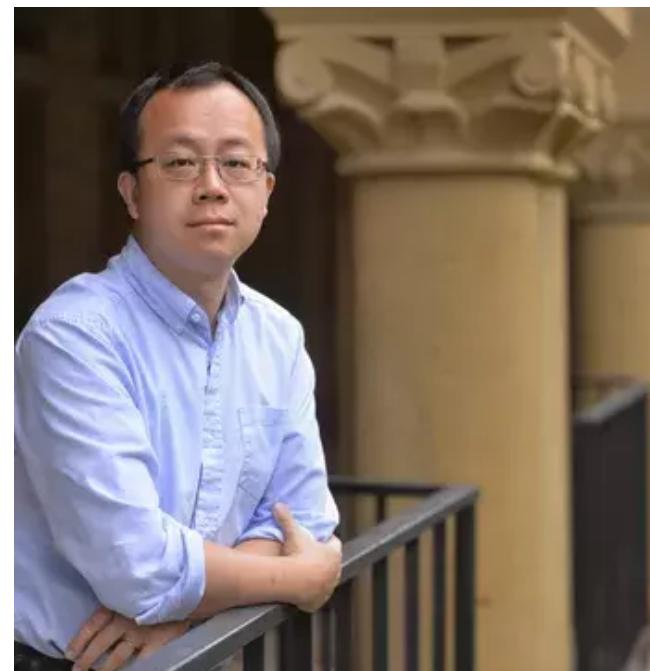


Two Tales, One Resolution for Physics-Informed Inference-time Scaling

Debiasing and Precondition

Yiping Lu

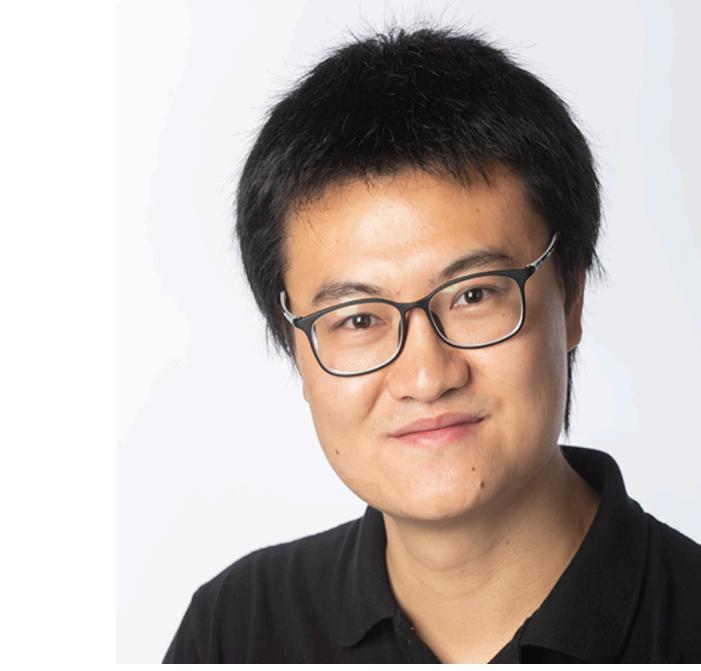
Northwestern | McCORMICK SCHOOL OF
ENGINEERING



Lexing Ying (Stanford)



Jose Blanchet (Stanford)



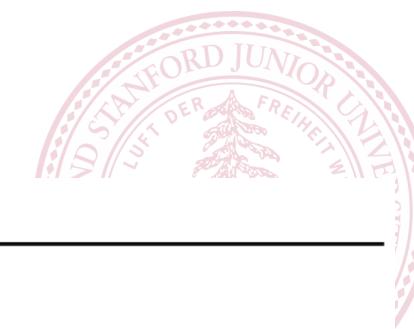
Shihao Yang (Gatech)



Ruihan Xu (Uchicago)

My Journey

- Undergrad: Peking University: 2015-2019
 - Work with Prof. Bin Dong and Prof. Liwei Wang
- Ph.D. Stanford University: 2019-2023
 - Work with Prof. Lexing Ying and Jose E



Beyond Finite Layer Neural Networks: Bridging Deep Architectures and Numerical Differential Equations

Yiping Lu¹ Aoxiao Zhong² Quanzheng Li^{2,3,4} Bin Dong^{5,6,4}

Abstract

Deep neural networks have become the state-of-the-art models in numerous machine learning tasks. However, general guidance to network architecture design is still missing. In our work, we bridge deep neural network design with numerical differential equations. We show that many effective networks, such as ResNet, PolyNet, FractalNet and RevNet, can be interpreted as different numerical discretizations of differential equations. This finding brings us a brand new perspective on the design of effective deep architectures. We can take advantage of the rich knowledge in numerical analysis to guide us in de-

s while maintaining a similar performance. This can be explained mathematically using the concept of modified equation from numerical analysis. Last but not least, we also establish a connection between stochastic control and noise injection in the training process which helps to improve generalization of the networks. Furthermore, by relating stochastic training strategy with stochastic dynamic system, we can easily apply stochastic training to the networks with the LM-architecture. As an example, we introduced stochastic depth to LM-ResNet and achieve significant improvement over the original LM-ResNet on CIFAR10.

PDE-NET: LEARNING PDES FROM DATA

Zichao Long*, Yiping Lu*

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Xianzhong Ma*

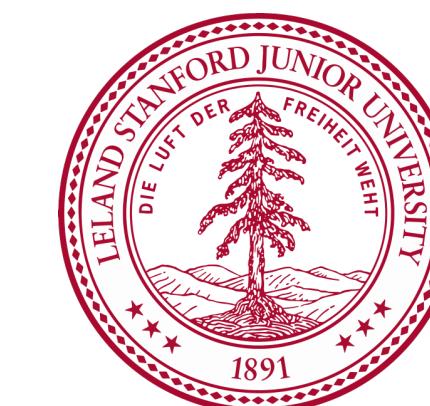
School of Mathematical Sciences, Peking University
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 - Work with Prof. Lexing Ying and Jose Blanchet



Machine Learning For Elliptic PDEs: Fast Rate Generalization Bound, Neural Scaling Law and Minimax Optimality

Yiping Lu*, Haoxuan Chen[†], Jianfeng Lu [‡], Lexing Ying[§], and Jose Blanchet [¶]

Abstract. In this paper, we study the statistical limits of deep learning techniques for solving elliptic partial differential equations (PDEs) from random samples using the Deep Ritz Method (DRM) and Physics-Informed Neural Networks (PINNs). To simplify the problem, we focus on a prototype elliptic PDE: the Schrödinger equation on a hypercube with zero Dirichlet boundary condition, which is applied in quantum-mechanical systems. We establish upper and lower bounds for both methods, which improve upon concurrently developed upper bounds for this problem via a fast rate generalization bound. We discover that the current Deep Ritz Method is sub-optimal and propose a modified version of it. We also prove that PINN and the modified version of DRM can achieve minimax optimal bounds over Sobolev spaces. Empirically, following recent work which has shown that the deep model accuracy will improve with growing training sets according to a power law, we supply computational experiments to show similar behavior of dimension dependent power law for deep PDE solvers.

Minimax Optimal Kernel Operator Learning via Multilevel Training

Jikai Jin,^{*,1} Yiping Lu,^{*,2} Jose Blanchet,^{*,3} and Lexing Ying^{*,4}

¹School of Mathematical Sciences, Peking University, Beijing, China

²ICME, Stanford University, CA, USA

³Management Science & Engineering, Stanford University, CA, USA

⁴Department of Mathematics, Stanford University, Stanford, CA, USA

*Corresponding author. Email: jkjin@pku.edu.cn; yplu@stanford.edu; jose.blanchet@stanford.edu; lexing@stanford.edu

Do you trust your theorem?

Big Constant

Theorem If you randomly collect () data, then you can achieve () accuracy with your AI!

Relu network is **optimal** for function regression

PINN is **optimal** for differential equation solving

Diffusion Model is **optimal** for density estimation

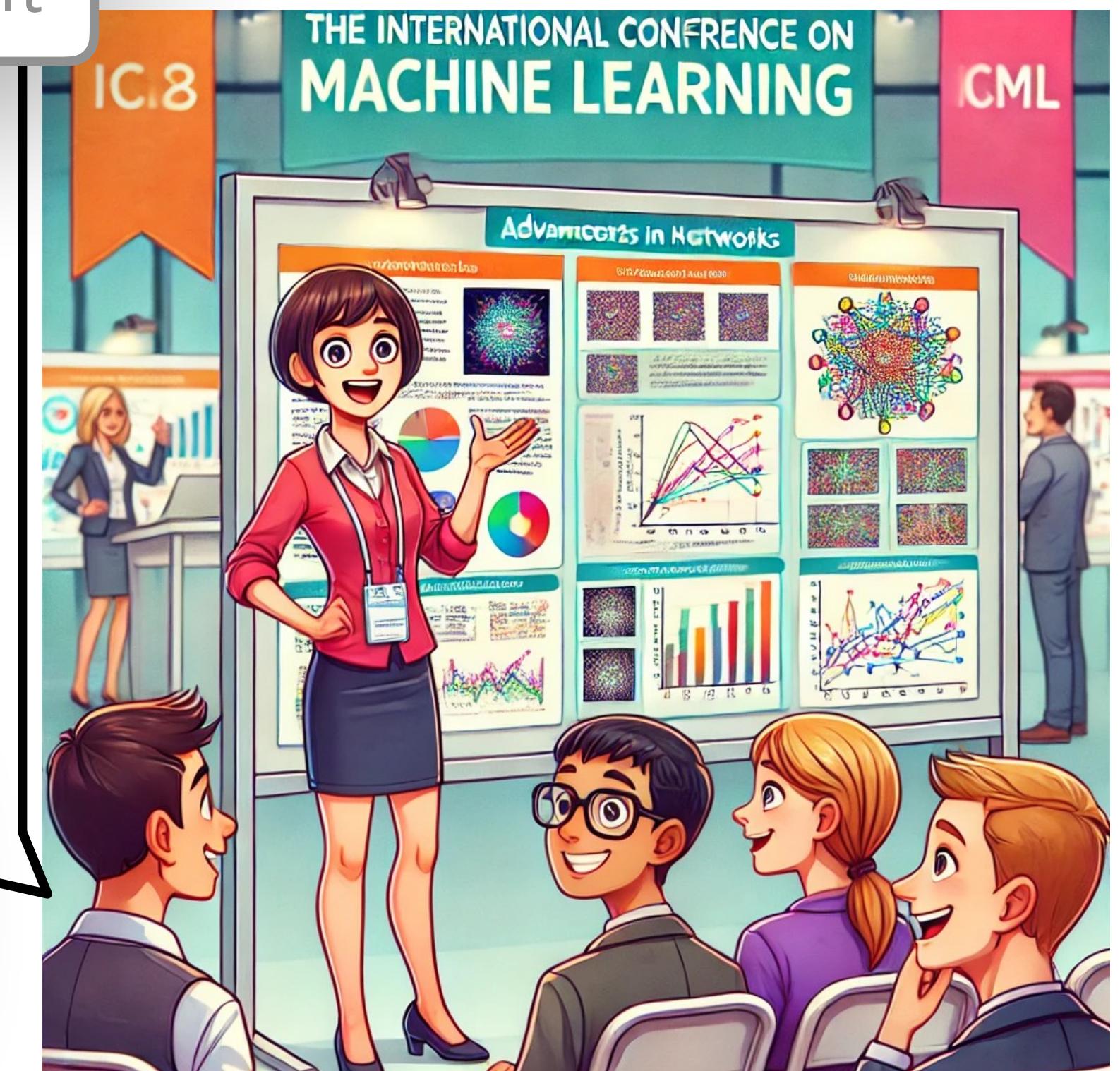
Transformer is **optimal** for in-context learning

...

Global optimization, the structure of target function,
The smoothness of ReLU, lower bound of $p(x)$...



Does Optimal results really gives me guarantee?



Consider How you use ChatGPT...

What is the most efficient route from Times Square to Chelsea Market that minimizes walking?

Option 1: Subway

1. **Walk to a Nearby Subway Entrance:** Head to the Times Sq–42nd Street station.
2. **Board the E Train:** Catch an E train heading downtown (toward World Trade Center).
3. **Ride to 8th Avenue–14th Street Station:** Get off at this station (roughly 4 stops).
4. **Walk to Chelsea Market:** Exit the station and walk east on 14th Street for a few blocks until you reach Chelsea Market at 75 9th Avenue.



double-check
with a map

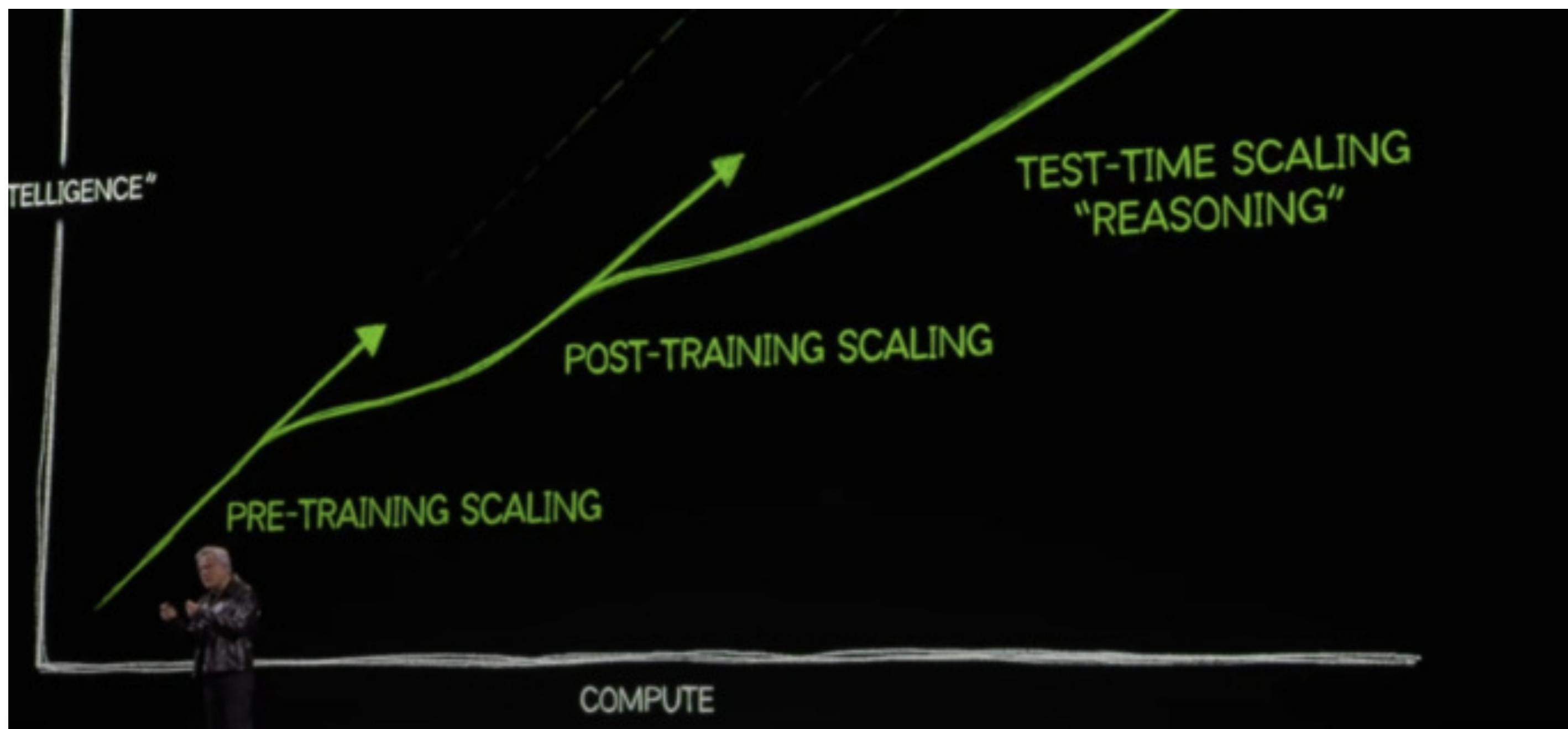
Port Authority

2 stops for A
3 stops for C/E

1. **At Times Square:** Enter the Times Square–42nd Street station.
2. **Board the 1 Train:** Hop on a downtown 1 train (the red line).
3. **Ride to 14th Street:** Stay on until you reach the 14th Street station.
4. **Exit Appropriately:** Use the exit that leads toward 9th Avenue—this drop-off point is just a short walk from Chelsea Market (located at 75 9th Ave).

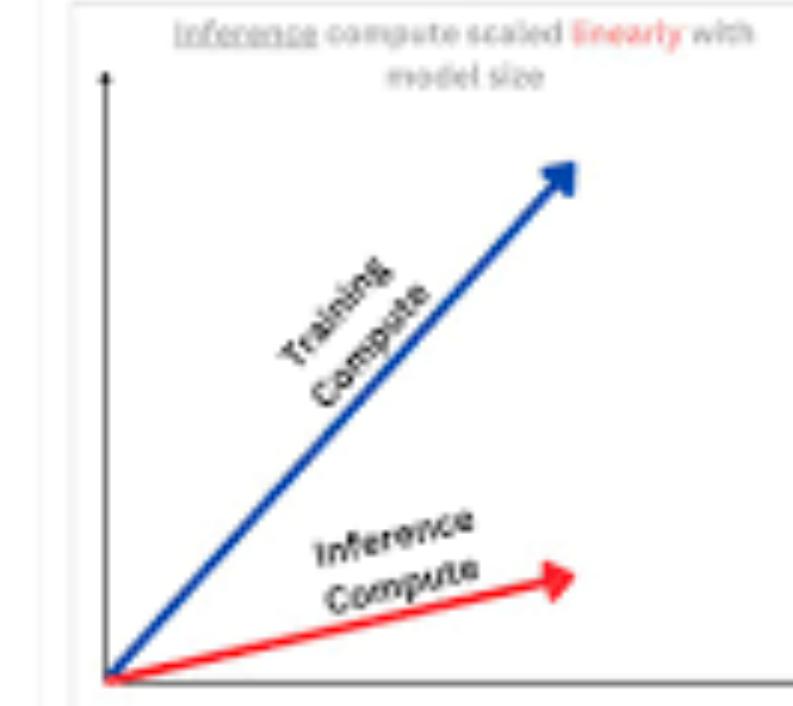


Inference Time Scaling Law

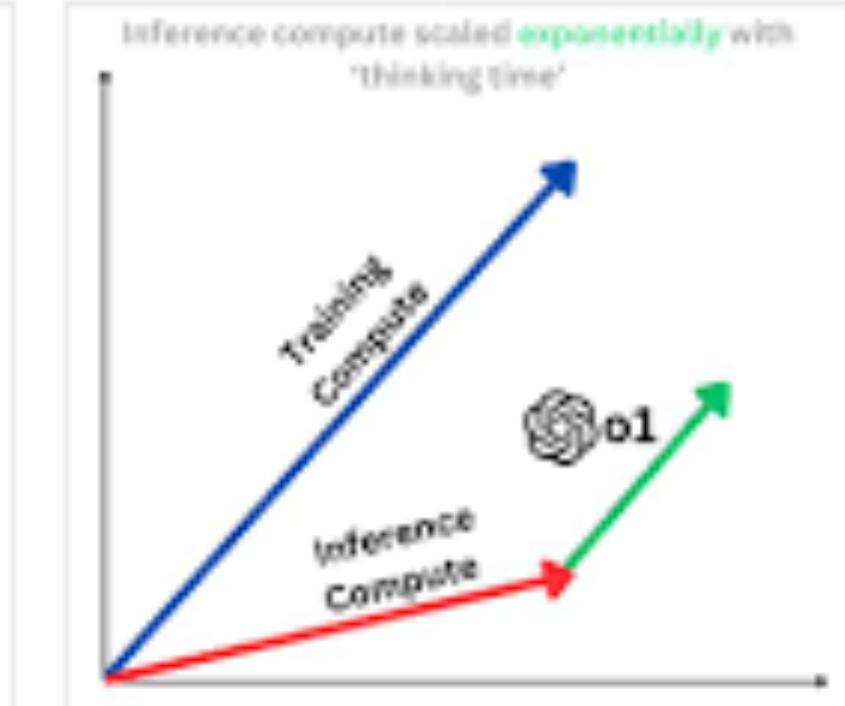


New scaling law: why OpenAI's o1 model matters
OpenAI created a new way to scale - through more compute during generation

Before OpenAI o1



After OpenAI o1



Sources: Peter Grossette (<https://www.linkedin.com/in/peter-grossette/>)

How can we perform Inference-Time Scaling for Scientific Machine Learning?

With trustworthy guarantee

don't fine-tune/retrain/add a new surrogate model

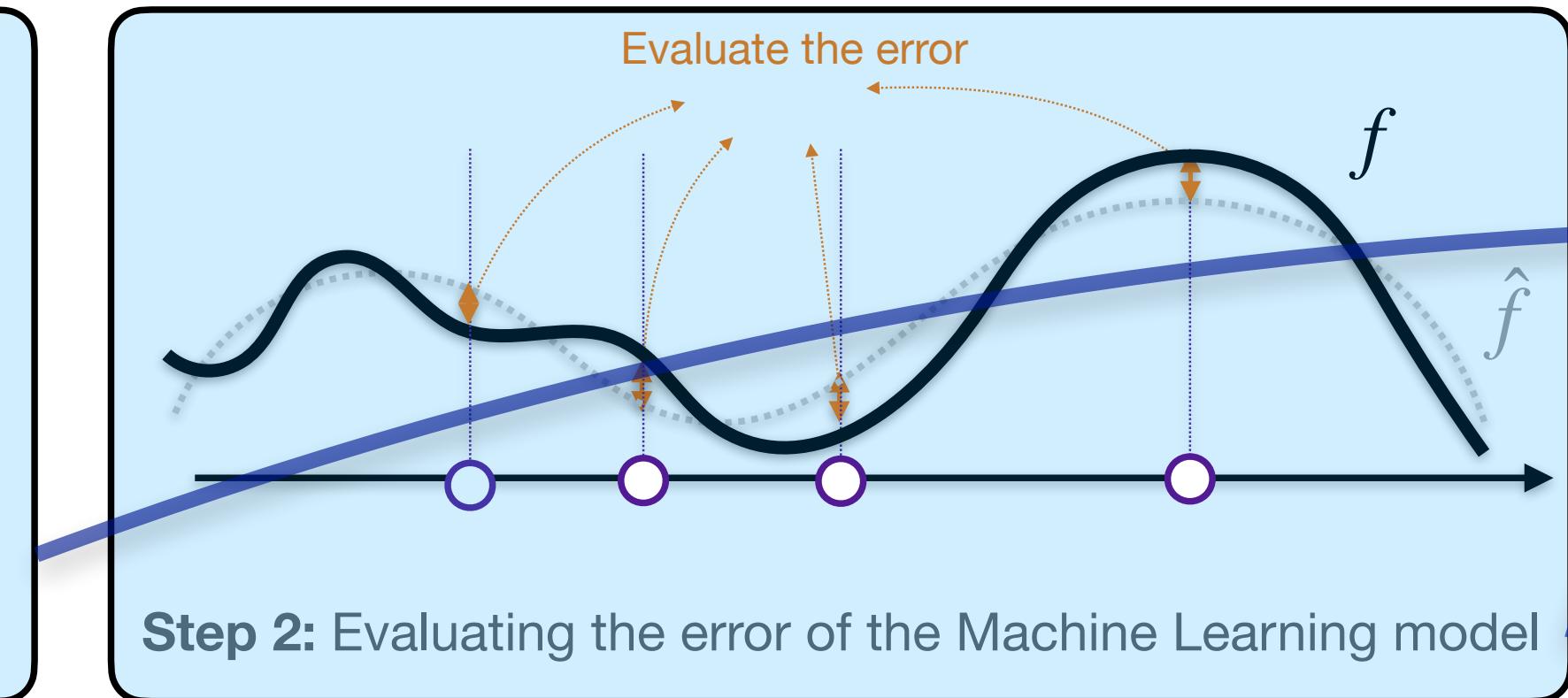
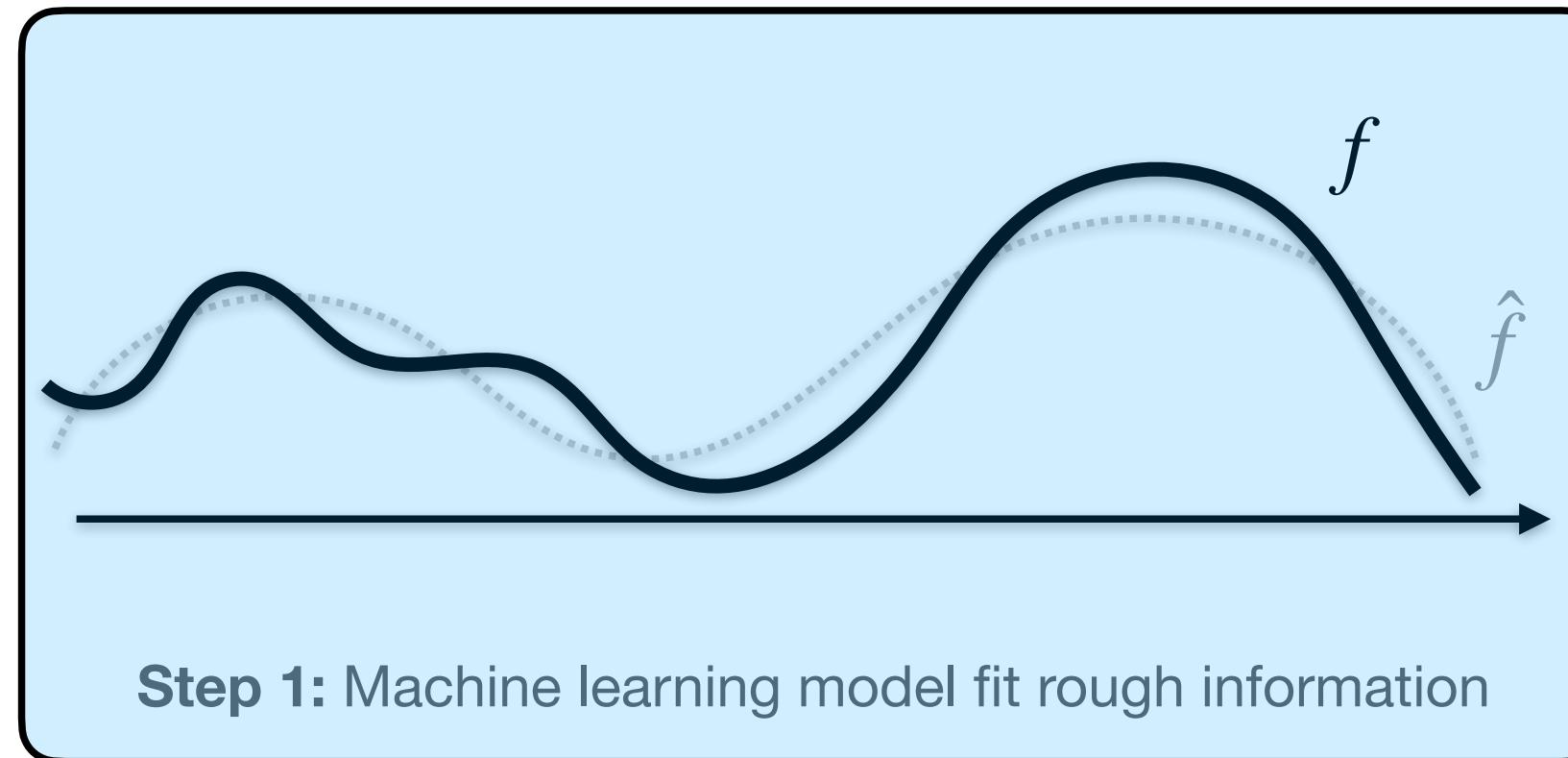
How can we perform Inference-Time Scaling for Scientific Machine Learning?

