

# Trace, Metric, and Reality: Notes on Abstract Linear Algebra

Adam Coffman

DEPARTMENT OF MATHEMATICAL SCIENCES  
INDIANA UNIVERSITY - PURDUE UNIVERSITY FORT WAYNE  
2101 E. COLISEUM BLVD.  
FORT WAYNE, IN 46805-1499  
*E-mail address:* `CoffmanA@ipfw.edu`  
*URL:* `http://ipfw.edu/math/`

2010 *Mathematics Subject Classification.* Primary 15-02  
Secondary 15A69, 15A63, 53B30, 53C15

*Key words and phrases.* Trace, Hilbert-Schmidt, indefinite metric, tensor,  
vector-valued form, complex structure operator.

This project was supported in part by a 1998 Purdue Research Foundation  
Summer Faculty Grant and sabbatical semesters at IPFW in 2005 and 2015.

ABSTRACT. Elementary properties of the trace operator, and of some natural vector-valued generalizations, are given basis-free statements and proofs, using canonical maps from abstract linear algebra. Properties of contraction with respect to a non-degenerate (but possibly indefinite) metric are similarly analyzed. Several identities are stated geometrically, in terms of the Hilbert-Schmidt trace metric on spaces of linear maps, and metrics induced by tensor products and direct sums.

# Contents

Preface	v
Chapter 1. Introduction and Notation	1
1.1. Overview	1
1.2. Spaces of linear maps	2
1.3. Tensor products	4
1.4. Direct sums	9
1.5. Idempotents and involutions	15
Chapter 2. A Survey of Trace Elements	27
2.1. Endomorphisms: the scalar-valued trace	27
2.2. The generalized trace	33
2.3. Vector-valued trace	48
2.4. Equivalence of alternative definitions	54
Chapter 3. Bilinear Forms	77
3.1. Metrics	77
3.2. Isometries	81
3.3. Trace, with respect to a metric	84
3.4. The Hilbert-Schmidt metric	86
3.5. Orthogonal direct sums	88
3.6. Miscellaneous results	90
Chapter 4. Vector-valued Bilinear Forms	115
4.1. Symmetric forms	115
4.2. Vector-valued trace with respect to a metric	120
4.3. Partially symmetric forms	126
4.4. Revisiting the generalized trace	130
Chapter 5. Complex Structures	137
5.1. Complex Structure Operators	137
5.2. Complex linear and antilinear maps	140
5.3. Commuting Complex Structure Operators	143
5.4. Real trace with complex vector values	166
Bibliography	171



## Preface

These notes are a mostly self-contained collection of some theorems of linear algebra that arise in geometry, particularly results about the trace and bilinear forms. Many results are stated with complete proofs, the main method of proof being the use of canonical maps from abstract linear algebra. So, the content of these notes is highly dependent on the notation for these maps developed in Chapter 1.

This notation will be used in all the subsequent Chapters, which appear in a logical order, but for  $1 < m < n$ , it is possible to follow Chapter 1 immediately by Chapter  $n$ , with only a few citations of Chapter  $m$ .

In forming such a collection of results, there will be several statements which, while interesting, will not be needed in later Lemmas, Theorems, or Examples, and can be skipped without losing any logical steps. Such statements, when proved, will be labeled “Proposition,” and otherwise labeled “Exercise,” when a short proof follows from a “Hint” or is left to the reader entirely.

Adam Coffman  
July, 2000.



## CHAPTER 1

# Introduction and Notation

### 1.1. Overview

The goal of these notes is to present a basis-free treatment of linear maps and multilinear maps, and spaces of such maps. The unifying theme is the trace operator on these spaces, and generalizations of the trace, including vector-valued traces, and traces with respect to non-degenerate inner products. The emphasis is on the canonical nature of the objects involved. The definition of the trace (Definition 2.3) is essentially the “conceptual” approach of Mac Lane-Birkhoff ([**MB**] §IX.10) and Bourbaki ([**B**]). This approach also is taken in disciplines using linear algebra as a tool, for example, representation theory and mathematical physics. ([**FH**] §13.1, [**Geroch**] Chapter 14, [**K**]) In most cases, it is not difficult to translate the results into the usual statements about matrices and tensors, and in some cases, the proofs are more economical than choosing a basis and using matrices. In particular, no unexpected deviations from matrix theory arise.

Part of the motivation for this approach is a study of vector-valued Hermitian forms, with respect to abstractly defined complex and real structures. The conjugate-linear nature of these objects necessitates careful treatment of scalar multiplication, duality of vector spaces and maps, and tensor products of vector spaces and maps ([**GM**], [**P**]). The study of Hermitian forms seems to require a preliminary investigation into the fundamentals of the theory of bilinear forms, which now forms the first half of these notes. The payoff from the detailed treatment of bilinear forms will be the natural way in which the Hermitian case follows, in the second half.

The following Sections in this Chapter introduce some canonical linear maps, and basic concepts which will be used in all the subsequent Chapters. Chapter 2 reviews the usual trace of a map  $V \rightarrow V$ , and states a definition of the generalized trace of maps  $V \otimes U \rightarrow V \otimes W$ , or  $V \rightarrow V \otimes W$ , whose output is an element of  $\text{Hom}(U, W)$ , or  $W$ , respectively. Many of the theorems can be viewed as linear algebra versions of more general statements in category theory, as considered by [**JSV**], [**Maltsiniotis**], [**K**], [**PS**], [**Stolz-Teichner**].

Chapter 3 offers a similar basis-free approach to definitions, properties, and examples of a metric on a vector space, and the trace, or contraction, with respect to a metric. The metrics are assumed to be non-degenerate, and finite-dimensionality is a consequence. The main construction is a generalization of the well-known inner product  $\text{Tr}(A^T \cdot B)$  on the space of matrices (Theorem 3.39). This could be called the “Hilbert-Schmidt” metric on  $\text{Hom}(U, V)$ , induced by arbitrary metrics on  $U$  and  $V$ , and  $\text{Hom}(U, V)$  is shown to be isometric to  $U^* \otimes V$  with the induced tensor product metric. Chapter 4 develops the  $W$ -valued case of the trace with respect to a metric.

The basis-free approach is also motivated by an attempt to describe vector bundles and structures on them, including bilinear and Hermitian forms, and almost complex structures. An important geometric application would be real vector bundles with Riemannian or pseudo-Riemannian metrics, since definiteness is not assumed. The linear maps can be directly replaced by bundle morphisms, “distinguished non-zero element” by “nonvanishing section,” and in some cases, “ $\mathbb{K}$ ” by “trivial line bundle.”

The plan is to proceed at an elementary pace, so that if the first few Lemmas make sense to the reader, then nothing more advanced will be encountered after that. In particular, the relationships with differential geometry and category theory can be ignored entirely by the uninterested reader and are mentioned here only in optional “Remarks.” It will be pointed out when the finite-dimensionality is used—for example, in the Theorems in Chapter 2 about the vector-valued trace  $\text{Tr}_{V;U,W}$ ,  $V$  must be finite-dimensional, but  $U$  and  $W$  need not be.

## 1.2. Spaces of linear maps

Vector spaces are considered over an arbitrary field  $(\mathbb{K}, +, \cdot, 0, 1)$ , and  $+$  and  $\cdot$  will also refer to addition and scalar multiplication in the vector space. A  $\mathbb{K}$ -linear map  $A$  with domain  $U$  and target  $V$ , such that  $A(u) = v$ , will be written as  $A : U \rightarrow V : u \mapsto v$ , or the  $A$  may appear near the arrow when several maps are combined in a diagram.

NOTATION 1.1. The set of all  $\mathbb{K}$ -linear maps from  $U$  to  $V$  is denoted  $\text{Hom}(U, V)$ .

$\text{Hom}(U, V)$  is itself a vector space over  $\mathbb{K}$ , which is finite-dimensional when  $U$  and  $V$  are.

NOTATION 1.2.  $\text{Hom}(V, V)$  is abbreviated  $\text{End}(V)$ , the space of endomorphisms of  $V$ . The identity map  $v \mapsto v$  is denoted  $\text{Id}_V \in \text{End}(V)$ .

NOTATION 1.3.  $\text{Hom}(V, \mathbb{K})$  is abbreviated  $V^*$ , the dual space of  $V$ .

DEFINITION 1.4. For maps  $A : U' \rightarrow U$  and  $B : V \rightarrow V'$ , define

$$\text{Hom}(A, B) : \text{Hom}(U, V) \rightarrow \text{Hom}(U', V')$$

so that for  $F : U \rightarrow V$ ,

$$\text{Hom}(A, B)(F) = B \circ F \circ A : U' \rightarrow V'.$$

LEMMA 1.5. ([B] §II.1.2) *If  $A : U \rightarrow V$ ,  $B : V \rightarrow W$ ,  $C : X \rightarrow Y$ ,  $D : Y \rightarrow Z$ , then*

$$\text{Hom}(A, D) \circ \text{Hom}(B, C) = \text{Hom}(B \circ A, D \circ C) : \text{Hom}(W, X) \rightarrow \text{Hom}(U, Z).$$

■

DEFINITION 1.6. For any vector spaces  $U, V, W$ , define a generalized transpose map,

$$t_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}(V, W), \text{Hom}(U, W)),$$

so that for  $A : U \rightarrow V$ ,  $B : V \rightarrow W$ ,

$$t_{UV}^W(A) = \text{Hom}(A, \text{Id}_W) : B \mapsto B \circ A.$$

NOTATION 1.7. In the special case  $W = \mathbb{K}$ ,  $t_{UV}^{\mathbb{K}}$  is a canonical transpose map from  $\text{Hom}(U, V)$  to  $\text{Hom}(V^*, U^*)$ , and it will be abbreviated  $t_{UV}^{\mathbb{K}} = t_{UV}$ .



NOTATION 1.8.  $t_{UV}(A) = \text{Hom}(A, Id_{\mathbb{K}})$  will be abbreviated by  $A^* : V^* \rightarrow U^*$ , so that for  $\phi \in V^*$ ,  $A^*(\phi)$  is the map  $\phi \circ A : U \rightarrow \mathbb{K}$ , i.e.,  $(A^*(\phi))(u) = \phi(A(u))$ .

By Lemma 1.5,  $(B \circ A)^* = A^* \circ B^*$ , and if  $A$  is invertible, then  $(A^{-1})^* = (A^*)^{-1}$ . Also,  $Id_{V^*} = Id_V^*$ .

DEFINITION 1.9. For any vector spaces  $V, W$ , define

$$d_{VW} : V \rightarrow \text{Hom}(\text{Hom}(V, W), W)$$

so that for  $v \in V$ ,  $H \in \text{Hom}(V, W)$ ,

$$(d_{VW}(v)) : H \mapsto H(v).$$

LEMMA 1.10. For any vector spaces  $U, V, W$ , and any  $H : U \rightarrow V$ ,

$$(t_{\text{Hom}(V, W), \text{Hom}(U, W)}^W(t_{UV}^W(H))) \circ d_{UW} = d_{VW} \circ H.$$

PROOF. By definition of the  $t$  maps,

$$t_{\text{Hom}(V, W), \text{Hom}(U, W)}^W(t_{UV}^W(H)) = \text{Hom}(\text{Hom}(H, Id_W), Id_W).$$

Let  $u \in U$ ,  $A : V \rightarrow W$ .

$$\begin{aligned} (d_{VW}(H(u)))(A) &= A(H(u)), \\ (\text{Hom}(\text{Hom}(H, Id_W), Id_W)(d_{UW}(u)))(A) &= ((d_{UW}(u)) \circ \text{Hom}(H, Id_W))(A) \\ &= (d_{UW}(u))(A \circ H) = A(H(u)). \end{aligned}$$

■

NOTATION 1.11. In the special case  $W = \mathbb{K}$ , abbreviate  $d_{V\mathbb{K}}$  by  $d_V$ . It is the canonical double duality map  $d_V : V \rightarrow V^{**}$ , defined by  $(d_V(v))(\phi) = \phi(v)$ .

Lemma 1.10 gives the identity ([B] §II.2.7, [AF] §20)

$$d_V \circ A = A^{**} \circ d_U,$$

where for  $A : U \rightarrow V$ ,  $A^{**}$  abbreviates  $t_{V^*U^*}(t_{UV}(A))$ .

LEMMA 1.12. ([B] §II.7.5).  $d_V$  is one-to-one.  $d_V$  is invertible if and only if  $V$  is finite-dimensional.

PROOF. The one-to-one property is easily checked. ■

LEMMA 1.13.  $\text{Hom}(d_{VW}, Id_W) \circ d_{\text{Hom}(V, W), W} = Id_{\text{Hom}(V, W)}$ .

PROOF. For  $v \in V$ ,  $K : V \rightarrow W$ ,

$$\begin{aligned} ((\text{Hom}(d_{VW}, Id_W) \circ d_{\text{Hom}(V, W), W})(K))(v) &= (d_{\text{Hom}(V, W), W}(K))(d_{VW}(v)) \\ &= (d_{VW}(v))(K) = K(v). \end{aligned}$$

■

In the  $W = \mathbb{K}$  case, this gives the identity ([AF] §20)

$$d_V^* \circ d_{V^*} = Id_{V^*}.$$

REMARK 1.14. The vector space  $V$  is often identified with a subspace of  $V^{**}$ , and the map  $d_V$  ignored, but it is less trouble than might be expected to keep  $V$  and  $V^{**}$  distinct, and always accounting for  $d_V$  will turn out to be convenient bookkeeping.

### 1.3. Tensor products

DEFINITION 1.15. A function  $A$  from the cartesian product  $U \times V$  to a vector space  $W$  is a bilinear map means: for any  $\lambda \in \mathbb{K}$ ,  $u_1, u_2 \in U$ ,  $v_1, v_2 \in V$ ,

$$A(\lambda \cdot u_1, v_1) = A(u_1, \lambda \cdot v_1) = \lambda \cdot A(u_1, v_1),$$

$$A(u_1 + u_2, v_1) = A(u_1, v_1) + A(u_2, v_1),$$

and

$$A(u_1, v_1 + v_2) = A(u_1, v_1) + A(u_1, v_2).$$

DEFINITION 1.16. A tensor product of spaces  $U, V$ , is a vector space  $U \otimes V$ , together with a bilinear map  $\tau : U \times V \rightarrow U \otimes V$ , denoted  $\tau(u, v) = u \otimes v$ , such that for any bilinear map  $A : U \times V \rightarrow W$ , there exists a unique  $\mathbb{K}$ -linear map  $\alpha : U \otimes V \rightarrow W$  such that  $A = \alpha \circ \tau$ .

LEMMA 1.17. *For any  $U, V$ , there exists a tensor product  $U \otimes V$  and a map  $\tau$ . This space is unique up to a canonical invertible  $\mathbb{K}$ -linear map, and spanned by elements of the form  $u \otimes v$ , so that any linear map  $B : U \otimes V \rightarrow W$  is uniquely determined by its values on the set of elements  $u \otimes v$ . If  $U$  and  $V$  are finite-dimensional, then so is  $U \otimes V$ . ■*

EXAMPLE 1.18. The scalar multiplication map from  $\mathbb{K} \times U$  to  $U$  is bilinear, and induces a map

$$l : \mathbb{K} \otimes U \rightarrow U : \lambda \otimes u \mapsto \lambda \cdot u.$$

The map  $l$  is invertible, with inverse  $l^{-1}(u) = 1 \otimes u$ .

EXAMPLE 1.19. The switching map  $U \times V \rightarrow V \times U$  induces an invertible  $\mathbb{K}$ -linear map

$$s : U \otimes V \rightarrow V \otimes U : u \otimes v \mapsto v \otimes u.$$

LEMMA 1.20. *The spaces  $(U \otimes V) \otimes W$  and  $U \otimes (V \otimes W)$  are related by a canonical, invertible  $\mathbb{K}$ -linear map. ■*

REMARK 1.21. The above statements follow the standard treatments of tensor products. ([MB] §IX.8, [AF] §19, [K] §II.1) From this point, cartesian products  $U \times V$ , and maps  $\tau$  will not re-appear, unless the reader wants to verify certain maps are well-defined. Lemma 1.17 will be used frequently and without comment when defining maps or proving equality of two maps. The notion of a multilinear map is also standard, and the spaces from Lemma 1.20 will be identified with yet another space, a triple tensor product  $U \otimes V \otimes W$ , and similarly, the associativity for tensor products of more than three spaces will be implicitly assumed.

A tensor product of  $\mathbb{K}$ -linear maps canonically induces a  $\mathbb{K}$ -linear map on a tensor product, as follows.

DEFINITION 1.22. For any vector spaces  $U_1, U_2, V_1, V_2$ , define

$$j : \text{Hom}(U_1, V_1) \otimes \text{Hom}(U_2, V_2) \rightarrow \text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2),$$

so that for  $A : U_1 \rightarrow V_1$ ,  $B : U_2 \rightarrow V_2$ ,  $u \in U_1$ , and  $v \in U_2$ , the map  $j(A \otimes B)$  acts as:

$$(j(A \otimes B)) : u \otimes v \mapsto (A(u)) \otimes (B(v)).$$

LEMMA 1.23. ([B] §II.7.7, [K] §II.2)  *$j$  is one-to-one. If one of the ordered pairs  $(U_1, U_2)$ ,  $(U_1, V_1)$ , or  $(U_2, V_2)$  consists of finite-dimensional spaces, then  $j$  is invertible.* ■

Although it frequently occurs in the literature that  $j(A \otimes B)$  and  $A \otimes B$  are identified, here in Section 2.2 and thereafter,  $j$  will appear in diagrams as a map in its own right, and it will be useful to keep track of it everywhere, instead of only ignoring it sometimes. However, the notation  $j(A \otimes B)$  is not always as convenient as the following abbreviation.

NOTATION 1.24. Let  $[A \otimes B]$  denote  $j(A \otimes B)$ , so that, for example, the equation

$$(j(A \otimes B))(u \otimes v) = (A(u)) \otimes (B(v))$$

will appear as

$$[A \otimes B](u \otimes v) = (A(u)) \otimes (B(v)).$$

LEMMA 1.25. ([B] §II.3.2, [K] §II.6)

$$[A \otimes B] \circ [E \otimes F] = [(A \circ E) \otimes (B \circ F)].$$

■

Note that the brackets conveniently establish an order of operations, and appear three times in the Lemma, but may stand for three distinct canonical  $j$  maps, depending on the domains of  $A$ ,  $B$ ,  $E$ , and  $F$ . When it will be necessary or convenient to keep track of different maps, the  $j$  symbols will be used instead of the brackets, and will sometimes be labeled with subscripts, primes, etc., as in the following Lemma.

LEMMA 1.26. *For maps  $A_1 : U_3 \rightarrow U_1$ ,  $A_2 : U_4 \rightarrow U_2$ ,  $B_1 : V_1 \rightarrow V_3$ ,  $B_2 : V_2 \rightarrow V_4$ , the following diagram is commutative:*

$$\begin{array}{ccc} \text{Hom}(U_1, V_1) \otimes \text{Hom}(U_2, V_2) & \xrightarrow{a_1} & \text{Hom}(U_3, V_3) \otimes \text{Hom}(U_4, V_4) \\ \downarrow j' & & \downarrow j'' \\ \text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2) & \xrightarrow{a_2} & \text{Hom}(U_3 \otimes U_4, V_3 \otimes V_4) \end{array}$$

where the horizontal arrows are

$$\begin{aligned} a_1 &= [\text{Hom}(A_1, B_1) \otimes \text{Hom}(A_2, B_2)] \\ a_2 &= \text{Hom}([A_1 \otimes A_2], [B_1 \otimes B_2]). \end{aligned}$$

PROOF. For  $E : U_1 \rightarrow V_1$  and  $F : U_2 \rightarrow V_2$ ,

$$\begin{aligned} E \otimes F &\mapsto (j'' \circ [\text{Hom}(A_1, B_1) \otimes \text{Hom}(A_2, B_2)])(E \otimes F) \\ &= j''((\text{Hom}(A_1, B_1)(E)) \otimes (\text{Hom}(A_2, B_2)(F))) \\ &= j''((B_1 \circ E \circ A_1) \otimes (B_2 \circ F \circ A_2)), \\ E \otimes F &\mapsto (\text{Hom}([A_1 \otimes A_2], [B_1 \otimes B_2]) \circ j')(E \otimes F) \\ &= [B_1 \otimes B_2] \circ (j'(E \otimes F)) \circ [A_1 \otimes A_2] \\ &= j''((B_1 \circ E \circ A_1) \otimes (B_2 \circ F \circ A_2)). \end{aligned}$$

■

DEFINITION 1.27. For any vector spaces  $U, V$ , define

$$k_{UV} : U^* \otimes V \rightarrow \text{Hom}(U, V)$$

so that for  $\xi \in U^*$ ,  $v \in V$ , and  $u \in U$ ,

$$(k_{UV}(\xi \otimes v)) : u \mapsto \xi(u) \cdot v \in V.$$

LEMMA 1.28. ([B] §II.7.7, [K] §II.2) *The canonical map  $k_{UV}$  is one-to-one, and invertible if  $U$  or  $V$  is finite-dimensional.* ■

LEMMA 1.29. *For maps  $A : U' \rightarrow U$ ,  $B : V \rightarrow V'$ , the following diagram is commutative:*

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{k_{UV}} & \text{Hom}(U, V) \\ [A^* \otimes B] \downarrow & & \downarrow \text{Hom}(A, B) \\ U'^* \otimes V' & \xrightarrow{k_{U'V'}} & \text{Hom}(U', V') \end{array}$$

PROOF. For  $\phi \otimes v \in U^* \otimes V$ ,  $u \in U'$ ,

$$\begin{aligned} (\text{Hom}(A, B) \circ k_{UV})(\phi \otimes v) &= B \circ (k_{UV}(\phi \otimes v)) \circ A : \\ u &\mapsto B(\phi(A(u)) \cdot v) = \phi(A(u)) \cdot B(v), \\ (k_{U'V'} \circ [A^* \otimes B])(\phi \otimes v) &= k_{U'V'}((A^*(\phi)) \otimes (B(v))) : \\ u &\mapsto (A^*(\phi))(u) \cdot B(v) = \phi(A(u)) \cdot B(v). \end{aligned}$$

■

DEFINITION 1.30. For any vector spaces  $U, V, W$  define

$$e_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}(V, W) \otimes U, W)$$

so that for  $A : U \rightarrow V$ ,  $B : V \rightarrow W$ , and  $u \in U$ ,

$$e_{UV}^W(A) : B \otimes u \mapsto B(A(u)) \in W.$$

NOTATION 1.31. For the special case  $W = \mathbb{K}$ , abbreviate  $e_{UV}^{\mathbb{K}} = e_{UV}$ :

$$e_{UV} : \text{Hom}(U, V) \rightarrow (V^* \otimes U)^*$$

so that for  $A : U \rightarrow V$ ,  $\phi \in V^*$ , and  $u \in U$ ,

$$e_{UV}(A) : \phi \otimes u \mapsto \phi(A(u)) \in \mathbb{K}.$$

LEMMA 1.32.  $e_{UV}^W$  is one-to-one. If  $U$  and  $V$  are finite-dimensional, then  $e_{UV}$  is invertible.

PROOF. It is straightforward to check that the kernel of  $e_{UV}^W$  is  $\{0_{\text{Hom}(U, V)}\}$ . ■

NOTATION 1.33. The composite of the  $e$  and  $k$  maps is denoted

$$f_{UV} = e_{UV} \circ k_{UV} : U^* \otimes V \rightarrow (V^* \otimes U)^*.$$

The image  $f_{UV}(\phi \otimes v)$  acts on  $\xi \otimes u$  to give  $\phi(u) \cdot \xi(v) \in \mathbb{K}$ :

$$(e_{UV}(k_{UV}(\phi \otimes v)))(\xi \otimes u) = \xi((k_{UV}(\phi \otimes v))(u)) = \xi(\phi(u) \cdot v) = \phi(u) \cdot \xi(v).$$

LEMMA 1.34.  $f_{UV}^* \circ d_{V^* \otimes U} = f_{VU} : V^* \otimes U \rightarrow (U^* \otimes V)^*$ .

PROOF. For  $\xi \otimes u \in V^* \otimes U$ , and  $\phi \otimes v \in U^* \otimes V$ ,

$$\begin{aligned} (f_{UV}^*(d_{V^* \otimes U}(\xi \otimes u)))(\phi \otimes v) &= (d_{V^* \otimes U}(\xi \otimes u))(f_{UV}(\phi \otimes v)) \\ &= (f_{UV}(\phi \otimes v))(\xi \otimes u) \\ &= \phi(u) \cdot \xi(v) \\ &= (f_{VU}(\xi \otimes u))(\phi \otimes v). \end{aligned}$$

■

REMARK 1.35. The canonical maps used appearing in this Section are well-known in abstract linear algebra, but also appear in concrete matrix algebra and applications: see [Magnus], [G<sub>2</sub>] (particularly §I.8 and §VI.3), [Graham], and [HJ] (Chapter 4), which also has historical references. For example, the map  $k_{UV}^{-1} : \text{Hom}(U, V) \rightarrow U^* \otimes V$  is a “vectorization,” or “*vec*” operator. The  $t_{UV}$  map, of course, is analogous to the transpose operation ( $A \mapsto A'$ , in the notation of [Magnus]) and the identity  $\text{vec}(ABC) = (C' \otimes A)\text{vec}B$  ([Nissen], [HJ], [Magnus] §1.10) is the analogue of Lemma 1.29.

NOTATION 1.36. For any vector spaces  $U, V$ , the following composite will be abbreviated:

$$p = [d_V \otimes Id_{V^*}] \circ s : U^* \otimes V \rightarrow V^{**} \otimes U^*.$$

So, for  $\phi \in U^*, v \in V$ ,  $p(\phi \otimes v) = (d_V(v)) \otimes \phi$ .

LEMMA 1.37. For any  $U, V$ , the following diagram is commutative.

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{k_{UV}} & \text{Hom}(U, V) \\ p \downarrow & & \downarrow t_{UV} \\ V^{**} \otimes U^* & \xrightarrow{k_{V^* U^*}} & \text{Hom}(V^*, U^*) \end{array}$$

PROOF.

$$\begin{aligned} \phi \otimes v &\mapsto (k_{V^* U^*} \circ p)(\phi \otimes v) = k_{V^* U^*}((d_V(v)) \otimes \phi) : \\ \xi &\mapsto (k_{V^* U^*}((d_V(v)) \otimes \phi))(\xi) = (d_V(v))(\xi) \cdot \phi = \xi(v) \cdot \phi : \\ u &\mapsto \xi(v) \cdot \phi(u), \\ \phi \otimes v &\mapsto (t_{UV} \circ k_{UV})(\phi \otimes v) : \\ \xi &\mapsto (t_{UV}(k_{UV}(\phi \otimes v)))(\xi) = \xi \circ (k_{UV}(\phi \otimes v)) : \\ u &\mapsto \xi((k_{UV}(\phi \otimes v))(u)) = \xi(\phi(u) \cdot v) = \phi(u) \cdot \xi(v). \end{aligned}$$

■

REMARK 1.38. The map  $p$  also corresponds to a well-known object in matrix algebra, the “commutation matrix”  $K$  in [Magnus] §3.1. The identity  $K \text{vec} A = \text{vec}(A')$  corresponds to the identity  $k \circ p = t \circ k$ . It has also been called the “shuffle matrix” ([L], [HJ]).

DEFINITION 1.39. For arbitrary vector spaces  $U, V, W$  define

$$q : \text{Hom}(V, \text{Hom}(U, W)) \rightarrow \text{Hom}(V \otimes U, W)$$

so that for  $K : V \rightarrow \text{Hom}(U, W)$ ,  $v \in V$ ,  $u \in U$ ,

$$(q(K))(v \otimes u) = (K(v))(u).$$

LEMMA 1.40. ([**AF**] §19, [**B**] §II.4.1, [**MB**] §IX.11, [**K**] §II.1)  $q$  is invertible.

PROOF. For  $D \in \text{Hom}(V \otimes U, W)$ , check that

$$q^{-1}(D) : v \mapsto (u \mapsto D(v \otimes u))$$

defines an inverse. ■

LEMMA 1.41. For any  $U, V, W$ , the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}(U, V) & \xrightarrow{e_{UV}^W} & \text{Hom}(\text{Hom}(V, W) \otimes U, W) \\ t_{UV}^W \downarrow & \nearrow q & \\ \text{Hom}(\text{Hom}(V, W), \text{Hom}(U, W)) & & \end{array}$$

PROOF.

$$q \circ t_{UV}^W : A \mapsto q(\text{Hom}(A, \text{Id}_W)) : B \otimes u \mapsto (B \circ A)(u) = (e_{UV}^W(A))(B \otimes u).$$
■

LEMMA 1.42. ([**AF**] §20) For any maps  $D : V_2 \rightarrow V_1$ ,  $E : U_2 \rightarrow U_1$ ,  $F : W_1 \rightarrow W_2$ , the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}(V_1, \text{Hom}(U_1, W_1)) & \xrightarrow{q_1} & \text{Hom}(U_1 \otimes V_1, W_1) \\ \downarrow \text{Hom}(D, \text{Hom}(E, F)) & & \downarrow \text{Hom}([E \otimes D], F) \\ \text{Hom}(V_2, \text{Hom}(U_2, W_2)) & \xrightarrow{q_2} & \text{Hom}(U_2 \otimes V_2, W_2) \end{array}$$

PROOF. Starting with any  $G : V_1 \rightarrow \text{Hom}(U_1, W_1)$ ,

$$\begin{aligned} (\text{Hom}([E \otimes D], F) \circ q_1)(G) : u \otimes v &\mapsto (F \circ (q_1(G)) \circ [E \otimes D])(u \otimes v) \\ &= F((G(D(v)))(E(u))), \\ (q_2 \circ \text{Hom}(D, \text{Hom}(E, F)))(G) : u \otimes v &\mapsto (q_2(\text{Hom}(E, F) \circ G \circ D))(u \otimes v) \\ &= ((\text{Hom}(E, F) \circ G \circ D)(v))(u) \\ &= (F \circ (G(D(v))) \circ E)(u) \\ &= F((G(D(v)))(E(u))). \end{aligned}$$
■

### 1.4. Direct sums

DEFINITION 1.43. Given vector spaces  $V, V_1, V_2$ , and ordered pairs of maps  $(P_1, P_2)$  and  $(Q_1, Q_2)$ , where  $P_i : V \rightarrow V_i$ ,  $Q_i : V_i \rightarrow V$  for  $i = 1, 2$ ,  $V$  is a direct sum of  $V_1$  and  $V_2$  means:

$$Q_1 \circ P_1 + Q_2 \circ P_2 = Id_V$$

and

$$P_i \circ Q_I = \begin{cases} Id_{V_i} & \text{if } i = I \\ 0_{\text{Hom}(V_I, V_i)} & \text{if } i \neq I \end{cases}.$$

This data will sometimes be abbreviated  $V = V_1 \oplus V_2$ , when the maps  $P_i$  (called projections or projection operators) and  $Q_i$  (inclusions or inclusion operators) are understood. The generalization to direct sums of finitely many spaces is straightforward. (see also [AF] §6.)

EXAMPLE 1.44. If  $H : U \rightarrow V$  is an invertible map between arbitrary vector spaces, and  $V = V_1 \oplus V_2$ , then  $U$  is a direct sum of  $V_1$  and  $V_2$ , with projections  $P_i \circ H : U \rightarrow V_i$  for  $i = 1, 2$ , and inclusions  $H^{-1} \circ Q_i : V_i \rightarrow U$ .

EXAMPLE 1.45. If  $V = V_1 \oplus V_2$ , and  $U$  is any vector space, then  $V \otimes U$  is a direct sum of  $V_1 \otimes U$  and  $V_2 \otimes U$ . The projection and inclusion operators are  $[P_i \otimes Id_U] : V \otimes U \rightarrow V_i \otimes U$ , and  $[Q_i \otimes Id_U] : V_i \otimes U \rightarrow V \otimes U$ . There is an analogous direct sum  $U \otimes V = U \otimes V_1 \oplus U \otimes V_2$ .

EXAMPLE 1.46. If  $V = V_1 \oplus V_2$ , and  $U$  is any vector space, then  $\text{Hom}(U, V)$  is a direct sum of  $\text{Hom}(U, V_1)$  and  $\text{Hom}(U, V_2)$ . The projection and inclusion operators are  $\text{Hom}(Id_U, P_i) : \text{Hom}(U, V) \rightarrow \text{Hom}(U, V_i)$ , and  $\text{Hom}(Id_U, Q_i) : \text{Hom}(U, V_i) \rightarrow \text{Hom}(U, V)$ .

EXAMPLE 1.47. If  $V = V_1 \oplus V_2$ , and  $U$  is any vector space, then  $\text{Hom}(V, U)$  is a direct sum of  $\text{Hom}(V_1, U)$  and  $\text{Hom}(V_2, U)$ . The projection operators are  $\text{Hom}(Q_i, Id_U) : \text{Hom}(V, U) \rightarrow \text{Hom}(V_i, U)$ , and the inclusion operators are

$$\text{Hom}(P_i, Id_U) : \text{Hom}(V_i, U) \rightarrow \text{Hom}(V, U).$$

EXAMPLE 1.48. As the  $U = \mathbb{K}$  special case of the previous Example, if  $V = V_1 \oplus V_2$ , then  $V^* = V_1^* \oplus V_2^*$ , with projections  $Q_i^*$  and inclusions  $P_i^*$ .

LEMMA 1.49. *Given  $V = V_1 \oplus V_2$ , the image of  $Q_2$ , i.e., the subspace  $Q_2(V_2)$  of  $V$ , is equal to the subspace  $\ker(P_1)$ , the kernel of  $P_1$ , which is also equal to  $\ker(Q_1 \circ P_1)$ .*

PROOF. The second equality follows from  $\ker(P_1) \subseteq \ker(Q_1 \circ P_1) \subseteq \ker(P_1 \circ Q_1 \circ P_1) = \ker(P_1)$ . It follows from  $P_1 \circ Q_2 = 0_{\text{Hom}(V_2, V_1)}$  that  $Q_2(V_2) \subseteq \ker(P_1)$ , and if  $P_1(v) = 0_{V_1}$ , then  $v = (Q_1 \circ P_1 + Q_2 \circ P_2)(v) = Q_2(P_2(v))$ , so  $\ker(P_1) \subseteq Q_2(V_2)$ . ■

LEMMA 1.50. *Given  $U = U_1 \oplus U_2$  and  $V = V_1 \oplus V_2$ , with projections and inclusions  $P'_i, Q'_i, P_i, Q_i$ , respectively, suppose the maps  $A_1 : U_1 \rightarrow V_1$  and  $A_2 : U_2 \rightarrow V_2$  are both invertible. Then the map  $Q_1 \circ A_1 \circ P'_1 + Q_2 \circ A_2 \circ P'_2 : U \rightarrow V$  is invertible.*

PROOF. The inverse is  $Q'_1 \circ A_1^{-1} \circ P_1 + Q'_2 \circ A_2^{-1} \circ P_2$ . ■

LEMMA 1.51. *Given a direct sum  $V = V_1 \oplus V_2$  as in Definition 1.43, another direct sum  $U = U_1 \oplus U_2$ , with operators  $P'_i$  and  $Q'_i$ , and  $H : U \rightarrow V$ , the following are equivalent:*

- (1)  $Q_1 \circ P_1 \circ H = H \circ Q'_1 \circ P'_1$ ;
- (2)  $Q_2 \circ P_2 \circ H = H \circ Q'_2 \circ P'_2$ ;
- (3)  $P_1 \circ H \circ Q'_2 = 0_{\text{Hom}(U_2, V_1)}$  and  $P_2 \circ H \circ Q'_1 = 0_{\text{Hom}(U_1, V_2)}$ ;
- (4) *There exist maps  $H_1 : U_1 \rightarrow V_1$  and  $H_2 : U_2 \rightarrow V_2$  such that  $H = Q_1 \circ H_1 \circ P'_1 + Q_2 \circ H_2 \circ P'_2$ .*

PROOF. First, for (1)  $\iff$  (2),

$$\begin{aligned} H &= (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H = H \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \\ &= Q_1 \circ P_1 \circ H + Q_2 \circ P_2 \circ H = H \circ Q'_1 \circ P'_1 + H \circ Q'_2 \circ P'_2. \end{aligned}$$

Applying either equality (1) or (2), then subtracting, gives the other equality. For (1)  $\implies$  (3),

$$\begin{aligned} P_1 \circ H \circ Q'_2 &= P_1 \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H \circ Q'_2 \\ &= P_1 \circ H \circ Q'_1 \circ P'_1 \circ Q'_2 = 0_{\text{Hom}(U_2, V_1)}, \\ P_2 \circ H \circ Q'_1 &= P_2 \circ H \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ Q'_1 \\ &= P_2 \circ Q_1 \circ P_1 \circ H \circ Q'_1 = 0_{\text{Hom}(U_1, V_2)}. \end{aligned}$$

Next, to show that (3) implies (1) or (2), let  $i = 1$  or  $2$ :

$$\begin{aligned} Q_i \circ P_i \circ H &= Q_i \circ P_i \circ H \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \\ &= Q_i \circ P_i \circ H \circ Q'_i \circ P'_i \\ &= (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H \circ Q'_i \circ P'_i = H \circ Q'_i \circ P'_i. \end{aligned}$$

The construction in (4) is the same as in Lemma 1.50 (without requiring invertibility). The implication (4)  $\implies$  (1) is straightforward. To show that (1) and (2) imply (4), let  $H_1 = P_1 \circ H \circ Q'_1$  and  $H_2 = P_2 \circ H \circ Q'_2$ . Then,

$$\begin{aligned} Q_1 \circ H_1 \circ P'_1 + Q_2 \circ H_2 \circ P'_2 &= Q_1 \circ P_1 \circ H \circ Q'_1 \circ P'_1 + Q_2 \circ P_2 \circ H \circ Q'_2 \circ P'_2 \\ &= Q_1 \circ P_1 \circ Q_1 \circ P_1 \circ H + Q_2 \circ P_2 \circ Q_2 \circ P_2 \circ H \\ &= (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H = H. \end{aligned}$$

■

DEFINITION 1.52. For  $V = V_1 \oplus V_2$ ,  $U = U_1 \oplus U_2$ , and  $H : U \rightarrow V$ ,  $H$  respects the direct sums means:  $H$  satisfies any of the equivalent conditions from Lemma 1.51. For such a map and  $i = 1, 2$ , the composites  $P_i \circ H \circ Q'_i : U_i \rightarrow V_i$  are said to be induced by  $H$ .

LEMMA 1.53. *If  $H : U \rightarrow V$  is an invertible map which respects the direct sums as in Definition 1.52, then  $H^{-1}$  also respects the direct sums, and for each  $i = 1, 2$ , the induced map  $P_i \circ H \circ Q'_i : U_i \rightarrow V_i$  is invertible, with inverse  $P'_i \circ H^{-1} \circ Q_i$ .*

PROOF.

$$\begin{aligned} (P_i \circ H \circ Q'_i) \circ (P'_i \circ H^{-1} \circ Q_i) &= P_i \circ Q_i \circ P_i \circ H \circ H^{-1} \circ Q_i \\ &= P_i \circ Q_i = \text{Id}_{V_i} \\ (P'_i \circ H^{-1} \circ Q_i) \circ (P_i \circ H \circ Q'_i) &= P'_i \circ H^{-1} \circ H \circ Q'_i \circ P'_i \circ Q_i \\ &= P'_i \circ Q_i \end{aligned}$$



If  $i = I$ , this shows  $P_i \circ H \circ Q'_i$  is invertible. If  $i \neq I$ , then  $P'_I \circ H^{-1} \circ Q_i = 0_{\text{Hom}(V_i, U_I)}$ , so  $H^{-1}$  respects the direct sums. ■

LEMMA 1.54. *Given  $U = U_1 \oplus U_2$ ,  $V = V_1 \oplus V_2$ ,  $W = W_1 \oplus W_2$ , if  $H : U \rightarrow V$  respects the direct sums and  $H' : V \rightarrow W$  respects the direct sums, then  $H' \circ H : U \rightarrow W$  respects the direct sums. A map induced by the composite is equal to the composite of the corresponding induced maps.* ■

LEMMA 1.55. *Suppose  $V = V_1 \oplus V_2$ ,  $U = U_1 \oplus U_2$ , and  $H : U \rightarrow V$  respects the direct sums, inducing maps  $P_i \circ H \circ Q'_i$ . Then, for any  $A : W \rightarrow X$ , the map  $\text{Hom}(A, H) : \text{Hom}(X, U) \rightarrow \text{Hom}(W, V)$  respects the direct sums*

$$\text{Hom}(X, U_1) \oplus \text{Hom}(X, U_2) \rightarrow \text{Hom}(W, V_1) \oplus \text{Hom}(W, V_2)$$

*from Example 1.46. The induced map  $\text{Hom}(Id_W, P_i) \circ \text{Hom}(A, H) \circ \text{Hom}(Id_X, Q'_i)$  is equal to  $\text{Hom}(A, P_i \circ H \circ Q'_i)$ . Analogously, the map  $\text{Hom}(H, A) : \text{Hom}(V, W) \rightarrow \text{Hom}(U, X)$  respects the direct sums*

$$\text{Hom}(V_1, W) \oplus \text{Hom}(V_2, W) \rightarrow \text{Hom}(U_1, X) \oplus \text{Hom}(U_2, X)$$

*from Example 1.47, and the induced map  $\text{Hom}(Q'_i, Id_W) \circ \text{Hom}(H, A) \circ \text{Hom}(P_i, Id_X)$  is equal to  $\text{Hom}(P_i \circ H \circ Q'_i, A)$ .*

PROOF. All the claims follow immediately from Lemma 1.5. ■

NOTATION 1.56. Given a direct sum  $V = V_1 \oplus V_2$  with operator pairs  $(P_1, P_2)$ ,  $(Q_1, Q_2)$ , the pairs in the other order,  $(P_2, P_1)$ ,  $(Q_2, Q_1)$ , also satisfy the definition of direct sum. The notation  $V = V_2 \oplus V_1$  refers to these re-ordered pairs.

EXAMPLE 1.57. A map  $H \in \text{End}(V)$  which satisfies  $Q_2 \circ P_2 \circ H = H \circ Q_1 \circ P_1$  respects the direct sums  $H : V_1 \oplus V_2 \rightarrow V_2 \oplus V_1$  (and so  $H$  also satisfies the other identities from Lemma 1.51).

LEMMA 1.58. *Given a vector space  $V$  that admits two direct sums,  $V = V_1 \oplus V_2$ ,  $V = V_1'' \oplus V_2''$  with projections and inclusions  $P_i, Q_i, P_i'', Q_i''$ , respectively, the following are equivalent:*

- (1) *The identity map  $Id_V : V_1 \oplus V_2 \rightarrow V_1'' \oplus V_2''$  respects the direct sums;*
- (2)  $Q_1 \circ P_1 = Q_1'' \circ P_1''$ ;
- (3)  $Q_2 \circ P_2 = Q_2'' \circ P_2''$ ;
- (4)  $P_I'' \circ Q_i = 0_{\text{Hom}(V_i, V_I'')}$  for  $i \neq I$ ;
- (5)  $P_I \circ Q_i'' = 0_{\text{Hom}(V_i'', V_I)}$  for  $i \neq I$ .

PROOF. The first statement is, by Definition 1.52 and Lemma 1.51, equivalent to any of the next three statements. The equivalence with the last statement follows from Lemma 1.53. ■

DEFINITION 1.59. Given  $V$ , two direct sums  $V = V_1 \oplus V_2$  and  $V = V_1'' \oplus V_2''$  are equivalent direct sums means: they satisfy any of the properties from Lemma 1.58.

For a fixed  $V$ , this notion is clearly an equivalence relation on direct sum decompositions of  $V$ .

EXAMPLE 1.60. If  $V = V_1 \oplus V_2$  and  $H_1 : U_1 \rightarrow V_1$  and  $H_2 : U_2 \rightarrow V_2$  are invertible, then  $V$  is a direct sum  $U_1 \oplus U_2$ , with projections  $H_i^{-1} \circ P_i$  and inclusions  $Q_i \circ H_i$ . The direct sums  $V = V_1 \oplus V_2$  and  $V = U_1 \oplus U_2$  are equivalent.

LEMMA 1.61. *Given  $U$  and  $V$ , a direct sum  $U = U_1 \oplus U_2$ , and a map  $H : U \rightarrow V$ , suppose  $V = V_1 \oplus V_2$  and  $V = V_1'' \oplus V_2''$  are equivalent direct sums. Then  $H$  respects the direct sums  $U_1 \oplus U_2 \rightarrow V_1 \oplus V_2$  if and only if  $H$  respects the direct sums  $U_1 \oplus U_2 \rightarrow V_1'' \oplus V_2''$ . Similarly, a map  $A : V \rightarrow U$  respects the direct sums  $V_1 \oplus V_2 \rightarrow U_1 \oplus U_2$  if and only if  $A$  respects the direct sums  $V_1'' \oplus V_2'' \rightarrow U_1 \oplus U_2$ . ■*

LEMMA 1.62. *Given  $V = V_1 \oplus V_2$  and  $V = V_1'' \oplus V_2''$  with operators  $P_i, Q_i, P_i'', Q_i''$ , if  $P_1 \circ Q_2'' = 0_{\text{Hom}(V_2'', V_1)}$ , and  $P_1'' \circ Q_2 = 0_{\text{Hom}(V_2, V_1'')}$ , then  $P_1'' \circ Q_1 : V_1 \rightarrow V_1''$  and  $P_2'' \circ Q_2 : V_2 \rightarrow V_2''$  are both invertible.*

PROOF. The inverse of  $P_i'' \circ Q_i$  is  $P_i \circ Q_i'' : V_i'' \rightarrow V_i$ . ■

As a special case of both Lemma 1.62 and Lemma 1.53, if  $V = V_1 \oplus V_2$  and  $V = V_1'' \oplus V_2''$  are equivalent direct sums, then there are canonically induced invertible maps  $P_i'' \circ Q_i : V_i \rightarrow V_i'', i = 1, 2$ .

LEMMA 1.63. *Suppose  $\phi \in V^*$ , and  $\phi \neq 0_{V^*}$ . Then there exists a direct sum  $V = \mathbb{K} \oplus \ker(\phi)$ .*

PROOF.  $\ker(\phi)$  is the “kernel” of  $\phi$ , the subspace  $\{w \in V : \phi(w) = 0\}$ . Let  $Q_2$  be the inclusion of this subspace in  $V$ . Since  $\phi \neq 0_{V^*}$ , there exists some  $v \in V$  so that  $\phi(v) \neq 0_V$ . Let  $\alpha, \beta \in \mathbb{K}$  be any constants so that  $\alpha \cdot \beta \cdot \phi(v) = 1$ . Define  $Q_1^\beta : \mathbb{K} \rightarrow V$  so that for  $\gamma \in \mathbb{K}$ ,  $Q_1^\beta(\gamma) = \beta \cdot \gamma \cdot v$ . Define  $P_1^\alpha = \alpha \cdot \phi : V \rightarrow \mathbb{K}$ . Then,

$$P_1^\alpha \circ Q_1^\beta : \gamma \mapsto \alpha \cdot \phi(\beta \cdot \gamma \cdot v) = \alpha \cdot \beta \cdot \phi(v) \cdot \gamma = \gamma.$$

For any  $w \in \ker(\phi)$ ,  $(P_1^\alpha \circ Q_2)(w) = \alpha \cdot \phi(w) = 0$ . Define  $P_2 = Id_V - Q_1^\beta \circ P_1^\alpha$ , which is a map from  $V$  to  $\ker(\phi)$ : if  $u \in V$ , then

$$\begin{aligned} (\phi \circ P_2)(u) &= (\phi \circ Id_V - \phi \circ Q_1^\beta \circ P_1^\alpha)(u) \\ &= \phi(u) - \phi(Q_1^\beta(\alpha \cdot \phi(u))) \\ &= \phi(u) - \phi(\beta \cdot \alpha \cdot \phi(u) \cdot v) \\ &= \phi(u) - \alpha \cdot \beta \cdot \phi(v) \cdot \phi(u) = 0. \end{aligned}$$

Also,

$$P_2 \circ Q_1^\beta = Id_V \circ Q_1^\beta - Q_1^\beta \circ P_1^\alpha \circ Q_1^\beta = Q_1^\beta - Q_1^\beta \circ Id_{\mathbb{K}} = 0_{\text{Hom}(\mathbb{K}, \ker(\phi))},$$

$$P_2 \circ Q_2 = (Id_V - Q_1^\beta \circ P_1^\alpha) \circ Q_2 = Q_2 - Q_1^\beta \circ 0_{(\ker(\phi))^*} = Id_{\ker(\phi)}.$$

■

Given  $V$  and  $\phi$ , the direct sum from the previous Lemma is generally not unique, nor are two such direct sums, depending on  $v, \alpha, \beta$ , even equivalent in general. However, in some later Examples, there will be a canonical element  $v \notin \ker(\phi)$ , and in such a case, different choices of  $\alpha, \beta$  give equivalent direct sums.

LEMMA 1.64. *Given  $V, \phi \in V^*$ , and  $v \in V$  so that  $\phi(v) \neq 0$ , let  $\alpha, \beta, \alpha', \beta' \in \mathbb{K}$  be any constants so that  $\alpha \cdot \beta \cdot \phi(v) = \alpha' \cdot \beta' \cdot \phi(v) = 1$ . Then the direct sum  $V = \mathbb{K} \oplus \ker(\phi)$  constructed in the Proof of Lemma 1.63 is equivalent to the analogous direct sum with operators  $Q_1^{\beta'} : \gamma \mapsto \beta' \cdot \gamma \cdot v, P_1^{\alpha'} = \alpha' \cdot \phi, Q_2$ , and  $P_2' = Id_V - Q_1^{\beta'} \circ P_1^{\alpha'}$ . ■*

DEFINITION 1.65. A map  $C \in \text{Hom}(X, Y)$  is a linear monomorphism means:  $C$  has the following cancellation property for any compositions with linear maps  $A$  and  $B$  (which are well-defined in the sense that  $X$  is the target space of both  $A$  and  $B$ ),

$$C \circ A = C \circ B \implies A = B.$$

DEFINITION 1.66. A map  $C \in \text{Hom}(X, Y)$  is a linear epimorphism means:  $C$  has the following cancellation property for any compositions with linear maps  $A$  and  $B$ ,

$$A \circ C = B \circ C \implies A = B.$$

The following result will be used as a step in Theorem 4.30.

THEOREM 1.67. Suppose  $U = U_1 \oplus U_2$  is a direct sum with projection and inclusion operators  $P_i, Q_i$ , and that there are vector spaces  $V, V_1, V_2$ , and maps  $P'_1 : V \rightarrow V_1, Q'_2 : V_2 \rightarrow V, H : U \rightarrow V, H_1 : U_1 \rightarrow V_1, H_2 : U_2 \rightarrow V_2$ , such that  $P'_1 \circ H = H_1 \circ P_1, Q'_2 \circ H_2 = H \circ Q_2$ , and  $P'_1 \circ Q'_2 = 0_{\text{Hom}(V_2, V_1)}$ . Suppose further that  $H$  and  $H_1$  are invertible, and that  $Q'_2$  is a linear monomorphism. Then, there exist maps  $Q'_1 : V_1 \rightarrow V$  and  $P'_2 : V \rightarrow V_2$  such that  $V$  is a direct sum of  $V_1$  and  $V_2$ . Also,  $H$  respects the direct sums, and  $H_2$  is invertible.

PROOF. Let  $Q'_1 = H \circ Q_1 \circ H_1^{-1}$ . Then

$$P'_1 \circ Q'_1 = P'_1 \circ H \circ Q_1 \circ H_1^{-1} = H_1 \circ P_1 \circ Q_1 \circ H_1^{-1} = \text{Id}_{V_1}.$$

Let  $P'_2 = H_2 \circ P_2 \circ H^{-1}$ . Then

$$P'_2 \circ Q'_1 = H_2 \circ P_2 \circ H^{-1} \circ H \circ Q_1 \circ H_1^{-1} = 0_{\text{Hom}(V_1, V_2)}.$$

$$\begin{aligned} Q'_1 \circ P'_1 + Q'_2 \circ P'_2 &= H \circ Q_1 \circ H_1^{-1} \circ P'_1 + Q'_2 \circ H_2 \circ P_2 \circ H^{-1} \\ &= H \circ Q_1 \circ P_1 \circ H^{-1} + H \circ Q_2 \circ P_2 \circ H^{-1} \\ &= \text{Id}_V. \end{aligned}$$

$$Q'_2 \circ P'_2 \circ Q'_2 = (\text{Id}_V - Q'_1 \circ P'_1) \circ Q'_2 = Q'_2,$$

so  $P'_2 \circ Q'_2 = \text{Id}_{V_2}$ , by the monomorphism property.  $H$  respects the direct sums:

$$P'_1 \circ H \circ Q_2 = H_1 \circ P_1 \circ Q_2 = 0_{\text{Hom}(U_2, V_1)}$$

$$P'_2 \circ H \circ Q_1 = H_2 \circ P_2 \circ H^{-1} \circ H \circ Q_1 = 0_{\text{Hom}(U_1, V_2)}.$$

By Lemma 1.53,  $P'_2 \circ H \circ Q_2 = H_2 \circ P_2 \circ H^{-1} \circ H \circ Q_2 = H_2$  has inverse  $P_2 \circ H^{-1} \circ Q'_2$ . ■

EXERCISE 1.68. Let  $V = V_1 \oplus V_2$  be a direct sum, as in Definition 1.43, with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ . Suppose there is some  $i$  and some map  $P' : V \rightarrow V_i$  so that  $P' \circ Q_i = \text{Id}_{V_i}$ , and  $P' \circ Q_I = 0_{\text{Hom}(V_I, V_i)}$  for  $I \neq i$ . Then,  $P' = P_i$ .

HINT. In the  $i = 1$  case,

$$\begin{aligned} P' &= P' \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \\ &= P' \circ Q_1 \circ P_1 + P' \circ Q_2 \circ P_2 \\ &= \text{Id}_{V_1} \circ P_1 + 0_{\text{Hom}(V_2, V_1)} \circ P_2 = P_1. \end{aligned}$$
■

So, given all the  $Q_i$  in a direct sum, each map  $P_i$  is unique. There is a similar uniqueness result for the  $Q_i$ , given all the  $P_i$ .

EXERCISE 1.69. Let  $V = V_1 \oplus V_2$  be a direct sum, as in Definition 1.43, with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ . Suppose there is some  $i$  and some map  $Q' : V_i \rightarrow V$  so that  $P_i \circ Q' = Id_{V_i}$ , and  $P_I \circ Q' = 0_{\text{Hom}(V_i, V_I)}$  for  $I \neq i$ . Then,  $Q' = Q_i$ . ■

EXERCISE 1.70. Let  $V = V_1 \oplus V_2$  be a direct sum, as in Definition 1.43, with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ , and let  $A : V_1 \rightarrow V_2$ . Then the following operators  $(P'_1, P'_2)$ ,  $(Q'_1, Q'_2)$  also define a direct sum.

$$\begin{aligned} Q'_1 &= Q_1 + Q_2 \circ A : V_1 \rightarrow V \\ Q'_2 &= Q_2 : V_2 \rightarrow V \\ P'_1 &= P_1 : V \rightarrow V_1 \\ P'_2 &= P_2 - A \circ P_1 : V \rightarrow V_2. \end{aligned}$$

This is equivalent to the original direct sum if and only if  $A = 0_{\text{Hom}(V_1, V_2)}$ .

HINT. The definition of direct sum requires checking five equations. One of them is:

$$\begin{aligned} Q'_1 \circ P'_1 + Q'_2 \circ P'_2 &= (Q_1 + Q_2 \circ A) \circ P_1 + Q_2 \circ (P_2 - A \circ P_1) \\ &= Q_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ P_2 - Q_2 \circ A \circ P_1 \\ &= Id_V. \end{aligned}$$

The compositions  $P'_I \circ Q'_i$  are also easy to check. If the direct sums are equivalent, then  $0_{\text{Hom}(V_1, V_2)} = P_2 \circ Q'_1 = P_2 \circ (Q_1 + Q_2 \circ A) = A$ , and conversely. ■

The direct sum  $P'_i, Q'_i$  is the graph of  $A$ . In a certain sense, Exercise 1.70 has a converse: if a space  $V$  decomposes in two ways as a direct sum, with the same inclusion  $Q_2$ , then the two direct sums are related using the graph construction, up to equivalence.

EXERCISE 1.71. Given  $V, V_1, V_2$ , suppose the pairs  $(P_1, P_2)$  and  $(Q_1, Q_2)$  define a direct sum  $V = V_1 \oplus V_2$ , and the pairs  $(P'_1, P'_2)$ ,  $(Q'_1, Q'_2)$  also satisfy Definition 1.43. If  $Q_2 = Q'_2$ , then there exists a map  $A : V_1 \rightarrow V_2$ , and there exist  $(P''_1, P''_2)$ ,  $(Q''_1, Q''_2)$  which define a third direct sum, and which satisfy:

$$\begin{aligned} Q''_1 &= Q_1 + Q_2 \circ A : V_1 \rightarrow V \\ Q''_2 &= Q_2 : V_2 \rightarrow V \\ P''_1 &= P_1 : V \rightarrow V_1 \\ P''_2 &= P_2 - A \circ P_1 : V \rightarrow V_2. \end{aligned}$$

This  $(P''_1, P''_2)$ ,  $(Q''_1, Q''_2)$  direct sum is equivalent to the  $(P'_1, P'_2)$ ,  $(Q'_1, Q'_2)$  direct sum.

HINT. Note that by Exercise 1.69, the hypothesis  $Q_2 = Q'_2$  is equivalent to assuming that  $Q'_2$  satisfies  $P_2 \circ Q'_2 = Id_{V_2}$  and  $P_1 \circ Q'_2 = 0_{\text{Hom}(V_2, V_1)}$ . Choosing  $Q''_2 = Q_2$  gives the identity  $P'_1 \circ Q''_2 = 0_{\text{Hom}(V_2, V_1)}$ .

The above four equations are the properties defining a graph. The claimed existence follows from checking that the following choices have the claimed properties.

$$\begin{aligned} A &= -P'_2 \circ Q_1 : V_1 \rightarrow V_2 \\ Q''_1 &= Q'_1 \circ P'_1 \circ Q_1 : V_1 \rightarrow V \\ P''_2 &= P'_2 : V \rightarrow V_2. \end{aligned}$$

The equivalence of direct sums as in Definition 1.59 is verified by checking  $P'_2 \circ Q''_1 = 0_{\text{Hom}(V_1, V_2)}$ . ■

### 1.5. Idempotents and involutions

DEFINITION 1.72. An element  $P \in \text{End}(V)$  is an idempotent means:  $P \circ P = P$ .

LEMMA 1.73. Given  $V$  and  $P_1, P_2 \in \text{End}(V)$ , any three out of the following four properties (1) – (4) imply the remaining fourth:

- (1)  $P_1$  is an idempotent;
- (2)  $P_2$  is an idempotent;
- (3)  $P_1 \circ P_2 + P_2 \circ P_1 = 0_{\text{End}(V)}$ ;
- (4)  $P_1 + P_2$  is an idempotent.

Property (4) is equivalent to:

- (5) There exists  $P_3 \in \text{End}(V)$  such that  $P_3$  is an idempotent and  $P_1 + P_2 + P_3 = \text{Id}_V$ .

If, further, either  $\frac{1}{2} \in \mathbb{K}$  or  $P_1 + P_2 = \text{Id}_V$ , then  $P_1, P_2, P_3$  satisfying properties (1) – (5) also satisfy:

- (6) For distinct  $i_1, i_2 \in \{1, 2, 3\}$ ,  $P_{i_1} \circ P_{i_2} = 0_{\text{End}(V)}$ .

Conversely, if  $P_1, P_2, P_3 \in \text{End}(V)$  satisfy (6) and  $P_1 + P_2 + P_3 = \text{Id}_V$ , then  $P_1, P_2, P_3$  are idempotents satisfying (1) – (5). ■

EXAMPLE 1.74. An idempotent  $P$  defines a direct sum structure as follows. Let  $V_1 = P(V)$ , the subspace of  $V$  which is the image of  $P$ . Define  $Q_1 : V_1 \rightarrow V$  to be the subspace inclusion map, and define  $P_1 : V \rightarrow V_1$  by restricting the target of  $P : V \rightarrow V$  to get  $P_1 : V \rightarrow V_1$  with  $P_1(v) = P(v)$  for all  $v \in V$ . Then  $P = Q_1 \circ P_1 : V \rightarrow V$  by construction. The map  $\text{Id}_V - P$  is also an idempotent (this is a special case of Lemma 1.73), so proceeding analogously, define  $V_2 = (\text{Id}_V - P)(V)$ , the image of the linear map  $\text{Id}_V - P : V \rightarrow V$ . Again, let  $Q_2 : V_2 \rightarrow V$  be the subspace inclusion, and define  $P_2 = \text{Id}_V - P$ , with its target space restricted to  $V_2$ , so that  $Q_2 \circ P_2 = \text{Id}_V - P$  by construction, and  $Q_1 \circ P_1 + Q_2 \circ P_2 = P + (\text{Id}_V - P) = \text{Id}_V$ . To show  $V = V_1 \oplus V_2$ , it remains only to check that these maps satisfy the other parts of Definition 1.43. For  $v_1 \in V_1$ ,  $Q_1(v_1) = v_1 = P(w_1)$  for some  $w_1 \in V$ , so  $(P_1 \circ Q_1)(v_1) = P_1(P(w_1)) = P(P(w_1)) = P(w_1) = v_1$ , and  $(P_2 \circ Q_1)(v_1) = (\text{Id}_V - P)(P(w_1)) = P(w_1) - P(P(w_1)) = 0_{V_2}$ . Similarly, for  $Q_2(v_2) = v_2 = (\text{Id}_V - P)(w_2)$ ,  $(P_1 \circ Q_2)(v_2) = P_1(w_2 - P(w_2)) = P(w_2) - P(P(w_2)) = 0_{V_1}$  and  $(P_2 \circ Q_2)(v_2) = (\text{Id}_V - P)((\text{Id}_V - P)(w_2)) = (\text{Id}_V - P)(w_2) = v_2$ . This construction of the direct sum  $V = V_1 \oplus V_2$  is only canonical up to re-ordering.

The statement of Lemma 1.49 in the special case of Example 1.74 is that the image of  $\text{Id}_V - P$  is the kernel of  $P$ , and the image of  $P$  is the kernel of  $\text{Id}_V - P$ .

EXAMPLE 1.75. Given any direct sum  $V = U_1 \oplus U_2$  as in Definition 1.43 with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ , the composite  $Q_1 \circ P_1 : V \rightarrow V$  is an idempotent, and so is  $Id_V - Q_1 \circ P_1 = Q_2 \circ P_2$ . This is a converse to the construction of Example 1.74; any direct sum canonically defines an unordered pair of two idempotents. For  $P = Q_1 \circ P_1$ , the direct sum  $V = V_1 \oplus V_2$  constructed in Example 1.74 is equivalent, as in Definition 1.59, to the original direct sum.

LEMMA 1.76. *Given idempotents  $P : V \rightarrow V$ ,  $P' : U \rightarrow U$  defining direct sums  $V_1 \oplus V_2$  and  $U_1 \oplus U_2$  as in Example 1.74, a map  $H : U \rightarrow V$  respects the direct sums (as in Definition 1.52) if and only if  $H \circ P' = P \circ H$ . ■*

DEFINITION 1.77. An element  $K \in \text{End}(V)$  is an involution means:  $K \circ K = Id_V$ .

LEMMA 1.78. *If  $\frac{1}{2} \in \mathbb{K}$  and  $K \in \text{End}(V)$  is an involution, then  $P = \frac{1}{2}(Id_V + K)$  is an idempotent, and  $Id_V - P = \frac{1}{2}(Id_V - K)$  is also an idempotent. ■*

LEMMA 1.79. *For an involution  $K \in \text{End}(V)$ , let  $V_1 = \{v \in V : K(v) = v\}$ , and  $V_2 = \{v \in V : K(v) = -v\}$ . If  $\frac{1}{2} \in \mathbb{K}$ , then  $V = V_1 \oplus V_2$ , with  $Q_i$  the subspace inclusion maps, and projections:*

$$(1.1) \quad P_1 = \frac{1}{2} \cdot (Id_V + K),$$

$$(1.2) \quad P_2 = \frac{1}{2} \cdot (Id_V - K).$$

PROOF. This can be proved directly, but also follows from the construction of Example 1.74. It is easy to check that  $V_1$  is a subspace of  $V$ , equal to the image of the idempotent  $P$  from Lemma 1.78 and that  $V_2$  is equal to the image of  $Id_V - P$ . The composites  $Q_1 \circ P_1$ ,  $Q_2 \circ P_2 \in \text{End}(V)$  are also given by the formulas  $\frac{1}{2} \cdot (Id_V \pm K)$ . ■

NOTATION 1.80. We will refer to the construction of  $V = V_1 \oplus V_2$  as in Lemma 1.79 as the direct sum produced by the involution  $K$ . The subspaces  $V_1$ ,  $V_2$  and maps  $P_1$ ,  $P_2$  in (1.1), (1.2) are canonical, but Lemma 1.79 made a choice of order in the direct sum  $V = V_1 \oplus V_2$ . With this ordering convention, the involution  $-K$  produces the direct sum  $V = V_2 \oplus V_1$  as in Notation 1.56. For the projection maps defined by formulas (1.1), (1.2), the double arrow will appear in diagrams,  $P_1 : V \rightrightarrows V_1$ ,  $P_2 : V \rightrightarrows V_2$ , and the same arrow for composites of such projections. For the subspace inclusion maps as in Lemma 1.79, the hook arrow will appear:  $Q_1 : V_1 \hookrightarrow V$  for the fixed point subspace of  $K$ , and  $Q_2 : V_2 \hookrightarrow V$  for the fixed point subspace of  $-K$ , and similarly for composites of such inclusions.

EXAMPLE 1.81. Given any direct sum  $V = U_1 \oplus U_2$  as in Definition 1.43 with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ , the map  $Q_1 \circ P_1 - Q_2 \circ P_2 : V \rightarrow V$  is an involution. If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum produced by this involution,  $V = V_1 \oplus V_2$  as in Lemma 1.79, is equivalent, as in Definition 1.59, to the original direct sum. As in Lemma 1.53 and Lemma 1.62, there are invertible maps  $U_i \rightarrow V_i$ . If the direct sum operators  $Q_i$ ,  $P_i$  were defined by some involution  $K$  as in Lemma 1.79, then  $Q_1 \circ P_1 - Q_2 \circ P_2 = K$ .

LEMMA 1.82. *For an idempotent  $P : V \rightarrow V$ , let  $V = V_1 \oplus V_2$  be the direct sum from Example 1.74. The maps  $K = 2 \cdot P - \text{Id}_V : V \rightarrow V$  and  $\text{Id}_V - 2 \cdot P = -K$  are involutions, and both  $K$  and  $-K$  respect the direct sums  $V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ . If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum from Lemma 1.79 produced by  $K$  is the same as  $V = V_1 \oplus V_2$ .*

PROOF. The claim that  $K$  and  $-K$  respect the direct sums is a special case of Lemma 1.76.  $\blacksquare$

LEMMA 1.83. *Given  $\frac{1}{2} \in \mathbb{K}$  and two involutions  $K : V \rightarrow V$  and  $K' : U \rightarrow U$ , which produce direct sums  $V_1 \oplus V_2$ ,  $U_1 \oplus U_2$  as in Lemma 1.79, a map  $H : U \rightarrow V$  respects the direct sums  $U_1 \oplus U_2 \rightarrow V_1 \oplus V_2$  if and only if  $K \circ H = H \circ K'$ .*  $\blacksquare$

LEMMA 1.84. *Given  $V$  and a pair of involutions on  $V$ ,  $K_1$  and  $K_2$ , if  $\frac{1}{2} \in \mathbb{K}$ , then the following are equivalent:*

- (1) *The involutions commute, i.e.,  $K_1 \circ K_2 = K_2 \circ K_1$ ;*
- (2) *The composite  $K_1 \circ K_2$  is an involution;*
- (3)  *$K_2$  respects the direct sum  $V = V_1 \oplus V_2$  produced by  $K_1$ ;*
- (4)  *$K_1$  respects the direct sum  $V = V_3 \oplus V_4$  produced by  $K_2$ .*

PROOF. The equivalence (1)  $\iff$  (2) is elementary and does not require  $\frac{1}{2} \in \mathbb{K}$ . The direct sums in (3), (4) are as in Lemma 1.79. The equivalences (1)  $\iff$  (3) and (1)  $\iff$  (4) are special cases of Lemma 1.83.  $\blacksquare$

In statement (3) of Lemma 1.84,  $K_2$  induces an involution on both  $V_1$  and  $V_2$  as in Definition 1.52 and Lemma 1.53, and similarly for  $K_1$  in statement (4).

Given  $\frac{1}{2} \in \mathbb{K}$ , and  $V$  with commuting involutions  $K_1, K_2$  as in Lemma 1.84 and Lemma 1.85, and corresponding direct sums  $V = V_1 \oplus V_2$ ,  $V = V_3 \oplus V_4$ , respectively, as in Lemma 1.84, let  $V = V_5 \oplus V_6$  be the direct sum produced by the involution  $K_1 \circ K_2$ .

LEMMA 1.85. *Given  $V$ , subspaces  $V_1, \dots, V_5$  as above, and commuting involutions  $K_1, K_2 \in \text{End}(V)$ , for  $v \in V$ , any pair of two of the following three statements implies the remaining one:*

- (1)  $v = K_1(v) \in V_1$ ;
- (2)  $v = K_2(v) \in V_3$ ;
- (3)  $v = (K_1 \circ K_2)(v) \in V_5$ .

$\blacksquare$

It follows from Lemma 1.85 that these subspaces of  $V$  are equal:

$$(1.3) \quad V_1 \cap V_3 = V_1 \cap V_5 = V_3 \cap V_5 = V_1 \cap V_3 \cap V_5.$$

Let  $(P_5, P_6), (Q_5, Q_6)$  denote the projections and inclusions for the above direct sum  $V = V_5 \oplus V_6$  produced by  $K_1 \circ K_2$ . Since  $K_1 \circ Q_5 = K_2 \circ Q_5$ , the maps induced by  $K_1$  and  $K_2$  are equal:

$$(1.4) \quad P_5 \circ K_1 \circ Q_5 = P_5 \circ K_2 \circ Q_5 : V_5 \rightarrow V_5;$$

this map is a canonical involution on  $V_5$ , producing a direct sum  $V_5 = V'_5 \oplus V''_5$  with projection  $P'_5 : V_5 \rightarrow V'_5$  from (1.1). Similarly, there is an involution induced by  $K_1$  or  $K_1 \circ K_2$  on  $V_3$ , producing  $V_3 = V'_3 \oplus V''_3$ , and there is another involution induced by  $K_2$  or  $K_1 \circ K_2$  on  $V_1$ , producing  $V_1 = V'_1 \oplus V''_1$ . On the set  $V_6$ , the induced involutions are opposite:  $P_6 \circ K_1 \circ Q_6 = -P_6 \circ K_2 \circ Q_6$ ; if one produces a

direct sum  $V_6 = V'_6 \oplus V''_6$ , the other produces  $V_6 = V''_6 \oplus V'_6$ . Similarly, there are opposite induced involutions on  $V_2$  and  $V_4$ .

THEOREM 1.86. *Given  $\frac{1}{2} \in \mathbb{K}$ , and commuting involutions on  $V$  with the above notation,*

$$V'_5 = V'_3 = V'_1 = V_1 \cap V_3 \cap V_5.$$

*The composite projections are all equal:*

$$P'_5 \circ P_5 = P'_3 \circ P_3 = P'_1 \circ P_1 : V \twoheadrightarrow V_1 \cap V_3 \cap V_5.$$

*Also,  $V''_5 = V_2 \cap V_4$ ,  $V''_3 = V_2 \cap V_6$ , and  $V''_1 = V_4 \cap V_6$ .*

PROOF.  $V'_5$  is the set of fixed points  $v \in V_5$  of the involution  $P_5 \circ K_1 \circ Q_5$ . Denote the operators from Lemma 1.79  $P'_5$ ,  $Q'_5$ , so

$$Q'_5 \circ P'_5 = \frac{1}{2} \cdot (Id_{V_5} + P_5 \circ K_1 \circ Q_5).$$

To establish the first claim, it will be enough to show  $V'_5 = V_1 \cap V_3$ ; the claims  $V'_3 = V_1 \cap V_5$  and  $V'_1 = V_3 \cap V_5$  are similar, and then (1.3) applies. To show  $V'_5 \subseteq V_1$ , use the fact that  $K_1$  commutes with  $Q_5 \circ P_5 = \frac{1}{2} \cdot (Id_V + K_1 \circ K_2)$ ; if  $v \in V'_5 \subseteq V_5$ , then

$$v = Q_5(v) = (P_5 \circ K_1 \circ Q_5)(v) = Q_5((P_5 \circ K_1 \circ Q_5)(v)) = K_1(Q_5(v)),$$

so  $v \in V_1$ . Showing  $V'_5 \subseteq V_3$  is similar, so  $V'_5 \subseteq V_1 \cap V_3$ .

Another argument would be to consider the subspace  $V'_5$  as the image of  $Q_5 \circ Q'_5 \circ P'_5 \circ P_5$  in  $V$ . Then

$$\begin{aligned} Q_5 \circ Q'_5 \circ P'_5 \circ P_5 &= Q_5 \circ \frac{1}{2} \cdot (Id_{V_5} + P_5 \circ K_1 \circ Q_5) \circ P_5 \\ &= \frac{1}{2} \cdot Q_5 \circ P_5 + \frac{1}{2} \cdot K_1 \circ Q_5 \circ P_5 \\ (1.5) \qquad &= Q_1 \circ P_1 \circ Q_5 \circ P_5, \end{aligned}$$

which shows  $V'_5$  is contained in  $V_1$ , the image of  $Q_1$  in  $V$ .

Conversely, if  $v \in V_1 \cap V_3$ , then  $v = K_1(v) = Q_5(v) \in V_5$  (Lemma 1.85) and  $(P_5 \circ K_1 \circ Q_5)(v) = (P_5 \circ Q_5)(v) = v \in V'_5$ .

The equality of the composites of projections follows from using the commutativity of the involutions to get  $Q_1 \circ P_1 \circ Q_5 \circ P_5 = Q_5 \circ P_5 \circ Q_1 \circ P_1$ , and then (1.5) implies  $Q_5 \circ Q'_5 \circ P'_5 \circ P_5 = Q_1 \circ Q'_1 \circ P'_1 \circ P_1$ .

