# Math Notes

Chang Si Yuan

June 14, 2016

(ver 0.0.1a)

This compilation of notes are to be used as a reference for the GCE "A"-level Mathematics paper, both as a refresher in theories as well as for general descriptions of presentation form. These notes are meant for free, public use, but at the reader's own risk.
Good luck with your exams.

# 1 Assumed Knowledge

## 1.1 Algebra

### 1.1.1 Completing the Square

$$x^2 + bx + c = (x + \frac{b}{2})^2 + c - (\frac{b}{2})^2$$

### 1.1.2 Polynomial Expansions

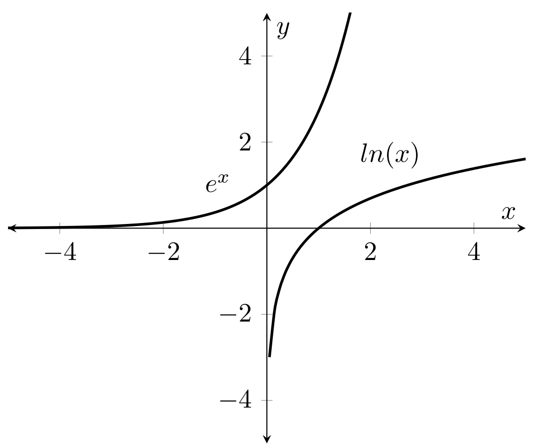
$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$a^2 - b^2 = (a+b)(a-b)$$
$$a^3 \pm b^3 = (a \pm b)(a^2 \mp 2ab + b^2)$$

### 1.1.3 Partal Fractions

$$\frac{f(x)}{(ax+b)(cx+d)}$$
$$= g(x) + \frac{A}{ax+b} + \frac{B}{cx+d}$$
$$\frac{f(x)}{(ax+b)(cx+d)^2}$$
$$= g(x) + \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$
$$\frac{f(x)}{(ax+b)(x^2+c)}$$
$$= g(x) + \frac{A}{ax+b} + \frac{Bx+C}{x^2+c}$$

### 1.1.4 Exponent and Logarithm

$$e^n = \underbrace{e \times e \times e \times ... \times e}_{\text{n times}}$$
$$e^{\frac{1}{2}} = \sqrt{e}$$
$$log_e(x) = ln(x)$$
$$= \text{how many times } e \text{ is multiplied by itself to get x}$$
$$log_{10}(x) = lg(x)$$
$$x = e^{ln(x)}$$
$$log_x(y) = \frac{log_{base}(y)}{log_{base}(x)}$$




## 1.2 Trigonometry

### 1.2.1 Sine and Cosine Rule

For any triangle with length of sides $a$, $b$ and $c$ and with opposite angles $A$ $B$ and $C$:

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)}$$
$$a^2 = b^2 + c^2 - 2bccos(A)$$
$$cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

### 1.2.2 Sum of Angles

$$sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B)$$
$$sin(2A) = 2sin(A)cos(A)$$
$$cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B)$$
$$cos(2A) = cos^2(A) - sin^2(A)$$
$$= 2cos^2(A) - 1$$
$$= 1 - 2sin^2(A)$$
$$tan(A \pm B) = \frac{tan(A) \pm tan(B)}{1 \mp tan(A)tan(B)}$$
$$tan(2A) = \frac{2tan(A)}{1 - tan^2(A)}$$
$$\text{Area of Triangle} = \tfrac{1}{2}absin(C)$$