“Physics-informed”

With trustworthy guarantee

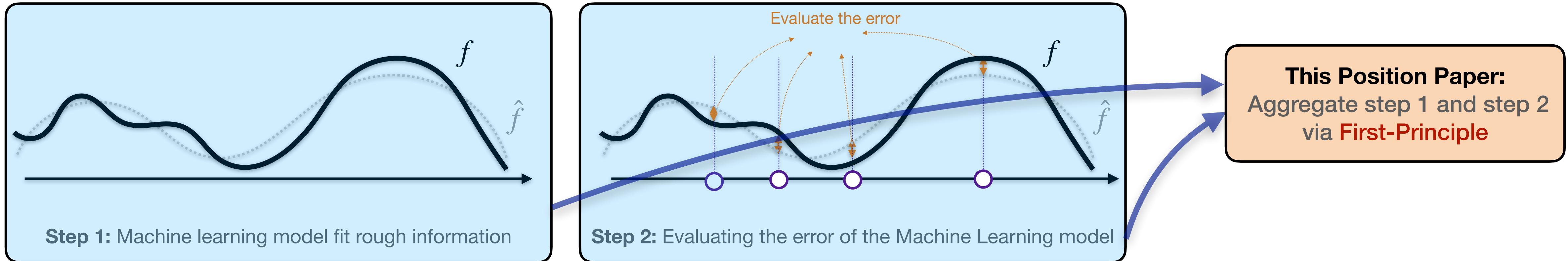
Tale 1: Debiasing Hybrid Scientific Computing and Machine Learning

Physics-Informed Inference Time Scaling



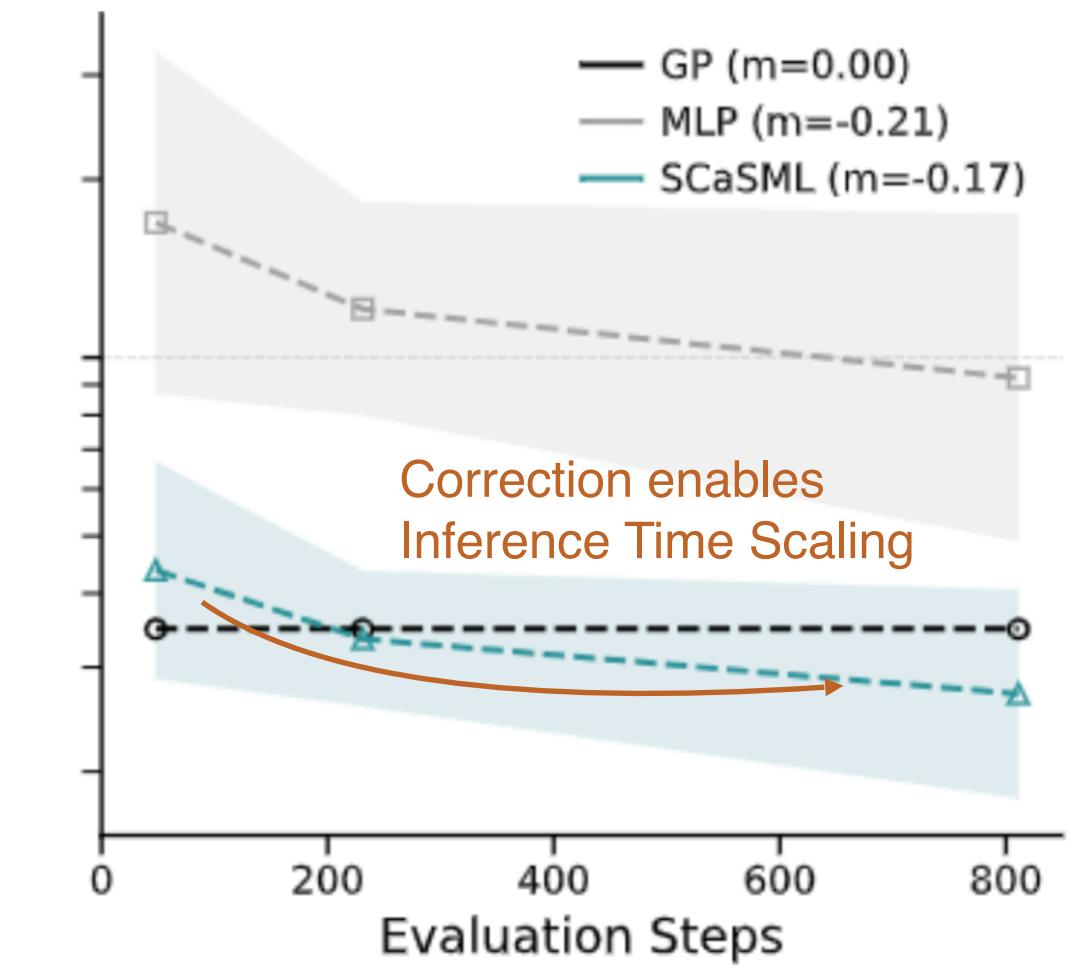
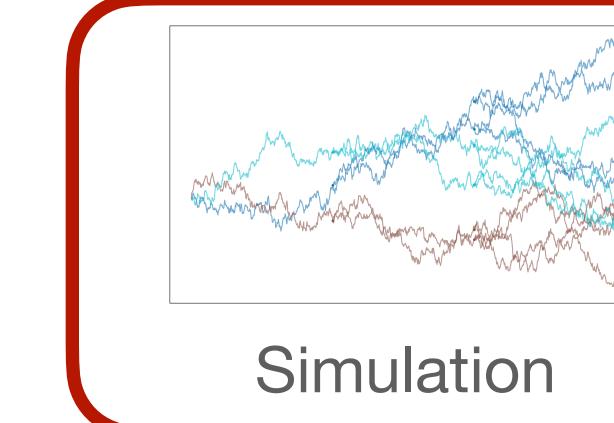
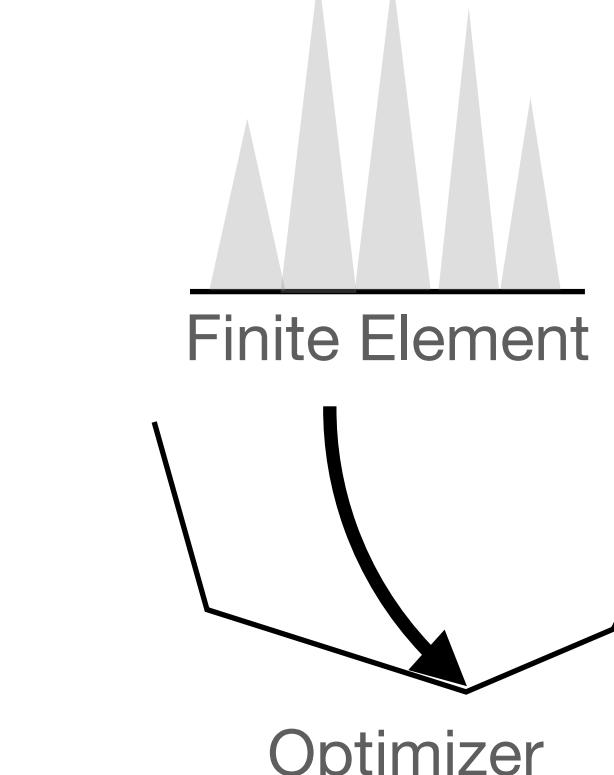
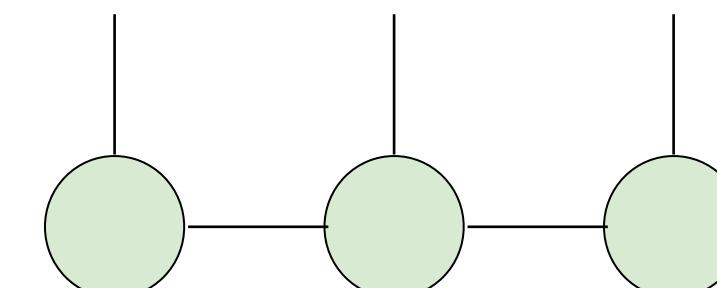
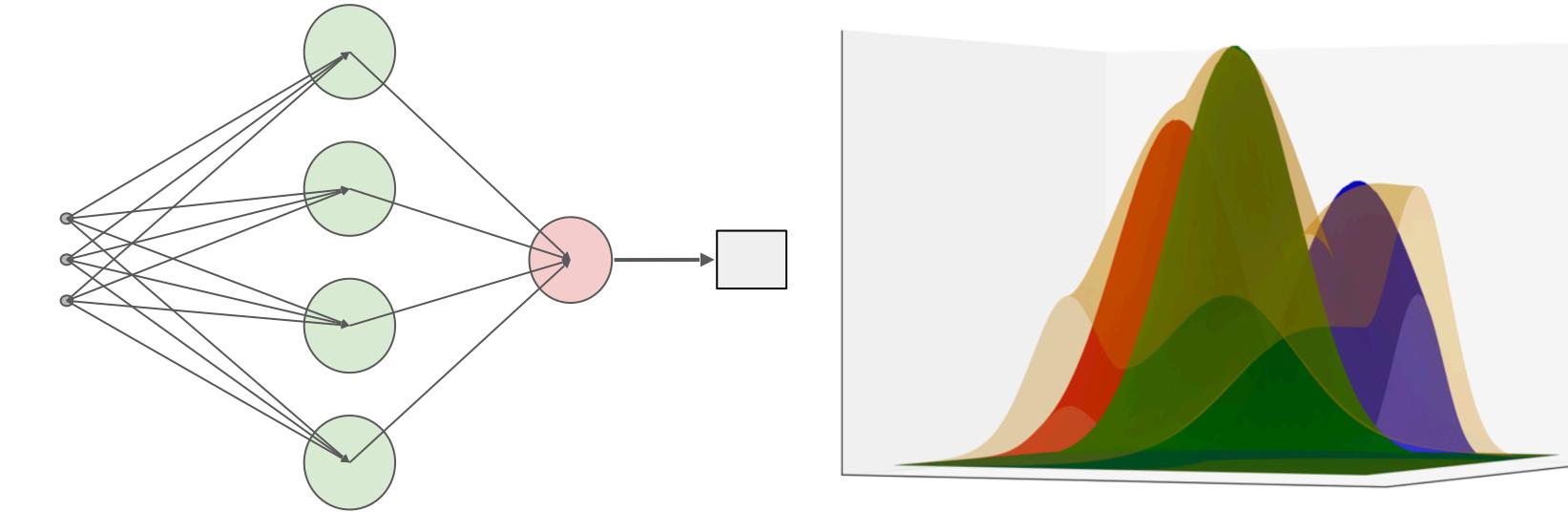
This Position Paper:
Aggregate step 1 and step 2
via First-Principle

Physics-Informed Inference Time Scaling



Step 2. Correct with a Trustworthy Solver

Step 1. Train a Surrogate (ML) Model



Our Framework

Step 1: Sceintific Computing as Machine Learning

$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

Function fitting

Example 2

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i)) \quad \text{Solving } \Delta u = f$$

Solving PDE

Example 3

$$\theta = A, \quad X_i = (x_i, Ax_i)$$

Estimation \hat{A} via Randomized SVD

Our Framework

Step 2: Consider a Downstream Application



Example 1

Scientific Machine Learning

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

Downstream application

$$\Phi(\theta) = \int f(x)dx$$

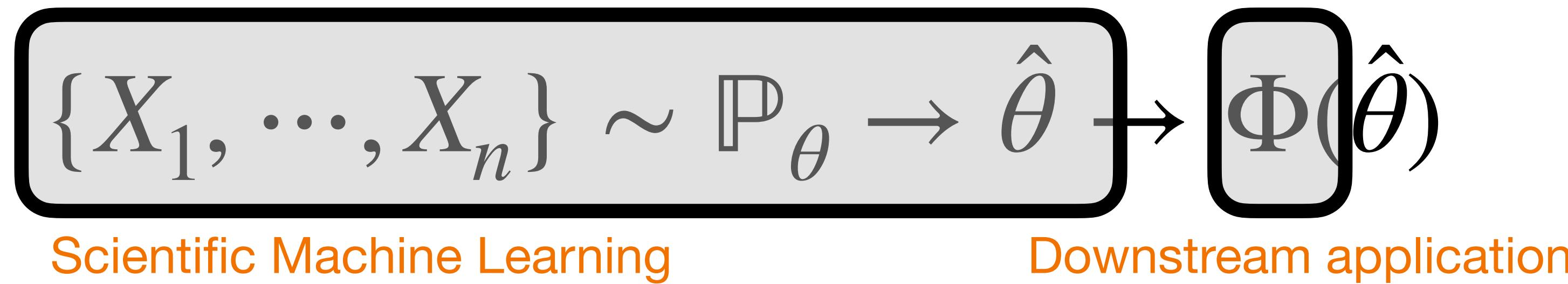
Example 2

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i)) \quad \Phi(\theta) = (\Delta^{-1}f)(x)$$

Example 3

$$\theta = A, \quad X_i = (x_i, Ax_i) \quad \Phi(\theta) = \text{tr}(A), \text{eigs}(A)$$

Our Framework



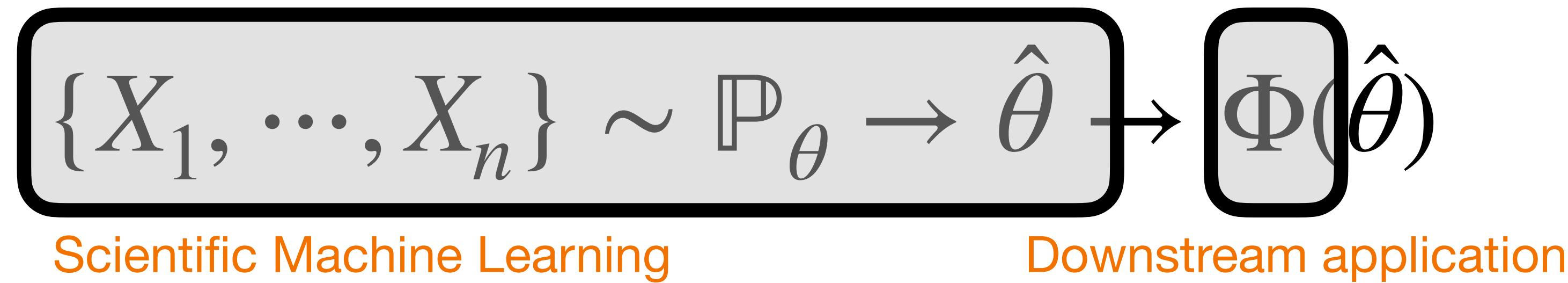
AIM: Unbiased prediction even with biased machine learning estimator

AIM: Compute $\Phi(\hat{\theta}) - \Phi(\theta)$ during Inference time



Using (stochastic) simulation to calibrate the (scientific) machine learning output !

Our Framework



AIM: Unbiased prediction even with biased machine learning estimator

How to estimate $\Phi(\hat{\theta}) - \Phi(\theta)$?

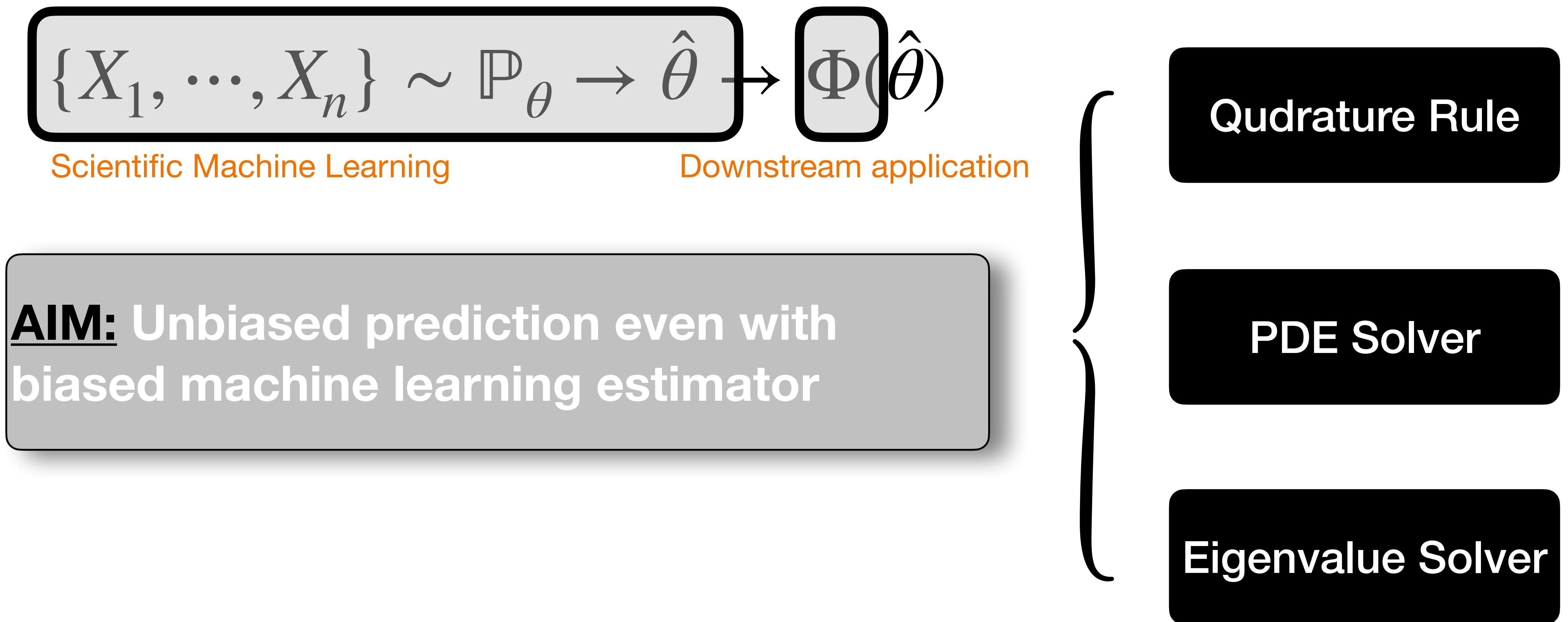


Physics-Informed! (Structure of Φ)

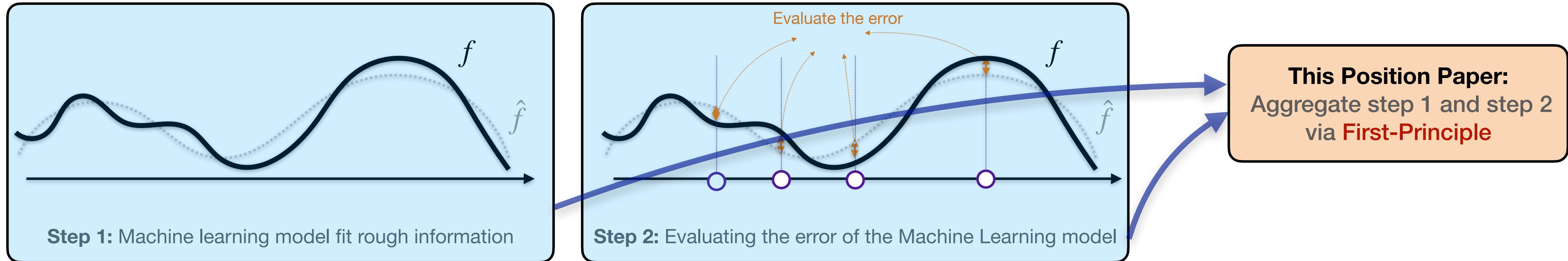
Why it is easier than directly estimate $\Phi(\theta)$?

Variance Reduction

Our Framework



Debiasing a Machine Learning Solution



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

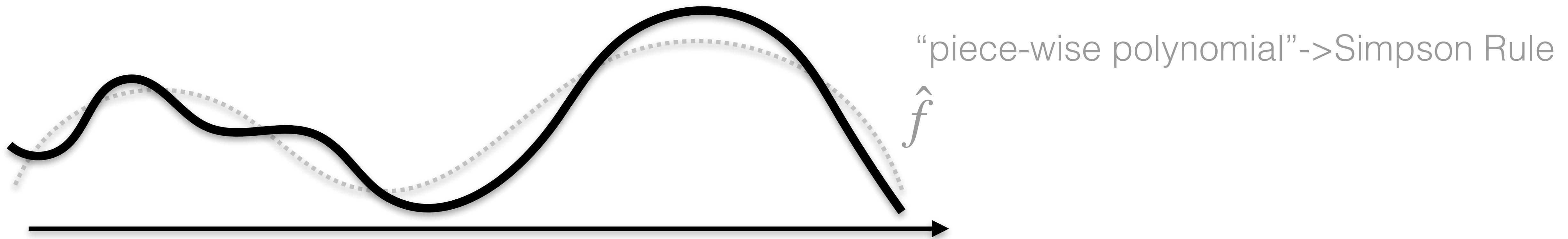
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

Debiasing a Machine Learning Solution



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \hat{\theta} \rightarrow \Phi(\hat{\theta})$$

Scientific Machine Learning

Downstream application

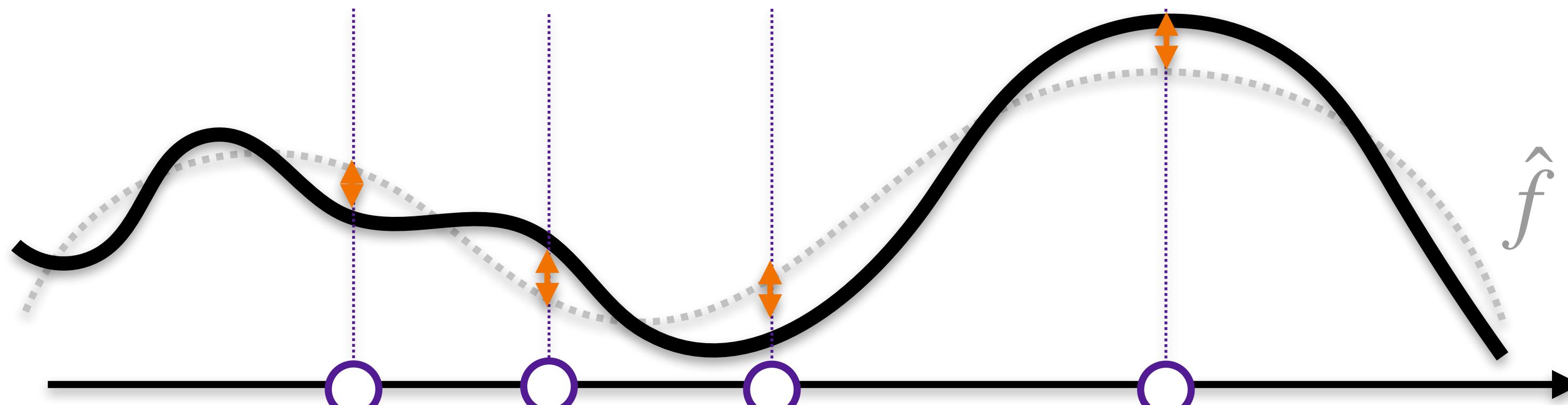
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

Debiasing a Machine Learning Solution



Our Approach

$$\text{Estimate } \mathbb{E}_P f \approx \mathbb{E}_P \hat{f} + \mathbb{E}_{\hat{P}} f - \hat{f}$$

An estimate to $\Phi(\hat{\theta}) - \Phi(\theta)$



Scientific Machine Learning

Downstream application

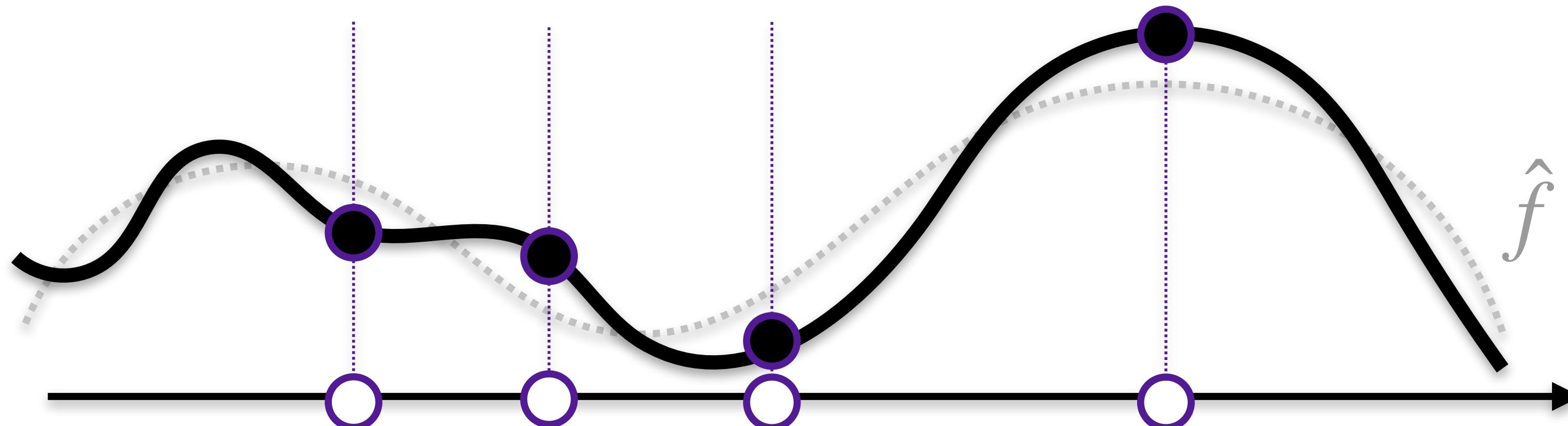
Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

Debiasing a Machine Learning Solution



Monte Carlo?

