Math Notes

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1 Assumed Knowledge

1.1 Algebra

1.1.1 Completing the Square

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - (\frac{b}{2})^2$$

1.1.2 Polynomial Expansions

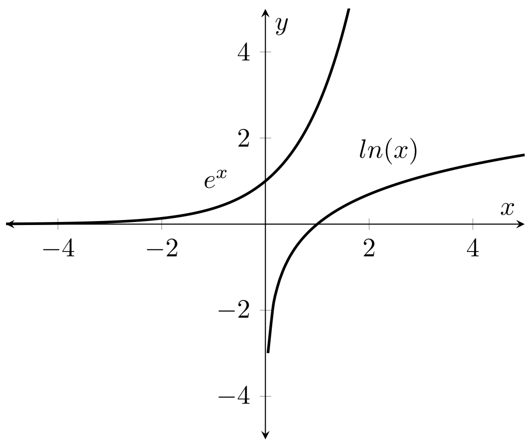
$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 \pm b^3 = (a \pm b)(a^2 \mp 2ab + b^2)$$

1.1.3 Partal Fractions

$$\frac{f(x)}{(ax+b)(cx+d)}$$
$$= g(x) + \frac{A}{ax+b} + \frac{B}{cx+d}$$
$$\frac{f(x)}{(ax+b)(cx+d)^2}$$
$$= g(x) + \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$
$$\frac{f(x)}{(ax+b)(x^2+c)}$$
$$= g(x) + \frac{A}{ax+b} + \frac{Bx+C}{x^2+c}$$

1.1.4 Exponent and Logarithm

$$e^n = \underbrace{e \times e \times e \times ... \times e}_{\text{n times}}$$
$$e^{\frac{1}{2}} = \sqrt{e}$$
$$log_e(x) = ln(x)$$
$= $ how many times e is multiplied by itself to get x
$$log_{10}(x) = lg(x)$$
$$x = e^{ln(x)}$$
$$log_x(y) = \frac{log_{base}(y)}{log_{base}(x)}$$



1.2 Trigonometry

1.2.1 Sine and Cosine Rule

For any triangle with length of sides a, b and c and with opposite angles A B and C:

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$
$$a^2 = b^2 + c^2 - 2bccos(A)$$
$$cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

1.2.2 Sum of Angles

$$sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B)$$
$$sin(2A) = 2sin(A)cos(A)$$
$$cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B)$$

$$\begin{aligned} cos(2A) &= cos^2(A) - sin^2(A) \\ &= 2cos^2(A) - 1 \\ &= 1 - 2sin^2(A) \end{aligned}$$

$$tan(A \pm B) = \frac{tan(A) \pm tan(B)}{1 \mp tan(A)tan(B)}$$
$$tan(2A) = \frac{2tan(A)}{1 - tan^2(A)}$$

$$\text{Area of Triangle} = \tfrac{1}{2}absin(C)$$

1.2.3 Factor and Reverse Factor Formula

$$sin(A) + sin(B) = 2sin(\tfrac{A+B}{2})cos(\tfrac{A-B}{2})$$
$$sin(A) - sin(B) = 2cos(\tfrac{A+B}{2})sin(\tfrac{A-B}{2})$$
$$cos(A) + cos(B) = 2cos(\tfrac{A+B}{2})cos(\tfrac{A-B}{2})$$
$$cos(A) - cos(B) = -2sin(\tfrac{A+B}{2})sin(\tfrac{A-B}{2})$$
$$sin(A)cos(B) = \tfrac{1}{2}[sin(A+B) + sin(A-B)]$$
$$cos(A)sin(B) = \tfrac{1}{2}[sin(A+B) - sin(A-B)]$$
$$cos(A)cos(B) = \tfrac{1}{2}[cos(A+B) + cos(A-B)]$$
$$sin(A)sin(B) = -\tfrac{1}{2}[cos(A+B) - cos(A-B)]$$

2 Inequalities

2.1 Inequalities

2.1.1 Properties of Inequalities

$$a > b, c > 0 \implies ac > bc$$
$$a > b, c < 0 \implies ac < bc$$
$$\frac{a}{b} > 0 \implies ab > 0$$
$$\frac{a}{b} < 0 \implies ab < 0$$

