NYU

Linear Algebra

LS $\xrightarrow{\text{Elimination}}$ Uppertriangular Form $\xrightarrow{\text{Solve}}$ Solution

Lecture 5
Inverse Matrices

LU Decomposition

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Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Idea of Inverse Matrices

The idea of Inverse Matrices

① A have an inverse A^{-1} mean

LS $Ax=b$ have a unique solution $x=A^{-1}b$ for all vector b

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

$$\underset{\mathbb{R}^{n\times n}}{A}\ \underset{\mathbb{R}^{n\times p}}{X} = \underset{\mathbb{R}^{n\times p}}{B}$$

$$X = A^{-1}B \ ✓ \qquad \cancel{BA^{-1}} \ ✗$$

$\mathbb{R}^{n\times n}$ $\mathbb{R}^{n\times p}$ $\mathbb{R}^{n\times p}$ $\mathbb{R}^{n\times n}$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

$$AX = B$$
$$\Downarrow$$
$$\underbrace{A^{-1}(A}_{I}X) = A^{-1}B$$
$$\Downarrow$$
$$X = A^{-1}B$$

$$Ax = b$$
$$\Downarrow$$
$$\underbrace{A^{-1}(A}_{I}x) = A^{-1}b \Rightarrow \underbrace{I \cdot x}_{\substack{\| \\ x}} = A^{-1}b$$

The idea of Inverse Matrices

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1}$$

! order

$(AB)^{-1}AB = I$

$= \underbrace{B^{-1}A^{-1}}\, AB$

$= B^{-1} \underbrace{A^{-1}A}_{I} B$

$= B^{-1}IB = \underbrace{B^{-1}B}_{I.} = I$

$AB(AB)^{-1} = I$

$\| $

$A\underbrace{BB^{-1}}_{I}A^{-1}$

$\|$

AA^{-1}

$\|$

I

$ABx = b$

$\underbrace{A^{-1}(AB}_{I}x) = A^{-1}b$

$\Rightarrow Bx = A^{-1}b$

$\Downarrow$

$x = B^{-1}(A^{-1}b)$

$= \underbrace{B^{-1}A^{-1}}_{(AB)^{-1}}b$

$\Rightarrow$

$A\overset{y}{Bx} = b$

$Ay = b$

$y = A^{-1}b$

Bx

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix $(A+B)$ is not necessarily invertible.

$x=1 \qquad y=-1 \qquad x+y=0$

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B=\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \qquad A+B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$ provided that $d_{ii} \neq 0$.

Inverse of an Elimination Matrix

Consider the elimination matrix

Lower Traingular Matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

a_{31}

E_{31} x

Replace x Row (3) with

$c \cdot$ Row (1) + Row (3) = Row (3')

$(E_{31})^{-1}$ change

$c \cdot$ Row (1) + Row (3) = Row (3')

back to Row (3)

inverse

which adds c copies of the first row to the third row. Then,

Subtract $c \cdot$ Row (1) from Row (3')