### 1.2.3 Factor and Reverse Factor Formula

$$sin(A) + sin(B) = 2sin(\tfrac{A+B}{2})cos(\tfrac{A-B}{2})$$
$$sin(A) - sin(B) = 2cos(\tfrac{A+B}{2})sin(\tfrac{A-B}{2})$$
$$cos(A) + cos(B) = 2cos(\tfrac{A+B}{2})cos(\tfrac{A-B}{2})$$
$$cos(A) - cos(B) = -2sin(\tfrac{A+B}{2})sin(\tfrac{A-B}{2})$$
$$sin(A)cos(B) = \tfrac{1}{2}[sin(A+B) + sin(A-B)]$$
$$cos(A)sin(B) = \tfrac{1}{2}[sin(A+B) - sin(A-B)]$$
$$cos(A)cos(B) = \tfrac{1}{2}[cos(A+B) + cos(A-B)]$$
$$sin(A)sin(B) = -\tfrac{1}{2}[cos(A+B) - cos(A-B)]$$

# 2 Inequalities

## 2.1 Inequalities

### 2.1.1 Properties of Inequalities

$$a > b, c > 0 \implies ac > bc$$
$$a > b, c < 0 \implies ac < bc$$
$$\frac{a}{b} > 0 \implies ab > 0$$
$$\frac{a}{b} < 0 \implies ab < 0$$

Positive sides of inequalities suggest that both terms share similar positive or negative signs, negative sides of inequalities suggest that both terms have opposite positive or negative signs.

### 2.1.2 Quardratic Inequalities

Find where $f(x) = 0$ by completing square or quadratic formula and sketch graph.

### 2.1.3 Inequality Reduction

For any inequality $\frac{f(x)}{g(x)} > or < 0$ where $f(x)$ or $g(x)$ is strictly positive or negative, reduce inequality to non-strictly positive/negative function and change sign accordingly.
Careful for elements of the form $(x+a)^2$, though these can be assumed to be strictly positive, the case where $(x+a)^2 = 0$ needs to be accounted for.

### 2.1.4 Modulus Inequalities

$$|x| < a \iff -a < x < a$$
$$|x| > a \iff x < -a \quad or \quad a < x$$
$$|x-a| < b \iff a-b < x < a+b$$
$$|x-a| > b \iff x < a-b \quad or \quad a+b < x$$

To solve inequalities, sketch and find intercept, then deduce suitable range of $x$.

# 3 Vectors

## 3.1 Representation

### 3.1.1 Point Representation

$O$ is always defined as the origin
Written: $\underline{\mathsf{r}}$ or $\overrightarrow{OR}$
Column: $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
Unit Vector: $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ where $\mathbf{i}$,$\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the x, y and z dimensions

### 3.1.2 Line Representation

Vector Equation:

$$l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \quad \lambda \in \mathbb{R}$$

Parametric Equation:

$$l : \begin{cases} x = a_x + \lambda b_x \\ y = a_y + \lambda b_y \\ z = a_z + \lambda b_z \end{cases} \quad \lambda \in \mathbb{R}$$

Cartesian Equation:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z}$$

### 3.1.3 Plane Representation

Vector Equation:

$$\pi : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

Scalar Product Equation:

$$\pi : \mathbf{r} \cdot \mathbf{n} = d$$

Cartesian Equation:

$$\pi : xn_x + y_n y + zn_z = d$$

## 3.2 Manipulation

### 3.2.1 Vector Algebra

Vector Addition: remove same inside or outside terms

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

Negative Vectors: reverse the points

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

Vector Subtraction: reverse points, then add

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{AO} = \overrightarrow{AB}$$

### 3.2.2 Vector Properties

Modulus / Magnitude : The total distance of a vector

$$\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector / Cap : A vector which defines a direction and has modulus of 1

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

### 3.2.3 Ratio Theorem

### 3.2.4 Scalar Product

Scalar/Dot product produces a scalar which is a representation of how inline two vectors are with each other.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|cos(\theta)$$
$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$
$$\mathbf{a} \cdot (\perp \mathbf{a}) = 0$$
$$\mathbf{a} \cdot (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
$$(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$$

### 3.2.5 Vector Product

Vector/Cross product produces a vector which has direction perpendicular to its input vectors and has magnitude similar to area subtended by its input vectors.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}}|\mathbf{a}||\mathbf{b}|sin(\theta)$$
$$\mathbf{a} \times \mathbf{a} = 0$$
$$\mathbf{a} \times (\perp \mathbf{a}) = \hat{\mathbf{n}}|\mathbf{a}||\mathbf{b}|$$
$$\mathbf{a} \times (\mathbf{b} \pm \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
$$(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$$

## 3.3 Angles Between Vectors

General formula:

$$cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Lesser used formula:

$$sin(\theta) = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

### 3.3.1 Point-Point Angle

Let $\mathbf{a}$ and $\mathbf{b}$ be point vectors.

