Sets

Sets - Basic operations

A set is just a collection of things. The things inside a set are called the elements of the set.

2 important points about a set

1. the order of elements does not matter.
2. repetition does not matter (some people do not allow repitition). When you talk about the number of elements of a set, it is implicit that you are talking about the distinct elements.

Examples- A set of numbers $\{1, 2, 3\}$

A set of MCIT lecturers $\{'Dave', 'Chris', 'Tom', 'Swap', Arvind'\}$

A set does not have to make any logical sense. As far as mathematics is concerned $\{3, 5.6, \frac{8}{2.7}\}$ is a set.

Notation

If S is a set and x is something in the set, we say $x \in S$.

So far, the notation we have introduced is called set-roster notation where we are listing out elements of the set. Often, sets are large and we do not want to explicitly spend time writing each element. $\{1, 2, 3, \ldots, 42\}$ is valid notation and is understood to mean all the integers from 1 to 42.

Remember that $\{0\}$ *is different from the number* 0*. One of them is a set containing the single element 0, the other is just the number 0.*

Conventionally, the variable used for representing a set is in upper case. The set S, the set T etc. The elements of the set are generally represented in lower case. Although not incorrect, it would be surprising to see $S \in x$.

Think of these conventions to be similar to the way you name your variables in Python (when you get to Java in 591, you will see more rules). They are established so that readers of your mathematical statements are less and less puzzled.

Question - How many elements are there in the set $\{1, \{1, 2\}\}$.

Answer - 2. The set has two things in it. One of them is a number and the other is a set. Does not matter how many elements are in the set inside the set.

The **empty set**(the set containing nothing) is denoted by the greek letter $\emptyset$. Please do not confuse the empty set with the set that contains 0. One of them has a single element, the other has no elements at all.

The special sets

Certain sets are used so often that they have special notation.

- $\mathbb{R}$ - real numbers (basically any number we touch in this course).
- $\mathbb{Q}$ - rational numbers. These are numbers of the form $\frac{p}{q}$ where $q \neq 0$. Remember that numbers with a decimal point can be expressed in this form as long as they either have a terminating decimal like 2.56 or they have a recurring decimal like 0.3333333.
- $\mathbb{Z}$ - integers. These are the positive and negative numbers that you are familiar with.
- $\mathbb{N}$ - natural numbers. All positive integers. Importantly we will not include 0 in this set.

Often to narrow down on the positive or negatives, the special set will have a superscript on it such as $\mathbb{Z}^+$, which means the positive integers. Or for instance, $\mathbb{R}^-$, which would mean the negative real numbers.

Set-builder

An alternative way of specifying a set is to start with one of the special sets and then pick elements that only satisfy certain properties.

What does $\{x \in \mathbb{Z} | -2 < x < 5\}$ correspond to?
It is equivalent to the set $\{-1, 0, 1, 2, 3, 4\}$.

The syntax is to first introduce the 'main' set (generally called the universe), then put down a | and then some kind of conditional statement (you have by now seen if statements in 591).

Subset

If A and B are sets, then A is called a subset of B, $A \subseteq B$ if and only if, every element in A is also an element in B.
If $x \in A$, this means that $x \in B$.
Importantly, if $A \subseteq B$, this does not necessarily mean $B \subseteq A$.
The (true) statement 'Every rational number is a real number' gets mathematically expressed as $\mathbb{Q} \subseteq \mathbb{R}$. But, we know that there are real numbers that are not rational numbers. Among others, π and e and $\sqrt{2}$. So, $\mathbb{R} \not\subseteq \mathbb{Q}$.

Element of and subset of

A very common confusion is that between the $\in$ and $\subseteq$. Here is an attempt at clarifying that via an example (the zybook has one too).

Consider the set $S = \{CIT, \pi, \{\text{apples}, \text{bananas}\}\}$
What are the elements of this set?
An element of the set is a single item inside the set. So the elements are

1. CIT
2. π
3. {apples, bananas}

And YES, one of the elements is this weird set of some fruit.

What are the subsets of this set?
This question becomes easier to answer now that we have answered the elements question. Subsets after all are made by collecting none, some or all of these elements and putting them between { and }, that is, making a set out of them.
So here are all the subsets

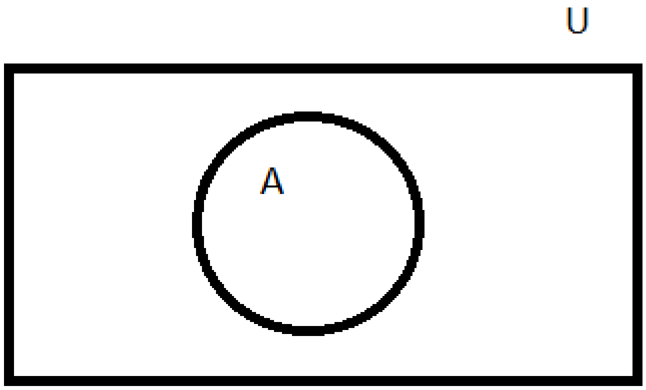
1. $\emptyset$
2. $\{CIT\}$
3. $\{\pi\}$
4. $\{\{apples, bananas\}\}$ - this one is slightly tricky, but just think of the apples and bananas set as this one single entity.
5. $\{CIT, \pi\}$
6. $\{\pi, \{\text{apples}, \text{bananas}\}\}$
7. $\{CIT, \{\text{apples}, \text{bananas}\}\}$
8. $\{CIT, \pi, \{\text{apples}, \text{bananas}\}\}$

