Lecture 2

Vectors and Spans

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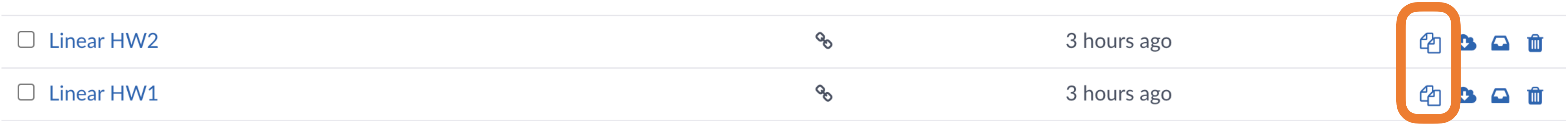
Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time).
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time).
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Latex -> Overleaf -> Copy



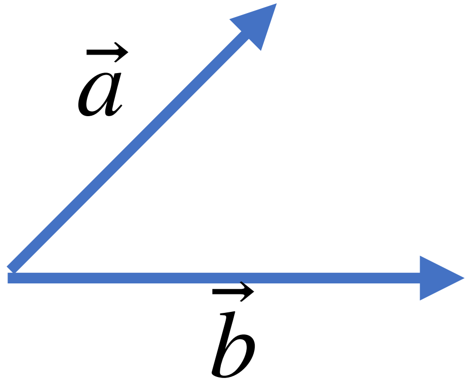
You can put what you want to recap in the [(anonymous) form].

ReCap

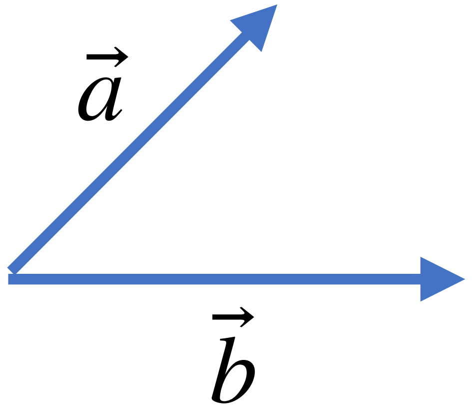
Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Vector Addition

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$



$$\vec{a} + \vec{b} = ?$$



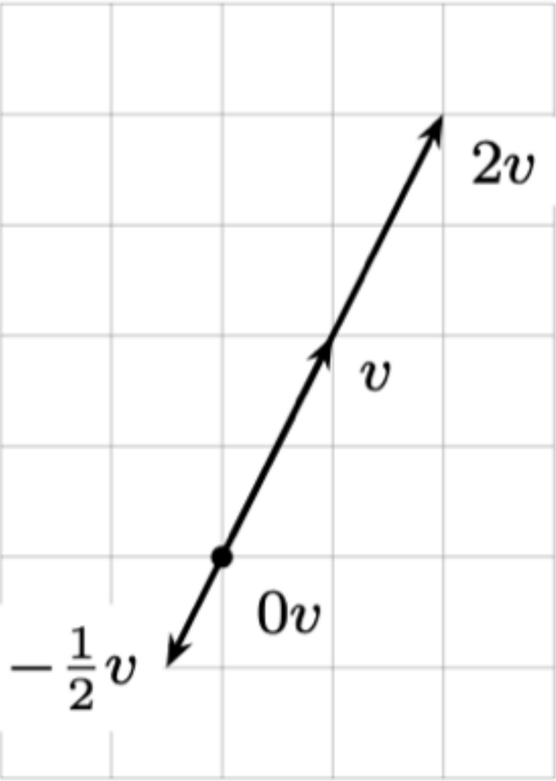
$$\vec{a} - \vec{b} = ?$$

$$\vec{a} - \vec{a} = ?$$

Scalar vector multiplication

$$\color{red}{c}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = $$

Some multiples of v.



Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

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Example 3 Dot products enter in economics and business. We have three goods to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. *Selling* q_1 *units at the price* p_1 *brings in* $q_1 p_1$. The total income (quantities q times prices p) is ***the dot product*** $\boldsymbol{q} \cdot \boldsymbol{p}$ ***in three dimensions***:

$$\textit{Income} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3 = \textit{dot product}.$$

Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

Length

Distance

$\|\vec{v}\|$ And $|c|$

Unit Vector:

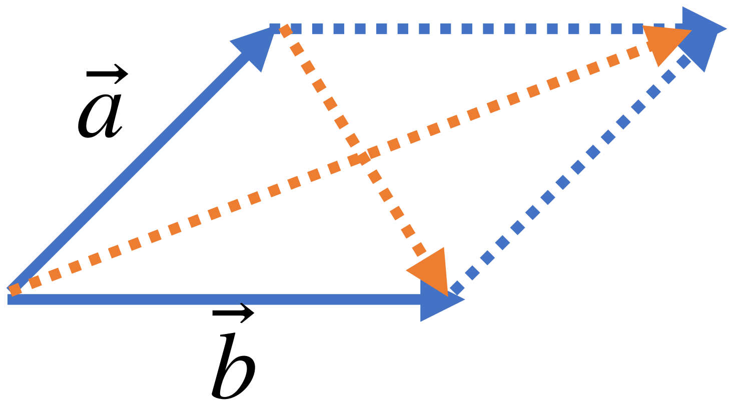
What is the unit vector of $(1,1)$?

Dot Product

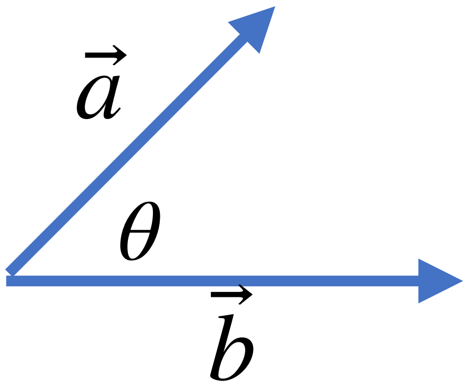
Communicative $\vec{a} \cdot \vec{b} =$

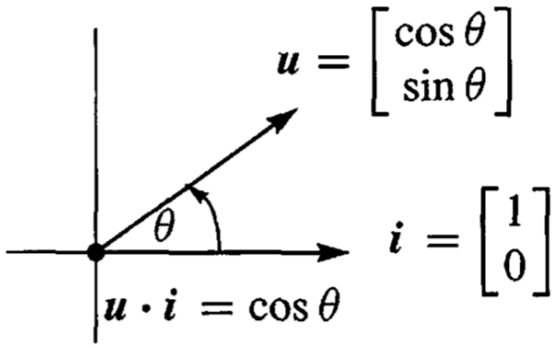
Distributive $(\vec{a} + \vec{b}) \cdot \vec{c} =$