### 3.3.2 Line-Line Angle

Let $\mathbf{a}$ and $\mathbf{b}$ be direction vectors of lines.

### 3.3.3 Line-Plane Angle

Let $\mathbf{a}$ be direction vector of line and $\mathbf{b}$ be normal vector of plane
Angle between line and plane will be $90° - \theta$.

### 3.3.4 Plane-Plane Angle

Let $\mathbf{a}$ and $\mathbf{b}$ be normal vectors of planes.

## 3.4 Intersection

### 3.4.1 Point-Line Intersection

Solve series of parametric equations or find $\lambda$ which lets point equal to point on line.

### 3.4.2 Line-Line Intersection

If direction vectors are scalar multiples of each other, lines are parallel.
Find a point which satisfies both lines, solving parametric equations of both line equations.

$$\mathbf{a_1} + \lambda\mathbf{b_1} = \mathbf{a_2} + \mu\mathbf{b_2} \quad \lambda, \mu \in \mathbb{R}$$

If lines are both non-parallel and non-intersecting, lines are skew.

### 3.4.3 Line-Plane Intersection

If dot product of direction vector of line and normal of plane equals to $0$, line is parallel to plane.
Substitute line equation into plane equation and expand to solve for $\lambda$.

$$\begin{aligned} p &= (\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + \lambda(\mathbf{b} \cdot \mathbf{n}) \end{aligned}$$

### 3.4.4 Plane-Plane Intersection

If normal of planes are scalar multiples of each other, planes are parallel.
Equating two planes results in a line.
Cross product of normal vectors of two planes produces the direction vector of line.
Position vector of line can be observed from equations, find a vector which satisfies both plane equations.

## 3.5 Projections

### 3.5.1 Point-Point Projection

To find distance $d$ of projection of point vector $\mathbf{a}$ on point vector $\mathbf{b}$ :

$$d = \mathbf{a} \cdot \hat{\mathbf{b}}$$

### 3.5.2 Point-Line Projection

To find distance $d$ of projection of point vector $\mathbf{a}$ on direction vector of line $\mathbf{b}$, similar to Point-Point Projection.

### 3.5.3 Point-Line Perpendicular

To find perpendicular of point vector $\mathbf{a}$ on line $l : \mathbf{r} = \mathbf{b} + \lambda\mathbf{c}$, let $\mathbf{P}$ be a point on the line such that $\overrightarrow{AP}$ is perpendicular to $\mathbf{c}$ and solve for $\lambda$ :

$$\begin{aligned} 0 &= \overrightarrow{AP} \cdot \mathbf{c} \\ &= (\mathbf{b} + \lambda\mathbf{c} - \mathbf{a}) \cdot \mathbf{c} \quad \lambda|\mathbf{c}|^2 = (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} \\ &= \lambda\mathbf{c} \cdot \mathbf{c} + (\mathbf{b} - \mathbf{a}) \cdot \mathbf{c} \end{aligned}$$