Venn diagrams (Pg 340 in book)

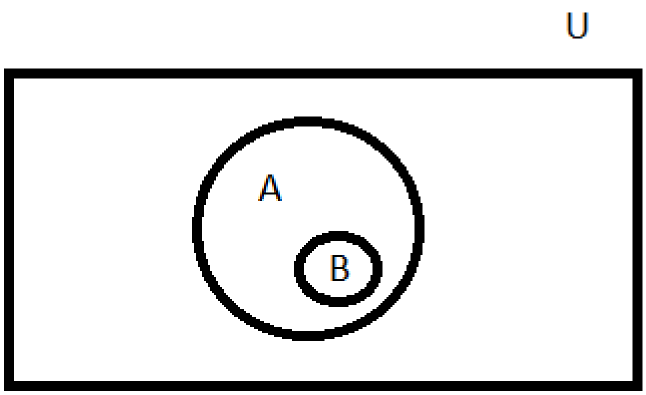
A Venn diagram is just a pictorial representation of a set.
Generally speaking, we draw one big box for the universe. The universe can be different depending upon the context being used. For instance, with the numbers (especially in this course), it makes sense to think of the universe as $\mathbb{R}$.

All sets that we speak of are contained in this universe. So a set A will be be a subset of U. $A \subseteq U$.

In a Venn diagram, the set A will be represented like this.



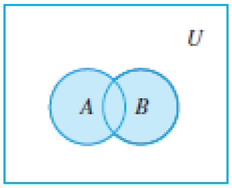
Using a Venn diagram, it becomes easy to represent a lot of things in set theory. For example, if $B \subseteq A$, it can be drawn like this



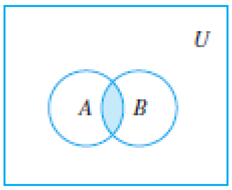
Set operations

In the following, the shaded region in each Venn diagram represents the operation we are talking about.

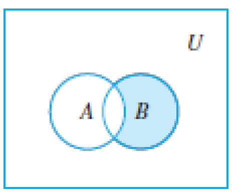
- Union - The union of two sets A and B is defined as the set of elements that are either in A or in B (elements that are in both sets are included!). It is represented as $A \cup B$



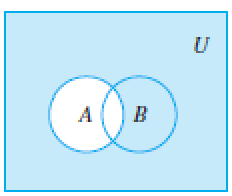
- Intersection - this is defined as the set of all elements that are in both A and in B. It is represented as $A \cap B$.



- Difference - The difference of B minus A, denoted $B - A$ refers to the set consisting of elements in B that are not in A.



- Complement - the complement of A, denoted as $\bar{A}$ is the set of all elements in U that are not in A.



It is also a good time to get used to writing these in mathematical manner.

$$\begin{aligned} A \cup B &= \{x \in U | x \in A \text{ or } x \in B\} \\ A \cap B &= \{x \in U | x \in A \text{ and } x \in B\} \\ B - A &= \{x \in B | x \notin A\} \\ \bar{A} &= \{x \in U | x \notin A\} \end{aligned}$$

Properties

- $A \cup B = B \cup A$ - commutative law
- $A \cap B = B \cap A$ - commutative law
- $A \cap (B \cap C) = (A \cap B) \cap C$ - associative law
- $A \cup (B \cup C) = (A \cup B) \cup C$ - associative law
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ - distributive law
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ - distributive law
- $A \cap \emptyset = \emptyset$
- $A \cup U = U$
- $A \cap U = A$
- $A \cup \emptyset = A$

De-Morgan's laws

De-Morgan's laws which you might have seen as part of logic, translate very easily into set theory.

$$\begin{aligned} \overline{(A \cup B)} &= \bar{A} \cap \bar{B} \\ \overline{(A \cap B)} &= \bar{A} \cup \bar{B} \end{aligned}$$

Examples

Show that $\overline{\bar{A} \cap U} = A$

We simplify this using the properties

$$\begin{aligned} \overline{\bar{A} \cap U} &= \bar{\bar{A}} \cup \bar{U} && \text{de-morgan's} \\ &= A \cup \bar{U} && \text{complement of a complement is the set itself} \\ &= A \cup \emptyset && \text{complement of the universe = empty!} \\ &= A \end{aligned}$$

Show that $(A \cap B) \cup \overline{(A \cup \bar{B})} = B$

Cardinality

The number of elements in a finite set is called the cardinality of that set. A is a set, the cardinality is denoted by $|A|$.

What is the cardinality of the set $S = \{\{1\}, \{1, 2\}\}$?

Be careful, this is a set of sets! Cardinality is 2 and not 3.

Power set

The power set of a set A is the set of all subsets of A and is denoted P(A).

An important thing to remember about the power set of A is that the empty set $\emptyset$ and the set A itself, are both subsets of A.

What do you think $|P(A)|$ is?

CS application

Databases - A lot of query syntax involves very basic set theory.

Made up Examples

There is an retirement agency that is interested in knowing the employees in MS and Apple that are close to retirement.

Actual SQL syntax would look something like

```
SELECT Name, EmployeeId
FROM MSEmployees
WHERE age > 58

UNION

SELECT Name, EmployeeId
FROM AppleEmployees
WHERE age > 58
```

Tell me what I ate for lunch but did not eat for dinner. Set difference!

```
SELECT item FROM Lunch
EXCEPT
SELECT item FROM Dinner
```