

Lecture 5
Inverse Matrices

Dr. Yiping Lu

Recap

Block multiplication If the cuts between columns of A match the cuts between rows of B , then block multiplication of AB is allowed:

$$\begin{matrix} m_1 \\ m_2 \end{matrix} \begin{matrix} n_1 & n_2 \\ \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \end{matrix} \begin{matrix} k_1 \\ \left[\begin{array}{cc} B_{11} & \cdots \\ B_{21} & \cdots \end{array} \right] \end{matrix} \begin{matrix} n_1 \\ n_2 \end{matrix} = \begin{matrix} n_1 & k_1 & n_2 & k_1 \\ \begin{matrix} m_1 \\ m_2 \end{matrix} \left[\begin{array}{cc} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{array} \right] \end{matrix}. \quad (1)$$



Strang Sections 2.5 – Inverse Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed),
N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by
Margalit and Rabinoff, in addition to our text



The Idea of Inverse Matrices

The idea of Inverse Matrices

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

The idea of Inverse Matrices

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.


Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

In this case, B is the **inverse** of A , and is written A^{-1} .

identity matrix


$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:



Properties of Inverses

Inverse of a Product

Theorem: If A and B are invertible, then AB is invertible, with

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of the sum of Matrices

In general, even if both A and B are invertible matrices of the same size, the matrix $(A + B)$ is not necessarily invertible.

Inverse of a Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix} \text{ provided that } d_{ii} \neq 0.$$

Inverse of an Elimination Matrix

Consider the elimination matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \textcolor{red}{c} & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which adds c copies of the first row to the third row. Then,

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -\textcolor{red}{c} & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse of a Permutation Matrix

The inverse of a permutation matrix is its transpose.

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \implies P_{34}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \implies P^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Inverse of upper triangular matrix

Block multiplication If the cuts between columns of A match the cuts between rows of B , then block multiplication of AB is allowed:

$$\begin{matrix} m_1 \\ m_2 \end{matrix} \begin{bmatrix} \overset{n_1}{A_{11}} & \overset{n_2}{A_{12}} \\ \overset{n_1}{A_{21}} & \overset{n_2}{A_{22}} \end{bmatrix} \begin{bmatrix} \overset{k_1}{B_{11}} & \cdots \\ B_{21} & \cdots \end{bmatrix} \overset{n_1}{=} \begin{bmatrix} \overset{n_1}{\overset{k_1}{A_{11}B_{11}}} + \overset{n_2}{\overset{k_1}{A_{12}B_{21}}} & \cdots \\ \overset{n_1}{\overset{k_1}{A_{21}B_{11}}} + \overset{n_2}{\overset{k_1}{A_{22}B_{21}}} & \cdots \end{bmatrix}. \quad (1)$$

Goal



Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

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Computing U – 2×2 case

We will start with a 2×2 matrix, then a 3×3 matrix, and then generalize to the $n \times n$ case.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$$

If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21} .

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If $a_{11} \neq 0$, then it is a pivot and we use it to eliminate a_{21} .

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} \end{bmatrix}$$

d

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d

If $a_{11} = 0$, but $a_{21} \neq 0$, we have to permute first. If both a_{11} and a_{21} are zero, then the matrix is already upper triangular.

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31} .

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31} .

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Computing U – 3×3 case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31} .

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

Computing U – 3×3 case

If $a_{11} \neq 0$, then we make it first pivot and use it to eliminate a_{21} and a_{31} .

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

If $b \neq 0$, then we make it second pivot and use it to eliminate d .

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{d}{b} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & d & e \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b & c \\ 0 & 0 & f \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

Computing U – General Case

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \xrightarrow{\text{goal}} U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & b & c & \dots & d \\ 0 & 0 & e & \dots & f \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & g \end{bmatrix}$$

$$a_{11} \text{ pivot} \longrightarrow E_{21}A \rightarrow E_{31}E_{21}A \rightarrow E_{41}E_{31}E_{21}A \rightarrow \underbrace{E_{n1} \dots E_{41}E_{31}E_{21}A}_B$$

$$b \text{ pivot} \longrightarrow E_{32}B \rightarrow E_{42}E_{32}B \rightarrow E_{52}E_{42}E_{32}B \rightarrow \underbrace{E_{n2} \dots E_{52}E_{42}E_{32}B}_C$$

$$e \text{ pivot} \longrightarrow E_{43}C \rightarrow E_{53}E_{43}C \rightarrow E_{63}E_{53}E_{43}C \rightarrow E_{n3} \dots E_{63}E_{53}E_{43}C$$

⋮

note that we're assuming we can find a pivot without having to use permutations

Computing L

2×2 case:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } U = E_{21}A.$$

$$\implies A = \underbrace{E_{21}^{-1}}_L U$$

3×3 case:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } U = E_{32}E_{31}E_{21}A.$$

$$\implies A = (E_{32}E_{31}E_{21})^{-1}U$$

$$= \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_L U$$



Questions?