Elimination Matrix

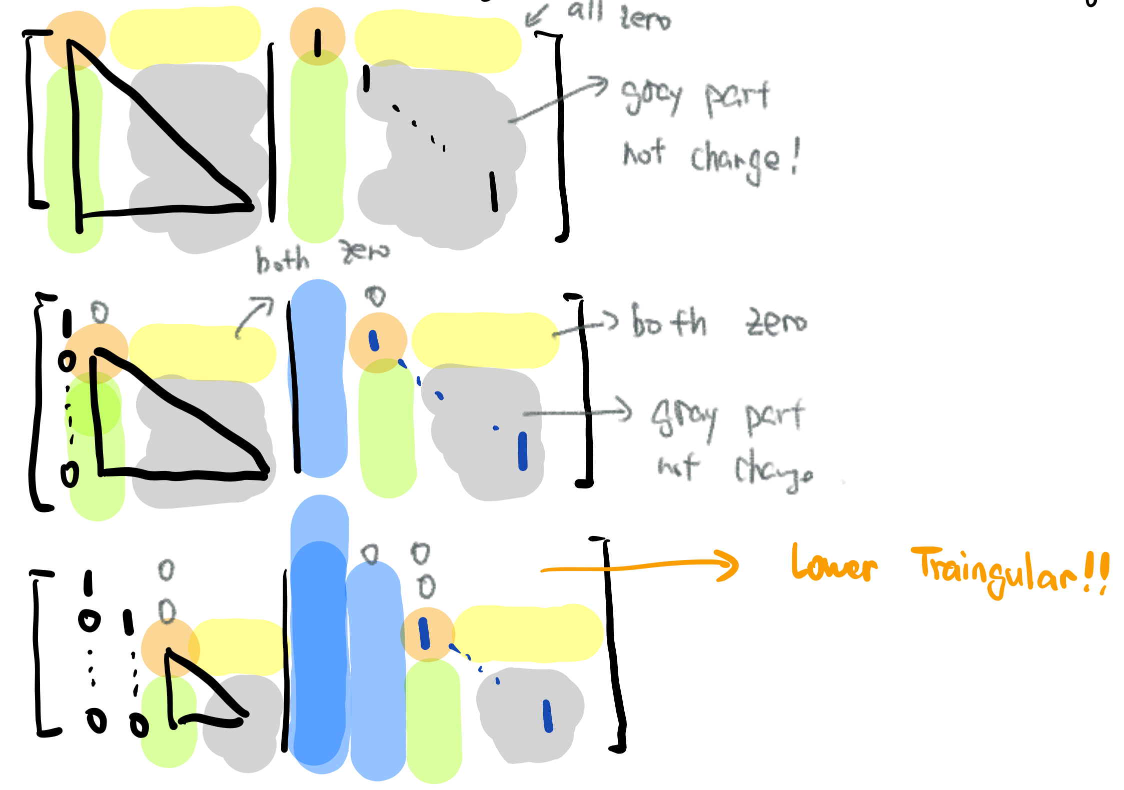
Lower - - - -

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -c & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

a_{31}

Goal

Inverse of Lower Traingular Matrix is a Lower Triangular Matrix



Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

Permute 2-Rows!

row (3) with row (4)

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Longrightarrow P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P_{34} = P_{34}^T$$

a_{34}, a_{43}

New!!

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P^T$$

$Px \Rightarrow$

row (1)	→ row (4)	→	$a_{14} = 1$
row (2)	(2)	→	$a_{22} = 1$
(3)	(1)	→	$a_{31} = 1$
(4)	(3)	→	$a_{43} = 1$

P^{-1}

row (4)	→ row (1)	→	$a_{41} = 1$
(2)	(2)		$a_{22} = 1$
(1)	(3)		$a_{13} = 1$
(3)	(4)		$a_{34} = 1$

More on the Transpose of a Matrix

Recall

The transpose of an $m \times n$ matrix A is denoted by A^T, and it has entries $a^T_{ij} = a_{ji}$. That is, the columns of A^T are the rows of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \implies A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Properties of the Transpose

sum: $(A + B)^T = A^T + B^T$ (v)

$$\underbrace{\mathbb{R}^{m\times n}}\quad \mathbb{R}^{n\times p}$$

product: $(AB)^T = B^T A^T$!order

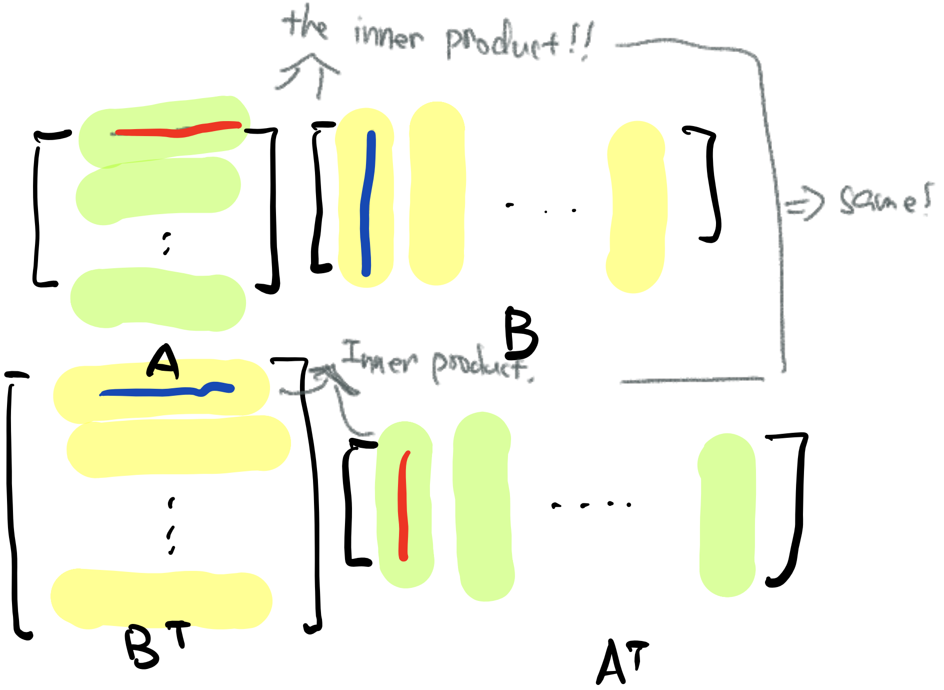
$$\underbrace{\mathbb{R}^{m\times p}}_{\mathbb{R}^{p\times m}} = \underbrace{\mathbb{R}^{p\times n}\ \mathbb{R}^{n\times m}}_{\mathbb{R}^{p\times m}}$$