The last claim of the Theorem follows from similar calculations. However, the three subspaces are in general not equal to each other. ■

The projection  $P_5 : V \twoheadrightarrow V_5$  satisfies  $P_5 \circ K_1 = (P_5 \circ K_1 \circ Q_5) \circ P_5$ , so Lemma 1.83 applies:  $P_5$  respects the direct sums  $V_1 \oplus V_2 \rightarrow V'_5 \oplus V''_5$  and the map  $V_1 \rightarrow V'_5$  induced by  $P_5$  is  $P'_5 \circ P_5 \circ Q_1$ . By Theorem 1.86,

$$(1.6) \qquad P'_5 \circ P_5 \circ Q_1 = P'_1 \circ P_1 \circ Q_1 = P'_1 : V_1 \twoheadrightarrow V'_1 = V'_5.$$

This gives an alternate construction of  $P'_1$  as a map induced by  $P_5$ , or similarly, any  $P'_i$  is equal to a map induced by  $P_I$  for any distinct  $i = 1, 3, 5$ ,  $I = 1, 3, 5$ .



**THEOREM 1.87.** *Given  $\frac{1}{2} \in \mathbb{K}$ , suppose  $K_V^1, K_V^2$  are commuting involutions on  $V$  as in Theorem 1.86. Similarly, let  $K_U^1, K_U^2$  be commuting involutions on  $U$ , with corresponding notation for the direct sums:  $U = U_1 \oplus U_2, U = U_3 \oplus U_4$ , etc. If a map  $H : U \rightarrow V$  satisfies  $H \circ K_U^1 = K_V^1 \circ H$  and  $H \circ K_U^2 = K_V^2 \circ H$ , then  $H$  respects the corresponding direct sums  $U_1 \oplus U_2 \rightarrow V_1 \oplus V_2$  and  $U_3 \oplus U_4 \rightarrow V_3 \oplus V_4$ . Further, the induced map  $P_V^1 \circ H \circ Q_U^1 : U_1 \rightarrow V_1$  respects the direct sums  $U'_1 \oplus U''_1 \rightarrow V'_1 \oplus V''_1$  and similarly for the maps  $U_3 \rightarrow V_3, U_5 \rightarrow V_5$  induced by  $H$ . The induced map  $U'_1 \rightarrow V'_1$  is equal to the map  $U'_3 \rightarrow V'_3$  induced by  $P_V^3 \circ H \circ Q_U^3 : U_3 \rightarrow V_3$ .*

**PROOF.** The fact that  $H$  respects each pair of direct sums is Lemma 1.83. The subspace  $U_1$  has a canonical involution  $P_U^1 \circ K_U^2 \circ Q_U^1$ , and since  $K_U^1, K_U^2$  commute,  $K_U^2$  also commutes with  $Q_U^1 \circ P_U^1 = \frac{1}{2} \cdot (Id_U + K_U^1)$ . The map induced by  $H$ ,  $P_V^1 \circ H \circ Q_U^1 : U_1 \rightarrow V_1$ , satisfies:

$$\begin{aligned} (P_V^1 \circ H \circ Q_U^1) \circ (P_U^1 \circ K_U^2 \circ Q_U^1) &= P_V^1 \circ H \circ K_U^2 \circ Q_U^1 \circ P_U^1 \circ Q_U^1 \\ &= P_V^1 \circ K_V^2 \circ H \circ Q_U^1 \\ &= (P_V^1 \circ K_V^2 \circ Q_V^1) \circ (P_V^1 \circ H \circ Q_U^1). \end{aligned}$$

It follows from Lemma 1.83 again that  $P_V^1 \circ H \circ Q_U^1$  respects the direct sums as claimed. The induced map is  $P_V^{1'} \circ (P_V^1 \circ H \circ Q_U^1) \circ Q_U^{1'} : U'_1 \rightarrow V'_1$ .

The last claim of the Theorem is that this induced map is equal to the map  $P_V^{3'} \circ (P_V^3 \circ H \circ Q_U^3) \circ Q_U^{3'} : U'_3 \rightarrow V'_3$ . The claim follows from the fact that the induced maps are essentially just restrictions of  $H$  to the same subspace  $U'_1 = U'_3 = U_1 \cap U_3$ , by Theorem 1.86. More specifically, the subspace inclusions are equal:  $Q_U^3 \circ Q_U^{3'} = Q_U^1 \circ Q_U^{1'} : U_1 \cap U_3 \hookrightarrow U$ , and the composites of projections are equal:  $P_V^{1'} \circ P_V^1 = P_V^{3'} \circ P_V^3$ .  $\blacksquare$

**LEMMA 1.88.** *Given  $\frac{1}{2} \in \mathbb{K}$  and two involutions  $K : V \rightarrow V$  and  $K' : U \rightarrow U$ , which produce direct sums  $V_1 \oplus V_2, U_1 \oplus U_2$  as in Lemma 1.79, a map  $H : U \rightarrow V$  respects the direct sums  $U_1 \oplus U_2 \rightarrow V_2 \oplus V_1$  if and only if  $K \circ H = -H \circ K'$ .*

**PROOF.** Note the order of the spaces  $V_2 \oplus V_1$  is different from that appearing in Lemma 1.83, so the notation refers to the identities  $Q_1 \circ P_1 \circ H = H \circ Q'_2 \circ P'_2$  and  $Q_2 \circ P_2 \circ H = H \circ Q'_1 \circ P'_1$ . The claims can be checked directly, but also follow from applying Lemma 1.83 to the involutions  $K$  and  $-K'$ .  $\blacksquare$

**THEOREM 1.89.** *Given  $V$ , the following statements (1) to (7) are equivalent, and any implies (8). Further, if  $\frac{1}{2} \in \mathbb{K}$ , then all eight statements are equivalent:*

- (1)  $V$  admits a direct sum of the form  $V = U \oplus U$ ;
- (2)  $V = U_1 \oplus U_2$  and there exist invertible maps  $A_1 : U_3 \rightarrow U_1$  and  $A_2 : U_3 \rightarrow U_2$ ;
- (3)  $V = U' \oplus U''$  and there exists an invertible map  $A : U'' \rightarrow U'$ ;
- (4)  $V = U' \oplus U''$  and there exists an involution  $K \in \text{End}(V)$  that respects the direct sums  $U' \oplus U'' \rightarrow U'' \oplus U'$ ;
- (5)  $V$  admits an idempotent  $P \in \text{End}(V)$  and an involution  $K \in \text{End}(V)$  such that  $P \circ K = K \circ (Id_V - P)$ ;
- (6)  $V = U' \oplus U''$  and there exists an invertible  $H \in \text{End}(V)$  that respects the direct sums  $U' \oplus U'' \rightarrow U'' \oplus U'$ ;
- (7)  $V$  admits an idempotent  $P \in \text{End}(V)$  and an invertible map  $H \in \text{End}(V)$  such that  $P \circ H = H \circ (Id_V - P)$ ;
- (8)  $V$  admits anticommuting involutions  $K_1, K_2$  (i.e.,  $K_1 \circ K_2 = -K_2 \circ K_1$ ).

PROOF. The implication (1)  $\implies$  (2) is canonical: let  $U_1 = U_2 = U_3 = U$ , and  $A_1 = A_2 = Id_U$ .

The implication (2)  $\implies$  (1) is canonical. Given  $V = U_1 \oplus U_2$  with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ , Example 1.60 applies. Let  $U = U_3$ , to get  $V = U \oplus U$  with projections  $(A_1^{-1} \circ P_1, A_2^{-1} \circ P_2)$  and inclusions  $(Q_1 \circ A_1, Q_2 \circ A_2)$ . The direct sums are equivalent.

The implication (1)  $\implies$  (3) is canonical: let  $U' = U'' = U$  and  $A = Id_U$ .

The implication (2)  $\implies$  (3) is canonical: let  $U' = U_1$ ,  $U'' = U_2$  and  $A = A_1 \circ A_2^{-1}$ .

For (3)  $\implies$  (1), there are two choices. Given  $V = U' \oplus U''$  with projections  $(P', P'')$  and inclusions  $(Q', Q'')$ , one choice is to let  $U = U'$ . Then  $V = U \oplus U$  with projections  $(P_1, A \circ P_2)$  and inclusions  $(Q_1, Q_2 \circ A^{-1})$ . The other choice is to let  $U = U''$ , with projections  $(A^{-1} \circ P_1, P_2)$  and inclusions  $(Q_1 \circ A, Q_2)$ . As in Example 1.60, either of the two constructions gives a direct sum equivalent to  $V = U' \oplus U''$ , so they are equivalent to each other.

For (3)  $\implies$  (2), there are two choices. One choice is to let  $U_3 = U'$ ,  $A_1 = Id_{U'}$ ,  $A_2 = A^{-1}$ . Applying the canonical (2)  $\implies$  (1) construction then gives projections  $(P_1, A \circ P_2)$  as in the first choice of the previous implication. The second choice is to let  $U_3 = U''$ ,  $A_1 = A$ ,  $A_2 = Id_{U''}$ , which similarly corresponds to the second choice in the previous implication.

The implication (3)  $\implies$  (4) is canonical. Given  $A : U'' \rightarrow U'$ , let

$$(1.7) \quad K = Q'' \circ A^{-1} \circ P' + Q' \circ A \circ P''.$$

It is straightforward to check that  $K$  is an involution, and  $Q' \circ P' \circ K = K \circ Q'' \circ P''$ , so  $K$  respects the direct sums as in Example 1.57.

The implication (4)  $\implies$  (3) is canonical. Given  $K$ , let  $A = P' \circ K \circ Q'' : U'' \rightarrow U'$ , which by Lemma 1.53 has inverse  $A^{-1} = P'' \circ K \circ Q'$ .

The implication (4)  $\implies$  (6) is canonical: let  $H = K$ .

For (6)  $\implies$  (3), there are two choices. One choice is to let  $A = P' \circ H \circ Q'' : U'' \rightarrow U'$ , so  $A^{-1} = P'' \circ H^{-1} \circ Q'$ . The canonical involution (1.7) from the implication (3)  $\implies$  (4) is then  $K = Q'' \circ P'' \circ H^{-1} \circ Q' \circ P' + Q' \circ P' \circ H \circ Q'' \circ P''$ . The second choice is to let  $A = P' \circ H^{-1} \circ Q''$ . This similarly leads to an involution  $Q'' \circ P'' \circ H \circ Q' \circ P' + Q' \circ P' \circ H^{-1} \circ Q'' \circ P''$ , which, unless  $H$  is an involution, may be different from the involution from the first choice.

For (4)  $\implies$  (5), and for (6)  $\implies$  (7), there are two choices:  $P = Q' \circ P'$ , or  $P = Q'' \circ P''$ . This choice between two idempotents was already mentioned in Example 1.75.

Conversely, for (5)  $\implies$  (4) (and similarly for (7)  $\implies$  (6)), there are two choices. For  $U' = \ker(P)$  and  $U'' = P(V)$ , as in Example 1.74, there are two ways to form a direct sum:  $V = U' \oplus U''$  or  $V = U'' \oplus U'$ . The map  $K$  (similarly  $H$ ) respects the direct sums as in Lemma 1.76.

For (4)  $\implies$  (8), which does not require  $\frac{1}{2} \in \mathbb{K}$ , there are two choices (assuming the ordering of the pair  $K_1, K_2$  does not matter). Given  $K$ , let  $K_1 = K$ . One choice is to let  $K_2 = Q' \circ P' - Q'' \circ P''$ , as in Example 1.81. It follows from  $K \circ Q' \circ P' = Q'' \circ P'' \circ K$  that  $K_1 \circ K_2 = -K_2 \circ K_1$ . The second choice is to let  $K_2 = -Q' \circ P' + Q'' \circ P''$ .

Similarly for (5)  $\implies$  (8), there are two choices. Given  $K$ , let  $K_1 = K$ . One choice is to let  $K_2 = 2 \cdot P - Id_V$ , as in Lemma 1.82. The second choice is to let  $K_2 = Id_V - 2 \cdot P$ .

For (8)  $\implies$  (4) using  $\frac{1}{2} \in \mathbb{K}$ , the involution  $K_1$  produces a direct sum  $V = V_1 \oplus V_2$  as in Lemma 1.79, with projections  $P_1 = \frac{1}{2}(Id_V + K_1)$ ,  $P_2 = \frac{1}{2}(Id_V - K_1)$  (the order of the direct sum could be chosen the other way,  $V_2 \oplus V_1$ ). By Lemma 1.88,  $K = K_2$  satisfies (4). In this case, the invertible map from (3) is, by Lemma 1.53, the composite

$$(1.8) \quad A = P_1 \circ K_2 \circ Q_2 : V_2 \rightarrow V_1,$$

with inverse  $P_2 \circ K_2 \circ Q_1 : V_1 \rightarrow V_2$ . Another choice for (8)  $\implies$  (4) is to use  $K_2$  to produce a different direct sum, and then let  $K = K_1$ .  $\blacksquare$

**THEOREM 1.90.** *Given  $\frac{1}{2} \in \mathbb{K}$  and two involutions  $K, K' \in \text{End}(V)$ , which produce direct sums  $V = V_1 \oplus V_2$ ,  $V = V'_1 \oplus V'_2$  as in Lemma 1.79, if  $K$  and  $K'$  anticommute, then for  $i = 1, 2$ ,  $I = 1, 2$ , and  $\beta \in \mathbb{K}$ ,  $\beta \neq 0$ , the map*

$$\beta \cdot P'_I \circ Q_i : V_i \rightarrow V'_I$$

*is invertible.*

**PROOF.** Consider  $P'_I \circ Q_i : V_i \rightarrow V'_I$  and  $P_i \circ Q'_I : V'_I \rightarrow V_i$ . Then

$$P'_I \circ Q_i \circ P_i \circ Q'_I = P'_I \circ \frac{1}{2} \cdot (Id_V \pm K) \circ Q'_I.$$

Since  $K$  respects the direct sums  $V'_1 \oplus V'_2 \rightarrow V'_2 \oplus V'_1$  by Lemma 1.88,  $P'_I \circ K \circ Q'_I = 0_{\text{End}(V'_I)}$  by Lemma 1.51. In the other order,

$$P_i \circ Q'_I \circ P'_I \circ Q_i = P_i \circ \frac{1}{2} \cdot (Id_V \pm K') \circ Q_i,$$

and similarly,  $P_i \circ K' \circ Q_i = 0_{\text{End}(V_i)}$ .

Since  $P'_I \circ Q'_I = Id_{V'_I}$  and  $P_i \circ Q_i = Id_{V_i}$ , the conclusion is that for any scalar  $\beta \in \mathbb{K}$ ,  $\beta \neq 0$ , the map  $\beta \cdot P'_I \circ Q_i$  has inverse  $\frac{2}{\beta} \cdot P_i \circ Q'_I$ .  $\blacksquare$

**LEMMA 1.91.** *Given involutions  $K_1, K_2, K_3 \in \text{End}(V)$ , any pair of two of the following three statements implies the remaining one:*

- (1)  $K_3$  commutes with  $K_1 \circ K_2$ ;
- (2)  $K_3$  anticommutes with  $K_1$ ;
- (3)  $K_3$  anticommutes with  $K_2$ .

$\blacksquare$

**EXERCISE 1.92.** If  $K_1, K_2, K_3$  are involutions such that  $K_1$  and  $K_2$  commute and  $K_3$  satisfies the three conditions from Lemma 1.91, then the set

$$\{\pm Id_V, \pm K_1, \pm K_2, \pm K_3, \pm K_2 \circ K_3, \pm K_1 \circ K_3, \pm K_1 \circ K_2, \pm K_1 \circ K_2 \circ K_3\}$$

is the image of a representation  $D_4 \times \mathbb{Z}_2 \rightarrow \text{End}(V)$ , where  $D_4$  is the eight-element dihedral group and  $\mathbb{Z}_2$  is the two-element group.  $\blacksquare$

For  $\frac{1}{2} \in \mathbb{K}$  and commuting involutions  $K_1, K_2$ , recall the direct sums  $V = V_1 \oplus V_2$ ,  $V = V_3 \oplus V_4$ ,  $V = V_5 \oplus V_6$  from Lemma 1.85 produced by  $K_1, K_2, K_1 \circ K_2$ . Further, suppose  $K_3$  is another involution satisfying the three conditions from Lemma 1.91, and let  $V = V_7 \oplus V_8$  and  $V = V_9 \oplus V_{10}$  be the direct sums produced by the involutions  $K_3$  and  $K_1 \circ K_2 \circ K_3$ . Theorem 1.86 applies to  $V_5$  twice: first, to the pair  $K_1, K_2$  to get the canonical involution  $P_5 \circ K_1 \circ Q_5$  from (1.4) producing  $V_5 = V'_5 \oplus V''_5$  with  $V'_5 = V_1 \cap V_3 \cap V_5$ , and second, to the other pair

$K_1 \circ K_2, K_3$  to get another involution  $P_5 \circ K_3 \circ Q_5 = P_5 \circ K_1 \circ K_2 \circ K_3 \circ Q_5$  as in (1.4), producing another direct sum  $V_5 = V_5''' \oplus V_5''''$ , with  $V_5''' = V_5 \cap V_7 \cap V_9$ .

COROLLARY 1.93. *Given  $\frac{1}{2} \in \mathbb{K}$ ,  $0 \neq \beta \in \mathbb{K}$ , commuting involutions  $K_1, K_2$ , and an involution  $K_3$  as in Lemma 1.91, the map*

$$\beta \cdot P_5''' \circ Q_5' : V_5' \rightarrow V_5''''$$

*is invertible.*

PROOF.  $Q_5'$  is as in the Proof of Theorem 1.86. The projection  $P_5''' : V_5 \rightarrow V_5'''$  is from the direct sum  $V_5 = V_5''' \oplus V_5''''$  produced by  $P_5 \circ K_3 \circ Q_5$ . The involutions  $P_5 \circ K_1 \circ Q_5, P_5 \circ K_3 \circ Q_5 \in \text{End}(V_5)$  anticommute, and Theorem 1.90 applies. ■

Using a step analogous to (1.5), the image of  $v \in V_5'$  under the above invertible map can be written as:

$$\begin{aligned} \beta \cdot P_5''' \circ Q_5' : v &\mapsto \beta \cdot (P_5''' \circ Q_5')(v) \\ (1.9) \quad &= Q_5(Q_5'''(\beta \cdot (P_5''' \circ (P_5 \circ Q_5) \circ Q_5')(v))) \\ &= \beta \cdot ((Q_5 \circ Q_5''' \circ P_5''' \circ P_5) \circ Q_5 \circ Q_5')(v) \\ &= \beta \cdot ((Q_7 \circ P_7 \circ Q_5 \circ P_5) \circ Q_5 \circ Q_5')(v) \\ &= \beta \cdot (Q_7 \circ P_7 \circ Q_5 \circ Q_5')(v) \\ &= \frac{\beta}{2} \cdot (v + K_3(v)). \end{aligned}$$

Corollary 1.93 could be re-stated as constructing an invertible map between these subspaces of  $V_5 = \{v \in V : v = (K_1 \circ K_2)(v)\}$ :

$$\{v = K_1(v) = K_2(v)\} \rightarrow \{v = K_3(v) = (K_1 \circ K_2 \circ K_3)(v)\}.$$

Two more subspaces of  $V_5$ , from Theorem 1.86, are:

$$\begin{aligned} V_5'' &= \{v \in V : v = -K_1(v) = -K_2(v)\} = V_2 \cap V_4, \\ V_5'''' &= \{v \in V : v = -K_3(v) = -(K_1 \circ K_2 \circ K_3)(v)\} = V_8 \cap V_{10}, \end{aligned}$$

and Theorem 1.90 also gives a construction of invertible maps:  $V_5' \rightarrow V_5''''$ ,  $V_5'' \rightarrow V_5'''$ , and  $V_5''' \rightarrow V_5''$ .

EXAMPLE 1.94. Given any spaces  $V$  and  $W$ , and an involution  $K$  on  $V$ , the map  $[Id_W \otimes K]$  is an involution on  $W \otimes V$ . If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum produced by  $[Id_W \otimes K]$  has projections  $\frac{1}{2} \cdot (Id_{W \otimes V} \pm [Id_W \otimes K])$ . For the direct sum  $V = V_1 \oplus V_2$  as in Lemma 1.79, there is also a direct sum  $W \otimes V = W \otimes V_1 \oplus W \otimes V_2$  as in Example 1.45, with projections  $[Id_W \otimes \frac{1}{2} \cdot (Id_V \pm K)]$ . The two constructions lead to the same formula for the projection operators, so the projections are canonical and  $K$  produces a direct sum  $W \otimes V = W \otimes V_1 \oplus W \otimes V_2$ . The space  $W \otimes V_1$  can be considered as a subspace of  $W \otimes V$ , equal to the fixed point set of  $[Id_W \otimes K]$ , and similarly  $W \otimes V_2$  is the fixed point subspace of  $-[Id_W \otimes K]$ . The space  $V \otimes W$  admits an analogous involution and direct sum.

EXAMPLE 1.95. Given any spaces  $U, W$  with involutions  $K_U$  on  $U$  and  $K_W$  on  $W$ , the involutions  $[Id_U \otimes K_W]$  and  $[K_U \otimes Id_W]$  on  $U \otimes W$  commute, so Lemma 1.85 applies, and if  $\frac{1}{2} \in \mathbb{K}$ , then Lemma 1.84 and Theorem 1.86 apply. For the direct sums  $U = U_1 \oplus U_2$  and  $W = W_1 \oplus W_2$  produced as in Lemma 1.79,  $[K_U \otimes Id_W]$  respects the direct sum  $U \otimes W_1 \oplus U \otimes W_2$  from Example 1.94; the induced involution on  $U \otimes W_1$  is exactly  $[K_U \otimes Id_{W_1}]$ , so  $U \otimes W_1$  admits a direct sum  $U_1 \otimes W_1 \oplus U_2 \otimes W_1$ .

$W_1$ . Similarly,  $[Id_U \otimes K_W]$  induces an involution on  $U_1 \otimes W$  and a direct sum  $U_1 \otimes W_1 \oplus U_1 \otimes W_2$ . The subspace  $U_1 \otimes W_1$  appears in two different ways, but there is no conflict in naming it: by Theorem 1.86,  $U_1 \otimes W_1 = (U \otimes W_1) \cap (U_1 \otimes W)$ .

EXAMPLE 1.96. Given any spaces  $V$  and  $W$ , and an involution  $K$  on  $V$ , the map  $\text{Hom}(Id_W, K)$  is an involution on  $\text{Hom}(W, V)$ . If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum produced by  $\text{Hom}(Id_W, K)$  has projections

$$\frac{1}{2} \cdot (Id_{\text{Hom}(W, V)} \pm \text{Hom}(Id_W, K)) : A \mapsto \frac{1}{2} \cdot (A \pm K \circ A).$$

For the direct sum  $V = V_1 \oplus V_2$  as in Lemma 1.79, there is also a direct sum  $\text{Hom}(W, V) = \text{Hom}(W, V_1) \oplus \text{Hom}(W, V_2)$  as in Example 1.46, with projections

$$\text{Hom}(Id_W, P_i) : \text{Hom}(W, V) \rightarrow \text{Hom}(W, V_i) : A \mapsto P_i \circ A = \frac{1}{2} \cdot (Id_V \pm K) \circ A.$$

The two constructions lead to the same formula for the projection operators. The only difference is in the target space: the fixed point set of  $\text{Hom}(Id_W, K)$  is the set of maps  $A : W \rightarrow V$  such that  $A = K \circ A$ , while the image of the projection  $\text{Hom}(Id_W, P_1)$  is a set of maps with domain  $W$  and target  $V_1 = \{v \in V : v = K(v)\}$ , which is a subspace of  $V$ . It will not cause any problems to consider  $\text{Hom}(W, V_i)$  as a subspace of  $\text{Hom}(W, V)$ ; more precisely, in the case where  $V = V_1 \oplus V_2$  is a direct sum produced by an involution, the operator  $\text{Hom}(Id_W, Q_i)$  from Example 1.46 can be regarded as a subspace inclusion as in Lemma 1.79, so  $A$  and  $Q_i \circ A$  are identified. Then the above two direct sum constructions have the same projection and inclusion operators, so the projections are canonical and  $K$  produces a direct sum  $\text{Hom}(W, V) = \text{Hom}(W, V_1) \oplus \text{Hom}(W, V_2)$ .

EXAMPLE 1.97. Given any spaces  $V$  and  $W$ , and an involution  $K$  on  $V$ , the map  $\text{Hom}(K, Id_W)$  is an involution on  $\text{Hom}(V, W)$ . If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum produced by  $\text{Hom}(K, Id_W)$  has projections

$$(1.10) \quad \frac{1}{2} \cdot (Id_{\text{Hom}(V, W)} \pm \text{Hom}(K, Id_W)) : A \mapsto \frac{1}{2} \cdot (A \pm A \circ K).$$

For the direct sum  $V = V_1 \oplus V_2$  as in Lemma 1.79, there is also a direct sum  $\text{Hom}(V, W) = \text{Hom}(V_1, W) \oplus \text{Hom}(V_2, W)$  as in Example 1.47, with projections

$$\text{Hom}(Q_i, Id_W) : \text{Hom}(V, W) \rightarrow \text{Hom}(V_i, W) : A \mapsto A \circ Q_i.$$

Unlike Example 1.96, the two constructions lead to different formulas for the projection operators. The fixed point set of  $\text{Hom}(K, Id_W)$  is the set of maps  $A : V \rightarrow W$  such that  $A = A \circ K$ , while the image of the projection  $\text{Hom}(Q_1, Id_W)$  is a set of maps with domain  $V_1$  and target  $W$  — which does not look like a subspace of  $\text{Hom}(V, W)$ . The conclusion is that the two direct sum constructions are different. However, they are equivalent, as in Definition 1.59. Checking statements (2) and (3) of Lemma 1.58,

$$\text{Hom}(P_i, Id_W) \circ \text{Hom}(Q_i, Id_W) : A \mapsto A \circ Q_i \circ P_i = A \circ \frac{1}{2} \cdot (Id_V \pm K),$$

which is the same as (1.10).

EXAMPLE 1.98. Given  $\frac{1}{2} \in \mathbb{K}$ , any spaces  $U$  and  $W$ , and involutions  $K_U \in \text{End}(U)$  and  $K_W \in \text{End}(W)$ , let  $U = U_1 \oplus U_2$  be the direct sum produced by  $K_U$ , with projections  $(P_1, P_2)$  and inclusions  $(Q_1, Q_2)$ , and let  $W = W_1 \oplus W_2$  be the direct sum produced by  $K_W$ , with operators  $(P'_1, P'_2)$ ,  $(Q'_1, Q'_2)$ . Then there are

commuting involutions  $\text{Hom}(Id_U, K_W)$ ,  $\text{Hom}(K_U, Id_W)$ , on  $\text{Hom}(U, W)$ , and their composite  $\text{Hom}(K_U, K_W)$  is another involution. Theorem 1.86 applies, and we re-use its  $V_1, \dots, V_6$  notation for the produced direct sums. As in Example 1.96, there is a direct sum

$$\begin{aligned} \text{Hom}(U, W) &= \{A = K_W \circ A\} \oplus \{A = -K_W \circ A\} \\ &= \text{Hom}(U, W_1) \oplus \text{Hom}(U, W_2) \\ &= V_1 \oplus V_2. \end{aligned}$$

As in Example 1.97, there are two different but equivalent direct sums, the first is

$$\text{Hom}(U, W) = \{A = A \circ K_U\} \oplus \{A = -A \circ K_U\} = V_3 \oplus V_4,$$

with projections and inclusions denoted  $(P_3, P_4)$ ,  $(Q_3, Q_4)$ , and the second is

$$\text{Hom}(U, W) = \text{Hom}(U_1, W) \oplus \text{Hom}(U_2, W),$$

with projections and inclusions

$$(\text{Hom}(Q_1, Id_W), \text{Hom}(Q_2, Id_W)), (\text{Hom}(P_1, Id_W), \text{Hom}(P_2, Id_W)).$$

From Lemma 1.62, there are canonical invertible maps

$$\begin{aligned} P_3 \circ \text{Hom}(P_1, Id_W) &: \text{Hom}(U_1, W) \rightarrow V_3, \\ \text{Hom}(Q_1, Id_W) \circ Q_3 &: V_3 \rightarrow \text{Hom}(U_1, W). \end{aligned}$$

There is also the direct sum produced by the composite involution,

$$\begin{aligned} \text{Hom}(U, W) &= \{A = K_W \circ A \circ K_U\} \oplus \{A = -K_W \circ A \circ K_U\} \\ &= V_5 \oplus V_6. \end{aligned}$$

It follows from Theorem 1.86 that  $V_1$ ,  $V_3$ , and  $V_5$  admit canonical involutions and direct sums. For example,  $P_3 \circ \text{Hom}(Id_U, K_W) \circ Q_3$  is the involution on  $V_3$ , and it produces the direct sum  $V_3 = V'_3 \oplus V''_3$ , where  $V'_3 = V_1 \cap V_3 \cap V_5$ . The above invertible map  $\text{Hom}(Q_1, Id_W) \circ Q_3 : V_3 \rightarrow \text{Hom}(U_1, W)$  satisfies

$$\begin{aligned} &\text{Hom}(Id_{U_1}, K_W) \circ (\text{Hom}(Q_1, Id_W) \circ Q_3) \\ &= \text{Hom}(Q_1, K_W) \circ Q_3 \\ &= (\text{Hom}(Q_1, Id_W) \circ Q_3) \circ (P_3 \circ \text{Hom}(Id_U, K_W) \circ Q_3), \end{aligned}$$

so by Lemma 1.83, it respects the direct sums

$$V'_3 \oplus V''_3 \rightarrow \text{Hom}(U_1, W_1) \oplus \text{Hom}(U_1, W_2).$$

By Lemma 1.53, there is a canonical invertible map from

$$V'_3 = V_1 \cap V_3 \cap V_5 = \{A : U \rightarrow W : A = K_W \circ A = A \circ K_U = K_W \circ A \circ K_U\}$$

to  $\text{Hom}(U_1, W_1)$ , specifically, the map

$$\begin{aligned} A &\mapsto P'_1 \circ (Q_3(Q'_3(A))) \circ Q_1 \\ &= P'_1 \circ A \circ Q_1 = \frac{1}{2} \cdot (Id_W + K_W) \circ A \circ Q_1 = A \circ Q_1. \end{aligned}$$

The inverse is defined for  $B \in \text{Hom}(U_1, W_1)$  by

$$B \mapsto P'_3(P_3(Q'_1 \circ B \circ P_1)),$$

which, since  $Q'_1 \circ B \circ P_1$  is an element of the subspace  $V_1 \cap V_3$ , simplifies to

$$Q'_1 \circ B \circ P_1 = B \circ P_1 = B \circ \frac{1}{2} \cdot (Id_U + K_U).$$

EXAMPLE 1.99. For  $\frac{1}{2} \in \mathbb{K}$ , and involutions  $K_U$  on  $U$  and  $K_W$  on  $W$  as in the previous Example, suppose  $K_3$  is an involution on  $\text{Hom}(U, W)$  that commutes with  $\text{Hom}(K_U, K_W)$  and anticommutes with either  $\text{Hom}(K_U, Id_W)$  or  $\text{Hom}(Id_U, K_W)$ . Then Lemma 1.91 and Corollary 1.93 apply. Continuing with the  $V_1, \dots, V_6$  notation from Theorem 1.86 and Example 1.98, and also the  $V_5 = V_5''' \oplus V_5''''$  and  $V_7, \dots, V_{10}$  notation from Corollary 1.93, the result of the Corollary is that for  $0 \neq \beta \in \mathbb{K}$ , there is an invertible map:

$$\beta \cdot P_5''' \circ Q_5' : V_5' \rightarrow V_5'''$$

which maps

$$\{A \in \text{Hom}(U, W) : A = A \circ K_U = K_W \circ A\}$$

to

$$\{A \in \text{Hom}(U, W) : A = K_W \circ A \circ K_U = K_3(A)\}.$$

There is also the canonical invertible map from Example 1.98,

$$P_3' \circ (P_3 \circ \text{Hom}(P_1, Id_W)) \circ \text{Hom}(Id_{U_1}, Q_1'),$$

which maps  $\text{Hom}(U_1, W_1)$  to  $V_3' = V_5'$ . The composite of these maps is an invertible map  $\text{Hom}(U_1, W_1) \rightarrow V_5'''$ :

$$(\beta \cdot P_5''' \circ Q_5') \circ (P_3' \circ (P_3 \circ \text{Hom}(P_1, Id_W)) \circ \text{Hom}(Id_{U_1}, Q_1')).$$

For  $B \in \text{Hom}(U_1, W_1)$ , its image in  $V_5''' \subseteq \text{Hom}(U, W)$  simplifies as follows, using the equality of subspace inclusions  $Q_5 \circ Q_5' = Q_3 \circ Q_3'$  and steps similar to (1.9):

$$\begin{aligned} B &\mapsto ((\beta \cdot P_5''' \circ Q_5') \circ (P_3' \circ P_3 \circ \text{Hom}(P_1, Q_1')))(B) \\ &= Q_5(Q_5'''(\beta \cdot (P_5''' \circ (P_5 \circ Q_5) \circ Q_5' \circ P_3' \circ P_3 \circ \text{Hom}(P_1, Q_1'))(B))) \\ &= \beta \cdot ((Q_5 \circ Q_5''' \circ P_5''' \circ P_5) \circ (Q_3 \circ Q_3' \circ P_3' \circ P_3) \circ \text{Hom}(P_1, Q_1'))(B) \\ &= \beta \cdot ((Q_7 \circ P_7 \circ Q_5 \circ P_5) \circ (Q_5 \circ P_5 \circ Q_3 \circ P_3) \circ \text{Hom}(P_1, Q_1'))(B), \end{aligned}$$

which, since  $Q_1' \circ B \circ P_1$  is an element of the subspace  $V_3 \cap V_5$ , simplifies to

$$\beta \cdot (Q_7 \circ P_7)(Q_1' \circ B \circ P_1) = \frac{\beta}{2} \cdot (Q_1' \circ B \circ P_1 + K_3(Q_1' \circ B \circ P_1)).$$

The inverse map

$$\{A = K_W \circ A \circ K_U = K_3(A)\} \rightarrow \text{Hom}(U_1, W_1)$$

is the composite:

$$\begin{aligned} &(\text{Hom}(Id_{U_1}, P_1') \circ (\text{Hom}(Q_1, Id_W) \circ Q_3) \circ Q_3') \circ (\frac{2}{\beta} \cdot P_5' \circ Q_5''') \\ &= \frac{2}{\beta} \cdot \text{Hom}(Q_1, P_1') \circ (Q_5 \circ Q_5') \circ P_5' \circ (P_5 \circ Q_5) \circ Q_5''' \\ &= \frac{2}{\beta} \cdot \text{Hom}(Q_1, P_1') \circ (Q_3 \circ P_3 \circ Q_5 \circ P_5) \circ Q_5 \circ Q_5''' \\ &= \frac{2}{\beta} \cdot \text{Hom}(Q_1, P_1') \circ Q_3 \circ P_3 \circ Q_5 \circ Q_5'', \end{aligned}$$

which acts as  $A \mapsto \frac{2}{\beta} \cdot P_1' \circ (\frac{1}{2} \cdot (A + A \circ K_U)) \circ Q_1 = \frac{2}{\beta} \cdot A \circ Q_1$ .



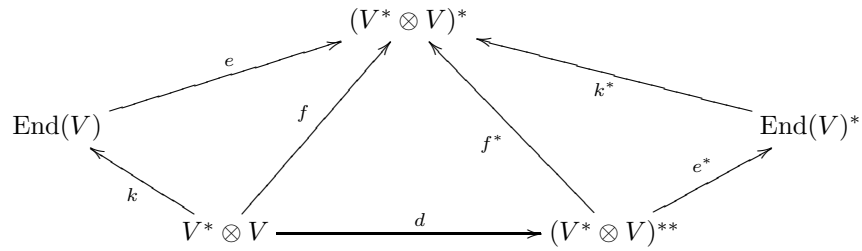


## A Survey of Trace Elements

### 2.1. Endomorphisms: the scalar-valued trace

In the following diagram, the canonical maps  $k_{VV}$ ,  $e_{VV}$ , and  $f_{VV}$  are abbreviated  $k$ ,  $e$ ,  $f$ , and the double duality  $d_{V^* \otimes V}$  is abbreviated  $d$ .

LEMMA 2.1. *For any vector space  $V$ , the following diagram is commutative.*



PROOF. The right triangle is commutative by Lemma 1.5, and the middle by Lemma 1.34. ■

The spaces  $\text{End}(V)$  and  $(V^* \otimes V)^*$  each have the interesting property of containing a distinguished element, which is nonzero when  $V$  has nonzero dimension. The identity  $Id_V : v \mapsto v$  is the distinguished element of  $\text{End}(V)$ .

DEFINITION 2.2. The distinguished element of  $(V^* \otimes V)^*$  is the evaluation operator,  $Ev_V : \phi \otimes v \mapsto \phi(v)$ .

These two elements are related by  $e : Id_V \mapsto Ev_V$  since  $(e(Id_V))(\phi \otimes v) = \phi(Id_V(v))$ .

DEFINITION 2.3. For finite-dimensional  $V$ , define the trace operator by

$$Tr_V = (k^*)^{-1}(Ev_V) \in \text{End}(V)^*.$$

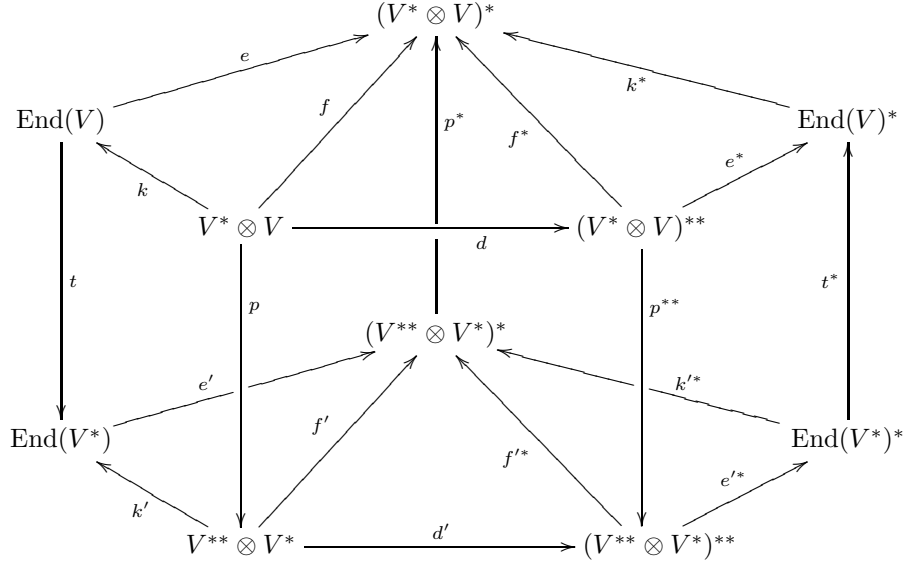
This distinguished element is the image of the two previously mentioned distinguished elements under any path in the above diagram leading to  $\text{End}(V)^*$ , by Lemma 2.1, and the fact that all the arrows are invertible when  $V$  is finite-dimensional. At least one arrow in any path taking  $Id_V$  or  $Ev_V$  to  $Tr_V$  is the inverse of one of the arrows indicated in the diagram.

REMARK 2.4. Using Definition 2.3 as the definition of trace, so that  $Tr_V(A) = Ev_V(k^{-1}(A))$ , is exactly the approach of [MB], [B] §II.4.3, and [K] §II.3. In [G<sub>2</sub>] §I.8, this formula is stated as a consequence of a different definition of trace.

LEMMA 2.5. *For finite-dimensional  $V$ , and  $H \in \text{End}(V)$ ,*

$$\text{Tr}_{V^*}(H^*) = \text{Tr}_V(H).$$

PROOF. In this case,  $H^*$  is  $t_{VV}(H)$ . In the following diagram,  $t_{VV}$ ,  $k_{V^*V^*}$ ,  $e_{V^*V^*}$ ,  $f_{V^*V^*}$ , and  $d_{V^{**} \otimes V^*}$  are abbreviated  $t$ ,  $k'$ ,  $e'$ ,  $f'$ , and  $d'$ . There is also a map  $p : V^* \otimes V \rightarrow V^{**} \otimes V^*$  from Notation 1.36, and  $p^*$  maps the distinguished element  $Ev_{V^*} \in (V^{**} \otimes V^*)^*$  to  $Ev_V$ :



The commutativity of some of the squares in the diagram, for example,  $k' \circ p = t \circ k$  from Lemma 1.37, is enough to imply the lemma:

$$t^*(\text{Tr}_{V^*}) = (t^* \circ (k'^*)^{-1})(Ev_{V^*}) = ((k^*)^{-1} \circ p^*)(Ev_{V^*}) = (k^*)^{-1}(Ev_V) = \text{Tr}_V.$$

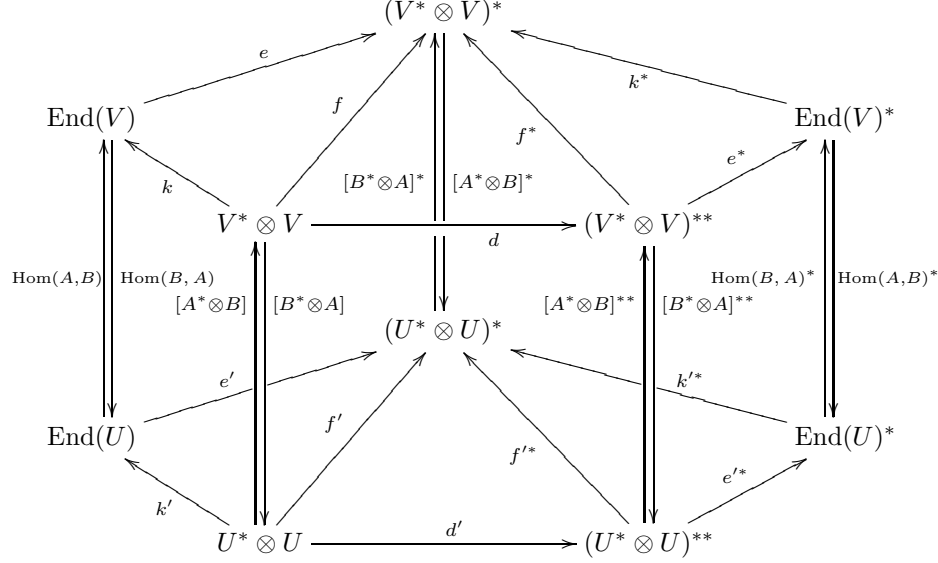
■

The equality  $q \circ t = e$  from Lemma 1.41, which could fit in the back left square of the above diagram, shows that  $q$  maps the distinguished element  $Id_{V^*}$  to  $Ev_V$ . So, there is another formula for the trace,

$$(2.1) \quad \text{Tr}_V = ((k^*)^{-1} \circ q)(Id_{V^*}).$$

LEMMA 2.6. For maps  $A : V \rightarrow U$  and  $B : U \rightarrow V$  between vector spaces of finite, but possibly different, dimensions,  $Tr_V(B \circ A) = Tr_U(A \circ B)$ .

PROOF. Abbreviated names for maps are used again in the following diagram, with primes in the lower pentagon.



Some of the squares in the diagram are commutative, for example,

$$\begin{aligned}
 (e \circ \text{Hom}(A, B))(H) &= e(B \circ H \circ A) : \\
 \phi \otimes v &\mapsto \phi(B(H(A(v)))) , \\
 ([B^* \otimes A]^* \circ e')(H) &= (e'(H)) \circ [B^* \otimes A] : \\
 \phi \otimes v &\mapsto (e'(H))((B^*(\phi)) \otimes (A(v))) \\
 &= (B^*(\phi))(H(A(v))) = \phi(B(H(A(v)))) ,
 \end{aligned}$$

and also  $\text{Hom}(B, A) \circ k = k' \circ [B^* \otimes A]$ , by Lemma 1.29. The equality follows:

$$\begin{aligned}
 Tr_U(A \circ B) &= Tr_U(\text{Hom}(B, A)(Id_V)) = (\text{Hom}(B, A)^*(Tr_U))(Id_V) \\
 &= ((\text{Hom}(B, A)^* \circ (k')^{-1} \circ e')(Id_U))(Id_V) \\
 &= ((e^* \circ d \circ k^{-1} \circ \text{Hom}(A, B))(Id_U))(Id_V) \\
 &= (((k^*)^{-1} \circ e)(Id_V))(\text{Hom}(A, B)(Id_U)) = Tr_V(B \circ A).
 \end{aligned}$$

■

EXAMPLE 2.7. In the case  $V = \mathbb{K}$ ,  $k(Id_{\mathbb{K}} \otimes 1) = Id_{\mathbb{K}} \in \text{End}(\mathbb{K}) = \mathbb{K}^*$ . The trace is  $Tr_{\mathbb{K}} = e^*(d(Id_{\mathbb{K}} \otimes 1)) \in \mathbb{K}^{**}$ , and for  $\phi \in \mathbb{K}^*$ ,  $Tr_{\mathbb{K}}(\phi) = (e(\phi))(Id_{\mathbb{K}} \otimes 1) = Id_{\mathbb{K}}(\phi(1)) = \phi(1)$ . So,  $Tr_{\mathbb{K}} = d_{\mathbb{K}}(1)$ , and in particular,  $Tr_{\mathbb{K}}(Id_{\mathbb{K}}) = 1$ .

EXAMPLE 2.8. If  $V$  is a direct sum of  $n$  copies of  $\mathbb{K}$ ,  $V = \mathbb{K} \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K}$ , then  $Tr_V(Id_V) = Tr_V(\sum Q_i \circ P_i) = \sum Tr_{\mathbb{K}}(P_i \circ Q_i) = \sum 1$ , which is  $n$  in fields of characteristic 0.

EXAMPLE 2.9. Assume  $Tr_V(Id_V) \neq 0$ . Let  $\text{End}_0(V)$  denote the kernel of  $Tr_V$ , i.e., the subspace of trace 0 endomorphisms. Recall from Lemma 1.63 that there exists a direct sum  $\text{End}(V) = \mathbb{K} \oplus \text{End}_0(V)$ , and in particular, there exist constants

$\alpha, \beta \in \mathbb{K}$  so that  $\alpha \cdot \beta \cdot \text{Tr}_V(\text{Id}_V) = 1$ , and a direct sum is defined by  $P_1^\alpha = \alpha \cdot \text{Tr}_V$ ,  $Q_1^\beta : \mathbb{K} \rightarrow \text{End}(V) : \gamma \mapsto \beta \cdot \gamma \cdot \text{Id}_V$ ,  $P_2 = \text{Id}_{\text{End}(V)} - Q_1^\beta \circ P_1^\alpha$ , and the inclusion map  $Q_2 : \text{End}_0(V) \rightarrow \text{End}(V)$ . Such a direct sum admits a free parameter and is generally not unique, but since  $\text{Id}_V$  is a canonical element of  $\text{End}(V)$  and is not in  $\ker(\text{Tr}_V)$  by assumption, Lemma 1.64 applies, and any choice of constants  $\alpha, \beta$  leads to an equivalent direct sum. So, any endomorphism  $H$  can be written as a sum of a scalar multiple of  $\text{Id}_V$ , and a trace zero endomorphism:

$$H = (H - \frac{\text{Tr}_V(H)}{\text{Tr}_V(\text{Id}_V)} \cdot \text{Id}_V) + \frac{\text{Tr}_V(H)}{\text{Tr}_V(\text{Id}_V)} \cdot \text{Id}_V,$$

and this decomposition of  $H$  is canonical.

THEOREM 2.10. *For  $V$  finite-dimensional, and  $A \in \text{End}(V)$ ,*

$$\text{Tr}_V(A) = (\text{Ev}_V \circ [\text{Id}_{V^*} \otimes A] \circ k^{-1})(\text{Id}_V).$$

PROOF. By Lemma 1.29,

$$[\text{Id}_{V^*} \otimes A] \circ k^{-1} = [\text{Id}_{V^*} \otimes A] \circ k^{-1} = k^{-1} \circ \text{Hom}(\text{Id}_V, A),$$

so

$$(\text{Ev}_V \circ [\text{Id}_{V^*} \otimes A] \circ k^{-1})(\text{Id}_V) = \text{Ev}_V(k^{-1}(A)) = \text{Tr}_V(A).$$

■

REMARK 2.11. The idea of Theorem 2.10 (as in [K] §II.3) is that the trace of  $A$  is the image of the distinguished element  $k^{-1}(\text{Id}_V)$  under the composite of maps in this diagram:

$$V^* \otimes V \xrightarrow{[\text{Id}_{V^*} \otimes A]} V^* \otimes V \xrightarrow{\text{Ev}_V} \mathbb{K}.$$

The statement of Theorem 2.10 could also be written as

$$\text{Tr}_V(A) = ((d_{\text{End}(V)}(\text{Id}_V)) \circ \text{Hom}(k^{-1}, \text{Ev}_V) \circ j)(\text{Id}_{V^*} \otimes A).$$

In terms of the scalar multiplication map  $l$  from Example 1.18 and an inclusion from Example 2.9,  $Q_1^1 : \mathbb{K} \rightarrow \text{End}(V^*) : 1 \mapsto \text{Id}_{V^*}$ , the composite  $[Q_1^1 \otimes \text{Id}_{\text{End}(V)}] \circ l^{-1} : \text{End}(V) \rightarrow \text{End}(V^*) \otimes \text{End}(V)$  takes  $A$  to  $\text{Id}_{V^*} \otimes A$ , so

$$\text{Tr}_V = (d_{\text{End}(V)}(\text{Id}_V)) \circ \text{Hom}(k^{-1}, \text{Ev}_V) \circ j \circ [Q_1^1 \otimes \text{Id}_{\text{End}(V)}] \circ l^{-1}.$$

PROPOSITION 2.12. *For  $V = V_1 \oplus V_2$ ,  $A \in \text{End}(V_1)$ , and  $B \in \text{End}(V_2)$ , let  $A \oplus B$  be the element of  $\text{End}(V)$  defined by  $A \oplus B = Q_1 \circ A \circ P_1 + Q_2 \circ B \circ P_2$ . If  $V$  is finite-dimensional, then*

$$\text{Tr}_V(A \oplus B) = \text{Tr}_{V_1}(A) + \text{Tr}_{V_2}(B).$$

PROOF. The construction of  $A \oplus B$  is as in Lemma 1.50.

$$\begin{aligned} \text{Tr}_V(A \oplus B) &= \text{Tr}_V(Q_1 \circ A \circ P_1) + \text{Tr}_V(Q_2 \circ B \circ P_2) \\ &= \text{Tr}_{V_1}(P_1 \circ Q_1 \circ A) + \text{Tr}_{V_2}(P_2 \circ Q_2 \circ B) \\ &= \text{Tr}_{V_1}(A) + \text{Tr}_{V_2}(B), \end{aligned}$$

using Lemma 2.6.

■

PROPOSITION 2.13. *If  $V$  is finite-dimensional,  $V = V_1 \oplus V_2$ , and  $K \in \text{End}(V)$ , then*

$$\text{Tr}_V(K) = \text{Tr}_{V_1}(P_1 \circ K \circ Q_1) + \text{Tr}_{V_2}(P_2 \circ K \circ Q_2).$$

PROOF. Using Lemma 2.6,

$$\text{Tr}_V(K) = \text{Tr}_V((Q_1 \circ P_1 + Q_2 \circ P_2) \circ K) = \text{Tr}_{V_1}(P_1 \circ K \circ Q_1) + \text{Tr}_{V_2}(P_2 \circ K \circ Q_2). \quad \blacksquare$$

EXERCISE 2.14. Given  $V$  finite-dimensional and  $A, P \in \text{End}(V)$ , suppose  $P$  is an idempotent with image subspace  $V_1 = P(V)$ , and let  $P_1$  and  $Q_1$  be the projection and inclusion operators from Example 1.74. Then

$$\text{Tr}_V(P \circ A) = \text{Tr}_{V_1}(P_1 \circ A \circ Q_1).$$

In particular, for  $A = \text{Id}_V$ ,  $\text{Tr}_V(P) = \text{Tr}_{V_1}(\text{Id}_{V_1})$ .  $\blacksquare$

PROPOSITION 2.15. ([B] §II.10.11) *Suppose  $V$  is finite-dimensional and  $\Phi \in \text{End}(V)^*$ . Then there exists  $F \in \text{End}(V)$  such that  $\Phi(A) = \text{Tr}_V(F \circ A)$  for all  $A \in \text{End}(V)$ .*

PROOF. Let  $F = e^{-1}(k^*(\Phi))$ . The result follows from the commutativity of the appropriate paths in the diagram for the Lemma 2.6 in the case  $U = V$ .

$$\begin{aligned} \Phi(A) &= (\text{Hom}(\text{Id}_V, A)^*(\Phi))(\text{Id}_V) \\ &= ((e^* \circ d \circ k^{-1} \circ \text{Hom}(A, \text{Id}_V) \circ e^{-1} \circ k^*)(\Phi))(\text{Id}_V) \\ &= ((k^{-1})^*(e(\text{Id}_V))) (\text{Hom}(A, \text{Id}_V)(e^{-1}(k^*(\Phi)))) \\ &= \text{Tr}_V(F \circ A). \end{aligned} \quad \blacksquare$$

PROPOSITION 2.16. ([G<sub>1</sub>] §IV.7) *If  $A \in \text{End}(V)$  and  $\text{Tr}_V(A \circ B) = 0$  for all  $B$ , then  $A = 0_{\text{End}(V)}$ .*

PROOF.

$$\begin{aligned} \text{Hom}(\text{Id}_V, A)^*(\text{Tr}_V) &= (\text{Hom}(\text{Id}_V, A)^* \circ (k^*)^{-1} \circ e)(\text{Id}_V) \\ &= ((k^*)^{-1} \circ e \circ \text{Hom}(A, \text{Id}_V))(\text{Id}_V) \\ &= ((k^*)^{-1} \circ e)(A), \end{aligned}$$

by the commutativity of the diagram from Lemma 2.6, with  $U = V$ . If

$$\text{Tr}_V(A \circ B) = (\text{Hom}(\text{Id}_V, A)^*(\text{Tr}_V))(B) = (((k^*)^{-1} \circ e)(A))(B)$$

is always zero, then  $((k^*)^{-1} \circ e)(A) = 0_{\text{End}(V)^*}$ , and since  $(k^*)^{-1} \circ e$  has zero kernel,  $A$  must be  $0_{\text{End}(V)}$ .  $\blacksquare$

PROPOSITION 2.17. ([B] §II.10.11) *Suppose  $V$  is finite-dimensional, and  $\Phi \in \text{End}(V)^*$  satisfies  $\Phi(A \circ B) = \Phi(B \circ A)$  for all  $A, B \in \text{End}(V)$ . Then there exists  $\lambda \in \mathbb{K}$  such that  $\Phi = \lambda \cdot \text{Tr}_V$ .*

PROOF. By Proposition 2.15,  $\Phi(A \circ B) = \text{Tr}_V(F \circ A \circ B) = \Phi(B \circ A) = \text{Tr}_V(F \circ B \circ A)$ . By Lemma 2.6,  $\text{Tr}_V(F \circ B \circ A) = \text{Tr}_V(A \circ B \circ F)$  for all  $B$ , so  $\text{Hom}(A, F)^*(\text{Tr}_V) = \text{Hom}(F, A)^*(\text{Tr}_V)$ . Then

$$\begin{aligned} \text{Hom}(A, F)^*((k^*)^{-1}(e(\text{Id}_V))) &= \text{Hom}(F, A)^*((k^*)^{-1}(e(\text{Id}_V))) \\ (k^*)^{-1}(e(\text{Hom}(F, A)(\text{Id}_V))) &= (k^*)^{-1}(e(\text{Hom}(A, F)(\text{Id}_V))) \\ ((k^*)^{-1} \circ e)(A \circ F) &= ((k^*)^{-1} \circ e)(F \circ A), \end{aligned}$$

so  $A \circ F = F \circ A$  for all  $A$ , and so  $F = \lambda \cdot \text{Id}_V$  ([B] Exercise II.1.26). ■

PROPOSITION 2.18. ([G<sub>1</sub>] §IV.7) *Suppose  $\text{Tr}_V(\text{Id}_V) \neq 0$ , and that the operator  $\Omega \in \text{End}(\text{End}(V))$  satisfies  $\Omega(A \circ B) = (\Omega(A)) \circ (\Omega(B))$  for all  $A, B$ , and  $\Omega(\text{Id}_V) = \text{Id}_V$ . Then  $\text{Tr}_V(\Omega(H)) = \text{Tr}_V(H)$  for all  $H \in \text{End}(V)$ .*

PROOF.

$$\begin{aligned} \text{Tr}_V(\Omega(A \circ B)) &= \text{Tr}_V((\Omega(A)) \circ (\Omega(B))) \\ &= \text{Tr}_V((\Omega(B)) \circ (\Omega(A))) \\ &= \text{Tr}_V(\Omega(B \circ A)), \end{aligned}$$

so  $\Omega^*(\text{Tr}_V)$  satisfies the hypothesis of the previous Proposition and  $\text{Tr}_V(\Omega(H)) = \lambda \cdot \text{Tr}_V(H)$ . The second property of  $\Omega$  implies  $\text{Tr}_V(\Omega(\text{Id}_V)) = \text{Tr}_V(\text{Id}_V) = \lambda \cdot \text{Tr}_V(\text{Id}_V)$ , so either  $\text{Tr}_V(\text{Id}_V) = 0$ , or  $\lambda = 1$  and  $\text{Tr}_V(\Omega(H)) = \text{Tr}_V(H)$  for all  $H$ . ■

REMARK 2.19. The following Proposition is a generalization of Example 2.7, motivated by a property of a “line bundle.”

PROPOSITION 2.20. *Suppose that  $L$  is a finite-dimensional vector space, and that the evaluation map  $\text{Ev}_L : L^* \otimes L \rightarrow \mathbb{K}$  is invertible. Then  $\text{Tr}_L(\text{Id}_L) = 1$ .*

PROOF.  $k = k_{LL}$  and  $e = e_{LL}$  are invertible.  $\text{Ev}_L \neq 0_{(L^* \otimes L)^*}$ , so  $\text{Id}_L = e^{-1}(\text{Ev}_L) \neq 0_{\text{End}(L)}$ .  $\text{Tr}_L = \text{Ev}_L \circ k^{-1}$  is invertible, so  $\text{Tr}_L(\text{Id}_L) \neq 0$ , and Example 2.9 applies. In particular, there is some  $\beta \in \mathbb{K}$  so that  $\beta \cdot \text{Tr}_L(\text{Id}_L) = 1$ , and a map  $Q_1^\beta : \mathbb{K} \rightarrow \text{End}(L) : \gamma \mapsto \beta \cdot \gamma \cdot \text{Id}_L$  so that  $\text{Tr}_L \circ Q_1^\beta = \text{Id}_{\mathbb{K}}$ . It follows that

$$(\text{Ev}_L \circ k^{-1}) \circ Q_1^\beta = \text{Id}_{\mathbb{K}} \implies Q_1^\beta \circ \text{Ev}_L = k.$$

There is also some  $v \in L$ , and there is some  $\phi \in L^*$ , so that  $\text{Ev}_L(\phi \otimes v) \neq 0$ , and so  $\phi(v) \neq 0$ , and  $v \neq 0_L$ . Then,

$$k(\phi \otimes v) = (Q_1^\beta \circ \text{Ev}_L)(\phi \otimes v) = \beta \cdot \phi(v) \cdot \text{Id}_L,$$

so

$$(k(\phi \otimes v))(v) = \phi(v) \cdot v = (\beta \cdot \phi(v) \cdot \text{Id}_L)(v) = \beta \cdot \phi(v) \cdot v,$$

which implies  $\beta = 1$ . ■

## 2.2. The generalized trace

The trace  $Tr_V : \text{End}(V) \rightarrow \mathbb{K}$  generalizes to a map

$$Tr_{V;U,W} : \text{Hom}(V \otimes U, V \otimes W) \rightarrow \text{Hom}(U, W),$$

constructed using canonical  $j$  maps. The following particular cases of  $j$  maps will be used repeatedly:

$$j_1 : \text{End}(V)^* \otimes \text{End}(\text{Hom}(U, W)) \rightarrow \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W))$$

$$j_2 : \text{End}(V) \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(V \otimes U, V \otimes W).$$

When  $V$  is finite-dimensional, both  $j_1$  and  $j_2$  are invertible, by Lemma 1.23. Denote by  $l_1$  the scalar multiplication map  $\mathbb{K} \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(U, W)$ . The domain of  $j_1$  contains the distinguished element  $Tr_V \otimes Id_{\text{Hom}(U, W)}$ .

DEFINITION 2.21. For finite-dimensional  $V$ , define

$$Tr_{V;U,W} = (\text{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(U, W)}).$$

Note that the finite-dimensionality of  $V$  is used in the Definition, since  $j_2$  must be invertible, but  $U$  and  $W$  may be arbitrary vector spaces.

EXAMPLE 2.22. A map of the form  $j_2(A \otimes B) : V \otimes U \rightarrow V \otimes W$ , for  $A : V \rightarrow V$  and  $B : U \rightarrow W$ , has trace

$$Tr_{V;U,W}(j_2(A \otimes B)) = l_1((j_1(Tr_V \otimes Id_{\text{Hom}(U, W)}))(A \otimes B)) = Tr_V(A) \cdot B.$$

In the  $V = \mathbb{K}$  case, the trace is an invertible map. Define scalar multiplication maps  $l_W : \mathbb{K} \otimes W \rightarrow W$  and  $l_U : \mathbb{K} \otimes U \rightarrow U$ .

THEOREM 2.23.  $Tr_{\mathbb{K};U,W} = \text{Hom}(l_U^{-1}, l_W) : \text{Hom}(\mathbb{K} \otimes U, \mathbb{K} \otimes W) \rightarrow \text{Hom}(U, W)$ .

PROOF. For any  $\phi \in \mathbb{K}^*$ ,  $F \in \text{Hom}(U, W)$ , the following square is commutative:

$$\begin{array}{ccc} \mathbb{K} \otimes U & \xrightarrow{j_2(\phi \otimes F)} & \mathbb{K} \otimes W \\ \downarrow l_U & & \downarrow l_W \\ U & \xrightarrow{\phi(1) \cdot F} & W \end{array}$$

$$\begin{aligned} (l_W \circ (j_2(\phi \otimes F)))(\lambda \otimes u) &= \phi(\lambda) \cdot F(u), \\ ((\phi(1) \cdot F) \circ l_U)(\lambda \otimes u) &= (\phi(1) \cdot F)(\lambda \cdot u) = \phi(\lambda) \cdot F(u). \end{aligned}$$

So,

$$\text{Hom}(l_U^{-1}, l_W) \circ j_2 = l_1 \circ (j_1((d_{\mathbb{K}}(1)) \otimes Id_{\text{Hom}(U, W)})) : \phi \otimes F \mapsto \phi(1) \cdot F.$$

The equality follows from Example 2.7, where  $Tr_{\mathbb{K}} = d_{\mathbb{K}}(1)$ :

$$\text{Hom}(l_U^{-1}, l_W) = l_1 \circ (j_1((d_{\mathbb{K}}(1)) \otimes Id_{\text{Hom}(U, W)})) \circ j_2^{-1} = Tr_{\mathbb{K};U,W}.$$

■

REMARK 2.24. The next Theorems in this Section are straightforward linear algebra identities for the generalized trace. Versions of these results are stated in a more general context of category theory, and given different proofs, in [Maltiniotis] §3.5 or [JSV] §2. The result of Theorem 2.23 is related to a property called “vanishing” by [JSV], and Theorems 2.25 and 2.26 are “naturality” properties ([JSV]).

An analogue of Lemma 2.6 applies to maps  $A : V \rightarrow V'$  and  $B : V' \otimes U \rightarrow V \otimes W$ , using the canonical maps

$$\begin{aligned} j_U &: \text{Hom}(V, V') \otimes \text{End}(U) \rightarrow \text{Hom}(V \otimes U, V' \otimes U), \\ j_W &: \text{Hom}(V, V') \otimes \text{End}(W) \rightarrow \text{Hom}(V \otimes W, V' \otimes W). \end{aligned}$$

THEOREM 2.25. *For finite-dimensional  $V, V'$ ,*

$$\text{Tr}_{V;U,W}(B \circ (j_U(A \otimes \text{Id}_U))) = \text{Tr}_{V';U,W}((j_W(A \otimes \text{Id}_W)) \circ B).$$

PROOF. In the following diagram,

$$\begin{array}{ccccc} M_{11} & \xrightarrow{a_1} & M_{12} & \xleftarrow{a_2} & M_{13} \\ \downarrow j_1 & & \downarrow j_1'' & & \downarrow j_1' \\ M_{21} & \xrightarrow{a_3} & M_{22} & \xleftarrow{a_4} & M_{23} \\ \text{Hom}(j_2, l_1^{-1}) \updownarrow \text{Hom}(j_2^{-1}, l_1) & & \text{Hom}(j_2'', l_1^{-1}) \updownarrow \text{Hom}((j_2'')^{-1}, l_1) & & \text{Hom}(j_2', l_1^{-1}) \updownarrow \text{Hom}((j_2')^{-1}, l_1) \\ M_{31} & \xrightarrow{a_5} & M_{32} & \xleftarrow{a_6} & M_{33} \end{array}$$

the objects are

$$\begin{aligned} M_{11} &= \text{End}(V)^* \otimes \text{End}(\text{Hom}(U, W)) \\ M_{21} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{31} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(U, W)) \\ M_{12} &= \text{Hom}(V', V)^* \otimes \text{End}(\text{Hom}(U, W)) \\ M_{22} &= \text{Hom}(\text{Hom}(V', V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{32} &= \text{Hom}(\text{Hom}(V' \otimes U, V \otimes W), \text{Hom}(U, W)) \\ M_{13} &= \text{End}(V')^* \otimes \text{End}(\text{Hom}(U, W)) \\ M_{23} &= \text{Hom}(\text{End}(V') \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{33} &= \text{Hom}(\text{Hom}(V' \otimes U, V' \otimes W), \text{Hom}(U, W)), \end{aligned}$$

where the left and right columns are the maps from the definition of trace and the horizontal arrows in the diagram are

$$\begin{aligned} a_1 &= [\text{Hom}(A, \text{Id}_V)^* \otimes \text{Id}_{\text{End}(\text{Hom}(U, W))}] \\ a_2 &= [\text{Hom}(\text{Id}_{V'}, A)^* \otimes \text{Id}_{\text{End}(\text{Hom}(U, W))}] \\ a_3 &= \text{Hom}([\text{Hom}(A, \text{Id}_V) \otimes \text{Id}_{\text{Hom}(U, W)}], \text{Id}_{\mathbb{K} \otimes \text{Hom}(U, W)}) \\ a_4 &= \text{Hom}([\text{Hom}(\text{Id}_{V'}, A) \otimes \text{Id}_{\text{Hom}(U, W)}], \text{Id}_{\mathbb{K} \otimes \text{Hom}(U, W)}) \\ a_5 &= \text{Hom}(\text{Hom}(j_U(A \otimes \text{Id}_U), \text{Id}_{V \otimes W}), \text{Id}_{\text{Hom}(U, W)}) \\ a_6 &= \text{Hom}(\text{Hom}(\text{Id}_{V' \otimes U}, j_W(A \otimes \text{Id}_W)), \text{Id}_{\text{Hom}(U, W)}). \end{aligned}$$

The two quantities in the statement of the Theorem are

$$\begin{aligned} \text{Tr}_{V;U,W}(B \circ (j_U(A \otimes \text{Id}_U))) &= (a_5(\text{Tr}_{V;U,W}))(B), \\ \text{Tr}_{V';U,W}((j_W(A \otimes \text{Id}_W)) \circ B) &= (a_6(\text{Tr}_{V';U,W}))(B). \end{aligned}$$



Each of the squares in the diagram is commutative, by Lemmas 1.5 and 1.26. By Lemma 2.6,  $\text{Hom}(A, \text{Id}_V)^*(\text{Tr}_V) = \text{Hom}(\text{Id}_{V'}, A)^*(\text{Tr}_{V'})$ , so

$$\begin{aligned} a_1(\text{Tr}_V \otimes \text{Id}_{\text{Hom}(U, W)}) &= (\text{Hom}(A, \text{Id}_V)^*(\text{Tr}_V)) \otimes \text{Id}_{\text{Hom}(U, W)} \\ &= (\text{Hom}(\text{Id}_{V'}, A)^*(\text{Tr}_{V'})) \otimes \text{Id}_{\text{Hom}(U, W)} \\ &= a_2(\text{Tr}_{V'} \otimes \text{Id}_{\text{Hom}(U, W)}). \end{aligned}$$

The Theorem follows from the commutativity of the diagram:

$$\begin{aligned} a_5(\text{Tr}_{V;U,W}) &= (a_5 \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1)(\text{Tr}_V \otimes \text{Id}_{\text{Hom}(U, W)}) \\ &= (\text{Hom}((j_2'')^{-1}, l_1) \circ j_1'' \circ a_1)(\text{Tr}_V \otimes \text{Id}_{\text{Hom}(U, W)}) \\ &= (\text{Hom}((j_2'')^{-1}, l_1) \circ j_1'' \circ a_2)(\text{Tr}_{V'} \otimes \text{Id}_{\text{Hom}(U, W)}) \\ &= (a_6 \circ \text{Hom}((j_2')^{-1}, l_1) \circ j_1')(\text{Tr}_{V'} \otimes \text{Id}_{\text{Hom}(U, W)}) \\ &= a_6(\text{Tr}_{V';U,W}). \end{aligned}$$

■

The general strategy for the preceding proof will be repeated in some subsequent proofs. To derive an equality involving the generalized trace, a diagram is set up with the maps from Definition 2.21 on the left and right. The bottom row will be the desired theorem, and the top row is the “key step,” which is either obvious, or which uses the previously derived properties of the scalar-valued trace. There will be little choice in selecting canonical maps as horizontal arrows, and the commutativity of the diagram will give the theorem as a consequence of the key step.

**THEOREM 2.26.** *For  $A : V \otimes U \rightarrow V \otimes W$ ,  $B : W \rightarrow W'$ , and  $C : U' \rightarrow U$ , the composite  $[\text{Id}_V \otimes B] \circ A \circ [\text{Id}_V \otimes C] : V \otimes U' \rightarrow V \otimes W'$  has trace*

$$\text{Tr}_{V;U',W'}([\text{Id}_V \otimes B] \circ A \circ [\text{Id}_V \otimes C]) = B \circ (\text{Tr}_{V;U,W}(A)) \circ C.$$

**PROOF.** In the following diagram,

$$\begin{array}{ccccc} M_{11} & \xrightarrow{a_1} & M_{12} & \xleftarrow{a_2} & M_{13} \\ \downarrow j_1 & & \downarrow j_1'' & & \downarrow j_1' \\ M_{21} & \xrightarrow{a_3} & M_{22} & \xleftarrow{a_4} & M_{23} \\ \text{Hom}(j_2, l_1^{-1}) \updownarrow & & \text{Hom}(j_2^{-1}, l_1) \updownarrow & & \text{Hom}(j_2, (l_1')^{-1}) \updownarrow \\ M_{31} & \xrightarrow{a_5} & M_{32} & \xleftarrow{a_6} & M_{33} \end{array}$$

the objects are

$$\begin{aligned}
M_{11} &= \text{End}(V)^* \otimes \text{End}(\text{Hom}(U, W)) \\
M_{21} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\
M_{31} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(U, W)) \\
M_{12} &= \text{End}(V)^* \otimes \text{Hom}(\text{Hom}(U, W), \text{Hom}(U', W')) \\
M_{22} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U', W')) \\
M_{32} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(U', W')) \\
M_{13} &= \text{End}(V)^* \otimes \text{End}(\text{Hom}(U', W')) \\
M_{23} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U', W'), \mathbb{K} \otimes \text{Hom}(U', W')) \\
M_{33} &= \text{Hom}(\text{Hom}(V \otimes U', V \otimes W'), \text{Hom}(U', W')),
\end{aligned}$$

where the left and right columns are the maps from the definition of trace and the horizontal arrows in the diagram are

$$\begin{aligned}
a_1 &= [Id_{\text{End}(V)^*} \otimes \text{Hom}(Id_{\text{Hom}(U, W)}, \text{Hom}(C, B))] \\
a_2 &= [Id_{\text{End}(V)^*} \otimes \text{Hom}(\text{Hom}(C, B), Id_{\text{Hom}(U', W')})] \\
a_3 &= \text{Hom}(Id_{\text{End}(V) \otimes \text{Hom}(U, W)}, [Id_{\mathbb{K}} \otimes \text{Hom}(C, B)]) \\
a_4 &= \text{Hom}([Id_{\text{End}(V)} \otimes \text{Hom}(C, B)], Id_{\mathbb{K} \otimes \text{Hom}(U', W')}) \\
a_5 &= \text{Hom}(Id_{\text{Hom}(V \otimes U, V \otimes W)}, \text{Hom}(C, B)) \\
a_6 &= \text{Hom}(\text{Hom}([Id_V \otimes C], [Id_V \otimes B]), Id_{\text{Hom}(U', W')}).
\end{aligned}$$

The two quantities in the statement of the Theorem are

$$\begin{aligned}
Tr_{V;U',W'}([Id_V \otimes B] \circ A \circ [Id_V \otimes C]) &= (a_6(Tr_{V;U',W'}))(A) \\
B \circ (Tr_{V;U,W}(A)) \circ C &= (a_5(Tr_{V;U,W}))(A).
\end{aligned}$$

Squares except the lower left commute by Lemmas 1.5 and 1.26; for the remaining square, let  $\lambda \in \mathbb{K}$ ,  $E \in \text{Hom}(U, W)$ :

$$\begin{aligned}
(l'_1 \circ [Id_{\mathbb{K}} \otimes \text{Hom}(C, B)])(\lambda \otimes E) &= l'_1(\lambda \otimes (B \circ E \circ C)) = \lambda \cdot (B \circ E \circ C) \\
(\text{Hom}(C, B) \circ l_1)(\lambda \otimes E) &= B \circ (\lambda \cdot E) \circ C = \lambda \cdot (B \circ E \circ C).
\end{aligned}$$

The “key step” uses a property of the identity map, and not any properties of the trace:

$$a_1(Tr_V \otimes Id_{\text{Hom}(U, W)}) = Tr_V \otimes \text{Hom}(C, B) = a_2(Tr_V \otimes Id_{\text{Hom}(U', W')})$$

The Theorem follows from the commutativity of the diagram:

$$\begin{aligned}
a_5(Tr_{V;U,W}) &= (a_5 \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(U, W)}) \\
&= (\text{Hom}(j_2^{-1}, l'_1) \circ j''_1 \circ a_1)(Tr_V \otimes Id_{\text{Hom}(U, W)}) \\
&= (\text{Hom}(j_2^{-1}, l'_1) \circ j''_1 \circ a_2)(Tr_V \otimes Id_{\text{Hom}(U', W')}) \\
&= (a_6 \circ \text{Hom}((j'_2)^{-1}, l'_1) \circ j'_1)(Tr_V \otimes Id_{\text{Hom}(U', W')}) \\
&= a_6(Tr_{V;U',W'}).
\end{aligned}$$

■

LEMMA 2.27. *The following diagram is commutative.*

$$\begin{array}{ccc}
V_1^* \otimes V_2^* \otimes W_1 \otimes W_2 & \xrightarrow{s} & V_1^* \otimes W_1 \otimes V_2^* \otimes W_2 \\
\downarrow [j \otimes Id_{W_1 \otimes W_2}] & & \downarrow [k_{V_1 W_1} \otimes k_{V_2 W_2}] \\
\text{Hom}(V_2 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) \otimes W_1 \otimes W_2 & & \text{Hom}(V_1, W_1) \otimes \text{Hom}(V_2, W_2) \\
\downarrow [\text{Hom}(Id_{V_1 \otimes V_2}, l) \otimes Id_{W_1 \otimes W_2}] & & \downarrow j \\
(V_1 \otimes V_2)^* \otimes W_1 \otimes W_2 & \xrightarrow{k_{V_1 \otimes V_2, W_1 \otimes W_2}} & \text{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)
\end{array}$$

PROOF. The map  $s$  is an involution switching the middle two factors of the tensor product (as in Example 1.19), and  $l$  is multiplication of elements of  $\mathbb{K}$ .

$$\begin{aligned}
\phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2 &\mapsto (j \circ [k_{V_1 W_1} \otimes k_{V_2 W_2}] \circ s)(\phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2) \\
&= [(k_{V_1 W_1}(\phi_1 \otimes w_1)) \otimes (k_{V_2 W_2}(\phi_2 \otimes w_2))]: \\
v_1 \otimes v_2 &\mapsto ((\phi(v_1)) \cdot w_1) \otimes ((\phi(v_2)) \cdot w_2), \\
\phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2 &\mapsto k_{V_1 \otimes V_2, W_1 \otimes W_2}((l \circ [\phi_1 \otimes \phi_2]) \otimes w_1 \otimes w_2): \\
v_1 \otimes v_2 &\mapsto \phi_1(v_1) \cdot \phi_2(v_2) \cdot w_1 \otimes w_2.
\end{aligned}$$

■

REMARK 2.28. The above result appears in [K] §II.2, and is related to a matrix algebra equation in [Magnus] §3.6.