Estimate $\mathbb{E}_P f \approx \mathbb{E}_{\hat{P}} \hat{f}$



Scientific Machine Learning

Downstream application

Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Temperature, overall velocity...

Debiasing a Machine Learning Solution



Regression-adjusted Control Variates

Doubly Robust Estimator

Multi-fidelity monte carlo

...

- Investigated the **optimality** of the SCaSML Framework

- Jose Blanchet, Haoxuan Chen, Yiping Lu, Lexing Ying. When can Regression-Adjusted Control Variates Help? Rare Events, Sobolev Embedding and Minimax Optimality Neurips 2023

- Extend to **nonlinear** functional estimation using iterative methods

Later



Scientific Machine Learning

Downstream application

Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

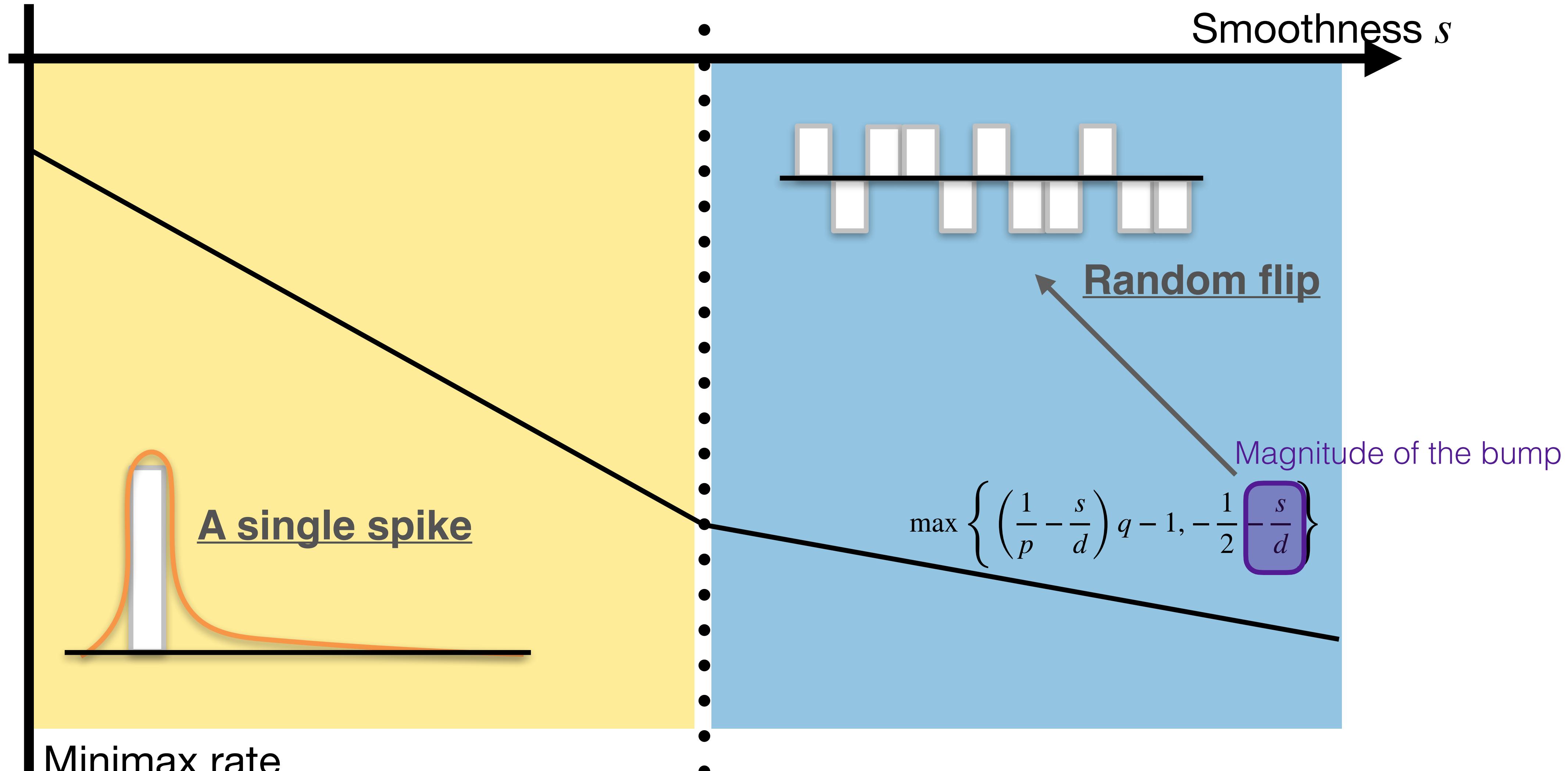
Temperature, overall velocity...