Positive sides of inequalities suggest that both terms share similar positive or negative signs, negative sides of inequalities suggest that both terms have opposite positive or negative signs

2.1.2 Quardratic Inequalities

Find where $f(x) = 0$ by completing square or quadratic formula and sketch graph

2.1.3 Inequality Reduction

For any inequality $\frac{f(x)}{g(x)} > or < 0$ where $f(x)$ or $g(x)$ is strictly positive or negative, reduce inequality to non-strictly positive/negative function and change sign accordingly

2.1.4 Modulus Inequalities

$$|x| < a \iff -a < x < a$$
$$|x| > a \iff x < -a \quad or \quad a < x$$
$$|x - a| < b \iff a - b < x < a + b$$
$$|x - a| > b \iff x < a - b \quad or \quad a + b < x$$

To solve inequalities, sketch and find intercept, then deduce suitable range of x

3 Vectors

3.1 Representation

3.1.1 Point Representation

O is always defined as the origin
Written: $\underline{r}$ or $\overrightarrow{OR}$
Column: $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
Unit Vector: $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ where $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the x, y and z dimensions

3.1.2 Line Representation

Vector Equation:

$$l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \quad \lambda \in \mathbb{R}$$

Parametric Equation:

$$l : \begin{cases} x = a_x + \lambda b_x \\ y = a_y + \lambda b_y \\ z = a_z + \lambda b_z \end{cases} \lambda \in \mathbb{R}$$

Cartesian Equation:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z}$$

3.1.3 Plane Representation

Vector Equation:

$$\pi : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

Scalar Product Equation:

$$\pi : \mathbf{r} \cdot \mathbf{n} = d$$

Cartesian Equation:

$$\pi : xn_x + y_n y + zn_z = d$$

3.2 Manipulation

3.2.1 Vector Algebra

Vector Addition: remove same inside or outside terms

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

Negative Vectors: reverse the points

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

Vector Subtraction: reverse points, then add

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{AO} = \overrightarrow{AB}$$

3.2.2 Vector Properties

Modulus / Magnitude : The total distance of a vector

$$\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector / Cap : A vector which defines a direction and has modulus of 1

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

3.2.3 Ratio Theorem

3.2.4 Scalar Product

Scalar/Dot product produces a scalar which is a representation of how inline two vectors are with each other

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|cos(\theta)$$

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

$$\mathbf{a} \cdot (\perp \mathbf{a}) = 0$$

$$\mathbf{a} \cdot (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$$

3.2.5 Vector Product

Vector/Cross product produces a vector which has direction perpendicular to its input vectors and has magnitude similar to area subtended by its input vectors

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}}|\mathbf{a}||\mathbf{b}|sin(\theta)$$

$$\mathbf{a} \times \mathbf{a} = 0$$

$$\mathbf{a} \times (\perp \mathbf{a}) = \hat{\mathbf{n}}|\mathbf{a}||\mathbf{b}|$$

$$\mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$$

3.3 Angles Between Vectors

General formula:

$$cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Lesser used formula:

$$sin(\theta) = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

3.3.1 Point-Point Angle

Let **a** and **b** be point vectors

3.3.2 Line-Line Angle

Let **a** and **b** be direction vectors of lines

3.3.3 Line-Plane Angle

Let **a** be direction vector of line and **b** be normal vector of plane
Angle between line and plane will be $90° - \theta$

3.3.4 Plane-Plane Angle

Let $\mathbf{a}$ and $\mathbf{b}$ be normal vectors of planes

3.4 Intersection

3.4.1 Point-Line Intersection

Solve series of parametric equations or find λ which lets point equal to point on line

3.4.2 Line-Line Intersection

If direction vectors are scalar multiples of each other, lines are parallel
Find a point which satisfies both lines, solving parametric equations of both line equations

$$\mathbf{a_1} + \lambda \mathbf{b_1} = \mathbf{a_2} + \mu \mathbf{b_2} \quad \lambda, \mu \in \mathbb{R}$$