Example $\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 =$



Angle

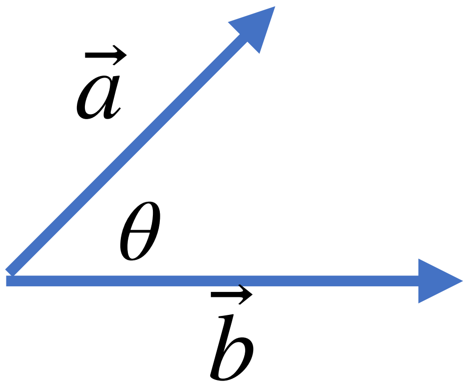


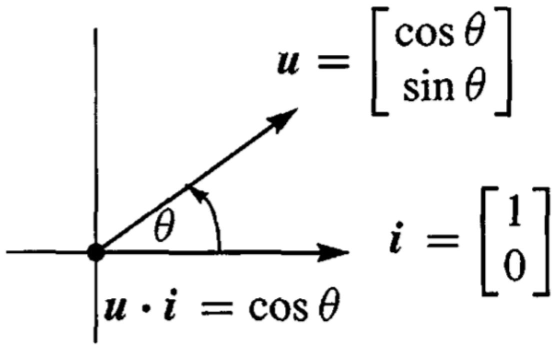


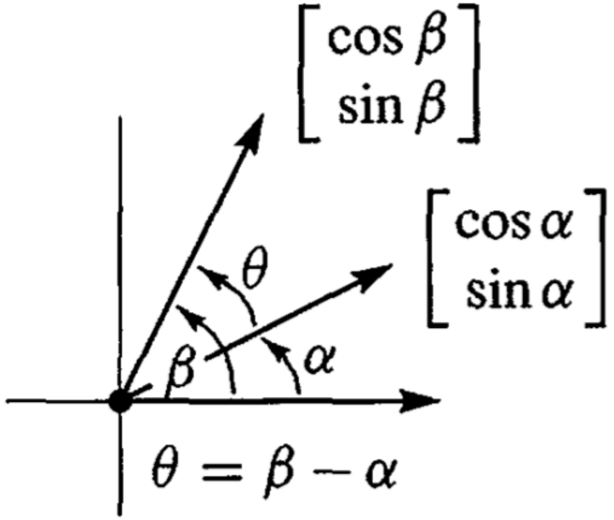
How to decide whether θ is larger than $\frac{\pi}{2}$

What is the unit vector parallel/orthogonal to $(4,3)$?

Angle







calculate $\cos(\beta - \alpha)$

Example

1.2 C Find a vector $\boldsymbol{x} = (c, d)$ that has dot products $\boldsymbol{x} \cdot \boldsymbol{r} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{s} = 0$ with the given vectors $\boldsymbol{r} = (2, -1)$ and $\boldsymbol{s} = (-1, 2)$.

How is this question related to Example **1.1 C**, which solved $c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b} = (1, 0)$?

1.1 C Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Inequalities

SCHWARZ INEQUALITY $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$

TRIANGLE INEQUALITY $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

Example 6 The dot product of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality in this case says that $2ab \leq a^2 + b^2$.

Reminder: Linear Combination

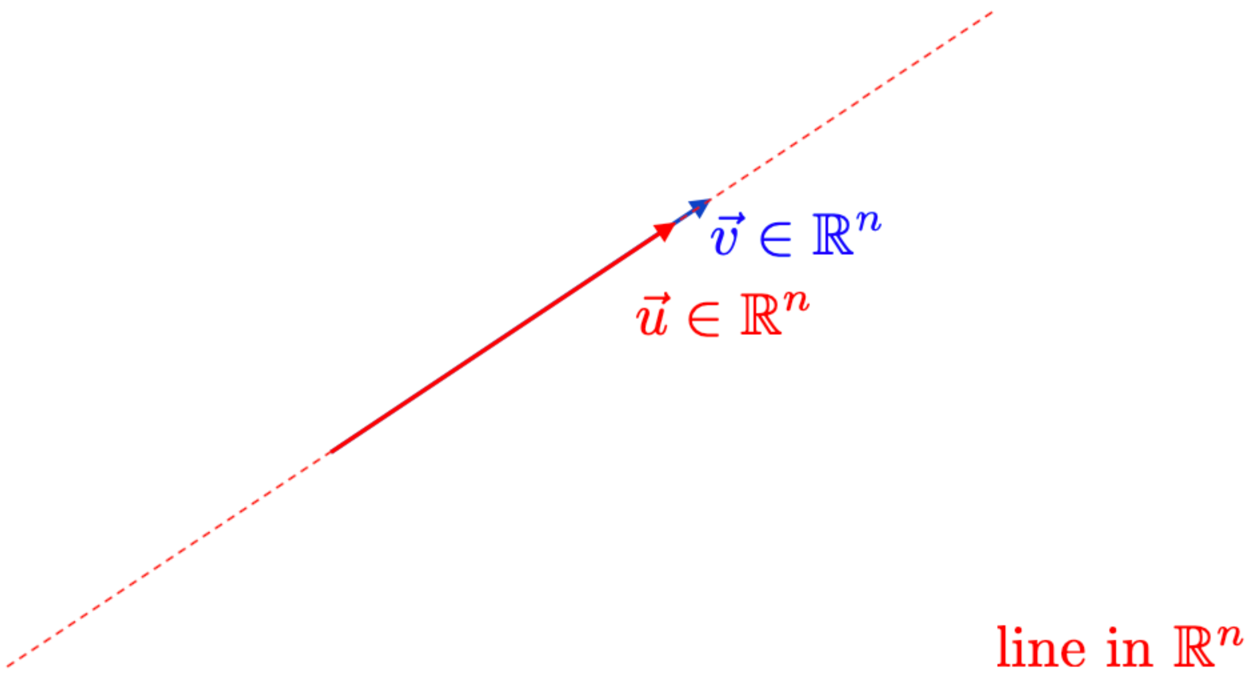
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