THEOREM 2.29. *For finite-dimensional  $V$  and  $U$ , and  $A : V \otimes U \rightarrow V \otimes U$ ,*

$$Tr_U(Tr_{V;U,U}(A)) = Tr_{V \otimes U}(A).$$

PROOF. As in Lemma 2.6, the maps  $k_{VV}$  and  $k_{UU}$  are abbreviated  $k$  and  $k'$ , and the corresponding map for  $V \otimes U$  is denoted  $k'' : (V \otimes U)^* \otimes V \otimes U \rightarrow \text{End}(V \otimes U)$ . By Lemma 2.27, these  $k$  maps are related by the following commutative diagram.

$$\begin{array}{ccc}
V^* \otimes V \otimes U^* \otimes U & \xrightarrow{[k \otimes k']} & \text{End}(V) \otimes \text{End}(U) \\
\downarrow s & & \downarrow j_2 \\
V^* \otimes U^* \otimes V \otimes U & & \\
\downarrow [j \otimes Id_{V \otimes U}] & & \\
\text{Hom}(V \otimes U, \mathbb{K} \otimes \mathbb{K}) \otimes V \otimes U & & \\
\downarrow [\text{Hom}(Id_{V \otimes U}, l) \otimes Id_{V \otimes U}] & & \\
(V \otimes U)^* \otimes V \otimes U & \xrightarrow{k''} & \text{End}(V \otimes U)
\end{array}$$

The composite of maps in the left column is abbreviated  $a_1$ . In particular, since  $U$  and  $V$  are assumed finite-dimensional, all the arrows in the square are invertible.

In the following diagram,

$$\begin{array}{ccc}
\text{End}(V)^* \otimes \text{End}(\text{End}(U)) & \xrightleftharpoons[a_2^{-1}]{a_2} & (V^* \otimes V)^* \otimes \text{End}(U^* \otimes U) \\
\downarrow j_1 & & \downarrow j'_1 \\
\text{Hom}(\text{End}(V) \otimes \text{End}(U), \mathbb{K} \otimes \text{End}(U)) & \xrightarrow{a_3} & \text{Hom}(V^* \otimes V \otimes U^* \otimes U, \mathbb{K} \otimes U^* \otimes U) \\
\text{Hom}(j_2, l_1^{-1}) \updownarrow \text{Hom}(j_2^{-1}, l_1) & & \text{Hom}(a_1, (l'_1)^{-1} \circ (k')^{-1}) \updownarrow \text{Hom}(a_1^{-1}, k' \circ l'_1) \\
\text{Hom}(\text{End}(V \otimes U), \text{End}(U)) & \xrightarrow{a_4} & \text{Hom}((V \otimes U)^* \otimes V \otimes U, \text{End}(U)) \\
\downarrow \text{Hom}(Id_{\text{End}(V \otimes U)}, Tr_U) & & \downarrow \text{Hom}(Id_{(V \otimes U)^* \otimes V \otimes U}, Tr_U) \\
\text{End}(V \otimes U)^* & \xrightarrow{k''^*} & ((V \otimes U)^* \otimes V \otimes U)^*
\end{array}$$

the horizontal arrows are

$$\begin{aligned}
a_2 &= [k^* \otimes \text{Hom}(k', (k')^{-1})] \\
a_2^{-1} &= [(k^*)^{-1} \otimes \text{Hom}((k')^{-1}, k')] \\
a_3 &= \text{Hom}([k \otimes k'], [Id_{\mathbb{K}} \otimes (k')^{-1}]) \\
a_4 &= \text{Hom}(k'', Id_{\text{End}(U)}),
\end{aligned}$$

and the statement of the Theorem is that

$$\text{Hom}(Id_{\text{End}(V \otimes U)}, Tr_U)(Tr_{V \otimes U}) = Tr_{V \otimes U}.$$

The top square is commutative by Lemma 1.26, and Lemma 1.5 applies easily to the commutativity of the bottom square, and to that of the middle square using  $k'' \circ a_1 = j_2 \circ [k \otimes k']$  (from the first diagram) and  $l_1 \circ [Id_{\mathbb{K}} \otimes k'] = k' \circ l'_1$ .

The commutativity of this square,

$$\begin{array}{ccc}
V^* \otimes V \otimes U^* \otimes U & \xrightarrow{l'_1 \circ (j'_1(Ev_V \otimes Id_{U^* \otimes U}))} & U^* \otimes U \\
\downarrow a_1 & & \downarrow Ev_U \\
(V \otimes U)^* \otimes V \otimes U & \xrightarrow{Ev_{V \otimes U}} & \mathbb{K}
\end{array}$$

$$\begin{aligned}
(Ev_{V \otimes U} \circ a_1)(\phi \otimes v \otimes \xi \otimes u) &= Ev_{V \otimes U}((l \circ [\phi \otimes \xi]) \otimes v \otimes u) \\
&= \phi(v) \cdot \xi(u) \\
(Ev_U \circ l'_1 \circ (j'_1(Ev_V \otimes Id_{U^* \otimes U}))) (\phi \otimes v \otimes \xi \otimes u) &= Ev_U(\phi(v) \cdot \xi \otimes u) \\
&= \phi(v) \cdot \xi(u),
\end{aligned}$$

implies that the distinguished elements  $Ev_V \otimes Id_{U^* \otimes U}$  and  $Ev_{V \otimes U}$  are related by the right column of maps in the second diagram:

$$\begin{aligned}
Ev_V \otimes Id_{U^* \otimes U} &\mapsto Tr_U \circ k' \circ l'_1 \circ (j'_1(Ev_V \otimes Id_{U^* \otimes U})) \circ a_1^{-1} \\
&= Ev_U \circ l'_1 \circ (j'_1(Ev_V \otimes Id_{U^* \otimes U})) \circ a_1^{-1} \\
&= Ev_{V \otimes U}.
\end{aligned}$$

The above equation used the definition of  $Tr_U$ . Along the top row, the key step uses the definition of  $Tr_V$ :

$$a_2^{-1}(Ev_V \otimes Id_{U^* \otimes U}) = ((k^*)^{-1}(Ev_V)) \otimes (k' \circ Id_{U^* \otimes U} \circ (k')^{-1}) = Tr_V \otimes Id_{\text{End}(U)}.$$

The Theorem follows:

$$\begin{aligned}
Tr_U \circ Tr_{V;U,U} &= (\text{Hom}(Id_{\text{End}(V \otimes U)}, Tr_U) \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{End}(U)}) \\
&= (\text{Hom}(Id_{\text{End}(V \otimes U)}, Tr_U) \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1 \circ a_2^{-1})(Ev_V \otimes Id_{U^* \otimes U}) \\
&= (k''^*)^{-1}(Ev_{V \otimes U}) \\
&= Tr_{V \otimes U}.
\end{aligned}$$

■

REMARK 2.30. The previous Theorem appears in slightly different form in [K] §II.3. The following Corollary is a well-known identity for the (scalar-valued) trace ([B] §II.4.4, [Magnus] §1.10, [K] §II.6).

COROLLARY 2.31. *For  $A : V \rightarrow V$ ,  $B : U \rightarrow U$ ,*

$$Tr_{V \otimes U}(j_2(A \otimes B)) = Tr_V(A) \cdot Tr_U(B).$$

PROOF. As in Example 2.22,

$$\begin{aligned}
Tr_{V \otimes U}(j_2(A \otimes B)) &= Tr_U(Tr_{V;U,U}(j_2(A \otimes B))) \\
&= Tr_U(Tr_V(A) \cdot B) \\
&= Tr_V(A) \cdot Tr_U(B).
\end{aligned}$$

■

The result of Corollary 2.31 could also be proved directly using methods similar to the previous proof, and could be stated as the equality

$$j_2^*(Tr_{V \otimes U}) = (\text{Hom}(Id_{\text{End}(V) \otimes \text{End}(U)}, l) \circ j)(Tr_V \otimes Tr_U) \in (\text{End}(V) \otimes \text{End}(U))^*,$$

or

$$Tr_{V \otimes U} \circ j_2 = l \circ [Tr_V \otimes Tr_U].$$

COROLLARY 2.32. ([G<sub>2</sub>] §I.5) *For  $A : V \rightarrow V$ ,  $B : U \rightarrow U$ ,*

$$Tr_{\text{Hom}(V,U)}(\text{Hom}(A, B)) = Tr_V(A) \cdot Tr_U(B).$$

PROOF. By Lemma 1.29,  $\text{Hom}(A, B) = k_{VU} \circ [A^* \otimes B] \circ k_{VU}^{-1}$ , so Lemma 2.6, Corollary 2.31, and Lemma 2.5 apply:

$$\begin{aligned}
Tr_{\text{Hom}(V,U)}(\text{Hom}(A, B)) &= Tr_{\text{Hom}(V,U)}(k_{VU} \circ [A^* \otimes B] \circ k_{VU}^{-1}) \\
&= Tr_{V^* \otimes U}([A^* \otimes B]) \\
&= Tr_{V^*}(A^*) \cdot Tr_U(B) \\
&= Tr_V(A) \cdot Tr_U(B).
\end{aligned}$$

■

THEOREM 2.33. *For finite-dimensional  $V$  and  $V'$ , and  $A : V \otimes V' \otimes U \rightarrow V \otimes V' \otimes W$ ,*

$$Tr_{V \otimes V'; U, W}(A) = Tr_{V'; U, W}(Tr_{V; V' \otimes U, V' \otimes W}(A)).$$

PROOF. In the following diagram,

$$\begin{array}{ccc} M_{11} & \xrightarrow{a_1} & M_{12} \\ \downarrow j'_1 & & \downarrow j''_1 \\ M_{21} & \xrightarrow{a_2} & M_{22} \\ \text{Hom}(j'_2, l_1^{-1}) \updownarrow \text{Hom}((j'_2)^{-1}, l_1) & & \text{Hom}(j''_2, l_1^{-1}) \updownarrow \text{Hom}((j''_2)^{-1}, l_1) \\ M_{31} & \xrightarrow{a_3} & M_{32} \end{array}$$

the objects are

$$\begin{aligned} M_{11} &= \text{End}(V')^* \otimes \text{End}(\text{Hom}(U, W)) \\ M_{21} &= \text{Hom}(\text{End}(V') \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{31} &= \text{Hom}(\text{Hom}(V' \otimes U, V' \otimes W), \text{Hom}(U, W)) \\ M_{12} &= \text{End}(V \otimes V')^* \otimes \text{End}(\text{Hom}(U, W)) \\ M_{22} &= \text{Hom}(\text{End}(V \otimes V') \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{32} &= \text{Hom}(\text{Hom}(V \otimes V' \otimes U, V \otimes V' \otimes W), \text{Hom}(U, W)), \end{aligned}$$

the horizontal arrows are

$$\begin{aligned} a_1 &= [(Tr_{V; V', V'})^* \otimes Id_{\text{End}(\text{Hom}(U, W))}] \\ a_2 &= \text{Hom}([Tr_{V; V', V'} \otimes Id_{\text{Hom}(U, W)}], Id_{\mathbb{K} \otimes \text{Hom}(U, W)}) \\ a_3 &= \text{Hom}(Tr_{V; V' \otimes U, V' \otimes W}, Id_{\text{Hom}(U, W)}), \end{aligned}$$

and the statement of the Theorem is that

$$a_3(Tr_{V'; U, W}) = Tr_{V \otimes V'; U, W}.$$

The commutativity of this square,

$$\begin{array}{ccc} \text{End}(V) \otimes \text{End}(V') \otimes \text{Hom}(U, W) & \xrightarrow{[j_2''' \otimes Id_{\text{Hom}(U, W)}]} & \text{End}(V \otimes V') \otimes \text{Hom}(U, W) \\ \downarrow [Id_{\text{End}(V)} \otimes j_2'] & & \downarrow j_2'' \\ \text{End}(V) \otimes \text{Hom}(V' \otimes U, V' \otimes W) & \xrightarrow{j_2} & \text{Hom}(V \otimes V' \otimes U, V \otimes V' \otimes W) \end{array}$$

is easy to check, and together with that of the diagram:

$$\begin{array}{ccc} \text{End}(V) \otimes \text{End}(V') \otimes \text{Hom}(U, W) & \xrightarrow{[j_2''' \otimes Id_{\text{Hom}(U, W)}]} & \text{End}(V \otimes V') \otimes \text{Hom}(U, W) \\ \downarrow [Id_{\text{End}(V)} \otimes j_2'] & & \downarrow [Tr_{V; V', V'} \otimes Id_{\text{Hom}(U, W)}] \\ \text{End}(V) \otimes \text{Hom}(V' \otimes U, V' \otimes W) & & \text{End}(V') \otimes \text{Hom}(U, W) \\ \downarrow j_1(Tr_V \otimes Id_{\text{Hom}(V' \otimes U, V' \otimes W)}) & & \downarrow j_2' \\ \mathbb{K} \otimes \text{Hom}(V' \otimes U, V' \otimes W) & \xrightarrow{l_1'} & \text{Hom}(V' \otimes U, V' \otimes W) \end{array}$$

$$\begin{aligned}
D \otimes E \otimes F &\mapsto (j'_2 \circ [(Tr_{V;V',V'} \circ j_2''') \otimes Id_{\text{Hom}(U,W)}])(D \otimes E \otimes F) \\
&= j'_2((Tr_{V;V',V'}(j_2'''(D \otimes E))) \otimes F) \\
&= (Tr_V(D)) \cdot j'_2(E \otimes F) \\
D \otimes E \otimes F &\mapsto (l'_1 \circ (j(Tr_V \otimes j'_2)))(D \otimes E \otimes F) \\
&= l'_1((Tr_V(D)) \otimes (j'_2(E \otimes F))) \\
&= (Tr_V(D)) \cdot j'_2(E \otimes F),
\end{aligned}$$

implies

$$\begin{aligned}
Tr_{V;V' \otimes U, V' \otimes W} \circ j_2'' &= l'_1 \circ (j_1(Tr_V \otimes Id_{\text{Hom}(V' \otimes U, V' \otimes W)})) \circ j_2^{-1} \circ j_2'' \\
&= l'_1 \circ (j_1(Tr_V \otimes Id_{\text{Hom}(V' \otimes U, V' \otimes W)})) \\
&\quad \circ [Id_{\text{End}(V)} \otimes j_2'] \circ [(j_2''')^{-1} \otimes Id_{\text{Hom}(U,W)}] \\
&= j_2' \circ [Tr_{V;V',V'} \otimes Id_{\text{Hom}(U,W)}],
\end{aligned}$$

which is what is needed to show that the lower square of the first diagram is commutative. Its upper square is commutative by Lemma 1.26, and the distinguished elements in the top row are related by Theorem 2.29:

$$a_1(Tr_{V'} \otimes Id_{\text{Hom}(U,W)}) = (Tr_{V'} \circ Tr_{V;V',V'}) \otimes Id_{\text{Hom}(U,W)} = Tr_{V \otimes V'} \otimes Id_{\text{Hom}(U,W)}.$$

The Theorem follows:

$$\begin{aligned}
a_3(Tr_{V;U,W}) &= (a_3 \circ \text{Hom}((j_2')^{-1}, l_1) \circ j_1')(Tr_{V'} \otimes Id_{\text{Hom}(U,W)}) \\
&= (\text{Hom}((j_2'')^{-1}, l_1) \circ j_1'' \circ a_1)(Tr_{V'} \otimes Id_{\text{Hom}(U,W)}) \\
&= (\text{Hom}((j_2'')^{-1}, l_1) \circ j_1'')(Tr_{V \otimes V'} \otimes Id_{\text{Hom}(U,W)}) \\
&= Tr_{V \otimes V';U,W}.
\end{aligned}$$

■

REMARK 2.34. The result of the above Theorem is another “vanishing” property of the generalized trace ([JSV]).

The maps

$$\begin{aligned}
j_3 &: \text{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2 \otimes U_2, V_2 \otimes W_2) \\
&\rightarrow \text{Hom}(V_1 \otimes U_1 \otimes V_2 \otimes U_2, V_1 \otimes W_1 \otimes V_2 \otimes W_2), \\
j_4 &: \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2) \rightarrow \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)
\end{aligned}$$

appear in the following Theorem comparing the trace of a tensor product to the tensor product of traces. There are also some switching maps, as in Theorem 2.29,

$$\begin{aligned}
s_1 &: V_1 \otimes W_1 \otimes V_2 \otimes W_2 \rightarrow V_1 \otimes V_2 \otimes W_1 \otimes W_2 \\
s_2 &: V_1 \otimes V_2 \otimes U_1 \otimes U_2 \rightarrow V_1 \otimes U_1 \otimes V_2 \otimes U_2.
\end{aligned}$$

THEOREM 2.35. *For finite-dimensional  $V_1, V_2$ , and maps  $A : V_1 \otimes U_1 \rightarrow V_1 \otimes W_1$  and  $B : V_2 \otimes U_2 \rightarrow V_2 \otimes W_2$ ,*

$$Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}(s_1 \circ (j_3(A \otimes B)) \circ s_2) = j_4((Tr_{V_1; U_1, W_1}(A)) \otimes (Tr_{V_2; U_2, W_2}(B))).$$

PROOF. In the following diagram,

$$\begin{array}{ccccccc}
 M_{11} & \xrightarrow{a_1} & M_{12} & \xleftarrow{a_2} & M_{13} & \xleftarrow{[j \otimes j] \circ s_3} & M_{14} \\
 \downarrow j_1 & & \downarrow j_5 & & \downarrow j_6 & & \downarrow [j_1^1 \otimes j_1^2] \\
 M_{21} & \xrightarrow{a_3} & M_{22} & \xleftarrow{a_4} & M_{23} & \xleftarrow{\text{Hom}(s_4, s_5) \circ j_7} & M_{24} \\
 \uparrow a_5 & & \uparrow a_6 & & \uparrow a_7 & & \uparrow a_8 \\
 a_5^{-1} \downarrow & & a_6^{-1} \downarrow & & a_7^{-1} \downarrow & & a_8^{-1} \downarrow \\
 M_{31} & \xrightarrow{a_9} & M_{32} & \xleftarrow{a_{10}} & M_{33} & \xleftarrow{j_8} & M_{34}
 \end{array}$$

the objects are

$$\begin{aligned}
 M_{11} &= \text{End}(V_1 \otimes V_2)^* \otimes \text{End}(\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{21} &= \text{Hom}(\text{End}(V_1 \otimes V_2) \otimes \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2), \\
 &\quad \mathbb{K} \otimes \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{31} &= \text{Hom}(\text{Hom}(V_1 \otimes V_2 \otimes U_1 \otimes U_2, V_1 \otimes V_2 \otimes W_1 \otimes W_2), \\
 &\quad \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{12} &= (\text{End}(V_1) \otimes \text{End}(V_2))^* \otimes \\
 &\quad \text{Hom}(\text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2), \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{22} &= \text{Hom}(\text{End}(V_1) \otimes \text{End}(V_2) \otimes \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2), \\
 &\quad \mathbb{K} \otimes \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{32} &= \text{Hom}(\text{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \\
 &\quad \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
 M_{13} &= \text{Hom}(\text{End}(V_1) \otimes \text{End}(V_2), \mathbb{K} \otimes \mathbb{K}) \otimes \text{End}(\text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2)) \\
 M_{23} &= \text{Hom}(\text{End}(V_1) \otimes \text{End}(V_2) \otimes \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2), \\
 &\quad \mathbb{K} \otimes \mathbb{K} \otimes \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2)) \\
 M_{33} &= \text{Hom}(\text{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \\
 &\quad \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2)) \\
 M_{14} &= \text{End}(V_1)^* \otimes \text{End}(\text{Hom}(U_1, W_1)) \otimes \text{End}(V_2)^* \otimes \text{End}(\text{Hom}(U_2, W_2)) \\
 M_{24} &= \text{Hom}(\text{End}(V_1) \otimes \text{Hom}(U_1, W_1), \mathbb{K} \otimes \text{Hom}(U_1, W_1)) \otimes \\
 &\quad \text{Hom}(\text{End}(V_2) \otimes \text{Hom}(U_2, W_2), \mathbb{K} \otimes \text{Hom}(U_2, W_2)) \\
 M_{34} &= \text{Hom}(\text{Hom}(V_1 \otimes U_1, V_1 \otimes W_1), \text{Hom}(U_1, W_1)) \otimes \\
 &\quad \text{Hom}(\text{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \text{Hom}(U_2, W_2));
 \end{aligned}$$



the left, right columns define  $Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}$  and  $Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2}$ . The arrows are

$$\begin{aligned}
a_1 &= [(j'_2)^* \otimes \text{Hom}(j_4, Id_{\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)})] \\
a_2 &= [\text{Hom}(Id_{\text{End}(V_1) \otimes \text{End}(V_2)}, l_{\mathbb{K}}) \otimes \text{Hom}(Id_{\text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2)}, j_4)] \\
a_3 &= \text{Hom}([j'_2 \otimes j_4], Id_{\mathbb{K} \otimes \text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
a_4 &= \text{Hom}(Id_{\text{End}(V_1) \otimes \text{End}(V_2) \otimes \text{Hom}(U_1, W_1) \otimes \text{Hom}(U_2, W_2)}, [l_{\mathbb{K}} \otimes j_4]) \\
a_5 &= \text{Hom}(j_2, l_1^{-1}) \\
a_6 &= \text{Hom}([j_2^1 \otimes j_2^2] \circ s_4, l_1^{-1}) \\
a_7 &= \text{Hom}([j_2^1 \otimes j_2^2] \circ s_4, (l \circ l)^{-1}) \\
a_8 &= [\text{Hom}(j_2^1, (l_1^1)^{-1}) \otimes \text{Hom}(j_2^2, (l_1^2)^{-1})] \\
a_9 &= \text{Hom}(\text{Hom}(s_2, s_1) \circ j_3, Id_{\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
a_{10} &= \text{Hom}(Id_{\text{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2 \otimes U_2, V_2 \otimes W_2)}, j_4).
\end{aligned}$$

The Theorem claims the two maps

$$\begin{aligned}
Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2} \circ \text{Hom}(s_2, s_1) \circ j_3 &= a_9(Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}), \\
j_4 \circ (j_8(Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2})) &= (a_{10} \circ j_8)(Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2})
\end{aligned}$$

are equal. The diagram is commutative— all six squares are easy to check, for example, the upper left and upper middle follow from Lemma 1.26, and each of the remaining four involves two arrows with switching maps. The equality along the top row,

$$\begin{aligned}
a_1 &: Tr_{V_1 \otimes V_2} \otimes Id_{\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)} \mapsto (Tr_{V_1 \otimes V_2} \circ j'_2) \otimes j_4, \\
a_2 \circ [j \otimes j] \circ s_3 &: Tr_{V_1} \otimes Id_{\text{Hom}(U_1, W_1)} \otimes Tr_{V_2} \otimes Id_{\text{Hom}(U_2, W_2)} \\
&\mapsto (l_{\mathbb{K}} \circ [Tr_{V_1} \otimes Tr_{V_2}]) \otimes j_4,
\end{aligned}$$

follows from Corollary 2.31. This key step, together with the commutativity of the diagram, proves the Theorem:

$$\begin{aligned}
a_9 &: (Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}) \\
&\mapsto (a_9 \circ a_5^{-1} \circ j_1)(Tr_{V_1 \otimes V_2} \otimes Id_{\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
&= (a_6^{-1} \circ j_5 \circ a_1)(Tr_{V_1 \otimes V_2} \otimes Id_{\text{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
&= (a_6^{-1} \circ j_5 \circ a_2 \circ [j \otimes j] \circ s_3)(Tr_{V_1} \otimes Id_{\text{Hom}(U_1, W_1)} \otimes Tr_{V_2} \otimes Id_{\text{Hom}(U_2, W_2)}) \\
&= (a_{10} \circ j_8 \circ a_8^{-1} \circ [j_1^1 \otimes j_1^2])(Tr_{V_1} \otimes Id_{\text{Hom}(U_1, W_1)} \otimes Tr_{V_2} \otimes Id_{\text{Hom}(U_2, W_2)}) \\
&= (a_{10} \circ j_8)(Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2}).
\end{aligned}$$

■

REMARK 2.36. The maps  $j_4$ , from the previous Theorem, and

$$j_0 : \text{Hom}(V \otimes U_1, V \otimes W_1) \otimes \text{Hom}(U_2, W_2) \rightarrow \text{Hom}(V \otimes U_1 \otimes U_2, V \otimes W_1 \otimes W_2)$$

appear in the following Corollary about the compatibility of the trace and the tensor product, related to a “superposing” identity of [JSV].

COROLLARY 2.37. For  $A : V \otimes U_1 \rightarrow V \otimes W_1$  and  $B : U_2 \rightarrow W_2$ ,

$$\text{Tr}_{V;U_1 \otimes U_2, W_1 \otimes W_2}(j_0(A \otimes B)) = j_4((\text{Tr}_{V;U_1, W_1}(A)) \otimes B).$$

PROOF. It can be checked that the following diagram is commutative:

$$\begin{array}{ccc} V \otimes U_1 \otimes \mathbb{K} \otimes U_2 & \xrightarrow{j_3(A \otimes (l_{W_2}^{-1} \circ B \circ l_{U_2}))} & V \otimes W_1 \otimes \mathbb{K} \otimes W_2 \\ \uparrow s_2 & & \downarrow s_1 \\ V \otimes \mathbb{K} \otimes U_1 \otimes U_2 & & V \otimes \mathbb{K} \otimes W_1 \otimes W_2 \\ \downarrow [l_V \otimes \text{Id}_{U_1 \otimes U_2}] & & \downarrow [l_V \otimes \text{Id}_{W_1 \otimes W_2}] \\ V \otimes U_1 \otimes U_2 & \xrightarrow{j_0(A \otimes B)} & V \otimes W_1 \otimes W_2 \end{array}$$

Theorem 2.25, the diagram, the previous Theorem, and finally Theorem 2.23 apply:

$$\begin{aligned} LHS &= \text{Tr}_{V;U_1 \otimes U_2, W_1 \otimes W_2}((j_0(A \otimes B)) \circ [l_V \otimes \text{Id}_{U_1 \otimes U_2}] \circ [l_V^{-1} \otimes \text{Id}_{U_1 \otimes U_2}]) \\ &= \text{Tr}_{V \otimes \mathbb{K}; U_1 \otimes U_2, W_1 \otimes W_2}([l_V^{-1} \otimes \text{Id}_{W_1 \otimes W_2}] \circ (j_0(A \otimes B)) \circ [l_V \otimes \text{Id}_{U_1 \otimes U_2}]) \\ &= \text{Tr}_{V \otimes \mathbb{K}; U_1 \otimes U_2, W_1 \otimes W_2}(s_1 \circ (j_3(A \otimes (l_{W_2}^{-1} \circ B \circ l_{U_2}))) \circ s_2) \\ &= j_4((\text{Tr}_{V;U_1, W_1}(A)) \otimes (\text{Tr}_{\mathbb{K}; U_2, W_2}(\text{Hom}(l_{U_2}, l_{W_2}^{-1})(B)))) \\ &= j_4((\text{Tr}_{V;U_1, W_1}(A)) \otimes B). \end{aligned}$$

■

NOTATION 2.38. Denote the composite maps

$$\begin{aligned} \tilde{j}_U &= \text{Hom}(\text{Id}_{V \otimes U}, l) \circ j : V^* \otimes U^* \rightarrow (V \otimes U)^* \\ \tilde{j}_W &= \text{Hom}(\text{Id}_{V \otimes W}, l) \circ j : V^* \otimes W^* \rightarrow (V \otimes W)^*, \end{aligned}$$

where  $l$  is multiplication  $\mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$ .

These  $\tilde{j}$  maps are invertible if  $V$  is finite-dimensional. They appear in the next Theorem, relating the trace of the transpose to the transpose of the trace, and already appeared in Lemma 2.27. Such composites appear again in the next Chapter, but the tilde notation will only be used when abbreviating is more useful than not.

THEOREM 2.39. For finite-dimensional  $V$ ,  $H : V \otimes U \rightarrow V \otimes W$ , and  $t_{UV}$ ,  $t_{V \otimes U, V \otimes W}$  as in Notation 1.7,

$$\text{Tr}_{V^*; W^*, U^*}(\tilde{j}_U^{-1} \circ (t_{V \otimes U, V \otimes W}(H)) \circ \tilde{j}_W) = t_{UW}(\text{Tr}_{V; U, W}(H)).$$

PROOF. In the following diagram,

$$\begin{array}{ccccc} M_{11} & \xrightarrow{a_1} & M_{12} & \xleftarrow{a_2} & M_{13} \\ \downarrow j_1 & & \downarrow j_1'' & & \downarrow j_1' \\ M_{21} & \xrightarrow{a_3} & M_{22} & \xleftarrow{a_4} & M_{23} \\ \text{Hom}(j_2, l_1^{-1}) \updownarrow & & \text{Hom}(j_2^{-1}, l_1) \updownarrow & & \text{Hom}(j_2, (l_1')^{-1}) \updownarrow \\ M_{31} & \xrightarrow{a_5} & M_{32} & \xleftarrow{a_6} & M_{33} \end{array}$$

$\text{Hom}(j_2^{-1}, l_1') \updownarrow \quad \text{Hom}(j_2', (l_1')^{-1}) \updownarrow \quad \text{Hom}((j_2')^{-1}, l_1') \updownarrow$

the objects are

$$\begin{aligned}
M_{11} &= \text{End}(V)^* \otimes \text{End}(\text{Hom}(U, W)) \\
M_{21} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\
M_{31} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(U, W)) \\
M_{12} &= \text{End}(V)^* \otimes \text{Hom}(\text{Hom}(U, W), \text{Hom}(W^*, U^*)) \\
M_{22} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(W^*, U^*)) \\
M_{32} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(W^*, U^*)) \\
M_{13} &= \text{End}(V^*)^* \otimes \text{End}(\text{Hom}(W^*, U^*)) \\
M_{23} &= \text{Hom}(\text{End}(V^*) \otimes \text{Hom}(W^*, U^*), \mathbb{K} \otimes \text{Hom}(W^*, U^*)) \\
M_{33} &= \text{Hom}(\text{Hom}(V^* \otimes W^*, V^* \otimes U^*), \text{Hom}(W^*, U^*)),
\end{aligned}$$

the arrows are

$$\begin{aligned}
a_1 &= [Id_{\text{End}(V)^*} \otimes \text{Hom}(Id_{\text{Hom}(U, W)}, t_{UW})] \\
a_2 &= [t_{VV}^* \otimes \text{Hom}(t_{UW}, Id_{\text{Hom}(W^*, U^*)})] \\
a_3 &= \text{Hom}(Id_{\text{End}(V) \otimes \text{Hom}(U, W)}, [Id_{\mathbb{K}} \otimes t_{UW}]) \\
a_4 &= \text{Hom}([t_{VV} \otimes t_{UW}], Id_{\mathbb{K} \otimes \text{Hom}(W^*, U^*)}) \\
a_5 &= \text{Hom}(Id_{\text{Hom}(V \otimes U, V \otimes W)}, t_{UW}) \\
a_6 &= \text{Hom}(\text{Hom}(\tilde{j}_W, \tilde{j}_U^{-1}) \circ t_{V \otimes U, V \otimes W}, Id_{\text{Hom}(W^*, U^*)}),
\end{aligned}$$

and the two quantities in the statement of the Theorem are

$$\begin{aligned}
t_{UW}(Tr_{V;U,W}(H)) &= (a_5(Tr_{V;U,W}))(H) \\
Tr_{V^*;W^*,U^*}(\text{Hom}(\tilde{j}_W, \tilde{j}_U^{-1})(t_{V \otimes U, V \otimes W}(H))) &= (a_6(Tr_{V^*;W^*,U^*}))(H).
\end{aligned}$$

The diagram is commutative, for example, the lower right square:

$$\begin{aligned}
E \otimes F &\mapsto (\text{Hom}(Id_{V^* \otimes W^*}, \tilde{j}_U) \circ j_2' \circ [t_{VV} \otimes t_{UW}])(E \otimes F) \\
&= \tilde{j}_U \circ (j_2'((t_{VV}(E)) \otimes (t_{UW}(F)))) : \\
\phi \otimes \xi &\mapsto \tilde{j}_U((E^*(\phi)) \otimes (F^*(\xi))) : \\
v \otimes u &\mapsto \phi(E(v)) \cdot \xi(F(u)), \\
E \otimes F &\mapsto (\text{Hom}(\tilde{j}_W, Id_{V^* \otimes U^*}) \circ t_{V \otimes U, V \otimes W} \circ j_2)(E \otimes F) \\
&= (t_{V \otimes U, V \otimes W}(j_2(E \otimes F))) \circ \tilde{j}_W : \\
\phi \otimes \xi &\mapsto (\tilde{j}_W(\phi \otimes \xi)) \circ (j_2(E \otimes F)) : \\
v \otimes u &\mapsto \phi(E(v)) \cdot \xi(F(u)).
\end{aligned}$$

Lemma 2.5 implies the equality of the images of the distinguished elements in the top row:

$$\begin{aligned}
a_1(Tr_V \otimes Id_{\text{Hom}(U, W)}) &= Tr_V \otimes t_{UW} \\
a_2(Tr_{V^*} \otimes Id_{\text{Hom}(W^*, U^*)}) &= (t_{VV}^*(Tr_{V^*})) \otimes t_{UW} \\
&= Tr_V \otimes t_{UW},
\end{aligned}$$

and the Theorem follows from the commutativity of the diagram:

$$\begin{aligned}
 a_5(Tr_{V;U,W}) &= (a_5 \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(U,W)}) \\
 &= (\text{Hom}(j_2^{-1}, l_1') \circ j_1'' \circ a_1)(Tr_V \otimes Id_{\text{Hom}(U,W)}) \\
 &= (\text{Hom}(j_2^{-1}, l_1') \circ j_1'' \circ a_2)(Tr_{V^*} \otimes Id_{\text{Hom}(W^*, U^*)}) \\
 &= (a_6 \circ \text{Hom}((j_2')^{-1}, l_1') \circ j_1')(Tr_{V^*} \otimes Id_{\text{Hom}(W^*, U^*)}) \\
 &= a_6(Tr_{V^*;W^*,U^*}).
 \end{aligned}$$

■

EXERCISE 2.40. Given a direct sum  $V = V_1 \oplus V_2$ , and  $A : V_1 \otimes U \rightarrow V_1 \otimes W$ , and  $B : V_2 \otimes U \rightarrow V_2 \otimes W$ , define  $A \oplus B : V \otimes U \rightarrow V \otimes W$  using the projections and inclusions from Example 1.45:

$$A \oplus B = [Q_1 \otimes Id_W] \circ A \circ [P_1 \otimes Id_U] + [Q_2 \otimes Id_W] \circ B \circ [P_2 \otimes Id_U].$$

If  $V$  is finite-dimensional, then

$$Tr_{V;U,W}(A \oplus B) = Tr_{V_1;U,W}(A) + Tr_{V_2;U,W}(B).$$

HINT. The proof proceeds exactly as in Proposition 2.12, using Theorem 2.25.

■

EXERCISE 2.41. For  $V = V_1 \oplus V_2$  as above, and  $K : V \otimes U \rightarrow V \otimes W$ ,

$$\begin{aligned}
 Tr_{V;U,W}(K) &= Tr_{V_1;U,W}([P_1 \otimes Id_W] \circ K \circ [Q_1 \otimes Id_U]) \\
 &\quad + Tr_{V_2;U,W}([P_2 \otimes Id_W] \circ K \circ [Q_2 \otimes Id_U]).
 \end{aligned}$$

HINT. Using Theorem 2.25 and Lemma 1.25,

$$Tr_{V_i;U,W}([P_i \otimes Id_W] \circ K \circ [Q_i \otimes Id_U]) = Tr_{V;U,W}([(Q_i \circ P_i) \otimes Id_W] \circ K).$$

The proof proceeds exactly as in Proposition 2.13.

■

PROPOSITION 2.42. For finite-dimensional  $V_1, V_2$ , maps  $A : V_1 \otimes U_1 \rightarrow V_2 \otimes W_2$ ,  $B : V_2 \otimes U_2 \rightarrow V_1 \otimes W_1$ , and switching maps as in the following diagrams,

$$\begin{array}{ccc}
 V_1 \otimes U_2 \otimes U_1 & & V_2 \otimes U_2 \otimes U_1 \\
 \downarrow s_1 & & \downarrow [B \otimes Id_{U_1}] \\
 V_1 \otimes U_1 \otimes U_2 & & V_1 \otimes W_1 \otimes U_1 \\
 \downarrow [A \otimes Id_{U_2}] & & \downarrow s_3 \\
 V_2 \otimes W_2 \otimes U_2 & & V_1 \otimes U_1 \otimes W_1 \\
 \downarrow s_2 & & \downarrow [A \otimes Id_{W_1}] \\
 V_2 \otimes U_2 \otimes W_2 & & V_2 \otimes W_2 \otimes W_1 \\
 \downarrow [B \otimes Id_{W_2}] & & \downarrow s_4 \\
 V_1 \otimes W_1 \otimes W_2 & & V_2 \otimes W_1 \otimes W_2
 \end{array}$$

the traces of the composites are equal:

$$\begin{aligned}
 &Tr_{V_1;U_2 \otimes U_1, W_1 \otimes W_2}([B \otimes Id_{W_2}] \circ s_2 \circ [A \otimes Id_{U_2}] \circ s_1) \\
 &= Tr_{V_2;U_2 \otimes U_1, W_1 \otimes W_2}(s_4 \circ [A \otimes Id_{W_1}] \circ s_3 \circ [B \otimes Id_{U_1}]).
 \end{aligned}$$

PROOF. Since  $j : \text{Hom}(V_2, V_1) \otimes \text{Hom}(U_2, W_1) \rightarrow \text{Hom}(V_2 \otimes U_2, V_1 \otimes W_1)$  is invertible by the finite-dimensionality hypothesis and Lemma 1.23, it is enough, by the linearity of the above expressions, to check the claim for maps  $B$  of the form  $j(B_1 \otimes B_2)$ , for  $B_1 : V_2 \rightarrow V_1$ ,  $B_2 : U_2 \rightarrow W_1$ .

The following easily checked diagram shows the  $B_2$  factor commutes with  $A$ .

$$\begin{array}{ccc}
 V_1 \otimes U_2 \otimes U_1 & \xrightarrow{[Id_{V_1} \otimes [B_2 \otimes Id_{U_1}]]} & V_1 \otimes W_1 \otimes U_1 \\
 \downarrow s_1 & & \downarrow s_3 \\
 V_1 \otimes U_1 \otimes U_2 & \xrightarrow{[Id_{V_1} \otimes U_1 \otimes B_2]} & V_1 \otimes U_1 \otimes W_1 \\
 \downarrow [A \otimes Id_{U_2}] & & \downarrow [A \otimes Id_{W_1}] \\
 V_2 \otimes W_2 \otimes U_2 & \xrightarrow{[Id_{V_2} \otimes W_2 \otimes B_2]} & V_2 \otimes W_2 \otimes W_1 \\
 \downarrow s_2 & & \downarrow s_4 \\
 V_2 \otimes U_2 \otimes W_2 & \xrightarrow{[Id_{V_2} \otimes [B_2 \otimes Id_{W_2}]]} & V_2 \otimes W_1 \otimes W_2
 \end{array}$$

Using Theorem 2.25,

$$\begin{aligned}
 & LHS \\
 &= Tr_{V_1; U_2 \otimes U_1, W_1 \otimes W_2} ([ [B_1 \otimes B_2] \otimes Id_{W_2} ] \circ s_2 \circ [A \otimes Id_{U_2}] \circ s_1 ) \\
 &= Tr_{V_1; U_2 \otimes U_1, W_1 \otimes W_2} ([B_1 \otimes Id_{W_1 \otimes W_2}] \circ [Id_{V_2} \otimes [B_2 \otimes Id_{W_2}]] \circ s_2 \circ [A \otimes Id_{U_2}] \circ s_1 ) \\
 &= Tr_{V_1; U_2 \otimes U_1, W_1 \otimes W_2} ([B_1 \otimes Id_{W_1 \otimes W_2}] \circ s_4 \circ [A \otimes Id_{W_1}] \circ s_3 \circ [Id_{V_1} \otimes [B_2 \otimes Id_{U_1}]] ) \\
 &= Tr_{V_2; U_2 \otimes U_1, W_1 \otimes W_2} (s_4 \circ [A \otimes Id_{W_1}] \circ s_3 \circ [Id_{V_1} \otimes [B_2 \otimes Id_{U_1}]] \circ [B_1 \otimes Id_{U_2 \otimes U_1}]) \\
 &= Tr_{V_2; U_2 \otimes U_1, W_1 \otimes W_2} (s_4 \circ [A \otimes Id_{W_1}] \circ s_3 \circ [[B_1 \otimes B_2] \otimes Id_{U_1}]) \\
 &= RHS.
 \end{aligned}$$

■

EXERCISE 2.43. Using various switching maps, Proposition 2.42 can be used to prove related identities. For example, given maps:

$$\begin{aligned}
 A' : U_1 \otimes V_1 &\rightarrow V_2 \otimes W_2 \\
 B' : U_2 \otimes V_2 &\rightarrow V_1 \otimes W_1 \\
 s_5 : V_1 \otimes U_2 \otimes U_1 &\rightarrow U_2 \otimes U_1 \otimes V_1 \\
 s_6 : V_2 \otimes U_2 \otimes U_1 &\rightarrow U_1 \otimes U_2 \otimes V_2,
 \end{aligned}$$

the following identity can be proved as a consequence of Proposition 2.42:

$$\begin{aligned}
 & Tr_{V_1; U_2 \otimes U_1, W_1 \otimes W_2} ([B' \otimes Id_{W_2}] \circ [Id_{U_2} \otimes A'] \circ s_5) \\
 (2.2) \quad &= Tr_{V_2; U_2 \otimes U_1, W_1 \otimes W_2} (s_4 \circ [A' \otimes Id_{W_1}] \circ [Id_{U_1} \otimes B'] \circ s_6).
 \end{aligned}$$

HINT. Let  $A = A' \circ s$  and  $B = B' \circ s$  for appropriate switching maps  $s$ . ■

REMARK 2.44. Equation (2.2) is related to [PS] Proposition 2.7, on the “cyclicity” of the generalized trace.

### 2.3. Vector-valued trace

When there is no space  $U$ , the “vector-valued” or “ $W$ -valued” trace of a map  $V \rightarrow V \otimes W$  should be an element of  $W$ . The results on the generalized trace have analogues in this case, but the construction uses different canonical maps.

DEFINITION 2.45. For any vector space  $W$ , define  $m : W \rightarrow \text{Hom}(\mathbb{K}, W)$  so that for  $w \in W$ ,

$$m(w) : \lambda \mapsto \lambda \cdot w.$$

LEMMA 2.46.  $m$  is invertible.

PROOF. An inverse is  $m^{-1} = d_{\mathbb{K}W}(1) : \text{Hom}(\mathbb{K}, W) \rightarrow W$ , so that  $m^{-1} : A \mapsto A(1)$ ,  $((m \circ m^{-1})(A))(\lambda) = \lambda \cdot A(1) = A(\lambda)$ , and  $(m^{-1} \circ m)(w) = (m(w))(1) = 1 \cdot w = w$ .  $\blacksquare$

DEFINITION 2.47. For arbitrary vector spaces  $U, V, W$ , define

$$n : \text{Hom}(U, V) \otimes W \rightarrow \text{Hom}(U, V \otimes W)$$

so that for  $A : U \rightarrow V$ ,  $w \in W$ ,  $u \in U$ ,

$$n(A \otimes w) : u \mapsto (A(u)) \otimes w.$$

LEMMA 2.48. ([B] §II.7.7, [AF] §20) If  $U$  or  $W$  is finite-dimensional, then  $n$  is invertible.  $\blacksquare$

The  $n$  map is related to a canonical  $j$  map:

LEMMA 2.49. The following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}(U, V) \otimes W & \xrightarrow{n} & \text{Hom}(U, V \otimes W) \\ \downarrow [Id_{\text{Hom}(U, V)} \otimes m] & & \downarrow \text{Hom}(l_U, Id_{V \otimes W}) \\ \text{Hom}(U, V) \otimes \text{Hom}(\mathbb{K}, W) & \xrightarrow{j} & \text{Hom}(U \otimes \mathbb{K}, V \otimes W) \end{array}$$

PROOF. Starting with  $A \otimes w \in \text{Hom}(U, V) \otimes W$ ,

$$\begin{aligned} A \otimes w &\mapsto (j \circ [Id_{\text{Hom}(U, V)} \otimes m])(A \otimes w) \\ &= j(A \otimes (m(w))) : \\ u \otimes \lambda &\mapsto (A(u)) \otimes (\lambda \cdot w) = \lambda \cdot (A(u)) \otimes w, \\ A \otimes w &\mapsto (\text{Hom}(l_U, Id_{V \otimes W}) \circ n)(A \otimes w) \\ &= (n(A \otimes w)) \circ l_U : \\ u \otimes \lambda &\mapsto (A(\lambda \cdot u)) \otimes w = \lambda \cdot (A(u)) \otimes w. \end{aligned}$$

$\blacksquare$

The  $W$ -valued trace  $Tr_{V;W} : \text{Hom}(V, V \otimes W) \rightarrow W$  is defined for arbitrary  $W$  and finite-dimensional  $V$ , using the canonical maps

$$\begin{aligned} n &: \text{End}(V) \otimes W \rightarrow \text{Hom}(V, V \otimes W) \\ j'_1 &: \text{End}(V)^* \otimes \text{End}(W) \rightarrow \text{Hom}(\text{End}(V) \otimes W, \mathbb{K} \otimes W), \end{aligned}$$

where  $j'_1$  is another canonical  $j$  map in analogy with  $j_1$  from Definition 2.21, and  $n$  is invertible.

DEFINITION 2.50.  $Tr_{V;W} = (\text{Hom}(n^{-1}, l_W) \circ j'_1)(Tr_V \otimes Id_W)$ .

EXAMPLE 2.51. A map of the form  $n(A \otimes w) : V \rightarrow V \otimes W$ , for  $A : V \rightarrow V$  and  $w \in W$ , has trace

$$Tr_{V;W}(n(A \otimes w)) = l_W((j'_1(Tr_V \otimes Id_W))(A \otimes w)) = Tr_V(A) \cdot w.$$

The map  $q$  from Definition 1.39 and the following map will be used to relate Definition 2.50 to Definition 2.21.

NOTATION 2.52. Let  $n' : V \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(U, V \otimes W)$  be another  $n$  map, so that

$$(n'(v \otimes E)) : u \mapsto v \otimes (E(u)).$$

The order of the factors differs in this case from Definition 2.47, but such maps will still have “ $n$ ” labels.

Consider the vector-valued trace, in the case where  $W$  is replaced by the vector space  $\text{Hom}(U, W)$ .

THEOREM 2.53. For finite-dimensional  $V$ ,  $K : V \rightarrow V \otimes \text{Hom}(U, W)$ , and a map  $q : \text{Hom}(V, \text{Hom}(U, V \otimes W)) \rightarrow \text{Hom}(V \otimes U, V \otimes W)$ ,

$$Tr_{V; \text{Hom}(U, W)}(K) = Tr_{V; U, W}(q(n' \circ K)).$$

PROOF. In the following diagram,

$$\begin{array}{ccc} M_{11} & \xrightleftharpoons[\text{Hom}(n, l_1^{-1})]{\text{Hom}(n^{-1}, l_1)} & M_{12} \\ \text{Hom}(j_2, l_1^{-1}) \updownarrow & \text{Hom}(j_2^{-1}, l_1) & \uparrow \text{Hom}(\text{Hom}(Id_V, n'), Id_{\text{Hom}(U, W)}) \\ M_{21} & \xrightarrow{\text{Hom}(q, Id_{\text{Hom}(U, W)})} & M_{22} \end{array}$$

the objects are

$$\begin{aligned} M_{11} &= \text{Hom}(\text{End}(V) \otimes \text{Hom}(U, W), \mathbb{K} \otimes \text{Hom}(U, W)) \\ M_{21} &= \text{Hom}(\text{Hom}(V \otimes U, V \otimes W), \text{Hom}(U, W)) \\ M_{12} &= \text{Hom}(\text{Hom}(V, V \otimes \text{Hom}(U, W)), \text{Hom}(U, W)) \\ M_{22} &= \text{Hom}(\text{Hom}(V, \text{Hom}(U, V \otimes W)), \text{Hom}(U, W)). \end{aligned}$$

The square is commutative since

$$\begin{aligned} E \otimes F &\mapsto (q \circ \text{Hom}(Id_V, n') \circ n)(E \otimes F) = q(n' \circ (n(E \otimes F))) : \\ v \otimes u &\mapsto (n'((n(E \otimes F))(v)))(u) \\ &= (n'((E(v)) \otimes F))(u) \\ &= (E(v)) \otimes (F(u)) \\ &= (j_2(E \otimes F))(v \otimes u). \end{aligned}$$

Definitions 2.21 and 2.50 are related in this case by  $l_W = l_1$  and  $j'_1 = j_1$ . The Theorem follows:

$$\begin{aligned} Tr_{V; \text{Hom}(U, W)} &= (\text{Hom}(n^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(U, W)}) \\ &= (\text{Hom}(j_2^{-1} \circ q \circ \text{Hom}(Id_V, n'), l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(U, W)}) \\ &= \text{Hom}(q \circ \text{Hom}(Id_V, n'), Id_{\text{Hom}(U, W)})(Tr_{V; U, W}). \end{aligned}$$

■

Definition 2.50 is related to the original definition of trace when  $W = \mathbb{K}$ .

THEOREM 2.54. *For  $H : V \rightarrow V \otimes \mathbb{K}$ ,  $Tr_V(l_V \circ H) = Tr_{V;\mathbb{K}}(H)$ .*

PROOF. Let  $l_2 : \text{End}(V)^* \otimes \mathbb{K} \rightarrow \text{End}(V)^*$  be another scalar multiplication map. The following diagram is commutative:

$$\begin{array}{ccc}
 \text{End}(V)^* \otimes \mathbb{K}^* & \xleftarrow{a_1} & \text{End}(V)^* \otimes \mathbb{K} \\
 \downarrow j'_1 & & \downarrow l_2 \\
 \text{Hom}(\text{End}(V) \otimes \mathbb{K}, \mathbb{K} \otimes \mathbb{K}) & & \\
 \uparrow \text{Hom}(n, l_{\mathbb{K}}^{-1}) \quad \downarrow \text{Hom}(n^{-1}, l_{\mathbb{K}}) & & \\
 \text{Hom}(V, V \otimes \mathbb{K})^* & \xleftarrow{a_2} & \text{End}(V)^*
 \end{array}$$

$$\begin{aligned}
 a_1 &= [Id_{\text{End}(V)^*} \otimes m] \\
 a_2 &= \text{Hom}(Id_V, l_V)^*.
 \end{aligned}$$

For  $\lambda, \mu \in \mathbb{K}$ ,

$$\begin{aligned}
 \Phi \otimes \lambda &\mapsto (j'_1 \circ a_1)(\Phi \otimes \lambda) = j'_1(\Phi \otimes (m(\lambda))) : \\
 A \otimes \mu &\mapsto (\Phi(A)) \otimes (\mu \cdot \lambda), \\
 \Phi \otimes \lambda &\mapsto (\text{Hom}(n, l_{\mathbb{K}}^{-1}) \circ a_2 \circ l_2)(\Phi \otimes \lambda) = l_{\mathbb{K}}^{-1} \circ ((\lambda \cdot \Phi) \circ \text{Hom}(Id_V, l_V)) \circ n : \\
 A \otimes \mu &\mapsto l_{\mathbb{K}}^{-1}((\lambda \cdot \Phi)(l_V \circ (n(A \otimes \mu)))) = 1 \otimes (\lambda \cdot \Phi(\mu \cdot A)),
 \end{aligned}$$

since  $(l_V \circ (n(A \otimes \mu))) : v \mapsto l_V((A(v)) \otimes \mu) = (\mu \cdot A)(v)$ . The Theorem follows from  $a_1(Tr_V \otimes 1) = Tr_V \otimes Id_{\mathbb{K}}$ :

$$\begin{aligned}
 a_2(Tr_V) &= (a_2 \circ l_2)(Tr_V \otimes 1) \\
 &= (\text{Hom}(n^{-1}, l_{\mathbb{K}}) \circ j'_1 \circ a_1)(Tr_V \otimes 1) \\
 &= Tr_{V;\mathbb{K}}.
 \end{aligned}$$

■

Definitions 2.21 and 2.50 are related in the case  $U = \mathbb{K}$ :

THEOREM 2.55. *For  $H : V \rightarrow V \otimes W$ ,  $Tr_{V;\mathbb{K},W}(H \circ l_V) : 1 \mapsto Tr_{V;W}(H)$ .*

PROOF. The following diagram is commutative:

$$\begin{array}{ccc}
 \text{End}(V)^* \otimes \text{End}(\text{Hom}(\mathbb{K}, W)) & \xrightarrow{a_1} & \text{End}(V)^* \otimes \text{End}(W) \\
 \downarrow j_1 & & \downarrow j'_1 \\
 \text{Hom}(\text{End}(V) \otimes \text{Hom}(\mathbb{K}, W), \mathbb{K} \otimes \text{Hom}(\mathbb{K}, W)) & \xrightarrow{a_2} & \text{Hom}(\text{End}(V) \otimes W, \mathbb{K} \otimes W) \\
 \uparrow \text{Hom}(j_2, l_1^{-1}) \quad \downarrow \text{Hom}(j_2^{-1}, l_1) & & \uparrow \text{Hom}(n, l_W^{-1}) \quad \downarrow \text{Hom}(n^{-1}, l_W) \\
 \text{Hom}(\text{Hom}(V \otimes \mathbb{K}, V \otimes W), \text{Hom}(\mathbb{K}, W)) & \xrightarrow{a_3} & \text{Hom}(\text{Hom}(V, V \otimes W), W)
 \end{array}$$

where the horizontal arrows are

$$\begin{aligned}
 a_1 &= [Id_{\text{End}(V)^*} \otimes \text{Hom}(m, m^{-1})] \\
 a_2 &= \text{Hom}([Id_{\text{End}(V)} \otimes m], [Id_{\mathbb{K}} \otimes m^{-1}]) \\
 a_3 &= \text{Hom}(\text{Hom}(l_V, Id_{V \otimes W}), m^{-1}).
 \end{aligned}$$



The statement of the Theorem becomes

$$a_3(Tr_{V;\mathbb{K},W}) = Tr_{V;W}.$$

The top square is commutative by Lemma 1.26. The lower square is commutative by Lemma 2.49, and the identity  $m \circ l_W = l_1 \circ [Id_{\mathbb{K}} \otimes m] : \mathbb{K} \otimes W \rightarrow \text{Hom}(\mathbb{K}, W)$ . The Theorem follows from  $a_1(Tr_V \otimes Id_{\text{Hom}(\mathbb{K}, W)}) = Tr_V \otimes Id_W$ :

$$\begin{aligned} a_3(Tr_{V;\mathbb{K},W}) &= (a_3 \circ \text{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\text{Hom}(\mathbb{K}, W)}) \\ &= (\text{Hom}(n^{-1}, l_W) \circ j'_1 \circ a_1)(Tr_V \otimes Id_{\text{Hom}(\mathbb{K}, W)}) \\ &= (\text{Hom}(n^{-1}, l_W) \circ j'_1)(Tr_V \otimes Id_W) \\ &= Tr_{V;W}. \end{aligned}$$

■

The following results on the  $W$ -valued trace are corollaries of results from the previous Section. In most cases, Theorem 2.55 applies, leading to a straightforward calculation.

COROLLARY 2.56. For  $A : V \rightarrow V'$ ,  $B : V' \rightarrow V \otimes W$ ,

$$Tr_{V;W}(B \circ A) = Tr_{V';W}((j_W(A \otimes Id_W)) \circ B).$$

PROOF. Theorems 2.25 and 2.55 apply, using the map  $l_{V'} : V' \otimes \mathbb{K} \rightarrow V'$ , and the scalar multiplication identity  $A \circ l_V = l_{V'} \circ (j_{\mathbb{K}}(A \otimes Id_{\mathbb{K}}))$ .

$$\begin{aligned} Tr_{V;W}(B \circ A) &= (Tr_{V';\mathbb{K},W}(B \circ A \circ l_V))(1) \\ &= (Tr_{V;\mathbb{K},W}(B \circ l_{V'} \circ (j_{\mathbb{K}}(A \otimes Id_{\mathbb{K}}))))(1) \\ &= (Tr_{V';\mathbb{K},W}((j_W(A \otimes Id_W)) \circ B \circ l_{V'}))(1) \\ &= Tr_{V';W}((j_W(A \otimes Id_W)) \circ B). \end{aligned}$$

■

EXERCISE 2.57. For  $H : V \rightarrow V$ ,  $Tr_V(H) = (Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V))(1)$ .

HINT. By Theorems 2.54, 2.55,

$$Tr_V = \text{Hom}(Id_V, l_V^{-1})^*(\text{Hom}(\text{Hom}(l_V, Id_{V \otimes \mathbb{K}}), m^{-1})(Tr_{V;\mathbb{K},\mathbb{K}})).$$

The result could be proved directly by similar methods.

■

COROLLARY 2.58. For  $H : \mathbb{K} \rightarrow \mathbb{K} \otimes W$ ,  $Tr_{\mathbb{K};W}(H) = l_W(H(1))$ .

PROOF. Theorem 2.55 applies, with  $l_{\mathbb{K}} : \mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$ :

$$Tr_{\mathbb{K};W}(H) = (Tr_{\mathbb{K};\mathbb{K},W}(H \circ l_{\mathbb{K}}))(1).$$

By Theorem 2.23, this quantity is equal to

$$(\text{Hom}(l_{\mathbb{K}}^{-1}, l_W)(H \circ l_{\mathbb{K}}))(1) = l_W(H(1)).$$

■

COROLLARY 2.59. For  $A : V \rightarrow V \otimes W$ ,  $B : W \rightarrow W'$ ,

$$Tr_{V;W'}([Id_V \otimes B] \circ A) = B(Tr_{V;W}(A)).$$

PROOF. By Theorems 2.26 and 2.55,

$$\begin{aligned} Tr_{V;W'}([Id_V \otimes B] \circ A) &= (Tr_{V;\mathbb{K},W'}([Id_V \otimes B] \circ A \circ l_V))(1) \\ &= (B \circ (Tr_{V;\mathbb{K},W}(A \circ l_V)))(1) = B(Tr_{V;W}(A)). \end{aligned}$$

■

LEMMA 2.60. For a direct sum  $W = W_1 \oplus W_2$  with operators  $P_i, Q_i$ , there is also a direct sum  $\text{Hom}(V, V \otimes W) = \text{Hom}(V, V \otimes W_1) \oplus \text{Hom}(V, V \otimes W_2)$ , with projections  $\text{Hom}(Id_V, [Id_V \otimes P_i])$  and inclusions  $\text{Hom}(Id_V, [Id_V \otimes Q_i])$  as in Examples 1.45 and 1.46. The map  $Tr_{V;W} : \text{Hom}(V, V \otimes W) \rightarrow W$  respects the direct sums, and the induced map is equal to the  $W_i$ -valued trace:

$$Tr_{V;W_i} = P_i \circ Tr_{V;W} \circ \text{Hom}(Id_V, [Id_V \otimes Q_i]) : \text{Hom}(V, V \otimes W_i) \rightarrow W_i.$$

PROOF. Both results follow from Definition 1.52 and Corollary 2.59. ■

COROLLARY 2.61. For  $A : V \otimes V' \rightarrow V \otimes V' \otimes W$ ,

$$Tr_{V \otimes V';W}(A) = Tr_{V';W}(Tr_{V;V',V' \otimes W}(A)).$$

PROOF. Using Theorems 2.26, 2.33, and 2.55, and the scalar multiplication identity  $l_{V \otimes V'} = [Id_V \otimes l_{V'}] : V \otimes V' \otimes \mathbb{K} \rightarrow V \otimes V'$ ,

$$\begin{aligned} Tr_{V \otimes V';W}(A) &= (Tr_{V \otimes V';\mathbb{K},W}(A \circ l_{V \otimes V'}))(1) \\ &= (Tr_{V \otimes V';\mathbb{K},W}(A \circ [Id_V \otimes l_{V'}]))(1) \\ &= (Tr_{V';\mathbb{K},W}(Tr_{V;V' \otimes \mathbb{K},V' \otimes W}(A \circ [Id_V \otimes l_{V'}])))(1) \\ &= (Tr_{V';\mathbb{K},W}((Tr_{V;V',V' \otimes W}(A)) \circ l_{V'}))(1) \\ &= Tr_{V';W}(Tr_{V;V',V' \otimes W}(A)). \end{aligned}$$

■

EXERCISE 2.62. Denote by  $n''$  the map

$$\begin{aligned} n'' &: W \otimes \text{Hom}(U, W') \rightarrow \text{Hom}(U, W \otimes W') \\ &: w \otimes E \mapsto (u \mapsto w \otimes (E(u))). \end{aligned}$$

Then, for  $A : V \rightarrow V \otimes W$ ,  $B : U \rightarrow W'$ ,

$$Tr_{V;U,W \otimes W'}(j'_0(A \otimes B)) = n''((Tr_{V;W}(A)) \otimes B).$$

HINT. By Theorems 2.26, 2.55, Corollary 2.37, and the identity

$$(j'_0(A \otimes B)) \circ [Id_V \otimes l_U] = [A \otimes (B \circ l_U)] = [(A \circ l_V) \otimes B],$$

$$\begin{aligned} (Tr_{V;U,W \otimes W'}(j'_0(A \otimes B))) \circ l_U &= Tr_{V;\mathbb{K} \otimes U,W \otimes W'}([A \otimes (B \circ l_U)]) \\ &= Tr_{V;\mathbb{K} \otimes U,W \otimes W'}([(A \circ l_V) \otimes B]) \\ &= j_4((Tr_{V;\mathbb{K},W}(A \circ l_V)) \otimes B) \\ \implies Tr_{V;U,W \otimes W'}(j'_0(A \otimes B)) &= (j_4((Tr_{V;\mathbb{K},W}(A \circ l_V)) \otimes B)) \circ l_U^{-1} : \\ u &\mapsto ((Tr_{V;\mathbb{K},W}(A \circ l_V))(1)) \otimes (B(u)) \\ &= (Tr_{V;W}(A)) \otimes (B(u)) \\ &= (n''((Tr_{V;W}(A)) \otimes B))(u). \end{aligned}$$

COROLLARY 2.63. *For finite-dimensional  $V_1, V_2$ , maps  $A : V_1 \rightarrow V_1 \otimes W_1$ ,  $B : V_2 \rightarrow V_2 \otimes W_2$ , the switching involution  $s_1$ , and a canonical  $j$  map, as in Theorem 2.35:*

$$j'_3 : \text{Hom}(V_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2, V_2 \otimes W_2) \rightarrow \text{Hom}(V_1 \otimes V_2, V_1 \otimes V_2 \otimes W_1 \otimes W_2),$$

the following identity holds:

$$\text{Tr}_{V_1 \otimes V_2; W_1 \otimes W_2}(s_1 \circ (j'_3(A \otimes B))) = (\text{Tr}_{V_1; W_1}(A)) \otimes (\text{Tr}_{V_2; W_2}(B)) \in W_1 \otimes W_2.$$

PROOF. In the following diagram,

$$\begin{array}{ccccc} M_{11} & \xrightarrow{a_1} & M_{12} & \xrightarrow{j_3} & M_{13} \\ \downarrow j'_3 & & & & \downarrow a_2 \\ M_{21} & \xrightarrow{a_3} & M_{22} & \xrightarrow{a_4} & M_{23} \end{array}$$

the objects are

$$\begin{aligned} M_{11} &= \text{Hom}(V_1, V_1 \otimes W_1) \otimes \text{Hom}(V_2, V_2 \otimes W_2) \\ M_{21} &= \text{Hom}(V_1 \otimes V_2, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{12} &= \text{Hom}(V_1 \otimes \mathbb{K}, V_1 \otimes W_1) \otimes \text{Hom}(V_2 \otimes \mathbb{K}, V_2 \otimes W_2) \\ M_{22} &= \text{Hom}(V_1 \otimes V_2 \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{13} &= \text{Hom}(V_1 \otimes \mathbb{K} \otimes V_2 \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{23} &= \text{Hom}(V_1 \otimes V_2 \otimes \mathbb{K} \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2), \end{aligned}$$

and the arrows are

$$\begin{aligned} a_1 &= [\text{Hom}(l_{V_1}, \text{Id}_{V_1 \otimes W_1}) \otimes \text{Hom}(l_{V_2}, \text{Id}_{V_2 \otimes W_2})] \\ a_2 &= \text{Hom}(s_2, \text{Id}_{V_1 \otimes W_1 \otimes V_2 \otimes W_2}) \\ a_3 &= \text{Hom}(l_{V_1 \otimes V_2}, \text{Id}_{V_1 \otimes W_1 \otimes V_2 \otimes W_2}) \\ a_4 &= \text{Hom}([\text{Id}_{V_1 \otimes V_2} \otimes l_{\mathbb{K}}], \text{Id}_{V_1 \otimes W_1 \otimes V_2 \otimes W_2}). \end{aligned}$$

The diagram is commutative, by Lemma 1.26 and a scalar multiplication identity. By Theorems 2.55, 2.26, 2.35, and the diagram,

$$\begin{aligned} LHS &= (\text{Tr}_{V_1 \otimes V_2; \mathbb{K}, W_1 \otimes W_2}(s_1 \circ (j'_3(A \otimes B))) \circ l_{V_1 \otimes V_2})(1) \\ &= (\text{Tr}_{V_1 \otimes V_2; \mathbb{K}, W_1 \otimes W_2}(s_1 \circ (j_3((A \circ l_{V_1}) \otimes (B \circ l_{V_2}))) \circ s_2 \circ [\text{Id}_{V_1 \otimes V_2} \otimes l_{\mathbb{K}}^{-1}])(1) \\ &= ((\text{Tr}_{V_1 \otimes V_2; \mathbb{K} \otimes \mathbb{K}, W_1 \otimes W_2}(s_1 \circ (j_3((A \circ l_{V_1}) \otimes (B \circ l_{V_2}))) \circ s_2)) \circ l_{\mathbb{K}}^{-1})(1) \\ &= (j_4((\text{Tr}_{V_1; \mathbb{K}, W_1}(A \circ l_{V_1})) \otimes (\text{Tr}_{V_2; \mathbb{K}, W_2}(B \circ l_{V_2}))))(1 \otimes 1) \\ &= (\text{Tr}_{V_1; W_1}(A)) \otimes (\text{Tr}_{V_2; W_2}(B)) = RHS. \end{aligned}$$

EXERCISE 2.64. For a direct sum  $V = V_1 \oplus V_2$  as in Definition 1.43, and maps  $A : V_1 \rightarrow V_1 \otimes W$ ,  $B : V_2 \rightarrow V_2 \otimes W$ , define  $A \oplus B : V \rightarrow V \otimes W$ , using the inclusions from Example 1.45:

$$A \oplus B = [Q_1 \otimes Id_W] \circ A \circ P_1 + [Q_2 \otimes Id_W] \circ B \circ P_2.$$

If  $V$  is finite-dimensional, then

$$Tr_{V;W}(A \oplus B) = Tr_{V_1;W}(A) + Tr_{V_2;W}(B).$$

HINT. The proof proceeds exactly as in Proposition 2.12, using Corollary 2.56. ■

EXERCISE 2.65. For  $V = V_1 \oplus V_2$ , and  $K : V \rightarrow V \otimes W$ ,

$$Tr_{V;W}(K) = Tr_{V_1;W}([P_1 \otimes Id_W] \circ K \circ Q_1) + Tr_{V_2;W}([P_2 \otimes Id_W] \circ K \circ Q_2).$$

HINT. Using Corollary 2.56 and Lemma 1.25,

$$Tr_{V_i;W}([P_i \otimes Id_W] \circ K \circ Q_i) = Tr_{V;W}([(Q_i \circ P_i) \otimes Id_W] \circ K).$$

The proof proceeds exactly as in Proposition 2.13. ■

## 2.4. Equivalence of alternative definitions

In [JSV] §3, the canonical trace of a map  $F : V \otimes U \rightarrow V \otimes W$  is defined in terms of category theory. In the context and notation of these notes, the definition of [JSV] can be interpreted as saying that  $Tr_{V;U,W}(F)$  is the following composite map from  $U$  to  $W$ :

$$(2.3) \quad U \longrightarrow V^* \otimes V \otimes U \xrightarrow{[Id_V^* \otimes F]} V^* \otimes V \otimes W \xrightarrow{l_W \circ [Ev_V \otimes Id_W]} W,$$

where the first arrow is defined for  $u \in U$  by:

$$u \mapsto (k^{-1}(Id_V)) \otimes u.$$

As mentioned after Theorem 2.10, this map could be expressed in terms of maps  $l_U : \mathbb{K} \otimes U \rightarrow U$  and an inclusion  $Q_1^1 : \mathbb{K} \rightarrow \text{End}(V) : 1 \mapsto Id_V$  as in Example 2.9.

NOTATION 2.66. For finite-dimensional  $V$ ,  $k : V^* \otimes V \rightarrow \text{End}(V)$ , a switching map  $s : V^* \otimes V \rightarrow V \otimes V^*$ , and an inclusion  $Q_1^1 : \mathbb{K} \rightarrow \text{End}(V) : 1 \mapsto Id_V$  as in Example 2.9, define  $\eta : \mathbb{K} \rightarrow V \otimes V^*$  by:

$$\eta = s \circ k^{-1} \circ Q_1^1.$$

The switching map is included for later convenience in Theorem 2.87. The arrow in (2.3) can then be described as follows:

$$(2.4) \quad \begin{aligned} & [s^{-1} \otimes Id_U] \circ [\eta \otimes Id_U] \circ l_U^{-1} \\ & = [k^{-1} \otimes Id_U] \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1} : U \rightarrow V^* \otimes V \otimes U \\ & \quad : u \mapsto (k^{-1}(Id_V)) \otimes u. \end{aligned}$$

The following Theorem shows that the formula (2.3) for  $Tr_{V;U,W}(F)$  coincides with Definition 2.21.  $V$  must be finite-dimensional, but  $U$  and  $W$  may be arbitrary.

THEOREM 2.67. *For finite-dimensional  $V$ ,  $F : V \otimes U \rightarrow V \otimes W$ , and  $u \in U$ ,*  
 $(Tr_{V;U,W}(F))(u) = (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes F] \circ [k^{-1} \otimes Id_U])(Id_V \otimes u).$

PROOF. The following diagram is commutative, where the top arrow is  $a_1 = [Id_{\text{End}(V^*)} \otimes j_2]$ .

$$\begin{array}{ccc}
\text{End}(V^*) \otimes \text{Hom}(V \otimes U, V \otimes W) & \xleftarrow{a_1} & \text{End}(V^*) \otimes \text{End}(V) \otimes \text{Hom}(U, W) \\
\downarrow j & & \downarrow [j \otimes Id_{\text{Hom}(U,W)}] \\
\text{Hom}(V^* \otimes V \otimes U, V^* \otimes V \otimes W) & \xleftarrow{j} & \text{End}(V^* \otimes V) \otimes \text{Hom}(U, W) \\
\downarrow \text{Hom}([k^{-1} \otimes Id_U], [Ev_V \otimes Id_W]) & & \downarrow [\text{Hom}(k^{-1}, Ev_V) \otimes Id_{\text{Hom}(U,W)}] \\
\text{Hom}(\text{End}(V) \otimes U, \mathbb{K} \otimes W) & \xleftarrow{j} & \text{End}(V)^* \otimes \text{Hom}(U, W) \\
\downarrow \text{Hom}(Id_{\text{End}(V) \otimes U}, l_W) & & \downarrow [(d_{\text{End}(V)}(Id_V)) \otimes Id_{\text{Hom}(U,W)}] \\
\text{Hom}(\text{End}(V) \otimes U, W) & & \mathbb{K} \otimes \text{Hom}(U, W) \\
\downarrow d_{\text{End}(V) \otimes U, W}(Id_V \otimes u) & & \downarrow l_1 \\
W & \xleftarrow{d_{UW}(u)} & \text{Hom}(U, W)
\end{array}$$

The upper and lower squares are easy to check (the maps  $d_{UW}$ ,  $d_{\text{End}(V) \otimes U, W}$  are as in Definition 1.9), and the middle square is commutative by Lemma 1.26. Starting with  $Id_{V^*} \otimes F$  in the upper left corner, the RHS of the Theorem is the image under the left column. The top three arrows in the right column come from the construction in Theorem 2.10,

$$Tr_V(A) = ((d_{\text{End}(V)}(Id_V)) \circ \text{Hom}(k^{-1}, Ev_V) \circ j)(Id_{V^*} \otimes A),$$

so that the composition of  $a_1^{-1} = [Id_{\text{End}(V^*)} \otimes j_2^{-1}]$  with the right column of maps takes  $Id_{V^*} \otimes F$  to  $Tr_{V;U,W}(F) = l_1((j_1(Tr_V \otimes Id_{\text{Hom}(U,W)}))(j_2^{-1}(F)))$ . The lowest arrow plugs  $u$  into  $Tr_{V;U,W}(F)$ , giving the LHS of the Theorem, so the equality follows directly from the commutativity of the diagram.  $\blacksquare$

COROLLARY 2.68. *For finite-dimensional  $V$  and  $A : V \rightarrow V \otimes W$ ,*

$$Tr_{V;W}(A) = (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes A] \circ k^{-1})(Id_V).$$

PROOF. By Theorems 2.67 and 2.55,

$$\begin{aligned}
LHS &= (Tr_{V;\mathbb{K},W}(A \circ l_V))(1) \\
&= (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes (A \circ l_V)] \circ [k^{-1} \otimes Id_{\mathbb{K}}])(Id_V \otimes 1) \\
&= RHS.
\end{aligned}$$

$\blacksquare$

This shows, in analogy with Theorem 2.10 and (2.3) from Theorem 2.67, that the  $W$ -valued trace of  $A$  is the image of the distinguished element  $k^{-1}(Id_V)$  under the composite map

$$(2.5) \quad V^* \otimes V \xrightarrow{[Id_{V^*} \otimes A]} V^* \otimes V \otimes W \xrightarrow{l_W \circ [Ev_V \otimes Id_W]} W.$$

So, Corollary 2.68 could be used as an alternative, but equivalent, definition of vector-valued trace. This Section continues with some identities for the vector-valued trace, some of which (Theorem 2.71, Corollary 2.81, Corollary 2.99) could also be used in alternative approaches to the definition of  $Tr_{V;W}$ .