To find distance $d$ between point vector $\mathbf{a}$ and its perpendicular on line with direction vector $\mathbf{b}$, find magnitude of the cross product of $\mathbf{a}$ and unit vector of $\mathbf{b}$ :

$$d = |\mathbf{a} \times \hat{\mathbf{b}}|$$

### 3.5.4 Point-Plane Perpendicular

To find perpendicular of point vector $\mathbf{a}$ on plane $\pi : \mathbf{r} \cdot \mathbf{n} = d$, consider a line containing $\mathbf{a}$ and with direction vector $\mathbf{n}$, equate the two equations and then solve for $\lambda$:

$$l = \mathbf{a} + \lambda\mathbf{n}$$

$$\begin{aligned} d &= (\mathbf{a} + \lambda\mathbf{n}) \cdot \mathbf{n} \\ &= \mathbf{a} \cdot \mathbf{n} + \lambda\mathbf{n} \cdot \mathbf{n} \end{aligned} \lambda|\mathbf{n}|^2 = d - \mathbf{a} \cdot \mathbf{n}$$

To find distance $d$ between point vector $\mathbf{a}$ and plane with normal vector $\mathbf{n}$, find projection of $\mathbf{a}$ on unit vector of $\mathbf{n}$:

$$d = |\mathbf{a} \cdot \hat{\mathbf{n}}|$$

# 4 Complex Numbers

## 4.1 Imaginary Numbers

$$i = \sqrt{(} - 1)$$
$$i^2 = -1$$
$$i^4 = i^2 \times i^2 = 1$$

## 4.2 Cartesian Representation

### 4.2.1 Complex Numbers

Complex numbers are written in the form:

$$z = a + ib \quad a, b \in \mathbb{R}$$
$$\text{Where} \quad Re(z) = a \quad Im(z) = b$$

And populate the set $\mathbb{C}$

### 4.2.2 Conjugates

$$\text{For} \quad w = a + ib \quad z = c + id$$
$$w^* = a - ib$$
$$ww^* = a^2 + b^2 = |w|^2$$
$$(w + z)^* = w^* + z^*$$
$$(wz)^* = w^* z^*$$

### 4.2.3 Algebraic Manipulation

$$\text{For} \quad w = a + ib \quad z = c + id$$
$$w = z \implies a = c, b = d \quad \textbf{IMPT}$$
$$w + z = (a + c) + i(b + d)$$
$$w - z = (a - c) + i(b - d)$$
$$w * z = (ac - bd) + i(ad + bc)$$
$$|w| = sqrt(a^2 + b^2)$$
$$sqrt(w) \quad \text{occurs with a} \pm \text{sign}$$

For division, remove $i$ from denominator by multiplying numerator and denominator by conjugate and then solve:

$$\frac{w}{z} = \frac{w}{z} \times \frac{z^*}{z^*} = \frac{wz^*}{zz^*} = \frac{wz^*}{c^2 + d^2}$$

## 4.3 Complex Polynomial Roots

### 4.3.1 Theorem of Algebra

A polynomial of degree $n$ has $n$ real or complex roots.

### 4.3.2 Finding Complex Roots

If a polynomial has all real coefficients, complex roots occur in conjugate pairs. Write as such in exams:

$$f(x) \quad \text{has real coefficients}$$

$$a + ib \quad \text{is a root} \implies a - ib \quad \text{is a root}$$

For a polynomial with complex coefficients, use quadratic general formula. Note that a $\pm$ will still be present somewhere.

## 4.4 Polar Representation

### 4.4.1 Polar Representation

$$z = re^{i\theta} \quad r \in \mathbb{R}_0^+ \quad -\pi < \theta \leq \pi$$
$$|z| = r \quad arg(z) = \theta = tan^{-1}(\frac{a}{b})$$
$$Re(z) = rcos(\theta)$$
$$Im(z) = rsin(\theta)$$
$$re^{i\pi} = -1 \quad re^{i0} = 1$$
$$re^{i\frac{\pi}{2}} = i \quad re^{i\frac{-\pi}{2}} = -i$$