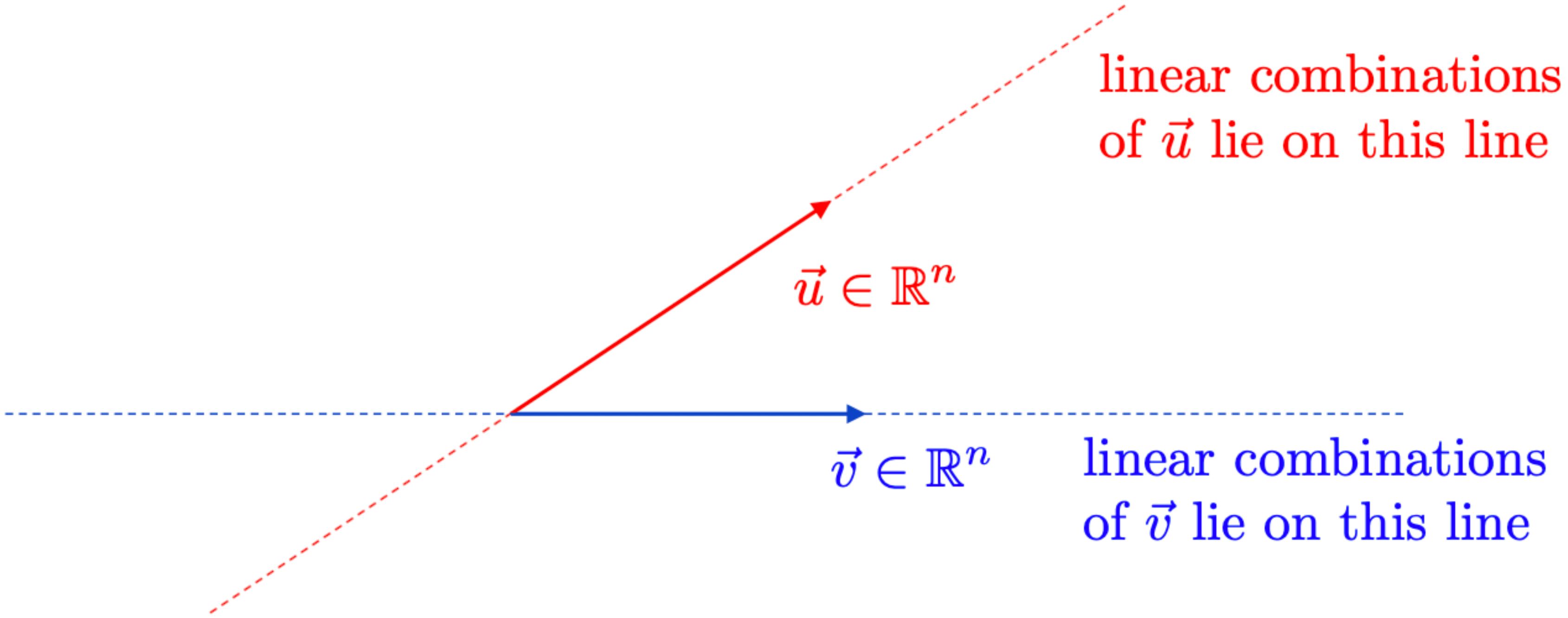
Definition

We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Geometric Interpretation of Linear Combinations



Geometric Interpretation of Linear Combinations



linear combinations of $\vec{u}$ and $\vec{v}$ lie on a plane in $\mathbb{R}^n$

Transfer Linear Equation to a Linear Combination Problem

$$2x + y = 1$$
$$x + y = 1$$

Spans

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Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

For example, what is the span of $(2, -4)$ and $(1, 1)$?

Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

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Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

Is $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

More Precise Definition

"the set of" "such that"

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

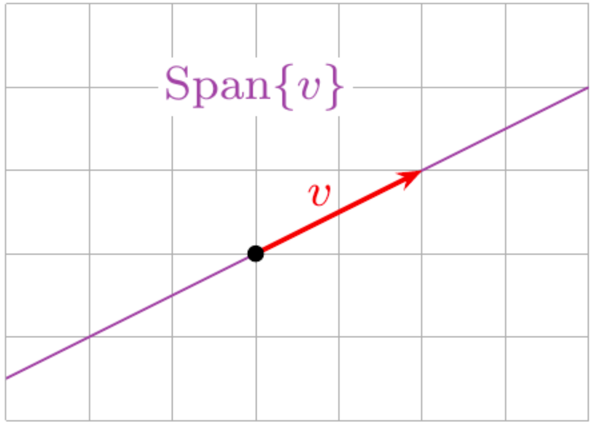
$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1v_1 + x_2v_2 + \cdots + x_pv_p \;\middle|\; x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \right\}.}$$

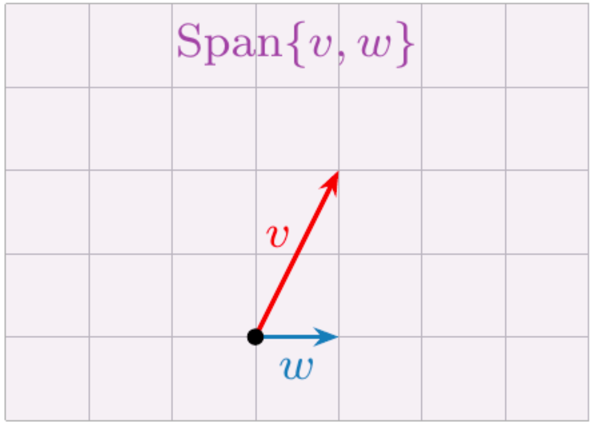
Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

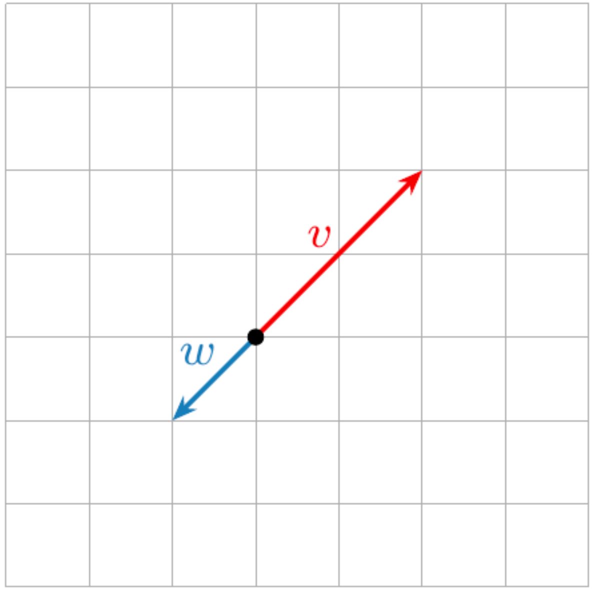
This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Span in $\mathbb{R}^2$

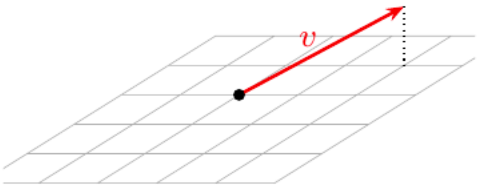
Drawing a picture of $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.







Span in $\mathbb{R}^3$



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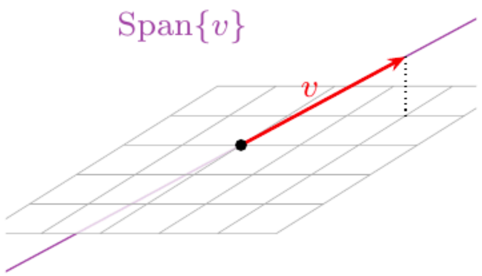

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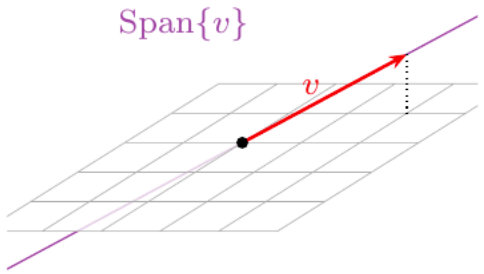