If lines are both non-parallel and non-intersecting, lines are skew

3.4.3 Line-Plane Intersection

If dot product of direction vector of line and normal of plane equals to 0, line is parallel to plane
Substitute line equation into plane equation and expand to solve for λ

$$\begin{aligned} p &= (\mathbf{a} + \lambda \mathbf{b}) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + \lambda (\mathbf{b} \cdot \mathbf{n}) \end{aligned}$$

3.4.4 Plane-Plane Intersection

If normal of planes are scalar multiples of each other, planes are parallel
Equating two planes results in a line
Cross product of normal vectors of two planes produces the direction vector of line
Position vector of line can be observed from equations, find a vector which satisfies both plane equations

3.5 Projections

3.5.1 Point-Point Projection

To find distance d of projection of point vector $\mathbf{a}$ on point vector $\mathbf{b}$:

$$d = \mathbf{a} \cdot \hat{\mathbf{b}}$$

3.5.2 Point-Line Projection

To find distance d of projection of point vector $\mathbf{a}$ on direction vector of line $\mathbf{b}$, similar to Point-Point Projection

3.5.3 Point-Line Perpendicular

To find perpendicular of point vector $\mathbf{a}$ on line $l : \mathbf{r} = \mathbf{b} + \lambda \mathbf{c}$, let $\mathbf{P}$ be a point on the line such that $\overrightarrow{AP}$ is perpendicular to $\mathbf{c}$ and solve for λ :

$$\begin{aligned} 0 &= \overrightarrow{AP} \cdot \mathbf{c} \\ &= (\mathbf{b} + \lambda \mathbf{c} - \mathbf{a}) \cdot \mathbf{c} \qquad \lambda |\mathbf{c}|^2 = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} \\ &= \lambda \mathbf{c} \cdot \mathbf{c} + (\mathbf{b} - \mathbf{a}) \cdot \mathbf{c} \end{aligned}$$

To find distance d between point vector $\mathbf{a}$ and its perpendicular on line with direction vector $\mathbf{b}$, find magnitude of the cross product of $\mathbf{a}$ and unit vector of $\mathbf{b}$:

$$d = |\mathbf{a} \times \hat{\mathbf{b}}|$$

3.5.4 Point-Plane Perpendicular

To find perpendicular of point vector $\mathbf{a}$ on plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$, consider a line containing $\mathbf{a}$ and with direction vector $\mathbf{n}$, equate the two equations and then solve for λ

$$l = \mathbf{a} + \lambda \mathbf{n}$$

$$\begin{aligned} d &= (\mathbf{a} + \lambda \mathbf{n}) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + \lambda \mathbf{n} \cdot \mathbf{n} \end{aligned} \lambda |\mathbf{n}|^2 = d - \mathbf{a} \cdot \mathbf{n}$$

To find distance d between point vector $\mathbf{a}$ and plane with normal vector $\mathbf{n}$, find projection of $\mathbf{a}$ on unit vector of $\mathbf{n}$:

$$d = |\mathbf{a} \cdot \hat{\mathbf{n}}|$$

4 Complex Numbers

4.1 Imaginary Numbers

$$i = sqrt(-1)$$

$$i^2 = -1$$

$$i^4 = i^2 \times i^2 = 1$$

4.2 Cartesian Representation

4.2.1 Complex Numbers

Complex numbers are written in the form:

$$z = a + ib \quad a, b \in \mathbb{R}$$
$$\text{Where} \quad Re(z) = a \quad Im(z) = b$$

And populate the set $\mathbb{C}$

4.2.2 Conjugates

$$\text{For} \quad w = a + ib \quad z = c + id$$
$$w^* = a - ib$$
$$ww^* = a^2 + b^2 = |w|^2$$
$$(w + z)^* = w^* + z^*$$
$$(wz)^* = w^* z^*$$

4.2.3 Algebraic Manipulation

$$\text{For} \quad w = a + ib \quad z = c + id$$
$$w = z \implies a = c, b = d \quad \textbf{IMPT}$$
$$w + z = (a + c) + i(b + d)$$
$$w - z = (a - c) + i(b - d)$$
$$w * z = (ac - bd) + i(ad + bc)$$
$$|w| = sqrt(a^2 + b^2)$$
$$sqrt(w) \quad \text{occurs with a} \pm \text{sign}$$