### 2.4.1. A generalized evaluation.

DEFINITION 2.69. The distinguished element

$$Ev_{VW} \in \text{Hom}(\text{Hom}(V, W) \otimes V, W)$$

is defined by

$$Ev_{VW}(A \otimes v) = A(v).$$

In the  $W = \mathbb{K}$  case,  $Ev_{V\mathbb{K}}$  is the distinguished element  $Ev_V \in (V^* \otimes V)^*$  from Definition 2.2. These evaluation maps are related as follows.

LEMMA 2.70. *The following diagram is commutative.*

$$\begin{array}{ccc} V^* \otimes V \otimes W & \xrightarrow{[Ev_V \otimes Id_W]} & \mathbb{K} \otimes W \\ \downarrow [Id_{V^*} \otimes s] & & \downarrow l \\ V^* \otimes W \otimes V & & \\ \downarrow [k_{VW} \otimes Id_V] & & \\ \text{Hom}(V, W) \otimes V & \xrightarrow{Ev_{VW}} & W \end{array}$$

PROOF.

$$\begin{aligned} \phi \otimes v \otimes w &\mapsto (l \circ [Ev_V \otimes Id_W])(\phi \otimes v \otimes w) \\ &= \phi(v) \cdot w, \\ \phi \otimes v \otimes w &\mapsto (Ev_{VW} \circ [k_{VW} \otimes Id_V] \circ [Id_{V^*} \otimes s])(\phi \otimes v \otimes w) \\ &= Ev_{VW}((k_{VW}(\phi \otimes w)) \otimes v) \\ &= \phi(v) \cdot w. \end{aligned}$$

■

THEOREM 2.71. *For finite-dimensional  $V$ , and a map*

$$n' : \text{Hom}(V, W) \otimes V \rightarrow \text{Hom}(V, V \otimes W)$$

*as in Definition 2.47,*

$$Tr_{V;W}(n'(A \otimes v)) = A(v),$$

*or equivalently,*

$$Tr_{V;W} \circ n' = Ev_{VW}.$$

PROOF. The equality  $Tr_{V;W} \circ n' = Ev_{VW}$  follows from the commutativity of the diagram.

$$\begin{array}{ccccc}
 \text{End}(V) \otimes W & \xleftarrow{[k_{VV} \otimes Id_W]} & V^* \otimes V \otimes W & & \\
 \downarrow n & \searrow [Tr_V \otimes Id_W] \quad \swarrow [Ev_V \otimes Id_W] & \downarrow s & & \\
 & \mathbb{K} \otimes W & V^* \otimes W \otimes V & & \\
 & \downarrow l & \downarrow [k_{VW} \otimes Id_V] & & \\
 \text{Hom}(V, V \otimes W) & \xleftarrow{n'} & \text{Hom}(V, W) \otimes V & & \\
 \searrow Tr_{V;W} & & \swarrow Ev_{VW} & & \\
 & W & & & 
 \end{array}$$

The left square is Definition 2.50, and the top triangle is Definition 2.3, together with Lemma 1.25. The right square is exactly Lemma 2.70. The back square is commutative:

$$\begin{aligned}
 \phi \otimes v \otimes w &\mapsto (n \circ [k_{VV} \otimes Id_W])(\phi \otimes v \otimes w) \\
 &= n((k_{VV}(\phi \otimes v)) \otimes w) : \\
 u &\mapsto \phi(u) \cdot v \otimes w, \\
 \phi \otimes v \otimes w &\mapsto (n' \circ [k_{VW} \otimes Id_V] \circ s)(\phi \otimes v \otimes w) \\
 &= n'((k_{VW}(\phi \otimes w)) \otimes v) : \\
 u &\mapsto v \otimes (\phi(u) \cdot w).
 \end{aligned}$$

■

In the following diagram, the left triangle is commutative by Lemma 1.41.

$$\begin{array}{ccc}
 & \text{Hom}(\text{Hom}(V, W) \otimes V, W) & \\
 e_{VV}^W \nearrow & & \nwarrow \text{Hom}(n', Id_W) \\
 \text{End}(V) & & \text{Hom}(\text{Hom}(V, V \otimes W), W) \\
 t_{VV}^W \downarrow & q \nearrow & \\
 \text{End}(\text{Hom}(V, W)) & & 
 \end{array}$$

Every space in the diagram contains a distinguished element:

$$Ev_{VW} = q(Id_{\text{Hom}(V, W)}) = q(t_{VV}^W(Id_V)) = e_{VV}^W(Id_V),$$

and Theorem 2.71 gives this analogue of Equation (2.1) from Lemma 2.5:

$$Tr_{V;W} = ((\text{Hom}(n', Id_W))^{-1} \circ q)(Id_{\text{Hom}(V, W)}) = ((\text{Hom}(n', Id_W))^{-1} \circ e_{VV}^W)(Id_V).$$

THEOREM 2.72. For finite-dimensional  $V$ , let

$$n_1 : \text{End}(V) \otimes U \rightarrow \text{Hom}(V, V \otimes U).$$

Then, for any  $F : V \otimes U \rightarrow V \otimes W$  and  $u \in U$ ,

$$(Tr_{V;U,W}(F))(u) = Tr_{V;W}(F \circ (n_1(Id_V \otimes u))).$$

PROOF. Consider the following diagram.

$$\begin{array}{ccccccc}
 U & \xrightarrow{\quad} & V^* \otimes V \otimes U & \xrightarrow{[Id_V^* \otimes F]} & V^* \otimes V \otimes W & \xrightarrow{[Ev_V \otimes Id_W]} & \mathbb{K} \otimes W \\
 \uparrow l_U & & \downarrow [k \otimes Id_U] & & \downarrow [Id_V^* \otimes s] & & \downarrow l_W \\
 \mathbb{K} \otimes U & \xrightarrow{[Q_1^1 \otimes Id_U]} & \text{End}(V) \otimes U & & V^* \otimes W \otimes V & & \\
 & \searrow n_1 & \swarrow k_{V,V \otimes W} & & \downarrow [k_{VW} \otimes Id_V] & & \\
 \text{Hom}(V, V \otimes U) & \xrightarrow{\text{Hom}(Id_V, F)} & \text{Hom}(V, V \otimes W) & \xleftarrow{n'} & \text{Hom}(V, W) \otimes V & \xrightarrow{Ev_{VW}} & W
 \end{array}$$

The composition from  $U$  to  $W$  along the top row gives  $Tr_{V;U,W}(F)$  by Theorem 2.67. The left square is from (2.4), and the right block is Lemma 2.70. The  $n'$  map is also from Theorem 2.71. The commutativity of the middle blocks is easily checked, so the claim follows from Theorem 2.71:

$$LHS = Ev_{VW}((n')^{-1}(F \circ (n_1(Id_V \otimes u)))) = RHS.$$

■

Corollary 2.68 and Theorem 2.71 are related by the following commutative diagram.

$$\begin{array}{ccccccc}
 V^* \otimes V & \xrightarrow{[Id_V^* \otimes A]} & V^* \otimes V \otimes W & \xrightarrow{[Ev_V \otimes Id_W]} & \mathbb{K} \otimes W & \xrightarrow{l_W} & W \\
 \downarrow k & & \downarrow k_{V,V \otimes W} & & \nearrow Ev_{VW} & & \\
 \text{End}(V) & \xrightarrow{\text{Hom}(Id_V, A)} & \text{Hom}(V, V \otimes W) & \xleftarrow{n'} & \text{Hom}(V, W) \otimes V & & 
 \end{array}$$

The left square is commutative by Lemma 1.29, and the right block is from the Proof of Theorem 2.72. Starting with  $Id_V \in \text{End}(V)$ , the composition along the top row gives  $Tr_{V;W}(A) \in W$  as in (2.5) from Corollary 2.68, and along the bottom row gives  $Ev_{VW}((n')^{-1}(A))$ , which also equals  $Tr_{V;W}(A)$  by Theorem 2.71.

LEMMA 2.73. For a switching map  $s : V \otimes V \rightarrow V \otimes V$ , if  $k : V^* \otimes V \rightarrow \text{End}(V)$  is invertible, then the following map:

$$v \mapsto (l_V \circ [Ev_V \otimes Id_V] \circ [Id_V^* \otimes s] \circ [k^{-1} \otimes Id_V])(Id_V \otimes v)$$

is equal to the identity map  $Id_V$ .

PROOF. By the special case  $W = V$  of Lemma 2.70, the above composite map is  $Ev_{VV}$ , so the given expression is

$$v \mapsto Ev_{VV}(Id_V \otimes v) = Id_V(v) = v.$$

■



EXAMPLE 2.74. If  $V$  is finite-dimensional, then the generalized trace of the switching map  $s : V \otimes V \rightarrow V \otimes V$  is:

$$(2.6) \quad \text{Tr}_{V;V,V}(s) = \text{Id}_V,$$

by the formula from Theorem 2.67 and Lemma 2.73:

$$\text{Tr}_{V;V,V}(s) : v \mapsto (l_V \circ [Ev_V \otimes \text{Id}_V] \circ [\text{Id}_{V^*} \otimes s] \circ [k^{-1} \otimes \text{Id}_V])(\text{Id}_V \otimes v) = v.$$

REMARK 2.75. Equation (2.6) is related to the “yanking” property of [JSV].

The following formula is an analogue of the formula from Lemma 2.73 for  $\phi \in V^*$  instead of  $v \in V$ , but using only  $k = k_{V^*}$  instead of  $k_{V^*V^*}$ .

LEMMA 2.76. *For a switching involution*

$$s' : V^* \otimes V \otimes V^* : \phi \otimes v \otimes \psi \mapsto \psi \otimes v \otimes \phi,$$

*if  $k : V^* \otimes V \rightarrow \text{End}(V)$  is invertible, then the following map:*

$$\phi \mapsto (l_{V^*} \circ [Ev_V \otimes \text{Id}_V] \circ s' \circ [k^{-1} \otimes \text{Id}_{V^*}])(\text{Id}_V \otimes \phi)$$

*is equal to the identity map  $\text{Id}_{V^*}$ .*

PROOF. The following diagram is commutative, and similar to the diagram from Lemma 2.70. Abbreviate  $t = t_{VV}$  as in Lemma 2.5.

$$\begin{array}{ccc} V^* \otimes V \otimes V^* & \xrightarrow{[Ev_V \otimes \text{Id}_{V^*}]} & \mathbb{K} \otimes V^* \\ \downarrow s' & & \downarrow l_{V^*} \\ V^* \otimes V \otimes V^* & & \\ \downarrow [k \otimes \text{Id}_{V^*}] & & \\ \text{End}(V) \otimes V^* & & \\ \downarrow [t \otimes \text{Id}_{V^*}] & & \\ \text{End}(V^*) \otimes V^* & \xrightarrow{Ev_{V^*V^*}} & V^* \end{array}$$

Starting with  $\text{Id}_V \otimes \phi$ , the commutativity of the diagram and the existence of  $k^{-1}$  give:

$$(2.7) \quad \begin{aligned} & (l_{V^*} \circ [Ev_V \otimes \text{Id}_V] \circ s' \circ [k^{-1} \otimes \text{Id}_{V^*}])(\text{Id}_V \otimes \phi) \\ &= Ev_{V^*V^*}((t(\text{Id}_V)) \otimes \phi) = \text{Id}_V^*(\phi) = \phi. \end{aligned}$$

■

THEOREM 2.77. *If  $k : V^* \otimes V \rightarrow \text{End}(V)$  is invertible, then  $d_V : V \rightarrow V^{**}$  is invertible.*

PROOF. The following map, temporarily denoted  $B : V^{**} \rightarrow V$ , is an inverse:

$$B : \Phi \mapsto l_V([\Phi \otimes \text{Id}_V](k^{-1}(\text{Id}_V))).$$

For any  $V, W$ , and  $v \in V$ , the following diagram is commutative.

$$\begin{array}{ccccc} V^* \otimes W & \xrightarrow{k_{VW}} & \text{Hom}(V, W) & \xleftarrow{l} & \text{Hom}(V, W) \otimes \mathbb{K} \\ \downarrow [(d_V(v)) \otimes \text{Id}_W] & & & & \downarrow [\text{Id}_{\text{Hom}(V,W)} \otimes (m(v))] \\ \mathbb{K} \otimes W & \xrightarrow{l_W} & W & \xleftarrow{Ev_{VW}} & \text{Hom}(V, W) \otimes V \end{array}$$

In the case  $W = V$ , starting with  $Id_V$  in the top middle gives the following equality:

$$\begin{aligned} (B \circ d_V)(v) &= l_V([(d_V(v)) \otimes Id_V](k^{-1}(Id_V))) \\ &= Ev_{VV}(Id_V \otimes v) = v. \end{aligned}$$

To check the composite in the other order, in the second diagram,  $s''$  is another switching involution,  $\eta$  and  $s$  are as in (2.4), the block is commutative, and the composition in the left column acts as the identity map, by (2.7) from Lemma 2.76.

$$\begin{array}{ccc} V^* & & \\ \downarrow l_{V^*}^{-1} & & \\ \mathbb{K} \otimes V^* & & \\ \downarrow [\eta \otimes Id_{V^*}] & & \\ V \otimes V^* \otimes V^* & & \\ \downarrow [s \otimes Id_{V^*}] & & \\ V^* \otimes V \otimes V^* & \xrightarrow{[(\Phi \otimes Id_V) \otimes Id_{V^*}]} & \mathbb{K} \otimes V \otimes V^* \\ \downarrow s' & & \downarrow s'' \\ V^* \otimes V \otimes V^* & & V^* \otimes \mathbb{K} \otimes V \\ \downarrow [Ev_V \otimes Id_{V^*}] & & \downarrow [Id_{V^*} \otimes l_V] \\ \mathbb{K} \otimes V^* & & V^* \otimes V \\ \downarrow l_{V^*} & & \downarrow Ev_V \\ V^* & \xrightarrow{\Phi} & \mathbb{K} \end{array}$$

The conclusion is:

$$\begin{aligned} \Phi(\phi) &= (Ev_V \circ [Id_{V^*} \otimes l_V] \circ s'' \circ [(\Phi \otimes Id_V) \otimes Id_{V^*}]((k^{-1}(Id_V)) \otimes \phi)) \\ &= Ev_V(\phi \otimes (l_V([\Phi \otimes Id_V](k^{-1}(Id_V))))) \\ &= \phi(l_V([\Phi \otimes Id_V](k^{-1}(Id_V)))) \\ &= \phi(B(\Phi)) = ((d_V \circ B)(\Phi))(\phi). \end{aligned}$$

■

PROPOSITION 2.78. *For finite-dimensional  $V$ , any map  $G : U \rightarrow \text{Hom}(V, W)$ , and a map*

$$q : \text{Hom}(U, \text{Hom}(V, W)) \rightarrow \text{Hom}(U \otimes V, W)$$

*as in Definition 1.39,*

$$(2.8) \quad q(G) = Tr_{V;W} \circ n' \circ [G \otimes Id_V].$$

PROOF.  $n'$  is as in Theorem 2.71, which gives, for  $u \otimes v \in U \otimes V$ ,

$$\begin{aligned} (Tr_{V;W} \circ n' \circ [G \otimes Id_V])(u \otimes v) &= (Ev_{VW} \circ [G \otimes Id_V])(u \otimes v) \\ &= (G(u))(v) \\ &= (q(G))(u \otimes v). \end{aligned}$$

■

Equation (2.8) from Proposition 2.78 can also be re-written, for  $F = q(G) \in \text{Hom}(U \otimes V, W)$ ,

$$F = \text{Ev}_{VW} \circ [(q^{-1}(F)) \otimes \text{Id}_V] = \text{Tr}_{V;W} \circ n' \circ [(q^{-1}(F)) \otimes \text{Id}_V].$$

THEOREM 2.79. *For finite-dimensional  $V$ , and maps*

$$\begin{aligned} n_2 : \text{Hom}(U, \text{Hom}(V, W)) \otimes V &\rightarrow \text{Hom}(U, \text{Hom}(V, W) \otimes V) \\ n_3 : \text{Hom}(V \otimes U, W) \otimes V &\rightarrow \text{Hom}(V \otimes U, V \otimes W) \\ q : \text{Hom}(U, \text{Hom}(V, W)) &\rightarrow \text{Hom}(V \otimes U, W) \\ F : V \otimes U &\rightarrow V \otimes W, \end{aligned}$$

the following diagram is commutative.

$$\begin{array}{ccc} U & \xrightarrow{\text{Tr}_{V;U,W}(F)} & W \\ & \searrow (n_2 \circ [q^{-1} \otimes \text{Id}_V] \circ n_3^{-1})(F) & \uparrow \text{Ev}_{VW} \\ & & \text{Hom}(V, W) \otimes V \end{array}$$

PROOF.  $n_3$  is invertible by Lemma 2.48. The claim follows from the commutativity of this diagram.

$$\begin{array}{ccc} \text{Hom}(V \otimes U, V \otimes W) & \xleftarrow{n_3} & \text{Hom}(V \otimes U, W) \otimes V \\ \downarrow \text{Tr}_{V;U,W} & & \uparrow [q \otimes \text{Id}_V] \\ \text{Hom}(U, W) & \xleftarrow{\text{Hom}(\text{Id}_U, \text{Ev}_{VW})} & \text{Hom}(U, \text{Hom}(V, W) \otimes V) \\ & & \downarrow n_2 \\ & & \text{Hom}(U, \text{Hom}(V, W)) \otimes V \end{array}$$

Starting with  $A \otimes v_1 \in \text{Hom}(U, \text{Hom}(V, W)) \otimes V$ ,

$$\begin{aligned} (\text{Hom}(\text{Id}_V, \text{Ev}_{VW}) \circ n_2)(A \otimes v_1) : u &\mapsto \text{Ev}_{VW}((n_2(A \otimes v_1))(u)) \\ &= \text{Ev}_{VW}((A(u)) \otimes v_1) = (A(u))(v_1). \end{aligned}$$

Going the other way around the diagram,

$$\begin{aligned} \text{Tr}_{V;U,W}(n_3((q(A)) \otimes v_1)) : u &\mapsto \text{Tr}_{V;W}((n_3((q(A)) \otimes v_1)) \circ (n_1(\text{Id}_V \otimes u))) \\ &= \text{Tr}_{V;W}(n'((A(u)) \otimes v_1)) \\ &= (A(u))(v_1). \end{aligned}$$

The first step uses Theorem 2.72 and its  $n_1$  map, and the last step uses Theorem 2.71 and its  $n'$  map. The middle step uses the following calculation:

$$\begin{aligned} (n_3((q(A)) \otimes v_1)) \circ (n_1(\text{Id}_V \otimes u)) : v_2 &\mapsto (n_3((q(A)) \otimes v_1))(v_2 \otimes u) \\ &= v_1 \otimes ((q(A))(v_2 \otimes u)) \\ &= v_1 \otimes ((A(u))(v_2)) \\ &= (n'((A(u)) \otimes v_1))(v_2). \end{aligned}$$

■

THEOREM 2.80. For finite-dimensional  $V$  and any vector space  $Z$ , let  $s_1 : V \otimes V \rightarrow V \otimes V$  be the switching map and let

$$n : \text{End}(V) \otimes V \otimes Z \rightarrow \text{Hom}(V, V \otimes V \otimes Z).$$

Then, for any  $B \in V \otimes Z$ ,

$$\text{Tr}_{V;V \otimes Z}([s_1 \otimes \text{Id}_Z] \circ (n(\text{Id}_V \otimes B))) = B.$$

PROOF. It is straightforward to check that the following diagram is commutative.

$$\begin{array}{ccc}
 \text{End}(V) \otimes V \otimes Z & \xleftarrow{[k_{VV} \otimes \text{Id}_{V \otimes Z}]} & V^* \otimes V \otimes V \otimes Z \\
 \downarrow n & & \downarrow [[\text{Id}_{V^*} \otimes s_1] \otimes \text{Id}_Z] \\
 \text{Hom}(V, V \otimes V \otimes Z) & & V^* \otimes V \otimes V \otimes Z \\
 \downarrow \text{Hom}(\text{Id}_V, [s_1 \otimes \text{Id}_Z]) & \swarrow k_{V, V \otimes V \otimes Z} & \downarrow [k_{VV} \otimes \text{Id}_{V \otimes Z}] \\
 \text{Hom}(V, V \otimes V \otimes Z) & \xleftarrow{n} & \text{End}(V) \otimes V \otimes Z
 \end{array}$$

So the LHS of the claim is:

$$\begin{aligned}
 & \text{Tr}_{V;V \otimes Z}([s_1 \otimes \text{Id}_Z] \circ (n(\text{Id}_V \otimes B))) \\
 &= (\text{Tr}_{V;V \otimes Z} \circ \text{Hom}(\text{Id}_V, [s_1 \otimes \text{Id}_Z]) \circ n)(\text{Id}_V \otimes B) \\
 &= (\text{Tr}_{V;V \otimes Z} \circ n \circ [k_{VV} \otimes \text{Id}_{V \otimes Z}] \\
 & \quad \circ [[\text{Id}_{V^*} \otimes s_1] \otimes \text{Id}_Z] \circ [k_{VV}^{-1} \otimes \text{Id}_{V \otimes Z}])(\text{Id}_V \otimes B).
 \end{aligned}$$

Let  $s_3 : V \otimes Z \rightarrow Z \otimes V$  be another switching, and denote  $V_1 = V_2 = V$ , so that the labels explain the switching

$$s_2 : V^* \otimes V_1 \otimes (V_2 \otimes Z) \rightarrow V^* \otimes (V_2 \otimes Z) \otimes V_1.$$

Using  $W = V \otimes Z$  and this  $s_2$  in the role of  $s$  from the diagram from the Proof of Theorem 2.71, the commutativity of that diagram continues the chain of equalities:

$$\begin{aligned}
 &= (\text{Ev}_{V;V \otimes Z} \circ [k_{V;V \otimes Z} \otimes \text{Id}_V] \circ s_2 \\
 & \quad \circ [[\text{Id}_{V^*} \otimes s_1] \otimes \text{Id}_Z] \circ [k_{VV}^{-1} \otimes \text{Id}_{V \otimes Z}])(\text{Id}_V \otimes B) \\
 &= (\text{Ev}_{V;V \otimes Z} \circ [k_{V;V \otimes Z} \otimes \text{Id}_V] \circ [\text{Id}_{V^*} \otimes V \circ s_3] \circ [k_{VV}^{-1} \otimes \text{Id}_{V \otimes Z}])(\text{Id}_V \otimes B) \\
 &= (\text{Ev}_{V;V \otimes Z} \circ [k_{V;V \otimes Z} \otimes \text{Id}_V] \circ [k_{VV}^{-1} \otimes s_3])(\text{Id}_V \otimes B).
 \end{aligned}$$

The commutativity of the following diagram:

$$\begin{array}{ccc}
 V^* \otimes V \otimes Z \otimes V & \xrightarrow{[k_{V;V \otimes Z} \otimes \text{Id}_V]} & \text{Hom}(V, V \otimes Z) \otimes V \\
 \downarrow [k_{VV} \otimes s_3^{-1}] & & \downarrow \text{Ev}_{V;V \otimes Z} \\
 \text{End}(V) \otimes V \otimes Z & \xrightarrow{[\text{Ev}_{V;V} \otimes \text{Id}_Z]} & V \otimes Z
 \end{array}$$

leads the equalities to the conclusion:

$$= [\text{Ev}_{V;V} \otimes \text{Id}_Z](\text{Id}_V \otimes B) = B.$$

■

The switching  $s_3 : V \otimes Z \rightarrow Z \otimes V$  from Theorem 2.80 appears in the following Corollaries, which are related to constructions in [Stolz-Teichner].

**COROLLARY 2.81.** *Given  $V$  finite-dimensional and a map  $A : V \rightarrow V \otimes W$ , if there exist a vector space  $Z$  and a factorization of the form*

$$A = [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1},$$

*for  $l : \mathbb{K} \otimes V \rightarrow V$ ,  $B_1 : \mathbb{K} \rightarrow V \otimes Z$ , and  $B_2 : Z \otimes V \rightarrow W$ , then*

$$Tr_{V;W}(A) = (B_2 \circ s_3 \circ B_1)(1).$$

**PROOF.** It is straightforward to check that the following diagram is commutative. The  $s_1$  is the switching from Theorem 2.80, and as in the above Proof, denote  $V_1 = V_2 = V$ , to keep track of switching, so that  $s_4 : V_1 \otimes (V_2 \otimes Z) \rightarrow (V_2 \otimes Z) \otimes V_1$ .

$$\begin{array}{ccccc}
 V_1 & \xleftarrow{l} & \mathbb{K} \otimes V_1 & & \\
 \downarrow n(Id_V \otimes (B_1(1))) & & \downarrow [B_1 \otimes Id_V] & & \\
 & & V_2 \otimes Z \otimes V_1 & & \\
 & \nearrow s_4 & & \nearrow [Id_V \otimes s_3] & \\
 V_1 \otimes V_2 \otimes Z & & & & V_2 \otimes V_1 \otimes Z \\
 & \searrow [s_1 \otimes Id_Z] & & & 
 \end{array}$$

The following equalities use the commutativity of the diagram, Corollary 2.59 twice, and Theorem 2.80 applied to  $B = B_1(1) \in V \otimes Z$ .

$$\begin{aligned}
 Tr_{V;W}(A) &= Tr_{V;W}([Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1}) \\
 &= B_2(Tr_{V;Z \otimes V}([B_1 \otimes Id_V] \circ l^{-1})) \\
 &= B_2(Tr_{V;Z \otimes V}([Id_V \otimes s_3] \circ [s_1 \otimes Id_Z] \circ (n(Id_V \otimes (B_1(1))))) \\
 &= (B_2 \circ s_3)(Tr_{V;V \otimes Z}([s_1 \otimes Id_Z] \circ (n(Id_V \otimes (B_1(1))))) \\
 &= (B_2 \circ s_3)(B_1(1)).
 \end{aligned}$$

■

**COROLLARY 2.82.** *Given  $V$  finite-dimensional and a map  $A : V \rightarrow V$ , if there exist a vector space  $Z$  and a factorization of the form*

$$A = l_V \circ [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1},$$

*for  $l_V : V \otimes \mathbb{K} \rightarrow V$ ,  $l : \mathbb{K} \otimes V \rightarrow V$ ,  $B_1 : \mathbb{K} \rightarrow V \otimes Z$ , and  $B_2 : Z \otimes V \rightarrow \mathbb{K}$ , then*

$$Tr_V(A) = (B_2 \circ s_3 \circ B_1)(1).$$

**PROOF.** Theorem 2.54 and Corollary 2.81 apply.

■

COROLLARY 2.83. *Given  $V$  finite-dimensional and a map  $A : V \otimes U \rightarrow V \otimes W$ , if there exist a vector space  $Z$  and a factorization of the form*

$$A = [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ s_5,$$

*for  $s_5 : V \otimes U \rightarrow U \otimes V$ ,  $B_1 : U \rightarrow V \otimes Z$ , and  $B_2 : Z \otimes V \rightarrow W$ , then*

$$Tr_{V;U,W}(A) = B_2 \circ s_3 \circ B_1.$$

PROOF. Theorem 2.26 applies:

$$\begin{aligned} Tr_{V;U,W}(A) &= Tr_{V;U,W}([Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ s_5) \\ &= B_2 \circ (Tr_{V;U,Z \otimes V}([B_1 \otimes Id_V] \circ s_5)), \end{aligned}$$

so to prove the claim it is enough to check

$$(2.9) \quad Tr_{V;U,Z \otimes V}([B_1 \otimes Id_V] \circ s_5) = s_3 \circ B_1$$

for  $B_1 \in \text{Hom}(U, V \otimes Z)$ . Let

$$n'' : V \otimes \text{Hom}(U, Z) \rightarrow \text{Hom}(U, V \otimes Z)$$

be as in Notation 2.52; by the finite-dimensionality of  $V$  and Lemma 2.48,  $n''$  is invertible. So, to prove (2.9), it is, further, enough to check the case where  $B_1 = n''(v_0 \otimes B_3)$  for some  $v_0 \in V$ ,  $B_3 : U \rightarrow Z$ . Let

$$L = [B_1 \otimes Id_V] \circ s_5 = [(n''(v_0 \otimes B_3)) \otimes Id_V] \circ s_5 \in \text{Hom}(V \otimes U, V \otimes Z \otimes V);$$

then (2.9) becomes:

$$Tr_{V;U,Z \otimes V}(L) = s_3 \circ (n''(v_0 \otimes B_3)).$$

The following canonical maps are needed, from Definitions 2.47, 1.39:

$$\begin{aligned} n_1 : \text{Hom}(U, Z) \otimes V &\rightarrow \text{Hom}(U, Z \otimes V) \\ n' : V \otimes \text{Hom}(U, Z \otimes V) &\rightarrow \text{Hom}(U, V \otimes Z \otimes V) \\ q : \text{Hom}(V, \text{Hom}(U, V \otimes Z \otimes V)) &\rightarrow \text{Hom}(V \otimes U, V \otimes Z \otimes V). \end{aligned}$$

The  $n$  maps are invertible by Lemma 2.48, because  $V$  is finite-dimensional. By Lemma 1.40,  $q$  is invertible, and:

$$\begin{aligned} (q^{-1}(L))(v) : u &\mapsto L(v \otimes u) = (v_0 \otimes (B_3(u))) \otimes v \\ \implies (q^{-1}(L))(v) &= (n' \circ [Id_V \otimes n_1])(v_0 \otimes B_3 \otimes v) \\ \implies ((n')^{-1} \circ (q^{-1}(L)))(v) &= [Id_V \otimes n_1](v_0 \otimes B_3 \otimes v). \end{aligned}$$

Define a linear map  $B_4 = m(v_0 \otimes B_3) : \mathbb{K} \rightarrow V \otimes \text{Hom}(U, Z)$ , so that  $B_4(1) = v_0 \otimes B_3$ . Then there is a factorization:

$$(n')^{-1} \circ (q^{-1}(L)) = [Id_V \otimes n_1] \circ [B_4 \otimes Id_V] \circ l^{-1}.$$

Theorem 2.53 and Corollary 2.81 apply, with  $s'_3$  as in the following commutative diagram.

$$\begin{array}{ccc} V \otimes \text{Hom}(U, Z) & \xrightarrow{s'_3} & \text{Hom}(U, Z) \otimes V \\ \downarrow n'' & & \downarrow n_1 \\ \text{Hom}(U, V \otimes Z) & \xrightarrow{\text{Hom}(Id_V, s_3)} & \text{Hom}(U, Z \otimes V) \end{array}$$

$$\begin{aligned}
Tr_{V;U,Z \otimes V}(L) &= Tr_{V;\text{Hom}(U,Z \otimes V)}((n')^{-1} \circ (q^{-1}(L))) \\
&= Tr_{V;\text{Hom}(U,Z \otimes V)}([Id_V \otimes n_1] \circ [B_4 \otimes Id_V] \circ l^{-1}) \\
&= (n_1 \circ s'_3 \circ B_4)(1) \\
&= (\text{Hom}(Id_V, s_3) \circ n'')(v_0 \otimes B_3) \\
&= s_3 \circ (n''(v_0 \otimes B_3)) = s_3 \circ B_1.
\end{aligned}$$

■

EXAMPLE 2.84. Consider  $V = V_1 = V_2$  and  $s_4 : V_1 \otimes (V_2 \otimes Z) \rightarrow (V_2 \otimes Z) \otimes V_1$  as in the diagram from the Proof of Corollary 2.81. Then

$$Tr_{V;V \otimes Z, Z \otimes V}(s_4) = s_3.$$

This is a special case of Corollary 2.83, with  $U = V \otimes Z$ ,  $W = Z \otimes V$ ,  $s_5 = s_4$ ,  $B_1 = Id_{V \otimes Z}$ , and  $B_2 = Id_{Z \otimes V}$ .

EXAMPLE 2.85. Using Theorem 2.26 and Example 2.84,

$$\begin{aligned}
Tr_{V;Z \otimes V, V \otimes Z}(s_4^{-1}) &= Tr_{V;Z \otimes V, V \otimes Z}([Id_V \otimes s_3^{-1}] \circ s_4 \circ [Id_V \otimes s_3^{-1}]) \\
&= s_3^{-1} \circ (Tr_{V;V \otimes Z, Z \otimes V}(s_4)) \circ s_3^{-1} \\
&= s_3^{-1}.
\end{aligned}$$

EXAMPLE 2.86. Formula (2.6) from Example 2.74 also follows as a special case; for the switching map  $s_1 : V \otimes V \rightarrow V \otimes V$  as in Theorem 2.80,  $Tr_{V;V,V}(s_1) = Id_V$ . This is the case of Corollary 2.83 with  $U = V$ ,  $Z = \mathbb{K}$ ,  $B_1 = l_V^{-1}$ ,  $B_2 = l$ ,  $A = s_1 = s_5 = [Id_V \otimes l] \circ [l_V^{-1} \otimes Id_V] \circ s_5$ , and  $s_3 : V \otimes \mathbb{K} \rightarrow \mathbb{K} \otimes V$ , so

$$Tr_{V;V,V}(s_5) = l \circ s_3 \circ l_V^{-1} = Id_V.$$

### 2.4.2. Coevaluation and dualizability.

THEOREM 2.87. For finite-dimensional  $V$ ,  $\eta : \mathbb{K} \rightarrow V \otimes V^*$  as in Notation 2.66, and scalar multiplication maps  $l_V : \mathbb{K} \otimes V \rightarrow V$ ,  $l_{V^*} : \mathbb{K} \otimes V^* \rightarrow V^*$ ,  $l_1 : V \otimes \mathbb{K} \rightarrow V$ ,  $l_2 : V^* \otimes \mathbb{K} \rightarrow V^*$ ,

$$l_1 \circ [Id_V \otimes Ev_V] \circ [\eta \otimes Id_V] \circ l_V^{-1} = Id_V,$$

and

$$l_{V^*} \circ [Ev_V \otimes Id_{V^*}] \circ [Id_{V^*} \otimes \eta] \circ l_2^{-1} = Id_{V^*}.$$

PROOF. In the first diagram,  $V = V_1 = V_2 = V_3$ . The upper left square is from the formula (2.4) of Notation 2.66. The  $s_5$  notation in the right half is from Corollary 2.83 and Example 2.86. The first claim is that the lower left part of the

diagram is commutative.

$$\begin{array}{ccccc}
 \text{Hom}(V_2, V_1) \otimes V_3 & \xleftarrow{[k \otimes Id_{V_3}]} & V_2^* \otimes V_1 \otimes V_3 & \xrightarrow{[Id_{V_2^*} \otimes s_5]} & V_2^* \otimes V_3 \otimes V_1 \\
 \uparrow [Q_1^1 \otimes Id_{V_3}] & & \downarrow [s \otimes Id_{V_3}] & & \downarrow [Ev_V \otimes Id_{V_1}] \\
 \mathbb{K} \otimes V_3 & \xrightarrow{[\eta \otimes Id_{V_3}]} & V_1 \otimes V_2^* \otimes V_3 & & \mathbb{K} \otimes V_1 \\
 & \searrow l_V & \downarrow [Id_{V_1} \otimes Ev_V] & & \swarrow l_V \\
 & & V_1 \otimes \mathbb{K} & & \\
 & & \downarrow l_1 & & \\
 & & V & & 
 \end{array}$$

The commutativity of the right half of the diagram is easy to check. The first claim follows from checking that the identity map is equal to the composite of maps starting at the bottom and going clockwise. Lemma 2.73 applies.

$$\begin{aligned}
 (2.10) \quad & l_1 \circ [Id_V \otimes Ev_V] \circ [\eta \otimes Id_V] \circ l_V^{-1} \\
 &= l_V \circ [Ev_V \otimes Id_V] \circ [Id_{V^*} \otimes s_5] \circ [k^{-1} \otimes Id_V] \circ [Q_1^1 \otimes Id_V] \circ l_V^{-1} \\
 &= Id_V.
 \end{aligned}$$

The expression (2.10) is also equal to  $Tr_{V,V}(s_5) = Id_V$  as in Examples 2.74 and 2.86.

For the second claim, consider the second diagram, where  $s'$  is the switching involution from Lemma 2.76 and  $s'''$  is another switching involution so that the top block is easily seen to be commutative.

$$\begin{array}{ccccc}
 \text{Hom}(V_2, V_1) \otimes V_3^* & \xleftarrow{[k \otimes Id_{V_3^*}]} & V_2^* \otimes V_1 \otimes V_3^* & & \\
 \swarrow s''' & & \searrow s' & & \\
 V_3^* \otimes \text{Hom}(V_2, V_1) & \xleftarrow{[Id_{V_3^*} \otimes k]} & V_3^* \otimes V_2^* \otimes V_1 & & \\
 \uparrow [Id_{V_3^*} \otimes Q_1^1] & & \downarrow [Id_{V_3^*} \otimes s] & & \\
 V_3^* \otimes \mathbb{K} & \xrightarrow{[Id_{V_3^*} \otimes \eta]} & V_3^* \otimes V_1 \otimes V_2^* & & \\
 & \searrow l_2 & \downarrow [Ev_V \otimes Id_{V_2^*}] & & \\
 & & \mathbb{K} \otimes V_2^* & & \\
 & & \downarrow l_{V^*} & & \\
 & & V^* & & 
 \end{array}$$

As in the first diagram, the definition of  $\eta$  is used in the left square, and the second claim is that the lower left part of the second diagram is commutative. The calculation is again to check the clockwise composition, and Lemma 2.76 applies.

$$\begin{aligned}
 & l_{V^*} \circ [Ev_V \otimes Id_{V^*}] \circ [Id_{V^*} \otimes \eta] \circ l_2^{-1} \\
 &= l_{V^*} \circ [Ev_V \otimes Id_{V^*}] \circ s' \circ [k^{-1} \otimes Id_{V^*}] \circ s''' \circ [Id_{V^*} \otimes Q_1^1] \circ l_2^{-1} \\
 &= l_{V^*} \circ [Ev_V \otimes Id_{V^*}] \circ s' \circ [k^{-1} \otimes Id_{V^*}] \circ [Q_1^1 \otimes Id_{V^*}] \circ l_{V^*}^{-1} \\
 &= Id_{V^*}.
 \end{aligned}$$



DEFINITION 2.88. A vector space  $V$  is dualizable means: there exists  $(D, \epsilon, \eta)$ , where  $D$  is a vector space, and  $\epsilon : D \otimes V \rightarrow \mathbb{K}$  and  $\eta : \mathbb{K} \rightarrow V \otimes D$  are linear maps such that the following diagrams (involving various scalar multiplication maps) are commutative.

$$\begin{array}{ccc}
 \mathbb{K} \otimes V & \xrightarrow{[\eta \otimes Id_V]} & V \otimes D \otimes V \\
 \downarrow & & \downarrow [Id_V \otimes \epsilon] \\
 V & \xleftarrow{\quad} & V \otimes \mathbb{K}
 \end{array}
 \qquad
 \begin{array}{ccc}
 D \otimes \mathbb{K} & \xrightarrow{[Id_D \otimes \eta]} & D \otimes V \otimes D \\
 \downarrow & & \downarrow [\epsilon \otimes Id_D] \\
 D & \xleftarrow{\quad} & \mathbb{K} \otimes D
 \end{array}$$

EXAMPLE 2.89. Given  $V$  as in Theorem 2.87, the space  $D = V^*$  and the maps  $\epsilon = Ev_V$  and  $\eta = s \circ k^{-1} \circ Q_1^1$  satisfy the identities from Definition 2.88.

REMARK 2.90. In category theory and other generalizations of this construction ([**Stolz-Teichner**], [**PS**]),  $\eta$  is called a coevaluation map. A more general notion, with left and right duals, is considered by [**Maltsiniotis**].

LEMMA 2.91. *If  $V$  is dualizable, with duality data  $(D, \epsilon, \eta)$ , then there is an invertible map  $D \rightarrow V^*$ .*

PROOF. It is equivalent, by Example 1.18 and Lemma 2.46, to show there is an invertible map  $\text{Hom}(\mathbb{K}, D) \rightarrow \text{Hom}(\mathbb{K} \otimes V, \mathbb{K})$ . Denote:

$$\begin{aligned}
 (2.11) \quad A : \text{Hom}(\mathbb{K}, D) &\rightarrow \text{Hom}(\mathbb{K} \otimes V, \mathbb{K}) \\
 \delta &\mapsto (\lambda \otimes v \mapsto \epsilon((\delta(\lambda)) \otimes v)), \\
 B : \text{Hom}(\mathbb{K} \otimes V, \mathbb{K}) &\rightarrow \text{Hom}(\mathbb{K}, D) \\
 \phi &\mapsto (\lambda \mapsto (l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta)(\lambda)),
 \end{aligned}$$

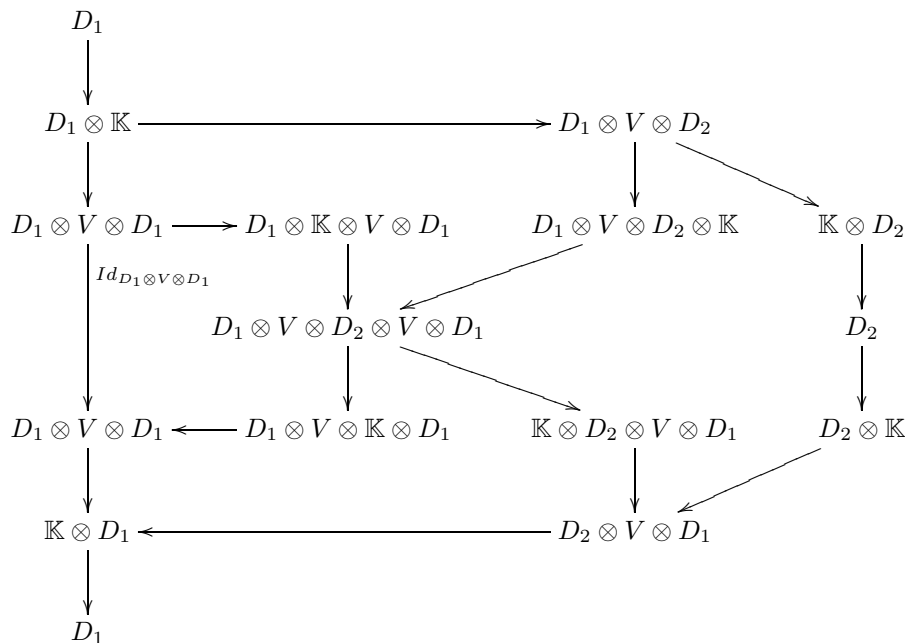
where  $l$  denotes various scalar multiplications. The following diagrams are commutative, where unlabeled arrows are scalar multiplications or their inverses:

$$\begin{array}{ccc}
 \mathbb{K} & \xrightarrow{\delta} & D \\
 \downarrow \eta & & \downarrow \\
 V \otimes D & & D \otimes \mathbb{K} \\
 \downarrow & & \downarrow [Id_D \otimes \eta] \\
 \mathbb{K} \otimes V \otimes D & \xrightarrow{[[\delta \otimes Id_V] \otimes Id_D]} & D \otimes V \otimes D \\
 \searrow [(A(\delta)) \otimes Id_D] & & \downarrow [\epsilon \otimes Id_D] \\
 & & \mathbb{K} \otimes D \\
 & & \downarrow \\
 & & D
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{K} \otimes V & & \\
 \downarrow [\eta \otimes Id_V] & & \\
 V \otimes D \otimes V & \xrightarrow{\quad} & \mathbb{K} \otimes V \otimes D \otimes V \\
 \downarrow [Id_V \otimes \epsilon] & & \downarrow [[\phi \otimes Id_D] \otimes Id_V] \\
 V \otimes \mathbb{K} & & \mathbb{K} \otimes D \otimes V \\
 \downarrow & & \downarrow \\
 V & & D \otimes V \\
 \downarrow & & \downarrow \epsilon \\
 \mathbb{K} \otimes V & \xrightarrow{\phi} & \mathbb{K}
 \end{array}$$

In the left diagram, the top square is easily checked and the lower triangle uses the formula for  $A$ . The composition in the right column gives the identity map for  $D$  by Definition 2.88, so  $Id_D \circ \delta = B(A(\delta)) : \mathbb{K} \rightarrow D$ .

$$\begin{aligned}
(A \circ B)(\phi) : \lambda \otimes v &\mapsto \epsilon(((B(\phi))(\lambda)) \otimes v) \\
&= \epsilon(((l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta)(\lambda)) \otimes v) \\
&= (\epsilon \circ [(l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta) \otimes Id_V])(\lambda \otimes v) \\
&= (\phi \circ Id_{\mathbb{K} \otimes V})(\lambda \otimes v).
\end{aligned}$$
$$D_1 \longrightarrow D_1 \otimes \mathbb{K} \xrightarrow{[Id_{D_1} \otimes \eta_2]} D_1 \otimes V \otimes D_2 \xrightarrow{[\epsilon_1 \otimes Id_{D_2}]} \mathbb{K} \otimes D_2 \longrightarrow D_2$$
$$D_2 \longrightarrow D_2 \otimes \mathbb{K} \xrightarrow{[Id_{D_2} \otimes \eta_1]} D_2 \otimes V \otimes D_1 \xrightarrow{[\epsilon_2 \otimes Id_{D_1}]} \mathbb{K} \otimes D_1 \longrightarrow D_1$$

PROOF. Some of the arrows in the following diagram are left unlabeled, but they involve only identity maps, scalar multiplications and their inverses, and the given  $\eta_1, \eta_2, \epsilon_1, \epsilon_2$  maps.



The composition in the left column gives the identity map  $D_1 \rightarrow D_1$ , and the center left square is commutative, involving the composite  $[Id_V \otimes \epsilon_2] \circ [\eta_2 \otimes Id_V]$ . The commutativity of the top, right, and bottom blocks is easy to check. The composite  $D_1 \rightarrow D_2 \rightarrow D_1$  clockwise from top to bottom is equal to  $a_{21} \circ a_{12}$ , and the commutativity of the diagram establishes the claim that  $a_{21} \circ a_{12} = Id_{D_1}$ ; checking the inverse in the other order follows from relabeling the subscripts.

For the identity  $[Id_V \otimes a_{12}] \circ \eta_1 = \eta_2$ , consider the following diagram.

$$\begin{array}{ccccc}
 \mathbb{K} & \xrightarrow{\eta_2} & V \otimes D_2 & \xrightarrow{\quad} & \mathbb{K} \otimes V \otimes D_2 \\
 \downarrow \eta_1 & \nearrow [Id_V \otimes a_{12}] & \uparrow & \searrow [[\eta_1 \otimes Id_V] \otimes Id_{D_2}] & \downarrow \\
 & & V \otimes \mathbb{K} \otimes D_2 & & \\
 & & \nwarrow [Id_V \otimes [\epsilon_1 \otimes Id_{D_2}]] & & \\
 V \otimes D_1 & \xrightarrow{\quad} & V \otimes D_1 \otimes \mathbb{K} & \xrightarrow{[Id_V \otimes [Id_{D_1} \otimes \eta_2]]} & V \otimes D_1 \otimes V \otimes D_2 \\
 & & & & \downarrow
 \end{array}$$

The lower block uses the definition of  $a_{12}$ . The right block involves  $\eta_1$  and  $\epsilon_1$  so that one of the identities from Definition 2.88 applies. The claim is that the left triangle is commutative, and this follows from the easily checked commutativity of the outer rectangle.

Similarly for the identity  $\epsilon_2 \circ [a_{12} \otimes Id_V] = \epsilon_1$ , consider the following diagram.

$$\begin{array}{ccccc}
 D_1 \otimes V \otimes D_2 \otimes V & \xrightarrow{[Id_{D_1} \otimes [Id_V \otimes \epsilon_2]]} & D_1 \otimes V \otimes \mathbb{K} & \xrightarrow{\quad} & D_1 \otimes V \\
 \downarrow & \nearrow [[Id_{D_1} \otimes \eta_2] \otimes Id_V] & \searrow & \nearrow [a_{12} \otimes Id_V] & \downarrow \epsilon_1 \\
 & & D_1 \otimes \mathbb{K} \otimes V & & \\
 & & \nwarrow [[\epsilon_1 \otimes Id_{D_2}] \otimes Id_V] & & \\
 \mathbb{K} \otimes D_2 \otimes V & \xrightarrow{\quad} & D_2 \otimes V & \xrightarrow{\epsilon_2} & \mathbb{K}
 \end{array}$$

The left block uses the definition of  $a_{12}$ . The top block involves  $\eta_2$  and  $\epsilon_2$  so that one of the identities from Definition 2.88 applies. The claim is that the right triangle is commutative, and this follows from the easily checked commutativity of the outer rectangle.  $\blacksquare$

In the case  $D = V^*$  from Example 2.89, the maps from Lemma 2.91 and 2.92 agree (up to composition with trivial invertible maps as in the following Exercise) and so they are canonical.

**EXERCISE 2.93.** Applying Lemma 2.91 to the triple  $(V^*, Ev_V, \eta)$  from Example 2.89 gives a map  $A$  such that the left diagram is commutative. If  $V$  is also dualizable with  $(D_2, \epsilon_2, \eta_2)$ , then the maps  $B$  from Lemma 2.91 and  $a_{12}$  from Lemma 2.92 make the right diagram commutative.

$$\begin{array}{ccc}
 V^* & \xrightarrow{Id_{V^*}} & V^* \\
 \downarrow m & & \downarrow \text{Hom}(l, Id_{\mathbb{K}}) \\
 \text{Hom}(\mathbb{K}, V^*) & \xrightarrow{A} & \text{Hom}(\mathbb{K} \otimes V^*, \mathbb{K})
 \end{array}
 \qquad
 \begin{array}{ccc}
 D_2 & \xleftarrow{a_{12}} & V^* \\
 \downarrow m & & \downarrow \text{Hom}(l, Id_{\mathbb{K}}) \\
 \text{Hom}(\mathbb{K}, D_2) & \xleftarrow{B} & \text{Hom}(\mathbb{K} \otimes V^*, \mathbb{K})
 \end{array}$$

HINT. The first claim is left as an exercise. For the second claim, consider  $\phi \in V^*$ ,  $\lambda \in \mathbb{K}$ ; the following quantities agree, showing the right diagram is commutative.

$$\begin{aligned}
(m \circ a_{12})(\phi) : \lambda &\mapsto (m(a_{12}(\phi)))(\lambda) = \lambda \cdot a_{12}(\phi) \\
&= \lambda \cdot (l \circ [Ev_V \otimes Id_{D_2}] \circ [Id_{V^*} \otimes \eta_2] \circ l^{-1})(\phi) \\
&= \lambda \cdot (l \circ [Ev_V \otimes Id_{D_2}])(\phi \otimes (\eta_2(1))), \\
(B \circ \text{Hom}(l, Id_{\mathbb{K}}))(\phi) : \lambda &\mapsto (B(\phi \circ l))(\lambda) \\
&= (l \circ [(\phi \circ l) \otimes Id_{D_2}] \circ l^{-1} \circ \eta_2)(\lambda) \\
&= (l \circ [(\phi \circ l) \otimes Id_{D_2}])(1 \otimes (\eta_2(\lambda))) \\
&= (l \circ [\phi \otimes Id_{D_2}])(\eta_2(\lambda)).
\end{aligned}$$

■

LEMMA 2.94. *If  $V$  is dualizable, with  $(D, \epsilon, \eta)$ , then  $D$  is dualizable, with duality data  $(V, \epsilon \circ s, s \circ \eta)$ , where  $s : V \otimes D \rightarrow D \otimes V$  is a switching map.*

PROOF. In the following diagram,  $V = V_1 = V_2$ :

$$\begin{array}{ccccc}
V_1 \otimes \mathbb{K} & \xrightarrow{[Id_V \otimes \eta]} & V_1 \otimes V_2 \otimes D & \xrightarrow{[Id_V \otimes s]} & V_1 \otimes D \otimes V_2 \\
\downarrow & & \downarrow s_1 & & \downarrow [s \otimes Id_V] \\
\mathbb{K} \otimes V_1 & \xrightarrow{[\eta \otimes Id_V]} & V_2 \otimes D \otimes V_1 & \xleftarrow{s_2} & D \otimes V_1 \otimes V_2 \\
\downarrow & & \downarrow [Id_V \otimes \epsilon] & & \downarrow [\epsilon \otimes Id_V] \\
V & \xleftarrow{\quad} & V_2 \otimes \mathbb{K} & \xleftarrow{\quad} & \mathbb{K} \otimes V_2
\end{array}$$

Unlabeled arrows are obvious switching or scalar multiplication. The  $s_1, s_2$  switchings are as indicated by the subscripts. The lower left square is commutative, by the first identity from Definition 2.88, and the other small squares are easy to check, so the large square is commutative, which is the second identity for  $(V, \epsilon \circ s, s \circ \eta)$  from Definition 2.88 applied to  $D$ .

Similarly, in the following diagram,  $D = D_1 = D_2$ :

$$\begin{array}{ccccc}
\mathbb{K} \otimes D_1 & \xrightarrow{[\eta \otimes Id_D]} & V \otimes D_2 \otimes D_1 & \xrightarrow{[s \otimes Id_D]} & D_2 \otimes V \otimes D_1 \\
\downarrow & & \downarrow s'_1 & & \downarrow [Id_D \otimes s] \\
D_1 \otimes \mathbb{K} & \xrightarrow{[Id_D \otimes \eta]} & D_1 \otimes V \otimes D_2 & \xleftarrow{s'_2} & D_2 \otimes D_1 \otimes V \\
\downarrow & & \downarrow [\epsilon \otimes Id_D] & & \downarrow [Id_D \otimes \epsilon] \\
D & \xleftarrow{\quad} & \mathbb{K} \otimes D_2 & \xleftarrow{\quad} & D \otimes \mathbb{K}
\end{array}$$

Again, the lower left square is commutative by hypothesis, and the commutativity of the large square is the first identity for  $(V, \epsilon \circ s, s \circ \eta)$  from Definition 2.88 applied to  $D$ . ■

LEMMA 2.95. *If  $V$  is dualizable, then  $d_V$  is invertible.*

PROOF. Let  $a_1 : D \rightarrow V^*$  be the invertible map from Lemma 2.91, defined in terms of  $\epsilon$  and  $A_1 = A$  from (2.11). The transposes of these maps appear in the right square of the diagram.

By Lemma 2.94,  $D$  is also dualizable, with an evaluation map  $\epsilon \circ s$ , which defines  $A_2$  as in (2.11) and an invertible map  $a_2 : V \rightarrow D^*$  from Lemma 2.91 again. These maps appear in the top square of the diagram.

$$\begin{array}{ccccc}
 \text{Hom}(\mathbb{K}, V) & \xrightarrow{A_2} & (\mathbb{K} \otimes D)^* & & \\
 m_V \uparrow & & l_D^* \uparrow & & \\
 V & \xrightarrow{a_2} & D^* & \xleftarrow{m_D^*} & (\text{Hom}(\mathbb{K}, D))^* \\
 & \searrow d_V & \uparrow a_1^* & & \uparrow A_1^* \\
 & & V^{**} & \xleftarrow{l_V^{**}} & (\mathbb{K} \otimes V)^{**}
 \end{array}$$

The two squares in the diagram are commutative by construction. The following calculation checks that  $a_1^* \circ d_V : V \rightarrow D^*$  is equal to  $a_2$ .

$$\begin{aligned}
 l_D^* \circ a_1^* \circ d_V &= l_D^* \circ m_D^* \circ A_1^* \circ (l_V^{**})^{-1} \circ d_V \\
 &= ((l_V^{-1})^* \circ A_1 \circ m_D \circ l_D)^* \circ d_V : \\
 v &\mapsto (d_V(v)) \circ ((l_V^{-1})^* \circ A_1 \circ m_D \circ l_D) : \\
 \lambda \otimes u &\mapsto (d_V(v))((A_1(m_D(\lambda \cdot u))) \circ l_V^{-1}) \\
 &= (A_1(m_D(\lambda \cdot u)))(1 \otimes v) \\
 &= \epsilon(((m_D(\lambda \cdot u))(1)) \otimes v) \\
 &= \epsilon((\lambda \cdot u) \otimes v), \\
 A_2 \circ m_V : v &\mapsto A_2(m_V(v)) : \\
 \lambda \otimes u &\mapsto (\epsilon \circ s)((m_V(v))(\lambda)) \otimes u \\
 &= (\epsilon \circ s)((\lambda \cdot v) \otimes u) \\
 &= \epsilon(u \otimes (\lambda \cdot v)).
 \end{aligned}$$

It follows that  $d_V = (a_1^*)^{-1} \circ a_2$  is invertible. ■

THEOREM 2.96. *Given  $V$ , the following are equivalent.*

- (1)  $k : V^* \otimes V \rightarrow \text{End}(V)$  is invertible;
- (2)  $V$  is dualizable;
- (3)  $d : V \rightarrow V^{**}$  is invertible;
- (4)  $V$  is finite-dimensional.

PROOF. The Proof of Theorem 2.87 only used the property that  $k$  is invertible to show that  $V$  is dualizable, with  $D = V^*$ ,  $\epsilon = Ev_V$ , and  $\eta = s \circ k^{-1} \circ Q_1^1$ ; this is the implication (1)  $\implies$  (2). Lemma 2.95 just showed (2)  $\implies$  (3), and Theorem 2.77 showed directly that (1)  $\implies$  (3). The implications (3)  $\implies$  (4)  $\implies$  (1) were stated without proof in Lemma 1.12 and Lemma 1.28. ■

REMARK 2.97. In the special case where  $(D, \epsilon, \eta) = (V^*, Ev_V, \eta)$  from Example 2.89, the map  $a_1$  from Lemma 2.95 is  $Id_{V^*}$  as in Exercise 2.93, and the map  $a_2$  is exactly  $d_V$ . This shows that Lemma 2.91 (establishing that  $A_2$  has an inverse,  $B$ ) is related to Theorem 2.77 (showing that  $d_V$  has an inverse); the second diagram from the Proof of Theorem 2.77 is similar to the right diagram from the Proof of Lemma 2.91.

The following result is a generalization of Theorem 2.67.

THEOREM 2.98. *If  $V$  is dualizable, with any triple  $(D_2, \epsilon_2, \eta_2)$  as in Definition 2.88, and  $s_2 : V \otimes D_2 \rightarrow D_2 \otimes V$  is the switching map, then for any  $F : V \otimes U \rightarrow V \otimes W$ ,*

$$(Tr_{V;U,W}(F))(u) = (l_W \circ [\epsilon_2 \otimes Id_W] \circ [Id_{D_2} \otimes F] \circ [(s_2 \circ \eta_2) \otimes Id_U] \circ l_U^{-1})(u).$$

PROOF. By Theorem 2.96,  $V$  must be finite-dimensional, so the trace exists. By Theorem 2.87 (the Proof of which uses Theorem 2.67) and Example 2.89, there is another triple  $(D_1, \epsilon_1, \eta_1) = (V^*, Ev_V, \eta)$  satisfying Definition 2.88. There is an invertible map  $a_{12} : V^* \rightarrow D_2$  by Lemma 2.92. Consider the following diagram.

$$\begin{array}{ccccccc}
 & & & & & & W \\
 & & & & & & \uparrow l_W \\
 & & & & & & \mathbb{K} \otimes W \\
 & & & & & & \uparrow [\epsilon_2 \otimes Id_W] \\
 \text{End}(V) \otimes U & \xleftarrow{[k \otimes Id_U]} & V^* \otimes V \otimes U & \xrightarrow{[Id_{V^*} \otimes F]} & V^* \otimes V \otimes W & \xrightarrow{[Ev_V \otimes Id_W]} & \\
 \uparrow [Q_1^1 \otimes Id_U] & & \downarrow [s \otimes Id_U] & & \searrow & & \\
 \mathbb{K} \otimes U & \xrightarrow{[\eta \otimes Id_U]} & V \otimes V^* \otimes U & & & & \\
 \uparrow l_U^{-1} & & \downarrow [[Id_V \otimes a_{12}] \otimes Id_U] & & & & \\
 U & \xrightarrow{[\eta_2 \otimes Id_U]} & V \otimes D_2 \otimes U & \xrightarrow{[s_2 \otimes Id_U]} & D_2 \otimes V \otimes U & \xrightarrow{[Id_{D_2} \otimes F]} & D_2 \otimes V \otimes W
 \end{array}$$

$[[a_{12} \otimes Id_V] \otimes Id_W]$  (diagonal arrow from  $V^* \otimes V \otimes U$  to  $D_2 \otimes V \otimes U$ )

The composition from  $U$  to  $W$  along the top row gives  $Tr_{V;U,W}(F)$  by Theorem 2.67. The left square is from Theorem 2.87 and the left and right triangles are commutative by Lemma 2.92. The RHS of the Theorem is the path from  $U$  to  $W$  along the bottom row, so the claimed equality follows from the easily checked commutativity of the middle block.  $\blacksquare$

COROLLARY 2.99. *If  $V$  is dualizable, with any triple  $(D, \epsilon, \eta)$  as in Definition 2.88, and  $s : V \otimes D \rightarrow D \otimes V$  is the switching map, then for any  $A : V \rightarrow V \otimes W$ ,*

$$Tr_{V;W}(A) = (l_W \circ [\epsilon \otimes Id_W] \circ [Id_D \otimes A] \circ s \circ \eta)(1).$$

PROOF. This follows from Theorem 2.98 in the same way that Corollary 2.68 follows from Theorem 2.67. By Theorem 2.55,

$$\begin{aligned}
 LHS &= (Tr_{V;\mathbb{K},W}(A \circ l_V))(1) \\
 &= (l_W \circ [\epsilon \otimes Id_W] \circ [Id_D \otimes (A \circ l_V)] \circ [(s \circ \eta) \otimes Id_{\mathbb{K}}] \circ l_{\mathbb{K}}^{-1})(1) \\
 &= RHS.
 \end{aligned}$$

$\blacksquare$

This generalizes Corollary 2.68 by showing that, for any duality data  $(D, \epsilon, \eta)$ , the  $W$ -valued trace of  $A$  is the image of 1 under the composite map

$$\mathbb{K} \xrightarrow{\eta} V \otimes D \xrightarrow{s} D \otimes V \xrightarrow{[Id_D \otimes A]} D \otimes V \otimes W \xrightarrow{l_W \circ [\epsilon \otimes Id_W]} W.$$

COROLLARY 2.100. *If  $V$  is dualizable, with any triple  $(D, \epsilon, \eta)$  as in Definition 2.88, and  $s : V \otimes D \rightarrow D \otimes V$  is the switching map, then for any  $A : V \rightarrow V$ ,*

$$Tr_V(A) = (\epsilon \circ [Id_D \otimes A] \circ s \circ \eta)(1).$$

PROOF. By Theorem 2.54 and Corollary 2.99,

$$\begin{aligned} LHS &= Tr_{V;\mathbb{K}}(l_V^{-1} \circ A) \\ &= (l_{\mathbb{K}} \circ [\epsilon \otimes Id_{\mathbb{K}}] \circ [Id_D \otimes (l_V^{-1} \circ A)] \circ s \circ \eta)(1) \\ &= RHS. \end{aligned}$$

■

This generalizes Theorem 2.10 by showing that for any  $(D, \epsilon, \eta)$ , the trace of  $A \in \text{End}(V)$  is the image of 1 under the composite map

$$\mathbb{K} \xrightarrow{\eta} V \otimes D \xrightarrow{s} D \otimes V \xrightarrow{[Id_D \otimes A]} D \otimes V \xrightarrow{\epsilon} \mathbb{K}.$$

EXERCISE 2.101. Given  $V$  dualizable with  $(D, \epsilon, \eta)$ , and any space  $U$ , the arrows in this diagram are invertible:

$$\begin{array}{c} \text{Hom}(D \otimes U, \text{Hom}(V, U)) \\ \downarrow \text{Hom}(s, Id_{\text{Hom}(V, U)}) \\ \text{Hom}(U \otimes D, \text{Hom}(V, U)) \\ \downarrow q \\ \text{Hom}(U \otimes D \otimes V, U) \\ \uparrow \text{Hom}(Id_{U \otimes D \otimes V}, l_U) \\ \text{Hom}(U \otimes D \otimes V, U \otimes \mathbb{K}) \end{array}$$

There is a distinguished element  $[Id_U \otimes \epsilon] \in \text{Hom}(U \otimes D \otimes V, U \otimes \mathbb{K})$ . In the case  $(D, \epsilon, \eta) = (V^*, Ev_V, \eta)$  from Example 2.89, the image of the distinguished element  $[Id_U \otimes Ev_V]$  under the upward composition is the distinguished element  $k_{VU} \in \text{Hom}(V^* \otimes U, \text{Hom}(V, U))$ . ■

BIG EXERCISE 2.102. For a dualizable space  $V$ , define (as in [PS] §2) the mate of a map  $A : V \otimes U \rightarrow V \otimes W$  with respect to duality data  $(D, \epsilon, \eta)$  as the map  $A^m : D \otimes U \rightarrow D \otimes W$  given by the composition in the following diagram:

$$\begin{array}{ccc}
 D \otimes V \otimes U \otimes D & \xrightarrow{[Id_D \otimes [A \otimes Id_D]]} & D \otimes V \otimes W \otimes D \\
 \uparrow [Id_D \otimes [s \otimes Id_D]] & & \downarrow [\epsilon \otimes Id_{W \otimes D}] \\
 D \otimes U \otimes V \otimes D & & \mathbb{K} \otimes W \otimes D \\
 \uparrow [Id_{D \otimes U} \otimes \eta] & & \downarrow l \\
 D \otimes U \otimes \mathbb{K} & & W \otimes D \\
 \uparrow l^{-1} & & \downarrow s \\
 D \otimes U & \xrightarrow{A^m} & D \otimes W
 \end{array}$$

Then (as in [PS] §7),

$$Tr_{D;U,W}(A^m) = Tr_{V;U,W}(A).$$

In particular, LHS does not depend on the choice of  $(D, \epsilon, \eta)$ .

HINT. By Lemma 2.94,  $D$  is dualizable, so by Theorem 2.98, LHS exists. The formula from Theorem 2.98 does not depend on the choice of duality data for  $D$ ; it is convenient to choose to use  $(V, \epsilon \circ s, s \circ \eta)$  from Lemma 2.94. In the following diagram, the lower middle block uses maps from the above definition of  $A^m$ , including

$$a = [Id_V \otimes [Id_D \otimes [A \otimes Id_D]]].$$

$$\begin{array}{ccccccc}
 V \otimes D \otimes U & \xrightarrow{[s \otimes Id_U]} & D \otimes V \otimes U & \xrightarrow{[Id_D \otimes A]} & D \otimes V \otimes W & \xrightarrow{[\epsilon \otimes Id_W]} & \mathbb{K} \otimes W \\
 \uparrow [\eta \otimes Id_U] & & \downarrow & & \uparrow & & \downarrow l \\
 \mathbb{K} \otimes U & & D \otimes \mathbb{K} \otimes V \otimes U & & \mathbb{K} \otimes D \otimes V \otimes W & & \\
 \uparrow & & \downarrow & & \uparrow & & \\
 U & & D \otimes V \otimes D \otimes V \otimes U & & D \otimes V \otimes D \otimes V \otimes W & & \\
 \downarrow & & \downarrow s_0 & & \uparrow & & \\
 \mathbb{K} \otimes U & & V \otimes D \otimes V \otimes U \otimes D & \xrightarrow{a} & V \otimes D \otimes V \otimes W \otimes D & & W \\
 \downarrow [(s \circ \eta) \otimes Id_U] & & \uparrow & & \downarrow & & \downarrow l \\
 D \otimes V \otimes U & \xrightarrow{[s^{-1} \otimes Id_U]} & V \otimes D \otimes U \otimes \mathbb{K} & \xrightarrow{[Id_V \otimes A^m]} & V \otimes D \otimes W & \xrightarrow{[(\epsilon \circ s) \otimes Id_W]} & \mathbb{K} \otimes W
 \end{array}$$

By Theorem 2.98, the path from  $U$  to  $W$  along the top row is  $Tr_{V;U,W}(A)$ , and from  $U$  to  $W$  along the bottom row is  $Tr_{D;U,W}(A^m)$ . The claimed equality follows from the commutativity of the diagram. The maps in the top middle block are as in



the next diagram, with notation  $D = D_1 = D_2$  and  $V = V_1 = V_2 = V_3$  to indicate various switching maps.

$$\begin{array}{ccc}
D_2 \otimes V_2 \otimes U & \xrightarrow{[Id_D \otimes A]} & D_1 \otimes V_3 \otimes W \\
\uparrow [l \otimes Id_{V \otimes U}] & \nearrow [l \otimes A] & \uparrow [l \otimes Id_W] \\
D_2 \otimes \mathbb{K} \otimes V_2 \otimes U & & \mathbb{K} \otimes D_1 \otimes V_3 \otimes W \\
\downarrow [[Id_D \otimes \eta] \otimes Id_{V \otimes U}] & \nearrow [[\epsilon \otimes Id_D] \otimes A] & \uparrow [\epsilon \otimes Id_{D \otimes V \otimes W}] \\
D_2 \otimes V_1 \otimes D_1 \otimes V_2 \otimes U & \xrightarrow{[Id_{D \otimes V \otimes D} \otimes A]} & D_2 \otimes V_1 \otimes D_1 \otimes V_3 \otimes W \\
\downarrow s_0 & & \downarrow s \\
V_1 \otimes D_1 \otimes V_2 \otimes U \otimes D_2 & \xrightarrow{a} & V_1 \otimes D_1 \otimes V_3 \otimes W \otimes D_2
\end{array}$$

The above diagram is commutative; the middle block involving both  $\eta$  and  $\epsilon$  uses one of the properties from Definition 2.88. Using these maps as specified in the above right column, it is also easy to check that the right block from the big diagram is commutative.

To check that the left block from the big diagram is commutative, note that starting with  $u \in U$ , the path going up to  $D \otimes V \otimes U$  has image  $((s \circ \eta)(1)) \otimes u$ , and the downward path also takes  $u$  to  $((s \circ \eta)(1)) \otimes u$ . So, it is enough to check that  $((s \circ \eta)(1)) \otimes u$  has the same image along the two paths leading to  $V \otimes D \otimes V \otimes U \otimes D$ . This is shown in the next diagram, where the numbering  $V = V_1 = V_2$  and  $D = D_1 = D_2$  is chosen to match the left column of the previous diagram.

$$\begin{array}{ccccc}
& & & & V_1 \otimes D_1 \otimes V_2 \otimes U \otimes D_2 \\
& & & \nearrow s_0 & \uparrow \\
& & & [Id_{V \otimes D} \otimes s] & [Id_V \otimes [Id_D \otimes [s \otimes Id_D]]] \\
D_2 \otimes V_1 \otimes D_1 \otimes V_2 \otimes U & \xleftarrow{[s \otimes Id_{V \otimes U}]} & V_1 \otimes D_1 \otimes D_2 \otimes V_2 \otimes U & & V_1 \otimes D_1 \otimes U \otimes V_2 \otimes D_2 \\
& \uparrow [Id_{V \otimes D} \otimes s_1^{-1}] & \nearrow [Id_{V \otimes D \otimes U} \otimes s] & & \uparrow \\
& [[Id_D \otimes \eta] \otimes Id_{V \otimes U}] & V_1 \otimes D_1 \otimes U \otimes D_2 \otimes V_2 & \nearrow [Id_V \otimes [Id_D \otimes v \otimes \eta]] & \\
& \uparrow [\eta \otimes s_1] & \uparrow [Id_{V \otimes D} \otimes s] & & \\
D \otimes \mathbb{K} \otimes V \otimes U & \xleftarrow{[s \otimes Id_{V \otimes U}]} & \mathbb{K} \otimes D \otimes V \otimes U & & V \otimes D \otimes U \otimes \mathbb{K} \\
& \uparrow [[Id_{\mathbb{K}} \otimes s] \otimes Id_U] & \uparrow [Id_V \otimes l] & & \downarrow [Id_V \otimes l] \\
& [l \otimes Id_{V \otimes U}] & \mathbb{K} \otimes V \otimes D \otimes U & \searrow l & \\
D \otimes V \otimes U & \xleftarrow{[s \otimes Id_U]} & V \otimes D \otimes U & & 
\end{array}$$

All the blocks in the last diagram are commutative, except for the center right block. However, starting with  $(\eta(1)) \otimes u \in V \otimes D \otimes U$  in the lower right corner, all paths leading upward to  $V \otimes D \otimes V \otimes U \otimes D$  give the same image. The upward

right column, and the clockwise path around the left side, correspond, respectively, to the lower half and the upper half of the left block in the big diagram. ■

## CHAPTER 3

### Bilinear Forms

A bilinear form over the field  $\mathbb{K}$  has as input two elements of a vector space  $V$  and as output an element of  $\mathbb{K}$ , and is  $\mathbb{K}$ -linear in either input when the other is fixed. This section examines the trace of a bilinear form on a finite-dimensional  $V$ , with respect to a metric  $g$  on  $V$ .

#### 3.1. Metrics

DEFINITION 3.1. A bilinear form  $h$  on a vector space  $V$  is a  $\mathbb{K}$ -linear map  $h : V \rightarrow V^*$ .

For vectors  $u, v \in V$ , a bilinear form  $h$  acts on  $u$  to give an element of the dual,  $h(u) \in V^*$ , which then acts on  $v$  to give  $(h(u))(v) \in \mathbb{K}$ .

DEFINITION 3.2. The transpose operator,  $T_V \in \text{End}(\text{Hom}(V, V^*))$ , is defined by  $T_V = \text{Hom}(d_V, Id_{V^*}) \circ t_{VV^*} : h \mapsto h^* \circ d_V$ .

The effect of this operator is to switch the two inputs:

$$((T_V(h))(u))(v) = ((h^* \circ d_V)(u))(v) = (d_V(u))(h(v)) = (h(v))(u).$$

LEMMA 3.3.  $T_V$  is an involution.

PROOF. The claim is obvious, but it could also be checked directly from the definition, or as a corollary of Lemma 4.3 or Lemma 4.5. ■

DEFINITION 3.4. A bilinear form  $h$  is symmetric means:  $h = T_V(h)$ .  $h$  is antisymmetric means:  $h = -T_V(h)$ .

