NYU

Lecture 7

Vector Spaces and Subspaces

Dr. Ralph Chikhany

If M and $N \in \mathbb{R}^{s \times s}$. $MN = N$, Then $M = I$ (X)

- $S=1$. $m, n \in \mathbb{R}$ $m \cdot n = n \rightarrow m = 1$

  this is wrong because n can be zero!

- General S, If N = all zero matrix. then M can be any matrix

  If N have an inverse matrix N^{-1}

  $$M \underbrace{N N^{-1}}_{I} = \underbrace{N N^{-1}}_{I} \Rightarrow M = I.$$

What will happen if N don't have an inverse!

What is all possible M. $MN = N$ $\quad \underbrace{MN} - \underbrace{N} = 0$

$MN - I \cdot N = (M - I) \cdot N$

$\Updownarrow$

$\vec{x} \cdot N = \vec{0}$ and what is all $\vec{x}$ Today!

Vector subspace!

Strang Sections 3.1 – Spaces of Vectors

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Vector Spaces

Vector Spaces

Closed in operation → all the linear combination!!

Informal V is a vector space

if $v_1, \dots, v_n \in V$

then $c_1 v_1 + \cdots + c_n v_n \in V$

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

(VS1) For all $v, w \in V$, $v + w = w + v$. communitative

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$. associative.

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

(VS4) For each element $v \in V$, there exists an element $w \in V$, s.t. $v + w = 0$.

$w = -v$

addition!!

$\mathbb{R}^d$ is a vector space!!!!

Vector Spaces

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

→ scalar - vector multiplication.

(VS5) For each element $v \in V$, $1v = v$.

(VS6) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$.

(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

distributive law

Vector Spaces

(VS1) For all $v, w \in V$, $v + w = w + v$.

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

(VS4) For each element $v \in V$, there exists an element $w \in V$, s.t. $v + w = 0$.

(VS5) For each element $v \in V$, $1v = v$.

(VS6) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$.

(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

Note: All elements in the field $\mathbb{R}$ are called scalars and all elements in the vector space V are called vectors.

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

all functions. will provide a vector space.

$$\overbrace{(f+g)}^{\text{function}}(s) = f(s) + g(s) \quad \text{and} \quad \overbrace{(cf)}^{\text{function}}(s) = c\boxed{f(s)}$$

for all $s \in S$. function f + function g scalar c × function f

VS 1. $f + g = g + f$ $(f+g)(s) = f(s) + g(s) = \underline{g(s) + f(s)} = (g+f)(s)$

VS 2. $(f+g) + h = f + (g+h)$ Similar ! !

zero function.

VS 3. $f + 0 = f$. for any s. $0(s) = 0$. $(f+0)(s) = f(s) + 0(s)$

$= f(s) + 0 = f(s)$

VS 4. there exist g s.t. $f + g = 0$.

define my g for any s. $g(s) = -f(s)$, $(f+g)(s) = f(s) + g(s)$

$= f(s) + (-f)(s) = 0 = 0(s)$

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

vector

Example. 1) $\{v \mid Av = 0\}$ is a vector space! A is a matrix

2) $\{v \mid Av = b\}$ is not a vector space! $b \neq \vec{0}$

if $v_1, v_2 \in V. \Rightarrow c_1 v_1 + c_2 v_2 \in V$

1) $v_1, v_2 \in V$ means $Av_1 = 0, \quad Av_2 = 0$

whether $c_1 v_1 + c_2 v_2 \in V \Leftrightarrow$ asking: whether $A(c_1 v_1 + c_2 v_2) = 0 \ \vec{b}$ is true

$A(c_1 v_1 + c_2 v_2) = c_1 \underbrace{Av_1}_{\vec{0}}^{\vec{b}} + c_2 \underbrace{Av_2}_{\vec{0}}^{\vec{b}} = c_1 \vec{0} + c_2 \vec{0} = \vec{0}$

$= (c_1 + c_2)\vec{b} \leftarrow$ not always true when $c_1 + c_2 \neq 1$

Subspaces

Solving Systems of Equations

A set $W \subset V$ is a subspace of a vector space V if for all vectors $v, w \in W$ and $c \in \mathbb{R}$ if

(1) $v + w \in W$

(2) $cv \in W$

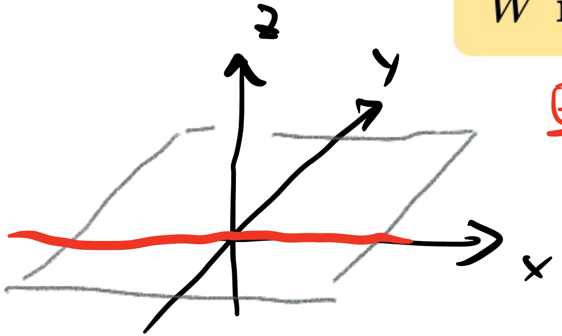
(3) $\mathbf{0} \in W$

$W \subseteq V$

vector space — subset — vector space

- W is a subspace of V

W itself is a vector space.



Ex. x-y plane is a vector space

x-axis is a vector space

$\Rightarrow$ x-axis is a subspace of x-y plane!

Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

2x2 Matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad c_1\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + c_2\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} c_1a_1 + c_2a_2 & c_1b_1 + \cdots \\ \cdots & \cdots \end{pmatrix}$

- all the 2x2 matrix is a vector space
- all the 2x2 diag is a vector space $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$$c_1\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} + c_2\begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} = \underbrace{\begin{pmatrix} c_1a_1 + c_2a_2 & 0 \\ 0 & c_1d_1 + c_2d_2 \end{pmatrix}}_{\text{diag Matrix}}$$

→ linear Comb of diag Matrix is diag matrix

- 2x2 diag matrix is a subspace of 2x2 matrix

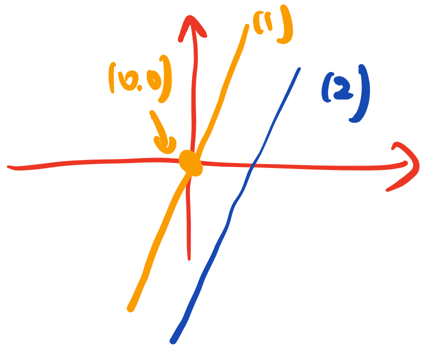
Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

Ex. $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $A \cdot v = v_1 + 2v_2$

1) $\left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| v_1 + 2v_2 = 0 \right\}$

2) $\left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| v_1 + 2v_2 = b \right\}$ $b \neq 0$



Vector space means "line/plane" go through the origin !!

[by definition. $\vec{0} \in V$)

VS 3 !

$v_1 \cdots v_n \in V$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n \in V$

Null space !

!!! $\{ x \mid Ax = 0 \}$ is a vector space

$\{ x \mid Ax = b \}$, $b \neq 0$ is not a Vector space

$\{ Ax \mid x \in \mathbb{R}^n \}$ is a vector space

$\parallel$

$\mathrm{Col}(A) = \mathrm{span}\{ \vec{a}_1 \cdots \vec{a}_n \}$, $A = [\vec{a}_1 \cdots \vec{a}_n]$

Column Space

Column Space

Column Vectors !!

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, such that $A = [\vec{a}_1 \;\; \vec{a}_2 \;\; \dots \;\; \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$. The column space of A consists of all possible linear combinations of the columns of A. That is,

$\vec{v}_1 = c_{11}\vec{a}_1 + \cdots + c_{1n}\vec{a}_n$

$\vec{v}_2 = c_{21}\vec{a}_1 + \cdots + c_{2n}\vec{a}_n$

$(\vec{v}_1, \vec{v}_2 \in \text{span}\{\vec{a}_1, \dots \vec{a}_n\})$

a vector space?

Column space. $\Rightarrow$

$$\text{Col}\, A = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$$

$\text{span}\{\vec{a}_1, \dots \vec{a}_n\} = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

$d_1\vec{v}_1 + d_2\vec{v}_2 = (d_1 c_{11} + d_2 c_{21})\vec{a}_1 + (d_1 c_{12} + d_2 c_{22})\vec{a}_2 + \cdots + (d_1 c_{1n} + d_2 c_{2n})\vec{a}_n$

To solve $A\vec{x} = \vec{b}$, you must express $\vec{b}$ as a linear combination of the columns of A. Thus, $\vec{b}$ has to be in the column space of A, otherwise we won't be able to find a solution for the system $A\vec{x} = \vec{b}$.

another linear combination of $\vec{a}_1 \dots \vec{a}_n$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$

all linear comb of column vectors

$\Rightarrow$ means $d_1\vec{v}_1 + d_2\vec{v}_2 \in \text{span}\{\vec{a}_1 \dots \vec{a}_n\}$

$\Rightarrow$ span is a vector space!

Column Space

The column space of A is a subspace of $\mathbb{R}^m$.

for linear system $Ax = b$ have solution $\Leftrightarrow$ b lies in the column space of A

$b \in \mathrm{Col}(A)$

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n] = \left[\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right]$$

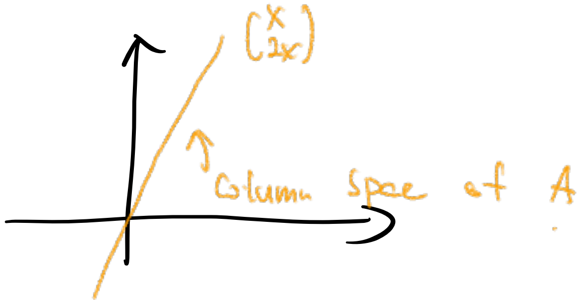
$$\text{Therefore, } \mathrm{Col}\, A = \mathrm{span} \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\} \subset \mathbb{R}^m.$$

Example – Describe the Column Space of A

Example.

$A = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$

$\vec{c_1} = \begin{pmatrix}1\\2\end{pmatrix}$ $\quad \vec{c_2} = \begin{pmatrix}-3\\-6\end{pmatrix}$ $\quad \vec{c_2} = -3\vec{c_1}$



$\text{span}\{\vec{c_1}, \vec{c_2}\} = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}, \Leftarrow \left\{\begin{pmatrix}x\\2x\end{pmatrix} \mid x \in \mathbb{R}^n\right\}$

$\| $

$\text{Col}(A)$

Example – Describe the Column Space of A

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

"upper traingular Form"

extra Vector

these are three pivots!!

What is the Column Space of A (?)

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \right\} = \mathbb{R}^3$$

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{matrix} \leftarrow b_1 \\ \leftarrow b_2 \\ \leftarrow b_3 \end{matrix}$$

$$\Rightarrow \text{Col}(A) = \mathbb{R}^3$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

} ⇒ Span all possible Values!

← My third axis is 0

$$\text{Col}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \middle| \; x_1, x_2 \in \mathbb{R} \right\}$$

In Quiz 2 If $Ax = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ have Solution? No

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \notin \text{Col}(A)$, $\Rightarrow Ax = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ don't have a solution.