det(AB) = \det(A)\det(B)$ $\det(c \cdot A) = c^n A$
- $\det(-A) = (-1)^n \det(A) = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$ $A \in \mathbb{R}^{n \times n}$

Thus

$$\det(A)^2 = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$$

which means

$$\det(A) = \begin{cases} 0, -1 & \text{if } n \text{ is odd} \\ 0, 1 & \text{if } n \text{ is even} \end{cases}$$

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Question No 5

Suppose an $m \times n$ matrix A has rank r. What are the ranks of

(a) A^T?

(b) AA^T?

(c) $AA^T + \lambda I$ ($\lambda > 0$)?

(d) $A^T AA^T$?

Solution

Answer 1

(A) r

(B) we showed in class it's r (page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

(C) it's a positive definite matrix with all eigenvalues lareger than λ, think why.

(D) r (similar page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

Answer 2 Using SVD

(A) $\mathrm{rank}(A^T) = \dim(\mathrm{row}(A^T)) = \dim(\mathrm{col}(A)) = \mathrm{rank}(A) = r$.

(B) Let $A = U\Sigma V^T$ be a full SVD. Then,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T.$$

Thus, $U\Sigma^2 U^T$ is a SVD of AA^T. If Σ has r positive singular values then so will Σ^2. Therefore, the rank of AA^T is r.

(C) Since $I_m = UU^T$, the equation above yields $AA^T + \lambda I = U\Sigma^2 U^T + \lambda I = U(\Sigma^2 + \lambda I_m)U^T$. Since $\Sigma^2 + \lambda I = \mathrm{diag}(\sigma_1^2 + \lambda, \ldots, \sigma_r^2 + \lambda, \ldots, \lambda)$, the rank is m.

(D) $A^T AA^T = (U\Sigma V^T)^T (U\Sigma V^T)(U\Sigma V^T)^T = V\Sigma^T U^T U\Sigma V^T U^T U\Sigma V^T = V\Sigma^T \Sigma\Sigma^T V^T = V\Sigma^3 V^T$. Σ^3 has r positive singular values as like Σ. Therefore, the rank is r.

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Question No 6

The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

For a matrix A, the eigenspace with eigenvalue λ is the kernel of the matrix $A - \lambda I$. Here we have $\lambda = 1$, so we subtract I from each of the matrices above:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and find the dimensions of the kernels.
The ranks of these matrices are 0, 2, 2, 1 respectively, so by the rank-nullity theorem the dimensions of the kernels are 3, 1, 1, 2.
Answer: 3, 1, 1, 2.

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Question No 7

For $A \in \mathbb{R}^{n\times n}$ has singular value $\sigma_1, \cdots, \sigma_n$ prove

- $\operatorname{tr}(A^\top A) = \sigma_1^2 + \cdots + \sigma_n^2$
- $\operatorname{tr}((A^\top A + \lambda I)^{-1} A^\top A) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$

Very similar to $A(A^TA)^{-1}A^T$

$(\Sigma^T\Sigma + \lambda I)^{-1} \to \begin{bmatrix} \frac{1}{\sigma_1^2+\lambda} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2+\lambda} \end{bmatrix}$

$\| \quad \begin{bmatrix} \sigma_1^2+\lambda & & \\ & \ddots & \\ & & \sigma_n^2+\lambda \end{bmatrix}$

Solution

$\Sigma^T\Sigma = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$

Using SVD $A = U\Sigma V^\top$ Then we have

≈ Similar ↓

- $A^\top A = V\Sigma^\top\Sigma V^\top$ so $\operatorname{tr}(A^\top A) = \operatorname{tr}(\Sigma^\top\Sigma) = \sigma_1^2 + \cdots + \sigma_n^2$
- $(A^\top A + \lambda I) = V(\Sigma^\top\Sigma + \lambda I)V^\top$, $(A^\top A + \lambda I)^{-1} = V(\Sigma^\top\Sigma + \lambda I)^{-1}V^\top$

  diag   diag

- $(A^\top A + \lambda I)^{-1}A^\top A = V(\Sigma^\top\Sigma + \lambda I)^{-1}\Sigma^\top\Sigma V^\top = V \begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix} V^\top$
-

$$\operatorname{trace}((A^\top A+\lambda I)^{-1}A^\top A) = \operatorname{trace}\left(\begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix}\right) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$$

Examination Requirements: NIL

Projection.