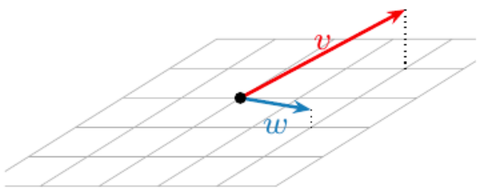

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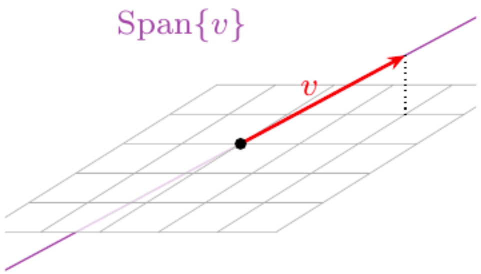

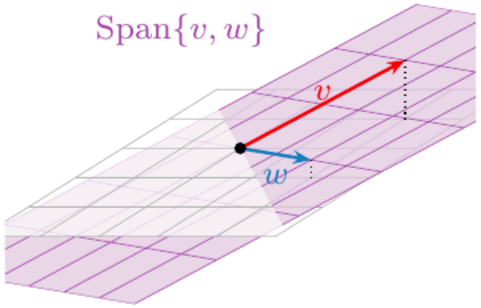

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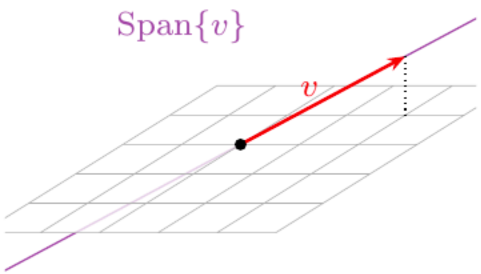
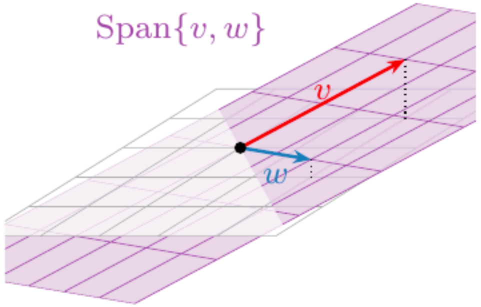
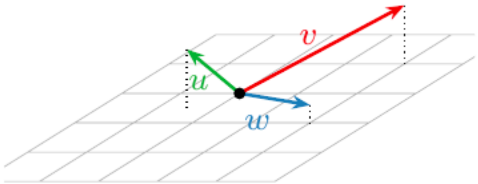
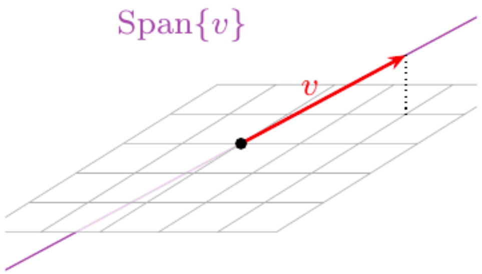
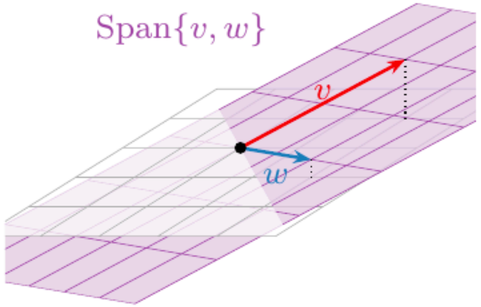