If  $h$  is symmetric, then  $(h(u))(v) = (h(v))(u)$ , and if  $h$  is antisymmetric, then  $(h(u))(v) = -(h(v))(u)$ .

If  $\frac{1}{2} \in \mathbb{K}$ , then any bilinear form  $h$  is the sum of a symmetric form and an antisymmetric form,

$$h = \frac{1}{2} \cdot (h + T_V(h)) + \frac{1}{2} \cdot (h - T_V(h)).$$

The space  $\text{Hom}(V, V^*)$  is thus, as in Lemma 1.79, a direct sum  $\text{Sym}(V) \oplus \text{Alt}(V)$  of the subspaces of symmetric and antisymmetric forms on  $V$ .

DEFINITION 3.5. A metric  $g$  on  $V$  is a symmetric, invertible map  $g : V \rightarrow V^*$ .

The invertibility property implies a non-degeneracy condition: for each non-zero  $v \in V$ , there exists a vector  $u \in V$  so that  $(g(v))(u) \neq 0$ . Also, by the following Theorem, a metric exists only on finite-dimensional vector spaces.

**THEOREM 3.6.** *Given a symmetric (or antisymmetric) bilinear form  $g : V \rightarrow V^*$ , the following are equivalent:*

- (1)  *$V$  is finite-dimensional and there exists  $P : V^* \rightarrow V$  such that  $P \circ g = Id_V$ ;*
- (2)  *$V$  is finite-dimensional and there exists  $Q : V^* \rightarrow V$  such that  $g \circ Q = Id_{V^*}$ ;*
- (3)  *$g$  is invertible.*

**PROOF.** Let  $g$  be symmetric; the antisymmetric case is similar.

If  $g$  is invertible, then from  $g^* \circ d_V = g$ ,

$$(g^{-1})^* \circ g = (g^*)^{-1} \circ g = d_V$$

is invertible, so  $V$  is finite-dimensional by Lemma 1.12, which implies (1) and (2).

Assuming (1),

$$Id_{V^*} = Id_{V^*}^* = (P \circ g)^* = g^* \circ P^* = g \circ d_V^{-1} \circ P^*,$$

so  $g$  has a left inverse and a right inverse, and (3) follows. Similarly, assuming (2),

$$Id_{V^{**}} = Id_{V^*}^* = (g \circ Q)^* = Q^* \circ g^* = Q^* \circ g \circ d_V^{-1},$$

so  $d_V = Q^* \circ g \implies Id_V = d_V^{-1} \circ Q^* \circ g$ , and  $g$  has a left inverse and a right inverse. ■

**THEOREM 3.7.** *Given a metric  $g$  on  $V$ , the form  $d_V \circ g^{-1} : V^* \rightarrow V^{**}$  is a metric on  $V^*$ .*

**PROOF.** To show  $d_V \circ g^{-1}$  is symmetric, use the definition of  $T_{V^*}$  and Lemma 1.13:

$$T_{V^*}(d_V \circ g^{-1}) = (d_V \circ g^{-1})^* \circ d_{V^*} = (g^{-1})^* \circ d_V^* \circ d_{V^*} = (g^*)^{-1} = d_V \circ g^{-1}.$$

The last step uses the symmetry of  $g$ . This map is obviously invertible. ■

The form  $d_V \circ g^{-1}$  could be called the metric induced by  $g$  on  $V^*$ , or the dual metric. It acts on elements  $\phi, \xi \in V^*$  as

$$((d_V \circ g^{-1})(\phi))(\xi) = \xi(g^{-1}(\phi)).$$

**NOTATION 3.8.** If an arbitrary vector space  $V$  is a direct sum of  $V_1$  and  $V_2$ , as in Definition 1.43, and  $h_1 : V_1 \rightarrow V_1^*$ ,  $h_2 : V_2 \rightarrow V_2^*$ , then

$$(3.1) \quad P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2 : V \rightarrow V^*$$

will be called the direct sum  $h_1 \oplus h_2$  of the bilinear forms  $h_1$  and  $h_2$ .

The expression (3.1) is the same construction as in Lemma 1.50, applied to the direct sum  $V^* = V_1^* \oplus V_2^*$  from Example 1.48.

**THEOREM 3.9.**  $T_V(h_1 \oplus h_2) = (T_{V_1}(h_1)) \oplus (T_{V_2}(h_2))$ .

**PROOF.**

$$\begin{aligned} LHS &= (P_1^* \circ h_1 \circ P_1)^* \circ d_V + (P_2^* \circ h_2 \circ P_2)^* \circ d_V \\ &= P_1^* \circ h_1^* \circ P_1^{**} \circ d_V + P_2^* \circ h_2^* \circ P_2^{**} \circ d_V \\ &= P_1^* \circ h_1^* \circ d_{V_1} \circ P_1 + P_2^* \circ h_2^* \circ d_{V_2} \circ P_2 \\ &= P_1^* \circ (T_{V_1}(h_1)) \circ P_1 + P_2^* \circ (T_{V_2}(h_2)) \circ P_2 = RHS, \end{aligned}$$

using Lemmas 1.5 and 1.10. ■

It follows that the direct sum of symmetric forms is symmetric, and that the direct sum of antisymmetric forms is antisymmetric.

**COROLLARY 3.10.** *If  $g_1$  is a metric on  $V_1$  and  $g_2$  is a metric on  $V_2$ , then  $g_1 \oplus g_2$  is a metric on  $V = V_1 \oplus V_2$ .*

**PROOF.**  $g_1 \oplus g_2$  is symmetric by the previous Theorem, and is invertible by Lemma 1.50. Specifically, the inclusion maps  $Q_1, Q_2$  are used to construct an inverse to the expression (3.1):

$$(3.2) \quad \begin{aligned} (Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) \circ (P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2) &= Id_V, \\ (P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2) \circ (Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) &= Id_{V^*}. \end{aligned}$$

■

The following Lemma will be convenient in some of the theorems about the tensor product of symmetric forms.

**LEMMA 3.11.** ([B] §II.4.4) *For  $A : U_1 \rightarrow U_2$  and  $B : V_1 \rightarrow V_2$ , the following diagram is commutative.*

$$\begin{array}{ccc} U_2^* \otimes V_2^* & \xrightarrow{[A^* \otimes B^*]} & U_1^* \otimes V_1^* \\ \downarrow j & & \downarrow j \\ \text{Hom}(U_2 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) & \xrightarrow{\text{Hom}([A \otimes B], Id_{\mathbb{K} \otimes \mathbb{K}})} & \text{Hom}(U_1 \otimes V_1, \mathbb{K} \otimes \mathbb{K}) \\ \downarrow \text{Hom}(Id_{U_2 \otimes V_2}, l) & & \downarrow \text{Hom}(Id_{U_1 \otimes V_1}, l) \\ (U_2 \otimes V_2)^* & \xrightarrow{[A \otimes B]^*} & (U_1 \otimes V_1)^* \end{array}$$

**PROOF.** The scalar multiplication  $\mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$  is denoted  $l$ . The top square is commutative by Lemma 1.26, and the lower one by Lemma 1.5. ■

**THEOREM 3.12.** *If  $h_1 : U \rightarrow U^*$  and  $h_2 : V \rightarrow V^*$ , then*

$$\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2] : (U \otimes V) \rightarrow (U \otimes V)^*$$

*is a bilinear form such that*

$$T_{U \otimes V}(\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2])$$

*is equal to*

$$\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [(T_U(h_1)) \otimes (T_V(h_2))].$$

**PROOF.** First, for any  $U, V$ , the following diagram is commutative:

$$\begin{array}{ccccc} U \otimes V & \xrightarrow{d_{U \otimes V}} & (U \otimes V)^{**} & \xrightarrow{\text{Hom}(Id_{U \otimes V}, l)^*} & \text{Hom}(U \otimes V, \mathbb{K} \otimes \mathbb{K})^* \\ \downarrow [d_U \otimes d_V] & & & & \downarrow j^* \\ U^{**} \otimes V^{**} & \xrightarrow{j} & \text{Hom}(U^* \otimes V^*, \mathbb{K} \otimes \mathbb{K}) & \xrightarrow{\text{Hom}(Id_{U^* \otimes V^*}, l)} & (U^* \otimes V^*)^* \end{array}$$

$$\begin{aligned}
u \otimes v &\mapsto (j^* \circ \text{Hom}(Id_{U \otimes V}, l)^* \circ d_{U \otimes V})(u \otimes v) : \\
\phi \otimes \xi &\mapsto (d_{U \otimes V}(u \otimes v))(\text{Hom}(Id_{U \otimes V}, l)([\phi \otimes \xi])) \\
&= l([\phi \otimes \xi](u \otimes v)) \\
&= \phi(u) \cdot \xi(v), \\
u \otimes v &\mapsto (\text{Hom}(Id_{U^* \otimes V}, l) \circ j \circ [d_U \otimes d_V])(u \otimes v) \\
&= l \circ [(d_U(u)) \otimes (d_V(v))] : \\
\phi \otimes \xi &\mapsto \phi(u) \cdot \xi(v).
\end{aligned}$$

Note that the bottom row of the diagram is one of the columns of the diagram in the previous lemma in the case  $U_2 = U^*$ ,  $V_2 = V^*$ . The statement of the Theorem follows, using the above commutativity and the previous lemma.

$$\begin{aligned}
T_{U \otimes V}(h_1 \otimes h_2) &= (\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2])^* \circ d_{U \otimes V} \\
&= [h_1 \otimes h_2]^* \circ j^* \circ \text{Hom}(Id_{U \otimes V}, l)^* \circ d_{U \otimes V} \\
&= [h_1 \otimes h_2]^* \circ \text{Hom}(Id_{U^* \otimes V^*}, l) \circ j \circ [d_U \otimes d_V] \\
&= \text{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1^* \otimes h_2^*] \circ [d_U \otimes d_V] \\
&= \text{Hom}(Id_{U \otimes V}, l) \circ j \circ [(T_U(h_1)) \otimes (T_V(h_2))].
\end{aligned}$$

■

NOTATION 3.13. The form  $\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2]$  from the above Theorem will be called the tensor product of bilinear forms, and denoted  $\{h_1 \otimes h_2\}$ , in analogy with the bracket notation. As defined, the tensor product form acts as

$$(\{h_1 \otimes h_2\}(u_1 \otimes v_1))(u_2 \otimes v_2) = (h_1(u_1))(u_2) \cdot (h_2(v_1))(v_2).$$

When  $h_1$  and  $h_2$  are symmetric forms, it is clear from this formula that  $\{h_1 \otimes h_2\}$  is also symmetric, but the above proof, using Definition 3.4, makes explicit the roles of the symmetry and the scalar multiplication. It also follows that the tensor product of antisymmetric forms is symmetric.

COROLLARY 3.14. *If  $g_1$  and  $g_2$  are metrics on  $U$  and  $V$ , then  $\{g_1 \otimes g_2\}$  is a metric on  $U \otimes V$ .*

PROOF.  $\{g_1 \otimes g_2\}$  is symmetric, by the Theorem,  $j$  is invertible by the finite-dimensionality, and the inverse of  $\text{Hom}(Id_{U \otimes V}, l) \circ j \circ [g_1 \otimes g_2] : U \otimes V \rightarrow (U \otimes V)^*$  is

$$[g_1^{-1} \otimes g_2^{-1}] \circ j^{-1} \circ \text{Hom}(Id_{U \otimes V}, l^{-1}).$$

■

There is a distributive law for the direct sum and tensor product of bilinear forms. Let  $V$  be a direct sum of  $V_1$  and  $V_2$ , and recall, from Example 1.45, that  $V \otimes U$  is a direct sum of  $V_1 \otimes U$  and  $V_2 \otimes U$ , with projection maps  $[P_i \otimes Id_U]$ .

**THEOREM 3.15.** *For bilinear forms  $h_1, h_2, g$  on arbitrary vector spaces  $V_1, V_2, U$ , the following bilinear forms on  $V \otimes U$  are equal:*

$$\{(h_1 \oplus h_2) \otimes g\} = \{h_1 \otimes g\} \oplus \{h_2 \otimes g\}.$$

**PROOF.** Unraveling the definitions, and applying Lemmas 3.11 and 1.25 gives the claimed equality:

$$\begin{aligned} RHS &= [P_1 \otimes Id_U]^* \circ \text{Hom}(Id_{V_1 \otimes U}, l) \circ j \circ [h_1 \otimes g] \circ [P_1 \otimes Id_U] \\ &\quad + [P_2 \otimes Id_U]^* \circ \text{Hom}(Id_{V_2 \otimes U}, l) \circ j \circ [h_2 \otimes g] \circ [P_2 \otimes Id_U] \\ &= \text{Hom}(Id_{V \otimes U}, l) \circ j \circ [P_1^* \otimes Id_U] \circ [h_1 \otimes g] \circ [P_1 \otimes Id_U] \\ &\quad + \text{Hom}(Id_{V \otimes U}, l) \circ j \circ [P_2^* \otimes Id_U] \circ [h_2 \otimes g] \circ [P_2 \otimes Id_U] \\ &= \text{Hom}(Id_{V \otimes U}, l) \circ j \circ [(P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2) \otimes g] \\ &= LHS. \end{aligned}$$

■

In particular, if  $h_1, h_2$ , and  $g$  are metrics, and  $h = h_1 \oplus h_2$ , then the induced tensor product metric  $\{h \otimes g\}$  on  $V \otimes U$  coincides with the induced direct sum metric on  $(V_1 \otimes U) \oplus (V_2 \otimes U)$ .

### 3.2. Isometries

**DEFINITION 3.16.** For a map  $H : U \rightarrow V$ , and a form  $h : V \rightarrow V^*$ , the map  $H^* \circ h \circ H$  is a form on  $U$ , called the pullback of  $h$ .

**LEMMA 3.17.** *For any map  $H : U \rightarrow V$ ,  $T_U(H^* \circ h \circ H) = H^* \circ (T_V(h)) \circ H$ .*

**PROOF.** Using Lemmas 1.5 and 1.10,

$$\begin{aligned} T_U(H^* \circ h \circ H) &= (H^* \circ h \circ H)^* \circ d_U = H^* \circ h^* \circ H^{**} \circ d_U \\ &= H^* \circ h^* \circ d_V \circ H = H^* \circ (T_V(h)) \circ H. \end{aligned}$$

■

So, if  $h \in \text{Sym}(V)$ , then  $H^* \circ h \circ H \in \text{Sym}(U)$ . The pullback of an antisymmetric form is similarly antisymmetric. Lemma 1.83 applies to the map  $h \mapsto H^* \circ h \circ H$ ; it respects the direct sums  $\text{Sym}(V) \oplus \text{Alt}(V) \rightarrow \text{Sym}(U) \oplus \text{Alt}(U)$ .

**EXAMPLE 3.18.** If  $h$  is a metric on  $V$ , and  $H : U \rightarrow V$  is invertible, then  $H^* \circ h \circ H$  is a metric on  $U$ , since it is symmetric by Lemma 3.17, and has inverse  $H^{-1} \circ h^{-1} \circ (H^*)^{-1}$ .

**REMARK 3.19.** The pullback of a metric  $h$  by an arbitrary linear map  $H$  need not be a metric, for example, the case where  $H$  is the inclusion of a lightlike line in Minkowski space. See also Definition 3.101.

**DEFINITION 3.20.** A  $\mathbb{K}$ -linear map  $H : U \rightarrow V$  is an isometry, with respect to metrics  $g$  on  $U$  and  $h$  on  $V$ , means:  $H$  is invertible, and  $\overline{g} = \overline{H^* \circ h \circ H}$ , so the diagram is commutative,

$$\begin{array}{ccc} U & \xrightarrow{H} & V \\ \downarrow g & & \downarrow h \\ U^* & \xleftarrow{H^*} & V^* \end{array}$$

This means that the metric  $g$  is equal to the pullback of  $h$  by  $H$ , and that for elements of  $U$ ,

$$(g(u_1))(u_2) = (h(H(u_1)))(H(u_2)).$$

It follows immediately from the definition that the composite of isometries is an isometry, that the inverse of an isometry is an isometry, and that  $Id_V$  and  $-Id_V$  are isometries.

REMARK 3.21. The equation  $g = H^* \circ h \circ H$  does not itself require that  $H^{-1}$  exists, and one could consider non-surjective “isometric embeddings,” but invertibility will be assumed as part of the definition, just for convenience.

EXERCISE 3.22. If  $h : V \rightarrow V^*$  and  $H : U \rightarrow V$ , and  $H^* \circ h \circ H : U \rightarrow U^*$  is invertible, then  $H$  is a linear monomorphism. ■

THEOREM 3.23. Any metric  $g : U \rightarrow U^*$  is an isometry with respect to itself,  $g$ , and the dual metric,  $d_U \circ g^{-1}$ .

PROOF. The pullback by  $g$  of the dual metric is

$$g^* \circ d_U \circ g^{-1} \circ g = g,$$

by the symmetry of  $g$ . ■

THEOREM 3.24.  $d_U : U \rightarrow U^{**}$  is an isometry with respect to  $g$  and the dual of the dual metric  $d_{U^*} \circ (d_U \circ g^{-1})^{-1} = d_{U^*} \circ g \circ d_U^{-1}$  on  $U^{**}$ .

PROOF. By the identity  $d_U^* \circ d_{U^*} = Id_{U^*}$  from Lemma 1.13,

$$g = d_U^* \circ d_{U^*} \circ g \circ d_U^{-1} \circ d_U.$$

■

THEOREM 3.25. If  $A : U_1 \rightarrow U_2$  and  $B : V_1 \rightarrow V_2$  are isometries, with respect to  $g_1, g_2, h_1$ , and  $h_2$ , then  $[A \otimes B] : U_1 \otimes V_1 \rightarrow U_2 \otimes V_2$  is an isometry with respect to the induced metrics.

PROOF. The statement of the Theorem is that

$$\{g_1 \otimes h_1\} = [A \otimes B]^* \circ \{g_2 \otimes h_2\} \circ [A \otimes B].$$

The RHS can be expanded, and then Lemma 3.11 applies:

$$\begin{aligned} RHS &= [A \otimes B]^* \circ \text{Hom}(Id_{U_2 \otimes V_2}, l) \circ j \circ [g_2 \otimes h_2] \circ [A \otimes B] \\ &= \text{Hom}(Id_{U_1 \otimes V_1}, l) \circ j \circ [A^* \otimes B^*] \circ [g_2 \otimes h_2] \circ [A \otimes B] \\ &= \text{Hom}(Id_{U_1 \otimes V_1}, l) \circ j \circ [g_1 \otimes h_1] = LHS. \end{aligned}$$

The last step uses Lemma 1.25 and  $g_1 = A^* \circ g_2 \circ A$ ,  $h_1 = B^* \circ h_2 \circ B$ . ■

LEMMA 3.26. For any  $U, V$ , the following diagram is commutative.

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{f_{UV}} & (V^* \otimes U)^* \\ \downarrow p & & \uparrow \text{Hom}(Id_{V^* \otimes U}, l) \\ V^{**} \otimes U^* & \xrightarrow{j} & \text{Hom}(V^* \otimes U, \mathbb{K} \otimes \mathbb{K}) \end{array}$$



PROOF. Recall  $f$  and  $p$  are the maps from Notation 1.33 and Notation 1.36.

$$\begin{aligned}\phi \otimes v &\mapsto (\text{Hom}(Id_{V^* \otimes U}, l) \circ j \circ p)(\phi \otimes v) = l \circ [(d_V(v)) \otimes \phi] : \\ \xi \otimes u &\mapsto \xi(v) \cdot \phi(u) \\ &= (f(\phi \otimes v))(\xi \otimes u).\end{aligned}$$

■

LEMMA 3.27. *If  $F : V_1 \rightarrow V_2^*$ ,  $E : U_1^* \rightarrow U_2$ , and  $V_1$  is finite-dimensional, then the following diagram is commutative.*

$$\begin{array}{ccc} U_1^* \otimes V_1 & \xrightarrow{p_1} & V_1^{**} \otimes U_1^* \\ \downarrow [(d_{U_2} \circ E) \otimes F] & & \downarrow [(F \circ d_{V_1}^{-1}) \otimes E] \\ U_2^{**} \otimes V_2^* & \xleftarrow{p_2} & V_2^* \otimes U_2 \end{array}$$

PROOF.

$$\begin{aligned}\phi \otimes v &\mapsto (p_2 \circ [(F \circ d_{V_1}^{-1}) \otimes E] \circ p_1)(\phi \otimes v) \\ &= (p_2 \circ [(F \circ d_{V_1}^{-1}) \otimes E])((d_{V_1}(v)) \otimes \phi) \\ &= p_2((F(v)) \otimes (E(\phi))) \\ &= (d_{V_2}(E(\phi))) \otimes (F(v)) = [(d_{V_2} \circ E) \otimes F](\phi \otimes v).\end{aligned}$$

■

THEOREM 3.28. *The canonical map  $f_{UV} : U^* \otimes V \rightarrow (V^* \otimes U)^*$  is an isometry, with respect to metrics induced by a metric  $g$  on  $U$  and a metric  $h$  on  $V$ .*

PROOF. The diagram is commutative, where the compositions in the left and right columns define the induced metrics, and there are two different  $p$  maps:

$$\begin{array}{ccc} U^* \otimes V & \xrightarrow{f_{UV}} & (V^* \otimes U)^* \\ \downarrow [(d_U \circ g^{-1}) \otimes h] & \searrow p & \downarrow \text{Hom}(Id_{V^* \otimes U}, l) \\ U^{**} \otimes V^* & & V^{**} \otimes U^* \\ \downarrow j & \swarrow p' & \downarrow [(d_V \circ h^{-1}) \otimes g] \\ \text{Hom}(U^* \otimes V, \mathbb{K} \otimes \mathbb{K}) & & V^* \otimes U \\ \downarrow \text{Hom}(Id_{U^* \otimes V}, l) & \swarrow f_{VU} & \downarrow d_{V^* \otimes U} \\ (U^* \otimes V)^* & \xleftarrow{f_{UV}^*} & (V^* \otimes U)^{**} \end{array}$$

$\text{Hom}(Id_{V^* \otimes U}, l) \updownarrow \text{Hom}(Id_{V^* \otimes U}, l^{-1})$   
 $\text{Hom}(V^* \otimes U, \mathbb{K} \otimes \mathbb{K})$   
 $j \updownarrow j^{-1}$   
 $[(h \circ d_V^{-1}) \otimes g^{-1}]$

The lower triangle is commutative by Lemma 1.34. Two of the loops in the diagram are commutative by Lemma 3.26, and the block in the middle is commutative by Lemma 3.27. ■

LEMMA 3.29. *Let  $U = U_1 \oplus U_2$ , with operators  $Q_i, P_i$  and metric  $g$ , and let  $V = V_1 \oplus V_2$ , with operators  $Q'_i, P'_i$ , and metric  $h$ . Suppose that for  $i = 1$  or  $2$ , the form  $(Q'_i)^* \circ h \circ Q'_i$  is a metric on  $V_i$ . If  $H : U \rightarrow V$  is an isometry that respects the direct sums, then the form  $Q_i^* \circ g \circ Q_i$  is a metric on  $U_i$ , and the induced map  $P'_i \circ H \circ Q_i : U_i \rightarrow V_i$  is an isometry.*

PROOF. The induced map  $P'_i \circ H \circ Q_i$  is invertible, as in Lemma 1.53. The following calculation (which uses the property that  $H$  respects the direct sums) shows that the form  $Q_i^* \circ g \circ Q_i$  is equal to the pullback of  $(Q'_i)^* \circ h \circ Q'_i$  by the map  $P'_i \circ H \circ Q_i$ , so it is a metric as in Example 3.18, and  $P'_i \circ H \circ Q_i$  is an isometry, by Definition 3.20.

$$\begin{aligned}
& (P'_i \circ H \circ Q_i)^* \circ ((Q'_i)^* \circ h \circ Q'_i) \circ (P'_i \circ H \circ Q_i) \\
&= Q_i^* \circ H^* \circ (P'_i)^* \circ (Q'_i)^* \circ h \circ Q'_i \circ P'_i \circ H \circ Q_i \\
&= (Q'_i \circ P'_i \circ H \circ Q_i)^* \circ h \circ Q'_i \circ P'_i \circ H \circ Q_i \\
&= (H \circ Q_i \circ P_i \circ Q_i)^* \circ h \circ H \circ Q_i \circ P_i \circ Q_i \\
&= Q_i^* \circ H^* \circ h \circ H \circ Q_i \\
&= Q_i^* \circ g \circ Q_i.
\end{aligned}$$

■

### 3.3. Trace, with respect to a metric

DEFINITION 3.30. With respect to a metric  $g$  on  $V$ , the trace of a bilinear form  $h$  on  $V$  is defined by

$$Tr_g(h) = Tr_V(g^{-1} \circ h).$$

By Lemma 2.6, this is the same as  $Tr_{V^*}(h \circ g^{-1})$ , and another way to write the definition is

$$Tr_g = \text{Hom}(Id_V, g^{-1})^*(Tr_V) \in \text{Hom}(V, V^*)^*.$$

THEOREM 3.31.  $Tr_g(Tr_V(h)) = Tr_g(h)$ .

PROOF.

$$\begin{aligned}
Tr_g(h^* \circ d_V) &= Tr_{V^*}(h^* \circ d_V \circ g^{-1}) = Tr_{V^*}(h^* \circ (g^{-1})^*) \\
&= Tr_{V^*}((g^{-1} \circ h)^*) = Tr_V(g^{-1} \circ h) = Tr_g(h),
\end{aligned}$$

using the symmetry of  $g$  and Lemma 2.5.

■

COROLLARY 3.32. *If  $\frac{1}{2} \in \mathbb{K}$ , then the trace of an antisymmetric form is 0 with respect to any metric  $g$ .* ■

THEOREM 3.33. *If  $Tr_V(Id_V) \neq 0$  and  $g$  is a metric on  $V$ , then  $\text{Hom}(V, V^*) = \mathbb{K} \oplus \ker(Tr_g)$ .*

PROOF. Since  $Tr_g(g) = Tr_V(Id_V) \neq 0$ , Lemma 1.63 applies. For any  $h : V \rightarrow V^*$  there is a canonical decomposition of  $h$  into a trace zero part, with respect to  $g$ , and a scalar multiple of  $g$ ,

$$h = (h - \frac{Tr_g(h)}{Tr_g(g)} \cdot g) + \frac{Tr_g(h)}{Tr_g(g)} \cdot g.$$

■

COROLLARY 3.34. *If both  $\frac{1}{2} \in \mathbb{K}$  and  $Tr_V(Id_V) \neq 0$ , then  $\text{Hom}(V, V^*)$  admits a direct sum  $\mathbb{K} \oplus \text{Sym}_0(V, g) \oplus \text{Alt}(V)$ , where  $\text{Sym}_0(V, g)$  is the kernel of the restriction of  $Tr_g$  to  $\text{Sym}(V)$ . ■*

PROPOSITION 3.35. *The trace is “invariant under pullback,” that is, for an invertible map  $H : U \rightarrow V$ , and a metric  $g$  on  $V$ ,*

$$Tr_{H^* \circ g \circ H}(H^* \circ h \circ H) = Tr_g(h).$$

PROOF.

$$\begin{aligned} Tr_{H^* \circ g \circ H}(H^* \circ h \circ H) &= Tr_U(H^{-1} \circ g^{-1} \circ (H^*)^{-1} \circ H^* \circ h \circ H) \\ &= Tr_U(H^{-1} \circ g^{-1} \circ h \circ H) \\ &= Tr_V(g^{-1} \circ h) = Tr_g(h), \end{aligned}$$

by Lemma 2.6. ■

PROPOSITION 3.36. *For  $V = V_1 \oplus V_2$ , bilinear forms  $h_1 : V_1 \rightarrow V_1^*$ ,  $h_2 : V_2 \rightarrow V_2^*$ , and metrics  $g_1, g_2$  on  $V_1, V_2$ ,*

$$Tr_{g_1 \oplus g_2}(h_1 \oplus h_2) = Tr_{g_1}(h_1) + Tr_{g_2}(h_2).$$

PROOF. Using the formula (3.2) for  $(g_1 \oplus g_2)^{-1}$  from Corollary 3.10, and Lemma 2.6,

$$\begin{aligned} LHS &= Tr_V((Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) \circ (P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2)) \\ &= Tr_V(Q_1 \circ g_1^{-1} \circ h_1 \circ P_1 + Q_2 \circ g_2^{-1} \circ h_2 \circ P_2) \\ &= Tr_{V_1}(P_1 \circ Q_1 \circ g_1^{-1} \circ h_1) + Tr_{V_2}(P_2 \circ Q_2 \circ g_2^{-1} \circ h_2) = RHS. \end{aligned}$$

PROPOSITION 3.37. *For bilinear forms  $E : U \rightarrow U^*$  and  $F : V \rightarrow V^*$ , and metrics  $g$  and  $h$  on  $U$  and  $V$ ,*

$$Tr_{\{g \otimes h\}}(\{E \otimes F\}) = Tr_g(E) \cdot Tr_h(F).$$

PROOF. Using the formula from Corollary 3.14, there is a convenient cancellation, and then Corollary 2.31 applies:

$$\begin{aligned} Tr_{\{g \otimes h\}}(\{E \otimes F\}) &= Tr_{U \otimes V}([g^{-1} \otimes h^{-1}] \circ j^{-1} \circ \text{Hom}(Id_{U \otimes V}, l^{-1}) \\ &\quad \circ \text{Hom}(Id_{U \otimes V}, l) \circ j \circ [E \otimes F]) \\ &= Tr_{U \otimes V}(j_2((g^{-1} \circ E) \otimes (h^{-1} \circ F))) \\ &= Tr_g(E) \cdot Tr_h(F). \end{aligned}$$

■

### 3.4. The Hilbert-Schmidt metric

DEFINITION 3.38. Given metrics  $g$  and  $h$  on  $U$  and  $V$ , define a bilinear form  $b$  on  $\text{Hom}(U, V)$ , acting on elements  $A, B : U \rightarrow V$  as:

$$(b(B))(A) = \text{Tr}_V(A \circ g^{-1} \circ B^* \circ h).$$

$b$  can be written as a composite:

$$b = \text{Hom}(Id_{\text{Hom}(U, V)}, \text{Tr}_V) \circ t_{VU}^V \circ \text{Hom}(h, g^{-1}) \circ t_{UV},$$

using the generalized transpose  $t_{VU}^V$  (from Definition 1.6). By Lemma 2.6,  $b$  can also be written as a trace with respect to  $g$ :

$$(b(B))(A) = \text{Tr}_U(g^{-1} \circ B^* \circ h \circ A) = \text{Tr}_g(B^* \circ h \circ A).$$

THEOREM 3.39. *The induced tensor product metric on  $U^* \otimes V$  is equal to the pullback of the form  $b$  by the canonical map  $k_{UV}$ .*

PROOF. The diagram is commutative, where the composition in the left column defines the induced metric (as in Theorem 3.28), and the composition in the right column defines the form  $b$ :

$$\begin{array}{ccccc}
 U^* \otimes V & \xrightarrow{k_{UV}} & \text{Hom}(U, V) & & \\
 \downarrow [(d_U \circ g^{-1}) \otimes h] & \searrow p & \downarrow t_{UV} & & \\
 & & V^{**} \otimes U^* & \xrightarrow{k_{V^*U^*}} & \text{Hom}(V^*, U^*) \\
 & & \downarrow [(h \circ d_V^{-1}) \otimes g^{-1}] & & \downarrow \text{Hom}(h, g^{-1}) \\
 U^{**} \otimes V^* & \xleftarrow{p'} & V^* \otimes U & \xrightarrow{k_{VU}} & \text{Hom}(V, U) \\
 \downarrow j & & \downarrow e_{VU} & & \downarrow t_{VU}^V \\
 \text{Hom}(U^* \otimes V, \mathbb{K} \otimes \mathbb{K}) & & & & \text{Hom}(\text{Hom}(U, V), \text{End}(V)) \\
 \downarrow \text{Hom}(Id_{U^* \otimes V}, l) & & & & \downarrow \text{Hom}(Id_{\text{Hom}(U, V)}, \text{Tr}_V) \\
 (U^* \otimes V)^* & \xleftarrow{k_{UV}^*} & \text{Hom}(U, V)^* & & 
 \end{array}$$

The three squares in the upper half of the diagram are commutative, by Lemma 3.27, Lemma 1.37, and Lemma 1.29 (with  $h \circ d_V^{-1} = h^*$  because  $h$  is symmetric). The left triangle in the lower half is commutative by Lemma 3.26 (using  $f_{VU} = e_{VU} \circ k_{VU}$ , from Notation 1.33). Checking the lower right triangle, starting with  $D \in \text{Hom}(V, U)$ , uses  $(k_{UV}(\phi \otimes v)) \circ D = k_{VV}((D^*(\phi)) \otimes v)$ , which follows from Lemma 1.29:

$$\text{Hom}(D, Id_V) \circ k_{UV} = k_{VV} \circ [D^* \otimes Id_V],$$

and the definition of the trace (Definition 2.21):

$$\begin{aligned}
D &\mapsto (k_{UV}^* \circ \text{Hom}(Id_{\text{Hom}(U,V)}, Tr_V) \circ t_{VU}^V)(D) \\
&= Tr_V \circ (t_{VU}^V(D)) \circ k_{UV} : \\
\phi \otimes v &\mapsto Tr_V((t_{VU}^V(D))(k_{UV}(\phi \otimes v))) \\
&= Tr_V((k_{UV}(\phi \otimes v)) \circ D) \\
&= ((k_{UV}^{-1})^*(Ev_V))(k_{UV}((D^*(\phi)) \otimes v)) \\
&= Ev_V((\phi \circ D) \otimes v) \\
&= \phi(D(v)) = (e_{VU}(D))(\phi \otimes v).
\end{aligned}$$

■

COROLLARY 3.40.  $b$  is a metric on  $\text{Hom}(U, V)$ .

PROOF. This follows from Example 3.18, where  $k_{UV}^{-1}$  is the invertible map relating the metric on  $U^* \otimes V$  to the form  $b$ , proving that  $b$  is symmetric and invertible, and  $k_{UV}$  and  $k_{UV}^{-1}$  are isometries. ■

COROLLARY 3.41. The canonical map  $e_{UV} : \text{Hom}(U, V) \rightarrow (V^* \otimes U)^*$  is an isometry, with respect to the induced metrics.

PROOF. This follows from Theorem 3.28, since  $e_{UV} = f_{UV} \circ k_{UV}^{-1}$ . ■

REMARK 3.42. The relationship between the Hilbert-Schmidt metric  $b$  and the tensor product metric seems to be well-known, although possibly not in this generality. A special case of a Hermitian version of Theorem 3.39 appears in [Bhatia] §I.4, and a positive definite version for endomorphisms in [G<sub>2</sub>] §III.4. Matrix versions of Theorem 3.39 appear in [Neudecker], [L], and [HJ] §4.2.

THEOREM 3.43. If  $A : U_2 \rightarrow U_1$  is an isometry with respect to metrics  $g_2, g_1$ , and  $B : V_1 \rightarrow V_2$  is an isometry with respect to metrics  $h_1, h_2$ , then  $\text{Hom}(A, B) : \text{Hom}(U_1, V_1) \rightarrow \text{Hom}(U_2, V_2)$  is an isometry with respect to the induced metrics.

PROOF. The hypotheses are  $h_1 = B^* \circ h_2 \circ B$ , and  $g_2 = A^* \circ g_1 \circ A$ . For  $E, F \in \text{Hom}(U_1, V_2)$ , the pullback of the induced metric on  $\text{Hom}(U_2, V_2)$  is

$$\begin{aligned}
(b(B \circ F \circ A))(B \circ E \circ A) &= Tr_{V_2}(B \circ E \circ A \circ g_2^{-1} \circ A^* \circ F^* \circ B^* \circ h_2) \\
&= Tr_{V_1}(E \circ g_1^{-1} \circ F^* \circ B^* \circ h_2 \circ B) \\
&= Tr_{V_1}(E \circ g_1^{-1} \circ F^* \circ h_1).
\end{aligned}$$

■

THEOREM 3.44. With respect to the  $b$  metrics induced by  $g_1, g_2, h_1, h_2$  on  $U_1, U_2, V_1, V_2$ , the map  $j : \text{Hom}(U_1, V_1) \otimes \text{Hom}(U_2, V_2) \rightarrow \text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2)$  is an isometry.

PROOF. For  $A_1, B_1 : U_1 \rightarrow V_1, A_2, B_2 : U_2 \rightarrow V_2$ , the statement of the Theorem is that the tensor product metric and pullback metric are equal:

$$(b(B_1))(A_1) \cdot (b(B_2))(A_2) = (b(j(B_1 \otimes B_2)))(j(A_1 \otimes A_2)).$$

Computing the RHS, using the metrics  $\{g_1 \otimes g_2\}$ , and  $\{h_1 \otimes h_2\}$ , gives:

$$\begin{aligned}
RHS &= Tr_{V_1 \otimes V_2}([A_1 \otimes A_2] \circ [g_1^{-1} \otimes g_2^{-1}] \circ j^{-1} \circ \text{Hom}(Id_{U_1 \otimes U_2}, l^{-1}) \\
&\quad \circ [B_1 \otimes B_2]^* \circ \text{Hom}(Id_{V_1 \otimes V_2}, l) \circ j \circ [h_1 \otimes h_2]) \\
&= Tr_{V_1 \otimes V_2}([A_1 \otimes A_2] \circ [g_1^{-1} \otimes g_2^{-1}] \circ [B_1^* \otimes B_2^*] \circ [h_1 \otimes h_2]) \\
&= Tr_{V_1 \otimes V_2}([(A_1 \circ g_1^{-1} \circ B_1^* \circ h_1) \otimes (A_2 \circ g_2^{-1} \circ B_2^* \circ h_2)]) \\
&= Tr_{V_1}(A_1 \circ g_1^{-1} \circ B_1^* \circ h_1) \cdot Tr_{V_2}(A_2 \circ g_2^{-1} \circ B_2^* \circ h_2).
\end{aligned}$$

The first step uses Lemma 3.11, and the second step uses Lemma 1.25, and finally Corollary 2.31 gives a product of traces equal to LHS.  $\blacksquare$

**THEOREM 3.45.** *With respect to the metrics induced by  $g$  on  $U$ ,  $h$  on  $V$ , and their duals, the map  $t_{UV} : \text{Hom}(U, V) \rightarrow \text{Hom}(V^*, U^*)$  is an isometry.*

**PROOF.** Calculating the pullback, for  $A, B \in \text{Hom}(U, V)$  gives

$$\begin{aligned}
(b(B^*))(A^*) &= Tr_{U^*}(A^* \circ (d_V \circ h^{-1})^{-1} \circ B^{**} \circ d_U \circ g^{-1}) \\
&= Tr_{U^*}(A^* \circ h \circ d_V^{-1} \circ B^{**} \circ d_U \circ g^{-1}) \\
&= Tr_{U^*}(A^* \circ h \circ B \circ g^{-1}) \\
&= Tr_V(B \circ g^{-1} \circ A^* \circ h) \\
&= (b(B))(A).
\end{aligned}$$

$\blacksquare$

**COROLLARY 3.46.**  *$T_V : \text{Hom}(V, V^*) \rightarrow \text{Hom}(V, V^*)$  is an isometry with respect to the  $b$  metric induced by a metric on  $V$ .*

**PROOF.** By Definition 3.2,  $T_V = \text{Hom}(d_V, Id_{V^*}) \circ t_{VV^*}$ , which is a composition of isometries, by Theorems 3.24, 3.43, and 3.45.  $\blacksquare$

### 3.5. Orthogonal direct sums

**DEFINITION 3.47.** A direct sum  $U = U_1 \oplus U_2 \oplus \cdots \oplus U_N$ , with inclusion maps  $Q_i$ , is orthogonal with respect to a metric  $g$  on  $U$  means:  $Q_I^* \circ g \circ Q_i = 0_{\text{Hom}(U_i, U_I^*)}$  for  $i \neq I$ .

Equivalently, the direct sum is orthogonal if and only if  $g : U \rightarrow U^*$  respects the direct sums (as in Definition 1.52), where the direct sum structure on  $U^*$  is as in Example 1.48.

**LEMMA 3.48.** *Given  $U$  with a metric  $g$ , if  $U = U_1 \oplus U_2$  and  $U = U'_1 \oplus U'_2$  are equivalent direct sums, and one is orthogonal with respect to  $g$ , then so is the other.*

**PROOF.** This follows from Lemma 1.61.  $\blacksquare$

**EXAMPLE 3.49.** If  $U = U_1 \oplus U_2$ , and  $g_1, g_2$  are metrics on  $U_1, U_2$ , then the direct sum is orthogonal with respect to the induced metric from Corollary 3.10,  $g = g_1 \oplus g_2 = P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2$ .

**THEOREM 3.50.** *If  $\text{Tr}_U(\text{Id}_U) \neq 0$  and  $g$  is any metric on  $U$ , then the direct sum  $\text{End}(U) = \mathbb{K} \oplus \text{End}_0(U)$  from Example 2.9 is orthogonal with respect to the  $b$  metric induced by  $g$ .*

**PROOF.** As noted in Lemma 1.63 and Example 2.9, any such direct sum is technically not unique, but equivalent to any other choice, so the non-uniqueness does not affect the claimed orthogonality by Lemma 3.48.

If  $A \in \text{End}_0(U) = \ker(\text{Tr}_U)$ , then the  $b$  metric applied to  $A$  and any element of the line spanned by  $\text{Id}_U$  is  $\text{Tr}_U(A \circ g^{-1} \circ (\lambda \cdot \text{Id}_U)^* \circ g) = \lambda \cdot \text{Tr}_U(A) = 0$ . ■

**THEOREM 3.51.** *If  $\text{Tr}_U(\text{Id}_U) \neq 0$ , and  $g$  is any metric on  $U$ , then the direct sum  $\text{Hom}(U, U^*) = \mathbb{K} \oplus \ker(\text{Tr}_g)$  from Theorem 3.33 is orthogonal with respect to the induced metric.*

**PROOF.** Such a direct sum is as in Lemmas 1.63 and 1.64.

If  $E \in \ker(\text{Tr}_g)$ , then the  $b$  metric applied to  $E$  and any scalar multiple of  $g$  is  $\text{Tr}_{U^*}(E \circ g^{-1} \circ (\lambda \cdot g)^* \circ d_U \circ g^{-1}) = \lambda \cdot \text{Tr}_{U^*}(E \circ g^{-1}) = 0$ . ■

**LEMMA 3.52.** *Suppose  $K \in \text{End}(U)$  is an involution and an isometry with respect to a metric  $g$  on  $U$ . If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum produced by  $K$ , as in Lemma 1.79, is orthogonal.*

**PROOF.** Using the isometry property,  $K^* \circ g \circ K = g$ , and using the involution property,  $K^* \circ g = g \circ K$ . Checking that  $g$  respects the direct sum, as in Definition 1.52, use  $Q_i \circ P_i = \frac{1}{2} \cdot (\text{Id}_U \pm K)$  as in Lemma 1.79:

$$g \circ Q_i \circ P_i = g \circ \frac{1}{2} \cdot (\text{Id}_U \pm K) = \frac{1}{2} \cdot (\text{Id}_{U^*} \pm K^*) \circ g = (Q_i \circ P_i)^* \circ g = P_i^* \circ Q_i^* \circ g. \quad \blacksquare$$

**THEOREM 3.53.** *If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum  $\text{Hom}(U, U^*) = \text{Sym}(U) \oplus \text{Alt}(U)$  is orthogonal with respect to the induced metric.*

**PROOF.** This follows from Lemmas 3.52 and 3.3, and Corollary 3.46. ■

**COROLLARY 3.54.** *If  $\text{Tr}_U(\text{Id}_U) \neq 0$  and  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum*

$$\text{Hom}(U, U^*) = \mathbb{K} \oplus \text{Sym}_0(U, g) \oplus \text{Alt}(U)$$

*is orthogonal with respect to the induced metric.* ■

There is a converse to the construction of Example 3.49: if a direct sum is orthogonal with respect to a given metric  $g$ , then metrics are induced on the summands.

**THEOREM 3.55.** *Given  $U = U_1 \oplus U_2$  with projection and inclusion operators  $P_i, Q_i$ , if the direct sum is orthogonal with respect to a metric  $g : U \rightarrow U^*$ , then each of the maps  $g_i = Q_i^* \circ g \circ Q_i : U_i \rightarrow U_i^*$  is a metric, and  $g = g_1 \oplus g_2$ .*

**PROOF.** The pullback form  $g_i$  has inverse  $P_i \circ g^{-1} \circ P_i^*$  by Lemma 1.53, and is symmetric by Lemma 3.17, so it is a metric. Since  $g$  respects the direct sums,  $P_i^* \circ Q_i^* \circ g = g \circ Q_i \circ P_i$ , so using the definition of direct sum of forms,

$$\begin{aligned} g_1 \oplus g_2 &= P_1^* \circ Q_1^* \circ g \circ Q_1 \circ P_1 + P_2^* \circ Q_2^* \circ g \circ Q_2 \circ P_2 \\ &= g \circ (Q_1 \circ P_1 + Q_2 \circ P_2) = g. \end{aligned} \quad \blacksquare$$

EXAMPLE 3.56. Theorem 3.55, applied to the above direct sums, demonstrates that under suitable hypotheses, a metric  $g$  on  $U$  induces metrics on  $\text{End}_0(U)$ ,  $\ker(\text{Tr}_g)$ ,  $\text{Sym}(U)$ ,  $\text{Sym}_0(U, g)$ , and  $\text{Alt}(U)$ .

THEOREM 3.57. *If  $U = U_1 \oplus U_2$  is an orthogonal direct sum with respect to  $g$ , and  $V$  has metric  $h$ , then the direct sum  $U \otimes V = (U_1 \otimes V) \oplus (U_2 \otimes V)$ , as in Example 1.45, is orthogonal with respect to the tensor product metric  $\{g \otimes h\}$ , and the metric on  $U_i \otimes V$  induced by the direct sum coincides with  $\{g_i \otimes h\}$ .*

PROOF. Using Lemma 3.11, and the inclusion operators  $[Q_i \otimes \text{Id}_V] : U_i \otimes V \rightarrow U \otimes V$ ,

$$\begin{aligned} & [Q_I \otimes \text{Id}_V]^* \circ \text{Hom}(\text{Id}_{U \otimes V}, l) \circ j \circ [g \otimes h] \circ [Q_i \otimes \text{Id}_V] \\ = & \text{Hom}(\text{Id}_{U_I \otimes V}, l) \circ j \circ [Q_I^* \otimes \text{Id}_V^*] \circ [g \otimes h] \circ [Q_i \otimes \text{Id}_V] \\ = & \text{Hom}(\text{Id}_{U_I \otimes V}, l) \circ j \circ [(Q_I^* \circ g \circ Q_i) \otimes h]. \end{aligned}$$

For  $i \neq I$ , the result is zero, showing the direct sum is orthogonal, and for  $i = I$ , the calculation shows that the tensor product of the induced metric  $g_i = Q_i^* \circ g \circ Q_i$  and  $h$  is equal to the metric induced by  $\{g \otimes h\}$  and  $[Q_i \otimes \text{Id}_V]$  on  $U_i \otimes V$ . ■

THEOREM 3.58. *If  $V = V_1 \oplus V_2$ , with operators  $Q'_i, P'_i$ , is an orthogonal direct sum with respect to a metric  $h$  on  $V$ , and  $U = U_1 \oplus U_2$ , and  $H : U \rightarrow V$  is an isometry with respect to  $g$  and  $h$  which respects the direct sums, then the direct sum  $U_1 \oplus U_2$  is orthogonal with respect to  $g$ , and  $P'_i \circ H \circ Q_i : U_i \rightarrow V_i$  is an isometry with respect to the induced metrics.*

PROOF. It is straightforward to check that  $H^* : V^* \rightarrow U^*$  respects the direct sums. It follows that  $g = H^* \circ h \circ H$  is a composite of maps that respect the direct sums, so  $U_1 \oplus U_2$  is orthogonal with respect to  $g$ . The induced metrics on  $U_i$  and  $V_i$  are as in Theorem 3.55, and the last claim is a special case of Lemma 3.29. ■

### 3.6. Miscellaneous results

The following facts about the trace, metrics, and direct sums are left as exercises; their proofs are short and lend themselves to the methods and notation of the previous sections. Many of the results on metrics are well-known properties of positive definite inner products on real vector spaces, suitably modified to extend to arbitrary metrics. A few of the results, labeled Lemmas, will be needed later.

#### 3.6.1. More isometries.

PROPOSITION 3.59. *Let  $U$  and  $V$  be vector spaces, and let  $h : V \rightarrow V^*$  be an invertible  $\mathbb{K}$ -linear map. Suppose  $H$  is just a function with domain  $U$  and target  $V$ , which is not necessarily linear, but which is an epimorphism in the category of sets ( $A \circ H = B \circ H \implies A = B$ , for any, not necessarily linear, functions  $A$  and  $B$ ). If there is some  $\mathbb{K}$ -linear map  $g : U \rightarrow U^*$  so that*

$$((h \circ H)(u)) \circ H = g(u)$$

*for all  $u \in U$ , then  $H$  is  $\mathbb{K}$ -linear.*



PROOF. For  $\mathbb{K}$ -linearity, two equations must hold. First, for any  $\lambda \in \mathbb{K}$ ,  $u \in U$ ,

$$\begin{aligned} ((h \circ H)(\lambda \cdot u)) \circ H = g(\lambda \cdot u) &= \lambda \cdot g(u) = \lambda \cdot ((h \circ H)(u)) \circ H \\ \implies (h \circ H)(\lambda \cdot u) &= \lambda \cdot (h \circ H)(u) \\ \implies h(H(\lambda \cdot u)) &= h(\lambda \cdot H(u)) \\ \implies H(\lambda \cdot u) &= \lambda \cdot H(u). \end{aligned}$$

Second, for any  $u_1, u_2 \in U$ ,

$$\begin{aligned} ((h \circ H)(u_1 + u_2)) \circ H = g(u_1 + u_2) &= g(u_1) + g(u_2) \\ &= ((h \circ H)(u_1)) \circ H + ((h \circ H)(u_2)) \circ H \\ &= ((h \circ H)(u_1) + (h \circ H)(u_2)) \circ H \\ \implies (h \circ H)(u_1 + u_2) &= (h \circ H)(u_1) + (h \circ H)(u_2) \\ \implies h(H(u_1 + u_2)) &= h(H(u_1) + H(u_2)) \\ \implies H(u_1 + u_2) &= H(u_1) + H(u_2). \end{aligned}$$

■

PROPOSITION 3.60. *Let  $U$  and  $V$  be vector spaces, and let  $g : U \rightarrow U^*$ ,  $h : V \rightarrow V^*$  be symmetric bilinear forms. Suppose  $H$  is just a function with domain  $U$  and target  $V$ , which is not necessarily linear. If  $\frac{1}{2} \in \mathbb{K}$  and  $H(0_U) = 0_V$  and  $H$  satisfies*

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u)$$

*for all  $u, v \in U$ , then  $H$  also satisfies*

$$((h \circ H)(u)) \circ H = g(u)$$

*for all  $u \in U$ .*

PROOF. Expanding the RHS of the hypothesis identity using the symmetric property of  $g$ ,

$$\begin{aligned} (g(v - u))(v - u) &= (g(v))(v) - (g(v))(u) - (g(u))(v) + (g(u))(u) \\ &= (g(v))(v) - 2(g(u))(v) + (g(u))(u). \end{aligned}$$

Expanding the LHS, using the symmetric property of  $h$  and  $H(0_U) = 0_V$ ,

$$\begin{aligned} (h(H(v) - H(u)))(H(v) - H(u)) &= (h(H(v)))(H(v)) \\ &\quad - (h(H(v)))(H(u)) - (h(H(u)))(H(v)) \\ &\quad + (h(H(u)))(H(u)) \\ &= (h(H(v) - H(0_U)))(H(v) - H(0_U)) \\ &\quad - 2(h(H(u)))(H(v)) \\ &\quad + (h(H(u) - H(0_U)))(H(u) - H(0_U)) \\ &= (g(v - 0_U))(v - 0_U) \\ &\quad - 2(h(H(u)))(H(v)) \\ &\quad + (g(u - 0_U))(u - 0_U). \end{aligned}$$

Setting the above quantities equal, cancelling like terms, and using  $\frac{1}{2} \in \mathbb{K}$ , the conclusion follows. ■

PROPOSITION 3.61. *Let  $U$  and  $V$  be vector spaces, and let  $g : U \rightarrow U^*$  be a metric on  $U$ . Suppose  $h$  is just a function with domain  $V$  and target  $V^*$ , which is not necessarily linear. If  $\frac{1}{2} \in \mathbb{K}$  and  $H : U \rightarrow V$  is a  $\mathbb{K}$ -linear map satisfying*

$$((h \circ H)(u))(H(u)) = (g(u))(u)$$

*for all  $u \in U$ , then  $H$  satisfies*

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u)$$

*for all  $u, v \in U$ , and  $\ker(H) = \{0_U\}$ .*

PROOF. To establish the claimed identity, use the linearity of  $H$ :

$$LHS = (h(H(v - u)))(H(v - u)) = RHS.$$

Suppose  $H(u) = 0_V$ . Then, for any  $v \in U$ ,

$$\begin{aligned} ((h \circ H)(v))(H(v)) &= (h(H(v) - H(u)))(H(v) - H(u)) \\ &= (g(v - u))(v - u) \\ &= (g(v))(v) - (g(v))(u) - (g(u))(v) + (g(u))(u) \\ &= ((h \circ H)(v))(H(v)) - 2(g(u))(v) + ((h \circ H)(u))(H(u)), \end{aligned}$$

the last step using the symmetric property of  $g$ . Using  $h(H(u)) \in V^*$  and  $H(u) = 0_V$ , the last term is 0, so cancelling like terms and using  $\frac{1}{2} \in \mathbb{K}$ , the conclusion is that  $(g(u))(v) = 0$ . Since this holds for all  $v$ ,  $g(u) = 0_{U^*}$ , and  $g$  is invertible, so  $u = 0_U$ . ■

COROLLARY 3.62. *Given a vector space  $V$  and metrics  $g$  and  $h$  on  $V$ , if  $\frac{1}{2} \in \mathbb{K}$  and  $H$  is just a function with domain  $V$  and target  $V$ , which is not necessarily linear, then the following are equivalent:*

(1)  $H$  is an epimorphism of sets, and for all  $u \in V$ ,

$$((h \circ H)(u)) \circ H = g(u);$$

(2)  $H$  is an epimorphism of sets,  $H(0_V) = 0_V$ , and for all  $u, v \in V$ ,

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u);$$

(3)  $H : V \rightarrow V$  is  $\mathbb{K}$ -linear, and for all  $u \in V$ ,

$$((h \circ H)(u))(H(u)) = (g(u))(u);$$

(4)  $H : V \rightarrow V$  is an isometry with respect to  $g$  and  $h$ .

PROOF. For (1)  $\implies$  (3), the linearity is Proposition 3.59 and the identity obviously follows. Since  $V$  is finite-dimensional, a  $\mathbb{K}$ -linear map  $V \rightarrow V$  with trivial kernel must be invertible (and therefore an epimorphism), so Proposition 3.61 gives (3)  $\implies$  (2). (2)  $\implies$  (1) is Proposition 3.60. It is immediate from Definition 3.20 that (4)  $\implies$  (1), and also (4)  $\implies$  (3). Finally, the linearity of (3), the identity of (1), and the above mentioned invertibility together imply (4). The implications (1)  $\iff$  (4)  $\implies$  (3) did not require  $\frac{1}{2} \in \mathbb{K}$ . ■

EXERCISE 3.63. The switching map  $s : U \otimes V \rightarrow V \otimes U : u \otimes v \mapsto v \otimes u$ , as in Example 1.19, is an isometry with respect to the induced tensor product metrics. ■

LEMMA 3.64. *Every map  $h : \mathbb{K} \rightarrow \mathbb{K}^*$  is of the form  $h^\nu$ , where for  $\nu \in \mathbb{K}$ ,  $(h^\nu(\lambda))(\mu) = \nu \cdot \lambda \cdot \mu$ . If  $\nu \neq 0$ , then  $h^\nu$  is a metric on  $\mathbb{K}$ , with inverse map  $\frac{1}{\nu} \cdot \text{Tr}_{\mathbb{K}}$ .*

PROOF. For any  $h$ , let  $\nu = (h(1))(1)$ ; then  $(h(\lambda))(\mu) = \lambda \cdot \mu \cdot (h(1))(1)$ .  $h^\nu$  is clearly symmetric, and if  $\nu \neq 0$ , then  $h^\nu$  is invertible, with inverse  $\frac{1}{\nu} \cdot \text{Tr}_{\mathbb{K}}$ , by Example 2.7:

$$((\frac{1}{\nu} \cdot \text{Tr}_{\mathbb{K}}) \circ h^\nu)(\lambda) = \frac{1}{\nu} \cdot \text{Tr}_{\mathbb{K}}(h^\nu(\lambda)) = \frac{1}{\nu} \cdot (h^\nu(\lambda))(1) = \frac{1}{\nu} \cdot \nu \cdot \lambda \cdot 1 = \lambda.$$

$$(h^\nu \circ (\frac{1}{\nu} \cdot \text{Tr}_{\mathbb{K}}))(A) = h^\nu(\frac{1}{\nu} \cdot A(1)) : \mu \mapsto \nu \cdot \frac{1}{\nu} \cdot A(1) \cdot \mu = A(\mu).$$

■

For all  $\nu$ ,  $h^\nu = \nu \cdot h^1$ , and a canonical choice of metric on  $\mathbb{K}$  is  $h^1 = (\text{Tr}_{\mathbb{K}})^{-1}$ .  $h^1$  is also equal to the map  $m : \mathbb{K} \rightarrow \text{Hom}(\mathbb{K}, \mathbb{K})$ .

EXERCISE 3.65. For three copies of the scalar field,  $\mathbb{K}_\alpha, \mathbb{K}_\beta, \mathbb{K}_\gamma$ , with metrics  $h^\alpha, h^\beta, h^\gamma$ , if  $\beta = \alpha \cdot \gamma \in \mathbb{K}$ , then the map  $\text{Tr}_{\mathbb{K}} : \text{Hom}(\mathbb{K}_\alpha, \mathbb{K}_\beta) \rightarrow \mathbb{K}_\gamma$  is an isometry with respect to the induced  $b$  metric and  $h^\gamma$ .

HINT. For  $A, B \in \text{End}(\mathbb{K})$ , the pullback metric is, by Example 2.7,

$$(h^\gamma(\text{Tr}_{\mathbb{K}}(A)))(\text{Tr}_{\mathbb{K}}(B)) = (h^\gamma(A(1)))(B(1)) = \gamma \cdot A(1) \cdot B(1).$$

The induced metric  $b$  on  $\text{Hom}(\mathbb{K}_\alpha, \mathbb{K}_\beta)$  gives

$$\begin{aligned} \text{Tr}_{\mathbb{K}}(A \circ (h^\alpha)^{-1} \circ B^* \circ h^\beta) &= (A \circ (h^\alpha)^{-1} \circ B^* \circ h^\beta)(1) \\ &= A(\alpha^{-1} \cdot \text{Tr}_{\mathbb{K}}(B^*(h^\beta(1)))) \\ &= A(\alpha^{-1} \cdot (h^\beta(1))(B(1))) \\ &= A(\alpha^{-1} \cdot \beta \cdot 1 \cdot B(1)) \\ &= \frac{\beta}{\alpha} \cdot A(1) \cdot B(1). \end{aligned}$$

■

LEMMA 3.66. *The scalar multiplication map  $l_U : U \otimes \mathbb{K} \rightarrow U$  is an isometry, with respect to the tensor product metric  $\{g \otimes h^\nu\}$  and  $\nu \cdot g$ .*

PROOF. Calculating the pullback of  $\nu \cdot g$  gives:

$$(\nu \cdot g(l_U(u_1 \otimes \lambda)))(l_U(u_2 \otimes \mu)) = \lambda \cdot \mu \cdot \nu \cdot (g(u_1))(u_2),$$

and the tensor product metric is

$$(\{g \otimes h^\nu\}(u_1 \otimes \lambda))(u_2 \otimes \mu) = \nu \cdot \lambda \cdot \mu \cdot (g(u_1))(u_2).$$

■

EXERCISE 3.67. The canonical map  $m : U \rightarrow \text{Hom}(\mathbb{K}, U)$ ,  $m(u) : \lambda \mapsto \lambda \cdot u$ , is an isometry with respect to  $g$  and the metric  $b$  induced by  $h^1$  and  $g$ .

HINT. The pullback of the metric  $b$  induced by the more general map  $h^\nu$  is, using Lemma 3.64,

$$\begin{aligned}
 (b(m(u_1)))(m(u_2)) &= \text{Tr}_{\mathbb{K}}((h^\nu)^{-1} \circ (m(u_1))^* \circ g \circ (m(u_1))) \\
 &= (h^\nu)^{-1}((m(u_1))^*(g((m(u_2))(1)))) \\
 &= \nu^{-1} \cdot \text{Tr}_{\mathbb{K}}((g((m(u_2))(1))) \circ (m(u_1))) \\
 &= \nu^{-1} \cdot (g((m(u_2))(1))((m(u_1))(1))) \\
 &= \nu^{-1} \cdot (g(u_2))(u_1).
 \end{aligned}$$

If  $U \neq \{0_U\}$ , then  $\nu = 1$  is necessary for equality. ■

LEMMA 3.68. *The canonical map  $n : \text{Hom}(U, V) \otimes W \rightarrow \text{Hom}(U, V \otimes W)$  is an isometry with respect to the induced metrics.*

PROOF. This follows from the fact that  $j$ ,  $l_U$ , and  $m$  are isometries, Lemma 2.49, where  $n = [Id_{\text{Hom}(U, V)} \otimes m^{-1}] \circ j \circ \text{Hom}(l_U, Id_{V \otimes W})$ , and Theorems 3.25 and 3.43. It could also be checked directly. ■

EXERCISE 3.69. The dual metric on  $U^*$ , induced by  $g$ , and the metric  $b$  on  $\text{Hom}(U, \mathbb{K})$ , induced by  $g$  and  $h^1$ , coincide.

HINT. For  $\phi \in U^*$ , and the more general metric  $h^\nu$  on  $\mathbb{K}$ , the identity

$$((h^\nu(1)) \circ \phi)(\lambda) = \nu \cdot 1 \cdot \phi(\lambda) = (\nu \cdot \phi)(\lambda)$$

is used in computing the  $b$  metric for  $\phi$ ,  $\xi \in U^*$ :

$$\begin{aligned}
 \text{Tr}_{\mathbb{K}}(\xi \circ g^{-1} \circ \phi^* \circ h^\nu) &= \xi(g^{-1}(\phi^*(h^\nu(1)))) = \xi(g^{-1}((h^\nu(1)) \circ \phi)) \\
 &= \xi(g^{-1}(\nu \cdot \phi)) = \nu \cdot \xi(g^{-1}(\phi)).
 \end{aligned}$$

The dual metric on  $U^*$  results in the quantity  $\xi(g^{-1}(\phi))$ , so  $\nu = 1$  is necessary for equality in the case  $U \neq \{0_U\}$ , and, in general, if  $h^\nu$  is the metric on  $\mathbb{K}$ , then the  $b$  metric on  $\text{Hom}(U, \mathbb{K})$  is equal to  $\nu \cdot d_U \circ g^{-1}$ . ■

EXERCISE 3.70. If  $A : U_2 \rightarrow U_1$  is an isometry, then  $A^* : U_1^* \rightarrow U_2^*$  is an isometry with respect to the dual metrics. ■

EXERCISE 3.71. The map  $p : U \otimes V \rightarrow V^{**} \otimes U$ , as in Notation 1.36, is an isometry with respect to the induced tensor product metrics. ■

EXERCISE 3.72. The map  $q : \text{Hom}(V, \text{Hom}(U, W)) \rightarrow \text{Hom}(V \otimes U, W)$ , as in Definition 1.39, is an isometry with respect to metrics induced by  $g, h, y$  on  $U, V, W$ .

HINT. All the maps in the following commutative diagram are isometries.

$$\begin{array}{ccc}
 V^* \otimes U^* \otimes W & \xrightarrow{[j \otimes Id_W]} & \text{Hom}(V \otimes U, \mathbb{K} \otimes \mathbb{K}) \otimes W \\
 \downarrow [Id_{V^*} \otimes k_{UW}] & & \downarrow [\text{Hom}(Id_{V \otimes U}, l) \otimes Id_W] \\
 V^* \otimes \text{Hom}(U, W) & & (V \otimes U)^* \otimes W \\
 \downarrow k_{V, \text{Hom}(U, W)} & & \downarrow k_{V \otimes U, W} \\
 \text{Hom}(V, \text{Hom}(U, W)) & \xrightarrow{q} & \text{Hom}(V \otimes U, W)
 \end{array}$$

■

PROPOSITION 3.73. Suppose  $V$  and  $L$  are finite-dimensional, and, as in Proposition 2.20,  $Ev_L : L^* \otimes L \rightarrow \mathbb{K}$  is invertible. Then  $d_{VL} : V \rightarrow \text{Hom}(\text{Hom}(V, L), L)$  is invertible. If, also, there is some metric  $y$  on  $L$ , and  $Ev_L$  is an isometry with respect to the induced metric on  $L^* \otimes L$  and  $h^1$  on  $\mathbb{K}$ , then  $d_{VL}$  is an isometry with respect to any metric  $g$  on  $V$ , and the induced metric on  $\text{Hom}(\text{Hom}(V, L), L)$ .

PROOF. Recall  $d_{VL}$  from Definition 1.9.

$$\begin{array}{ccc}
 V & \xrightarrow{d_{VL}} & \text{Hom}(\text{Hom}(V, L), L) \\
 d_V \downarrow & & \downarrow \text{Hom}(k_{VL}, Id_L) \\
 V^{**} & \xrightarrow{\text{Hom}(Id_{V^*}, Q_1^1)} & \text{Hom}(V^* \otimes L, L) \\
 \uparrow \text{Hom}(Id_{V^*}, Ev_L) & \searrow & \uparrow q \\
 \text{Hom}(V^*, L^* \otimes L) & \xrightarrow{\text{Hom}(Id_{V^*}, k_{LL})} & \text{Hom}(V^*, \text{End}(L))
 \end{array}$$

The lower triangle is commutative, as in Proposition 2.20, and so is the top part of the diagram:

$$\begin{aligned}
 v &\mapsto (q \circ \text{Hom}(Id_{V^*}, Q_1^1) \circ d_V)(v) \\
 &= q(Q_1^1 \circ (d_V(v))) : \\
 \phi \otimes u &\mapsto (Q_1^1(\phi(v)))(u) = (\phi(v) \cdot Id_L)(u) = \phi(v) \cdot u, \\
 v &\mapsto (\text{Hom}(k_{VL}, Id_L) \circ d_{VL})(v) \\
 &= (d_{VL}(v)) \circ k_{VL} : \\
 \phi \otimes u &\mapsto (k_{VL}(\phi \otimes u))(v) = \phi(v) \cdot u.
 \end{aligned}$$

$d_{VL}$  is invertible because all the other maps in the rectangle are invertible. By Theorem 3.43,  $\text{Hom}(Id_{V^*}, Ev_L)$  is an isometry with respect to the  $b$  metric on  $\text{Hom}(V^*, L^* \otimes L)$ , and the  $b$  metric on  $V^{**} = \text{Hom}(V^*, \mathbb{K})$ , induced by  $d_V \circ g^{-1}$  on  $V^*$  and  $h^1$  on  $\mathbb{K}$ .  $d_V$  is an isometry from  $V$  to  $V^{**}$  with respect to the dual metric, which by Exercise 3.69, is the same as the  $b$  metric on  $\text{Hom}(V^*, \mathbb{K})$ . So,  $d_{VL}$  is a composite of isometries. ■

### 3.6.2. Antisymmetric forms and symplectic forms.

EXERCISE 3.74. Given a bilinear form  $g : V \rightarrow V^*$ , if  $g$  satisfies  $(g(v))(v) = 0$  for all  $v \in V$ , then  $g$  is antisymmetric. If  $\frac{1}{2} \in \mathbb{K}$ , then, conversely, an antisymmetric form  $g$  satisfies  $(g(v))(v) = 0$ . ■

BIG EXERCISE 3.75. Given a bilinear form  $g : V \rightarrow V^*$ , the following are equivalent:

- (1) For all  $u, v \in V$ , if  $(g(u))(v) = 0$ , then  $(g(v))(u) = 0$ ;
- (2)  $g \in \text{Sym}(V) \cup \text{Alt}(V)$ .

HINT. (2)  $\implies$  (1) is easy; a proof of the well-known converse is given by [J] §6.1. A form satisfying either equivalent condition is variously described by the literature as “orthosymmetric” or “reflexive.” ■

DEFINITION 3.76. A bilinear form  $h : U \rightarrow U^*$  is a symplectic form means:  $h$  is antisymmetric and invertible.

Recall from Theorem 3.6 that the invertibility implies  $U$  is finite-dimensional.

EXERCISE 3.77. Given a symplectic form  $h$  on  $V$ , the form  $d_V \circ h^{-1} : V^* \rightarrow V^{**}$  is a symplectic form on  $V^*$ .

HINT. This is an analogue of Theorem 3.7. The antisymmetric property implies the equality  $d_V \circ h^{-1} = -(h^*)^{-1}$ . ■

Given a symplectic form  $h$  on  $V$ , the above Exercise suggests there are two opposite ways  $h$  could induce a symplectic form on  $V^*$ :

$$(3.3) \quad d_V \circ h^{-1} = -(h^*)^{-1},$$

$$(3.4) \quad -d_V \circ h^{-1} = (h^*)^{-1}.$$

EXERCISE 3.78. The tensor product of symplectic forms is a metric. ■

The following Definition is analogous to Definition 3.20.

DEFINITION 3.79. A map  $H : U \rightarrow V$  is a symplectic isometry, with respect to symplectic forms  $g$  on  $U$ , and  $h$  on  $V$ , means:  $H$  is invertible, and  $g = H^* \circ h \circ H$ .

LEMMA 3.80. A symplectic form  $h : U \rightarrow U^*$  is a symplectic isometry with respect to itself and the symplectic form  $-d_V \circ h^{-1}$  from (3.4). ■

EXERCISE 3.81. Given  $V$  with metric  $g$  and symplectic form  $h$ , the following are equivalent.

- (1)  $g$  is a symplectic isometry with respect to  $h$  and the symplectic form  $d_V \circ h^{-1}$  from (3.3);
- (2)  $h^{-1} \circ g \in \text{End}(V)$  is an involution.

■

EXERCISE 3.82. Given  $V$  with metric  $g$  and symplectic form  $h$ , the following are equivalent.

- (1)  $g$  is a symplectic isometry with respect to  $h$  and the symplectic form  $-d_V \circ h^{-1}$  from (3.4);
- (2)  $h$  is an isometry with respect to  $g$  and the dual metric  $d_V \circ g^{-1}$ ;
- (3)  $g^{-1} \circ h \in \text{End}(V)$  is an isometry with respect to  $g$ ;

(4)  $g^{-1} \circ h \in \text{End}(V)$  is a symplectic isometry with respect to  $h$ .

HINT. The equivalence of (2) and (3) follows from Theorem 3.23.  $\blacksquare$

EXERCISE 3.83. Given a symplectic form  $h$  on  $U$ , using either method (3.3) or (3.4) to induce a symplectic form on the dual space, the double dual  $U^{**}$  has a canonical symplectic form

$$d_{U^*} \circ (d_U \circ h^{-1})^{-1} = -d_{U^*} \circ (-d_U \circ h^{-1})^{-1} = d_{U^*} \circ h \circ d_U^{-1}.$$

The map  $d_U : U \rightarrow U^{**}$  is a symplectic isometry with respect to  $h$  and the above symplectic form.  $\blacksquare$

BIG EXERCISE 3.84. Several more of the elementary results on metrics can be adapted to symplectic forms.  $\blacksquare$

### 3.6.3. More direct sums and graphs.

EXERCISE 3.85. Given linear maps  $H : U \rightarrow V$  and  $h : V \rightarrow V^*$ , if  $H^* \circ h \circ H : U \rightarrow U^*$  is invertible, then there is a direct sum  $V = U \oplus \ker(H^* \circ h)$ . If, in addition,  $h$  is symmetric (or antisymmetric), then  $h : V \rightarrow V^*$  respects the induced direct sums and  $H^* \circ h \circ H : U \rightarrow U^*$  is a metric (respectively, symplectic form) on  $U$ . If, further,  $h$  is invertible, then  $h$  also induces a metric (respectively, symplectic form) on  $\ker(H^* \circ h)$ .

HINT.  $H$  is a linear monomorphism as in Exercise 3.22. Let  $Q_1 = H$ , and let  $P_1 = (H^* \circ h \circ H)^{-1} \circ H^* \circ h$ . Then  $P_1 \circ Q_1 = Id_U$ , and  $Q_1 \circ P_1 = H \circ (H^* \circ h \circ H)^{-1} \circ H^* \circ h$  is an idempotent on  $V$ . The kernel of  $Q_1 \circ P_1$  is equal to the kernel of  $H^* \circ h$ ; let  $Q_2$  denote the inclusion of this subspace in  $V$ , and define the projection  $P_2$  onto this subspace as in Example 1.74:  $P_2 = Id_V - Q_1 \circ P_1 = Q_2 \circ P_2$ .

The direct sum  $V = U \oplus \ker(H^* \circ h)$  induces a direct sum  $V^* = U^* \oplus (\ker(H^* \circ h))^*$  as in Example 1.48. If  $h$  is symmetric (or antisymmetric), then  $H^* \circ h \circ H$  is also symmetric (respectively, antisymmetric) by Lemma 3.17, and a metric (respectively, symplectic form) on  $U$ , so  $U$  is finite-dimensional and  $d_U$  is invertible. Consider the two expressions:

$$\begin{aligned} h \circ Q_1 \circ P_1 &= h \circ H \circ (H^* \circ h \circ H)^{-1} \circ H^* \circ h, \\ P_1^* \circ Q_1^* \circ h &= h^* \circ H^{**} \circ (H^* \circ h^* \circ H^{**})^{-1} \circ H^* \circ h. \end{aligned}$$

If  $h = \pm h^* \circ d_V$ , then, using Lemma 1.10,

$$\begin{aligned} h \circ Q_1 \circ P_1 &= \pm h^* \circ d_V \circ H \circ (H^* \circ (\pm h^* \circ d_V) \circ H)^{-1} \circ H^* \circ h \\ &= h^* \circ H^{**} \circ d_U \circ (H^* \circ h^* \circ H^{**} \circ d_U)^{-1} \circ H^* \circ h, \end{aligned}$$

so  $h \circ Q_1 \circ P_1 = P_1^* \circ Q_1^* \circ h$ , and  $h$  respects the direct sums.