Lower Bound

I Hardest Examples



SCaML

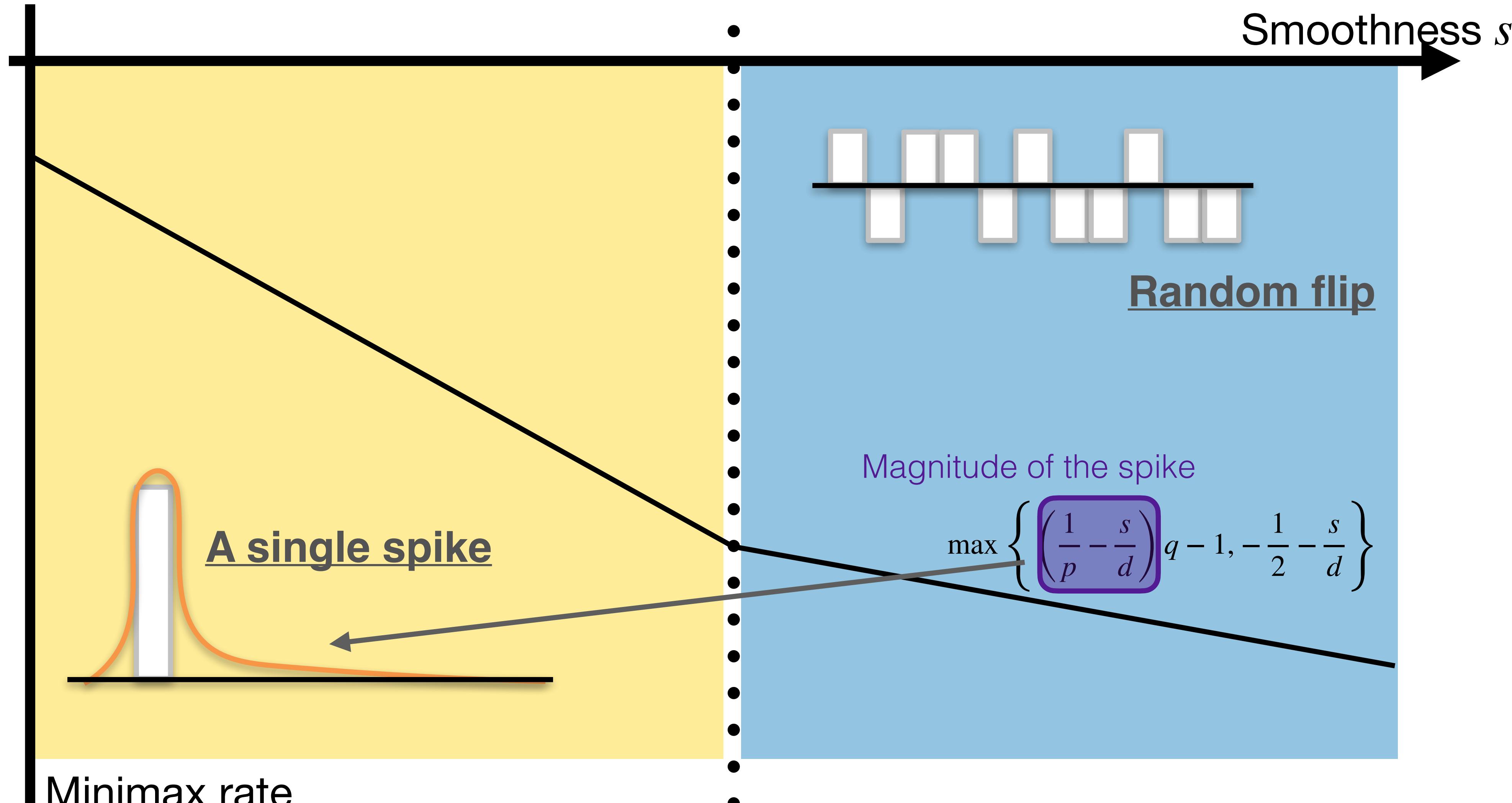


Minimax rate

Hardest Examples



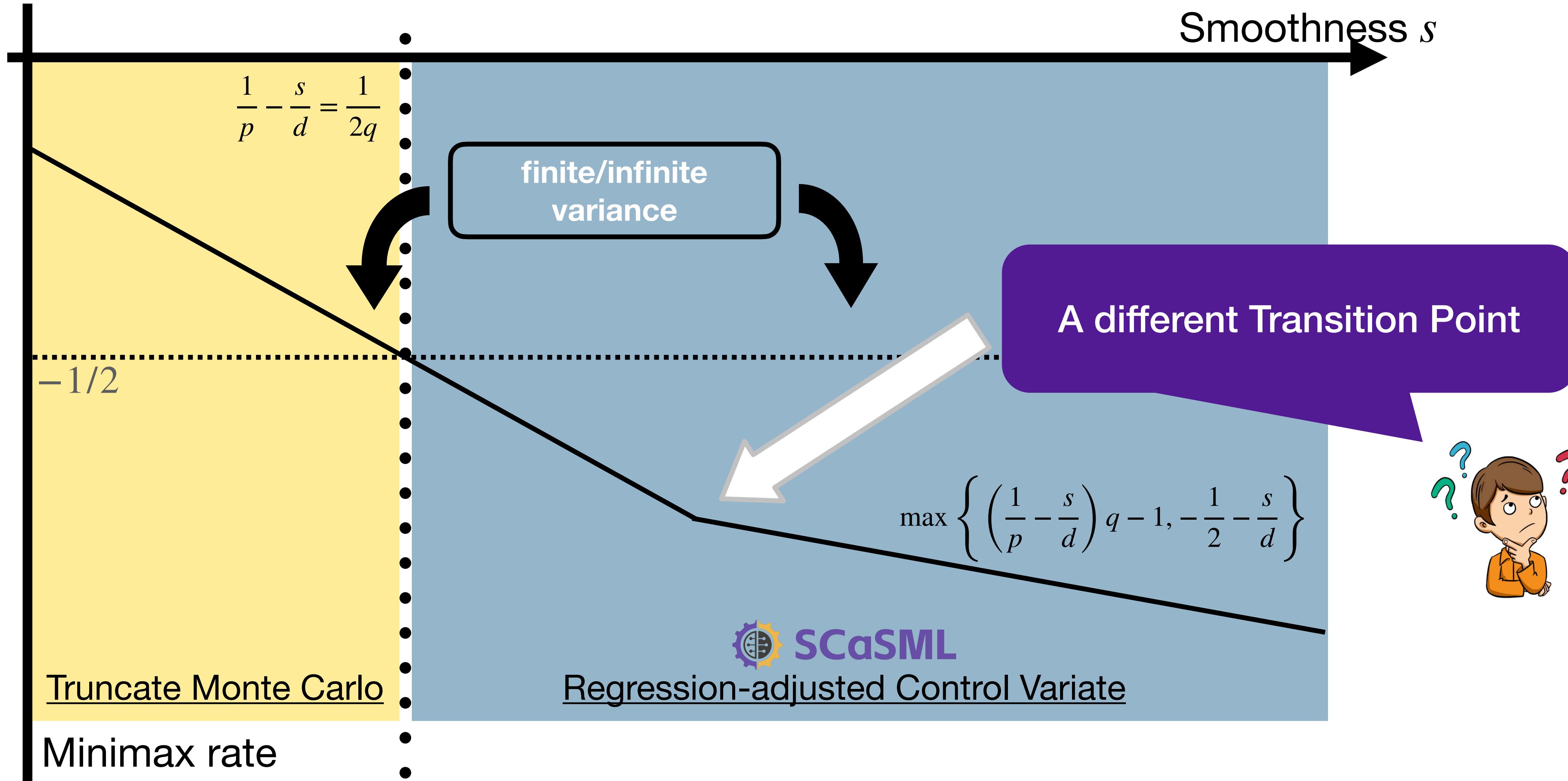
SCaML



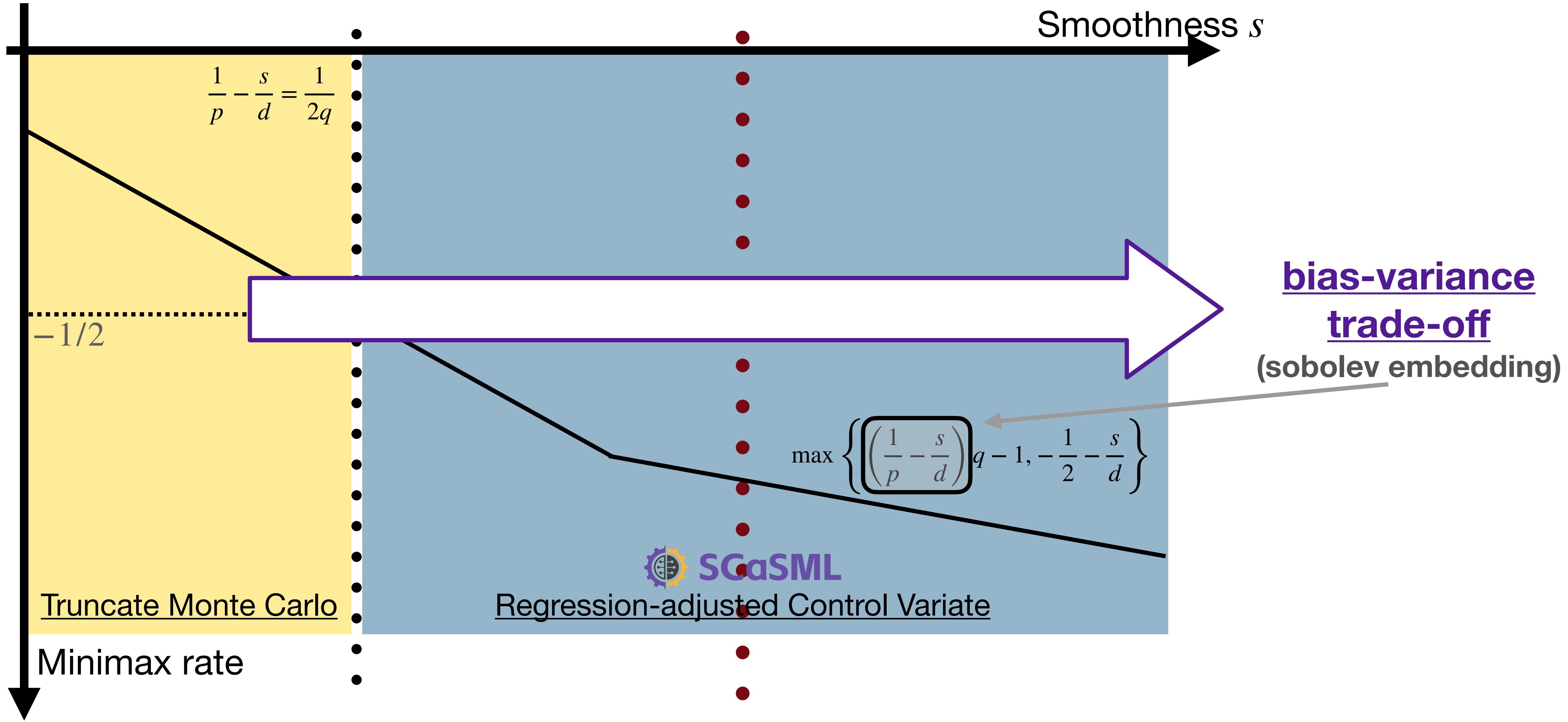
Minimax rate

Optimal Algorithms

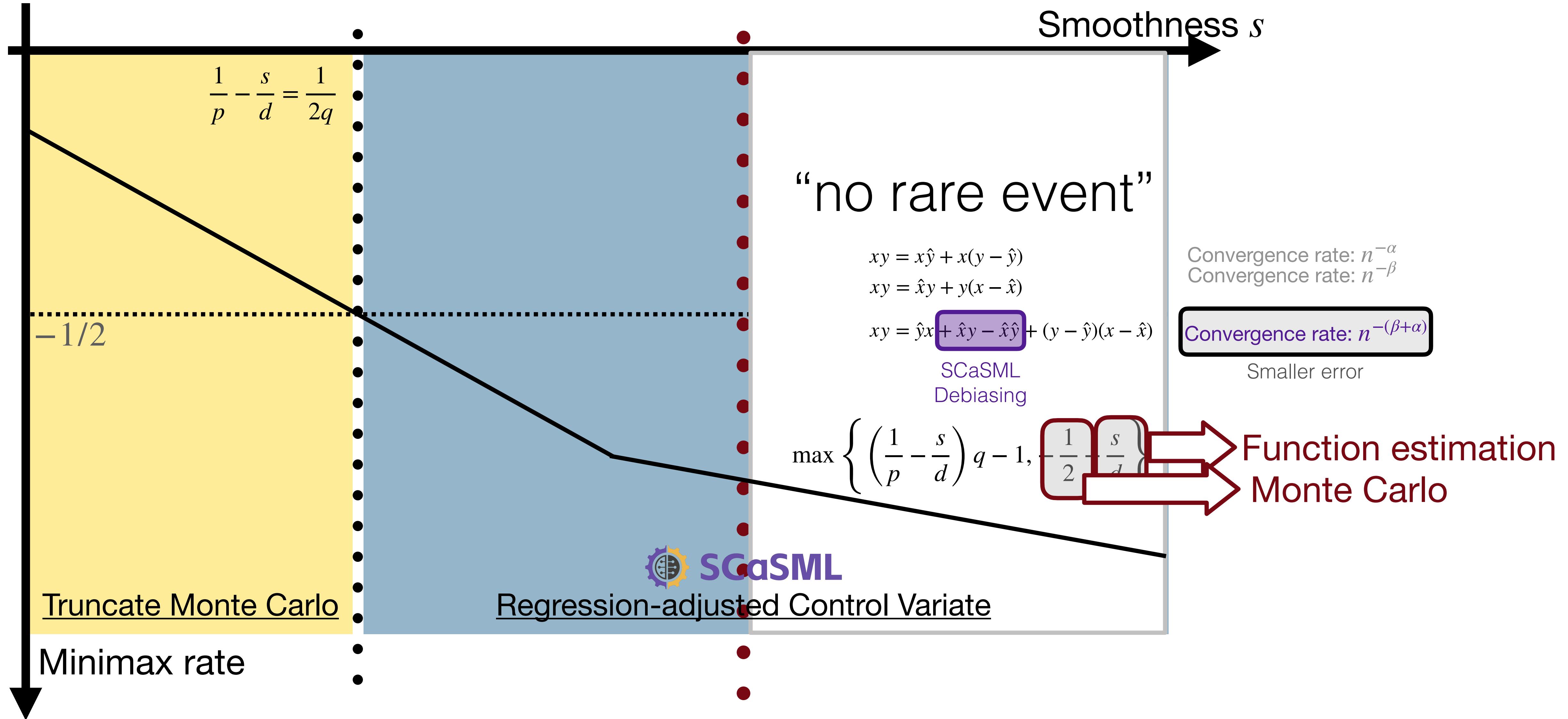
Easiest Understanding



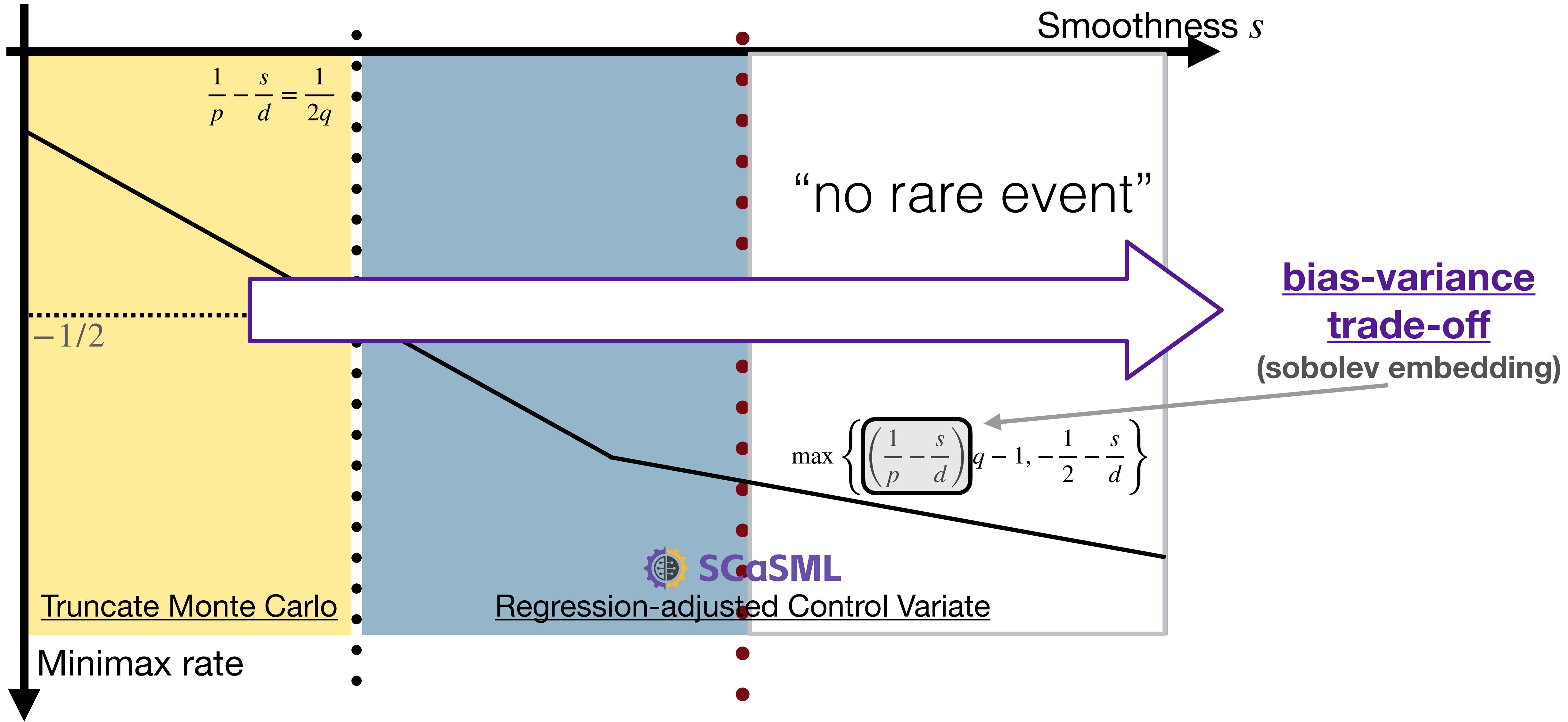
Easiest Understanding



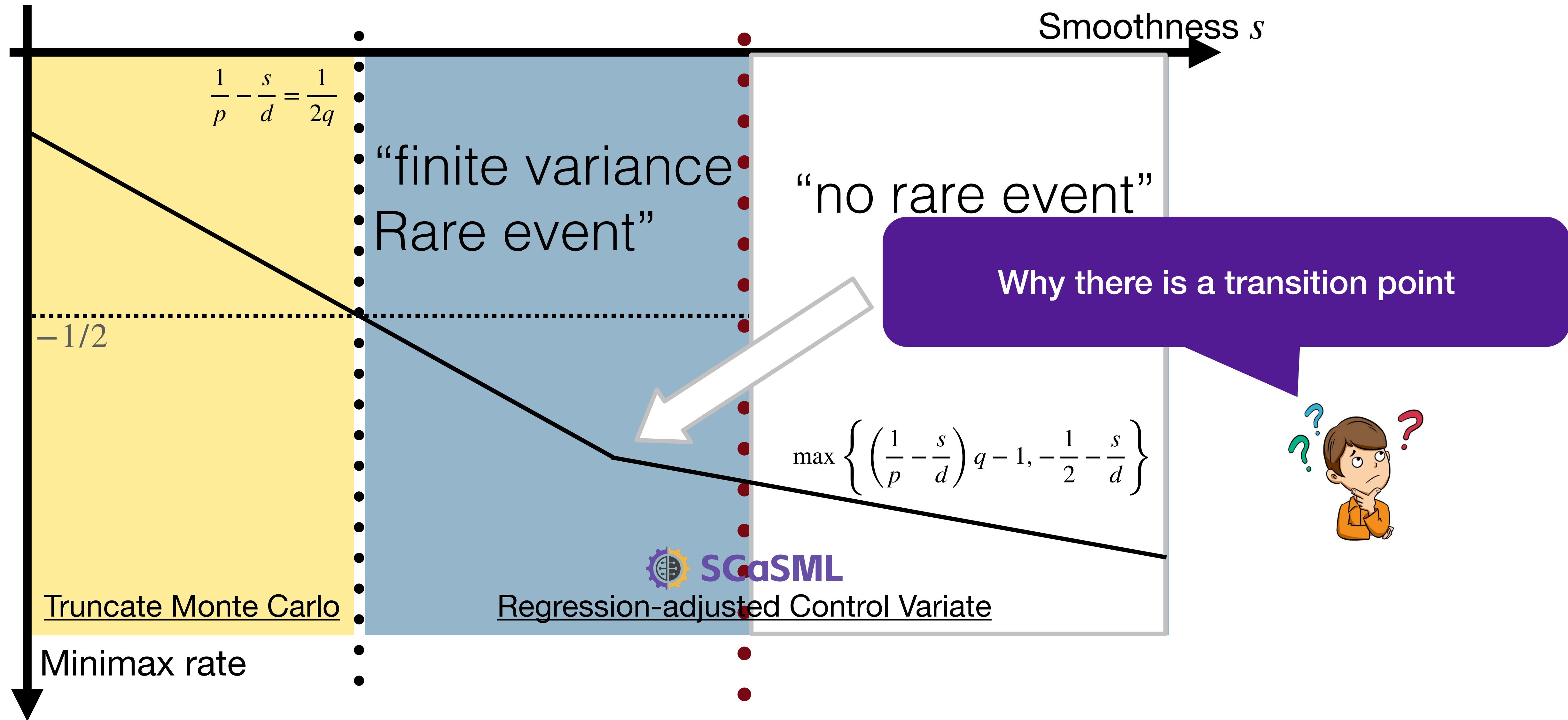
Easiest Understanding



Easiest Understanding



I Why...



Analysis of Error propagation



 **SCaML** estimate of $\mathbb{E}_P f^q, f \in W^{s,p}$

Step 1

Using half of the data to estimate \hat{f}

Step 2

$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$

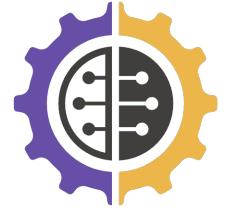


How does step 2 variance depend on estimation error?

Hardness = The variance of the debasing step

Analysis of Error propagation



 **SCaML** estimate of $\mathbb{E}_P f^q, f \in W^{s,p}$



Step 1

Using half of the data to estimate \hat{f}

Step 2

$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$

Low order term

$$f^{q-1}(f - \hat{f}) + (f - \hat{f})^q$$

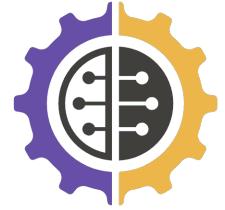
How does step2 variance
depend on estimation error?

“influence function” (gradient)

Error propagation

Analysis of Error propagation



 **SCaML** estimate of $\mathbb{E}_P f^q, f \in W^{s,p}$

Step 1

Using half of the data to estimate \hat{f}

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$$\mathbb{E}_P f^q = \mathbb{E}_P(\hat{f}^q) + \mathbb{E}_P(f^q - \hat{f}^q)$$

Low order term

$$f^{q-1}(f - \hat{f}) + (f - \hat{f})^q$$

“influence function” (gradient)

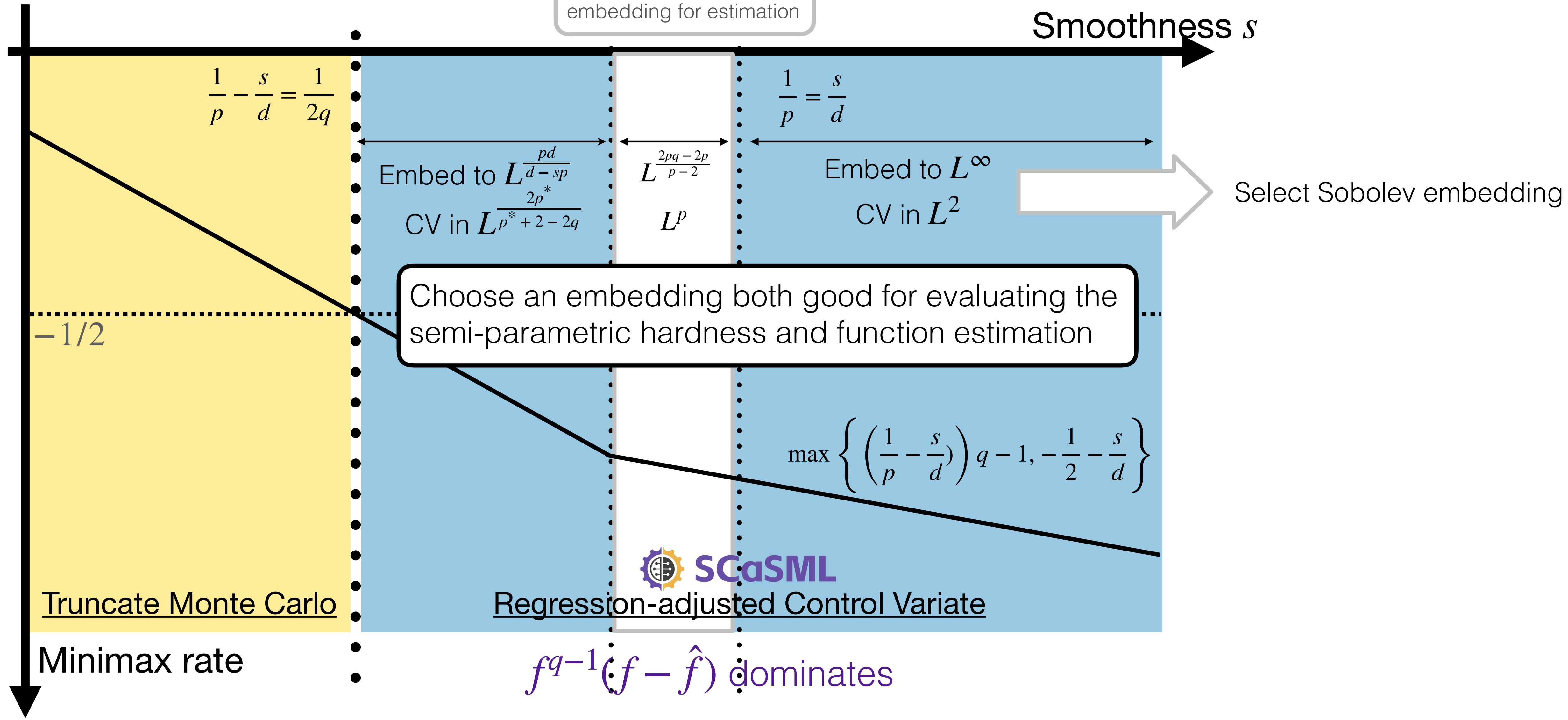
Error p

Embed f^{q-1} and $f - \hat{f}$ into “dual” space

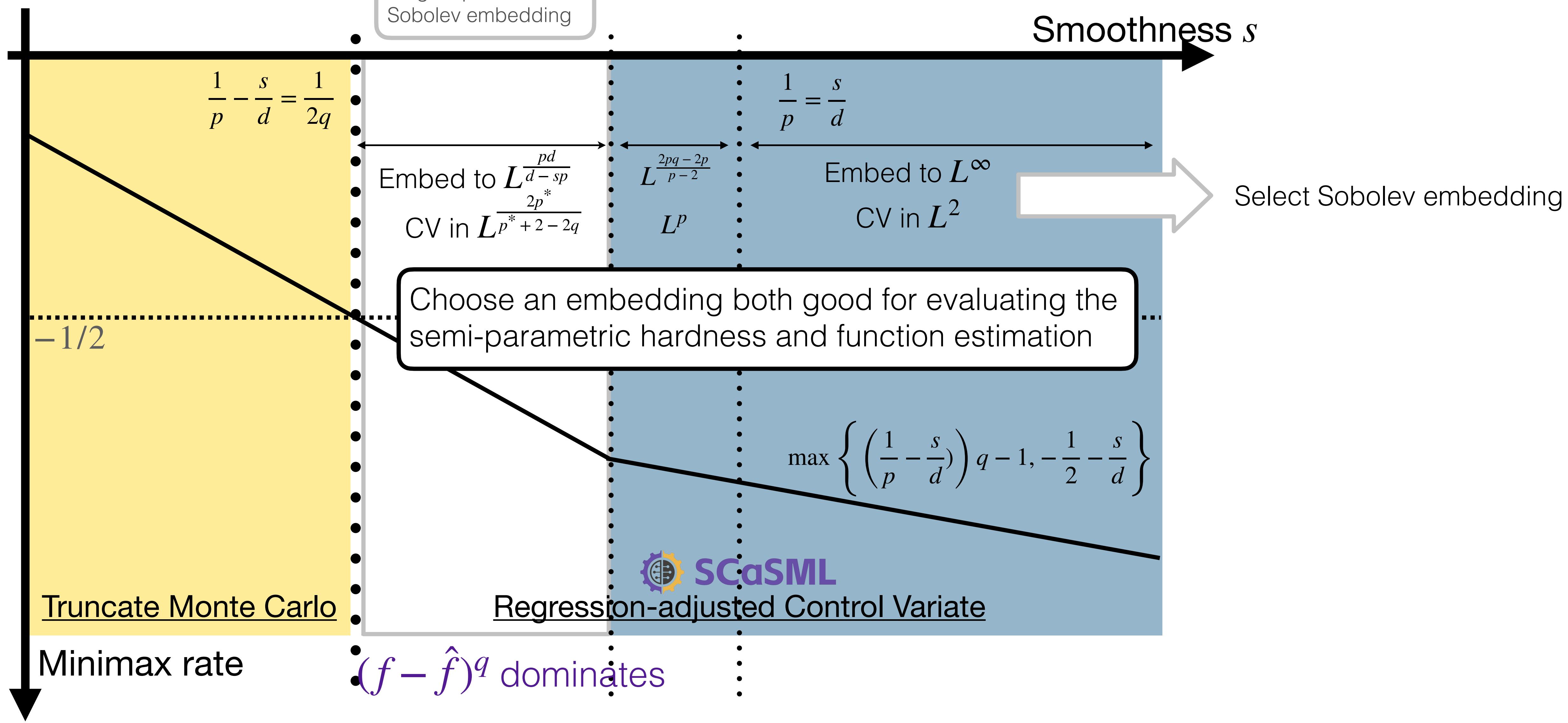
How to select the Sobolev embedding?



Selecting the Sobolev Embedding



Selecting the Sobolev Embedding



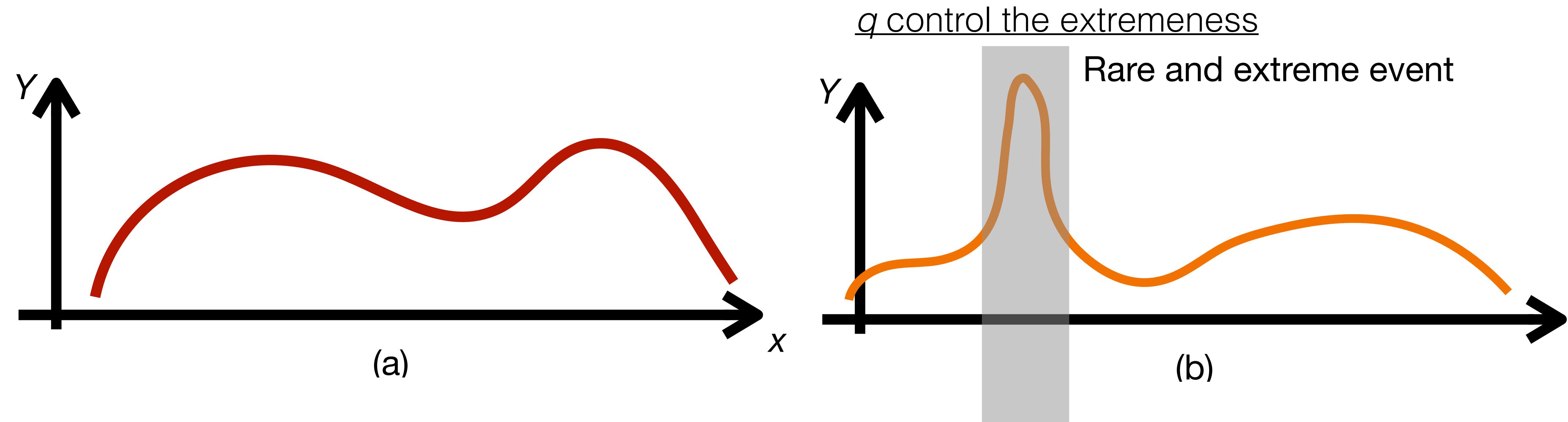
| Take Home Message on the Theory



SCaML

- a) Statistical optimal regression is the optimal control variate
- b) It helps only if there isn't a hard to simulate (infinite variance)

Rare and extreme event



When can Regression-Adjusted Control Variates Help? Rare Events, Sobolev Embedding and Minimax Optimality

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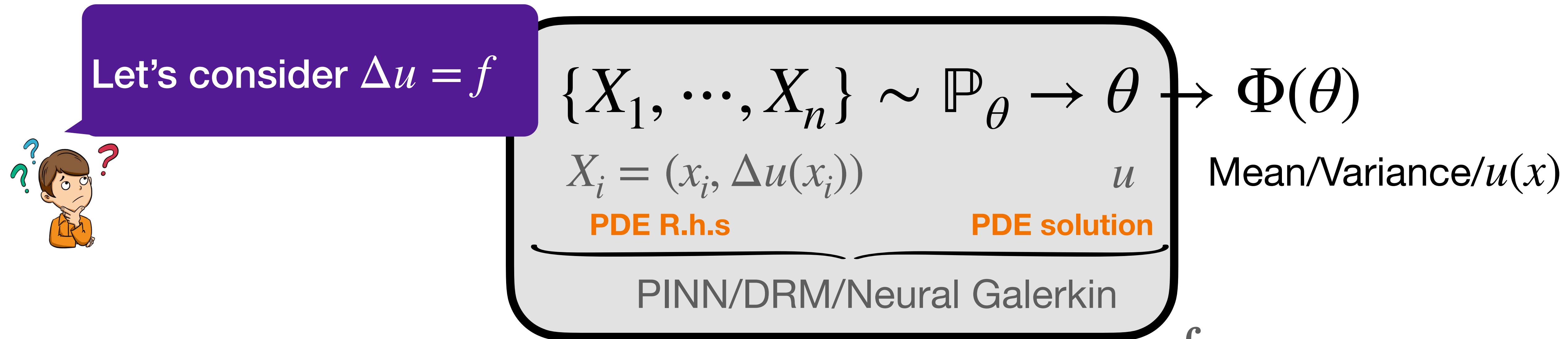
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PDE Solver

High Dimensional PDE-Solving

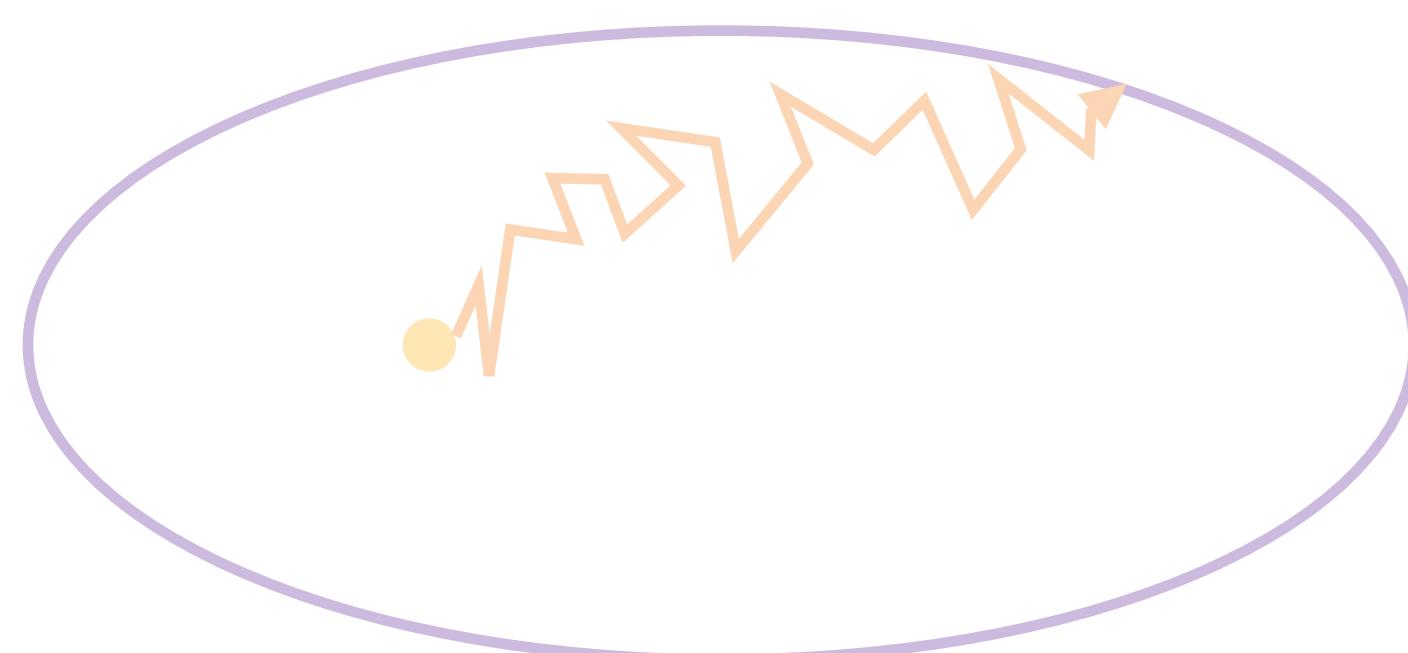


$$\Delta u = f$$

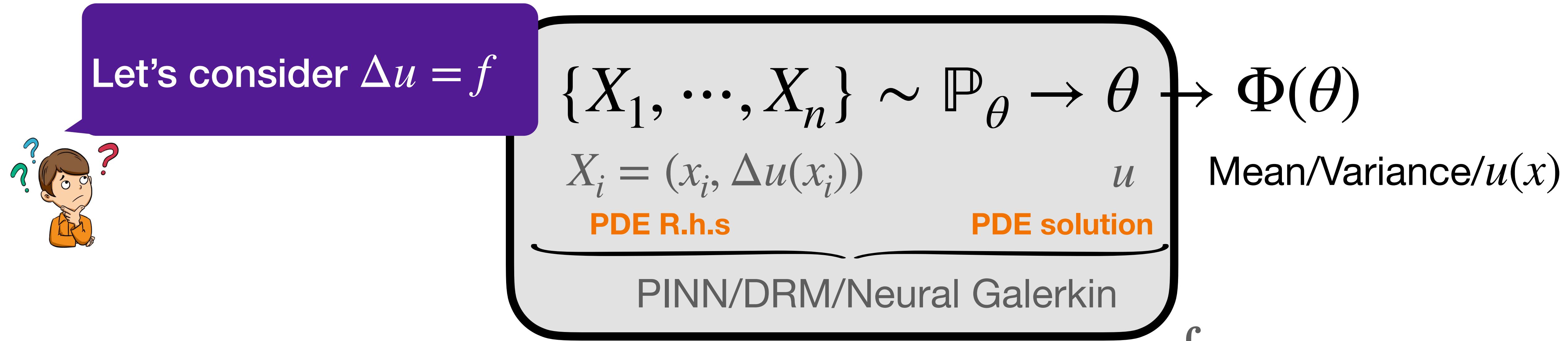
$$\Delta \hat{u} = \hat{f}$$

$$\Delta(u - \hat{u}) = f - \hat{f}$$

$$(u - \hat{u})(x) = \mathbb{E} \int (f - \hat{f})(X_t) dt$$



High Dimensional PDE-Solving

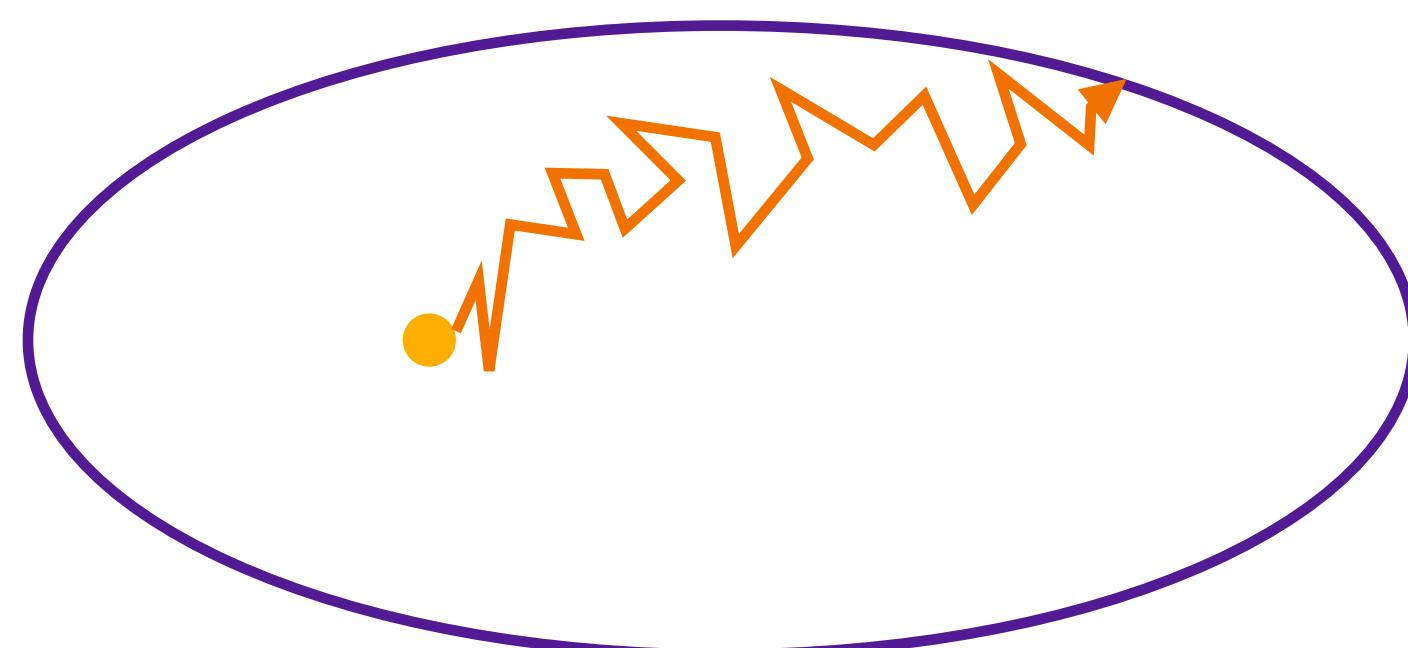


$$\Delta u = f$$

$$\Delta \hat{u} = \hat{f}$$

$$\Delta(u - \hat{u}) = f - \hat{f}$$

$$(u - \hat{u})(x) = \mathbb{E} \int (f - \hat{f})(X_t) dt$$



Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation



Can you do simulation
for nonlinear equation?



Δ is linear!

Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation

$$\frac{\partial \hat{U}}{\partial t}(x, t) + \boxed{\Delta \hat{U}(x, t)} + f(\hat{U}(x, t)) = g(x, t)$$

$g(x, t)$ is the error made by NN

Works for Semi-linear PDE

$$\frac{\partial U}{\partial t}(x, t) + \boxed{\Delta U(x, t)} + f(U(x, t)) = 0$$

Keeps the structure to enable brownian motion simulation

$$\frac{\partial \hat{U}}{\partial t}(x, t) + \boxed{\Delta \hat{U}(x, t)} + f(\hat{U}(x, t)) = g(x, t)$$

g(x, t) is the error made by NN

Subtract two equations

$$\frac{\partial(U - \hat{U})}{\partial t}(x, t) + \boxed{\Delta(U - \hat{U})(x, t)} + \underbrace{f(t, \hat{U}(x, t) + U(x, t) - \hat{U}(x, t)) - f(t, \hat{U}(x, t))}_{G(t, (U - \hat{U})(x, t))} = g(x, t).$$

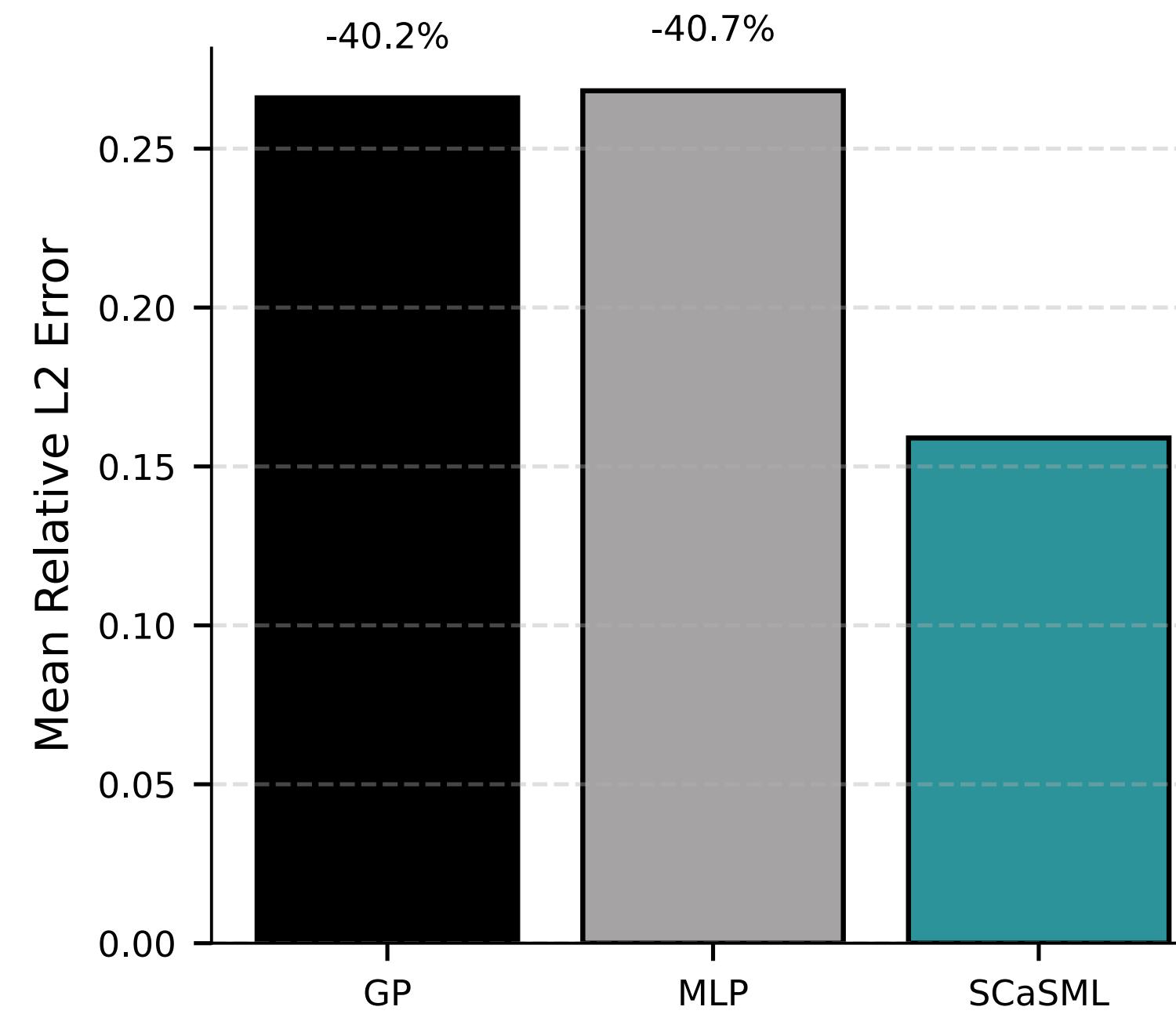
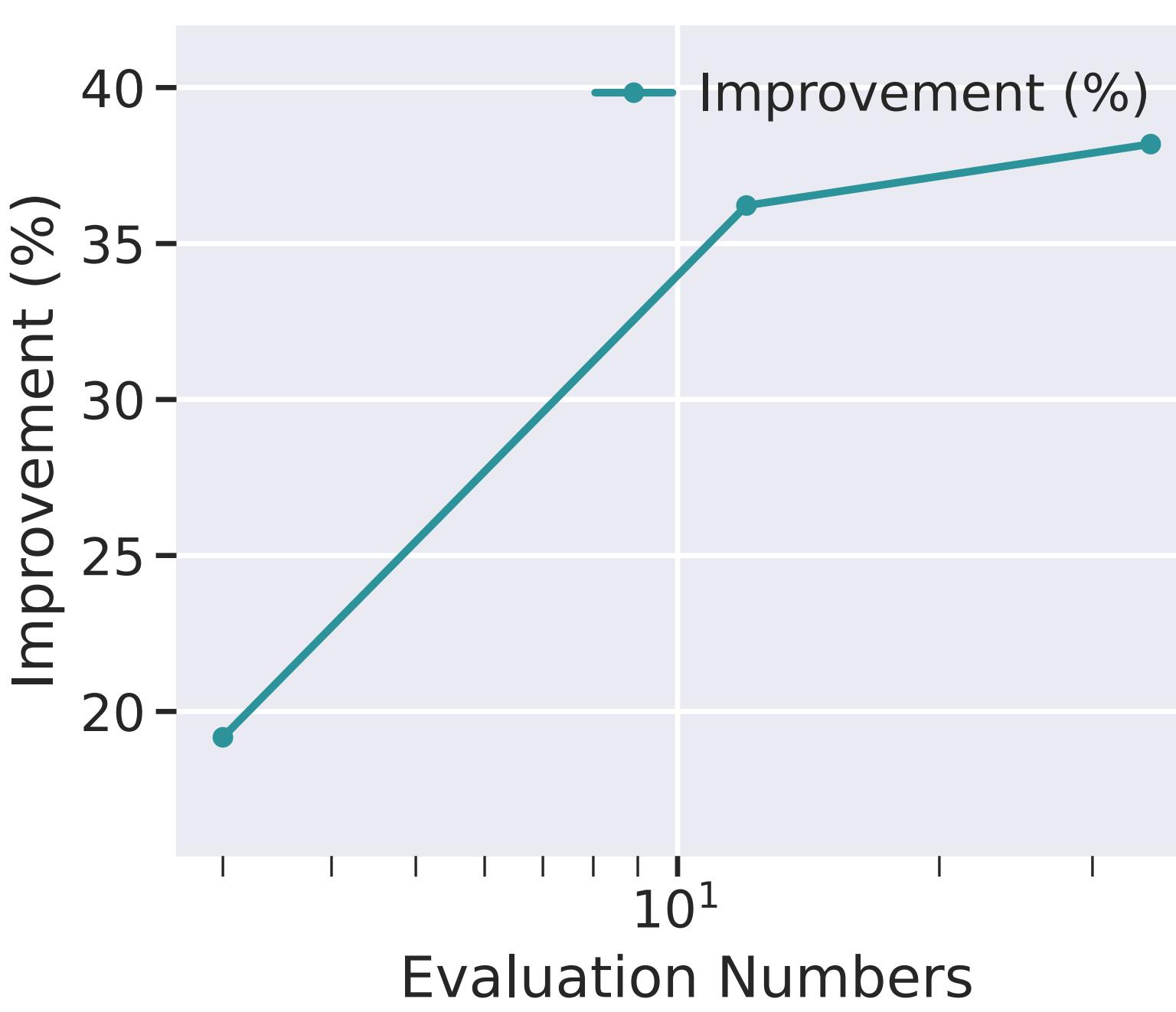
Numerical Results

		Time (s)			Relative L^2 Error			L^∞ Error			L^1 Error		
		SR	MLP	SCaSML	SR	MLP	SCaSML	SR	MLP	SCaSML	SR	MLP	SCaSML
LCD	10d	2.64	11.24	23.75	5.24E-02	2.27E-01	2.73E-02	2.50E-01	9.06E-01	1.61E-01	3.43E-02	1.67E-01	1.78E-02
	20d	1.14	7.35	17.59	9.09E-02	2.35E-01	4.73E-02	4.52E-01	1.35E+00	3.28E-01	9.47E-02	2.37E-01	4.52E-02
	30d	1.39	7.52	25.33	2.30E-01	2.38E-01	1.84E-01	4.73E+00	1.59E+00	1.49E+00	1.75E-01	2.84E-01	1.91E-01
	60d	1.13	7.76	35.58	3.07E-01	2.39E-01	1.32E-01	3.23E+00	2.05E+00	1.55E+00	5.24E-01	4.07E-01	2.06E-01
VB-PINN	20d	1.15	7.05	13.82	1.17E-02	8.36E-02	3.97E-03	3.16E-02	2.96E-01	2.16E-02	5.37E-03	3.39E-02	1.29E-03
	40d	1.18	7.49	16.48	3.99E-02	1.04E-01	2.85E-02	8.16E-02	3.57E-01	7.16E-02	1.97E-02	4.36E-02	1.21E-02
	60d	1.19	7.57	19.83	3.97E-02	1.17E-01	2.90E-02	8.10E-02	3.93E-01	7.10E-02	1.95E-02	4.82E-02	1.24E-02
	80d	1.32	7.48	21.99	6.78E-02	1.19E-01	5.68E-02	1.89E-01	3.35E-01	1.79E-01	3.24E-02	4.73E-02	2.49E-02
VB-GP	20d	1.97	10.66	65.46	1.47E-01	8.32E-02	5.52E-02	3.54E-01	2.22E-01	2.54E-01	7.01E-02	3.50E-02	1.91E-02
	40d	1.68	10.14	49.38	1.81E-01	1.05E-01	7.95E-02	4.01E-01	3.47E-01	3.01E-01	9.19E-02	4.25E-02	3.43E-02
	60d	1.01	7.25	35.14	2.40E-01	2.57E-01	1.28E-01	3.84E-01	9.50E-01	7.10E-02	1.27E-01	9.99E-02	6.11E-02
	80d	1.00	7.00	38.26	2.66E-01	3.02E-01	1.52E-01	3.62E-01	1.91E+00	2.62E-01	1.45E-01	1.09E-01	7.59E-02
LQG	100d	1.54	8.67	26.95	7.96E-02	5.63E+00	5.51E-02	7.78E-01	1.26E+01	6.78E-01	1.40E-01	1.21E+01	8.68E-02
	120d	1.25	8.17	27.46	9.37E-02	5.50E+00	6.64E-02	9.02E-01	1.27E+01	8.02E-01	1.73E-01	1.22E+01	1.05E-01
	140d	1.80	8.27	29.72	9.79E-02	5.37E+00	6.78E-02	1.00E+00	1.27E+01	9.00E-01	1.91E-01	1.23E+01	1.11E-01
	160d	1.74	9.07	32.08	1.11E-01	5.27E+00	9.92E-02	1.38E+00	1.28E+01	1.28E+00	2.15E-01	1.23E+01	1.79E-01
DR	100d	1.62	7.75	60.86	9.52E-03	8.99E-02	8.87E-03	7.51E-02	6.37E-01	6.51E-02	1.13E-02	9.74E-02	1.11E-02
	120d	1.26	7.28	65.66	1.11E-02	9.13E-02	9.90E-03	7.10E-02	5.74E-01	6.10E-02	1.40E-02	9.97E-02	1.23E-02
	140d	2.38	7.82	76.90	3.17E-02	8.97E-02	2.94E-02	1.79E-01	8.56E-01	1.69E-01	3.96E-02	9.77E-02	3.67E-02
	160d	1.75	7.42	82.40	3.46E-02	9.00E-02	3.23E-02	2.08E-01	8.02E-01	1.98E-01	4.32E-02	9.75E-02	4.02E-02

Inference-Time Scaling

$$\frac{\partial}{\partial t} u + \left[\sigma^2 u - \frac{1}{d} - \frac{\bar{\sigma}^2}{2} \right] (\nabla \cdot u) + \frac{\bar{\sigma}^2}{2} \Delta u = 0$$

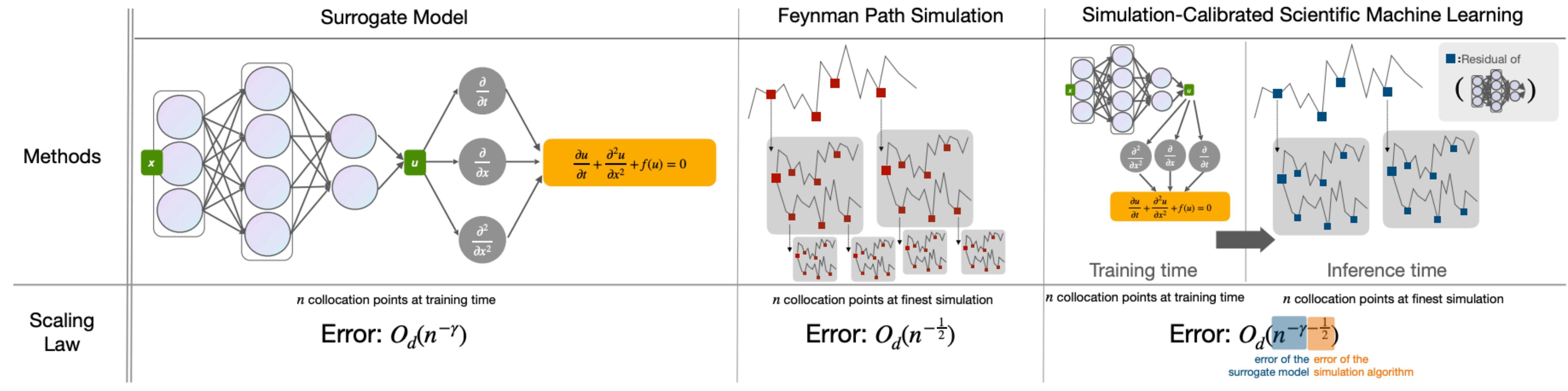
have closed-form solution $g(x) = \frac{\exp(T + \sum_i x_i)}{1 + \exp(T + \sum_i x_i)}$



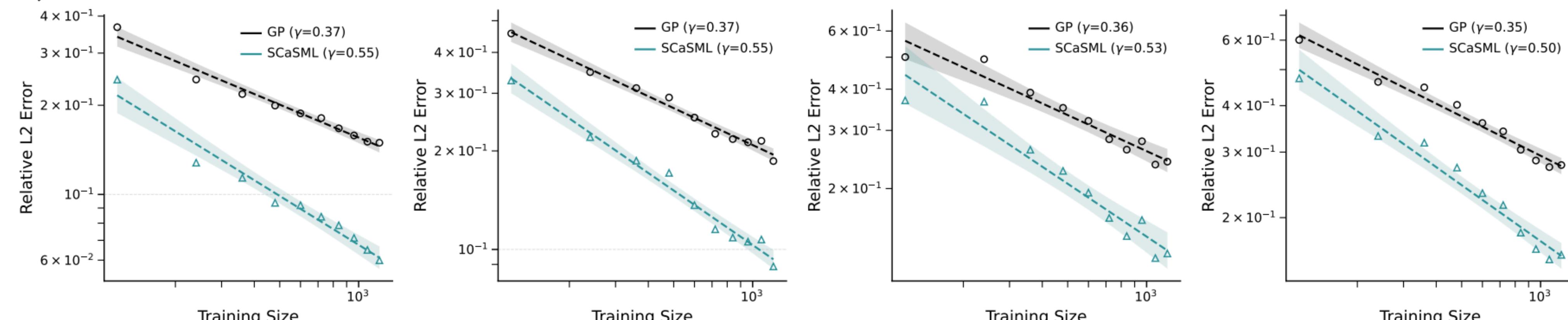
Method	Convergence Rate
PINN	$O(n^{-s/d})$
MLP	$O(n^{-1/4})$
SCaSML	$O(n^{-1/4-s/d})$

Better Scaling Law

a)



b)

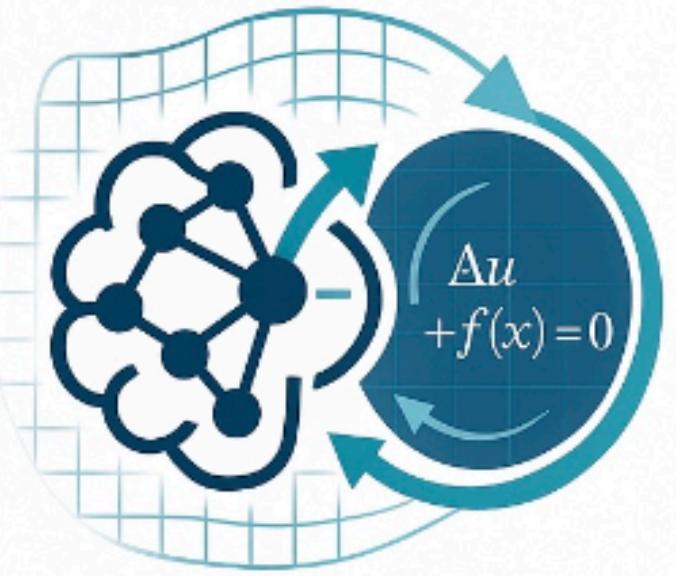


(a) $d = 20$

(b) $d = 40$

(c) $d = 60$

(d) $d = 80$



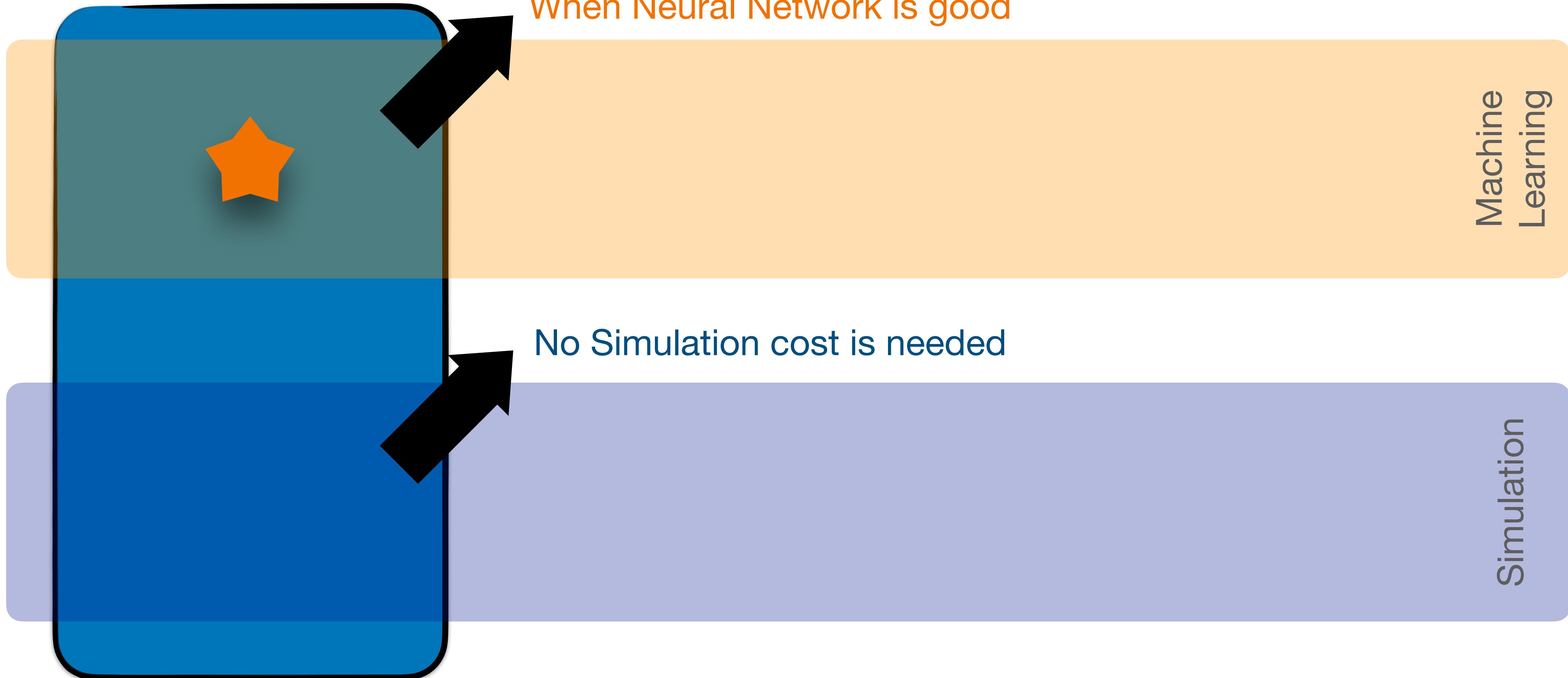
Physics-Informed Inference Time Scaling via Simulation-Calibrated Scientific Machine Learning

Zexi Fan¹, Yan Sun², Shihao Yang³, Yiping Lu^{*4}

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https://2prime.github.io/files/scasml_techreport.pdf

Our Aim Today : A Marriage



Our Aim Today : A Marriage

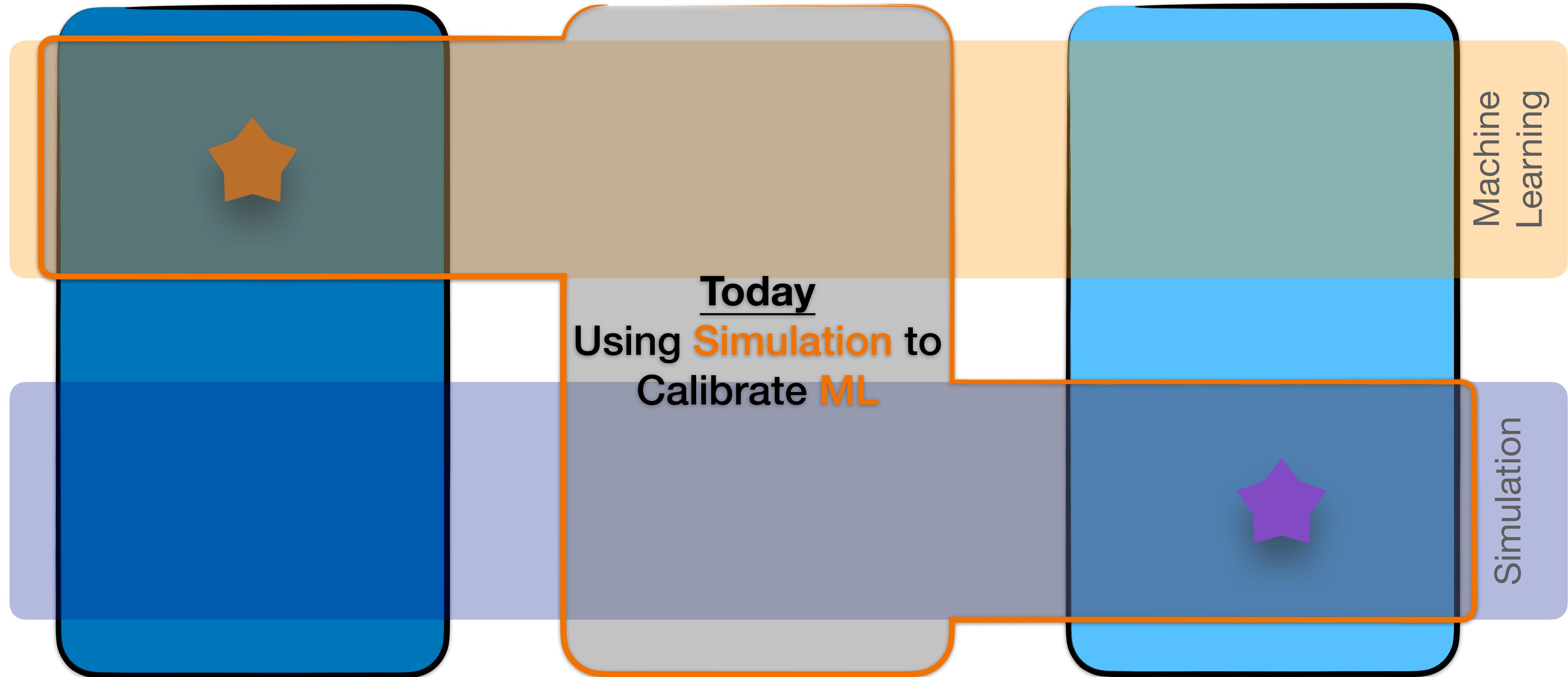
When Neural Network is bad

Provide pure Simulation solution

Machine
Learning

Simulation

Our AIM Today: A Marriage



More Examples...