inverse: $(A^T)^{-1} = (A^{-1})^T$

$(A^T)^{-1} \cdot A^T$

$(A^{-1})^T \cdot A^T = (\underbrace{A \cdot A^{-1}}_{I})^T = I^T = I$

$A^T \cdot (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I$



Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

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Goal

$$\underbrace{A = L \cdot U}$$

LU decomposition

L: lower triangular

U: upper triangular.

Example.

$$\begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}}^{\text{Elimination process}} \overbrace{\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix}}^{\text{upper triangular form}}$$

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

lower traingular Matrix

upper traingular!!

$$E_{21} A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

$\text{row}(2) \Leftarrow -\frac{a_{21}}{a_{11}} \cdot \text{row}(1) + \text{row}(2)$

$\underbrace{E_{21}}_{L^{-1}} A = U$

$A = E_{21}^{-1} U$

L

Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21}.

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}_{d} \end{bmatrix}$$

If $a_{11} = 0$, but $a_{21} \neq 0$, we have to permute first. If both a_{11} and a_{21} are zero, then the matrix is already upper triangular.

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$A \xrightarrow{1} E_{21}A \xrightarrow{2} E_{31}E_{21}A \xrightarrow{3} E_{32}E_{31}E_{21}A$$

$$E_{21} = \begin{bmatrix} 1 & & \\ -\frac{a_{21}}{a_{11}} & 1 & \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b_{32}}{b_{22}} & 1 \end{bmatrix}$$

b_{ij} is from $E_{31}E_{21}A$

$$E_{21}A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} * & * & * \\ 0 & b_{22} & * \\ 0 & b_{32} & * \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}}_{L^{-1}}A = \underbrace{\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}}_{U!}$$

$A = L \cdot U$!!!!

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

Computing U – 3×3 case

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31}.

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

If $b \neq 0$, then we make it second pivot and use it to eliminate d.

$$\underbrace{E_{32}E_{31}E_{21}}_{L^{-1}!}A = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{d}{b} & 1 \end{bmatrix}}^{E_{32}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & b & c & \cdots & d \\ 0 & 0 & e & \cdots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & g \end{bmatrix}$$

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & b & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & & & & \\ 0 & * & * & \cdots & * \end{bmatrix}$$

step2

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

Step1 : (n-1) Elimination Matrix : $E_{n1} \cdots E_{21}$!! order. operate first

a_{11} pivot $\longrightarrow E_{21}A \to E_{31}E_{21}A \to E_{41}E_{31}E_{21}A \to \underbrace{E_{n1}\dots E_{41}E_{31}E_{21}A}_{B}$

Step 2: (n-2) Elimination Matrix

b pivot $\longrightarrow E_{32}B \to E_{42}E_{32}B \to E_{52}E_{42}E_{32}B \to \underbrace{E_{n2}\dots E_{52}E_{42}E_{32}B}_{C}$

Step 3: (n-3) Elimination Matrix

e pivot $\longrightarrow E_{43}C \to E_{53}E_{43}C \to E_{63}E_{53}E_{43}C \to E_{n3}\dots E_{63}E_{53}E_{43}C$

$\vdots$

note that we're assuming we can find a pivot without having to use permutations

Computing L

2×2 case: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $U = E_{21}A$.

$$\implies A = \underbrace{E_{21}^{-1}}_{L} U$$

3×3 case: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $U = E_{32}E_{31}E_{21}A$.

$$\implies A = (E_{32}E_{31}E_{21})^{-1}U$$

$$= \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L} U$$

Goal

$$\begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_{11}x_1 \\ \vdots \\ d_{nn}x_n \end{pmatrix}$$

$$A = L\,D\,U$$

$$L = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \qquad D = \begin{pmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Lower Triangular but diag are 1

diagonal Matrix

Upper Triangular but diag are 1

Find LU decomposition first.

$$\begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ & 1 & 2 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & 1/3 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 & 0 \\ & 3/2 & 1 \\ & & 4/3 \end{bmatrix}} = L \underbrace{\begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 1/2 & \\ & 1 & 2/3 \\ & & 1 \end{bmatrix}}$$

LDU decomposition!

Questions?