If, further,  $h$  is invertible, then  $h$  is a metric (respectively, symplectic form) that respects the direct sums  $V \rightarrow V^*$ , so  $V = U \oplus \ker(H^* \circ h)$  is an orthogonal direct sum with respect to  $h$ , and Theorem 3.55 applies.  $\blacksquare$

EXERCISE 3.86. Given  $V = V_1 \oplus V_2$ ,  $U = U_1 \oplus U_2$ , with projection and inclusion maps  $P_i, Q_i$  on  $V$ ,  $P'_i, Q'_i$  on  $U$ , if  $A : U_1 \rightarrow V_1$  and  $B : U_2 \rightarrow V_2$  are isometries with respect to metrics  $g_i$  on  $U_i$ ,  $h_i$  on  $V_i$ , then

$$A \oplus B = Q_1 \circ A \circ P'_1 + Q_2 \circ B \circ P'_2 : U \rightarrow V$$

is an isometry with respect to the induced metrics.

HINT. The invertibility is by Lemma 1.50. The rest of the claim is that

$$g_1 \oplus g_2 = (A \oplus B)^* \circ (h_1 \oplus h_2) \circ (A \oplus B).$$

The RHS can be expanded:

$$\begin{aligned} RHS &= (P_1'^* \circ A^* \circ Q_1^* + P_2'^* \circ B^* \circ Q_2^*) \\ &\quad \circ (P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2) \\ &\quad \circ (Q_1 \circ A \circ P_1' + Q_2 \circ B \circ P_2') \\ &= P_1'^* \circ A^* \circ h_1 \circ A \circ P_1' + P_2'^* \circ B^* \circ h_2 \circ B \circ P_2' \\ &= P_1'^* \circ g_1 \circ P_1' + P_2'^* \circ g_2 \circ P_2' = LHS. \end{aligned}$$

The last step uses  $g_1 = A^* \circ h_1 \circ A$ ,  $g_2 = B^* \circ h_2 \circ B$ . ■

EXERCISE 3.87. If  $V_1$  and  $V_2$  have metrics  $g_1$  and  $g_2$ , and  $V = V_1 \oplus V_2$  and  $W = V_1 \oplus V_2$  are direct sums with operators  $P_i'$ ,  $Q_i'$  and  $P_i$ ,  $Q_i$ , respectively, then the map  $Q_1' \circ P_1 + Q_2' \circ P_2 : W \rightarrow V$  is an isometry with respect to the direct sum metrics from Corollary 3.10.

HINT. This is a special case of Exercise 3.86. The construction of the invertible map  $Q_1' \circ P_1 + Q_2' \circ P_2 : W \rightarrow V$  is a special case of the map from Lemma 1.50. ■

EXERCISE 3.88. Given  $V = V_1 \oplus V_2$  and metrics  $g_1$  and  $g_2$  on  $V_1$  and  $V_2$ , the dual of the metric  $g_1 \oplus g_2$  from Corollary 3.10 is  $d_V \circ (g_1 \oplus g_2)^{-1} : V^* \rightarrow V^{**}$ , as in Theorem 3.7. For the direct sum  $V^* = V_1^* \oplus V_2^*$  from Example 1.48, the direct sum of the dual metrics is  $(d_{V_1} \circ g_1^{-1}) \oplus (d_{V_2} \circ g_2^{-1})$ . These two metrics on  $V^*$  are equal.

HINT. Lemma 1.10 applies to the direct sum formula (3.1) and the inverse (3.2). ■

EXAMPLE 3.89. Given  $\frac{1}{2} \in \mathbb{K}$ , and given  $V$  with metric  $g$  and an involution  $K_1 : V \rightarrow V$ , producing a direct sum  $V_1 \oplus V_2$  as in Lemma 1.79, suppose the forms  $Q_i^* \circ g \circ Q_i$  are metrics for  $i = 1, 2$  (this is the case, for example, when  $K_1$  is an isometry, by Lemma 3.52 and Theorem 3.55). If  $K_2$  is another involution on  $V$  that is an isometry and anticommutes with  $K_1$ , then  $K_2$  respects the direct sums  $V_1 \oplus V_2 \rightarrow V_2 \oplus V_1$  as in Lemma 1.88, and the induced maps  $P_2 \circ K_2 \circ Q_1 : V_1 \rightarrow V_2$  and  $P_1 \circ K_2 \circ Q_2 : V_2 \rightarrow V_1$ , as in Theorem 1.89, are isometries by Lemma 3.29.

LEMMA 3.90. Given  $\frac{1}{2} \in \mathbb{K}$ , and given  $V$  with metric  $g$  and an involution  $K : V \rightarrow V$ , producing a direct sum  $V = V_1 \oplus V_2$  with operators  $P_i$ ,  $Q_i$  as in Lemma 1.79, suppose the direct sum is orthogonal with respect to  $g$  (this is the case, for example, when  $K$  is an isometry, by Lemma 3.52). Let  $K'$  be another involution on  $V$  that is an isometry and anticommutes with  $K$ , and which produces a direct sum  $V = V_1' \oplus V_2'$ , with operators  $P_i'$ ,  $Q_i'$ . If  $\beta \in \mathbb{K}$  satisfies  $\beta^2 = 2$ , then for  $i = 1, 2$ ,  $I = 1, 2$ , the map  $\beta \cdot P_I' \circ Q_i : V_i \rightarrow V_I'$  is an isometry.

PROOF. The map  $\beta \cdot P_I' \circ Q_i : V_i \rightarrow V_I'$  is invertible by Theorem 1.90. The induced metric on  $V_i$  is  $Q_i^* \circ g \circ Q_i$  and on  $V_I'$  is  $(Q_I')^* \circ g \circ Q_I'$ , by Lemma 3.52 and



Theorem 3.55. From the Proof of Lemma 3.52,  $g \circ Q'_I \circ P'_I = (P'_I)^* \circ (Q'_I)^* \circ g$ .

$$\begin{aligned}
 & (\beta \cdot P'_I \circ Q_i)^* \circ ((Q'_I)^* \circ g \circ Q'_I) \circ (\beta \cdot P'_I \circ Q_i) \\
 = & \beta^2 \cdot Q_i^* \circ (P'_I)^* \circ (Q'_I)^* \circ g \circ Q'_I \circ P'_I \circ Q_i \\
 = & \beta^2 \cdot Q_i^* \circ g \circ Q'_I \circ P'_I \circ Q_i \\
 = & \beta^2 \cdot Q_i^* \circ g \circ \frac{1}{2} \cdot (Id_V \pm K') \circ Q_i.
 \end{aligned}$$

By hypothesis,  $g$  respects the direct sum  $V_1 \oplus V_2$ , but  $K'$  reverses the direct sum as in Lemma 1.88. So,  $Q_i^* \circ g \circ K' \circ Q_i = 0_{\text{Hom}(V_i, V_i^*)}$  and the second term in the last line drops out. ■

EXERCISE 3.91. Let  $U = U_1 \oplus U_2$  and  $V = V_1 \oplus V_2$  be direct sums with operators  $P_i, Q_i, P'_i, Q'_i$ , and which are orthogonal with respect to metrics  $g, h$ . If  $H : U \rightarrow V$  is an isometry such that  $P'_2 \circ H \circ Q_1 = 0_{\text{Hom}(U_1, V_2)}$ , and  $P'_1 \circ H \circ Q_1$  is a linear epimorphism, then  $P'_1 \circ H \circ Q_2 = 0_{\text{Hom}(U_2, V_1)}$ , so  $H$  respects the direct sums.

HINT.

$$\begin{aligned}
 0_{\text{Hom}(U_1, U_2^*)} &= Q_2^* \circ g \circ Q_1 \\
 &= Q_2^* \circ H^* \circ h \circ H \circ Q_1 \\
 &= Q_2^* \circ H^* \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2)^* \circ h \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ H \circ Q_1 \\
 &= (P'_1 \circ H \circ Q_2)^* \circ Q_1^* \circ h \circ Q'_1 \circ P'_1 \circ H \circ Q_1.
 \end{aligned}$$

$Q_1^* \circ h \circ Q'_1$  is invertible by Theorem 3.55, so  $P'_1 \circ H \circ Q_2 = 0_{\text{Hom}(U_2, V_1)}$  by the linear epimorphism property (Definition 1.66). ■

EXERCISE 3.92. If  $U_1 = V_1$  in Exercise 3.91, then the epimorphism property is not needed in the hypothesis.

HINT.

$$\begin{aligned}
 (P_1 \circ H^{-1} \circ Q'_1) \circ (P'_1 \circ H \circ Q_1) &= P_1 \circ H^{-1} \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ H \circ Q_1 \\
 &= P_1 \circ Q_1 = Id_{V_1}.
 \end{aligned}$$

By the finite-dimensionality of  $V_1$ ,  $P'_1 \circ H \circ Q_1 \in \text{End}(V_1)$  is invertible. ■

EXERCISE 3.93. Given any vector space  $V$ , if  $U = U_1 \oplus U_2$  is a direct sum with projection operators  $P_i$  and inclusion operators  $Q_i$ , then as in Example 1.47,  $\text{Hom}(U, V) = \text{Hom}(U_1, V) \oplus \text{Hom}(U_2, V)$ , with projection operators  $\text{Hom}(Q_i, Id_V)$  and inclusion operators  $\text{Hom}(P_i, Id_V)$ . If  $U_1 \oplus U_2$  is an orthogonal direct sum with respect to  $g$ , and  $V$  has metric  $h$ , then  $\text{Hom}(U_1, V) \oplus \text{Hom}(U_2, V)$  is an orthogonal direct sum with respect to the induced  $b$  metric.

HINT. Consider  $A : U_i \rightarrow V, B : U_I \rightarrow V$ .

$$\begin{aligned}
 ((\text{Hom}(P_I, Id_V)^* \circ b \circ \text{Hom}(P_i, Id_V))(A))(B) &= (b(A \circ P_i))(B \circ P_I) \\
 &= Tr_V(B \circ P_I \circ g^{-1} \circ (A \circ P_i)^* \circ h) \\
 &= Tr_V(B \circ P_I \circ g^{-1} \circ P_i^* \circ A^* \circ h).
 \end{aligned}$$

By Lemma 1.53, since  $g : U \rightarrow U^*$  respects the direct sums, so does  $g^{-1} : U^* \rightarrow U$ , so for  $i \neq I$ ,  $P_I \circ g^{-1} \circ P_i^* = 0_{\text{Hom}(U_i^*, U_I)}$ . This makes  $(b(A \circ P_i))(B \circ P_I)$  equal to zero, proving orthogonality. ■

EXERCISE 3.94. Given any vector space  $U$ , if  $V = V_1 \oplus V_2$  is a direct sum with projection operators  $P_i$  and inclusion operators  $Q_i$ , then as in Example 1.46,  $\text{Hom}(U, V) = \text{Hom}(U, V_1) \oplus \text{Hom}(U, V_2)$ , with projection operators  $\text{Hom}(Id_U, P_i)$  and inclusion operators  $\text{Hom}(Id_U, Q_i)$ . If  $V_1 \oplus V_2$  is an orthogonal direct sum with respect to  $h$ , and  $U$  has metric  $g$ , then  $\text{Hom}(U, V_1) \oplus \text{Hom}(U, V_2)$  is an orthogonal direct sum with respect to the induced  $b$  metric.

HINT. Consider  $A : U \rightarrow V_i, B : U \rightarrow V_I$ .

$$\begin{aligned} ((\text{Hom}(Id_U, Q_I))^* \circ b \circ \text{Hom}(Id_U, Q_i))(A)(B) &= (b(Q_i \circ A))(Q_I \circ B) \\ &= \text{Tr}_V(Q_I \circ B \circ g^{-1} \circ (Q_i \circ A)^* \circ h) \\ &= \text{Tr}_{V_I}(B \circ g^{-1} \circ A^* \circ Q_i^* \circ h \circ Q_I). \end{aligned}$$

For  $i \neq I$ , this quantity is zero. ■

EXERCISE 3.95. If  $U = U_1 \oplus U_2$  is an orthogonal direct sum with respect to operators  $Q_i, P_i$ , and a metric  $g$  on  $U$ , and  $g_1, g_2$  are the metrics induced on  $U_1, U_2$  (from Theorem 3.55), and  $K : U \rightarrow U^*$ , then

$$\text{Tr}_g(K) = \text{Tr}_{g_1}(Q_1^* \circ K \circ Q_1) + \text{Tr}_{g_2}(Q_2^* \circ K \circ Q_2).$$

HINT. By Theorem 3.55,  $g_i^{-1} = P_i \circ g^{-1} \circ P_i^*$ , and from the hint for Exercise 3.93,  $P_I \circ g^{-1} \circ P_i^* = 0_{\text{Hom}(U_i^*, U_I)}$  for  $i \neq I$ . Using Lemma 2.6,

$$\begin{aligned} \text{Tr}_g(K) &= \text{Tr}_V(g^{-1} \circ (Q_1 \circ P_1 + Q_2 \circ P_2)^* \circ K \circ (Q_1 \circ P_1 + Q_2 \circ P_2)) \\ &= \text{Tr}_{V_1}(P_1 \circ g^{-1} \circ (P_1^* \circ Q_1^* + P_2^* \circ Q_2^*) \circ K \circ Q_1) \\ &\quad + \text{Tr}_{V_2}(P_2 \circ g^{-1} \circ (P_1^* \circ Q_1^* + P_2^* \circ Q_2^*) \circ K \circ Q_2) \\ &= \text{Tr}_{V_1}(P_1 \circ g^{-1} \circ P_1^* \circ Q_1^* \circ K \circ Q_1) \\ &\quad + \text{Tr}_{V_2}(P_2 \circ g^{-1} \circ P_2^* \circ Q_2^* \circ K \circ Q_2) \\ &= \text{Tr}_{V_1}(g_1^{-1} \circ Q_1^* \circ K \circ Q_1) + \text{Tr}_{V_2}(g_2^{-1} \circ Q_2^* \circ K \circ Q_2). \end{aligned}$$
■

LEMMA 3.96. Suppose the direct sum  $V = \mathbb{K} \oplus U$ , with inclusion operators  $Q_i$ , is orthogonal with respect to a metric  $g$ . Then, the induced metric  $Q_1^* \circ g \circ Q_1$  on  $\mathbb{K}$  is equal to  $h^\nu$ , for  $\nu = (g(Q_1(1)))(Q_1(1))$ .

PROOF.  $Q_1^* \circ g \circ Q_1 = h^\nu$  for some  $\nu \neq 0$ , by Lemma 3.64 and Theorem 3.55.

$$\nu = (h^\nu(1))(1) = ((Q_1^* \circ g \circ Q_1)(1))(1) = (g(Q_1(1)))(Q_1(1)).$$
■

EXERCISE 3.97. For an orthogonal direct sum  $V = \mathbb{K} \oplus U$  as in Lemma 3.96, if  $\alpha \in \mathbb{K}$  satisfies

$$(g(Q_1(\alpha)))(Q_1(\alpha)) = 1,$$

then

$$d_{\mathbb{K}U}(\alpha) : \text{Hom}(\mathbb{K}, U) \rightarrow U : A \mapsto A(\alpha)$$

is an isometry with respect to the induced metrics.

HINT. Let  $h^\nu$  and  $g_U$  be the metrics induced by  $g$  on  $\mathbb{K}$  and  $U$  from Lemma 3.96, so  $\nu = (g(Q_1(1)))(Q_1(1))$ . For  $A, B \in \text{Hom}(\mathbb{K}, U)$ , the pullback of  $g_U$  by  $d_{\mathbb{K}U}(\alpha)$  gives:

$$\begin{aligned} (g_U((d_{\mathbb{K}U}(\alpha))(A)))(d_{\mathbb{K}U}(\alpha)(B)) &= (g_U(A(\alpha)))(B(\alpha)) \\ &= \alpha^2 \cdot (g_U(A(1)))(B(1)). \end{aligned}$$

The calculation for the induced metric on  $\text{Hom}(\mathbb{K}, U)$  is:

$$\begin{aligned} (b(A))(B) &= \text{Tr}_{\mathbb{K}}((h^\nu)^{-1} \circ A^* \circ g_U \circ B) \\ &= (h^\nu)^{-1}((g_U(B(1))) \circ A) \\ &= \nu^{-1} \cdot (g_U(B(1)))(A(1)). \end{aligned}$$

If  $\alpha^2\nu = 1$ , then the outputs are equal; the converse holds for  $U \neq \{0_U\}$ . The above calculation works for any metric on  $U$ , and is similar to that from Exercise 3.67. ■

EXERCISE 3.98. For  $V = \mathbb{K} \oplus U$  and  $\nu$  as Lemma 3.96, the endomorphism

$$R = Id_V - \frac{2}{\nu} \cdot k(((g \circ Q_1)(1)) \otimes (Q_1(1))) \in \text{End}(V)$$

is an involution, and an isometry with respect to  $g$ . It respects the direct sum; in particular,  $R \circ Q_1 = -Q_1$ , and  $R \circ Q_2 = Q_2$ .

HINT. The equality  $R \circ Q_2 = Q_2$  follows from the orthogonality of the direct sum, and

$$\begin{aligned} (R \circ Q_1)(\lambda) &= (Id_V - \frac{2}{\nu} \cdot k(((g \circ Q_1)(1)) \otimes (Q_1(1))))(Q_1(\lambda)) \\ &= Q_1(\lambda) - \frac{2}{\nu} \cdot (g(Q_1(1)))(Q_1(\lambda)) \cdot Q_1(1) \\ &= Q_1(\lambda) - \frac{2}{\nu} \cdot \lambda \cdot \nu \cdot Q_1(1) \\ &= -Q_1(\lambda). \end{aligned}$$

It follows immediately that  $R$  respects the direct sums, and that  $R$  is an involution:

$$R \circ R = R \circ R \circ (Q_1 \circ P_1 + Q_2 \circ P_2) = R \circ (-Q_1) \circ P_1 + R \circ Q_2 \circ P_2 = Q_1 \circ P_1 + Q_2 \circ P_2.$$

A different approach would be to check that

$$\frac{1}{\nu} \cdot k(((g \circ Q_1)(1)) \otimes (Q_1(1))) \in \text{End}(V)$$

is an idempotent, so this is a special case of Lemma 1.82.  $R$  is an isometry, using the orthogonality of the direct sum again:

$$\begin{aligned} R^* \circ g \circ R &= (R \circ (Q_1 \circ P_1 + Q_2 \circ P_2))^* \circ g \circ (R \circ (Q_1 \circ P_1 + Q_2 \circ P_2)) \\ &= (-Q_1 \circ P_1 + Q_2 \circ P_2)^* \circ g \circ (-Q_1 \circ P_1 + Q_2 \circ P_2) \\ &= P_1^* \circ Q_1^* \circ g \circ Q_1 \circ P_1 + P_2^* \circ Q_2^* \circ g \circ Q_2 \circ P_2 \\ &\quad - P_1^* \circ Q_1^* \circ g \circ Q_2 \circ P_2 - P_2^* \circ Q_2^* \circ g \circ Q_1 \circ P_1 \\ &= g. \end{aligned}$$
■

EXERCISE 3.99. In an orthogonal direct sum  $\text{End}(V) = \mathbb{K} \oplus \text{End}_0(V)$  with inclusion operator  $Q_1^\beta$  as in Example 2.9 and Theorem 3.50, the induced metric on  $\mathbb{K}$  is  $h^\nu$ , where  $\nu = \beta^2 \cdot \text{Tr}_V(\text{Id}_V)$  does not depend on the choice of metric  $g$  on  $V$ . The reflection  $R$  from the previous Exercise does not depend on  $g$  or  $\beta$ .

HINT. Using Lemma 3.96,

$$\nu = (b(Q_1^\beta(1)))(Q_1^\beta(1)) = \text{Tr}_V(\beta \cdot \text{Id}_V \circ g^{-1} \circ (\beta \cdot \text{Id}_V)^* \circ g) = \beta^2 \cdot \text{Tr}_V(\text{Id}_V).$$

For  $A \in \text{End}(V)$ ,

$$\begin{aligned} R(A) &= A - \frac{2}{\nu} \cdot (b(Q_1^\beta(1)))(A) \cdot Q_1^\beta(1) \\ &= A - \frac{2}{\nu} \cdot \text{Tr}_V(A \circ g^{-1} \circ (\beta \cdot \text{Id}_V)^* \circ g) \cdot \beta \cdot \text{Id}_V \\ &= A - \frac{2 \cdot \text{Tr}_V(A) \cdot \beta^2}{\beta^2 \cdot \text{Tr}_V(\text{Id}_V)} \cdot \text{Id}_V. \end{aligned}$$

■

EXERCISE 3.100. Given  $V$  and a metric  $g$ , if  $v \in V$  satisfies  $(g(v))(v) \neq 0$ , then there exists an orthogonal direct sum  $V = \mathbb{K} \oplus \ker(g(v))$ , such that the image of  $Q_1$  is the line spanned by  $v$ , and for any  $A \in \text{End}(V)$ ,

$$\text{Tr}_V(Q_1 \circ P_1 \circ A) = \frac{(g(v))(A(v))}{(g(v))(v)}.$$

HINT. Since  $g(v) \neq 0_{V^*}$ , Lemmas 1.63 and 1.64 give a direct sum  $V = \mathbb{K} \oplus \ker(g(v))$ , which is canonical up to equivalence, as follows. Let  $Q_2$  be the inclusion of the subspace  $\ker(g(v))$  in  $V$ . For any  $\alpha, \beta \in \mathbb{K}$  with  $\alpha \cdot \beta \cdot (g(v))(v) = 1$ , define

$$\begin{aligned} Q_1^\beta : \mathbb{K} &\rightarrow V : \gamma \mapsto \beta \cdot \gamma \cdot v, \\ P_1^\alpha &= \alpha \cdot g(v) : V \rightarrow \mathbb{K}, \\ P_2 &= \text{Id}_V - Q_1^\beta \circ P_1^\alpha : V \rightarrow \ker(g(v)). \end{aligned}$$

For the orthogonality of the direct sum, it is straightforward to check, using the symmetric property of  $g$ , that  $(P_1^\alpha)^* \circ (Q_1^\beta)^* \circ g = g \circ Q_1^\beta \circ P_1^\alpha$ , or that  $(Q_1^\beta)^* \circ g \circ Q_2$  and  $Q_2^* \circ g \circ Q_1^\beta$  are both zero. This is also a special case of Exercise 3.85 with  $H = Q_1^\beta$ .

It is also easy to check that

$$Q_1^\beta \circ P_1^\alpha \circ A = k(\beta \cdot \alpha \cdot ((g(v)) \circ A) \otimes v) \in \text{End}(V),$$

so by the definition of trace,

$$(3.5) \quad \text{Tr}_V(Q_1^\beta \circ P_1^\alpha \circ A) = \text{Ev}_V(\beta \cdot \alpha \cdot ((g(v)) \circ A) \otimes v)$$

$$(3.6) \quad = \beta \cdot \alpha \cdot (g(v))(A(v)) = \frac{(g(v))(A(v))}{(g(v))(v)},$$

where the RHS of (3.6) does not depend on the choice of  $\alpha, \beta$ . Further, if operators  $P_i'', Q_i''$  define any direct sum equivalent to the above orthogonal sum, then  $Q_1^\beta \circ P_1^\alpha = Q_1'' \circ P_1''$  as in Lemma 1.58, and the LHS of (3.5) is invariant under equivalent direct sums.

The above argument did not use the invertibility of  $g$ .

■

DEFINITION 3.101. Given a bilinear form  $g : V \rightarrow V^*$ , a linear map  $A : U \rightarrow V$  is isotropic with respect to  $g$  means that the pullback of  $g$  by  $A$  is zero:

$$A^* \circ g \circ A = 0_{\text{Hom}(U, U^*)}.$$

EXERCISE 3.102. Given  $V = V_1 \oplus V_2$  with inclusion operators  $Q_i$ , bilinear forms  $g_1 : V_1 \rightarrow V_1^*$ ,  $g_2 : V_2 \rightarrow V_2^*$ , and a map  $A : V_1 \rightarrow V_2$ , the following are equivalent:

- (1)  $g_1 = A^* \circ g_2 \circ A$ ;
- (2) The map  $Q_1 + Q_2 \circ A : V_1 \rightarrow V$  is isotropic with respect to the bilinear form  $g_1 \oplus (-g_2)$ .

HINT. The first property is that  $g_1$  is the pullback of  $g_2$  by  $A$  as in Definition 3.16; special cases include  $A$  being an isometry (Definition 3.20) or a symplectic isometry (Definition 3.79).

The second property refers to the direct sum of forms as in (3.1) from Notation 3.8. The expression  $Q_1 + Q_2 \circ A$  is from the notion that a “graph” of a linear map can be defined in terms of a direct sum, as in Exercise 1.70. ■

LEMMA 3.103. Let  $V = U \oplus U^*$ , with operators  $P_i, Q_i$ . The direct sum induces a symmetric form on  $V$ ,

$$(3.7) \quad P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1.$$

If  $U$  is finite-dimensional, then this symmetric form is a metric.

PROOF.

$$\begin{aligned} (P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1)^* \circ d_V &= P_2^* \circ P_1^{**} \circ d_V + P_1^* \circ d_U^* \circ P_2^{**} \circ d_V \\ &= P_2^* \circ d_U \circ P_1 + P_1^* \circ d_U^* \circ d_{U^*} \circ P_2 \\ &= P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1. \\ (P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1) \circ (Q_1 \circ d_U^{-1} \circ Q_2^* + Q_2 \circ Q_1^*) &= P_2^* \circ Q_2^* + P_1^* \circ Q_1^* \\ &= Id_{V^*}. \\ (Q_1 \circ d_U^{-1} \circ Q_2^* + Q_2 \circ Q_1^*) \circ (P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1) &= Q_1 \circ P_1 + Q_2 \circ P_2 \\ &= Id_V. \end{aligned}$$

■

EXAMPLE 3.104. For  $V = U \oplus U^*$ , with operators  $P_i, Q_i$ , let  $g_U$  be a metric on  $U$ . Then, the direct sum of the metric  $g_U$  and its dual  $d_U \circ g_U^{-1}$  is a metric on  $V$ :

$$g_U \oplus g_{U^*} = P_1^* \circ g_U \circ P_1 + P_2^* \circ d_U \circ g_U^{-1} \circ P_2,$$

as in Theorem 3.7 and Corollary 3.10.

The map  $K = Q_2 \circ g_U \circ P_1 + Q_1 \circ g_U^{-1} \circ P_2$  is an involution on  $V$ , as in Equation (1.7) from Theorem 1.89, and it is an isometry with respect to both the above induced metric  $g_U \oplus g_{U^*}$ , and the metric  $g_V$  from (3.7) in Lemma 3.103. In particular, if  $\frac{1}{2} \in \mathbb{K}$ , then Lemma 3.52 applies, so that the direct sum  $V = V_1 \oplus V_2$ , where

$$\begin{aligned} P'_1 &= \frac{1}{2}(Id_V + K) = \frac{1}{2}(Id_V + Q_2 \circ g_U \circ P_1 + Q_1 \circ g_U^{-1} \circ P_2) \\ P'_2 &= \frac{1}{2}(Id_V - K) = \frac{1}{2}(Id_V - Q_2 \circ g_U \circ P_1 - Q_1 \circ g_U^{-1} \circ P_2), \end{aligned}$$

is orthogonal with respect to both metrics on  $V$ . Each of the two metrics on  $V$  induces a metric on  $V_1$  and on  $V_2$ .

EXERCISE 3.105. For  $V = U \oplus U^*$  and  $V = V_1 \oplus V_2$  as in the above Example, the two induced metrics on  $V_1$  are identical, while those on  $V_2$  are opposite.

HINT. It is more convenient to check the equality of the inverses of the induced metrics on  $V_1$ , using (3.2) from Corollary 3.10 and the formulas from Theorem 3.55:

$$\begin{aligned}
& P'_1 \circ g_V^{-1} \circ (P'_1)^* \\
&= \frac{1}{2}(Id_V + K) \circ (Q_1 \circ d_U^{-1} \circ Q_2^* + Q_2 \circ Q_1^*) \circ \frac{1}{2}(Id_V + K)^* \\
&= P'_1 \circ (g_U \oplus g_{U^*})^{-1} \circ (P'_1)^* \\
&= \frac{1}{2}(Id_V + K) \circ (Q_1 \circ g_U^{-1} \circ Q_1^* + Q_2 \circ g_U \circ d_U^{-1} \circ Q_2^*) \circ \frac{1}{2}(Id_V + K)^* \\
&= \frac{1}{2}(Q_1 \circ g_U^{-1} \circ Q_1^* + Q_2 \circ g_U^* \circ Q_2^* + Q_2 \circ Q_1^* + Q_1 \circ d_U^{-1} \circ Q_2^*) \\
&= \frac{1}{2}(g_V^{-1} + (g_U \oplus g_{U^*})^{-1}).
\end{aligned}$$

The calculations for the metrics induced on  $V_2$  are similar. ■

EXERCISE 3.106. ([LP]) Let  $V = U \oplus U^*$ . Given maps  $E : W \rightarrow U$  and  $h : U \rightarrow W^*$ , the following are equivalent:

- (1) The bilinear form  $h \circ E : W \rightarrow W^*$  is antisymmetric;
- (2) The map  $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \rightarrow V$  is isotropic with respect to the symmetric form (3.7) on  $V$ .

Further, if  $E$  is a linear monomorphism, then so is  $Q_1 \circ E + Q_2 \circ h^* \circ d_W$ .

HINT. By Definition 3.101, the second property is that the pullback of the symmetric form (3.7) on  $V = U \oplus U^*$  from Lemma 3.103 by the map  $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \rightarrow V$  is  $0_{\text{Hom}(W, W^*)}$ . The transpose  $T_W(h \circ E)$  is  $E^* \circ h^* \circ d_W$ .

$$\begin{aligned}
& (Q_1 \circ E + Q_2 \circ h^* \circ d_W)^* \circ (P'_1 \circ P_2 + P_2^* \circ d_U \circ P_1) \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \\
&= E^* \circ h^* \circ d_W + d_W^* \circ h^{**} \circ d_U \circ E \\
&= E^* \circ h^* \circ d_W + h \circ E.
\end{aligned}$$

For any maps  $F, G$ , if

$$(Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ F = (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ G,$$

then

$$\begin{aligned}
P_1 \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ F &= P_1 \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ G \\
&= E \circ F = E \circ G,
\end{aligned}$$

so if  $E$  is a linear monomorphism (Definition 1.65), then  $F = G$ , proving the second claim. ■

If  $W = U$  and  $E = Id_U$ , then this construction is exactly the graph of  $h^* \circ d_U$ , as in Exercise 1.70. A generalization of the construction appears in Section 4.1.

EXAMPLE 3.107. Let  $V = U \oplus U^*$ , with operators  $P_i, Q_i$ . The direct sum induces an antisymmetric form on  $V$ ,

$$(3.8) \quad P_2^* \circ d_U \circ P_1 - P_1^* \circ P_2.$$

If  $U$  is finite-dimensional, then this antisymmetric form is symplectic (Definition 3.76). The construction is similar to the induced symmetric form (3.7) from Lemma 3.103, and canonical up to sign (as in (3.3), (3.4)). The inverse of the symplectic form is  $Q_1 \circ d_U^{-1} \circ Q_2^* - Q_2 \circ Q_1^*$ .

EXERCISE 3.108. ([LP]) Let  $V = U \oplus U^*$ . Given maps  $E : W \rightarrow U$  and  $h : U \rightarrow W^*$ , the following are equivalent:

- (1) The bilinear form  $h \circ E : W \rightarrow W^*$  is symmetric;
- (2) The map  $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \rightarrow V$  is isotropic with respect to the antisymmetric form (3.8).

■

### 3.6.4. The adjoint.

DEFINITION 3.109. Metrics  $g, h$ , on  $U, V$  induce an adjoint map,

$$(3.9) \quad \text{Hom}(h, g^{-1}) \circ t_{UV} : \text{Hom}(U, V) \rightarrow \text{Hom}(V, U) : A \mapsto g^{-1} \circ A^* \circ h.$$

EXERCISE 3.110. The map (3.9) is an isometry with respect to the induced  $b$  metrics. Also, if  $A : U \rightarrow V$  is an isometry, then its adjoint is an isometry  $V \rightarrow U$ .

HINT. The first assertion follows from the fact that  $g, h$ , and  $t_{UV}$  are isometries. The second claim follows from the following equation, which uses the symmetry of  $g$  and  $h$ , and Lemma 1.10:

$$(3.10) \quad (g^{-1} \circ A^* \circ h)^* = h^* \circ A^{**} \circ (g^{-1})^* = h \circ d_V^{-1} \circ A^{**} \circ d_U \circ g^{-1} = h \circ A \circ g^{-1},$$

and the hypothesis  $g = A^* \circ h \circ A$ :

$$\begin{aligned} (g^{-1} \circ A^* \circ h)^* \circ g \circ (g^{-1} \circ A^* \circ h) &= (h \circ A \circ g^{-1}) \circ A^* \circ h \\ &= h \circ A \circ A^{-1} = h. \end{aligned}$$

■

LEMMA 3.111. *The composite of adjoint maps,*

$$(\text{Hom}(g, h^{-1}) \circ t_{VU}) \circ (\text{Hom}(h, g^{-1}) \circ t_{UV}) : \text{Hom}(U, V) \rightarrow \text{Hom}(U, V)$$

*is the identity. In particular, the adjoint map  $\text{Hom}(g, g^{-1}) \circ t_{UU} : \text{End}(U) \rightarrow \text{End}(U)$  is an involution.*

PROOF. Using (3.10),

$$h^{-1} \circ (g^{-1} \circ A^* \circ h)^* \circ g = h^{-1} \circ (h \circ A \circ g^{-1}) \circ g = A.$$

■

EXERCISE 3.112. If  $\text{Tr}_U(\text{Id}_U) \neq 0$ , then the adjoint map  $\text{Hom}(g, g^{-1}) \circ t_{UU}$  respects an orthogonal direct sum  $\text{End}(U) = \mathbb{K} \oplus \text{End}_0(U)$ . In particular, Theorem 3.58 applies, so that the adjoint map is an involution and an isometry when restricted to  $\text{End}_0(U)$ .

HINT. The direct sum refers to the construction of Example 2.9 and Theorem 3.50, and it is easily checked that  $P_I \circ \text{Hom}(g, g^{-1}) \circ t_{UU} \circ Q_i$  is zero for  $i \neq I$ . ■

THEOREM 3.113. *Let  $g$  be a metric on  $U$ . Then the following diagram is commutative, where  $s$  and  $s'$  are switching involutions.*

$$\begin{array}{ccccc}
 U \otimes U & \xrightarrow{s} & U \otimes U & & \\
 \downarrow [g \otimes g] & & \downarrow [g \otimes g] & & \\
 U^* \otimes U^* & \xrightarrow{s'} & U^* \otimes U^* & & \\
 \downarrow k_{UU^*} & & \downarrow k_{UU^*} & & \\
 \text{Hom}(U, U^*) & \xrightarrow{t_{UU^*}} & \text{Hom}(U^{**}, U^*) & \xrightarrow{\text{Hom}(d_U, Id_{U^*})} & \text{Hom}(U, U^*) \\
 \uparrow \text{Hom}(Id_U, g) & & \uparrow \text{Hom}(g^*, Id_{U^*}) & & \uparrow \text{Hom}(Id_U, g) \\
 \text{End}(U) & \xrightarrow{t} & \text{End}(U^*) & \xrightarrow{\text{Hom}(g, g^{-1})} & \text{End}(U) \\
 \uparrow k & & \uparrow k' & & \uparrow k \\
 U^* \otimes U & \xrightarrow{p} & U^{**} \otimes U^* & \xrightarrow{[g^* \otimes g^{-1}]} & U^* \otimes U \\
 \uparrow [g \otimes Id_U] & & & & \uparrow [g \otimes Id_U] \\
 U \otimes U & \xrightarrow{s} & U \otimes U & & 
 \end{array}$$

All the horizontal compositions of arrows define involutions, and if  $\frac{1}{2} \in \mathbb{K}$ , then they define direct sums on the spaces in the left column as in Lemma 1.79. These direct sums are all orthogonal.

PROOF. The second square from the top does not involve the metric  $g$ . The composite in the third row is  $T_U$ , and the composite in the fifth row,  $[g^* \otimes g^{-1}] \circ p$ , is the only involution not considered earlier. The commutativity of all the squares is easy to check. The orthogonality of the direct sum for  $\text{Hom}(U, U^*)$  was checked in Theorem 3.53, and the orthogonality of the other direct sums similarly follows from Lemma 3.52 since all the horizontal arrows are isometries and involutions, or from Theorem 3.58 since all the vertical arrows are isometries which respect the direct sums, by Lemma 1.83.  $\blacksquare$

DEFINITION 3.114. If  $\frac{1}{2} \in \mathbb{K}$ , and  $g$  is a metric on  $U$ , then the orthogonal direct sum on  $\text{End}(U)$ , produced by the involution  $\text{Hom}(g, g^{-1}) \circ t$  as in Theorem 3.113, defines subspaces of self-adjoint ( $A = g^{-1} \circ A^* \circ g$ ) and skew-adjoint ( $A = -g^{-1} \circ A^* \circ g$ ) endomorphisms.

EXAMPLE 3.115. Given  $\frac{1}{2} \in \mathbb{K}$ , and a metric  $g$  on  $U$ ,  $h : U \rightarrow U^*$  is a symmetric (or, antisymmetric) form if and only if  $g^{-1} \circ h \in \text{End}(U)$  is self-adjoint (respectively, skew-adjoint). This is the action of the the middle left vertical arrow, and its inverse, from Theorem 3.113, respecting the direct sums  $\text{Hom}(U, U^*) \rightarrow \text{End}(U)$ .

EXERCISE 3.116. Given a metric  $g$  on  $U$ , and any vector  $u \in U$ , the endomorphism

$$k_{UU}((g(u)) \otimes u) \in \text{End}(U)$$

is self-adjoint.



HINT. From the commutativity of the diagram in Theorem 3.113,

$$\begin{aligned}
 & (\text{Hom}(g, g^{-1}) \circ t_{UU})(k_{UU}((g(u)) \otimes u)) \\
 = & (\text{Hom}(g, g^{-1}) \circ t_{UU} \circ k_{UU} \circ [g \otimes Id_U])(u \otimes u) \\
 = & (k_{UU} \circ [g \otimes Id_U] \circ s)(u \otimes u) \\
 = & k_{UU}((g(u)) \otimes u).
 \end{aligned}$$

■

EXERCISE 3.117. Given metrics  $g, h$  on  $U, V$ , and any map  $A : U \rightarrow V$ ,

$$\text{Hom}(g^{-1} \circ A^* \circ h, A) : \text{End}(U) \rightarrow \text{End}(V)$$

respects the direct sum from Definition 3.114.

HINT. The following diagram is commutative, so Lemma 1.83 applies.

$$\begin{array}{ccccc}
 \text{End}(U) & \xrightarrow{t_{UU}} & \text{End}(U^*) & \xrightarrow{\text{Hom}(g, g^{-1})} & \text{End}(U) \\
 \downarrow \text{Hom}(g^{-1} \circ A^* \circ h, A) & & & & \downarrow \text{Hom}(g^{-1} \circ A^* \circ h, A) \\
 \text{End}(V) & \xrightarrow{t_{VV}} & \text{End}(V^*) & \xrightarrow{\text{Hom}(h, h^{-1})} & \text{End}(V)
 \end{array}$$

■

EXERCISE 3.118. Given metrics  $g, h$  on  $U, V$ , any vector  $u \in U$ , and any map  $A : U \rightarrow V$ , the two self-adjoint endomorphisms from Exercise 3.116 are related by the map from Exercise 3.117:

$$\text{Hom}(g^{-1} \circ A^* \circ h, A)(k_{UU}((g(u)) \otimes u)) = k_{VV}((h(A(u))) \otimes (A(u))).$$

HINT. The left square is commutative by Lemma 1.25, and the right square is commutative by Lemma 1.29 and Equation (3.10).

$$\begin{array}{ccccc}
 U \otimes U & \xrightarrow{[g \otimes Id_U]} & U^* \otimes U & \xrightarrow{k_{UU}} & \text{End}(U) \\
 \downarrow [A \otimes B] & & \downarrow [(h \circ A \circ g^{-1}) \otimes B] & & \downarrow \text{Hom}(g^{-1} \circ A^* \circ h, B) \\
 V \otimes V & \xrightarrow{[h \otimes Id_V]} & V^* \otimes V & \xrightarrow{k_{VV}} & \text{End}(V)
 \end{array}$$

The equality follows from the case where  $B = A$ , and starting with  $u \otimes u \in U \otimes U$ . ■

EXERCISE 3.119. Given a metric  $g$  on  $U$ , and an endomorphism  $A \in \text{End}(U)$ , any pair of two of the following three statements implies the remaining one:

- (1)  $A$  is an involution;
- (2)  $A$  is self-adjoint;
- (3)  $A$  is an isometry.

■

EXERCISE 3.120. Given a metric  $g$  on  $U$ , and an element  $u \in U$  such that  $(g(u))(u) = 1$ , the endomorphism

$$Id_U - 2 \cdot k((g(u)) \otimes u)$$

is an involution and an isometry. ■

**3.6.5. Eigenvalues.**

EXERCISE 3.121. Suppose  $h$  and  $g$  are bilinear forms on  $V$ , and  $g$  is symmetric. If  $h(v_1) = \lambda_1 \cdot g(v_1)$ , and  $(T_V(h))(v_2) = \lambda_2 \cdot g(v_2)$ , then either  $\lambda_1 = \lambda_2$ , or  $(g(v_1))(v_2) = 0$ .

HINT.

$$\begin{aligned} (\lambda_1 - \lambda_2) \cdot (g(v_1))(v_2) &= (\lambda_1 \cdot g(v_1))(v_2) - (\lambda_2 \cdot g(v_2))(v_1) \\ &= (h(v_1))(v_2) - ((T_V(h))(v_2))(v_1) = 0. \end{aligned}$$

■

EXERCISE 3.122. Suppose  $h$  and  $g$  are bilinear forms on  $V$ , and  $g$  is antisymmetric. If  $h(v_1) = \lambda_1 \cdot g(v_1)$ , and  $(T_V(h))(v_2) = \lambda_2 \cdot g(v_2)$ , then either  $\lambda_1 = -\lambda_2$ , or  $(g(v_1))(v_2) = 0$ . ■

EXERCISE 3.123. If  $h$  and  $g$  are both symmetric forms (or both antisymmetric), and  $h(v_1) = \lambda_1 \cdot g(v_1)$ , and  $h(v_2) = \lambda_2 \cdot g(v_2)$ , then either  $\lambda_1 = \lambda_2$ , or  $(g(v_1))(v_2) = 0$ . ■

EXERCISE 3.124. If  $g$  is a bilinear form on  $V$ , and  $E$  is an endomorphism of  $V$  such that  $g \circ E = E^* \circ g : V \rightarrow V^*$ , and  $E(v_1) = \lambda_1 \cdot v_1$ , and  $E(v_2) = \lambda_2 \cdot v_2$ , then either  $\lambda_1 = \lambda_2$ , or  $(g(v_1))(v_2) = 0$ .

HINT. When  $g$  is a metric, the hypothesis is that  $E$  is self-adjoint.

$$\begin{aligned} (\lambda_1 - \lambda_2) \cdot (g(v_1))(v_2) &= (\lambda_1 \cdot g(v_1))(v_2) - (\lambda_2 \cdot g(v_1))(v_2) \\ &= (g(E(v_1)))(v_2) - (g(v_1))(E(v_2)) \\ &= ((g \circ E)(v_1))(v_2) - ((E^* \circ g)(v_1))(v_2) = 0. \end{aligned}$$

■

EXERCISE 3.125. If  $g$  is a metric on  $U$ ,  $H : U \rightarrow U$  is a self-adjoint endomorphism, and  $v \in V$  is nonzero, then there exists  $\lambda \in \mathbb{K}$  such that  $H(v) = \lambda \cdot v$  if and only if  $H$  commutes with the endomorphism  $k((g(v)) \otimes v)$  from Exercise 3.116.

HINT. The diagram from Exercise 3.118 gives these two equalities:

$$\begin{aligned} H \circ (k((g(v)) \otimes v)) &= k((g(v)) \otimes (H(v))), \\ (k((g(v)) \otimes v)) \circ H &= k((g(H(v))) \otimes v). \end{aligned}$$

If  $H(v) = \lambda \cdot v$ , then the two quantities are equal. Conversely, if they are equal, then for any  $u \in U$ ,

$$\begin{aligned} (k((g(v)) \otimes (H(v))))(u) &= (k((g(H(v))) \otimes v))(u) \\ (g(v))(u) \cdot (H(v)) &= (g(H(v)))(u) \cdot v. \end{aligned}$$

Since  $v \neq 0_U$ , the non-degeneracy of  $g$  implies there is some  $u$  so that  $(g(v))(u) \neq 0$ . Let  $\lambda = \frac{(g(H(v)))(u)}{(g(v))(u)}$ . ■

EXERCISE 3.126. If  $g$  is a bilinear form on  $V$ , and  $E$  is an endomorphism of  $V$  such that  $E^* \circ g \circ E = g : V \rightarrow V^*$ , and  $E(v_1) = \lambda_1 \cdot v_1$ , and  $E(v_2) = \lambda_2 \cdot v_2$ , then either  $\lambda_1 \cdot \lambda_2 = 1$ , or  $(g(v_1))(v_2) = 0$ . In particular, either  $\lambda_1^2 = 1$ , or  $(g(v_1))(v_1) = 0$ .

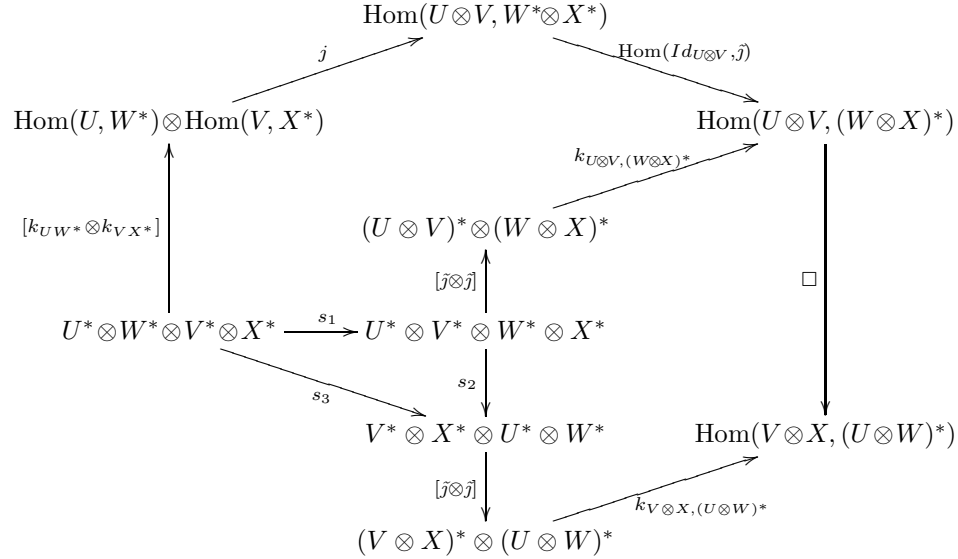
HINT.

$$\begin{aligned} (g(v_1))(v_2) &= ((E^* \circ g \circ E)(v_1))(v_2) = (g(E(v_1)))(E(v_2)) \\ &= \lambda_1 \cdot \lambda_2 \cdot (g(v_1))(v_2). \end{aligned}$$

■

### 3.6.6. Some formulas from applied mathematics.

In the following diagram,



$s_1$ ,  $s_2$ , and  $s_3 = s_2 \circ s_1$  are switching maps, and the various  $\tilde{j}$  maps are as in Notation 2.38. The top block is commutative, it is very similar to the diagram from Lemma 2.27.

NOTATION 3.127. If three of the four spaces  $U$ ,  $V$ ,  $W$ ,  $X$  are finite-dimensional, then all of the arrows in the above diagram are invertible. Define the map

$$\square : \text{Hom}(U \otimes V, (W \otimes X)^*) \rightarrow \text{Hom}(V \otimes X, (U \otimes W)^*)$$

to equal the composite going counter-clockwise around the lower right square in the diagram.

REMARK 3.128. The  $\square$  map is a version of the block vec matrix from [O].

EXERCISE 3.129. ([O] Theorem 1) If three of the four spaces  $U$ ,  $V$ ,  $W$ ,  $X$  are finite-dimensional, then for any  $A \in \text{Hom}(U \otimes V, (W \otimes X)^*)$ , the following are equivalent:

- (1) There exist  $h_1 \in \text{Hom}(U, W^*)$ ,  $h_2 \in \text{Hom}(V, X^*)$  such that

$$A = \tilde{j} \circ [h_1 \otimes h_2];$$

- (2) There exist  $\phi_1 \in (V \otimes X)^*$  and  $\phi_2 \in (U \otimes W)^*$  such that

$$\square(A) = k_{V \otimes X, (U \otimes W)^*}(\phi_1 \otimes \phi_2).$$

In the special case  $W = U$ ,  $X = V$ , (1) can be re-written using Notation 3.13:

(1') There exist  $h_1 \in \text{Hom}(U, U^*)$ ,  $h_2 \in \text{Hom}(V, V^*)$  such that  $A = \{h_1 \otimes h_2\}$ .

■

EXERCISE 3.130. ([O] Corollary 1) If  $V$  is finite-dimensional, then for any  $A \in \text{Hom}(V \otimes V, (V \otimes V)^*)$ , the following are equivalent:

- (1) There exists  $h \in \text{Hom}(V, V^*)$  such that  $A = \{h \otimes h\}$ ;
- (2) There exists  $\phi \in (V \otimes V)^*$  such that  $\square(A) = k_{V \otimes V, (V \otimes V)^*}(\phi \otimes \phi)$ .

Either (1) or (2) implies that the bilinear form  $\square(A)$  is symmetric. ■

REMARK 3.131. The following two Propositions relating the  $b$  metric to a trace on a tensor product space are analogous to a formula involving the “commutation matrix”  $K$ , which appears in [HJ] §4.3, and [Magnus] (exercise 3.9:  $\text{tr} K(A' \otimes B) = \text{tr} A'B$ ).

PROPOSITION 3.132. For  $A, B \in \text{Hom}(U, V)$ ,

$$\text{Tr}_{V^* \otimes U}([(h \circ d_V^{-1}) \otimes g^{-1}] \circ p \circ [B^* \otimes A]) = \text{Tr}_U(g^{-1} \circ B^* \circ h \circ A).$$

PROOF. In the following diagram,

$$\begin{array}{ccccc}
 U^* \otimes V \otimes U^* \otimes V & \xleftarrow{s} & U^* \otimes V \otimes U^* \otimes V & \xrightarrow{a_1} & \text{Hom}(U, V) \otimes \text{Hom}(U, V) \\
 \downarrow [[(d_U \circ g^{-1}) \otimes h] \otimes \text{Id}_{U^* \otimes V}] & & \downarrow [p \otimes \text{Id}_{U^* \otimes V}] & & \downarrow [t_{UV} \otimes \text{Id}_{\text{Hom}(U, V)}] \\
 U^{**} \otimes V^* \otimes U^* \otimes V & & V^{**} \otimes U^* \otimes U^* \otimes V & \xrightarrow{a_2} & \text{Hom}(V^*, U^*) \otimes \text{Hom}(U, V) \\
 \downarrow [j \otimes \text{Id}_{U^* \otimes V}] & & \downarrow [[\text{Id}_{V^{**}} \otimes g^{-1}] \otimes [\text{Id}_{U^*} \otimes h]] & & \downarrow j \\
 \text{Hom}(U^* \otimes V, \mathbb{K} \otimes \mathbb{K}) \otimes U^* \otimes V & & V^{**} \otimes U \otimes U^* \otimes V^* & & \downarrow j \\
 \downarrow [\text{Hom}(\text{Id}_{U^* \otimes V}, l) \otimes \text{Id}_{U^* \otimes V}] & & \downarrow s & & \downarrow j \\
 (U^* \otimes V)^* \otimes U^* \otimes V & & V^{**} \otimes V^* \otimes U^* \otimes U & & \text{Hom}(V^* \otimes U, U^* \otimes V) \\
 \downarrow \text{Ev}_{U^* \otimes V} & & \downarrow [k_{V^* V^*} \otimes k_{UU}] & & \downarrow \text{Hom}(\text{Id}_{V^* \otimes U}, p) \\
 \mathbb{K} & \nearrow l \circ [\text{Ev}_{V^*} \otimes \text{Ev}_U] & \text{End}(V^*) \otimes \text{End}(U) & \xrightarrow{j_2} & \text{Hom}(V^* \otimes U, V^{**} \otimes U^*) \\
 & \nearrow l \circ [\text{Tr}_{V^* \otimes U}] & \downarrow j_2 & & \downarrow a_3 \\
 & & \text{End}(V^* \otimes U) & \xleftarrow{a_3} & \text{Hom}(V^* \otimes U, V^{**} \otimes U^*) \\
 & & \text{Tr}_{V^* \otimes U} & & 
 \end{array}$$

the arrow  $s$  in the top row switches the two  $V$  factors, and the abbreviated arrow labels are

$$\begin{aligned}
 a_1 &= [k_{UV} \otimes k_{UV}] \\
 a_2 &= [k_{U^* V^*} \otimes k_{UV}] \\
 a_3 &= \text{Hom}(\text{Id}_{V^* \otimes U}, [(h \circ d_V^{-1}) \otimes g^{-1}]).
 \end{aligned}$$

The top right square is commutative by Lemmas 1.25 and 1.37. The lower left triangle is commutative by Corollary 2.31, and the triangle above that by the definition of trace. Starting with  $\Phi \otimes \phi \otimes \xi \otimes v \in V^{**} \otimes U^* \otimes U^* \otimes V$ , the lower

right square is commutative:

$$\begin{aligned}
\Phi \otimes \phi \otimes \xi \otimes v &\mapsto (a_3 \circ \text{Hom}(Id_{V^* \otimes U}, p) \circ j \circ a_2)(\Phi \otimes \phi \otimes \xi \otimes v) \\
&= [(h \circ d_V^{-1}) \otimes g^{-1}] \circ p \circ [(k_{U^* V^*}(\Phi \otimes \phi)) \otimes (k_{UV}(\xi \otimes v))] : \\
\psi \otimes u &\mapsto (h((\xi(u)) \cdot v)) \otimes (g^{-1}((\Phi(\psi)) \cdot \phi)), \\
\Phi \otimes \phi \otimes \xi \otimes v &\mapsto (j_2 \circ [k_{V^* V^*} \otimes k_{UU}] \circ s \circ [[Id_{V^*} \otimes g^{-1}] \otimes [Id_{U^*} \otimes h]])(\Phi \otimes \phi \otimes \xi \otimes v) \\
&= [(k_{V^* V^*}(\Phi \otimes (h(v)))) \otimes (k_{UU}(\xi \otimes (g^{-1}(\phi))))] : \\
\psi \otimes u &\mapsto (\Phi(\psi) \cdot h(v)) \otimes (\xi(u) \cdot g^{-1}(\phi)).
\end{aligned}$$

Starting with  $\phi \otimes w \otimes \xi \otimes v \in U^* \otimes V \otimes U^* \otimes V$ , the upper left square is commutative:

$$\begin{aligned}
\phi \otimes w \otimes \xi \otimes v &\mapsto (l \circ [Ev_{V^*} \otimes Ev_U] \circ s \circ [[Id_{V^*} \otimes g^{-1}] \otimes [Id_{U^*} \otimes h]] \circ [p \otimes Id_{U^* \otimes V}]) (\phi \otimes w \otimes \xi \otimes v) \\
&= (l \circ [Ev_{V^*} \otimes Ev_U] \circ s)((d_V(w)) \otimes (g^{-1}(\phi)) \otimes \xi \otimes (h(v))) \\
&= ((h(v))(w)) \cdot (\xi(g^{-1}(\phi))), \\
\phi \otimes w \otimes \xi \otimes v &\mapsto (Ev_{U^* \otimes V} \circ (\text{Hom}(Id_{U^* \otimes V}, l) \circ j \circ [(d_U \circ g^{-1}) \otimes h]) \otimes Id_{U^* \otimes V}) \circ s)(\phi \otimes w \otimes \xi \otimes v) \\
&= Ev_{U^* \otimes V}((l \circ [(d_U \circ g^{-1})(\phi)) \otimes (h(v))]) \otimes \xi \otimes w \\
&= (\xi(g^{-1}(\phi))) \cdot ((h(v))(w)).
\end{aligned}$$

This last quantity is also the result of the tensor product metric:

$$(\{(d_U \circ g^{-1}) \otimes h\}(\phi \otimes w))(\xi \otimes v) = (\xi(g^{-1}(\phi))) \cdot ((h(v))(w)),$$

from Corollary 3.14. So the claimed equality follows from the commutativity of the diagram, and the fact that  $k_{UV}^{-1}$  is an isometry (Theorem 3.39). Starting with  $B \otimes A \in \text{Hom}(U, V) \otimes \text{Hom}(U, V)$ :

$$\begin{aligned}
LHS &= (Tr_{V^* \otimes U} \circ a_3 \circ \text{Hom}(Id_{V^* \otimes U}, p) \circ j \circ [t_{UV} \otimes Id_{\text{Hom}(U, V)}])(B \otimes A) \\
&= (Ev_{U^* \otimes V} \circ (\text{Hom}(Id_{U^* \otimes V}, l) \circ j \circ [(d_U \circ g^{-1}) \otimes h]) \otimes Id_{U^* \otimes V}) \circ s \circ a_1^{-1})(B \otimes A) \\
&= (\{(d_U \circ g^{-1}) \otimes h\}(k_{UV}^{-1}(B)))(k_{UV}^{-1}(A)) \\
&= (b(B))(A) = RHS.
\end{aligned}$$

■

PROPOSITION 3.133. For  $A, B \in \text{Hom}(U, V)$ ,

$$Tr_{\{(d_V \circ h^{-1}) \otimes g\}}(f_{UV} \circ [B^* \otimes A]) = Tr_g(B^* \circ h \circ A).$$

PROOF. By Lemma 3.26 and the previous Proposition,

$$\begin{aligned}
LHS &= Tr_{V^* \otimes U}([(d_V \circ h^{-1}) \otimes g]^{-1} \circ j^{-1} \circ \text{Hom}(Id_{V^* \otimes U}, l)^{-1} \circ f_{UV} \circ [B^* \otimes A]) \\
&= Tr_{V^* \otimes U}([(h \circ d_V^{-1}) \otimes g^{-1}] \circ p \circ [B^* \otimes A]) \\
&= Tr_U(g^{-1} \circ B^* \circ h \circ A) = RHS.
\end{aligned}$$

■

### 3.6.7. Canonical metrics.

EXAMPLE 3.134. For finite-dimensional  $V$ , the canonical invertible map

$$(k^*)^{-1} \circ e : \text{End}(V) \rightarrow \text{End}(V)^*$$

from Lemma 2.1 is a metric on  $\text{End}(V)$ . It is symmetric by Lemma 1.10 and Lemma 2.1:

$$((k^*)^{-1} \circ e)^* \circ d_{\text{End}(V)} = e^* \circ (k^{-1})^{**} \circ d_{\text{End}(V)} = e^* \circ d \circ k^{-1} = (k^*)^{-1} \circ e.$$

This metric on  $\text{End}(V)$  should be called the “canonical” metric, to distinguish it from the  $b$  metric, induced by a choice of metric on  $V$ . The non-degeneracy of the metric was considered in Proposition 2.16, where it was also shown that for  $A, B \in \text{End}(V)$ ,

$$(((k^*)^{-1} \circ e)(A))(B) = \text{Tr}_V(A \circ B).$$

EXAMPLE 3.135. The canonical map  $f : V^* \otimes V \rightarrow (V^* \otimes V)^*$  is invertible for finite-dimensional  $V$ , and is symmetric by Lemma 1.34, so it is a metric on  $V^* \otimes V$ . The dual metric on  $(V^* \otimes V)^*$  is  $d \circ f^{-1} = (f^*)^{-1}$ .

This is another canonical metric, which is, in general, different from the tensor product metric induced by a choice of metric on  $V$ . By Lemma 3.26, the metric  $f$  is equal to the composite  $\text{Hom}(Id_{V^* \otimes V}, l) \circ j \circ p : V^* \otimes V \rightarrow (V^* \otimes V)^*$ .

EXERCISE 3.136. The dual metric  $d_{\mathbb{K}} \circ (h^1)^{-1}$  on  $\mathbb{K}^*$  coincides with the  $(k^*)^{-1} \circ e$  metric on  $\text{End}(\mathbb{K})$ .

HINT. For  $\phi, \xi \in \mathbb{K}^*$ ,

$$\begin{aligned} ((d_{\mathbb{K}} \circ (h^1)^{-1})(\phi))(\xi) &= \xi(\text{Tr}_{\mathbb{K}}(\phi)) = \xi(\phi(1)) = \phi(1) \cdot \xi(1) \\ (((k^*)^{-1} \circ e)(\phi))(\xi) &= \text{Tr}_{\mathbb{K}}(\phi \circ \xi) = \phi(\xi(1)) = \xi(1) \cdot \phi(1). \end{aligned}$$

■

EXERCISE 3.137. With respect to the canonical metrics  $(k^*)^{-1} \circ e$  on  $\text{End}(\mathbb{K})$  and  $h^1$  on  $\mathbb{K}$ ,  $\text{Tr}_{\mathbb{K}}$  is an isometry.

HINT. The canonical metric, applied to  $A, B \in \text{End}(\mathbb{K})$ , is:

$$(((k^*)^{-1} \circ e)(A))(B) = \text{Tr}_{\mathbb{K}}(A \circ B) = (A \circ B)(1) = A(B(1)) = B(1) \cdot A(1).$$

This coincides with the pullback:

$$(h^1(\text{Tr}_{\mathbb{K}}(A)))(\text{Tr}_{\mathbb{K}}(B)) = (h^1(A(1)))(B(1)) = A(1) \cdot B(1).$$

■

EXERCISE 3.138. The canonical map  $k : V^* \otimes V \rightarrow \text{End}(V)$  is an isometry with respect to the canonical metrics  $f$  and  $(k^*)^{-1} \circ e$ .

HINT. The pullback of  $(k^*)^{-1} \circ e$  by  $k$  agrees with  $f$ :

$$k^* \circ (k^*)^{-1} \circ e \circ k = f.$$

■

EXERCISE 3.139. The canonical map  $e : \text{End}(V) \rightarrow (V^* \otimes V)^*$  is an isometry with respect to the canonical metrics  $(k^*)^{-1} \circ e$  and  $(f^*)^{-1}$ .

HINT. The pullback of  $(f^*)^{-1}$  by  $e$  is:

$$e^* \circ (f^*)^{-1} \circ e = e^* \circ (e^*)^{-1} \circ (k^*)^{-1} \circ e = (k^*)^{-1} \circ e.$$

■

It follows that  $f_{VV}$  is an isometry, but this also follows from Theorem 3.23.

EXERCISE 3.140. ([G<sub>2</sub>] §I.8) If  $A : U \rightarrow V$  is an invertible map between finite-dimensional spaces, then  $\text{Hom}(A^{-1}, A) : \text{End}(U) \rightarrow \text{End}(V)$  is an isometry with respect to the  $(k^*)^{-1} \circ e$  metrics.

HINT. From the proof of Lemma 2.6:

$$\begin{aligned} (k'^*)^{-1} \circ e' &= (k'^*)^{-1} \circ e' \circ \text{Hom}(A, A^{-1}) \circ \text{Hom}(A^{-1}, A) \\ &= \text{Hom}(A^{-1}, A)^* \circ (k^*)^{-1} \circ e \circ \text{Hom}(A^{-1}, A). \end{aligned}$$

■

EXERCISE 3.141. For invertible  $A$  as in the previous Exercise,  $[(A^{-1})^* \otimes A] : U^* \otimes U \rightarrow V^* \otimes V$  is an isometry with respect to  $f_{UU}$  and  $f_{VV}$ .

HINT. This follows from Lemma 1.29:

$$[(A^{-1})^* \otimes A] = k_{VV}^{-1} \circ \text{Hom}(A^{-1}, A) \circ k_{UU},$$

and could also be checked directly.

■

EXERCISE 3.142. The transpose  $t : \text{End}(V) \rightarrow \text{End}(V^*) : A \mapsto A^*$  is an isometry with respect to the canonical  $(k^*)^{-1} \circ e$  metrics.

HINT. From the proof of Lemma 2.5:

$$t^* \circ (k'^*)^{-1} \circ e' \circ t = (k^*)^{-1} \circ e.$$

■

EXERCISE 3.143. The map  $j : \text{End}(U) \otimes \text{End}(V) \rightarrow \text{End}(U \otimes V)$  is an isometry with respect to the tensor product of canonical metrics, and the canonical metric on  $\text{End}(U \otimes V)$ .

HINT. By Corollary 2.31,

$$\begin{aligned} \text{Tr}_{U \otimes V}((j(A_1 \otimes B_1)) \circ (j(A_2 \otimes B_2))) &= \text{Tr}_{U \otimes V}(j((A_1 \circ A_2) \otimes (B_1 \circ B_2))) \\ &= \text{Tr}_U(A_1 \circ A_2) \cdot \text{Tr}_V(B_1 \circ B_2). \end{aligned}$$

■

EXERCISE 3.144. If  $\text{Tr}_U(\text{Id}_U) \neq 0$ , then the direct sum  $\text{End}(U) = \mathbb{K} \oplus \text{End}_0(U)$  is orthogonal with respect to the canonical metric  $(k^*)^{-1} \circ e$  on  $\text{End}(U)$ , and this induces a canonical metric on  $\text{End}_0(U)$ . ■

EXERCISE 3.145. The “adjoint” involution  $\text{Hom}(g, g^{-1}) \circ t_{UU}$  on  $\text{End}(U)$ , induced by a metric  $g$  on  $U$ , is an isometry with respect to the canonical metric. If  $\frac{1}{2} \in \mathbb{K}$ , then the direct sum decomposition into self-adjoint and skew-adjoint endomorphisms, from Definition 3.114, is orthogonal with respect to the canonical metric. On the space of self-adjoint endomorphisms, the metric induced by the canonical metric coincides with the metric induced by the induced  $b$  metric. On the space of skew-adjoint endomorphisms, the two induced metrics are opposite. ■



## Vector-valued Bilinear Forms

### 4.1. Symmetric forms

As a generalization of a “scalar-valued” bilinear form  $h : V \rightarrow V^*$ , a “vector-valued” (or “ $W$ -valued,” or “twisted”) form  $h : V \rightarrow \text{Hom}(V, W)$  could be considered. The space of such maps is  $\text{Hom}(V, \text{Hom}(V, W))$ . Most of the properties of the scalar-valued case generalize, but some of the canonical maps are different.

There would appear to be two ways to define a transpose operation on  $W$ -valued forms; one to transform  $\text{Hom}(V, \text{Hom}(V, W))$  into  $\text{Hom}(V, V^*) \otimes W$ , and then apply  $T_V \otimes Id_W$  (where  $T_V$  is the transpose for scalar-valued forms from Definition 3.2), and another to start from scratch with canonical maps from Chapter 1. Of course, the two approaches have the same result.

In the following Lemma,  $d_{VW}$  in the diagram is a generalized double duality from Definition 1.9, the  $t$  map is a generalized transpose from Definition 1.6, and  $n_1$  is a canonical  $n$  map from Definition 2.47, so that for  $F \otimes w \in \text{Hom}(V, V^*) \otimes W$ ,  $(n_1(F \otimes w))(v) = (F(v)) \otimes w$ .

LEMMA 4.1. *For any  $V, W$ , the following diagram is commutative.*

$$\begin{array}{ccccc}
 \text{Hom}(V, \text{Hom}(V, W)) & \xleftarrow{\text{Hom}(Id_V, k_{VW})} & \text{Hom}(V, V^* \otimes W) & \xleftarrow{n_1} & \text{Hom}(V, V^*) \otimes W \\
 \downarrow t_{V, \text{Hom}(V, W)}^W & & & & \downarrow [T_V \otimes Id_W] \\
 \text{Hom}(\text{Hom}(\text{Hom}(V, W), W), \text{Hom}(V, W)) & & & & \\
 \downarrow \text{Hom}(d_{VW}, Id_{\text{Hom}(V, W)}) & & & & \downarrow \\
 \text{Hom}(V, \text{Hom}(V, W)) & \xleftarrow{\text{Hom}(Id_V, k_{VW})} & \text{Hom}(V, V^* \otimes W) & \xleftarrow{n_1} & \text{Hom}(V, V^*) \otimes W
 \end{array}$$

PROOF. Starting with  $F \otimes w \in \text{Hom}(V, V^*) \otimes W$ ,

$$\begin{aligned}
 F \otimes w &\mapsto (\text{Hom}(d_{VW}, Id_{\text{Hom}(V, W)}) \circ t_{V, \text{Hom}(V, W)}^W \circ \text{Hom}(Id_V, k_{VW}) \circ n_1)(F \otimes w) : \\
 v &\mapsto ((t_{V, \text{Hom}(V, W)}^W(k_{VW} \circ (n_1(F \otimes w)))) \circ d_{VW})(v) \\
 &= (d_{VW}(v)) \circ k_{VW} \circ (n_1(F \otimes w)) : \\
 u &\mapsto (d_{VW}(v))(k_{VW}((F(u)) \otimes w)) \\
 &= (k_{VW}((F(u)) \otimes w))(v) = (F(u))(v) \cdot w, \\
 F \otimes w &\mapsto (\text{Hom}(Id_V, k_{VW}) \circ n_1 \circ [T_V \otimes Id_W])(F \otimes w) : \\
 v &\mapsto k_{VW}((n_1((T_V(F)) \otimes w))(v)) \\
 &= k_{VW}(((T_V(F))(v)) \otimes w) : \\
 u &\mapsto ((T_V(F))(v))(u) \cdot w = (F(u))(v) \cdot w.
 \end{aligned}$$

■

If  $V$  or  $W$  is finite-dimensional, then the  $k_{VW}$  and  $n_1$  maps in the above diagram are invertible.

DEFINITION 4.2. Let  $T_{V;W} = \text{Hom}(d_{VW}, Id_{\text{Hom}(V,W)}) \circ t_{V,\text{Hom}(V,W)}^W$ .

In the case  $W = \mathbb{K}$ ,  $T_{V;\mathbb{K}}$  is exactly  $T_V$ . The above Definition does not refer to the scalar field  $\mathbb{K}$ , scalar multiplication, or any dual space like  $V^*$ . It follows from the Definition that  $((T_{V;W}(h))(v))(u) = (h(u))(v)$ , and also that  $T_{V;W}$  is an involution:

LEMMA 4.3. For any  $V, W$ ,  $T_{V;W} \circ T_{V;W} = Id_{\text{Hom}(V,\text{Hom}(V,W))}$ .

PROOF.

$$\begin{aligned} T_{V;W}(T_{V;W}(h)) &= \text{Hom}((t_{V,\text{Hom}(V,W)}^W(h)) \circ d_{VW}, Id_W) \circ d_{VW} \\ &= \text{Hom}(d_{VW}, Id_W) \circ \text{Hom}(\text{Hom}(h, Id_W), Id_W) \circ d_{VW} \\ &= \text{Hom}(d_{VW}, Id_W) \circ d_{\text{Hom}(V,W),W} \circ h \\ &= h, \end{aligned}$$

using Lemmas 1.10, 1.5, and 1.13. ■

LEMMA 4.4. For  $h : V \rightarrow \text{Hom}(V, W)$ , and  $B : W \rightarrow W'$ ,

$$T_{V;W'}(\text{Hom}(Id_V, B) \circ h) = \text{Hom}(Id_V, B) \circ (T_{V;W}(h)).$$

PROOF. Using Definition 4.2 and Lemma 1.5,

$$\begin{aligned} LHS &= \text{Hom}(\text{Hom}(Id_V, B) \circ h, Id_{W'}) \circ d_{VW'} \\ &= \text{Hom}(h, Id_{W'}) \circ \text{Hom}(\text{Hom}(Id_V, B), Id_{W'}) \circ d_{VW'} \\ RHS &= \text{Hom}(Id_V, B) \circ \text{Hom}(h, Id_W) \circ d_{VW} \\ &= \text{Hom}(h, Id_{W'}) \circ \text{Hom}(Id_{\text{Hom}(V,W)}, B) \circ d_{VW}. \end{aligned}$$

The equality follows from the easily checked identity

$$\text{Hom}(Id_{\text{Hom}(V,W)}, B) \circ d_{VW} = \text{Hom}(\text{Hom}(Id_V, B), Id_{W'}) \circ d_{VW'}.$$
■

The involution  $T_{V;W}$  is also related to the switching involution  $s$ , as in the  $V = V_1 = V_2$  special case of the following Lemma.

LEMMA 4.5. For any  $V_1, V_2, W$ , the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}(V_1, \text{Hom}(V_2, W)) & \xrightarrow{q_1} & \text{Hom}(V_1 \otimes V_2, W) \\ \downarrow t_{V_1, \text{Hom}(V_2, W)}^W & & \downarrow \text{Hom}(s, Id_W) \\ \text{Hom}(\text{Hom}(\text{Hom}(V_2, W), W), \text{Hom}(V_1, W)) & & \\ \downarrow \text{Hom}(d_{V_2W}, Id_{\text{Hom}(V_1, W)}) & & \downarrow \\ \text{Hom}(V_2, \text{Hom}(V_1, W)) & \xrightarrow{q_2} & \text{Hom}(V_2 \otimes V_1, W) \end{array}$$
■

DEFINITION 4.6. A  $W$ -valued form  $h$  is symmetric means:  $h = T_{V;W}(h)$ .  $h$  is antisymmetric means:  $h = -T_{V;W}(h)$ . Let  $Sym(V;W)$  denote the subspace of symmetric forms, and  $Alt(V;W)$  the subspace of antisymmetric forms.

It follows from Lemmas 1.79 and 4.3 that if  $\frac{1}{2} \in \mathbb{K}$ , then  $\text{Hom}(V, \text{Hom}(V, W))$  splits as a direct sum  $\text{Sym}(V; W) \oplus \text{Alt}(V; W)$ . By Lemmas 1.83 and 4.4, the map  $\text{Hom}(Id_V, \text{Hom}(Id_V, B)) : \text{Hom}(V, \text{Hom}(V, W)) \rightarrow \text{Hom}(V, \text{Hom}(V, W'))$  respects the direct sums.

It follows from Lemma 4.1 that if  $h$  is of the form  $(\text{Hom}(Id_V, k_{VW}) \circ n_1)(E \otimes w)$ , for  $E : V \rightarrow V^*$ ,  $w \in W$ , and  $E$  is symmetric, or antisymmetric, then so is  $h$ .

DEFINITION 4.7. For any  $U, V, W$ , the pullback of a  $W$ -valued form  $h : V \rightarrow \text{Hom}(V, W)$  by a map  $H : U \rightarrow V$  is another  $W$ -valued form  $\text{Hom}(H, Id_W) \circ h \circ H : U \rightarrow \text{Hom}(U, W)$ .

In the case  $W = \mathbb{K}$ , this coincides with the previously defined pullback (Definition 3.16).