Scientific Machine Learning

Downstream application

Example 1

$$\theta = f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \int f^q(x) dx$$

Example 2

$$\theta = \Delta^{-1}f, \quad X_i = (x_i, f(x_i))$$

$$\Phi(\theta) = \theta(x)$$

Example 3

$$\theta = A, \quad X_i = (x_i, Ax_i)$$

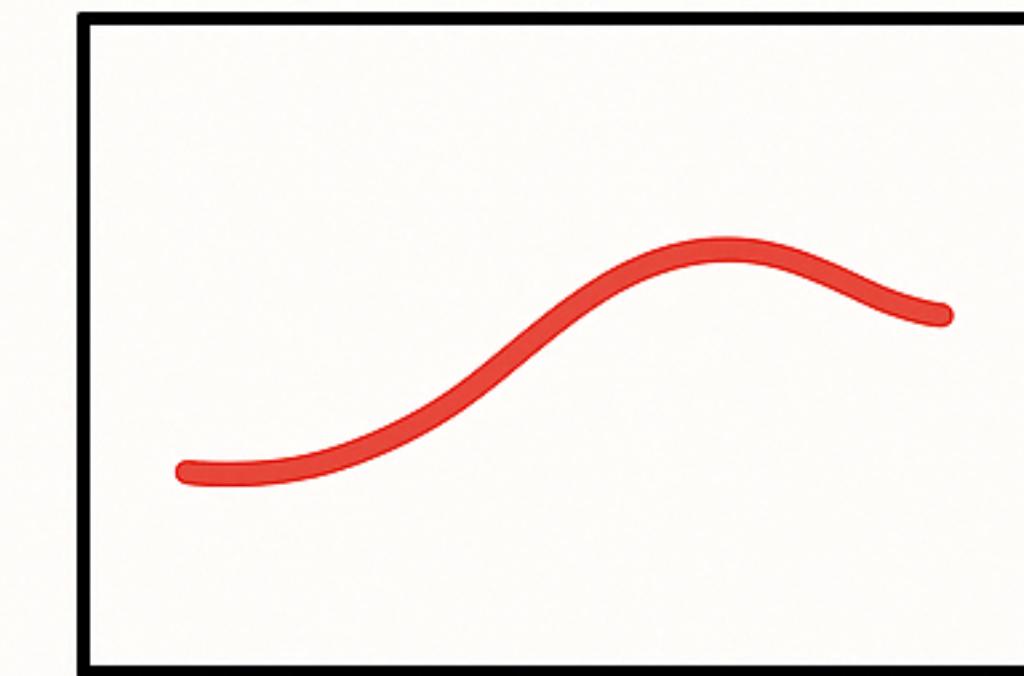
Estimation \hat{A} via Randomized SVD

$$\Phi(\theta) = \text{tr}(A)$$

Estimate $\text{tr}(A - \hat{A})$ via Hutchinson's estimator

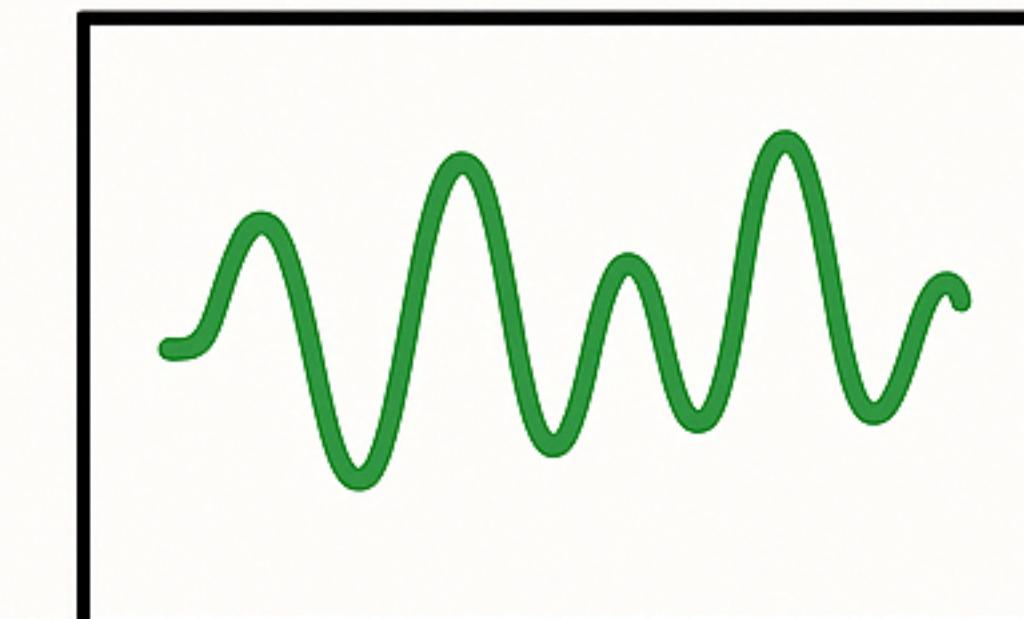
A multiscale view

Capture via surrogate model



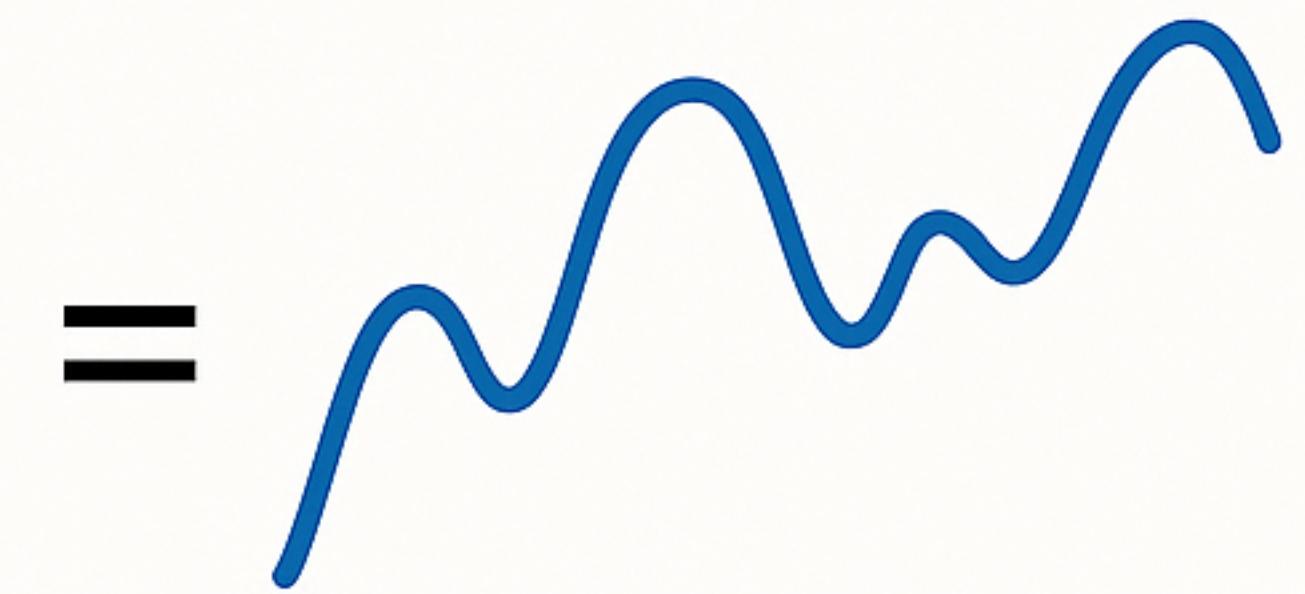
Coarse Scale

+



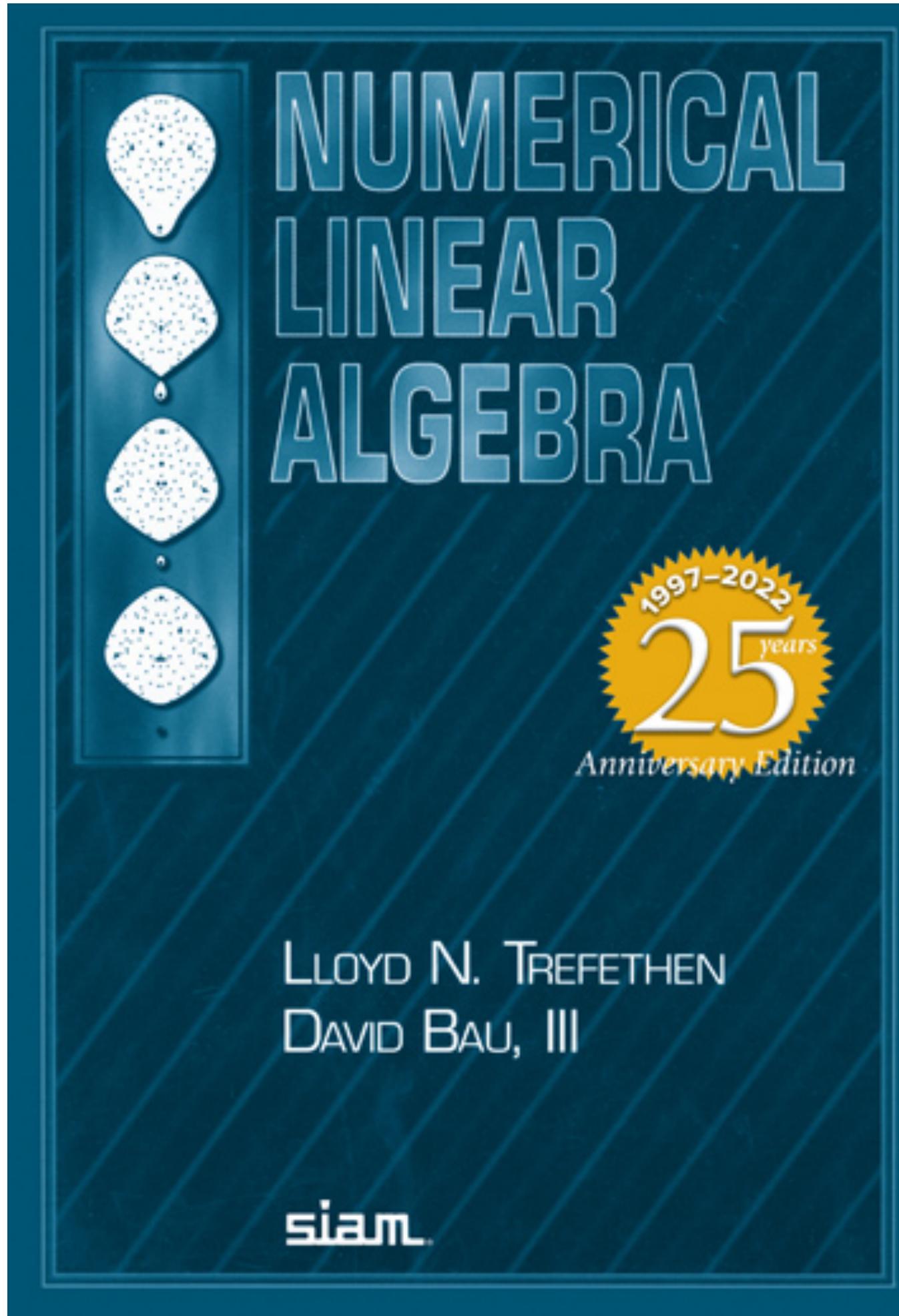
Fine Scale

True
Function



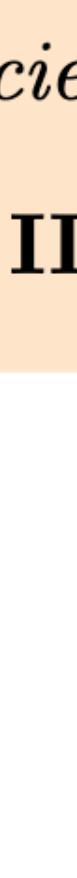
Tale 2: Pre-condition with a surprising connection with debiasing

Tale 2: Preconditioning



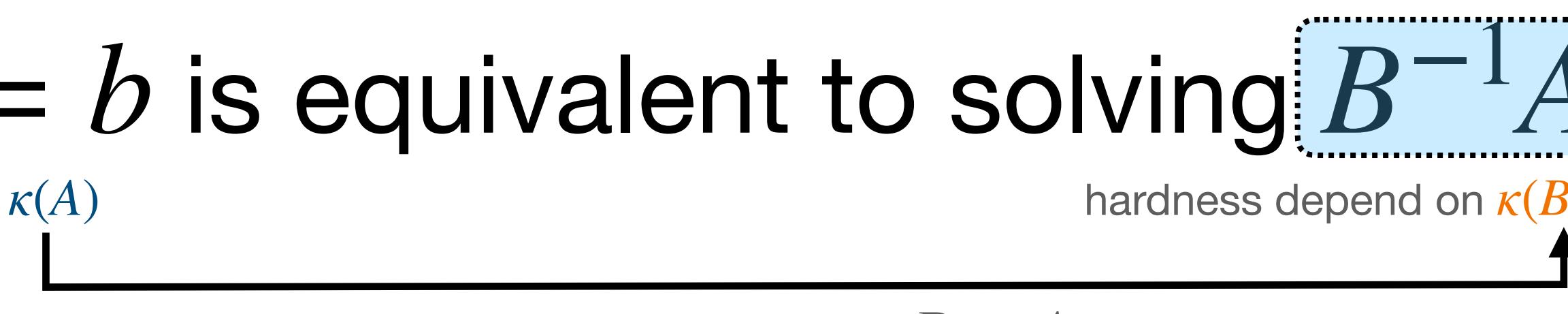
"In ending this book with the subject of preconditioners, we find ourselves at the philosophical center of the scientific computing of the future."

— L. N. Trefethen and D. Bau III, Numerical Linear Algebra [TB22]



Nothing will be more central to computational science in the next century than **the art of transforming a problem that appears intractable into another whose solution can be approximated rapidly**.

What is precondition

- Solving $Ax = b$ is equivalent to solving $B^{-1}Ax = B^{-1}b$
hardness depend on $\kappa(A)$ hardness depend on $\kappa(B^{-1}A)$


Become easier when $B \approx A$

A New Way to Implement Precondition

- Debiasing is a way of solving $Ax = b$
 - Using an approximate solver $Bx_1 = b$
- } Error depends on $\|A^{-1}(A - B)\|$

A New Way to Implement Precondition

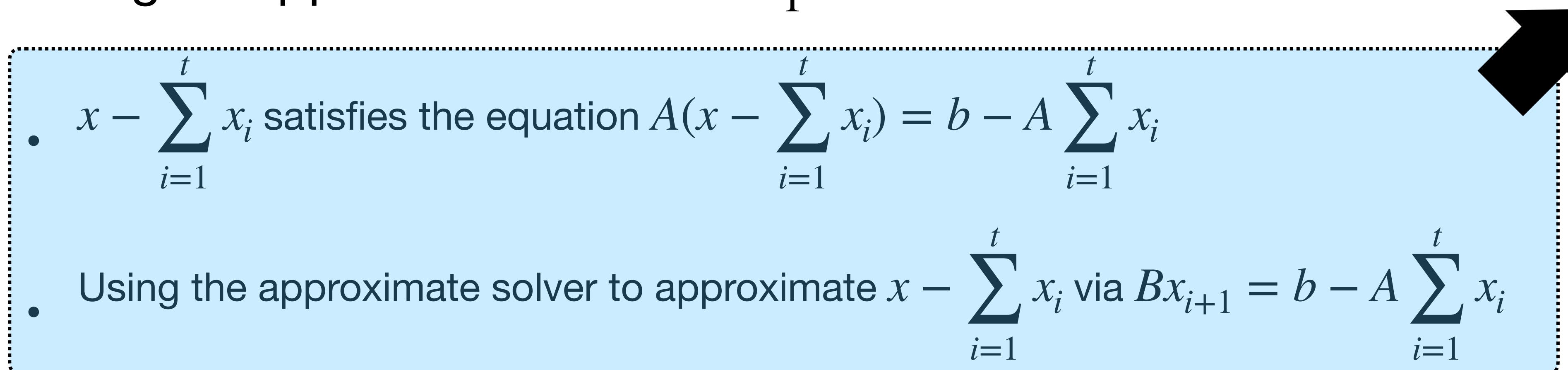
- Debiasing is a way of solving $Ax = b$
 - Using an approximate solver $Bx_1 = b$
 - $x - x_1$ satisfies the equation $A(x - x_1) = b - Ax_1$
 - Using the approximate solver to approximate $x - x_1$ via $Bx_2 = b - Ax_1$

Easy to solve for $b - Ax_1$ is small

A New Way to Implement Precondition

- Debiasing is a way of solving $Ax = b$
 - Using an approximate solver $Bx_1 = b$

Iterative Refinement Algorithm



A New Way to Implement Precondition

- Debiasing is a way of solving $Ax = b$

- Using an approximate solver $Bx_1 = b$

Iterative Refinement Algorithm

. $x - \sum_{i=1}^t x_i$ satisfies the equation $A(x - \sum_{i=1}^t x_i) = b - A \sum_{i=1}^t x_i$

. Using the approximate solver to approximate $x - \sum_{i=1}^t x_i$ via $Bx_{i+1} = b - A \sum_{i=1}^t x_i$

$$x_{i+1} = (I - B^{-1}A)x_i + B^{-1}b$$

Preconditioned Jacobi Iteration

This Talk: A New Way to Implement Precondition Via Debiasing

- **Step 1:** Aim to solve (potentially nonlinear) equation $A(u) = b$

use Machine Learning

- **Step 2:** Build an approximate solver $A(\hat{u}) \approx b$

Unreliable approximate
solver as preconditioner

- Via machine learning/sketching/finite element....

- **Step 3:** Solve $u - \hat{u}$



AIM: Debiasing a Learned Solution = Using Learned Solution as preconditioner!

Randomized NLA as Machine Learning

AIM: using matrix-vector multiplication to compute eigenvalue/least square problem

$$\underbrace{\{X_1, \dots, X_n\} \sim P_\theta \rightarrow \theta \rightarrow \Phi(\theta)}_{A} \quad \Phi(A) = \begin{cases} A^{-1}b \\ \text{Eigenvalue of } A \end{cases}$$

$X_i = (x, Ax)$

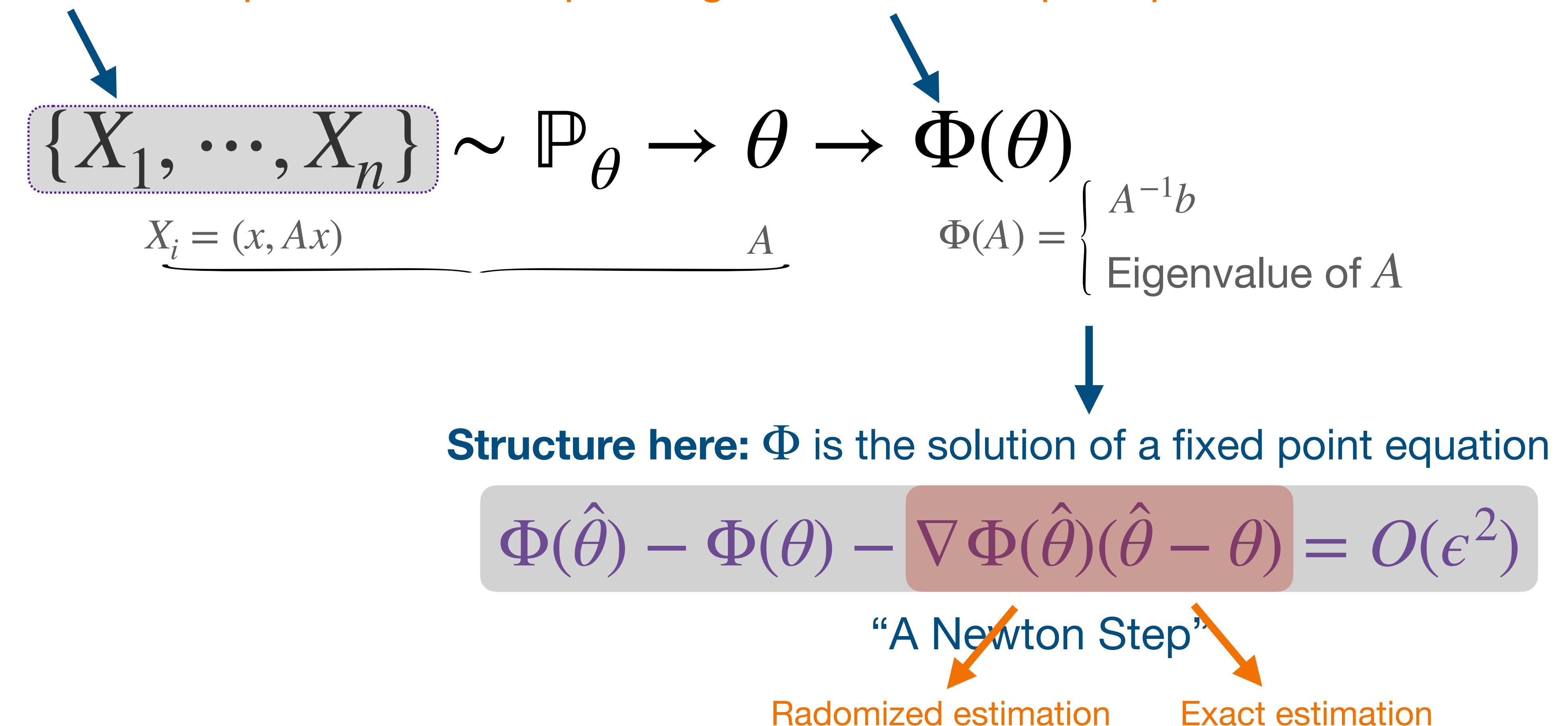
“Randomized Numerical Linear Algebra”/Sketching



“Sketch-and-Solve”

Randomized NLA as Machine Learning

AIM: using matrix-vector multiplication to compute eigenvalue/least square problem



(In)exact Sub-sample Newton Method/Sketch-and-Precondition

Relationship with Inverse Power Methods

(Approximate) Inverse Power Method	Our Method
$X_{n+1} = (\lambda I - A)^\dagger X_n$	$X_{n+1} = \boxed{(\lambda I - \hat{A})^\dagger}_{\nabla \Phi(\hat{\theta})} \boxed{(A - \hat{A})X_n}_{(\theta - \hat{\theta})}$

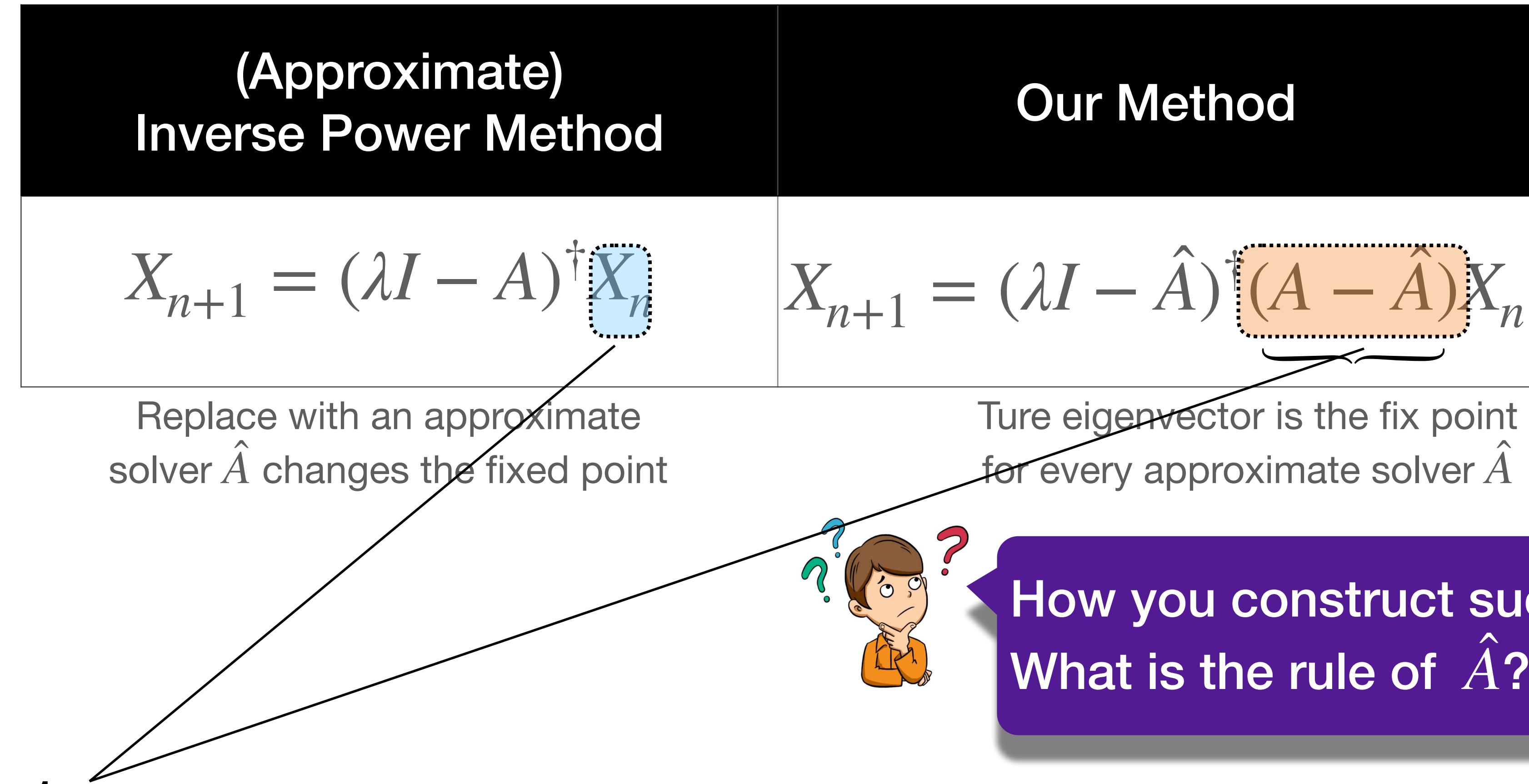
Relationship with Inverse Power Methods

(Approximate) Inverse Power Method	Our Method
$X_{n+1} = (\lambda I - A)^\dagger X_n$	$X_{n+1} = (\lambda I - \hat{A})^\dagger \underbrace{(A - \hat{A})}_{\text{True eigenvector is the fix point}} X_n$

Replace with an approximate
solver \hat{A} changes the fixed point

True eigenvector is the fix point
for every approximate solver \hat{A}

Relationship with Inverse Power Methods



Take Home Message 1:

Power the Residual but not Power the vector

Why better than Directly DMD

“Sketch-and-Solve” VS “Sketch-and-Precondition”

	Sketch-and-Solve	Sketch-and-Precondition
Least Square		Sketch-and-precondition, Sketch-and-project, Iterataive Sketching,
Low rank Approx	Idea 1: plug in a SVD Solver: Random SVD Idea 2: plug in a inverse power method	 <u>Our Work!</u>

Use sketched matrix \hat{A} as
an approximation to A

Use sketched matrix \hat{A} as
an precondition to the probelm



Sorry... but I can't see the
relationship....