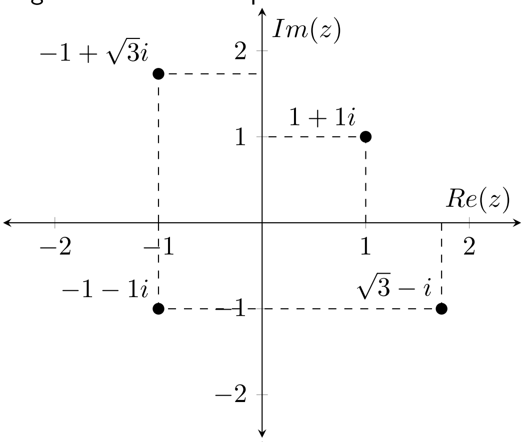
### 4.4.2 Algebraic Manipulation

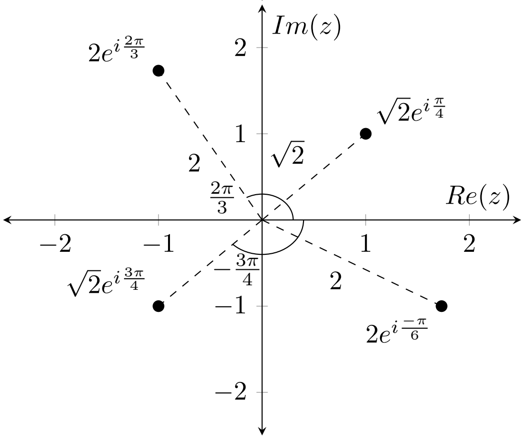
$$\text{For} \quad z = r_1 e^{i\theta_1} \quad z = r_2 e^{i\theta_2}$$
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$
$$z_1^n = r_1^n e^{in\theta_1}$$
$$\sqrt{z_1} = \sqrt{r_1} e^{i(\frac{\theta_1}{2})}, \sqrt{r_1} e^{i(\frac{\theta_1}{2} + \pi)}$$

## 4.5 Geometric Representation

### 4.5.1 Argand Diagrams

Cartesian form describes the x and y coordinate on the real and imaginary axis.
Polar form describes the distance between the point and the origin as well as the angle a line from the point to the origin makes with the positive real axis.



### 4.5.2 Geometric Manipulation

Multiplying by $re^{i\theta}$ scales the number by a factor of $r$ and rotates anticlockwise by an angle of $\theta$ about the origin.
Multiplying by $i$ rotates a complex number by $\frac{\pi}{2}$, or $90°$ anticlockwise.
The conjugate of a complex number is a reflection of the complex number on the x axis.

# 5 Functions

## 5.1 Properties of a Function

### 5.1.1 Individual Properties

A function $f$ is a relation which maps input of a set $D_f$ to outputs of a set $R_f$ using a certain rule.
Multiple elements in the input set can have the same output, but one single element in the input set can only have one output.

### 5.1.2 Function Presentation

When questions ask for functions in a similar form, be sure to maintain presentation.

$$f(x) = x^2 \quad x \in (-\infty, \infty)$$

$$f : x \mapsto x^2 \quad x \in (-\infty, \infty)$$

$$g(x) = \begin{cases} x^2 & x \in \mathbb{R}, x > 0 \\ -x & x \in \mathbb{R}, x < 0 \end{cases}$$

$$D_f = (-\infty, \infty) \quad R_f = [0, \infty)$$

Note: infinity is always written as non-inclusive

## 5.2 Inverse Functions

Inverse functions map the output of a function to the input of a function. Inverse functions only exist when each element of the output set of the original function is mapped to one and only one element in the input set, i.e. inverse functions only exist if a function is one-one.
Inverse functions are written as $f^{-1}$ and

$$D_{f^{-1}} = R_f \quad R_{f^{-1}} = D_f$$

### 5.2.1 Proving existence and inexistence

$f(x)$ cuts each line $y = k$, $k \in R_f$ at one and only one point, $f$ is one-one, hence $f^{-1}$ exists
Replace $R_f$ with the actual set
The line $y = k$ cuts $f(x)$ at more than one point, $f$ is not one-one, hence $f^{-1}$ does not exist .
Replace $k$ with the actual edge case.