LEMMA 4.8. For a map  $H : U \rightarrow V$ , and a form  $h : V \rightarrow \text{Hom}(V, W)$ ,

$$T_{U;W}(\text{Hom}(H, Id_W) \circ h \circ H) = \text{Hom}(H, Id_W) \circ (T_{V;W}(h)) \circ H.$$

PROOF. The following is the same proof as that of Lemma 3.17.

$$\begin{aligned} LHS &= (t_{U, \text{Hom}(U, W)}^W(\text{Hom}(H, Id_W) \circ h \circ H)) \circ d_{UW} \\ &= \text{Hom}(\text{Hom}(H, Id_W) \circ h \circ H, Id_W) \circ d_{UW} \\ &= \text{Hom}(H, Id_W) \circ \text{Hom}(h, Id_W) \circ \text{Hom}(\text{Hom}(H, Id_W), Id_W) \circ d_{UW} \\ &= \text{Hom}(H, Id_W) \circ (t_{V, \text{Hom}(V, W)}^W(h)) \circ d_{VW} \circ H \\ &= \text{Hom}(H, Id_W) \circ (T_{V;W}(h)) \circ H, \end{aligned}$$

using Lemmas 1.10 and 1.5. ■

So, the pullback by  $H : U \rightarrow V$  of a symmetric form  $h : V \rightarrow \text{Hom}(V, W)$  is a symmetric form  $U \rightarrow \text{Hom}(U, W)$ . It similarly follows that the pullback of an antisymmetric form is antisymmetric.

NOTATION 4.9. For  $h_1 : V_1 \rightarrow \text{Hom}(V_1, W)$ , and  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W)$ , and a direct sum  $V = V_1 \oplus V_2$ , let  $h_1 \oplus h_2 : V \rightarrow \text{Hom}(V, W)$  denote the form

$$\text{Hom}(P_1, Id_W) \circ h_1 \circ P_1 + \text{Hom}(P_2, Id_W) \circ h_2 \circ P_2.$$

In the  $W = \mathbb{K}$  case, this is exactly the construction of Notation 3.8.

LEMMA 4.10.  $T_{V;W}(h_1 \oplus h_2) = (T_{V_1;W}(h_1)) \oplus (T_{V_2;W}(h_2))$ .

PROOF. The proof proceeds exactly as in Theorem 3.9, using Lemmas 1.10 and 1.5. ■

It follows that the direct sum of symmetric  $W$ -valued forms is symmetric, and similarly, the direct sum of antisymmetric forms is antisymmetric.

Working with the tensor product of vector-valued forms is simpler than the scalar case (Notation 3.13), since the scalar multiplication is omitted. If  $h_1 : V_1 \rightarrow \text{Hom}(V_1, W_1)$  and  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W_2)$  are two vector valued forms, then the map

$$j \circ [h_1 \otimes h_2] : V_1 \otimes V_2 \rightarrow \text{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)$$

has output

$$((j \circ [h_1 \otimes h_2])(u_1 \otimes u_2))(v_1 \otimes v_2) = ((h_1(u_1))(v_1)) \otimes ((h_2(u_2))(v_2)) \in W_1 \otimes W_2,$$

so it is a  $W_1 \otimes W_2$ -valued form.

THEOREM 4.11.

$$T_{V_1 \otimes V_2, W_1 \otimes W_2}(j \circ [h_1 \otimes h_2]) = j \circ [(T_{V_1, W_1}(h_1)) \otimes (T_{V_2, W_2}(h_2))].$$

PROOF. In analogy with the proof of Theorem 3.12, the following diagram is commutative:

$$\begin{array}{ccc} V_1 \otimes V_2 & \xrightarrow{[d_{V_1 W_1} \otimes d_{V_2 W_2}]} & M_1 \\ \downarrow d_{V_1 \otimes V_2, W_1 \otimes W_2} & & \downarrow j' \\ \text{Hom}(\text{Hom}(V_1 \otimes V_2, W_1 \otimes W_2), W_1 \otimes W_2) & \xrightarrow{\text{Hom}(j, Id_{W_1 \otimes W_2})} & M_2 \end{array}$$

where

$$\begin{aligned} M_1 &= \text{Hom}(\text{Hom}(V_1, W_1), W_1) \otimes \text{Hom}(\text{Hom}(V_2, W_2), W_2) \\ M_2 &= \text{Hom}(\text{Hom}(V_1, W_1) \otimes \text{Hom}(V_2, W_2), W_1 \otimes W_2); \\ v_1 \otimes v_2 &\mapsto (\text{Hom}(j, Id_{W_1 \otimes W_2}) \circ d_{V_1 \otimes V_2, W_1 \otimes W_2})(v_1 \otimes v_2) \\ &= (d_{V_1 \otimes V_2, W_1 \otimes W_2}(v_1 \otimes v_2)) \circ j : \\ A \otimes B &\mapsto [A \otimes B](v_1 \otimes v_2) = (A(v_1)) \otimes (B(v_2)), \\ v_1 \otimes v_2 &\mapsto (j' \circ [d_{V_1 W_1} \otimes d_{V_2 W_2}])(v_1 \otimes v_2) \\ &= j'((d_{V_1 W_1}(v_1)) \otimes (d_{V_2 W_2}(v_2))) : \\ A \otimes B &\mapsto (A(v_1)) \otimes (B(v_2)). \end{aligned}$$

The statement of the Theorem follows from Lemma 1.5, the above diagram, and Lemmas 1.26 and 1.25:

$$\begin{aligned} LHS &= (t_{V_1 \otimes V_2, \text{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)}^{W_1 \otimes W_2}(j \circ [h_1 \otimes h_2])) \circ d_{V_1 \otimes V_2, W_1 \otimes W_2} \\ &= \text{Hom}([h_1 \otimes h_2], Id_{W_1 \otimes W_2}) \circ \text{Hom}(j, Id_{W_1 \otimes W_2}) \circ d_{V_1 \otimes V_2, W_1 \otimes W_2} \\ &= \text{Hom}([h_1 \otimes h_2], Id_{W_1 \otimes W_2}) \circ j' \circ [d_{V_1 W_1} \otimes d_{V_2 W_2}] \\ &= j \circ [\text{Hom}(h_1, Id_{W_1}) \otimes \text{Hom}(h_2, Id_{W_2})] \circ [d_{V_1 W_1} \otimes d_{V_2 W_2}] \\ &= j \circ [(T_{V_1, W_1}(h_1)) \otimes (T_{V_2, W_2}(h_2))]. \end{aligned}$$

■

It follows that the tensor product of symmetric forms is symmetric, as is the tensor product of antisymmetric forms.

In the  $W_1 = \mathbb{K}$  case, the tensor product of a scalar-valued form  $h_1 : V_1 \rightarrow V_1^*$  and a vector-valued form  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W)$  is a form  $j \circ [h_1 \otimes h_2]$  with values in  $\mathbb{K} \otimes W$ . The map  $\text{Hom}(Id_{V_1 \otimes V_2}, l_W) \circ j \circ [h_1 \otimes h_2]$  is a  $W$ -valued form.

COROLLARY 4.12. For  $h_1 : V_1 \rightarrow V_1^*$  and  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W)$ , the following  $W$ -valued forms are equal.

$$\begin{aligned} &T_{V_1 \otimes V_2, W}(\text{Hom}(Id_{V_1 \otimes V_2}, l_W) \circ j \circ [h_1 \otimes h_2]) \\ &= \text{Hom}(Id_{V_1 \otimes V_2}, l_W) \circ j \circ [(T_{V_1}(h_1)) \otimes (T_{V_2, W}(h_2))]. \end{aligned}$$

PROOF. The equality follows immediately from Lemma 4.4, the previous Theorem, and the equality  $T_{V, \mathbb{K}} = T_V$ . ■

EXERCISE 4.13. Let  $V = U \oplus \text{Hom}(U, L)$  be a direct sum with operators  $P_i, Q_i$ . Then,

$$(4.1) \quad \text{Hom}(P_1, Id_L) \circ P_2 + \text{Hom}(P_2, Id_L) \circ d_{UL} \circ P_1$$

is a symmetric  $L$ -valued form on  $V$ . If  $d_{UL}$  is invertible, then this form is also invertible.

HINT. The proof that the form (4.1) is symmetric is the same as the calculation from Lemma 3.103, but using Lemma 1.10 and Lemma 1.13 in their full generality. If  $d_{UL}$  is invertible (for example, as in Proposition 3.73), then the inverse of the form is

$$Q_1 \circ d_{UL}^{-1} \circ \text{Hom}(Q_2, Id_L) + Q_2 \circ \text{Hom}(Q_1, Id_L).$$

■

EXERCISE 4.14. Let  $V = U \oplus \text{Hom}(U, L)$  as in Exercise 4.13. Given maps  $E : W \rightarrow U$  and  $h : U \rightarrow \text{Hom}(W, L)$ , the following are equivalent:

- (1) The  $L$ -valued form  $h \circ E : W \rightarrow \text{Hom}(W, L)$  is antisymmetric;
- (2) The pullback of the symmetric  $L$ -valued form (4.1) by the map

$$Q_1 \circ E + Q_2 \circ \text{Hom}(h, Id_L) \circ d_{WL} : W \rightarrow V$$

is  $0_{\text{Hom}(W, \text{Hom}(W, L))}$ .

HINT. The statement is analogous to Exercise 3.106, but uses the generalized notion of pullback from Definition 4.7. ■

EXERCISE 4.15. Suppose  $V = V_1 \oplus V_2$ , and there is an invertible map  $g : V \rightarrow \text{Hom}(V, L)$  so that these pullbacks are zero:

$$\begin{aligned} \text{Hom}(Q_1, Id_L) \circ g \circ Q_1 &= 0_{\text{Hom}(V_1, \text{Hom}(V_1, L))}, \\ \text{Hom}(Q_2, Id_L) \circ g \circ Q_2 &= 0_{\text{Hom}(V_2, \text{Hom}(V_2, L))}. \end{aligned}$$

Then the maps

$$\begin{aligned} \text{Hom}(Q_1, Id_L) \circ g \circ Q_2 : V_2 &\rightarrow \text{Hom}(V_1, L), \\ \text{Hom}(Q_2, Id_L) \circ g \circ Q_1 : V_1 &\rightarrow \text{Hom}(V_2, L) \end{aligned}$$

are invertible.

HINT. The inverses are  $P_2 \circ g^{-1} \circ \text{Hom}(P_1, Id_L), P_1 \circ g^{-1} \circ \text{Hom}(P_2, Id_L)$ . ■

EXERCISE 4.16. ([EPW]) Let  $V = V_1 \oplus V_2$  and  $g : V \rightarrow \text{Hom}(V, L)$  be as in the previous Exercise. Then there is another direct sum  $V = V_1 \oplus \text{Hom}(V_1, L)$ , defined by operators  $P_1, Q_1$ , and

$$\begin{aligned} P'_2 &= \text{Hom}(Q_1, Id_L) \circ g : V \rightarrow \text{Hom}(V_1, L), \\ Q'_2 &= g^{-1} \circ \text{Hom}(P_1, Id_L) : \text{Hom}(V_1, L) \rightarrow V. \end{aligned}$$

If, also,  $g$  is symmetric, then  $g$  is equal to the  $L$ -valued form induced by this direct sum, as in Exercise 4.13:

$$g = \text{Hom}(P_1, Id_L) \circ P'_2 + \text{Hom}(P'_2, Id_L) \circ d_{V_1, L} \circ P_1.$$

HINT. It is easy to check that  $P'_2 \circ Q_1$  is zero, and  $P'_2 \circ Q'_2$  is the identity.

$$\begin{aligned}
& \text{Hom}(Q_2, Id_L) \circ g \circ Q_1 \circ P_1 \circ Q'_2 \\
&= \text{Hom}(Q_2, Id_L) \circ g \circ Q_1 \circ P_1 \circ g^{-1} \circ \text{Hom}(P_1 \circ Id_L) \\
&= \text{Hom}(Q_2, Id_L) \circ g \circ (Id_V - Q_2 \circ P_2) \circ g^{-1} \circ \text{Hom}(P_1, Id_L) \\
&= 0_{\text{Hom}(\text{Hom}(V_1, L), \text{Hom}(V_2, L))}.
\end{aligned}$$

By the previous Exercise,  $\text{Hom}(Q_2, Id_L) \circ g \circ Q_1$  is invertible, so

$$P_1 \circ Q'_2 = 0_{\text{Hom}(\text{Hom}(V_1, L), V_1)}.$$

$$\begin{aligned}
& Q_1 \circ P_1 + Q'_2 \circ P'_2 \\
&= Q_1 \circ P_1 + (Q_1 \circ P_1 + Q_2 \circ P_2) \circ Q'_2 \circ P'_2 \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \\
&= Q_1 \circ P_1 + Q_2 \circ P_2 \circ g^{-1} \circ \text{Hom}(P_1, Id_L) \circ \text{Hom}(Q_1, Id_L) \circ g \circ Q_2 \circ P_2 \\
&= Q_1 \circ P_1 + Q_2 \circ P_2 = Id_V,
\end{aligned}$$

using the inverse formula from the previous Exercise. As for the claimed equality,

$$\begin{aligned}
RHS &= \text{Hom}(P_1, Id_L) \circ \text{Hom}(Q_1, Id_L) \circ g \\
&\quad + \text{Hom}(\text{Hom}(Q_1, Id_L) \circ g, Id_L) \circ d_{V_1, L} \circ P_1 \\
&= \text{Hom}(P_1, Id_L) \circ \text{Hom}(Q_1, Id_L) \circ g \circ (Q_1 \circ P_1 + Q_2 \circ P_2) + \\
&\quad \text{Hom}(g, Id_L) \circ \text{Hom}(\text{Hom}(Q_1, Id_L), Id_L) \circ d_{V_1, L} \circ P_1 \\
&= \text{Hom}(P_1, Id_L) \circ \text{Hom}(Q_1, Id_L) \circ g \circ Q_2 \circ P_2 \\
&\quad + \text{Hom}(P_2, Id_L) \circ \text{Hom}(Q_2, Id_L) \circ g \circ Q_2 \circ P_2 \\
&\quad + \text{Hom}(g, Id_L) \circ d_{V_L} \circ Q_1 \circ P_1 \\
&= \text{Hom}(Q_1 \circ P_1 + Q_2 \circ P_2, Id_L) \circ g \circ Q_2 \circ P_2 + (T_{V_L}(g)) \circ Q_1 \circ P_1 \\
&= g \circ (Q_2 \circ P_2 + Q_1 \circ P_1) = g.
\end{aligned}$$

■

#### 4.2. Vector-valued trace with respect to a metric

A metric  $g$  on  $V$  suggests a trace operator with values in  $W$ , but at first there would appear to be two constructions of a map  $Tr_{g;W} : \text{Hom}(V, \text{Hom}(V, W)) \rightarrow W$ . One way would be to combine the previously constructed vector-valued trace  $Tr_{V^*,W}$  (Definition 2.50) and composition with  $g^{-1}$ , and another would be to start with the scalar trace  $Tr_g$  (Definition 3.30), and tensor with  $Id_W$ . Of course, the two approaches have the same result.

LEMMA 4.17. *For finite-dimensional  $V$ , the following diagram is commutative.*

$$\begin{array}{ccccc}
 \text{Hom}(V, \text{Hom}(V, W)) & \xrightarrow{\text{Hom}(Id_V, k_{VW}^{-1})} & \text{Hom}(V, V^* \otimes W) & \xrightleftharpoons[n_1]{n_1^{-1}} & \text{Hom}(V, V^*) \otimes W \\
 \downarrow \text{Hom}(g^{-1}, k_{VW}^{-1}) & \nearrow \text{Hom}(g^{-1}, Id_{V^* \otimes W}) & \searrow [\text{Hom}(g^{-1}, Id_{V^*}) \otimes Id_W] & & \\
 \text{Hom}(V^*, V^* \otimes W) & \xleftarrow{n} & \text{Hom}(V^*, V^*) \otimes W & & \\
 \downarrow Tr_{V^*; W} & & \downarrow [Tr_{V^*} \otimes Id_W] & \nearrow [Tr_g \otimes Id_W] & \\
 W & \xleftarrow{l_W} & \mathbb{K} \otimes W & & 
 \end{array}$$

PROOF. The upper left triangle commutes by Lemma 1.5. The lower square is the definition of  $Tr_{V^*; W}$ , and the right triangle uses the definition of  $Tr_g$ . The upper square is commutative:

$$\begin{aligned}
 h \otimes w &\mapsto (\text{Hom}(g^{-1}, Id_{V^* \otimes W}) \circ n_1)(h \otimes w) \\
 &= (n_1(h \otimes w)) \circ g^{-1} : \\
 \phi &\mapsto (n_1(h \otimes w))(g^{-1}(\phi)) = (h(g^{-1}(\phi))) \otimes w, \\
 h \otimes w &\mapsto (n \circ [\text{Hom}(g^{-1}, Id_V) \otimes Id_W])(h \otimes w) \\
 &= n((h \circ g^{-1}) \otimes w) : \\
 \phi &\mapsto (h(g^{-1}(\phi))) \otimes w.
 \end{aligned}$$

■

DEFINITION 4.18. Given a metric  $g$  on  $V$ , an arbitrary vector space  $W$ , and a  $W$ -valued form  $h : V \rightarrow \text{Hom}(V, W)$ , the  $W$ -valued trace with respect to  $g$  is the following element of  $W$ :

$$Tr_{g; W}(h) = Tr_{V^*; W}(k_{VW}^{-1} \circ h \circ g^{-1}).$$

Corollary 2.56 also gives the equality

$$Tr_{V^*; W}(k_{VW}^{-1} \circ h \circ g^{-1}) = Tr_{V; W}([g^{-1} \otimes Id_W] \circ k_{VW}^{-1} \circ h).$$

By the previous Lemma,

$$Tr_{g; W} = Tr_{V^*; W} \circ \text{Hom}(g^{-1}, k_{VW}^{-1}) = l_W \circ [Tr_g \otimes Id_W] \circ n_1^{-1} \circ \text{Hom}(Id_V, k_{VW}^{-1}).$$

EXAMPLE 4.19. If  $h$  is of the form  $h = (\text{Hom}(Id_V, k_{VW}) \circ n_1)(E \otimes w)$ , for  $E : V \rightarrow V^*$  and  $w \in W$ , then  $Tr_{g; W}(h) = Tr_g(E) \cdot w$ , and if  $Tr_g(E) = 0$ , then  $Tr_{g; W}(h) = 0_W$ .

The previously defined scalar-valued trace with respect to  $g$  (Definition 3.30) is exactly the  $W = \mathbb{K}$  case of the vector-valued case:

THEOREM 4.20. *For  $h : V \rightarrow V^*$ ,  $Tr_{g;\mathbb{K}}(h) = Tr_g(h)$ .*

PROOF. It is simple to verify  $k_{V\mathbb{K}} : V^* \otimes \mathbb{K} \rightarrow V^*$  is just the scalar multiplication appearing in Theorem 2.54, so that

$$Tr_{g;\mathbb{K}}(h) = Tr_{V^*;\mathbb{K}}(k_{V\mathbb{K}}^{-1} \circ h \circ g^{-1}) = Tr_{V^*}(h \circ g^{-1}) = Tr_g(h).$$

■

THEOREM 4.21. *For any metric  $h^\nu$  on  $\mathbb{K}$ , as in Lemma 3.64, and a form  $h : \mathbb{K} \rightarrow \text{Hom}(\mathbb{K}, W)$ ,*

$$Tr_{h^\nu;W}(h) = \frac{1}{\nu} \cdot (h(1))(1).$$

PROOF.

$$\begin{aligned} Tr_{h^\nu;W}(h) &= Tr_{\mathbb{K};W}([(h^\nu)^{-1} \otimes Id_W] \circ k_{\mathbb{K}W}^{-1} \circ h) \\ &= (l_W \circ [(h^\nu)^{-1} \otimes Id_W] \circ k_{\mathbb{K}W}^{-1} \circ h)(1) \\ &= \left(\frac{1}{\nu} \cdot m^{-1}\right)(h(1)) \\ &= \frac{1}{\nu}(h(1))(1), \end{aligned}$$

where the first step uses Corollary 2.58, and the last step uses the formula  $m^{-1} = d_{\mathbb{K}W}(1)$ , from Definition 2.45. The intermediate step uses the commutativity of the diagram

$$\begin{array}{ccc} W & \xleftarrow{m^{-1}} & \text{Hom}(\mathbb{K}, W) \\ \nu \cdot l_W \uparrow & & \uparrow k_{\mathbb{K}W} \\ \mathbb{K} \otimes W & \xrightarrow{[h^\nu \otimes Id_W]} & \mathbb{K}^* \otimes W \end{array}$$

$$\begin{aligned} \lambda \otimes w &\mapsto (m^{-1} \circ k_{\mathbb{K}W} \circ [h^\nu \otimes Id_W])(\lambda \otimes w) \\ &= (k_{\mathbb{K}W}((h^\nu(\lambda)) \otimes w))(1) \\ &= \nu \cdot \lambda \cdot 1 \cdot w \\ &= (\nu \cdot l_W)(\lambda \otimes w). \end{aligned}$$

■

THEOREM 4.22.  $Tr_{g;W}(T_{V;W}(h)) = Tr_{g;W}(h)$ .

PROOF. Since  $V$  must be finite-dimensional, Lemmas 4.1 and 4.17 apply.

$$\begin{aligned} Tr_{g;W} \circ T_{V;W} &= l_W \circ [Tr_g \otimes Id_W] \circ n_1^{-1} \circ \text{Hom}(Id_V, k_{VW}^{-1}) \\ &\quad \circ \text{Hom}(Id_V, k_{VW}) \circ n_1 \circ [T_V \otimes Id_W] \circ n_1^{-1} \circ \text{Hom}(Id_V, k_{VW}^{-1}) \\ &= l_W \circ [Tr_g \otimes Id_W] \circ [T_V \otimes Id_W] \circ n_1^{-1} \circ \text{Hom}(Id_V, k_{VW}^{-1}) \\ &= l_W \circ [Tr_g \otimes Id_W] \circ n_1^{-1} \circ \text{Hom}(Id_V, k_{VW}^{-1}) \\ &= Tr_{g;W}, \end{aligned}$$

by Lemma 1.25 and Theorem 3.31, which stated that  $Tr_g \circ T_V = Tr_g$ .

■



COROLLARY 4.23. *If  $h : V \rightarrow \text{Hom}(V, W)$  is antisymmetric and  $\frac{1}{2} \in \mathbb{K}$ , then  $\text{Tr}_{g;W}(h) = 0_W$ . ■*

PROPOSITION 4.24. *The  $W$ -valued trace is invariant under pullback, that is, if  $H : U \rightarrow V$  is invertible, then*

$$\text{Tr}_{H^* \circ g \circ H; W}(\text{Hom}(H, \text{Id}_W) \circ h \circ H) = \text{Tr}_{g; W}(h).$$

PROOF. Using Corollary 2.56 and Lemma 1.29,

$$\begin{aligned} LHS &= \text{Tr}_{U^*; W}(k_{UW}^{-1} \circ \text{Hom}(H, \text{Id}_W) \circ h \circ H \circ H^{-1} \circ g^{-1} \circ (H^*)^{-1}) \\ &= \text{Tr}_{V^*; W}([(H^*)^{-1} \otimes \text{Id}_W] \circ k_{UW}^{-1} \circ \text{Hom}(H, \text{Id}_W) \circ h \circ g^{-1}) \\ &= \text{Tr}_{V^*; W}(k_{VW}^{-1} \circ h \circ g^{-1}) = RHS. \end{aligned}$$

THEOREM 4.25. *For any map  $B : W \rightarrow W'$ ,*

$$\text{Tr}_{g; W'}(\text{Hom}(\text{Id}_V, B) \circ h) = B(\text{Tr}_{g; W}(h)).$$

PROOF. Using Corollary 2.59 and Lemma 1.29,

$$\begin{aligned} LHS &= \text{Tr}_{V^*; W'}(k_{VW'}^{-1} \circ \text{Hom}(\text{Id}_V, B) \circ h \circ g^{-1}) \\ &= \text{Tr}_{V^*; W'}([\text{Id}_{V^*} \otimes B] \circ k_{VW}^{-1} \circ h \circ g^{-1}) \\ &= B(\text{Tr}_{V^*; W}(k_{VW}^{-1} \circ h \circ g^{-1})) = RHS. \end{aligned}$$

PROPOSITION 4.26. *For metrics  $g_1, g_2$  on  $V_1, V_2$ , and  $W$ -valued forms  $h_1 : V_1 \rightarrow \text{Hom}(V_1, W)$ ,  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W)$ , and a direct sum  $V = V_1 \oplus V_2$ ,*

$$\text{Tr}_{g_1 \oplus g_2; W}(h_1 \oplus h_2) = \text{Tr}_{g_1; W}(h_1) + \text{Tr}_{g_2; W}(h_2) \in W.$$

PROOF. First, Lemmas 1.5 and 1.29 apply to simplify the following map from  $\text{Hom}(V_I, W)$  to  $V_i^* \otimes W$ :

$$\begin{aligned} &[Q_i^* \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ \text{Hom}(P_I, \text{Id}_W) \\ &= k_{V_i W}^{-1} \circ \text{Hom}(Q_i, \text{Id}_W) \circ \text{Hom}(P_I, \text{Id}_W) \\ &= k_{V_i W}^{-1} \circ \text{Hom}(P_I \circ Q_i, \text{Id}_W) \\ &= k_{V_i W}^{-1}, \text{ if } i = I, \text{ or } 0_{\text{Hom}(\text{Hom}(V_I, W), V_i^* \otimes W)} \text{ if } i \neq I. \end{aligned}$$

Then the formula (3.2) for  $(g_1 \oplus g_2)^{-1}$  from Corollary 3.10 applies:

$$\begin{aligned} LHS &= \text{Tr}_{V; W}([(g_1 \oplus g_2)^{-1} \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ (h_1 \oplus h_2)) \\ &= \text{Tr}_{V; W}([(Q_1 \circ g_1^{-1} \circ Q_1^*) \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ \text{Hom}(P_1, \text{Id}_W) \circ h_1 \circ P_1 \\ &\quad + [(Q_1 \circ g_1^{-1} \circ Q_1^*) \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ \text{Hom}(P_2, \text{Id}_W) \circ h_2 \circ P_2 \\ &\quad + [(Q_2 \circ g_2^{-1} \circ Q_2^*) \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ \text{Hom}(P_1, \text{Id}_W) \circ h_1 \circ P_1 \\ &\quad + [(Q_2 \circ g_2^{-1} \circ Q_2^*) \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ \text{Hom}(P_2, \text{Id}_W) \circ h_2 \circ P_2) \\ &= \text{Tr}_{V; W}([(Q_1 \circ g_1^{-1}) \otimes \text{Id}_W] \circ k_{V_1 W}^{-1} \circ h_1 \circ P_1) \\ &\quad + \text{Tr}_{V; W}([(Q_2 \circ g_2^{-1}) \otimes \text{Id}_W] \circ k_{V_2 W}^{-1} \circ h_2 \circ P_2) \\ &= \text{Tr}_{V_1; W}([(P_1 \circ Q_1 \circ g_1^{-1}) \otimes \text{Id}_W] \circ k_{V_1 W}^{-1} \circ h_1) \\ &\quad + \text{Tr}_{V_2; W}([(P_2 \circ Q_2 \circ g_2^{-1}) \otimes \text{Id}_W] \circ k_{V_2 W}^{-1} \circ h_2) \\ &= \text{Tr}_{g_1; W}(h_1) + \text{Tr}_{g_2; W}(h_2). \end{aligned}$$

The last steps used Corollary 2.56 and Lemma 1.25. ■

**THEOREM 4.27.** *For metrics  $g_1, g_2$  on  $V_1, V_2$ , and vector-valued forms  $h_1 : V_1 \rightarrow \text{Hom}(V_1, W_1), h_2 : V_2 \rightarrow \text{Hom}(V_2, W_2)$ ,*

$$\text{Tr}_{\{g_1 \otimes g_2\}; W_1 \otimes W_2}(j \circ [h_1 \otimes h_2]) = (\text{Tr}_{g_1; W_1}(h_1)) \otimes (\text{Tr}_{g_2; W_2}(h_2)) \in W_1 \otimes W_2.$$

**PROOF.** The following diagram is commutative:

$$\begin{array}{ccc}
 V_1 \otimes V_2 \otimes W_1 \otimes W_2 & \xleftarrow{s_1} & V_1 \otimes W_1 \otimes V_2 \otimes W_2 \\
 \downarrow [[g_1 \otimes g_2] \otimes \text{Id}_{W_1 \otimes W_2}] & & \downarrow [[g_1 \otimes \text{Id}_{W_1}] \otimes [g_2 \otimes \text{Id}_{W_2}]] \\
 V_1^* \otimes V_2^* \otimes W_1 \otimes W_2 & \xrightarrow{s} & V_1^* \otimes W_1 \otimes V_2^* \otimes W_2 \\
 \downarrow [j \otimes \text{Id}_{W_1 \otimes W_2}] & & \downarrow [k_{V_1 W_1} \otimes k_{V_2 W_2}] \\
 \text{Hom}(V_2 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) \otimes W_1 \otimes W_2 & & \text{Hom}(V_1, W_1) \otimes \text{Hom}(V_2, W_2) \\
 \downarrow [\text{Hom}(\text{Id}_{V_1 \otimes V_2}, l) \otimes \text{Id}_{W_1 \otimes W_2}] & & \downarrow j \\
 (V_1 \otimes V_2)^* \otimes W_1 \otimes W_2 & \xrightarrow{k_{V_1 \otimes V_2, W_1 \otimes W_2}} & \text{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)
 \end{array}$$

The commutativity of the lower part is exactly Lemma 2.27. The top square is easy to check, where the  $s_1$  map is as in Theorem 2.35. The statement of the Theorem follows from Corollary 2.63, using Lemma 1.25 and the formula for  $\{g_1 \otimes g_2\}^{-1}$  from Corollary 3.14:

$$\begin{aligned}
 LHS &= \text{Tr}_{V_1 \otimes V_2; W_1 \otimes W_2}([(g_1^{-1} \otimes g_2^{-1}) \circ j^{-1} \circ \text{Hom}(\text{Id}_{V_1 \otimes V_2}, l^{-1})] \otimes \text{Id}_{W_1 \otimes W_2}) \\
 &\quad \circ k_{V_1 \otimes V_2, W_1 \otimes W_2}^{-1} \circ j \circ [h_1 \otimes h_2]) \\
 &= \text{Tr}_{V_1 \otimes V_2; W_1 \otimes W_2}(s_1 \circ [[g_1^{-1} \otimes \text{Id}_{W_1}] \otimes [g_2^{-1} \otimes \text{Id}_{W_2}]] \\
 &\quad \circ [k_{V_1 W_1}^{-1} \otimes k_{V_2 W_2}^{-1}] \circ [h_1 \otimes h_2]) \\
 &= \text{Tr}_{V_1 \otimes V_2; W_1 \otimes W_2}(s_1 \circ (j'_3([g_1^{-1} \otimes \text{Id}_{W_1}] \circ k_{V_1 W_1}^{-1} \circ h_1) \\
 &\quad \otimes ([g_2^{-1} \otimes \text{Id}_{W_2}] \circ k_{V_2 W_2}^{-1} \circ h_2)))) \\
 &= (\text{Tr}_{V_1; W_1}([g_1^{-1} \otimes \text{Id}_{W_1}] \circ k_{V_1 W_1}^{-1} \circ h_1)) \\
 &\quad \otimes (\text{Tr}_{V_2; W_2}([g_2^{-1} \otimes \text{Id}_{W_2}] \circ k_{V_2 W_2}^{-1} \circ h_2)) = RHS.
 \end{aligned}$$
■

**COROLLARY 4.28.** *For metrics  $g_1, g_2$  on  $V_1, V_2$ , a scalar-valued form  $h_1 : V_1 \rightarrow V_1^*$ , and a  $W$ -valued form  $h_2 : V_2 \rightarrow \text{Hom}(V_2, W)$ ,*

$$\text{Tr}_{\{g_1 \otimes g_2\}; W}(\text{Hom}(\text{Id}_{V_1 \otimes V_2}, l_W) \circ j \circ [h_1 \otimes h_2]) = \text{Tr}_{g_1}(h_1) \cdot \text{Tr}_{g_2; W}(h_2).$$

**PROOF.** Using Theorem 4.25, the previous Theorem, and Theorem 4.20,

$$\begin{aligned}
 LHS &= l_W(\text{Tr}_{\{g_1 \otimes g_2\}; \mathbb{K} \otimes W}(j \circ [h_1 \otimes h_2])) \\
 &= l_W((\text{Tr}_{g_1; \mathbb{K}}(h_1)) \otimes (\text{Tr}_{g_2; W}(h_2))) = RHS.
 \end{aligned}$$
■

**THEOREM 4.29.** *If  $\frac{1}{2} \in \mathbb{K}$ , and  $g$  and  $y$  are metrics on  $V$  and  $W$ , then the direct sum  $Sym(V; W) \oplus Alt(V; W)$  is orthogonal with respect to the induced metric.*

**PROOF.** Since  $V$  is finite-dimensional, all the arrows in the diagram for Lemma 4.1 are invertible. Let  $H = \text{Hom}(Id_V, k_{VW}) \circ n_1$ , so  $H$  and  $H^{-1}$  are isometries by Theorems 3.39 and 3.43, and Lemma 3.68. Also,  $[T_V \otimes Id_W]$  is an isometry by Corollary 3.46 and Theorem 3.25, so by Lemma 4.1 and Definition 4.2,  $T_{V;W}$  is an isometry, and an involution by Lemma 4.3. Then Lemma 3.52 applies to the direct sum produced by  $T_{V;W}$ .  $\blacksquare$

By Theorems 3.53 and 3.57,  $\text{Hom}(V, V^*) \otimes W = (Sym(V) \otimes W) \oplus (Alt(V) \otimes W)$  is an orthogonal direct sum. Since  $H$  respects the direct sums, by Lemmas 4.1 and 1.83, it follows from Theorem 3.58 that the maps between  $Sym(V) \otimes W$  and  $Sym(V; W)$ , and between  $Alt(V) \otimes W$  and  $Alt(V; W)$ , are isometries.

**THEOREM 4.30.** *If  $Tr_V(Id_V) \neq 0$ , and  $g$  is a metric on  $V$ , then there is a direct sum  $\text{Hom}(V, \text{Hom}(V, W)) = W \oplus \ker(Tr_{g;W})$ . If  $y$  is a metric on  $W$ , then the direct sum is orthogonal with respect to the induced metric.*

**PROOF.** By Theorem 3.51,  $\text{Hom}(V, V^*) = \mathbb{K} \oplus \ker(Tr_g)$  is an orthogonal direct sum, with operators  $P_1'' = \alpha \cdot Tr_g$ ,  $Q_1'' : \lambda \mapsto \lambda \cdot \beta \cdot g$ , with  $\alpha \cdot \beta \cdot Tr_V(Id_V) = 1$  as in Example 2.9. Also,  $P_2'' = Id_{\text{Hom}(V, V^*)} - Q_1'' \circ P_1''$ , and  $Q_2''$  is just the inclusion of the subspace  $\ker(Tr_g)$  in  $\text{Hom}(V, V^*)$ . By Example 1.45,  $\text{Hom}(V, V^*) \otimes W$  is a direct sum of  $\mathbb{K} \otimes W$  and  $(\ker(Tr_g)) \otimes W$ , with operators  $P_i = [P_i'' \otimes Id_W]$ ,  $Q_i = [Q_i'' \otimes Id_W]$ . Let  $H = \text{Hom}(Id_V, k_{VW}) \circ n_1$ , and let  $P_1' = \alpha \cdot Tr_{g;W}$ , so that the following diagram is commutative by Lemma 4.17.

$$\begin{array}{ccc} \text{Hom}(V, \text{Hom}(V, W)) & \xleftarrow{H} & \text{Hom}(V, V^*) \otimes W \\ \downarrow \alpha \cdot Tr_{g;W} & & \downarrow [(\alpha \cdot Tr_g) \otimes Id_W] \\ W & \xleftarrow{l_W} & \mathbb{K} \otimes W \end{array}$$

Let  $Q_2'$  be the inclusion of  $\ker(Tr_{g;W})$  in  $\text{Hom}(V, \text{Hom}(V, W))$ , which is a linear monomorphism so that  $P_1' \circ Q_2' = 0_{\text{Hom}(\ker(Tr_{g;W}), W)}$ . Define  $H_2 : (\ker(Tr_g)) \otimes W \rightarrow \ker(Tr_{g;W})$  by  $H_2 = H \circ Q_2$ ; the image of  $H_2$  is contained in  $\ker(Tr_{g;W})$  by Lemma 4.17, so  $Q_2' \circ H_2 = H \circ Q_2$ . Theorem 1.67 applies, so that  $\text{Hom}(V, \text{Hom}(V, W))$  has a direct sum structure  $W \oplus \ker(Tr_{g;W})$ , and  $H$  respects the direct sums. By Theorem 3.57, if  $W$  has a metric  $y$ , then the direct sum  $\text{Hom}(V, V^*) \otimes W = (\mathbb{K} \otimes W) \oplus ((\ker(Tr_g)) \otimes W)$  is orthogonal with respect to the induced metric. Since  $H$  is an isometry (as mentioned in the proof of the previous Theorem), it follows from Theorem 3.58 that  $W \oplus \ker(Tr_{g;W})$  is orthogonal with respect to the induced metric.  $\blacksquare$

**COROLLARY 4.31.** *The metric induced on  $W$  by the direct sum from the previous Theorem is  $\beta^2 \cdot Tr_V(Id_V) \cdot y$ .*

**PROOF.** The induced metric on  $\mathbb{K} \otimes W$  is  $\{h^\nu \otimes y\}$ , for  $\nu = \beta^2 \cdot Tr_V(Id_V)$  by Theorem 3.57 and Lemma 3.96. By Theorem 3.58,  $P_1' \circ H \circ Q_1 = l_W \circ P_1 \circ Q_1 = l_W : \mathbb{K} \otimes W \rightarrow W$  is an isometry, so by Lemma 3.66, the metric in the target must be  $\nu \cdot y$ .  $\blacksquare$

COROLLARY 4.32. *If both  $\frac{1}{2} \in \mathbb{K}$  and  $\text{Tr}_V(\text{Id}_V) \neq 0$ , then there is a direct sum  $\text{Hom}(V, \text{Hom}(V, W)) = W \oplus \text{Sym}_0(g; W) \oplus \text{Alt}(V; W)$ , where  $\text{Sym}_0(g; W)$  is the kernel of the restriction of  $\text{Tr}_{g;W}$  to  $\text{Sym}(V; W)$ . If  $W$  has a metric  $y$ , then there is an orthogonal direct sum. ■*

EXERCISE 4.33. For  $K : V \rightarrow \text{Hom}(V, W)$ , an orthogonal direct sum  $V = V_1 \oplus V_2$  with respect to a metric  $g$  on  $V$ , and the induced metrics  $g_1, g_2$ , on  $V_1, V_2$ ,

$$\text{Tr}_{g;W}(K) = \text{Tr}_{g_1;W}(\text{Hom}(Q_1, \text{Id}_W) \circ K \circ Q_1) + \text{Tr}_{g_2;W}(\text{Hom}(Q_2, \text{Id}_W) \circ K \circ Q_2).$$

HINT. In analogy with Exercise 3.95, Lemma 1.29 and Corollary 2.56 apply:

$$\begin{aligned} LHS &= \text{Tr}_{V^*;W}(k_{VW}^{-1} \circ K \circ g^{-1}) \\ &= \text{Tr}_{V^*;W}(k_{VW}^{-1} \circ K \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \circ g^{-1} \circ (Q_1 \circ P_1 + Q_2 \circ P_2)^*) \\ &= \text{Tr}_{V^*;W}(k_{VW}^{-1} \circ K \circ Q_1 \circ P_1 \circ g^{-1} \circ P_1^* \circ Q_1^*) \\ &\quad + \text{Tr}_{V^*;W}(k_{VW}^{-1} \circ K \circ Q_2 \circ P_2 \circ g^{-1} \circ P_2^* \circ Q_2^*) \\ &= \text{Tr}_{V_1;W}([Q_1^* \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ K \circ Q_1 \circ g_1^{-1}) \\ &\quad + \text{Tr}_{V_2;W}([Q_2^* \otimes \text{Id}_W] \circ k_{VW}^{-1} \circ K \circ Q_2 \circ g_2^{-1}) \\ &= \text{Tr}_{V_1;W}(k_{V_1W}^{-1} \circ \text{Hom}(Q_1, \text{Id}_W) \circ K \circ Q_1 \circ g_1^{-1}) \\ &\quad + \text{Tr}_{V_2;W}(k_{V_2W}^{-1} \circ \text{Hom}(Q_2, \text{Id}_W) \circ K \circ Q_2 \circ g_2^{-1}) = RHS. \end{aligned}$$

■

### 4.3. Partially symmetric forms

Maps  $A \otimes V \otimes V \rightarrow F$  are called “trilinear  $F$ -forms” in [EHM], and are “partially symmetric” if they are invariant under transposition of the  $V$  factors. Such forms, of course, lie in the scope of these notes, and it will also be convenient to consider maps of the form

$$V \otimes U \rightarrow \text{Hom}(V, W),$$

as in the vector-valued forms of the previous two Sections, but with the domain twisted by  $U$ . The two notions are related by a canonical map.

NOTATION 4.34. For arbitrary  $V, U, W, X$ , define

$$q : \text{Hom}(X \otimes U, \text{Hom}(V, W)) \rightarrow \text{Hom}(V \otimes X \otimes U, W)$$

so that for  $G : X \otimes U \rightarrow \text{Hom}(V, W)$ ,  $v \in V$ ,  $x \in X$ , and  $u \in U$ ,

$$q(G) : v \otimes x \otimes u \mapsto (G(x \otimes u))(v).$$

The order of the factors differs in this case from Definition 1.39, but such maps will still have “ $q$ ” labels.

LEMMA 4.35. *The following diagram is commutative, where  $T_V$  is the transpose map from Definition 3.2, and  $q_1$  is the previously defined  $q$  map in the case  $X = V$ .*

$$\begin{array}{ccc}
 \text{Hom}(V, V^*) \otimes \text{Hom}(U, W) & \xrightarrow{[T_V \otimes Id_{\text{Hom}(U, W)}]} & \text{Hom}(V, V^*) \otimes \text{Hom}(U, W) \\
 \downarrow j & & \downarrow j \\
 \text{Hom}(V \otimes U, V^* \otimes W) & & \text{Hom}(V \otimes U, V^* \otimes W) \\
 \downarrow \text{Hom}(Id_{V \otimes U}, k_{VW}) & & \downarrow \text{Hom}(Id_{V \otimes U}, k_{VW}) \\
 \text{Hom}(V \otimes U, \text{Hom}(V, W)) & & \text{Hom}(V \otimes U, \text{Hom}(V, W)) \\
 \downarrow q_1 & & \downarrow q_1 \\
 \text{Hom}(V \otimes V \otimes U, W) & \xrightarrow{\text{Hom}([s \otimes Id_U], Id_W)} & \text{Hom}(V \otimes V \otimes U, W)
 \end{array}$$

■

DEFINITION 4.36. Define  $T_{V;U,W} \in \text{End}(\text{Hom}(V \otimes U, \text{Hom}(V, W)))$  by

$$T_{V;U,W} = q_1^{-1} \circ \text{Hom}([s \otimes Id_U], Id_W) \circ q_1.$$

If  $V$  is finite-dimensional, then  $j$  and  $k$  in the above diagram are invertible, and by the Lemma,

$$T_{V;U,W} = \text{Hom}(Id_{V \otimes U}, k_{VW}) \circ j \circ [T_V \otimes Id_{\text{Hom}(U, W)}] \circ j^{-1} \circ \text{Hom}(Id_{V \otimes U}, k_{VW}^{-1}).$$

As in Section 4.1,  $T_{V;U,W}$  is an involution, and if  $\frac{1}{2} \in \mathbb{K}$ , it defines a direct sum structure on  $\text{Hom}(V \otimes U, \text{Hom}(V, W))$ , by Lemma 1.79. The other two involutions in the above diagram also produce direct sums, and by Lemma 1.83, the maps  $q_1$  and  $\text{Hom}(Id_{V \otimes U}, k_{VW}) \circ j$  respect these direct sums.

EXERCISE 4.37. With respect to induced metrics,  $T_{V;U,W}$  is an isometry, and if  $\frac{1}{2} \in \mathbb{K}$ , it produces an orthogonal direct sum.

HINT. Use Definition 4.36 to show  $T_{V;U,W}$  is a composition of isometries. Then Lemma 3.52 applies, as in Theorem 4.29. ■

DEFINITION 4.38. A map  $G : V \otimes U \rightarrow \text{Hom}(V, W)$  is partially symmetric means:  $T_{V;U,W}(G) = G$ . More generally, a map  $G : X \otimes U \rightarrow \text{Hom}(V, W)$  is partially symmetric with respect to a map  $H : V \rightarrow X$  if  $G \circ [H \otimes Id_U] : V \otimes U \rightarrow \text{Hom}(V, W)$  is partially symmetric.

LEMMA 4.39. ([EHM]) For any  $V, U, W, X$ , and  $G : X \otimes U \rightarrow \text{Hom}(V, W)$ , the following diagram is commutative.

$$\begin{array}{ccc}
 \mathbb{K} \otimes X \otimes U & \xrightarrow{l} & X \otimes U \\
 \downarrow [Q_1^1 \otimes Id_{X \otimes U}] & & \downarrow G \\
 \text{End}(V) \otimes X \otimes U & & \\
 \downarrow n & \xrightarrow{\text{Hom}(Id_V, q(G))} & \text{Hom}(V, W) \\
 \text{Hom}(V, V \otimes X \otimes U) & & \\
 \uparrow k_{V, V \otimes X \otimes U} & \xrightarrow{[Id_V \otimes (q(G))]} & \uparrow k_{VW} \\
 V^* \otimes V \otimes X \otimes U & \xrightarrow{[Id_V \otimes (q(G))]} & V^* \otimes W
 \end{array}$$

PROOF. The lower square uses Lemma 1.29. In the upper square, the maps are  $n$  as in Definition 2.47, and an inclusion  $Q_1^1 : \lambda \rightarrow \lambda \cdot Id_V$  as in Example 2.9.

$$\begin{aligned}
 \lambda \otimes x \otimes u &\mapsto (\text{Hom}(Id_V, q(G)) \circ n \circ [Q_1^1 \otimes Id_{X \otimes U}])(\lambda \otimes x \otimes u) \\
 &= (q(G)) \circ (n(\lambda \cdot Id_V \otimes x \otimes u)) : \\
 v &\mapsto (q(G))(\lambda \cdot v \otimes x \otimes u) \\
 &= \lambda \cdot (G(x \otimes u))(v) = ((G \circ l)(\lambda \otimes x \otimes u))(v).
 \end{aligned}$$

■

THEOREM 4.40. For any spaces  $U, V, W, X, Y$ , and any maps  $G : X \otimes U \rightarrow \text{Hom}(V, W)$ ,  $M : Y \rightarrow X \otimes U$ , the following are equivalent.

$$\begin{aligned}
 (q(G)) \circ [Id_V \otimes M] &= 0_{\text{Hom}(V \otimes Y, W)} \\
 \iff G \circ M &= 0_{\text{Hom}(Y, \text{Hom}(V, W))}.
 \end{aligned}$$

PROOF. Let  $q$  be the map from Notation 4.34, and let  $q_2$  be another such map in the following diagram.

$$\begin{array}{ccc}
 \text{Hom}(X \otimes U, \text{Hom}(V, W)) & \xrightarrow{q} & \text{Hom}(V \otimes X \otimes U, W) \\
 \downarrow \text{Hom}(M, Id_{\text{Hom}(V, W)}) & & \downarrow \text{Hom}([Id_V \otimes M], Id_W) \\
 \text{Hom}(Y, \text{Hom}(V, W)) & \xrightarrow{q_2} & \text{Hom}(V \otimes Y, W)
 \end{array}$$

The diagram is commutative by Lemma 1.42. So,  $q_2(G \circ M) = (q(G)) \circ [Id_V \otimes M]$ . Since  $q_2$  is invertible (Lemma 1.40),  $G \circ M$  is zero if and only if its image under  $q_2$  is also zero. ■

EXAMPLE 4.41. Suppose there is some direct sum  $V \otimes X \otimes U = W_1 \oplus W_2$ , with projections  $P_i : V \otimes X \otimes U \rightarrow W_i$ . Then, for  $q_i : \text{Hom}(X \otimes U, \text{Hom}(V, W_i)) \rightarrow \text{Hom}(V \otimes X \otimes U, W_i)$  and  $M : Y \rightarrow X \otimes U$ ,

$$P_i \circ [Id_V \otimes M] = 0_{\text{Hom}(V \otimes Y, W_i)} \iff (q_i^{-1}(P_i)) \circ M = 0_{\text{Hom}(Y, \text{Hom}(V, W_i))}.$$

The following Theorem will use the direct sum

$$V \otimes V = S^2V \oplus \Lambda^2V,$$

produced by the switching involution  $s$  as in the proof of Lemma 4.35, with projections  $P_1 = \frac{1}{2} \cdot (Id_{V \otimes V} + s) : V \otimes V \twoheadrightarrow S^2V$  and  $P_2 = \frac{1}{2} \cdot (Id_{V \otimes V} - s) : V \otimes V \twoheadrightarrow \Lambda^2V$ .

THEOREM 4.42. Let  $H : V \rightarrow X$ , and suppose there is some direct sum

$$V \otimes X = Z_1 \oplus Z_2$$

with operators  $P'_i, Q'_i$ , such that  $[Id_V \otimes H] : V \otimes V \rightarrow V \otimes X$  respects the direct sums ( $P'_I \circ [Id_V \otimes H] \circ Q'_i$  is zero if  $i \neq I$ ), and  $P'_2 \circ [Id_V \otimes H] \circ Q'_2 : \Lambda^2 V \rightarrow Z_2$  is invertible. If  $G : X \otimes U \rightarrow \text{Hom}(V, W)$  is partially symmetric with respect to  $H$ , then

$$q(G) = (q(G)) \circ [(Q'_1 \circ P'_1) \otimes Id_U].$$

PROOF. The following diagram is commutative by Lemma 1.42, where  $q_1$  is as in Lemma 4.35.

$$\begin{array}{ccc} \text{Hom}(X \otimes U, \text{Hom}(V, W)) & \xrightarrow{q} & \text{Hom}(V \otimes X \otimes U, W) \\ \downarrow \text{Hom}([H \otimes Id_U], Id_{\text{Hom}(V, W)}) & & \downarrow \text{Hom}([Id_V \otimes [H \otimes Id_U]], Id_W) \\ \text{Hom}(V \otimes U, \text{Hom}(V, W)) & \xrightarrow{q_1} & \text{Hom}(V \otimes V \otimes U, W) \end{array}$$

Let  $P''_2$  denote the projection  $\frac{1}{2} \cdot (Id_{\text{Hom}(V \otimes U, \text{Hom}(V, W))} - T_{V;U,W})$ , so that if  $G$  is partially symmetric with respect to  $H$ , then

$$\begin{aligned} 0_{\text{Hom}(\Lambda^2 V \otimes U, W)} &= (q_1(P''_2(G \circ [H \otimes Id_U]))) \circ [Q_2 \otimes Id_U] \\ &= \left(\frac{1}{2} \cdot q_1(G \circ [H \otimes Id_U])\right) \circ [Q_2 \otimes Id_U] \\ &\quad - \left(\frac{1}{2} \cdot q_1(G \circ [H \otimes Id_U])\right) \circ [s \otimes Id_U] \circ [Q_2 \otimes Id_U] \\ &= (q_1(G \circ [H \otimes Id_U])) \circ [Q_2 \otimes Id_U] \circ [P_2 \otimes Id_U] \circ [Q_2 \otimes Id_U] \\ &= (q(G)) \circ [Id_V \otimes [H \otimes Id_U]] \circ [Q_2 \otimes Id_U] \\ &= (q(G)) \circ ([Q'_1 \otimes Id_U] \circ [P'_1 \otimes Id_U] + [Q'_2 \otimes Id_U] \circ [P'_2 \otimes Id_U]) \\ &\quad \circ [[Id_V \otimes H] \otimes Id_U] \circ [Q_2 \otimes Id_U] \\ &= (q(G)) \circ [Q'_2 \otimes Id_U] \circ [P'_2 \otimes Id_U] \circ [[Id_V \otimes H] \otimes Id_U] \circ [Q_2 \otimes Id_U]. \end{aligned}$$

Then, since  $[P'_2 \otimes Id_U] \circ [[Id_V \otimes H] \otimes Id_U] \circ [Q_2 \otimes Id_U]$  is invertible,

$$0_{\text{Hom}(Z_2 \otimes U, W)} = (q(G)) \circ [Q'_2 \otimes Id_U],$$

and it follows that

$$q(G) = (q(G)) \circ [(Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \otimes Id_U] = (q(G)) \circ [(Q'_1 \circ P'_1) \otimes Id_U].$$

■

It follows from the previous two Theorems that if  $M : Y \rightarrow X \otimes U$  and  $G$  is partially symmetric with respect to  $H$ , then

$$q_2(G \circ M) = (q(G)) \circ [(Q'_1 \circ P'_1) \otimes Id_U] \circ [Id_V \otimes M].$$

REMARK 4.43. The map  $H : V \rightarrow X$  could be an inclusion of a vector subspace, in which case the above  $Z_1$  corresponds to the space  $H.V$  in [EHM].

BIG EXERCISE 4.44. Given a metric  $g$  on  $V$ , there exists a trace operator

$$Tr_{g;U,W} : \text{Hom}(V \otimes U, \text{Hom}(V, W)) \rightarrow \text{Hom}(U, W)$$

having many nice properties which follow as corollaries of the results in Section 2.2.

■

#### 4.4. Revisiting the generalized trace

We return to some notions introduced in Section 2.4. Recall, from Notation 2.66, the map  $\eta = s \circ k^{-1} \circ Q_1^1 : \mathbb{K} \rightarrow V \otimes V^*$ .

NOTATION 4.45. For finite-dimensional  $V$ , consider the following diagram.

$$\begin{array}{ccccc}
 & & \text{End}(V) \otimes U & \xleftarrow{[k \otimes Id_U]} & V^* \otimes V \otimes U \\
 & \nearrow [Q_1^1 \otimes Id_U] & & \searrow [s \otimes Id_U] & \\
 \mathbb{K} \otimes U & \xrightarrow{[\eta \otimes Id_U]} & V \otimes V^* \otimes U & & \\
 \downarrow l_U & & \searrow n_1 & \nearrow k_{V, V \otimes U} & \\
 U & & & & \\
 \downarrow \eta_{VU} & & & & \\
 V \otimes \text{Hom}(V, U) & \xrightarrow{n_2} & \text{Hom}(V, V \otimes U) & & 
 \end{array}$$

$[Id_V \otimes k_{VU}]$  (diagonal arrow from  $\mathbb{K} \otimes U$  to  $\text{Hom}(V, V \otimes U)$ )

The top block is from (2.4), and the commutativity of the back triangle and right block are easily checked. So, a map  $\eta_{VU} : U \rightarrow V \otimes \text{Hom}(V, U)$  can be defined to be the following equal maps:

$$\begin{aligned}
 \eta_{VU} &= [Id_V \otimes k_{VU}] \circ [\eta \otimes Id_U] \circ l_U^{-1} \\
 &= n_2^{-1} \circ n_1 \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1}.
 \end{aligned}$$

With the above notation, Theorem 2.72 can be re-stated in terms of  $\eta_{VU}$ .

COROLLARY 4.46. For finite-dimensional  $V$ ,  $n_2$  as in the above diagram, any  $F : V \otimes U \rightarrow V \otimes W$ , and  $u \in U$ ,

$$(Tr_{V;U,W}(F))(u) = Tr_{V;W}(F \circ (n_2(\eta_{VU}(u)))).$$

PROOF. The following diagram is a modification of the diagram from the Proof of Theorem 2.72.

$$\begin{array}{ccccccc}
 \text{End}(V) \otimes U & \xleftarrow{[k \otimes Id_U]} & V^* \otimes V \otimes U & \xrightarrow{[Id_V \otimes F]} & V^* \otimes V \otimes W & \xrightarrow{[Ev_V \otimes Id_W]} & \mathbb{K} \otimes W \\
 & \nearrow [Q_1^1 \otimes Id_U] & \downarrow [s \otimes Id_U] & \searrow k_{V, V \otimes U} & \downarrow k_{V, V \otimes W} & & \downarrow l_W \\
 \mathbb{K} \otimes U & \xrightarrow{[\eta \otimes Id_U]} & V \otimes V^* \otimes U & & \text{Hom}(V, V \otimes W) & \xrightarrow{Tr_{V;W}} & W \\
 \downarrow l_U & & \downarrow [Id_V \otimes k_{VU}] & & \uparrow \text{Hom}(Id_V, F) & \nearrow n' & \uparrow Ev_{VW} \\
 U & \xrightarrow{\eta_{VU}} & V \otimes \text{Hom}(V, U) & \xrightarrow{n_2} & \text{Hom}(V, V \otimes U) & & \text{Hom}(V, W) \otimes V
 \end{array}$$

The diagram is commutative; the left blocks and lower middle triangle by the construction of  $\eta$  and  $\eta_{VU}$  in Notation 4.45, the upper middle triangle by Lemma 1.29, and the right block copied from the Proof of Theorem 2.72. The path from  $U$  to  $W$  along the top row is  $Tr_{V;U,W}(F)$  by Theorem 2.67, and equals the same composite map from  $U$  to  $W$  along the lower row, so

$$\begin{aligned}
 (4.2) \quad Tr_{V;U,W}(F) : u &\mapsto Ev_{VW}((n')^{-1}(F \circ (n_2(\eta_{VU}(u))))) \\
 &= Tr_{V;W}(F \circ (n_2(\eta_{VU}(u)))) \\
 &= Tr_{V;W}(F \circ (n_1(Id_V \otimes u))).
 \end{aligned}$$

■



The lower path in the diagram (the composition in (4.2)) has the interesting properties that it does not involve scalar multiplication or duals (except in the construction of  $\eta_{VU}$ ), and the maps  $\eta_{VU}$  and  $Ev_{VW}$  appear in symmetric roles.

The following pair of Theorems are analogues of Theorem 2.87; the idea is that  $\eta_{VU}$  and  $Ev_{VW}$  satisfy identities analogous to the abstractly defined evaluation and coevaluation maps as in Definition 2.88. Theorem 4.48 uses the transpose for vector-valued forms from Definition 4.2.

**THEOREM 4.47.** *For finite-dimensional  $V$ , and  $\eta_{VU} : U \rightarrow V \otimes \text{Hom}(V, U)$  as in Notation 4.45,*

$$[Id_V \otimes Ev_{VU}] \circ [\eta_{VU} \otimes Id_V] = s : U \otimes V \rightarrow V \otimes U.$$

**PROOF.** The claim is analogous to the first identity from Theorem 2.87, and the proof is also analogous. In the following diagram,  $V = V_1 = V_2 = V_3$ . The upper and middle left squares are from the diagram from Notation 4.45. The claim is that the lower left triangle in the following diagram is commutative.

$$\begin{array}{ccccc}
 \text{Hom}(V_2, V_1) \otimes U \otimes V_3 & \xleftarrow{[[k \otimes Id_U] \otimes Id_V]} & V_2^* \otimes V_1 \otimes U \otimes V_3 & \xrightarrow{[Id_{V^*} \otimes s_4^{-1}]} & V_2^* \otimes V_3 \otimes V_1 \otimes U \\
 \uparrow [[Q_1^1 \otimes Id_U] \otimes Id_V] & & \downarrow [[s \otimes Id_U] \otimes Id_V] & & \downarrow [Ev_V \otimes Id_{V \otimes U}] \\
 \mathbb{K} \otimes U \otimes V_3 & \xrightarrow{[[\eta \otimes Id_U] \otimes Id_V]} & V_1 \otimes V_2^* \otimes U \otimes V_3 & & \\
 \downarrow [l \otimes Id_V] & & \downarrow [[Id_V \otimes k_{VU}] \otimes Id_V] & & \downarrow \\
 U \otimes V_3 & \xrightarrow{[\eta_{VU} \otimes Id_V]} & V_1 \otimes \text{Hom}(V_2, U) \otimes V_3 & & \mathbb{K} \otimes V_1 \otimes U \\
 & \searrow s & \downarrow [Id_V \otimes Ev_{VU}] & \swarrow l & \\
 & & V \otimes U & & 
 \end{array}$$

The commutativity of the right block is easy to check, where the switching map  $s_4$  is as in Corollary 2.81. The composition starting at  $U \otimes V$  in the lower left and going all the way around the diagram clockwise to  $V \otimes U$  is the trace, as in Theorem 2.67, of  $s_4^{-1}$ , and the computation of Example 2.85 applies.

$$\begin{aligned}
 & [Id_V \otimes Ev_{VU}] \circ [\eta_{VU} \otimes Id_V] \\
 &= l \circ [Ev_V \otimes Id_{V \otimes U}] \circ [Id_{V^*} \otimes s_4^{-1}] \circ [(k^{-1} \circ Q_1^1) \otimes Id_{U \otimes V}] \circ l^{-1} \\
 &= Tr_{V; U \otimes V, V \otimes U}(s_4^{-1}) = s_3^{-1}.
 \end{aligned}$$

■

**THEOREM 4.48.** *For finite-dimensional  $V$ , and  $n$  maps as indicated in the following diagram,*

$$\begin{array}{ccccc}
 \text{Hom}(V, U) & \xrightarrow{\text{Hom}(Id_V, \eta_{VU})} & \text{Hom}(V, V \otimes \text{Hom}(V, U)) & \xleftarrow{n} & \text{Hom}(V, \text{Hom}(V, U)) \otimes V \\
 & & & & \downarrow [T_{V; U} \otimes Id_V] \\
 \text{Hom}(V, U) & \xleftarrow{\text{Hom}(Id_V, Ev_{VU})} & \text{Hom}(V, \text{Hom}(V, U) \otimes V) & \xleftarrow{n} & \text{Hom}(V, \text{Hom}(V, U)) \otimes V
 \end{array}$$

this composite map is equal to the identity map:

$$\text{Hom}(Id_V, Ev_{VU}) \circ n \circ [T_{V; U} \otimes Id_V] \circ n^{-1} \circ \text{Hom}(Id_V, \eta_{VU}) = Id_{\text{Hom}(V, U)}.$$

$$\begin{array}{c}
\begin{array}{c}
\text{Hom}(V_1, U) \\
\downarrow \text{Hom}(Id_V, \eta_{VU}) \\
\text{Hom}(V_1, V_3 \otimes \text{Hom}(V_2, U)) \\
\uparrow n \\
\text{Hom}(V_1, \text{Hom}(V_2, U)) \otimes V_3 \\
\downarrow [T_{V,U} \otimes Id_V] \\
\text{Hom}(V_2, \text{Hom}(V_1, U)) \otimes V_3 \\
\downarrow n \\
\text{Hom}(V_2, \text{Hom}(V_1, U) \otimes V_3) \\
\downarrow \text{Hom}(Id_V, Ev_{VU}) \\
\text{Hom}(V_2, U)
\end{array}
\quad
\begin{array}{c}
V_2^* \otimes V_3 \otimes V_1^* \otimes U \xrightarrow{[k \otimes Id_{V^* \otimes U}]} \text{Hom}(V_2, V_3) \otimes V_1^* \otimes U \\
\uparrow [s_4 \otimes Id_U] \quad \searrow [p \otimes Id_{V^* \otimes U}] \\
V_1^* \otimes V_2^* \otimes V_3 \otimes U \quad \text{Hom}(V_3^*, V_2^*) \otimes V_1^* \otimes U \\
\uparrow [s \otimes s] \quad \uparrow [k \otimes Id_{V^* \otimes U}] \\
V_2^* \otimes V_1^* \otimes U \otimes V_3 \xrightarrow{p} V_3^{**} \otimes V_2^* \otimes V_1^* \otimes U \\
\leftarrow [a_2 \otimes Id_V] \quad \downarrow [s_4 \otimes Id_V] \quad \downarrow [Id_{V^{**}} \otimes s_4] \\
V_1^* \otimes U \otimes V_2^* \otimes V_3 \xrightarrow{p} V_3^{**} \otimes V_1^* \otimes U \otimes V_2^* \\
\leftarrow [a_3 \otimes Id_V] \quad \downarrow [Ev_{V^*} \otimes Id_{U \otimes V^*}] \\
U \otimes V_2^*
\end{array}
\quad
\begin{array}{c}
V_1^* \otimes U \\
\uparrow l \\
\mathbb{K} \otimes V_1^* \otimes U \\
\downarrow [Q_1^1 \otimes Id_{V^* \otimes U}] \\
\text{Hom}(V_2, V_3) \otimes V_1^* \otimes U \\
\downarrow [t \otimes Id_{V^* \otimes U}] \\
\text{Hom}(V_3^*, V_2^*) \otimes V_1^* \otimes U \\
\uparrow [k \otimes Id_{V^* \otimes U}] \\
V_3^{**} \otimes V_2^* \otimes V_1^* \otimes U \\
\downarrow [Id_{V^{**}} \otimes s_4] \\
V_3^{**} \otimes V_1^* \otimes U \otimes V_2^* \\
\downarrow [Ev_{V^*} \otimes Id_{U \otimes V^*}] \\
\mathbb{K} \otimes U \otimes V_2^* \\
\downarrow l \\
U \otimes V_2^*
\end{array}
\end{array}$$
$$\begin{aligned} & l \circ [Ev_{V^*} \otimes Id_{U \otimes V^*}] \circ [Id_{V^{**}} \otimes s_4] \circ [(k^{-1} \circ \tilde{Q}_1^1) \otimes Id_{V^* \otimes U}] \circ l^{-1} \\ = & Tr_{V; V^* \otimes U, U \otimes V^*}(s_4) = s_3. \end{aligned}$$

The  $p$  maps are invertible by the finite-dimensionality of  $V$ , and the blocks with  $p$  and  $s$  maps are clearly commutative. The  $k$  and  $n$  maps are also invertible, and the center right triangle with  $k$ ,  $t$ , and  $p$  is commutative by Lemma 1.37. The claim of the Theorem is that the composition in the left column gives the identity map; this will follow if we can find invertible maps  $a_1$ ,  $a_2$ ,  $a_3$  that make the above diagram commutative.

The lower block of the following diagram matches the lower block of the first diagram.

$$\begin{array}{ccc}
(V_2 \otimes V_1)^* \otimes U \otimes V_3 & & \\
\downarrow [k_{V \otimes V, U} \otimes Id_V] & \swarrow [((\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U) \otimes Id_V] & \\
\text{Hom}(V_2 \otimes V_1, U) \otimes V_3 & & V_2^* \otimes V_1^* \otimes U \otimes V_3 \\
\uparrow [q \otimes Id_V] & & \downarrow [s_4 \otimes Id_V] \\
\text{Hom}(V_2, \text{Hom}(V_1, U)) \otimes V_3 & \xleftarrow{[a_3 \otimes Id_V]} & V_1^* \otimes U \otimes V_2^* \otimes V_3 \\
\downarrow n & & \downarrow p \\
\text{Hom}(V_2, \text{Hom}(V_1, U) \otimes V_3) & & V_3^{**} \otimes V_1^* \otimes U \otimes V_2^* \\
\downarrow \text{Hom}(Id_V, Ev_{VU}) & & \downarrow [Ev_{V^*} \otimes Id_{U \otimes V^*}] \\
\text{Hom}(V_2, U) & \xleftarrow{k_{VU} \circ s_3^{-1} \circ l} & \mathbb{K} \otimes U \otimes V_2^*
\end{array}$$

The maps appearing in the top block of the diagram are used to define the invertible map:

$$a_3 = q^{-1} \circ k_{V \otimes V, U} \circ [(\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U] \circ s_4^{-1}.$$

To check the commutativity of the diagram (and as a result, the commutativity of the lower block of the first diagram), it is enough to start with  $\phi \otimes \psi \otimes u \otimes v$  in the upper right corner, and compare the action of the images in  $\text{Hom}(V, U)$  on  $w \in V$ .

$$\begin{aligned}
w &\mapsto (Ev_{VU} \circ (n((q^{-1}(k_{V \otimes V, U}((l \circ [\phi \otimes \psi]) \otimes u))) \otimes v)))(w) \\
&= Ev_{VU}(((q^{-1}(k_{V \otimes V, U}((l \circ [\phi \otimes \psi]) \otimes u)))(w)) \otimes v) \\
&= ((q^{-1}(k_{V \otimes V, U}((l \circ [\phi \otimes \psi]) \otimes u)))(w))(v) \\
&= (k_{V \otimes V, U}((l \circ [\phi \otimes \psi]) \otimes u))(w \otimes v) \\
&= ((\phi(w)) \cdot (\psi(v))) \cdot u, \\
w &\mapsto ((k_{VU} \circ s_3^{-1} \circ l \circ [Ev_{V^*} \otimes Id_{U \otimes V^*}] \circ p \circ [s_4 \otimes Id_V])(\phi \otimes \psi \otimes u \otimes v))(w) \\
&= ((k_{VU} \circ s_3^{-1} \circ l \circ [Ev_{V^*} \otimes Id_{U \otimes V^*}])(d_V(v)) \otimes \psi \otimes u \otimes \phi)(w) \\
&= ((k_{VU} \circ s_3^{-1} \circ l)((\psi(v)) \otimes u \otimes \phi))(w) \\
&= (\psi(v)) \cdot ((\phi(w)) \cdot u).
\end{aligned}$$

In the following diagram, the composition going up the right column is the previously defined  $a_3$ .

$$\begin{array}{ccc}
\text{Hom}(V_1, \text{Hom}(V_2, U)) & \xrightarrow{T_{V,U}} & \text{Hom}(V_2, \text{Hom}(V_1, U)) \\
\downarrow q & & \downarrow q \\
\text{Hom}(V_1 \otimes V_2, U) & \xrightarrow{\text{Hom}(s_1, Id_U)} & \text{Hom}(V_2 \otimes V_1, U) \\
\uparrow k_{V \otimes V, U} & & \uparrow k_{V \otimes V, U} \\
(V_1 \otimes V_2)^* \otimes U & \xrightarrow{[s_1^* \otimes Id_U]} & (V_2 \otimes V_1)^* \otimes U \\
\uparrow [(\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U] & & \uparrow [(\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U] \\
V_1^* \otimes V_2^* \otimes U & \xrightarrow{[s \otimes Id_U]} & V_2^* \otimes V_1^* \otimes U \\
\uparrow [s \otimes Id_U] & & \downarrow s_4 \\
V_2^* \otimes V_1^* \otimes U & \xrightarrow{s_4} & V_1^* \otimes U \otimes V_2^*
\end{array}$$

The diagram is commutative: the top block by Lemma 4.5, the next lower block by Lemma 1.29, and the two lowest blocks are easy to check. Define the composition going up the left column:

$$a_2 = q^{-1} \circ k_{V \otimes V, U} \circ [(\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U] \circ [s \otimes Id_U].$$

Then  $a_2$  is invertible, and the commutativity of the corresponding center left square in the first diagram follows.

In the following diagram, the composition going up the right column is  $[a_2 \otimes Id_V]$ .

$$\begin{array}{ccc}
\text{Hom}(V_1, V_3 \otimes \text{Hom}(V_2, U)) & \xleftarrow{n} & \text{Hom}(V_1, \text{Hom}(V_2, U)) \otimes V_3 \\
\uparrow \text{Hom}(Id_V, [Id_V \otimes k_{VU}]) & & \downarrow [q \otimes Id_V] \\
\text{Hom}(V_1, V_3 \otimes V_2^* \otimes U) & & \text{Hom}(V_1 \otimes V_2, U) \otimes V_3 \\
\uparrow \text{Hom}(Id_V, [s \otimes Id_U]) & & \uparrow [k_{V \otimes V, U} \otimes Id_V] \\
\text{Hom}(V_1, V_2^* \otimes V_3 \otimes U) & & (V_1 \otimes V_2)^* \otimes U \otimes V_3 \\
\uparrow k_{V, V^* \otimes V \otimes U} & & \uparrow [(\text{Hom}(Id_{V \otimes V}, l) \circ j) \otimes Id_U] \otimes Id_V \\
V_1^* \otimes V_2^* \otimes V_3 \otimes U & \xleftarrow{[s \otimes s]} & V_1^* \otimes V_2^* \otimes U \otimes V_3 \\
& & \uparrow [[s \otimes Id_U] \otimes Id_V] \\
& & V_2^* \otimes V_1^* \otimes U \otimes V_3
\end{array}$$

The composition going up the left column defines the invertible map:

$$a_1 = \text{Hom}(Id_V, [Id_V \otimes k_{VU}]) \circ \text{Hom}(Id_V, [s \otimes Id_U]) \circ k_{V, V^* \otimes V \otimes U}.$$

To check the commutativity of the diagram, from which the commutativity of the corresponding center left square in the first diagram follows, start with this preliminary calculation for  $\phi, \psi \in V^*$ ,  $u, w, z \in V$ .

$$\begin{aligned} (q^{-1}(k_{V \otimes V, U}((l \circ [\psi \otimes \phi]) \otimes u)))(w) : z &\mapsto (k_{V \otimes V, U}((l \circ [\psi \otimes \phi]) \otimes u))(w \otimes z) \\ &= (\psi(w)) \cdot (\phi(z)) \cdot u \\ &= ((\psi(w)) \cdot (k_{VU}(\phi \otimes u)))(z). \end{aligned}$$

Then, starting with  $\phi \otimes \psi \otimes u \otimes v$  in the lower right corner of the diagram,

$$\begin{aligned} &(n \circ [a_2 \otimes Id_V])(\phi \otimes \psi \otimes u \otimes v) \\ &= (n \circ [q^{-1} \otimes Id_V] \circ [k_{V \otimes V} \otimes Id_V])((l \circ [\psi \otimes \phi]) \otimes u \otimes v) \\ &= n((q^{-1}(k_{V \otimes V}((l \circ [\psi \otimes \phi]) \otimes u))) \otimes v) : \\ w &\mapsto v \otimes ((q^{-1}(k_{V \otimes V, U}((l \circ [\psi \otimes \phi]) \otimes u)))(w)) \\ &= v \otimes ((\psi(w)) \cdot (k_{VU}(\phi \otimes u))), \\ &(a_1 \circ [s \otimes Id_U])(\phi \otimes \psi \otimes u \otimes v) \\ &= [Id_V \otimes k_{VU}] \circ [s \otimes Id_U] \circ (k_{V, V^* \otimes V \otimes U}(\psi \otimes \phi \otimes v \otimes u)) : \\ w &\mapsto [Id_V \otimes k_{VU}]([s \otimes Id_U]((\psi(w)) \cdot (\phi \otimes v \otimes u))) \\ &= (\psi(w)) \cdot v \otimes (k_{VU}(\phi \otimes u)). \end{aligned}$$

In the following diagram, the lowest two commutative squares are from the definition of  $\eta_{VU}$ .

$$\begin{array}{ccccc} & & \text{Hom}(V_2, V_3) \otimes V_1^* \otimes U & \xleftarrow{[k \otimes Id_{V^* \otimes U}]} & V_2^* \otimes V_3 \otimes V_1^* \otimes U \\ & \nearrow [Q_1^1 \otimes Id_{V^* \otimes U}] & \downarrow [s \otimes Id_U] & & \downarrow [s_4 \otimes Id_U] \\ \mathbb{K} \otimes V_1^* \otimes U & & V_1^* \otimes \text{Hom}(V_2, V_3) \otimes U & \xleftarrow{[[Id_{V^*} \otimes k] \otimes Id_U]} & V_1^* \otimes V_2^* \otimes V_3 \otimes U \\ & \downarrow l & \downarrow k_{V, \text{End}(V) \otimes U} & & \downarrow k_{V, V^* \otimes V \otimes U} \\ & V_1^* \otimes U & \text{Hom}(V_1, \text{Hom}(V_2, V_3) \otimes U) & \xleftarrow{\text{Hom}(Id_V, [k \otimes Id_U])} & \text{Hom}(V_1, V_2^* \otimes V_3 \otimes U) \\ & & \uparrow \text{Hom}(Id_V, [Q_1^1 \otimes Id_U]) & & \downarrow \text{Hom}(Id_V, [s \otimes Id_U]) \\ & & \text{Hom}(V_1, \mathbb{K} \otimes U) & \xrightarrow{\text{Hom}(Id_V, [\eta \otimes Id_U])} & \text{Hom}(V_1, V_3 \otimes V_2^* \otimes U) \\ & & \downarrow \text{Hom}(Id_V, l_U) & & \downarrow \text{Hom}(Id_V, [Id_V \otimes k_{VU}]) \\ & & \text{Hom}(V_1, U) & \xrightarrow{\text{Hom}(Id_V, \eta_{VU})} & \text{Hom}(V_1, V_3 \otimes \text{Hom}(V_2, U)) \end{array}$$

The commutativity of the two upper right squares is easy to check. The following calculation checks the commutativity of the left block. Starting with  $\alpha \otimes \phi \otimes u \in \mathbb{K} \otimes V^* \otimes U$ ,

$$\begin{aligned} &k_{V, \text{End}(V) \otimes U} \circ [s \otimes Id_U] \circ [Q_1^1 \otimes Id_{V^* \otimes U}] : \\ \alpha \otimes \phi \otimes v &\mapsto k_{V, \text{End}(V) \otimes U}(\phi \otimes (\alpha \cdot Id_V) \otimes u) : \\ v &\mapsto (\phi(v)) \cdot (\alpha \cdot Id_V) \otimes u, \\ &\text{Hom}(Id_V, [Q_1^1 \otimes Id_U]) \circ \text{Hom}(Id_V, l_U^{-1}) \circ k_{VU} \circ l : \\ \alpha \otimes \phi \otimes v &\mapsto [Q_1^1 \otimes Id_U] \circ l_U^{-1} \circ (k_{VU}(\alpha \cdot \phi \otimes u)) : \\ v &\mapsto (\alpha \cdot \phi(v) \cdot Id_V) \otimes u. \end{aligned}$$

The composition of the three downward arrows in the right column gives the previously defined  $a_1$ . So, the above calculation is enough to establish the commutativity of the top block in the first diagram:

$$\text{Hom}(Id_V, \eta_{VU}) \circ k_{VU} = a_1 \circ [s_4^{-1} \otimes Id_U] \circ [k^{-1} \otimes Id_{V^* \otimes U}] \circ [Q_1^1 \otimes Id_{V^* \otimes U}] \circ l^{-1}.$$

As mentioned earlier, this proves the claim of the Theorem. ■

## Complex Structures

At this point we abandon the general field  $\mathbb{K}$  and work almost exclusively with real number scalars and vector spaces over the field  $\mathbb{R}$ . Some of the objects could be considered vector spaces over the field of complex numbers, but for the sake of precision such objects will instead be considered as real vector spaces paired with some additional structure.

