1 Functions

1.1 Properties of a Function

1.1.1 Individual Properties

A function f is a relation which maps input of a set D_f to outputs of a set R_f using a certain rule
Multiple elements in the input set can have the same output, but one single element in the input set can only have one output.

1.1.2 Function Presentation

When questions ask for functions in a similar form, be sure to maintain presentation

$$f(x) = x^2 \quad x \in (-\infty, \infty)$$

$$f : x \mapsto x^2 \quad x \in (-\infty, \infty)$$

$$g(x) = \begin{cases} x^2 & x \in \mathbb{R}, x > 0 \\ -x & x \in \mathbb{R}, x < 0 \end{cases}$$

$$D_f = (-\infty, \infty) \quad R_f = [0, \infty)$$

Note: infinity is always written as non-inclusive

1.2 Inverse Functions

Inverse functions map the output of a function to the input of a function. Inverse functions only exist when each element of the output set of the original function is mapped to one and only one element in the input set, i.e. inverse functions only exist if a function is one-one
Inverse functions are written as f^{-1} and

$$D_{f^{-1}} = R_f \quad R_{f^{-1}} = D_f$$

1.2.1 Proving existence and inexistence

$f(x)$ cuts each line $y = k$, $k \in R_f$ at one and only one point, f is one-one, hence f^{-1} exists
Replace R_f with the actual set
The line $y = k$ cuts $f(x)$ at more than one point, f is not one-one, hence f^{-1} does not exist
Replace k with the actual edge case

1.2.2 Finding Inverse Functions

$$f(x) = x^2 + 1 \quad x \in [0, \infty) \text{Let} \quad y = f(x) = x^2 + 1$$

$$x^2 = y - 1$$

$$x = \sqrt{y - 1}$$

$$f^{-1}(y) = \sqrt{y - 1}$$

$$f^{-1}(x) = \sqrt{y - 1}$$

$$D_{f^{-1}} = R_f = [1, \infty)$$

$$R_{f^{-1}} = D_f = [0, \infty)$$

1.2.3 Graphical Relationships Between Functions and Inverse Functions

Inverse functions are, in essence, a function reflected along the line $y = x$
Intersections of functions with their inverse also satisfy the condition $f(x) = x$

1.3 Composite Functions

Considering two functions f and g, the composite function fg is obtained when inputs of g are mapped to their outputs of g, which are then used as inputs to f and mapped to outputs of f:

$$fg(x) = f(g(x))$$

1.3.1 Deriving Composite Functions

For the composite function fg to exist, $R_g \subseteq D_f$
The domain of function fg follows the domain of function g, i.e. $D_{fg} = D_g$
The rule of fg is obtained by substituting the rule of g into the rule of f
The range of fg is a subset of R_f and may be limited due to the fact that R_g may be smaller than D_f, hence R_{fg} must be reevaluated after creating its rule