For division, remove i from denominator by multiplying numerator and denominator by conjugate and then solve:

$$\begin{aligned} \frac{w}{z} &= \frac{w}{z} \times \frac{z^*}{z^*} \\ &= \frac{wz^*}{zz^*} \\ &= \frac{wz^*}{c^2 + d^2} \end{aligned}$$

4.3 Complex Polynomial Roots

4.3.1 Theorem of Algebra

A polynomial of degree n has n real or complex roots

4.3.2 Finding Complex Roots

If a polynomial has all real coefficients, complex roots occur in conjugate pairs. Write as such in exams:

$$f(x) \quad \text{has real coefficients}$$

$$a + ib \quad \text{is a root} \implies a - ib \quad \text{is a root}$$

For a polynomial with complex coefficients, use quadratic general formula. Note that $\pm$ will still be present somewhere

4.4 Polar Representation

4.4.1 Polar Representation

$$z = re^{i\theta} \quad r \in \mathbb{R}_0^+ \quad -\pi < \theta \leq \pi$$
$$|z| = r \quad arg(z) = \theta = tan^{-1}(\frac{a}{b})$$
$$Re(z) = rcos(\theta)$$
$$Im(z) = rsin(\theta)$$
$$re^{i\pi} = -1 \quad re^{i0} = 1$$
$$re^{i\frac{\pi}{2}} = i \quad re^{i\frac{-\pi}{2}} = -i$$

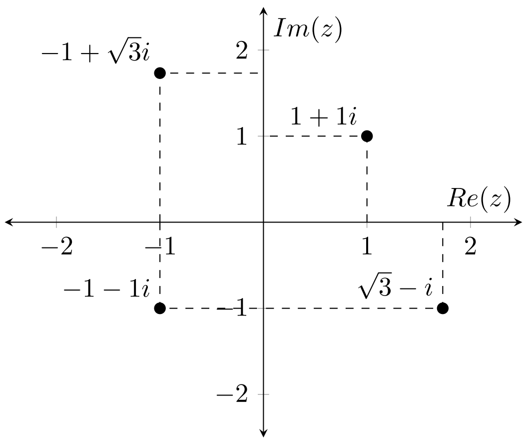
4.4.2 Algebraic Manipulation

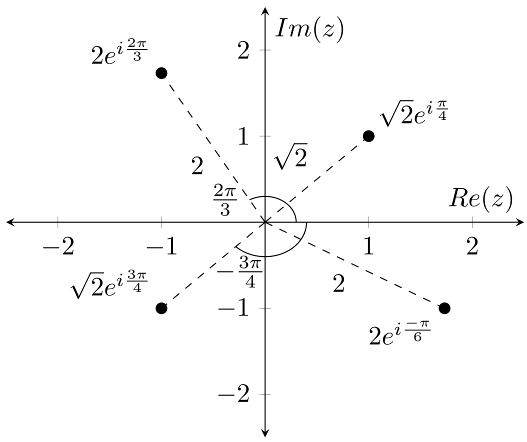
$$\text{For} \quad z = r_1 e^{i\theta_1} \quad z = r_2 e^{i\theta_2}$$
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$
$$z_1^n = r_1^n e^{in\theta_1}$$
$$\sqrt{z_1} = \sqrt{r_1} e^{i(\frac{\theta_1}{2})}, \sqrt{r_1} e^{i(\frac{\theta_1}{2} + \pi)}$$

4.5 Geometric Representation

4.5.1 Argand Diagrams

Cartesian form describes the x and y coordinate on the real and imaginary axis
Polar form describes the distance between the point and the origin as well as the angle a line from the point to the origin makes with the positive real axis





4.5.2 Geometric Manipulation

Multiplying by $re^{i\theta}$ scales the number by a factor of r and rotates anticlockwise by an angle of θ about the origin
Multiplying by i rotates a complex number by $\frac{\pi}{2}$, or 90° anticlockwise
The conjugate of a complex number is a reflection of the complex number on the x axis