Why better than Directly DMD

“Sketch-and-Solve” VS “Sketch-and-Precondition”

	Sketch-and-Solve	Sketch-and-Precondition
Least Square		Sketch-and-precondition, Sketch-and-project, Iterataive Sketching,
Low rank Approx	Idea 1: plug in a SVD Solver: Random SVD Idea 2: plug in a inverse power method	 <u>Our Work!</u>

Use sketched matrix \hat{A} as
an approximation to A

Use sketched matrix \hat{A} as
an precondition to the probelm



Idea: using (approximate) Newton method to solve the Lagrange from

$$\min u^\top A u - \lambda(x^\top x - 1)$$

Thus Our convergence is u linear-quadratic



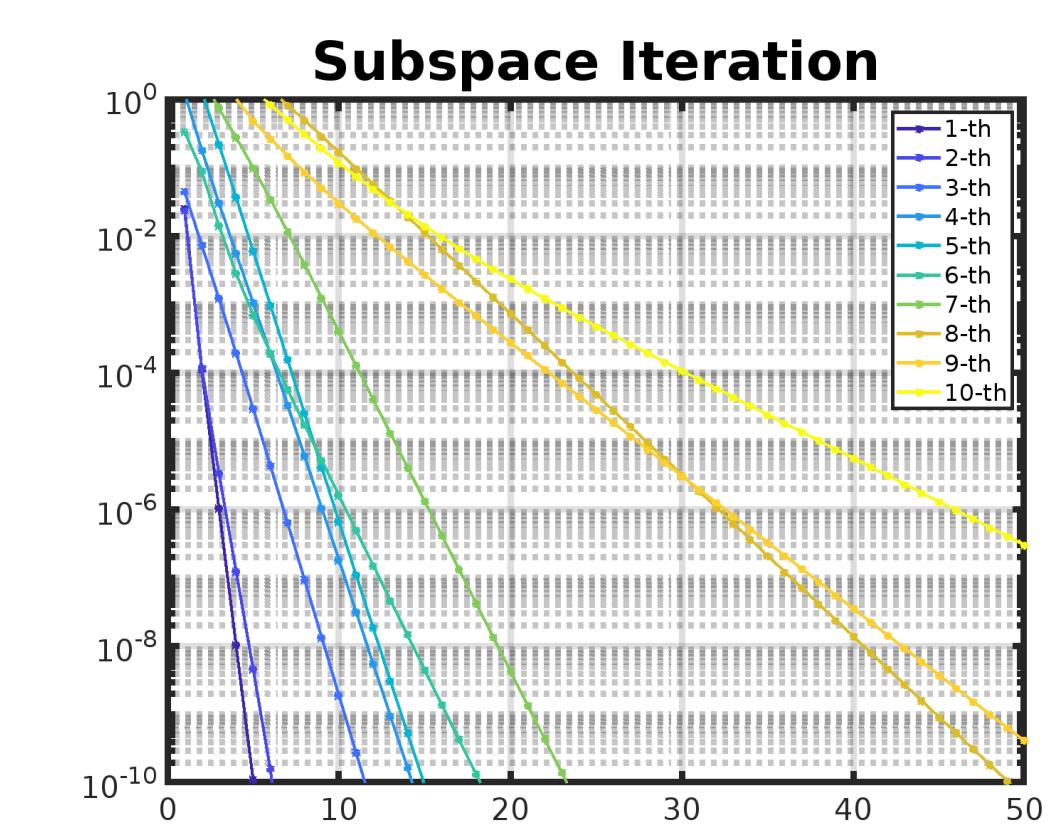
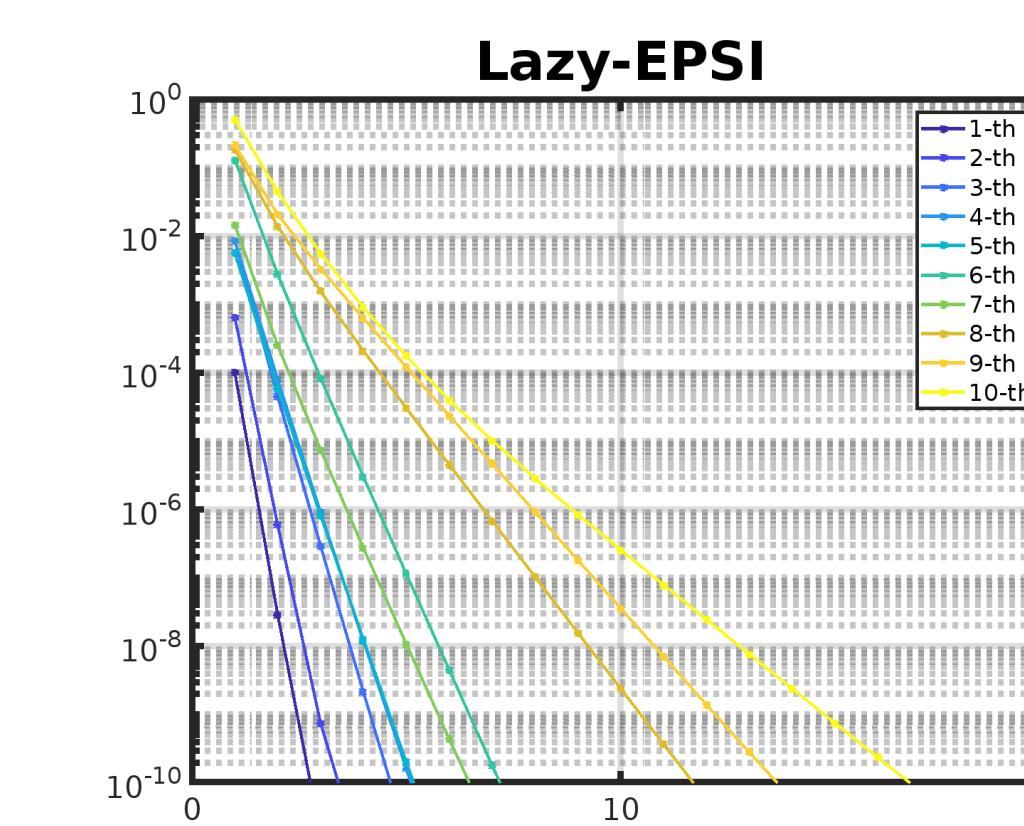
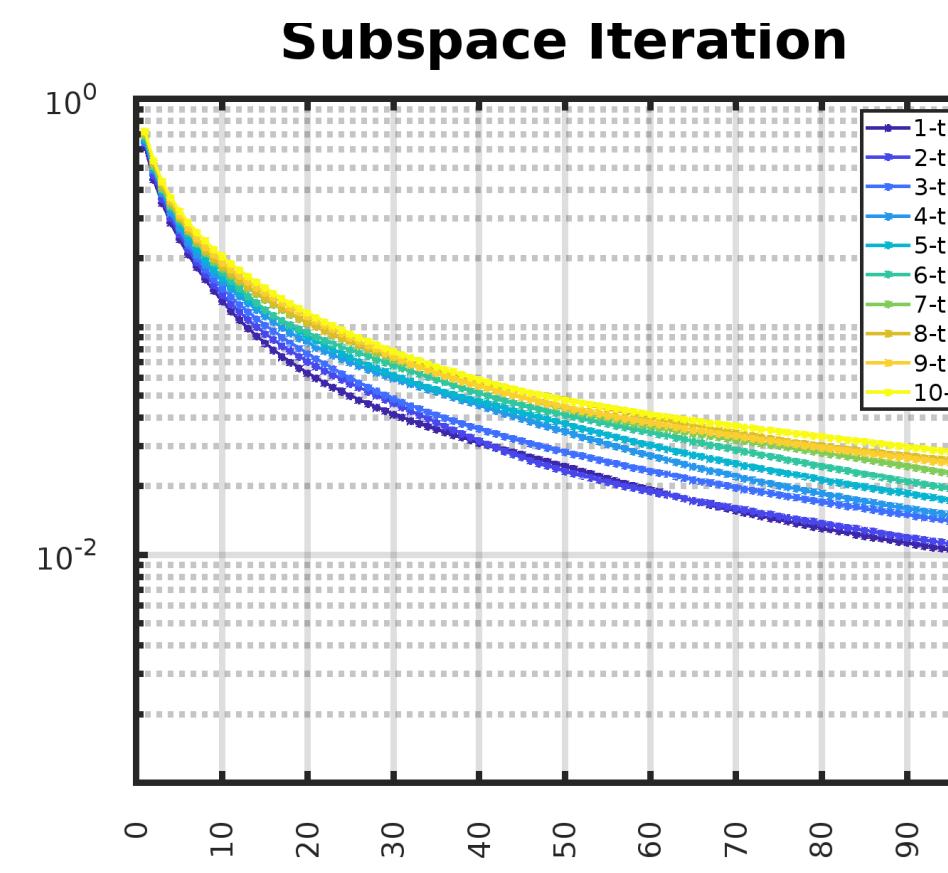
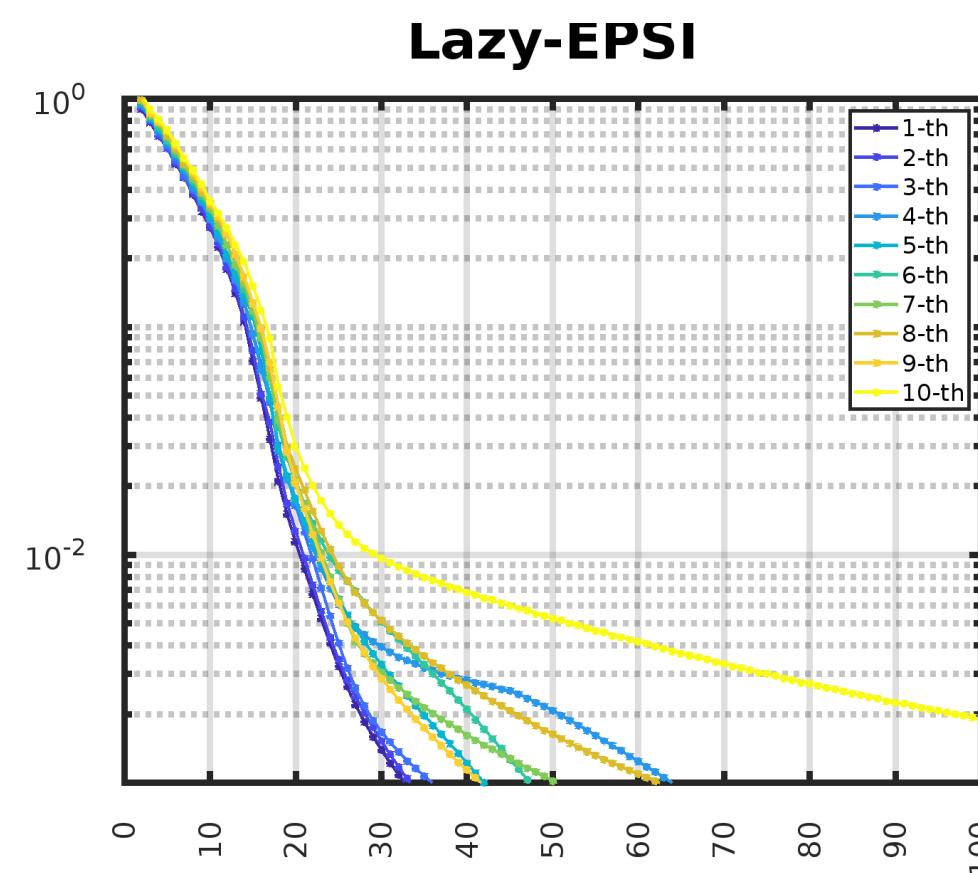
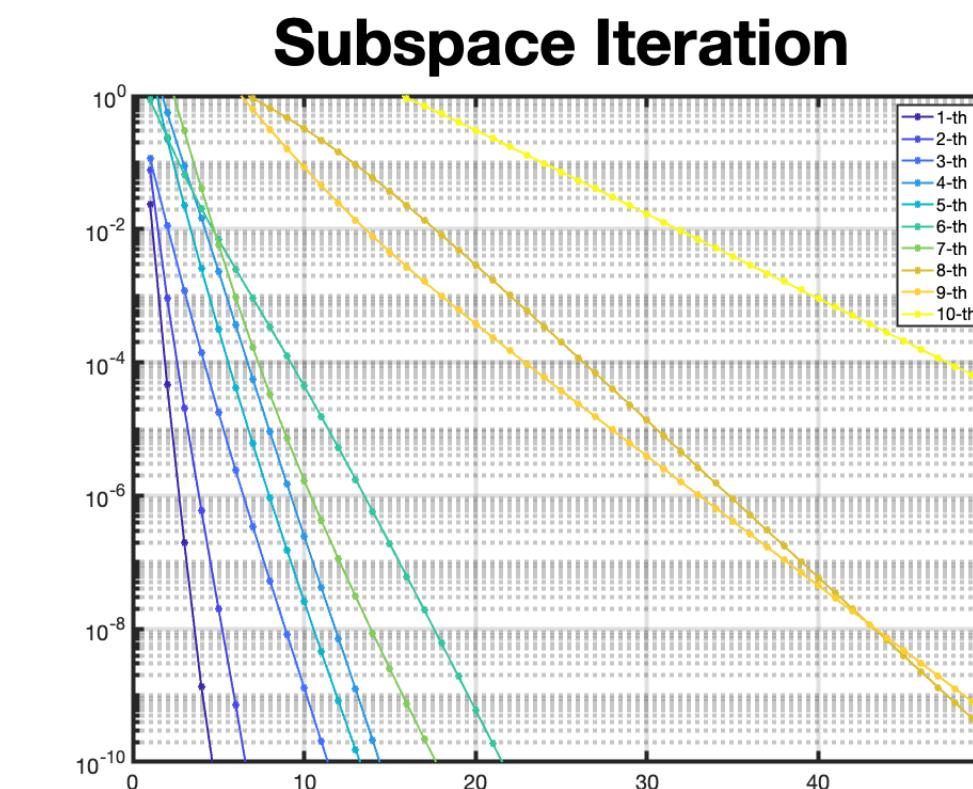
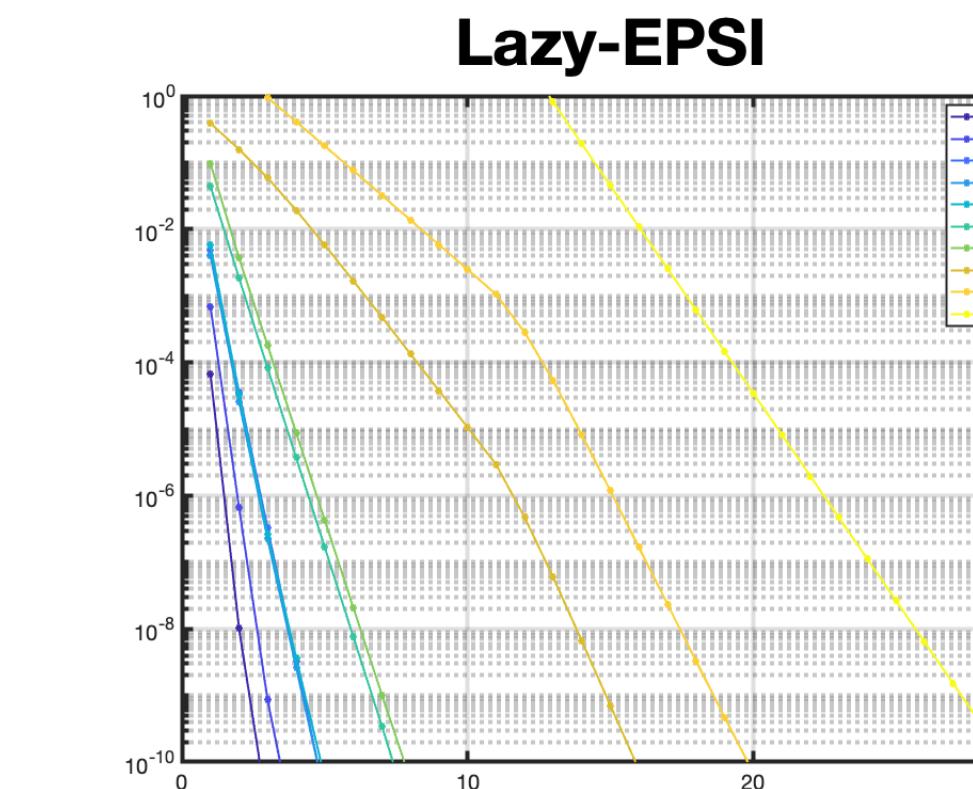
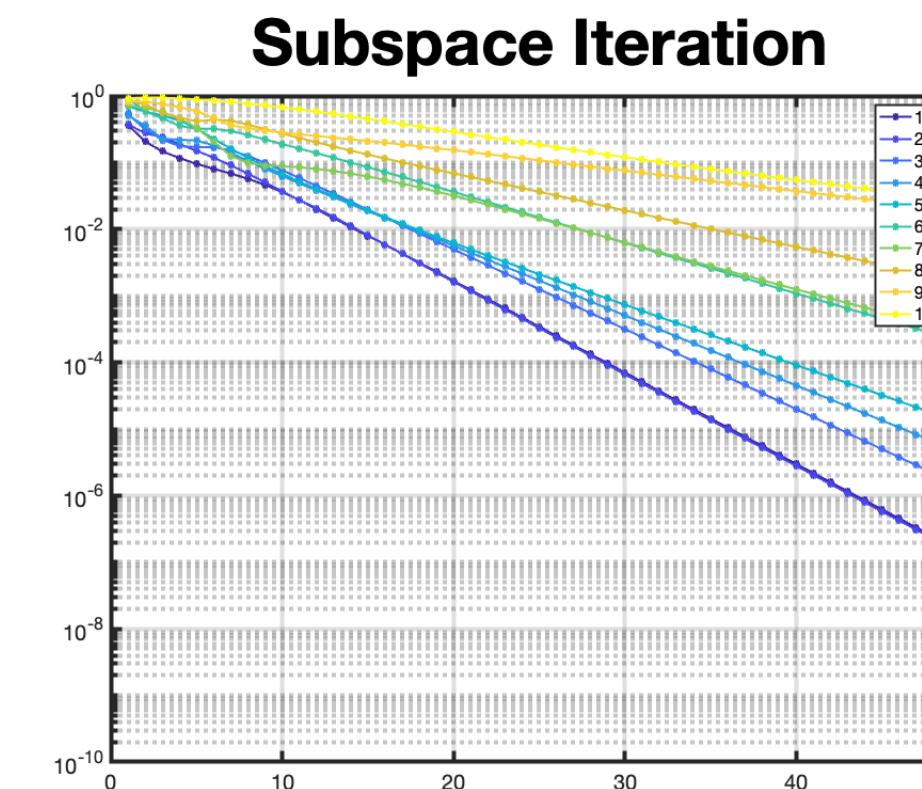
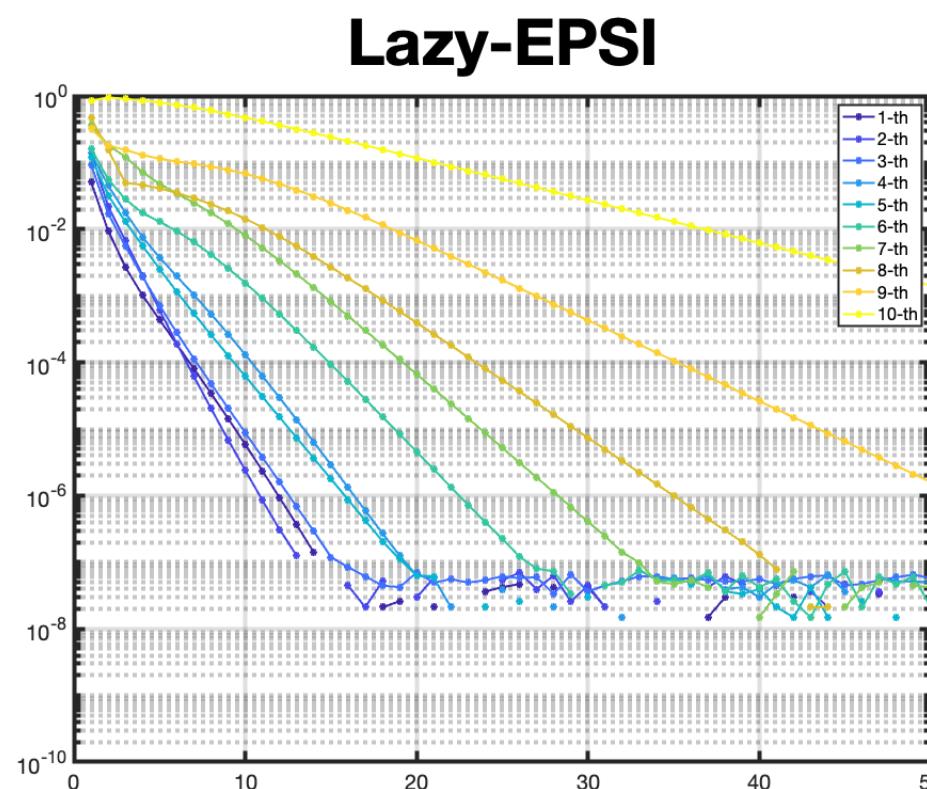
Contraction coefficient improves when sketching quality increases