### 5.2.2 Finding Inverse Functions

$$\begin{gathered} f(x) = x^2 + 1 \quad x \in [0, \infty)\text{Let} \quad y = f(x) = x^2 + 1 \\ x^2 = y - 1 \\ x = \sqrt{y-1} \\ f^{-1}(y) = \sqrt{y-1} \\ f^{-1}(x) = \sqrt{y-1} \\ D_{f^{-1}} = R_f = [1, \infty) \\ R_{f^{-1}} = D_f = [0, \infty) \end{gathered}$$

### 5.2.3 Graphical Relationships Between Functions and Inverse Functions

Inverse functions are, in essence, a function reflected along the line $y = x$.
Intersections of functions with their inverse also satisfy the condition $f(x) = x$.

## 5.3 Composite Functions

Considering two functions $f$ and $g$, the composite function $fg$ is obtained when inputs of g are mapped to their outputs of g, which are then used as inputs to f and mapped to outputs of f:

$$fg(x) = f(g(x))$$

### 5.3.1 Deriving Composite Functions

For the composite function $fg$ to exist, $R_g \subseteq D_f$.
The domain of function $fg$ follows the domain of function $g$, i.e. $D_{fg} = D_g$.
The rule of $fg$ is obtained by substituting the rule of $g$ into the rule of $f$.
The range of $fg$ is a subset of $R_f$ and may be limited due to the fact that $R_g$ may be smaller than $D_f$, hence $R_{fg}$ must be reevaluated after creating its rule.

# 6 APGP

## 6.1 Arithmetic Progression

### 6.1.1 Arithmetic Sequence

An arithmetic progression is a sequence of numbers which have the same difference between consecutive elements. Sequences are defined by their initial term $a$ and their constant difference $d$

$$\begin{gathered} u_1 = a = a + (1-1)d \\ u_2 = u_1 + d = a + d = a + (2-1)d \\ u_3 = u_2 + d = a + d + d = a + (3-1)d \\ \dots \\ u_n = a + (n-1)d \end{gathered}$$

### 6.1.2 Arithmetic Series

An Arithmetic Series is defined as the sum of a certain number of consecutive elements in an arithmetic sequence.

$$\begin{gathered} S_1 = u_1 \\ S_2 = S_1 + u_2 = u_1 + u_2 \\ S_3 = S_2 + u_3 = u_1 + u_2 + u_3 \\ ... \\ S_n = u_1 + u_2 + ... + u_{n-1} + u_n \end{gathered}$$

For an arithmetic sequence of known initial term and constant difference, the term $S_n$ can be derived from the equation

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(u_1 + u_n)$$

### 6.1.3 Proving a AP

## 6.2 Geometric Progression

### 6.2.1 Geometric Sequence

A geometric progression is a sequence of numbers which have the same constant ratio between consecutive elements. Sequences are defined by their initial term $a$ and their constant ratio $r$

$$\begin{gathered} u_1 = a = ar^{1-1} \\ u_2 = u_1 r = ar^{2-1} \\ u_3 = u_2 r = ar^{3-1} \\ ... \\ u_n = ar^{n-1} \end{gathered}$$

### 6.2.2 Geometric Series

A Geometric Series is defined as the sum of a certain number of consecutive elements in a geometric sequence.
For a geometric sequence of known initial term and constant difference, the term $S_n$ can be derived from the equation

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

The first equation is typically used for $r > 1$ while the second is used for $r < 1$.

### 6.2.3 Proving a GP

### 6.2.4 Convergence

For a $r < 1$, is can be proven that as $n$ tends to infinity, the value of $u_n$ tends to zero and the value of $S_n$ converges to a certain value and the series of this geometric sequence is said to be convergent. The value to which a series converges to is given by:

$$S_\infty = \frac{a}{1 - r}$$

The equation can be derived from the general formula of a geometric series as the numerator term $1 - r^n$ can be reduced to $1$ as $r^n$ tends to zero.