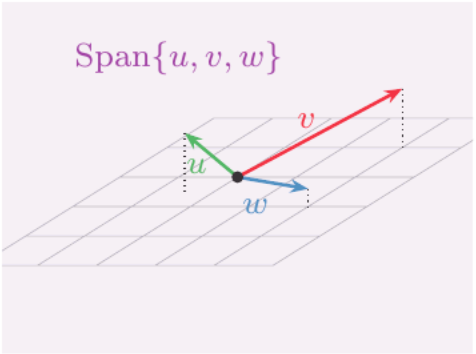




Span in $\mathbb{R}^3$







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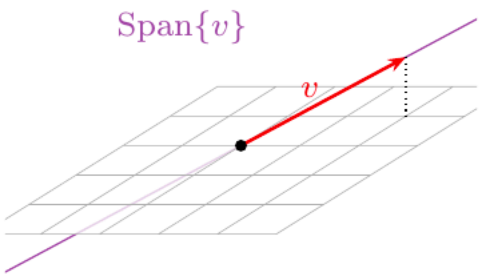
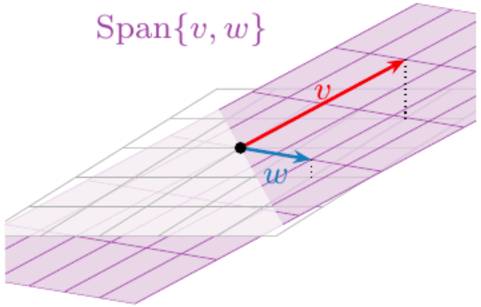
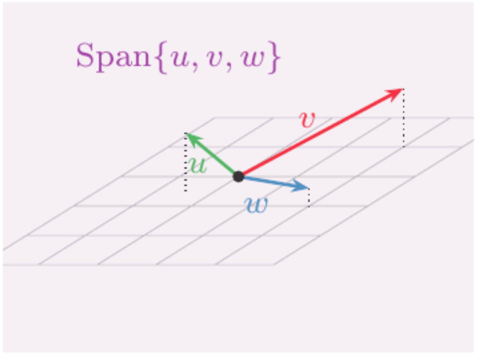
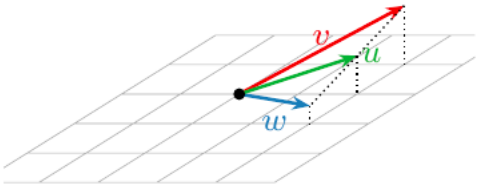
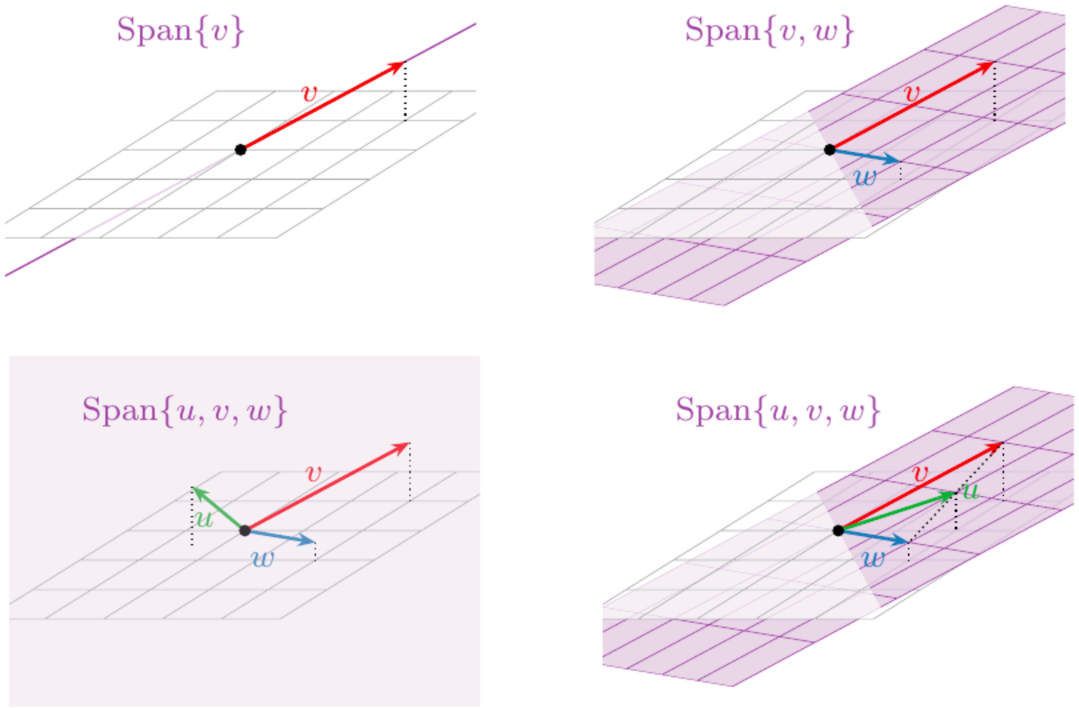








Span in $\mathbb{R}^3$



NYU

Questions?