$A\vec{x} = \vec{b}$ **Question** What is the nearest point of $\vec{b}$ in $Col(A)$

$A\vec{x} \in Col(A)$

answer. projection Matrix

$$P = A(A^TA)^{-1}A^T$$

behind. $A\vec{x} = \vec{b} \Rightarrow A^TA\vec{x} = A^T\vec{b} \Rightarrow \vec{x} = (A^TA)^{-1}A^T\vec{b}$ least square.

$\Downarrow$

$A\vec{x} = \underbrace{A(A^TA)^{-1}A^T}_{\text{projection matrix}}\vec{b}$

一. Hints (Exercise 2-4)

① $\|u\|=1$. $P = u(\underbrace{u^Tu}_{u^Tu=1})^{-1}u^T = uu^T$ $\|u\|=1$ $u^Tu = \|u\|^2 = 1$

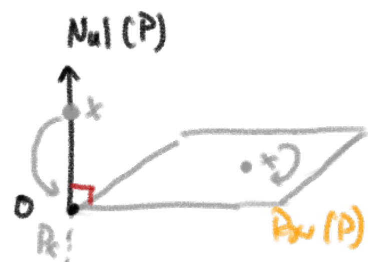
rank1 matrix. projection

② P is a Projection Matrix. $P^2 = P$ $P = P^T$

$P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{=I}A^T = A(A^TA)^{-1}A^T = P$

$P = A(A^TA)^{-1}A^T$

③ $P^2 = P$ means $\underbrace{P.(Px)}_{\text{projection 2 Time}} = \underbrace{Px}_{\text{Projection 1 time}}$. $\to Col(P)$



row (P)
$\|$
$- \; x \in Col(P)$ $Px = x$. x is eigenvector with eigenvalue 1

$Nul(P) \perp Row(P)$

$- \; x \in Nul(P)$ $Px = 0$
$\|$
left Nul (P) x is eigenvector with eigenvalue 0.

④ P only have eigenvalue 1 or 0

$P^2 = P \Rightarrow P^2 - P = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 1, 0$

△ If λ is the eigenvalue of P, Then $\lambda^2 - \lambda$ is the eigen $P^2 - P$

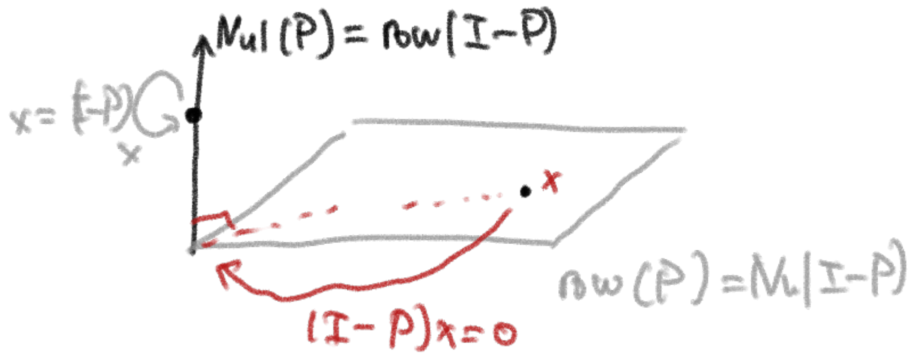
$Px = \lambda x \Rightarrow (P^2 - P)x = P^2x - Px = (\lambda^2x - \lambda x) = (\lambda^2 - \lambda)x$

⑤ $P = \underbrace{u_1u_1^T + u_2u_2^T + \cdots + u_ru_r^T}_{u_1 \cdots u_r \text{ is the orthonormal basis of } row(P)/Col(P)} + \underbrace{0 \cdot u_{r+1}u_{r+1}^T + \cdots + 0\, u_nu_n^T}_{u_{r+1} \cdots u_n \text{ is the orthonormal basis of } Nul(P)/\text{left Nul}(P)}$

⑥ $I = u_1 u_1^T + u_2 u_2^T + \cdots + u_r u_r^T + u_{r+1} u_{r+1}^T + \cdots + u_n u_n^T$

$u_1 \cdots u_n$ are I's eigenvector with eigenvalue 1

$$I - P = \underbrace{0 \cdot u_1 u_1^T + 0 \cdot u_2 u_2^T + \cdots + 0\, u_r u_r^T}_{u_1, u_2, \cdots, u_r \text{ is the orthonormal basis of } Nul(I-P) \,/\, \text{left } Nul(I-P)} + \underbrace{u_{r+1} u_{r+1}^T + \cdots + u_n u_n^T}_{u_{r+1} \cdots u_n \text{ is the orthonormal basis of } Col(I-P) \,/\, Row(I-P)}.$$



Gram - Schmidt Process.