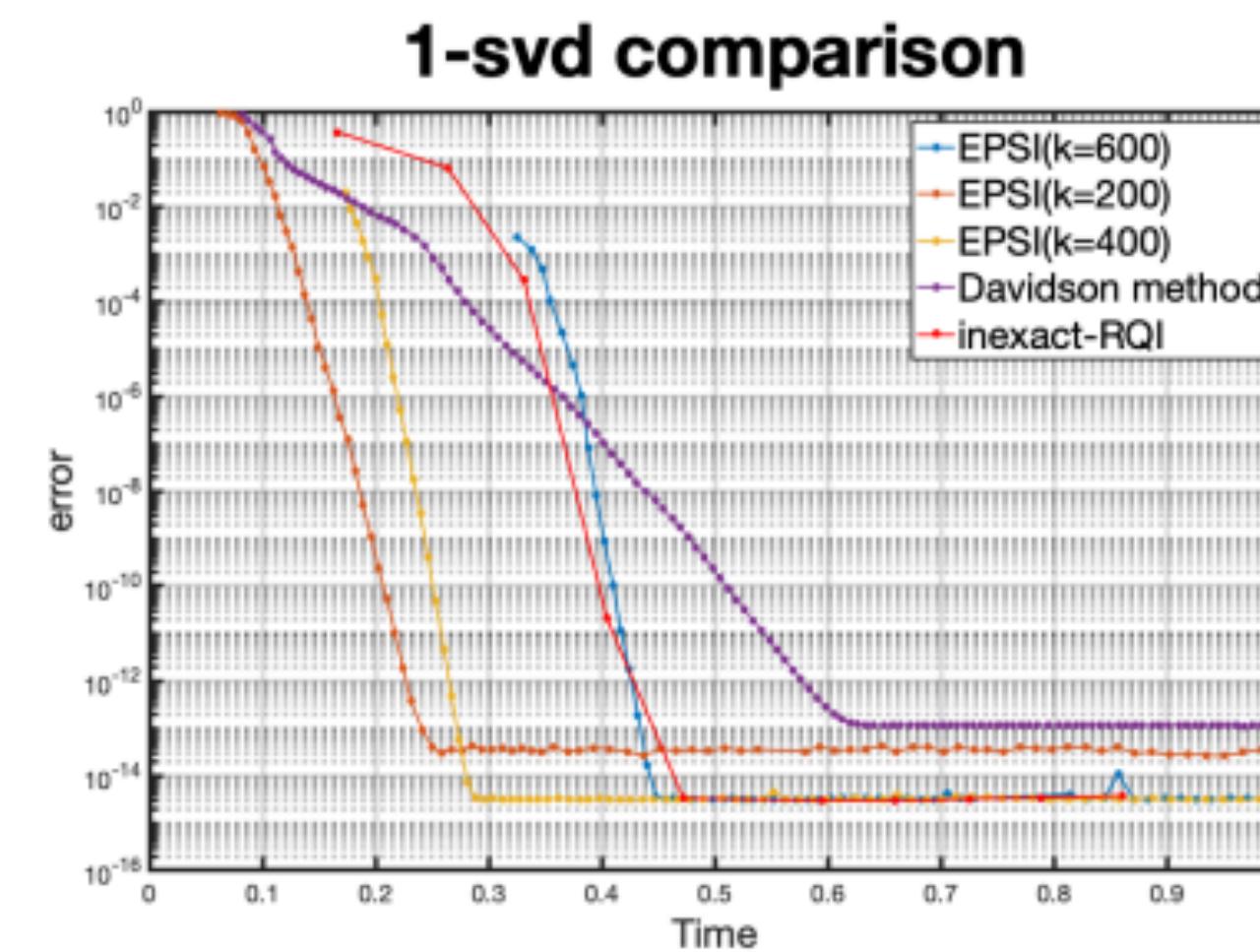
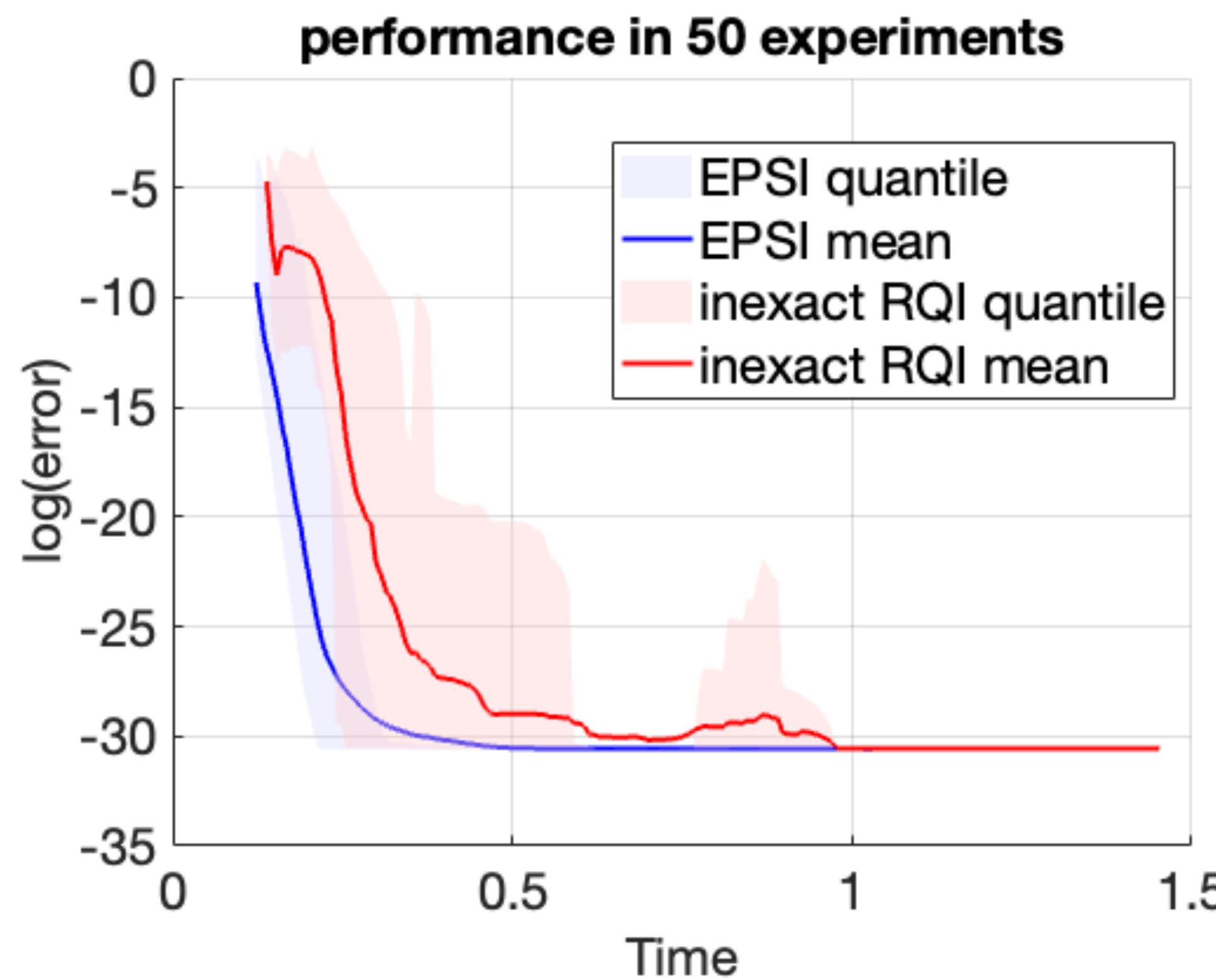
Look back to literature

	Overparameterized Least Square	Low Rank Approximation	
Sketch-and-Solve	[Sar06, DMMS11]	[LWM ⁺ 07, HMST11]	approximate in geometry-preserving, lower-dimensional space.
Randomized Initialization		[Gu15, MM15]	Initialize an (iterative) algorithm at a random point to make a favorable trajectory
Sketch-and-Precondition	[RT08, AMT10] [MSM14, DR24]	This work	boosting solver convergence while preserving the original problem's accuracy

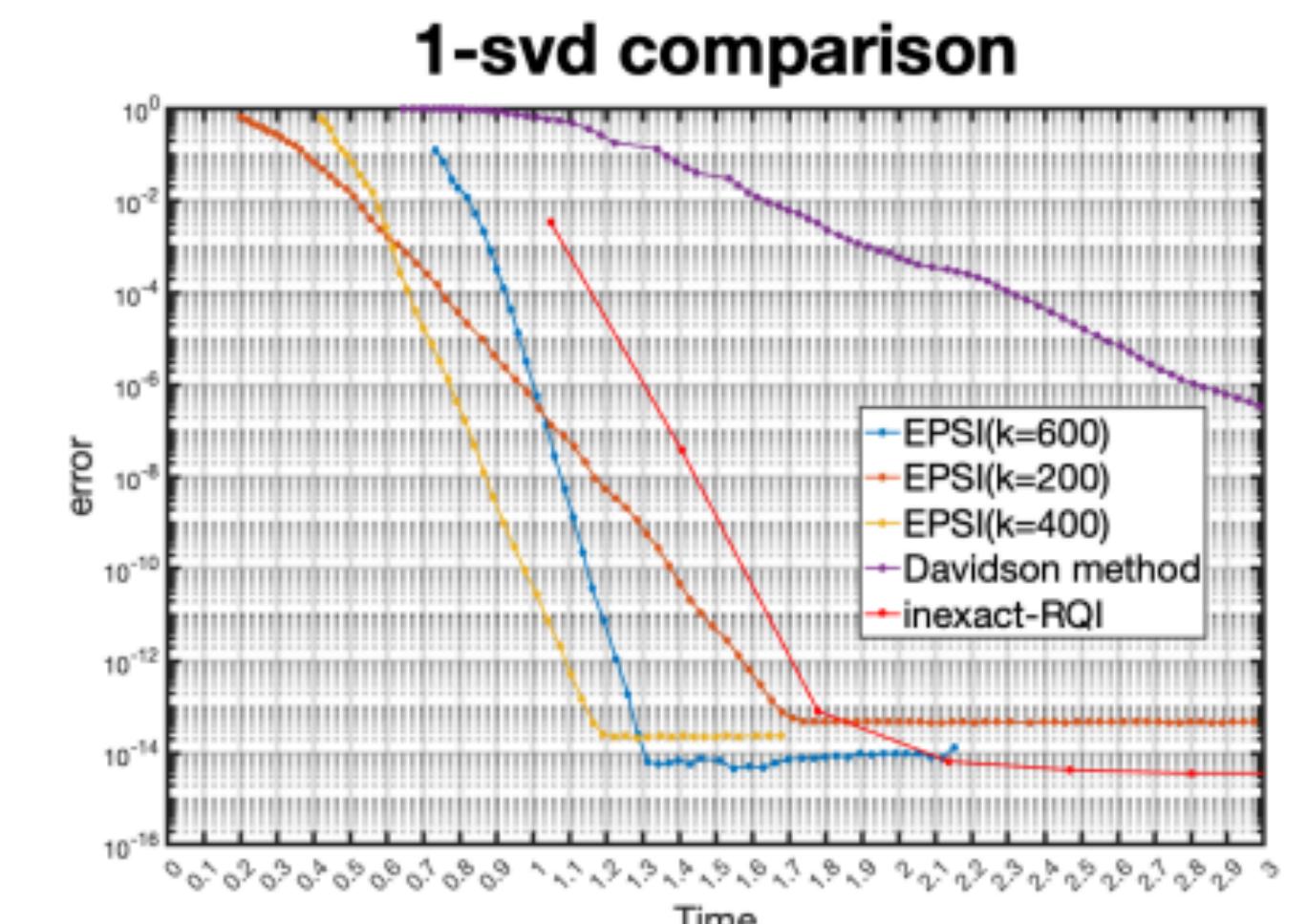
Eigenvalue Computation



Running Time



(b) $n = 2000, \kappa = 10^{-6}$



(c) $n = 4000, \kappa = 10^{-6}$

What is a Sketch-and-Precondition Derivation for Low-Rank Approximation? Inverse Power Error or Inverse Power Estimation?

Ruihan Xu *

Yiping Lu †

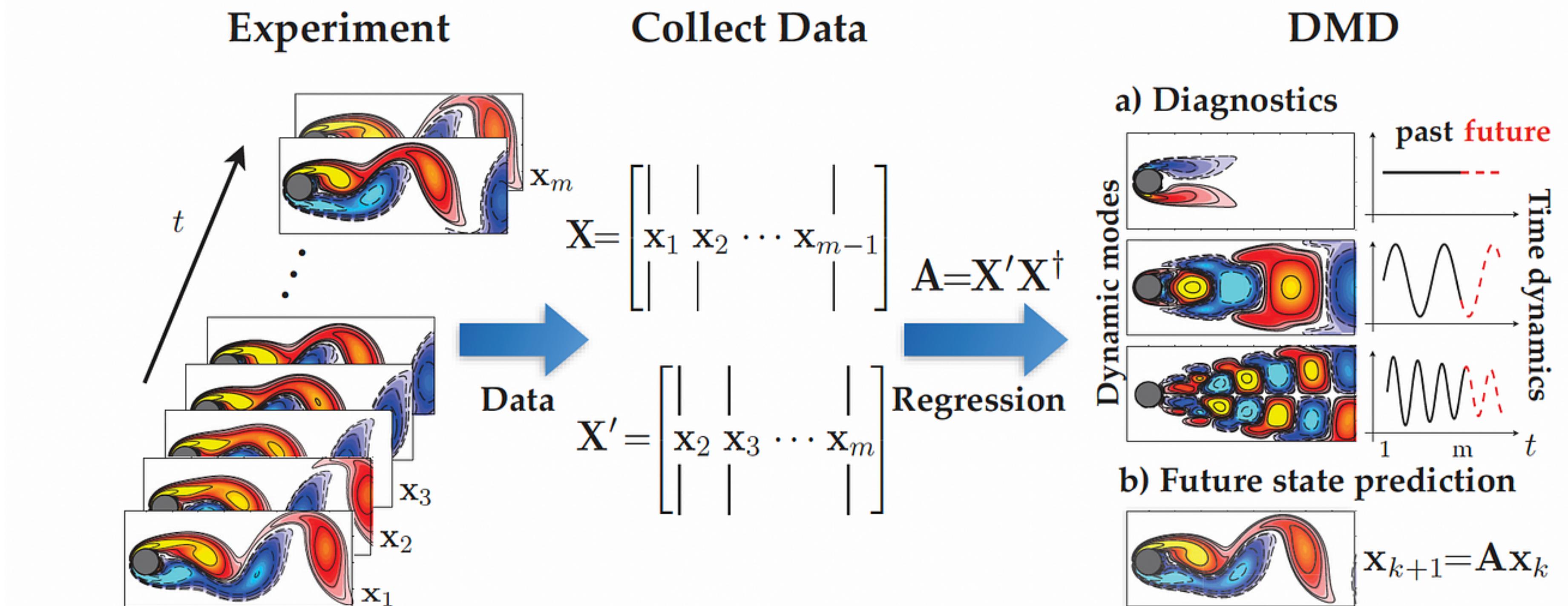
Abstract

Randomized sketching accelerates large-scale numerical linear algebra by reducing computational complexity. While the traditional sketch-and-solve approach reduces the problem size directly through sketching, the sketch-and-precondition method leverages sketching to construct a computational friendly preconditioner. This preconditioner improves the convergence speed of iterative solvers applied to the original problem, maintaining accuracy in the full space. Furthermore, the convergence rate of the solver improves at least linearly with the sketch size. Despite its potential, developing a sketch-and-precondition framework for randomized algorithms in low-rank matrix approximation remains an open challenge. We introduce the *Error-Powered Sketched Inverse Iteration* (EPSI) Method via run sketched Newton iteration for the Lagrange form as a sketch-and-precondition variant for randomized low-rank approximation. Our method achieves theoretical guarantees, including a convergence rate that improves at least linearly with the sketch size.

Another Supersing Fact...

Iteration lies in the Krylov Subspace

- enable dynamic mode decomposition
- Online fast update
- Much better than DMD



Another Supersing Fact...

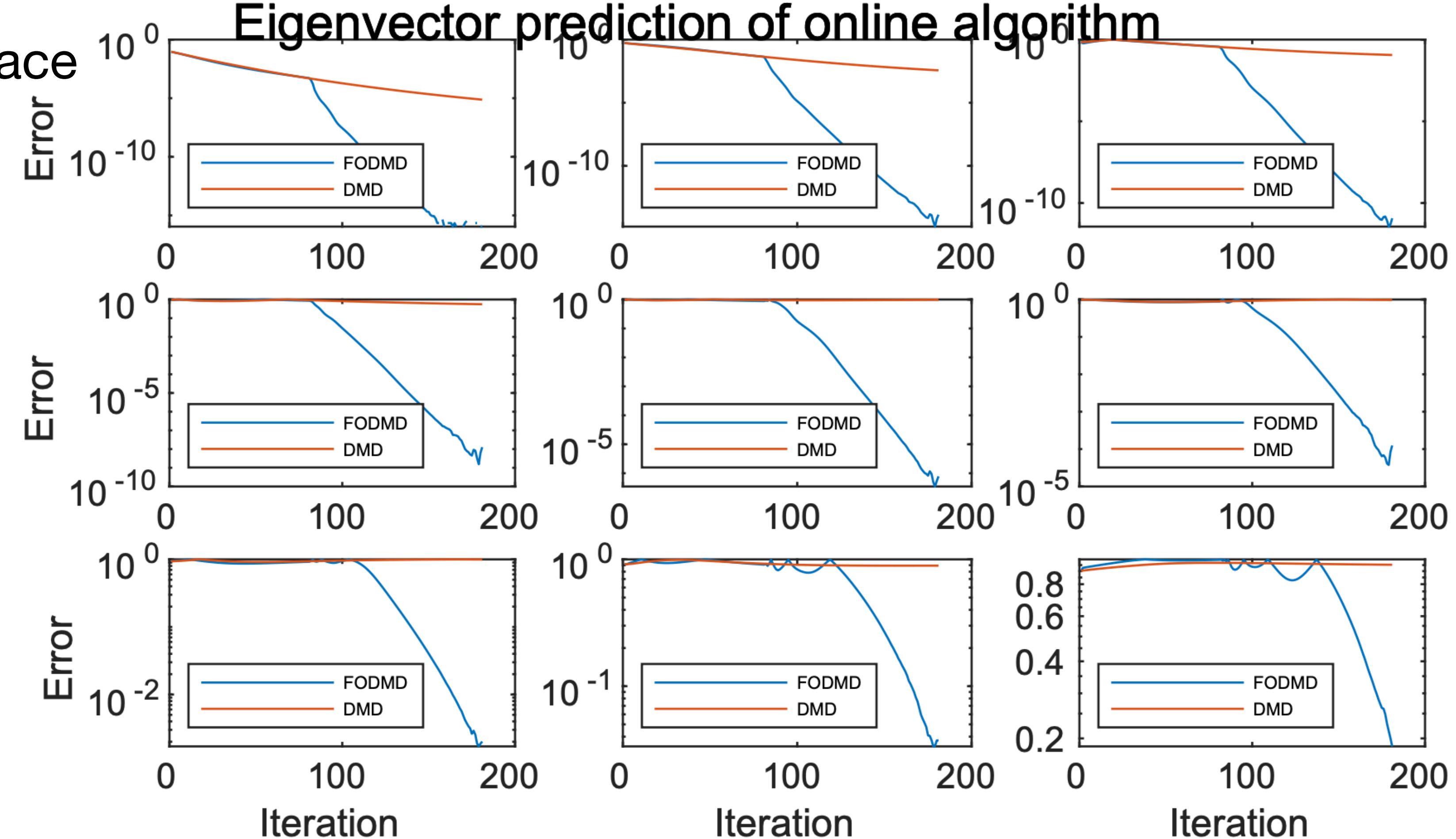
Iteration lies in the Krylov Subspace

- enable dynamic mode decomposition
- Online fast update
- Much better than DMD

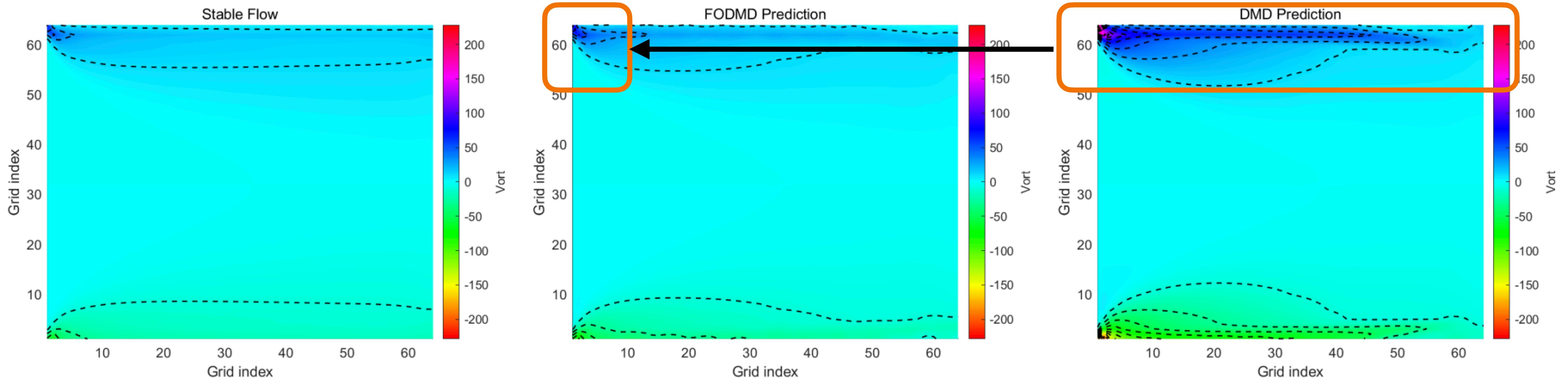
No matrix inverse, No SVD computation

Only a $n \times r$ QR iteration
(Everything has closed-form solution)

Eigenvector prediction of online algorithm



Prediction of Tube Flow



Open Question

Multiple Eigenvectors

Algorithm 2 Lazy-EPSI for Computing the First k Singular Vectors

Require: A , the input matrix; $k \in \mathbb{Z}^+$, the number of components; $q_{\max} \in \mathbb{Z}^+$, the maximum number of iterations.

```

1: for  $q = 1$  to  $q_{\max}$  do
2:    $U \leftarrow []$                                       $\triangleright$  Initialize  $U$  as an empty matrix.
3:   EPSI Iteration: Update first  $k$  eigenspace estimation  $U$ 
4:   for  $i = 1$  to  $k$  do
5:      $\hat{\sigma}_i \leftarrow \frac{(u_q^i)^\top A u_q^i}{(u_q^i)^\top u_q^i}$             $\triangleright$  Compute rayleigh quotient estimation  $\hat{\sigma}_i$  for the  $i$ th eigenvector.
6:      $\hat{u}_{q+1}^i \leftarrow ((I - UU^\top)\hat{A}(I - UU^\top) - \hat{\sigma}_i I)^{-1}((I - UU^\top)\hat{A}(I - UU^\top) - A)u_q^i$      $\triangleright$  Update  $\hat{u}_{q+1}^i$ .
7:      $U \leftarrow [U, \hat{u}_{q+1}^i]$                                       $\triangleright$  Append  $\hat{u}_{q+1}^i$  to  $U$ .
8:   end for
9:   Orthogonalization step: Use Estimated Eigenvector as Rangefinder
10:   $\Pi \leftarrow UU^\dagger$                                       $\triangleright$  Compute the projection matrix  $\Pi$ .
11:   $A_U \leftarrow \Pi A \Pi$                                       $\triangleright$  Compute the projected matrix  $A_U$ .
12:   $[u_{q+1}^1, u_{q+1}^2, \dots, u_{q+1}^k] \leftarrow \text{SVD}(A_U)$      $\triangleright$  Perform SVD to update  $u_{q+1}^i$ .
13: end for
14: return  $U$                                           $\triangleright$  Return the updated matrix  $U$ .

```

Update using EPSI

Project the matrix to
Subspace and
perform SVD

One more thing...

Iterative
debiasing

Easier for numerical stability computation
Algorithms can do **online** computation



Newton
Methods

Easier for convergence analysis



Randomized Iterative Solver as Iterative Refinement A Simple Fix Towards Backward Stability

Ruihan Xu
University of Chicago

Yiping Lu
Northwestern University

Abstract

Iterative sketching and sketch-and-precondition are well-established randomized algorithms for solving large-scale over-determined linear least-squares problems. In this paper, we introduce a new perspective that interprets Iterative Sketching and Sketching-and-Precondition as forms of Iterative Refinement. We also examine the numerical stability of two distinct refinement strategies: iterative refinement and recursive refinement, which progressively improve

tive tools for developing approximate matrix factorizations. These methods are remarkable for their simplicity and efficiency, often producing surprisingly accurate results.

In this paper, we consider randomized algorithms to solve the overdetermined linear least-squares problem

$$x = \arg \min_{y \in \mathbb{R}^n} \|b - Ay\| \quad (A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m) \quad (1)$$

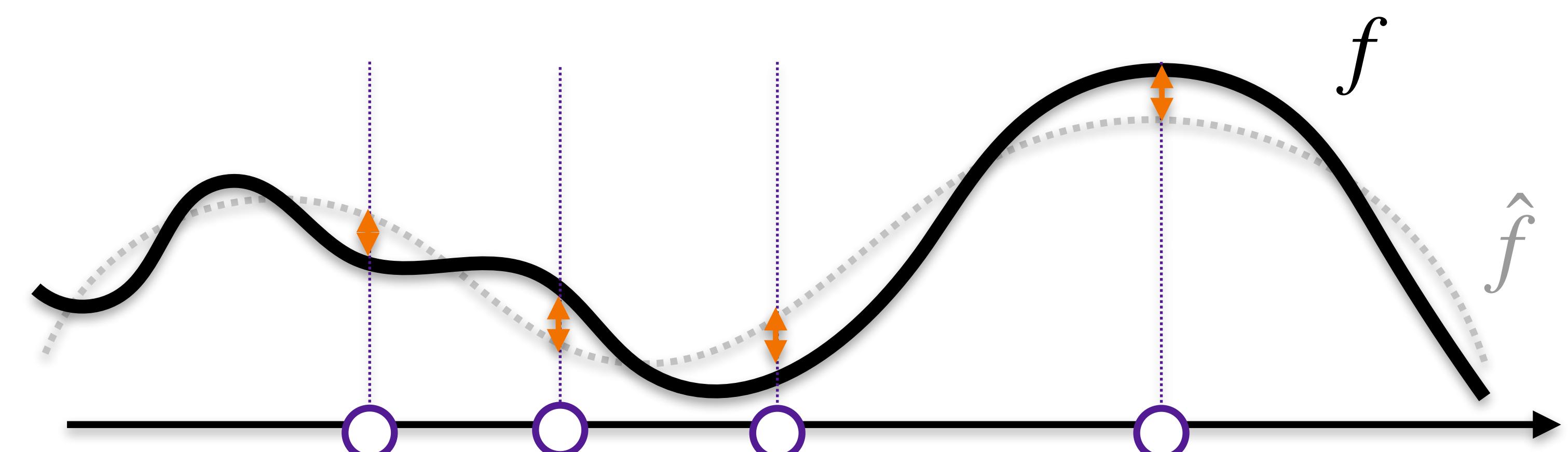
I What is SCaSML about?



$$\{X_1, \dots, X_n\} \sim \mathbb{P}_\theta \rightarrow \theta \rightarrow \Phi(\theta)$$

Step 1: Using Machine Learning to fit the rough function/environment

Step 2: Using validation dataset to know how much mistake machine learning algorithm has made



Step 3: Using Simulation algorithm to estimate $\Phi(\theta) - \Phi(\hat{\theta})$



Examples Later!

Using ML surrogate during inference time to improve ML solution