- $\{x_1 \cdots x_n\}$ basis → $\{u_1 \cdots u_n\}$ orthogonal,

Hint. Orthogonal matrix → Column is orthonormal

If you want orthogonal, you need to normalize your vector to unit vector!

- QR Decomposition. $R = Q^T A$

$A = QR$, $\quad A^T A = R^T \underbrace{Q^T Q}_{I} R = R^T R$

lower upper

But it's not LU Decomposition!

↓

LU need diag of L to be 1.

Diagonalization & Eigen Vectors.

$A = X \Lambda X^{-1}$ $A \in \mathbb{R}^{n \times n}$

- $X = [x_1 \cdots x_n]$ $x_1 \cdots x_n$ are linear independent Eigen vectors

This is not always true.

- In the case, $\lambda_1, \lambda_2, \cdots \lambda_n$ are all different numbers. this is true. (Excise 6)

- Example. $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $p(\lambda) = (\lambda - 1)^2$

$\lambda_1 = 1$ $\lambda_2 = 1$

$\Rightarrow A - I = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \rightarrow$ 1 dimension eigen space.

- $A^k = X \Lambda^k X^{-1}$ Excise 1

- $A = \mathbb{X} B \mathbb{X}^{-1}$ similar matrix

same eigenvale. $\rightarrow$ different eigenvectors.

same trace

same det.

$Ax = \lambda x$

$\mathbb{X} B \mathbb{X}^{-1} x = \lambda x$

$\Rightarrow B \mathbb{X}^{-1} x = \lambda \mathbb{X}^{-1} x$

$\Rightarrow \mathbb{X}^{-1} x$ is the eigen

$\Downarrow$

hint us the way to compute the matrix $\mathbb{X}$. (If we know A and B)

Symmetric Matrix

- Eigenvalue is real, can always be diagonalized
  Eigenvectors is orthogonal to each other.
  $Q^T = Q^{-1}$
- $A = Q \Lambda Q^T$ ⇒ A is similar to Λ

  Q is an orthogonal matrix

  eigenvalue eigenvectors
- $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$

  rank 1
  symmetric
  projection Matrix.

SVD

Any Matrix $A \in \mathbb{R}^{m \times n}$

- $\underset{m \times n}{A} = \underset{m \times m}{U} \, \underset{m \times n}{\Sigma} \, \underset{n \times n}{V^T}$

  orth diag orth

symmetric $AA^T = U \Sigma \Sigma^T U^T$, U : eigenvectors of AA^T

symmetric $A^TA = V \Sigma^T \Sigma V^T$, V : eigenvector of A^TA

- $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

⇒ U, V can provide orthonormal basis of the Four fundamental subspace of A.

- How to compute SVD
  - Use. U : eigenvector of AA^T or
    V : eigenvector of A^TA
  - use
    $u_1 = \frac{1}{\sigma_1} A v_1$ or $v_1 = \frac{1}{\sigma_1} A^T u_1$

- Properties of Det
- Cofactor.
- Check if a Transform is linear Transform (HW6)

Orthogonal Matrix $Q \in \mathbb{R}^{n \times n}$ square matrix

$$Q^T Q = I \iff Q Q^T = I$$

both means $Q^T = Q^{-1}$

Orthogonal Matrix $Q \in \mathbb{R}^{n \times m}$

means Q : orthonormal basis

$Q^T Q$ $m \times m$ matrix $= I$ eigen $\underbrace{1 \cdots 1}_{m}$

$\Downarrow$

$Q Q^T$ $n \times n$ matrix eigen $\underbrace{1 \cdots 1}_{n}$ $\underbrace{0 \cdots 0}_{m-n}$

means QQ^T is projection!