### 5.1. Complex Structure Operators

DEFINITION 5.1. Given a real vector space  $V$ , an endomorphism  $J \in \text{End}(V)$  is a complex structure operator means:  $J \circ J = -Id_V$ .

NOTATION 5.2. A complex structure operator is more briefly called a **CSO**. Sometimes a pair  $(V, J)$  will be denoted by a matching boldface letter,  $\mathbf{V}$ . Expressions such as  $v \in \mathbf{V}$ ,  $A : \mathbf{U} \rightarrow \mathbf{V}$ , etc., refer to the underlying real space  $v \in V$ ,  $A : U \rightarrow V$ , etc.

EXAMPLE 5.3. Given  $\mathbf{V} = (V, J_V)$  and another vector space  $U$ ,  $[Id_U \otimes J_V] \in \text{End}(U \otimes V)$  is a canonical CSO on  $U \otimes V$ , so we may denote  $U \otimes \mathbf{V} = (U \otimes V, [Id_U \otimes J_V])$ . Similarly, denote  $\mathbf{V} \otimes U = (V \otimes U, [J_V \otimes Id_U])$ .

EXAMPLE 5.4. Given a vector space  $V$  with CSO  $J_V$  and another vector space  $U$ ,  $\text{Hom}(Id_U, J_V) : A \mapsto J_V \circ A$  is a canonical CSO on  $\text{Hom}(U, V)$ , so we may denote  $\text{Hom}(U, \mathbf{V}) = (\text{Hom}(U, V), \text{Hom}(Id_U, J_V))$ . Similarly, denote  $\text{Hom}(\mathbf{V}, U) = (\text{Hom}(V, U), \text{Hom}(J_V, Id_U))$ .

EXAMPLE 5.5. Given  $V$ , a CSO  $J$  induces a CSO  $J^* = \text{Hom}(J, Id_{\mathbb{R}})$  on  $V^* = \text{Hom}(V, \mathbb{R})$ .

EXAMPLE 5.6. Given  $V$ , a CSO  $J \in \text{End}(V)$ , and any involution  $N$  that commutes with  $J$  (i.e.,  $N \in \text{End}(V)$  such that  $N \circ N = Id_V$  and  $N \circ J = J \circ N$ ),  $N \circ J$  is a CSO.

EXAMPLE 5.7. Given  $V \neq \{0_V\}$ , any CSO  $J \in \text{End}(V)$  is not unique, since  $-J$  is a different CSO. This is the  $N = -Id_V$  case from the previous Example.

EXAMPLE 5.8. Given  $V = V_1 \oplus V_2$ , suppose there is an invertible map  $A : V_2 \rightarrow V_1$ , as in (3) from Theorem 1.89. Then,  $V$  also admits a CSO,

$$(5.1) \quad J = Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2,$$

and its opposite,  $-J$ . In the special case (1) where  $V = U \oplus U$ ,  $A = Id_V$ , the above CSO is  $J = Q_2 \circ P_1 - Q_1 \circ P_2$ . In the equivalent situation, (8) from Theorem 1.89, where there exist anticommuting involutions  $K_1$  and  $K_2$  on  $V$ , the composite  $K_1 \circ K_2$  and its opposite are CSOs on  $V$ . Using  $K_1$  and  $K_2$  to define a direct sum and a map  $A$  as in (1.8), the above construction (5.1) gives the same pair of CSOs:  $\{\pm K_1 \circ K_2\} = \{\pm J\}$ .

LEMMA 5.9. *Given  $V$  with CSO  $J$  and  $v \in V$ , if  $v \neq 0_V$ , then the ordered list  $(v, J(v))$  is linearly independent.*

PROOF. “Independent” will always refer to independence over the scalar field  $\mathbb{R}$ . If  $J(v) = \alpha \cdot v$  for some  $\alpha \in \mathbb{R}$ , then  $J(J(v)) = (-1) \cdot v = \alpha^2 \cdot v$ ; there are no solutions for  $v \neq 0_V$  and  $\alpha \in \mathbb{R}$ . ■

EXERCISE 5.10. Not every real vector space admits a CSO. ■

EXERCISE 5.11. Given  $V$  with CSO  $J$ , if  $V \neq \{0_V\}$ , then the ordered list  $(Id_V, J)$  is linearly independent in  $\text{End}(V)$ . ■

LEMMA 5.12. *Given  $V$  with CSO  $J$  and  $v_1, \dots, v_\ell \in V$ , if the ordered list*

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}))$$

*is linearly independent, then so is the ordered list*

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell)).$$

PROOF. The  $\ell = 1$  case is Lemma 5.9. For  $\ell \geq 2$ , suppose there are real scalars  $\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_\ell$  such that

$$(5.2) \quad \alpha_1 \cdot v_1 + \dots + \alpha_\ell \cdot v_\ell + \beta_1 \cdot J(v_1) + \dots + \beta_\ell \cdot J(v_\ell) = 0_V.$$

Then,

$$(5.3) \quad \alpha_1 \cdot J(v_1) + \dots + \alpha_\ell \cdot J(v_\ell) - \beta_1 \cdot v_1 - \dots - \beta_\ell \cdot v_\ell = 0_V.$$

Subtracting  $\alpha_\ell$  times (5.2) minus  $\beta_\ell$  times (5.3), the  $J(v_\ell)$  terms cancel, and

$$\begin{aligned} &(\alpha_1 \alpha_\ell + \beta_1 \beta_\ell) \cdot v_1 + \dots + (\alpha_{\ell-1} \alpha_\ell + \beta_{\ell-1} \beta_\ell) \cdot v_{\ell-1} + (\alpha_\ell^2 + \beta_\ell^2) \cdot v_\ell \\ &+ (\alpha_\ell \beta_1 - \alpha_1 \beta_\ell) \cdot J(v_1) + \dots + (\alpha_\ell \beta_{\ell-1} - \alpha_{\ell-1} \beta_\ell) \cdot J(v_{\ell-1}) = 0_V. \end{aligned}$$

By the independence of the ordered list with  $2\ell - 1$  elements,  $\alpha_\ell^2 + \beta_\ell^2 = 0 \implies \alpha_\ell = \beta_\ell = 0$ . Then (5.2) and the independence hypothesis again (or Lemma 5.9 if  $\ell = 2$ ) imply  $\alpha_1 = \dots = \alpha_{\ell-1} = \beta_1 = \dots = \beta_{\ell-1} = 0$ . ■

DEFINITION 5.13. Given  $V$  with CSO  $J$ , a subspace  $H$  is  $J$ -invariant means:  $J(H) \subseteq H$ . Equivalently, because  $J$  is invertible,  $J(H) = H$ .

LEMMA 5.14. *Given  $V$  with CSO  $J$ , and a  $J$ -invariant subspace  $H$  of  $V$ , if  $0 < \dim(H) < \infty$ , then  $H$  admits an ordered basis of the form*

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell)).$$

PROOF. For  $H$  with  $\dim(H) > 0$ , there is some  $v_1 \in H$  with  $v_1 \neq 0_V$ , and by  $J$ -invariance,  $J(v_1) \in H$ , so by Lemma 5.9,  $(v_1, J(v_1))$  is a linearly independent ordered list of vectors in  $H$ . Suppose inductively that

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell))$$

is a linearly independent ordered list of vectors in  $H$ . If the ordered list spans  $H$ , it is an ordered basis; otherwise, there is some  $v_{\ell+1} \in H$  not in its span, so

$$(v_1, \dots, v_{\ell-1}, v_\ell, v_{\ell+1}, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell))$$

is an independent ordered list.  $J(v_{\ell+1}) \in H$  and Lemma 5.12 applies, so

$$(v_1, \dots, v_{\ell-1}, v_\ell, v_{\ell+1}, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell), J(v_{\ell+1}))$$

is an independent ordered list of elements in  $H$ . The construction eventually terminates by the finite dimension hypothesis. ■



DEFINITION 5.15. Given  $V$  with CSO  $J$ , a two-dimensional  $J$ -invariant subspace  $H = J(H) \subseteq V$  will be called a  $J$ -complex line in  $V$ .

By Lemma 5.14, a  $J$ -complex line  $H$  must be of the form  $\text{span}\{v, J(v)\}$  for some  $v \in H$ .

LEMMA 5.16. *Given  $V$  with CSO  $J$ , a  $J$ -complex line  $L$ , and a  $J$ -invariant subspace  $H \subseteq V$ , if there is a non-zero element  $v \in L \cap H$ , then  $L \subseteq H$ . In particular, if  $H$  is a  $J$ -complex line, then  $L = H$ .*

PROOF. By  $J$ -invariance,  $\{v, J(v)\} \subseteq L \cap H$ . By Lemma 5.9,  $(v, J(v))$  is an independent ordered list, so it is an ordered basis of  $L$  and its span is contained in  $H$ . Comment: the contrapositive can be stated: Given  $V$  with CSO  $J$ ,  $v \in V$ , and a  $J$ -invariant subspace  $H$ , if  $v \notin H$ , then  $H \cap \text{span}\{v, J(v)\} = \{0_V\}$ . ■

LEMMA 5.17. *Given  $V$  with CSO  $J$ , if  $L^1, L^2$  are distinct  $J$ -complex lines in  $V$ , then  $\dim(\text{span}(L^1 \cup L^2)) = 4$ . In particular, a subspace  $H \subseteq V$  with  $\dim(H) \leq 3$  can contain at most one  $J$ -complex line.*

PROOF. Suppose  $L_1$  is a  $J$ -complex line in  $V$ , with  $L_1 = \text{span}\{v, J(v)\}$ . If  $L_2$  is a  $J$ -complex line with  $L_2 \neq L_1$ , then  $L_2 \not\subseteq L_1$ , so there is some  $u \in L_2 \setminus L_1$ , and  $\{v, J(v), u\}$  is an independent set. Because  $L_2$  is  $J$ -invariant,  $J(u) \in L_2$ , so by Lemma 5.12,  $(v, J(v), u, J(u))$  is an independent ordered list of elements of  $L^1 \cup L^2$ , and an ordered basis of  $\text{span}(L^1 \cup L^2)$ . ■

LEMMA 5.18. *Given  $V$  with CSO  $J$ , an integer  $\ell$ , and a  $J$ -invariant subspace  $H$  of  $V$ , if  $\dim(H) \leq 2\ell \leq \dim(V)$ , then there exists a  $J$ -invariant subspace  $U$  of  $V$  with  $\dim(U) = 2\ell$  and  $H \subseteq U$ .*

PROOF. This is trivial for  $\dim(H) = 0$ ; otherwise, an ordered basis for  $H$  as in Lemma 5.14 can be extended by two more elements to an ordered basis of a  $J$ -invariant subspace as in the Proof of Lemma 5.14, and this can repeat until the dimension  $2\ell$  is reached. ■

LEMMA 5.19. *Given a vector space  $V_1$  with CSO  $J_1$  and an element  $v \in V_1$ , another vector space  $V_2$  with CSO  $J_2$ , and a real linear map  $A : V_1 \rightarrow V_2$ , the following are equivalent.*

- (1)  $A(J_1(v)) \in \text{span}\{A(v), J_2(A(v))\}$ ;
- (2)  $A$  maps the subspace  $\text{span}\{v, J_1(v)\} \subseteq V_1$  to the subspace  $\text{span}\{A(v), J_2(A(v))\} \subseteq V_2$ .

Further, if  $A$  and  $v$  satisfy (1) and  $A(J_1(v)) \neq 0_{V_2}$ , then  $A$  and  $J_1(v)$  satisfy (1):

$$A(J_1(J_1(v))) \in \text{span}\{A(J_1(v)), J_2(A(J_1(v)))\}.$$

PROOF. (2)  $\iff$  (1) is straightforward. If  $A(J_1(v)) = \alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))$  for some  $(\alpha_1, \alpha_2) \neq (0, 0)$ , then there is this linear combination.

$$\begin{aligned} & \frac{-\alpha_1}{\alpha_1^2 + \alpha_2^2} \cdot A(J_1(v)) + \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2} \cdot J_2(A(J_1(v))) \\ &= \frac{-\alpha_1}{\alpha_1^2 + \alpha_2^2} \cdot (\alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))) \\ & \quad + \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2} \cdot J_2(\alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))) \\ &= -A(v) = A(J_1(J_1(v))). \end{aligned}$$

■

The above notion for a real linear map  $A$  is slightly stronger than the statement that  $A$  maps the  $J_1$ -complex line  $\text{span}\{v, J_1(v)\}$  into some  $J_2$ -complex line; if  $A(v) = 0_{V_2}$ , condition (2) implies  $A$  maps  $\text{span}\{v, J_1(v)\}$  to the zero subspace.

## 5.2. Complex linear and antilinear maps

DEFINITION 5.20. For  $\mathbf{U} = (U, J_U)$ ,  $\mathbf{V} = (V, J_V)$ , a (real linear) map  $A \in \text{Hom}(U, V)$  is c-linear, with respect to the CSOs  $J_U, J_V$ , means:  $A \circ J_U = J_V \circ A$ . A map  $A \in \text{Hom}(U, V)$  is a-linear, with respect to the CSOs  $J_U, J_V$ , means:  $A \circ J_U = -J_V \circ A$ .

LEMMA 5.21. *If  $A : \mathbf{U} \rightarrow \mathbf{V}$  is c-linear (or a-linear) and invertible, then  $A^{-1}$  is also c-linear (or a-linear). The composite of two c-linear maps (or two a-linear maps) is c-linear.* ■

LEMMA 5.22. *Given  $\mathbf{V} = (V, J_V)$  and  $\mathbf{V}' = (V', J_{V'})$ , any map  $A : U' \rightarrow U$ , and a c-linear map  $B : \mathbf{V} \rightarrow \mathbf{V}'$ , the maps  $\text{Hom}(A, B) : \text{Hom}(U, \mathbf{V}) \rightarrow \text{Hom}(U', \mathbf{V}')$  and  $\text{Hom}(B, A) : \text{Hom}(\mathbf{V}', U') \rightarrow \text{Hom}(\mathbf{V}, U)$  are c-linear with respect to the induced CSOs from Example 5.4.* ■

LEMMA 5.23. *Given  $V$  and a CSO  $J \in \text{End}(V)$ , for any invertible  $A : U \rightarrow V$ , the composite  $A^{-1} \circ J \circ A \in \text{End}(U)$  is a CSO.  $A$  is c-linear with respect to  $A^{-1} \circ J \circ A$  and  $J$ . If  $B : U \rightarrow V$  is another invertible map and  $A^{-1} \circ J \circ A = B^{-1} \circ J \circ B$ , then  $A \circ B^{-1}$  is a c-linear endomorphism of  $(V, J)$ .* ■

The CSO  $A^{-1} \circ J \circ A$  is the pullback of  $J$ .

Given a direct sum  $V = V_1 \oplus V_2$ , recall from Definition 1.52 that a CSO  $J \in \text{End}(V)$  respects the direct sum if  $P_I \circ J \circ Q_i = 0_{\text{Hom}(V_i, V_I)}$  for  $i \neq I$ , or equivalently,  $Q_i \circ P_i \circ J = J \circ Q_i \circ P_i$  for  $i = 1$  or  $2$ , so the map  $Q_i \circ P_i$  is c-linear.

LEMMA 5.24. *Given  $V = V_1 \oplus V_2$  and CSOs  $J_{V_i} \in \text{End}(V_i)$  for  $i = 1, 2$ , then  $Q_1 \circ J_{V_1} \circ P_1 + Q_2 \circ J_{V_2} \circ P_2 \in \text{End}(V)$  is a CSO, and it respects the direct sum.* ■

LEMMA 5.25. *Given  $V = V_1 \oplus V_2$  and a CSO  $J \in \text{End}(V)$  that respects the direct sum, each induced map  $P_i \circ J \circ Q_i$  on  $V_i$  is a CSO.*

PROOF. The claim is easily checked. This Lemma is compatible with the previous one: given  $J$ , it is recovered by re-combining the induced maps as in Lemma 5.24:

$$Q_1 \circ (P_1 \circ J \circ Q_1) \circ P_1 + Q_2 \circ (P_2 \circ J \circ Q_2) \circ P_2 = J.$$

Conversely, given  $J_{V_1}, J_{V_2}$  as in Lemma 5.24, they agree with CSOs induced by  $Q_1 \circ J_{V_1} \circ P_1 + Q_2 \circ J_{V_2} \circ P_2$ :

$$P_i \circ (Q_1 \circ J_{V_1} \circ P_1 + Q_2 \circ J_{V_2} \circ P_2) \circ Q_i = J_{V_i}.$$

■

LEMMA 5.26. For  $V = V_1 \oplus V_2$  and  $V' = V'_1 \oplus V'_2$  and invertible maps  $A : V_2 \rightarrow V'_1$ ,  $A' : V'_2 \rightarrow V'_1$ , let  $J, J'$  be CSOs on  $V$  and  $V'$  constructed as in Example 5.8. Then, for  $H : V \rightarrow V'$ , the following are equivalent:

- (1)  $H$  is  $c$ -linear with respect to  $J$  and  $J'$ ;
- (2)  $A' \circ P'_2 \circ H \circ Q_2 = P'_1 \circ H \circ Q_1 \circ A$  and  $P'_1 \circ H \circ Q_2 = -A' \circ P'_2 \circ H \circ Q_1 \circ A$ .

HINT. To show (2)  $\implies$  (1), expand

$$H \circ J = (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ H \circ (Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2)$$

and similarly  $J' \circ H$ .

For (1)  $\implies$  (2), apply  $\text{Hom}(Q_1, P'_2)$  to both sides of  $H \circ J = J' \circ H$  to get one of the equations, and apply  $\text{Hom}(Q_2, P'_2)$  to get the other. ■

REMARK 5.27. The preceding Lemma can be considered as an algebraic version of the Cauchy-Riemann equations.

EXERCISE 5.28. Given  $(V_1, J_1)$  and  $(V_2, J_2)$  and  $V = V_1 \oplus V_2$ , a map  $A : V_1 \rightarrow V_2$  is  $a$ -linear if and only if

$$Q_1 \circ J_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ J_2 \circ P_2$$

is a CSO on  $V$ . This CSO respects the direct sum if and only if  $A = 0_{\text{Hom}(V_1, V_2)}$ , in which case it is the CSO constructed in Lemma 5.24. ■

EXERCISE 5.29. For  $V, \mathbf{V}_1, \mathbf{V}_2, A$  as in the previous Exercise, the CSO  $J_A = Q_1 \circ J_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ J_2 \circ P_2$  is similar to the direct sum CSO  $J_0 = Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2$ , in the sense that  $J_A = G^{-1} \circ J_0 \circ G$  for some invertible  $G \in \text{End}(V)$ , and  $G$  can be chosen so that  $P_2 \circ G \circ Q_2 = \text{Id}_{V_2}$ .

HINT. Let  $G = \text{Id}_V + \frac{1}{2} \cdot Q_2 \circ A \circ J_1 \circ P_1$ , then check  $G \circ J_A = J_0 \circ G$ , or use  $G^{-1} = \text{Id}_V - \frac{1}{2} \cdot Q_2 \circ A \circ J_1 \circ P_1$ . ■

EXERCISE 5.30. Given  $A \in \text{End}(V)$ , if there is some CSO  $J$  such that  $A$  is  $a$ -linear with respect to  $J$ , then  $\text{Tr}_V(A) = 0$ . ■

EXAMPLE 5.31. Given  $\mathbf{V} = (V, J_V)$ , the canonical maps  $l_1 : \mathbf{V} \otimes \mathbb{R} \rightarrow \mathbf{V}$  and  $l_2 : \mathbb{R} \otimes \mathbf{V} \rightarrow \mathbf{V}$  from Example 1.18 are  $c$ -linear, where the domains have CSO  $[J_V \otimes \text{Id}_{\mathbb{R}}]$ , or  $[\text{Id}_{\mathbb{R}} \otimes J_V]$ , from Example 5.3.

EXERCISE 5.32. Given  $W$  and  $\mathbf{V} = (V, J_V)$ , the canonical map (Definition 1.9)  $d_{VW} : \mathbf{V} \rightarrow \text{Hom}(\text{Hom}(\mathbf{V}, W), W)$  is  $c$ -linear with respect to the induced CSO as in Example 5.4. ■

EXERCISE 5.33. Given  $U, V, W$ , with  $\mathbf{U} = (U, J)$ , the canonical map (Definition 1.6)  $t_{UV}^W : \text{Hom}(\mathbf{U}, V) \rightarrow \text{Hom}(\text{Hom}(V, W), \text{Hom}(\mathbf{U}, W))$  is  $c$ -linear with respect to the induced CSOs. ■

EXERCISE 5.34. Given  $U, V, W$ , with  $\mathbf{V} = (V, J)$ ,  $t_{UV}^W : \text{Hom}(U, \mathbf{V}) \rightarrow \text{Hom}(\text{Hom}(\mathbf{V}, W), \text{Hom}(U, W))$  is  $c$ -linear with respect to the induced CSOs. ■

EXERCISE 5.35. Given  $U, V, W$ , with  $\mathbf{U} = (U, J)$ , the canonical map (Definition 1.30)  $e_{UV}^W : \text{Hom}(\mathbf{U}, V) \rightarrow \text{Hom}(\text{Hom}(V, W) \otimes \mathbf{U}, W)$  is  $c$ -linear with respect to the induced CSOs. ■

EXERCISE 5.36. Given  $U, V, W$ , with  $\mathbf{V} = (V, J)$ ,  $e_{UV}^W : \text{Hom}(U, \mathbf{V}) \rightarrow \text{Hom}(\text{Hom}(\mathbf{V}, W) \otimes U, W)$  is  $\mathbf{c}$ -linear with respect to the induced CSOs. ■

EXERCISE 5.37. Given  $V$  and  $\mathbf{W} = (W, J_W)$ , the involution

$$T_{V;W} = \text{Hom}(d_{VW}, Id_{\text{Hom}(V,W)}) \circ t_{V, \text{Hom}(V,W)}^W$$

on  $\text{Hom}(V, \text{Hom}(V, \mathbf{W}))$  from Definition 4.2 is  $\mathbf{c}$ -linear.

HINT. This follows from Lemma 4.4. ■

PROPOSITION 5.38. Given  $V$ , consider three elements  $A, J_1, J_2 \in \text{End}(V)$ . The following two statements are equivalent:

- (1)  $(J_1 + J_2) \circ A = J_1 - J_2$ ;
- (2)  $J_2 \circ (Id_V + A) = J_1 \circ (Id_V - A)$ .

The following two statements are also equivalent to each other:

- (1')  $A \circ (J_1 + J_2) = J_1 - J_2$ ;
- (2')  $(Id_V + A) \circ J_2 = (Id_V - A) \circ J_1$ .

If  $A, J_1, J_2$  satisfy either condition (1) or (1'), then any two of the following imply the remaining third:

- (3)  $J_1$  is invertible;
- (4)  $J_1 + J_2$  is invertible;
- (5)  $Id_V + A$  is invertible.

If  $A, J_1, J_2$  satisfy (1) or (1'), and also (5), then any two of the following imply the remaining third:

- (6)  $A \circ J_1 = -J_1 \circ A$ ;
- (7)  $J_1$  is a CSO;
- (8)  $J_2$  is a CSO.

PROOF. (1)  $\iff$  (2) by an elementary algebraic manipulation. The (1')  $\iff$  (2') and subsequent implications are analogous and left as an exercise. For (3), (4), (5), use (1) or (2) to establish

$$(J_1 + J_2) \circ (Id_V + A) = 2 \cdot J_1,$$

and the claim follows.

For (6), (7), (8), use (2) to establish

$$\begin{aligned} (5.4) \quad & (J_1 + J_2) \circ (A \circ J_1 + J_1 \circ A) \\ &= J_1 \circ A \circ J_1 + J_2 \circ A \circ J_1 + J_1 \circ J_1 \circ A + J_2 \circ J_1 \circ A \\ &= J_1 \circ A \circ J_1 + J_2 \circ (Id_V + A) \circ J_1 + J_1 \circ J_1 \circ A - J_2 \circ J_1 \circ (Id_V - A) \\ &= J_1 \circ A \circ J_1 + J_1 \circ (Id_V - A) \circ J_1 + J_1 \circ J_1 \circ A - J_2 \circ J_2 \circ (Id_V + A) \\ &= (J_1 \circ J_1 - J_2 \circ J_2) \circ (Id_V + A). \end{aligned}$$

Given (6), (8), the equation (5.4) becomes  $0_{\text{End}(V)} = (J_1 \circ J_1 + Id_V) \circ (Id_V + A)$ , and (7) follows from (5). Similarly, given (6), (7), (5.4) becomes  $0_{\text{End}(V)} = -(Id_V + J_2 \circ J_2) \circ (Id_V + A)$ , and (8) follows from (5). For (7), (8), note that (7) implies (3) and then (5) implies (4), so by (7) and (8), (5.4) becomes  $LHS = 0_{\text{End}(V)}$ , and (6) follows from (4). ■

Given  $V$  with a CSO  $J_1$ , Proposition 5.38 establishes a bijective correspondence between the set of CSOs  $J_2$  on  $V$  with  $J_1 + J_2$  invertible and the set of  $A \in \text{End}(V)$  with  $A$  a-linear (with respect to  $J_1$ ) and  $Id_V + A$  invertible, as follows. Since  $J_1$  is a CSO, (3) and (7) hold. For any a-linear  $A \in \text{End}(V)$  with  $Id_V + A$  invertible, (5) and (6) hold. If we define  $J_2$  by the similarity relation  $J_2 = (Id_V + A) \circ J_1 \circ (Id_V + A)^{-1}$ , then  $J_2 = J_1 \circ (Id_V - A) \circ (Id_V + A)^{-1}$ , so (2) holds, (1), (4), and (8) follow as consequences, and  $A$  satisfies  $A = (J_1 + J_2)^{-1} \circ (J_1 - J_2)$ . Conversely, for any CSO  $J_2$  with  $J_1 + J_2$  invertible, (4) and (8) hold. If we define

$$(5.5) \quad A = (J_1 + J_2)^{-1} \circ (J_1 - J_2),$$

then (1) holds, (2), (5), and (6) follow as consequences, and  $J_2$  satisfies  $J_2 = J_1 \circ (Id_V - A) \circ (Id_V + A)^{-1} = (Id_V + A) \circ J_1 \circ (Id_V + A)^{-1}$ .

**EXERCISE 5.39.** Given  $V$  and any two CSOs  $J_1, J_2$ , the map  $J_1 + J_2$  is c-linear with respect to  $J_1$  and  $J_2$ , and the maps  $\pm(J_1 - J_2)$  are a-linear with respect to  $J_1$  and  $J_2$ . ■

**BIG EXERCISE 5.40.** Given  $V_1, V_2$  with CSOs  $J_1, J_2$ , and a real linear map  $A : V_1 \rightarrow V_2$ , if  $\dim(A(V_1)) > 2$ , then the following are equivalent:

- (1) For each  $v \in V_1$ ,  $A(J_1(v)) \in \text{span}\{A(v), J_2(A(v))\}$ ;
- (2) For each  $v \in V_1$ ,  $A$  maps the subspace  $\text{span}\{v, J_1(v)\} \subseteq V_1$  to the subspace  $\text{span}\{A(v), J_2(A(v))\} \subseteq V_2$ ;
- (3) For each  $v \in V_1$ , either  $A(J_1(v)) = J_2(A(v))$  or  $A(J_1(v)) = -J_2(A(v))$ ;
- (4)  $A \circ J_1 = J_2 \circ A$  or  $A \circ J_1 = -J_2 \circ A$ .

**HINT.** The idea is that  $A$  takes  $J_1$ -complex lines to  $J_2$ -complex lines, and that this is equivalent to  $A$  being c-linear or a-linear. (See also [C] for other properties of  $A$  equivalent to (4).) ■

### 5.3. Commuting Complex Structure Operators

**LEMMA 5.41.** *Given  $V$  and two CSOs  $J_1, J_2$ , the following are equivalent:*

- (1)  $J_1$  and  $J_2$  commute (i.e.,  $J_1 \circ J_2 = J_2 \circ J_1$ );
- (2) The composite  $J_1 \circ J_2$  is an involution;
- (3) The composite  $-J_1 \circ J_2$  is an involution.

■

**EXAMPLE 5.42.** Given  $V$  with commuting CSOs  $J_1, J_2$ , Lemma 1.79 applies to the involution  $-J_1 \circ J_2$  as in Lemma 5.41, so there is a direct sum,  $V = V_c \oplus V_a$ , where

$$\begin{aligned} V_c &= \{v \in V : (-J_1 \circ J_2)(v) = v\} = \{v \in V : J_2(v) = J_1(v)\} \\ V_a &= \{v \in V : (-J_1 \circ J_2)(v) = -v\} = \{v \in V : J_2(v) = -J_1(v)\}, \end{aligned}$$

with projection operators

$$P_c = \frac{1}{2} \cdot (Id_V - J_1 \circ J_2) : V \rightarrow V_c, \quad P_a = \frac{1}{2} \cdot (Id_V + J_1 \circ J_2) : V \rightarrow V_a.$$

As remarked in the Proof of Lemma 1.79, the same formulas are also used for  $Q_c \circ P_c, Q_a \circ P_a \in \text{End}(V)$ .

Note that applying Lemma 1.79 to the involution  $J_1 \circ J_2$  would give the direct sum in the other order,  $V = V_a \oplus V_c$ .

LEMMA 5.43. *For  $V$ ,  $J_1$ ,  $J_2$  as in Lemma 5.41, and another space  $V'$  with commuting CSOs  $J'_1$ ,  $J'_2$ , a map  $H : V \rightarrow V'$  respects the direct sums  $V_c \oplus V_a$  and  $V'_c \oplus V'_a$  if and only if  $H \circ J_1 \circ J_2 = J'_1 \circ J'_2 \circ H$ .*

PROOF. This is an example of Lemma 1.83. ■

EXAMPLE 5.44. The previous Lemma applies to  $V' = V$  with either  $H = J_1$  or  $H = J_2$ , each of which induces a CSO on  $V_c$  and on  $V_a$  by Lemma 5.25. The subspace  $V_c$  has a canonical CSO, induced by either  $J_1$  or  $J_2$ . The maps  $P_c : V \rightarrow V_c$  and  $Q_c : V_c \hookrightarrow V$  are  $c$ -linear with respect to either  $J_1$  or  $J_2$ . The induced CSOs on the subspace  $V_a$  are opposite, and generally distinct.

LEMMA 5.45. *For  $V$ ,  $J_1$ ,  $J_2$  as in Lemma 5.41,  $\mathbf{U} = (U, J_U)$ , and a map  $H : U \rightarrow V$ , if  $H$  is  $c$ -linear with respect to both  $J_U$ ,  $J_1$  and  $J_U$ ,  $J_2$ , then the image of  $H$  is contained in  $V_c$ .*

PROOF. This can be checked by showing  $P_a \circ H = 0_{\text{Hom}(U, V_a)}$ . ■

EXAMPLE 5.46. Given  $V$  and three commuting CSOs  $J_1$ ,  $J_2$ ,  $J_3$ , consider an ordered triple  $(i_1, i_2, i_3)$  which is a permutation (no repeats) of the indices 1, 2, 3. For the first two indices, the two CSOs  $J_{i_1}$ ,  $J_{i_2}$  produce a direct sum  $V = V_{c(i_1 i_2)} \oplus V_{a(i_1 i_2)}$  with projection  $P_{c(i_1 i_2)} = \frac{1}{2} \cdot (Id_V - J_{i_1} \circ J_{i_2})$  as in Example 5.42. The ordering of the pair is irrelevant:  $V_{c(i_1 i_2)} = V_{c(i_2 i_1)}$ . The remaining CSO  $J_{i_3}$  respects this direct sum by Lemma 5.43, and by Lemma 5.25 induces a CSO  $P_{c(i_1 i_2)} \circ J_{i_3} \circ Q_{c(i_1 i_2)}$  on  $V_{c(i_1 i_2)}$ , which commutes with the CSO induced by  $J_{i_1}$  and  $J_{i_2}$ . So, again as in Example 5.42, there is a direct sum  $V_{c(i_1 i_2)} = (V_{c(i_1 i_2)})_c \oplus (V_{c(i_1 i_2)})_a$  with projection  $P_{c((i_1 i_2) i_3)} : V_{c(i_1 i_2)} \rightarrow (V_{c(i_1 i_2)})_c$ . Simplifying the composition of projections gives the formula

$$P_{c((i_1 i_2) i_3)} \circ P_{c(i_1 i_2)} = \frac{1}{4} \cdot (Id_V - J_1 \circ J_2 - J_2 \circ J_3 - J_1 \circ J_3).$$

Considering  $(V_{c(i_1 i_2)})_c$  as a subspace of  $V$ , the above formula shows that neither the composite map nor its image depends on the ordering of the three indices, and so  $(V_{c(i_1 i_2)})_c$  is equal to the subspace where all three CSOs coincide and it can be denoted  $V_{c(123)}$ .

The above Example is also a special case of Theorem 1.86: given three commuting CSOs, there are commuting involutions  $K_1 = -J_1 \circ J_2$ ,  $K_2 = -J_2 \circ J_3$ , and  $K_3 = -J_3 \circ J_1$ . The direct sums from Theorem 1.86 produced by these involutions are exactly  $V = V_{c(i_1 i_2)} \oplus V_{a(i_1 i_2)}$ , and each  $V_{c(i_1 i_2)}$  admits a canonically induced involution and direct sum. The conclusions of Theorem 1.86 are  $V_{c(12)} \cap V_{c(23)} \cap V_{c(13)} = V_{c(123)}$  and  $(V_{c(i_1 i_2)})_a = V_{a(i_1 i_3)} \cap V_{a(i_2 i_3)}$ . From (1.6), each projection  $P_{c(i_1 i_2) i_3}$  is equal to a map induced by  $P_{c(i_1 i_3)}$  and also to a map induced by  $P_{c(i_2 i_3)}$ .

EXAMPLE 5.47. Given  $V$  and four commuting CSOs  $J_1, J_2, J_3, J_4$ , the construction of Example 5.46 shows that for any ordered pair  $(i_1, i_2)$  selected without repeats from the indices 1, 2, 3, 4, there is a direct sum with projection  $P_{c(i_1 i_2)} : V \twoheadrightarrow V_{c(i_1 i_2)}$ , and for a third distinct index, another direct sum with projection  $P_{c((i_1 i_2) i_3)} : V_{c(i_1 i_2)} \twoheadrightarrow V_{c(i_1 i_2 i_3)}$ . Repeating the process for the remaining, fourth index, the fourth commuting CSO produces a direct sum  $V_{c(i_1 i_2 i_3)} = (V_{c(i_1 i_2 i_3)})_c \oplus (V_{c(i_1 i_2 i_3)})_a$ , with projection  $P_{c((i_1 i_2 i_3) i_4)} : V_{c(i_1 i_2 i_3)} \twoheadrightarrow (V_{c(i_1 i_2 i_3)})_c$ . As in the previous Example, the composition  $P_{c((i_1 i_2 i_3) i_4)} \circ P_{c((i_1 i_2) i_3)} \circ P_{c(i_1 i_2)}$  equals

$$\frac{1}{8} \cdot (Id_V - J_1 \circ J_2 - J_2 \circ J_3 - J_1 \circ J_3 - J_1 \circ J_4 - J_2 \circ J_4 - J_3 \circ J_4 + J_1 \circ J_2 \circ J_3 \circ J_4),$$

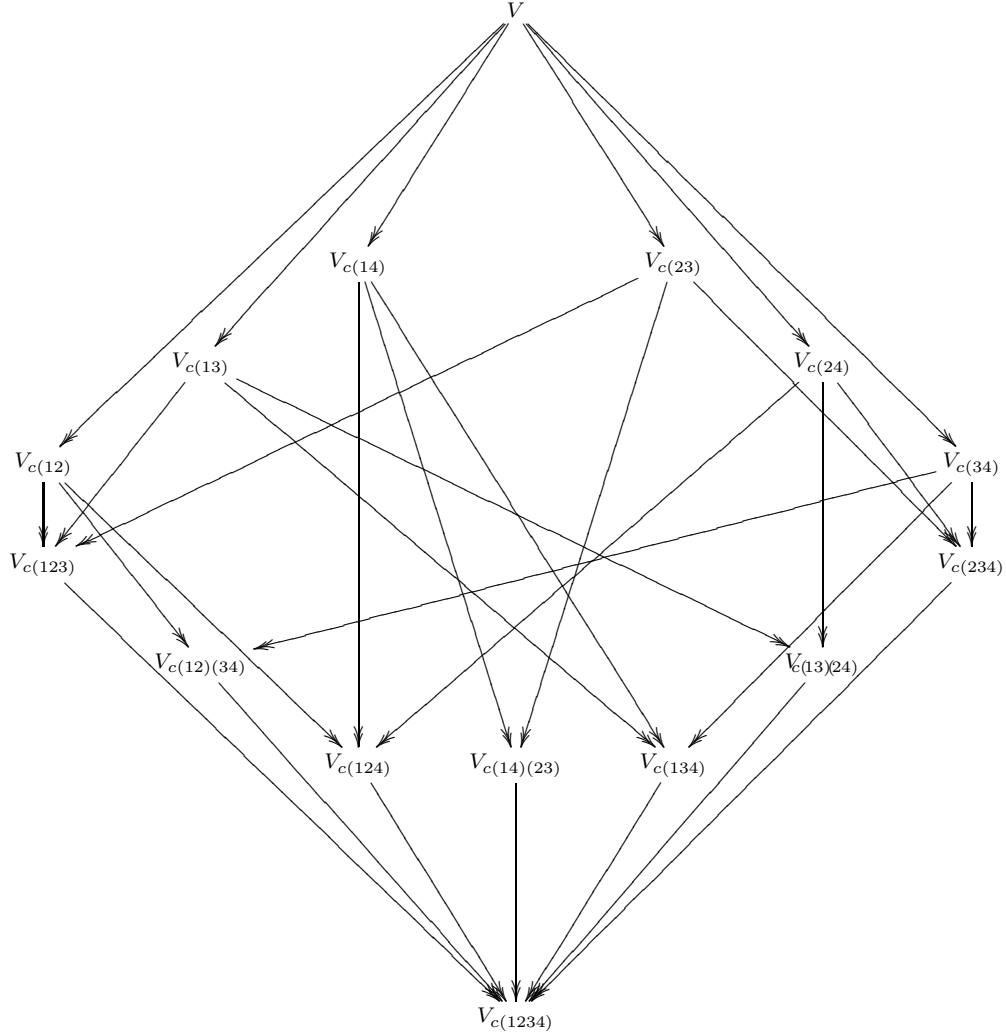
which shows the image of the last projection does not depend on the ordering of the four indices, so the subspace where all four CSOs coincide can be denoted  $V_{c(1234)}$ .

EXAMPLE 5.48. Given  $V$  and four commuting CSOs  $J_1, J_2, J_3, J_4$ , the space  $V_{c(i_1 i_2)}$  from the previous Example admits three commuting CSOs:  $P_{c(i_1 i_2)} \circ J_{i_1} \circ Q_{c(i_1 i_2)}$ ,  $P_{c(i_1 i_2)} \circ J_{i_3} \circ Q_{c(i_1 i_2)}$ ,  $P_{c(i_1 i_2)} \circ J_{i_4} \circ Q_{c(i_1 i_2)}$ . The previous Example considered pairing the first one with one of the other two to get two direct sums, but as in Example 5.46, there are three possible direct sums on  $V_{c(i_1 i_2)}$ , the third coming from  $P_{c(i_1 i_2)} \circ J_{i_3} \circ Q_{c(i_1 i_2)}$ ,  $P_{c(i_1 i_2)} \circ J_{i_4} \circ Q_{c(i_1 i_2)}$  to get a subspace denoted  $V_{c(i_1 i_2)(i_3 i_4)}$ , where  $J_{i_1} = J_{i_2}$  and  $J_{i_3} = J_{i_4}$ , equal to the subspace  $V_{c(i_3 i_4)(i_1 i_2)}$ . The composite projection  $V \twoheadrightarrow V_{c(i_1 i_2)(i_3 i_4)}$  is given by the formula

$$\frac{1}{4} \cdot (Id_V - J_1 \circ J_2 - J_3 \circ J_4 + J_1 \circ J_2 \circ J_3 \circ J_4).$$

The subspace  $V_{c(i_1 i_2)(i_3 i_4)}$  admits two commuting CSOs, the one induced by  $J_{i_1}$  and  $J_{i_2}$ , and the other by  $J_{i_3}$  and  $J_{i_4}$ , so there is a direct sum, and a projection onto the subspace  $V_{c(1234)}$  from Example 5.47.

THEOREM 5.49. *Given  $V$  and four commuting CSOs  $J_1, J_2, J_3, J_4$ , the following diagram is commutative, where the arrows are all the projections from direct sums produced by commuting CSOs described in Examples 5.47 and 5.48.*



PROOF. Some sub-diagrams were already considered in Examples 5.46, 5.47, 5.48. Some remain to be checked, for example, the equality of the composite projections  $V_{c(12)} \rightarrow V_{c(123)} \rightarrow V_{c(1234)}$  and  $V_{c(12)} \rightarrow V_{c(12)(34)} \rightarrow V_{c(1234)}$  follows from considering the three CSOs on  $V_{c(12)}$  as in Example 5.46. ■

LEMMA 5.50. *For  $V$  with commuting CSOs  $J_1, J_2$ , and  $V'$  with commuting CSOs  $J'_1, J'_2$  as in Lemma 5.43, if  $H : V \rightarrow V'$  is  $c$ -linear with respect to any of the pairs  $J_1, J'_1$  or  $J_1, J'_2$  or  $J_2, J'_1$  or  $J_2, J'_2$ , then  $P'_c \circ H \circ Q_c : V_c \rightarrow V'_c$  is  $c$ -linear with respect to the canonical CSOs. ■*



LEMMA 5.51. *Given  $V$  with commuting CSOs  $J_V^1, J_V^2$ , and  $U$  with commuting CSOs  $J_U^1, J_U^2$ , if  $H : U \rightarrow V$  satisfies both  $H \circ J_U^1 = J_V^1 \circ H$  and  $H \circ J_U^2 = J_V^2 \circ H$ , then  $H : U_c \oplus U_a \rightarrow V_c \oplus V_a$  respects the direct sums and the induced map  $P_c^V \circ H \circ Q_c^U : U_c \rightarrow V_c$  is  $c$ -linear with respect to the induced CSOs. If, also,  $H$  is invertible, then for  $i = c, a$ , the induced map  $P_i^V \circ H \circ Q_i^U : U_i \rightarrow V_i$  is invertible.*

PROOF. This follows from Lemma 5.43, Lemma 5.50, and Lemma 1.53. ■

We remark that the direct summands in Lemma 5.51 are all subspaces; if  $u = Q_c^U(u) \in U_c \subseteq U$ , then  $H(u) = H(Q_c^U(u)) \in V_c \subseteq V$ , so  $H(u)$  is in the fixed point set of the idempotent  $P_c^V$ :

$$u \in U_c \implies H(u) = P_c^V(H(u)) = H(Q_c^U(u)) = (P_c^V \circ H \circ Q_c^U)(u).$$

LEMMA 5.52. *Given  $V$  with commuting CSOs  $J_V^1, J_V^2$ , and  $U$  with commuting CSOs  $J_U^1, J_U^2$ , if  $H : U \rightarrow V$  satisfies both  $H \circ J_U^1 = -J_V^1 \circ H$  and  $H \circ J_U^2 = -J_V^2 \circ H$ , then  $H : U_c \oplus U_a \rightarrow V_c \oplus V_a$  respects the direct sums and the induced map  $P_c^V \circ H \circ Q_c^U : U_c \rightarrow V_c$  is  $a$ -linear with respect to the induced CSOs.*

PROOF. This is straightforward to check directly, or follows from Lemma 5.51 with  $J_V^1, J_V^2$  replaced by the opposite CSOs  $-J_V^1, -J_V^2$ . ■

The following Theorem weakens the hypotheses of Lemma 5.51.

THEOREM 5.53. *Given  $V$  with commuting CSOs  $J_V^1, J_V^2$ , and  $U$  with commuting CSOs  $J_U^1, J_U^2$ , if  $H : U \rightarrow V$  is  $c$ -linear with respect to  $J_U^2, J_V^2$ , then the kernel of the composite  $P_c^V \circ H \circ Q_a^U : U_a \rightarrow V_c$  is equal to the set  $\{u \in U_a : (H \circ J_U^1 \circ Q_a^U)(u) = (J_V^1 \circ H \circ Q_a^U)(u)\}$ .*

PROOF. Composing with  $Q_c^V$  does not change the kernel, so using the equalities  $J_U^1 \circ Q_a^U = -J_U^2 \circ Q_a^U$  and  $J_V^2 \circ H = H \circ J_U^2$ ,

$$\begin{aligned} Q_c^V \circ P_c^V \circ H \circ Q_a^U &= \frac{1}{2} \cdot (Id_V - J_V^1 \circ J_V^2) \circ H \circ Q_a^U \\ &= \frac{1}{2} \cdot (H - J_V^1 \circ H \circ J_U^2) \circ Q_a^U = \frac{1}{2} \cdot (H + J_V^1 \circ H \circ J_U^1) \circ Q_a^U, \end{aligned}$$

and the composite with the invertible map  $2 \cdot J_V^1$  has the same kernel:

$$2 \cdot J_V^1 \circ Q_c^V \circ P_c^V \circ H \circ Q_a^U = J_V^1 \circ H \circ Q_a^U - H \circ J_U^1 \circ Q_a^U.$$
■

LEMMA 5.54. *Given  $U$  with commuting CSOs  $J_1, J_2$ , if  $H \in \text{End}(U)$  is an involution such that  $H \circ J_1 = J_2 \circ H$ , then  $H$  respects the direct sum  $U_c \oplus U_a \rightarrow U_c \oplus U_a$ , and induces involutions on  $U_c$  and  $U_a$ . The induced involution  $P_c \circ H \circ Q_c$  is  $c$ -linear.*

PROOF. It follows from the involution property that  $H \circ J_1 = J_2 \circ H \implies J_1 \circ H = H \circ J_2 \implies H \circ J_1 \circ J_2 = J_1 \circ J_2 \circ H$ . So, Lemma 5.43 applies to show that  $H$  respects the direct sums, and induces involutions as in Lemma 1.84. The induced involution on  $U_c$  is  $c$ -linear by Lemma 5.50. This is also a special case of Lemma 5.51, with respect to  $(J_1, J_2)$  on the domain of  $H$  and  $(J_2, J_1)$  on the target. ■

LEMMA 5.55. *Given  $V$  with commuting CSOs  $J_V^1, J_V^2, J_V^3$ , and  $U$  with commuting CSOs  $J_U^1, J_U^2, J_U^3$ , if  $H : U \rightarrow V$  satisfies  $H \circ J_U^1 = J_V^1 \circ H$  and  $H \circ J_U^2 = J_V^2 \circ H$  and  $H \circ J_U^3 = J_V^3 \circ H$ , then  $H$  respects the corresponding direct sums from Example 5.46, and each induced map  $P_{c(i_1 i_2)}^V \circ H \circ Q_{c(i_1 i_2)}^U$  is  $c$ -linear with respect to the CSOs induced by  $J_U^1, J_V^1$  and also  $c$ -linear with respect to the CSOs induced by  $J_U^3, J_V^3$ . The induced map*

$$P_{c((i_1 i_2) i_3)}^V \circ P_{c(i_1 i_2)}^V \circ H \circ Q_{c(i_1 i_2)}^U \circ Q_{c((i_1 i_2) i_3)}^U : U_{c(123)} \rightarrow V_{c(123)}$$

*is  $c$ -linear and does not depend on the ordering  $(i_1, i_2, i_3)$ . If, also,  $H$  is invertible, then the induced maps are invertible.*

PROOF. This situation is a special case of Theorem 1.87, with commuting involutions on both  $V$  and  $U$  as in Example 5.46. In particular,  $H$  respects the direct sums

$$U_{c(i_1 i_2)} \oplus U_{a(i_1 i_2)} \rightarrow V_{c(i_1 i_2)} \oplus V_{a(i_1 i_2)}.$$

The induced map  $P_{c(i_1 i_2)}^V \circ H \circ Q_{c(i_1 i_2)}^U$  respects the direct sum

$$U_{c((i_1 i_2) i_3)} \oplus (U_{c(i_1 i_2)})_a \rightarrow V_{c((i_1 i_2) i_3)} \oplus (V_{c(i_1 i_2)})_a,$$

where  $U_{c((i_1 i_2) i_3)} = U_{c(123)}$  and  $(U_{c(i_1 i_2)})_a = U_{a(i_1 i_3)} \cap U_{a(i_2 i_3)}$  as in Example 5.46. The first claim of  $c$ -linearity follows from Lemma 5.51. The second  $c$ -linearity requires checking

$$P_{c(i_1 i_2)}^V \circ H \circ Q_{c(i_1 i_2)}^U \circ P_{c(i_1 i_2)}^U \circ J_U^{i_3} \circ Q_{c(i_1 i_2)}^U = P_{c(i_1 i_2)}^V \circ J_V^{i_3} \circ Q_{c(i_1 i_2)}^V \circ P_{c(i_1 i_2)}^V \circ H \circ Q_{c(i_1 i_2)}^U,$$

and then the last claims follow from Lemma 5.51 applied again to the commuting CSOs induced on  $U_{c(i_1 i_2)}, V_{c(i_1 i_2)}$ .  $\blacksquare$

LEMMA 5.56. *Given  $V$  with commuting CSOs  $J_V^1, J_V^2, J_V^3, J_V^4$ , and  $U$  with commuting CSOs  $J_U^1, J_U^2, J_U^3, J_U^4$ , if  $H : U \rightarrow V$  satisfies  $H \circ J_U^1 = J_V^1 \circ H$  and  $H \circ J_U^2 = J_V^2 \circ H$  and  $H \circ J_U^3 = J_V^3 \circ H$  and  $H \circ J_U^4 = J_V^4 \circ H$ , then  $H$  respects the corresponding direct sums from Examples 5.47 and 5.48, and induces maps  $U_{c(i_1 i_2)} \rightarrow V_{c(i_1 i_2)}, U_{c(i_1 i_2 i_3)} \rightarrow V_{c(i_1 i_2 i_3)}$ , and  $U_{c(i_1 i_2)(i_3 i_4)} \rightarrow V_{c(i_1 i_2)(i_3 i_4)}$ , which are  $c$ -linear with respect to all pairs of CSOs induced by  $J_U^i, J_V^i$ , and invertible if  $H$  is. The induced map  $U_{c(1234)} \rightarrow V_{c(1234)}$  is  $c$ -linear, and invertible if  $H$  is.*

PROOF. All the claims follow from Lemmas 5.51, 5.55. As in Lemma 5.55, the map  $U_{c(1234)} \rightarrow V_{c(1234)}$  is canonically induced, not depending on the ordering of the indices.  $\blacksquare$

EXAMPLE 5.57. Given  $U$  with commuting CSOs  $J_1, J_2, J_3$ , suppose  $H$  is an involution such that  $H \circ J_1 = J_2 \circ H$  and  $H \circ J_3 = J_3 \circ H$ . By Lemma 5.54,  $H$  respects the direct sum  $U_{c(12)} \oplus U_{a(12)} \rightarrow U_{c(12)} \oplus U_{a(12)}$ , and induces involutions on  $U_{c(12)}$  and  $U_{a(12)}$ . Lemma 5.55 applies to the triple  $(J_1, J_2, J_3)$  on the domain of  $H$  and  $(J_2, J_1, J_3)$  on the target. So,  $H$  respects the direct sums:

$$\begin{aligned} U_{c(13)} \oplus U_{a(13)} &\rightarrow U_{c(23)} \oplus U_{a(23)} \\ U_{c(23)} \oplus U_{a(23)} &\rightarrow U_{c(13)} \oplus U_{a(13)}, \end{aligned}$$

and the induced maps  $U_{c(13)} \rightarrow U_{c(23)} \rightarrow U_{c(13)}$  are  $c$ -linear and mutually inverses. The induced maps  $U_{a(13)} \rightarrow U_{a(23)} \rightarrow U_{a(13)}$  are also mutually inverses. The subspace  $U_{c(12)}$  admits commuting CSOs  $P_{c(12)} \circ J_1 \circ Q_{c(12)} = P_{c(12)} \circ J_2 \circ Q_{c(12)}$  and  $P_{c(12)} \circ J_3 \circ Q_{c(12)}$  as in Example 5.46, producing the direct sum  $U_{c(123)} \oplus (U_{c(12)})_a$ .

$U_{c(12)}$  also admits commuting,  $c$ -linear involutions  $P_{c(12)} \circ (-J_1 \circ J_3) \circ Q_{c(12)}$  and  $P_{c(12)} \circ H \circ Q_{c(12)}$ , so Lemma 1.84 and Theorem 1.86 apply. The sixteen operators

$$\{\pm Id_U, \pm H, \pm J_1 \circ J_2, \pm J_1 \circ J_3, \pm J_2 \circ J_3, \pm H \circ J_1 \circ J_2, \pm H \circ J_1 \circ J_3, \pm H \circ J_2 \circ J_3\}$$

form a group which is the image of a representation of  $D_4 \otimes \mathbb{Z}_2$ . Unlike the group from Exercise 1.92, there is no pair of anticommuting elements.

EXAMPLE 5.58. Given  $U$  and  $V$ , suppose there are commuting CSOs  $J_1, J_2$  on  $V$ . Then the CSOs  $[J_1 \otimes Id_U]$  and  $[J_2 \otimes Id_U]$  on  $V \otimes U$ , from Example 5.3, also commute. As in Example 5.42, this gives a direct sum  $V \otimes U = (V \otimes U)_c \oplus (V \otimes U)_a$  with projections

$$\frac{1}{2} \cdot (Id_{V \otimes U} \pm [J_1 \otimes Id_U] \circ [J_2 \otimes Id_U]).$$

$V \otimes U$  also admits a direct sum  $(V_c \otimes U) \oplus (V_a \otimes U)$  as in Example 1.45, with projections

$$[(\frac{1}{2} \cdot (Id_V \pm J_1 \circ J_2)) \otimes Id_U] = [P_i \otimes Id_U],$$

for  $P_i : V \rightarrow V_i$ ,  $i = c, a$ , as in Example 5.42. This is a special case of Example 1.94; the pairs of projections are identical (using the linearity of  $j$  and Lemma 1.25), so the direct sums are the same and we have the equalities of subspaces  $(V \otimes U)_c = V_c \otimes U$  and  $(V \otimes U)_a = V_a \otimes U$ . Similarly,  $(U \otimes V)_c = U \otimes V_c$  and  $(U \otimes V)_a = U \otimes V_a$ .

EXAMPLE 5.59. Given  $\mathbf{U} = (U, J_U)$  and  $\mathbf{V} = (V, J_V)$ , the two CSOs  $[Id_U \otimes J_V]$ ,  $[J_U \otimes Id_V] \in \text{End}(U \otimes V)$  commute, so this is a special case of Example 5.42. The direct sum so produced is denoted

$$U \otimes V = (U \otimes_c V) \oplus (U \otimes_a V).$$

As in Example 5.44, the subspace

$$U \otimes_c V = \{w \in U \otimes V : [Id_U \otimes J_V](w) = [J_U \otimes Id_V](w)\}$$

has a canonical CSO, induced by either of the CSOs, so we may denote it  $\mathbf{U} \otimes_c \mathbf{V}$ . The CSOs on  $U \otimes V$  induce opposite CSOs on the subspace

$$U \otimes_a V = \{w \in U \otimes V : [Id_U \otimes J_V](w) = -[J_U \otimes Id_V](w)\}.$$

EXAMPLE 5.60. For  $\mathbf{U} = (U, J_U)$ ,  $\mathbf{V} = (V, J_V)$ , and  $\mathbf{W} = (W, J_W)$ , the space  $U \otimes V \otimes W$  admits three commuting CSOs  $[J_U \otimes Id_{V \otimes W}]$ ,  $\dots$ ,  $[Id_{U \otimes V} \otimes J_W]$ . The subspaces  $(\mathbf{U} \otimes_c \mathbf{V}) \otimes_c \mathbf{W}$  and  $\mathbf{U} \otimes_c (\mathbf{V} \otimes_c \mathbf{W})$  are equal, as a special case of Example 5.46.

EXAMPLE 5.61. For  $\mathbf{U} = (U, J_U)$  and commuting CSOs  $J_1, J_2$  on  $V$ ,  $U \otimes V$  has three commuting CSOs:  $[J_U \otimes Id_V]$ ,  $[Id_U \otimes J_1]$ ,  $[Id_U \otimes J_2]$ . This is another special case of Example 5.46. The projection  $P_{c(23)} : U \otimes V \rightarrow (U \otimes V)_{c(23)} = U \otimes V_c$  is as in Example 5.58, so  $P_{c(23)} = [Id_U \otimes P_c]$ , where  $P_c : V \rightarrow V_c$  is as in Example 5.42. The subspace where all three CSOs agree is  $(U \otimes V)_{c(123)} = \mathbf{U} \otimes_c V_c$ .

EXAMPLE 5.62. For  $c$ -linear maps  $A : \mathbf{U} \rightarrow \mathbf{U}'$  and  $B : \mathbf{V} \rightarrow \mathbf{V}'$ , the map

$$[A \otimes B] : U \otimes V \rightarrow U' \otimes V'$$

respects the direct sums from Example 5.59. The induced map

$$P'_c \circ [A \otimes B] \circ Q_c : \mathbf{U} \otimes_c \mathbf{V} \rightarrow \mathbf{U}' \otimes_c \mathbf{V}'$$

is  $c$ -linear. As remarked after Lemma 5.51, the induced map equals  $[A \otimes B]$ , restricted to the domain and target subspaces.

EXERCISE 5.63. For  $a$ -linear maps  $A : \mathbf{U} \rightarrow \mathbf{U}'$  and  $B : \mathbf{V} \rightarrow \mathbf{V}'$ , the map

$$[A \otimes B] : U \otimes V \rightarrow U' \otimes V'$$

respects the direct sums. The induced map

$$P'_c \circ [A \otimes B] \circ Q_c : \mathbf{U} \otimes_c \mathbf{V} \rightarrow \mathbf{U}' \otimes_c \mathbf{V}'$$

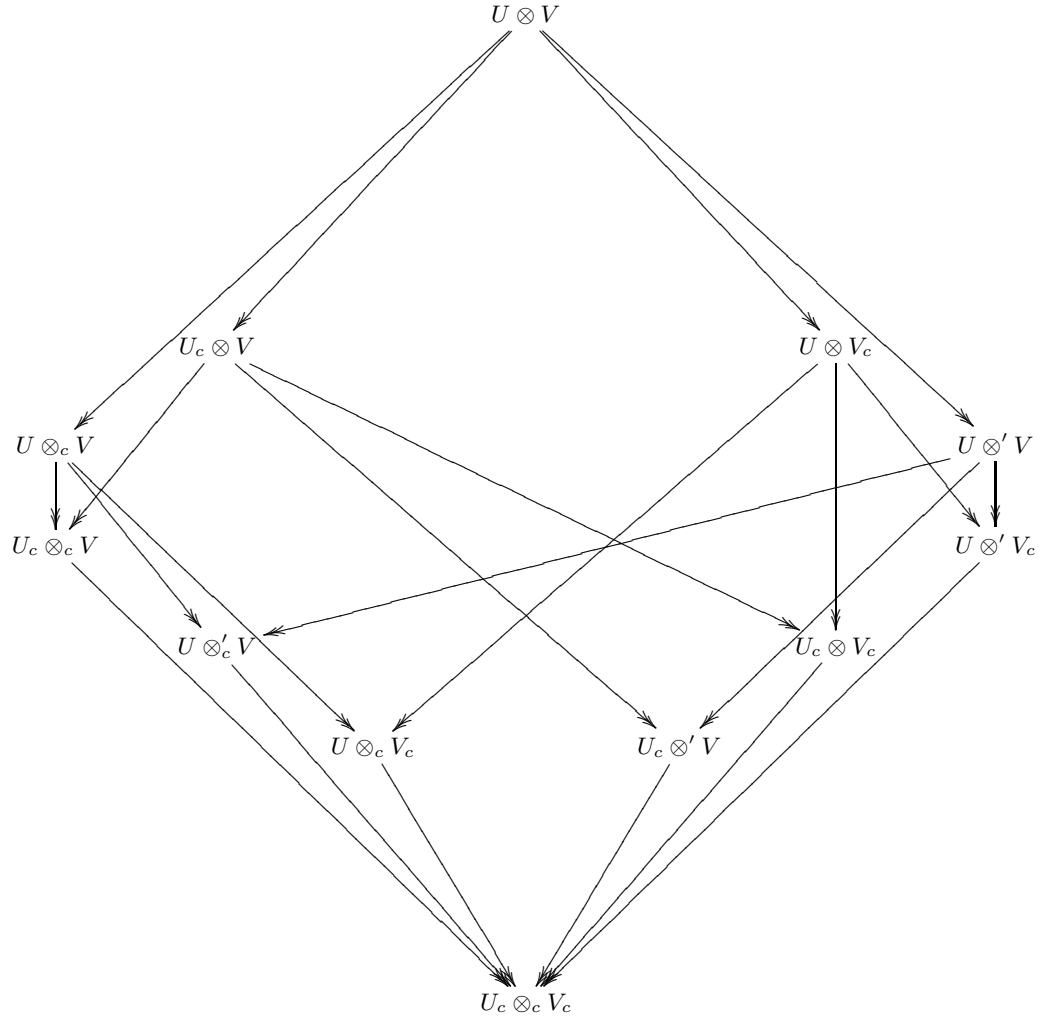
is  $a$ -linear. ■

EXAMPLE 5.64. For  $V$  with commuting CSOs  $J_V, J'_V$ , and  $U$  with commuting CSOs  $J_U, J'_U$ , the space  $U \otimes V$  has four commuting CSOs:

$$J_1 = [J_U \otimes Id_V], J_2 = [Id_U \otimes J_V], J_3 = [J'_U \otimes Id_V], J_4 = [Id_U \otimes J'_V].$$

Theorem 5.49 applies, to give a collection of subspaces of  $U \otimes V$ . From Example 5.58,  $(U \otimes V)_{c(13)} = U_c \otimes V$  and  $(U \otimes V)_{c(24)} = U \otimes V_c$ . From Example 5.59,  $(U \otimes V)_{c(12)} = U \otimes_c V$ , and we (temporarily) denote a similar construction  $(U \otimes V)_{c(34)} = U \otimes'_c V$ . The subspaces  $(U \otimes V)_{c(14)}$  and  $(U \otimes V)_{c(23)}$  did not appear in previous Examples, and are omitted from the following commutative diagram, where the positions of

the objects match the corresponding positions in the diagram from Theorem 5.49.



For example, the set  $U \otimes'_c V$  corresponds to  $(U \otimes V)_{c(12)(34)}$ , where  $J_1 = J_2$  and  $J_3 = J_4$ . The subspaces  $U_c \otimes_c V$ ,  $U \otimes_c V_c$ ,  $U_c \otimes' V$ , and  $U \otimes' V_c$  are as in Example 5.61.

EXAMPLE 5.65. Given  $\mathbf{U} = (U, J_U)$  and  $\mathbf{V} = (V, J_V)$ , the two CSOs  $\text{Hom}(Id_U, J_V)$ ,  $\text{Hom}(J_U, Id_V) \in \text{End}(\text{Hom}(U, V))$  commute, so this is a special case of Example 5.42. The direct sum so produced is denoted

$$\text{Hom}(U, V) = \text{Hom}_c(U, V) \oplus \text{Hom}_a(U, V).$$

The projection  $P_c : \text{Hom}(U, V) \rightarrow \text{Hom}_c(U, V)$  is defined by

$$\frac{1}{2} \cdot (Id_{\text{Hom}(U, V)} - \text{Hom}(Id_U, J_V) \circ \text{Hom}(J_U, Id_V)) = \frac{1}{2} \cdot (Id_{\text{Hom}(U, V)} - \text{Hom}(J_U, J_V)).$$

As in Example 5.44, the subspace

$$\begin{aligned} \text{Hom}_c(U, V) &= \{A \in \text{Hom}(U, V) : \text{Hom}(Id_U, J_V)(A) = \text{Hom}(J_U, Id_V)(A)\} \\ &= \{A \in \text{Hom}(U, V) : J_V \circ A = A \circ J_U\} \end{aligned}$$

has a canonical CSO, induced by either of the CSOs, so we may denote it  $\text{Hom}_c(\mathbf{U}, \mathbf{V})$ . The CSOs on  $\text{Hom}(U, V)$  induce opposite CSOs on the subspace

$$\begin{aligned}\text{Hom}_a(U, V) &= \{A \in \text{Hom}(U, V) : \text{Hom}(Id_U, J_V)(A) = -\text{Hom}(J_U, Id_V)(A)\} \\ &= \{A \in \text{Hom}(U, V) : J_V \circ A = -A \circ J_U\}.\end{aligned}$$

EXAMPLE 5.66. Given  $\mathbf{V} = (V, J_V)$ , there is a direct sum

$$\text{End}(\mathbf{V}) = \text{End}_c(\mathbf{V}) \oplus \text{End}_a(\mathbf{V})$$

as in Example 5.65, where  $\text{End}_c(\mathbf{V})$  admits a canonical CSO. The identity element  $Id_V \in \text{End}_c(\mathbf{V}) \subseteq \text{End}(\mathbf{V})$  is  $c$ -linear, and so is  $J_V$ .

EXERCISE 5.67. Given  $V \neq \{0_V\}$  and  $\mathbf{V} = (V, J_V)$  as in Example 5.66, the elements  $Id_V$  and  $J_V$  in  $\text{End}_c(\mathbf{V}) \subseteq \text{End}(\mathbf{V})$  form a linearly independent list  $(Id_V, J_V)$ . ■

LEMMA 5.68. For  $c$ -linear maps  $A : \mathbf{U}' \rightarrow \mathbf{U}$  and  $B : \mathbf{V} \rightarrow \mathbf{V}'$ , the map

$$\text{Hom}(A, B) : \text{Hom}(U, V) \rightarrow \text{Hom}(U', V')$$

respects the direct sums from Example 5.65. The induced map

$$P_c \circ \text{Hom}(A, B) \circ Q_c : \text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\mathbf{U}', \mathbf{V}') : F \mapsto B \circ F \circ A,$$

is  $c$ -linear. ■

HINT. This is a special case of Lemma 5.51. ■

NOTATION 5.69. The induced map from Lemma 5.68 can be denoted

$$\text{Hom}_c(A, B) = P_c \circ \text{Hom}(A, B) \circ Q_c : \text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\mathbf{U}', \mathbf{V}').$$

EXERCISE 5.70. For  $a$ -linear maps  $A : \mathbf{U}' \rightarrow \mathbf{U}$  and  $B : \mathbf{V} \rightarrow \mathbf{V}'$ , the map

$$\text{Hom}(A, B) : \text{Hom}(U, V) \rightarrow \text{Hom}(U', V')$$

respects the direct sums. The induced map

$$P_c \circ \text{Hom}(A, B) \circ Q_c : \text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\mathbf{U}', \mathbf{V}') : F \mapsto B \circ F \circ A$$

is  $a$ -linear. ■

EXAMPLE 5.71. Given  $U$  and  $V$ , suppose there are commuting CSOs  $J_1, J_2$  on  $V$ . Then the CSOs  $\text{Hom}(Id_U, J_1)$  and  $\text{Hom}(Id_U, J_2)$  on  $\text{Hom}(U, V)$  also commute. As in Example 5.42, this gives a direct sum  $\text{Hom}(U, V) = (\text{Hom}(U, V))_c \oplus (\text{Hom}(U, V))_a$  with projections

$$\frac{1}{2} \cdot (Id_{\text{Hom}(U, V)} \pm \text{Hom}(Id_U, J_1) \circ \text{Hom}(Id_U, J_2)).$$

$\text{Hom}(U, V)$  also admits a direct sum  $\text{Hom}(U, V_c) \oplus \text{Hom}(U, V_a)$  as in Example 1.46, with projections

$$(5.6) \quad \text{Hom}(Id_U, \frac{1}{2} \cdot (Id_V \pm J_1 \circ J_2)).$$

The pairs of projections are identical: this is a special case of Example 1.96. We can identify  $\text{Hom}(U, V_c) = (\text{Hom}(U, V))_c$ , and also identify  $\text{Hom}(Id_U, Q_c)$  with the inclusion of the subspace  $(\text{Hom}(U, V))_c$  in  $\text{Hom}(U, V)$ ; similarly,  $(\text{Hom}(U, V))_a = \text{Hom}(U, V_a)$ . More specifically, the set  $(\text{Hom}(U, V))_c$  is defined as  $\{A \in \text{Hom}(U, V) : J_1 \circ A = J_2 \circ A\}$ , while elements of  $\text{Hom}(U, V_c)$  are maps  $A$  such that for any  $u \in U$ ,  $A(u) \in V_c \subseteq V$ , meaning  $J_1(A(u)) = J_2(A(u))$ .

EXAMPLE 5.72. Given  $U$  and  $V$ , suppose there are commuting CSOs  $J_1, J_2$  on  $V$ . Then the CSOs  $\text{Hom}(J_1, Id_U)$  and  $\text{Hom}(J_2, Id_U)$  on  $\text{Hom}(V, U)$  also commute. As in Example 5.42, this gives a direct sum  $\text{Hom}(V, U) = (\text{Hom}(V, U))_c \oplus (\text{Hom}(V, U))_a$  with projections

$$\frac{1}{2} \cdot (Id_{\text{Hom}(V, U)} \pm \text{Hom}(J_1, Id_U) \circ \text{Hom}(J_2, Id_U)).$$

$\text{Hom}(V, U)$  also admits a direct sum  $\text{Hom}(V_c, U) \oplus \text{Hom}(V_a, U)$  as in Example 1.47, with projections  $\text{Hom}(Q_c, Id_U)$ ,  $\text{Hom}(Q_a, Id_U)$ . Unlike Examples 5.58 and 5.71, these pairs of projections are not obviously identical, and in fact this is a special case of Example 1.97. The set  $(\text{Hom}(V, U))_c$  is defined as  $\{A \in \text{Hom}(V, U) : A \circ J_1 = A \circ J_2\}$ , while elements of  $\text{Hom}(V_c, U)$  are maps  $A$  defined only on the subspace of  $v \in V$  such that  $J_1(v) = J_2(v)$ . The two direct sums are different but equivalent, as discussed in Example 1.97. Specifically, if, for  $i = c, a$ ,  $P_i'', Q_i''$  denote the projections and inclusions for the direct sum  $\text{Hom}(V, U) = (\text{Hom}(V, U))_c \oplus (\text{Hom}(V, U))_a$ , then

$$Q_i'' \circ P_i'' = \text{Hom}(P_i, Id_U) \circ \text{Hom}(Q_i, Id_U) : \text{Hom}(V, U) \rightarrow \text{Hom}(V, U),$$

and as in Lemma 1.62,

$$P_i'' \circ \text{Hom}(P_i, Id_U) : \text{Hom}(V_i, U) \rightarrow (\text{Hom}(V, U))_i$$

is invertible with inverse  $\text{Hom}(Q_i, Id_U) \circ Q_i''$ , and for  $i = c$ ,  $c$ -linear.

LEMMA 5.73. *Given  $U$  and  $V$ , with CSOs  $J_1, J_2 \in \text{End}(V)$ , let  $W$  be a space admitting a direct sum  $W_1 \oplus W_2$ . If  $H : W \rightarrow \text{Hom}(V, U)$  respects one of the two direct sums from Example 5.72, then  $H$  also respects the other direct sum. If, further,  $H$  is invertible, then both induced maps  $W_1 \rightarrow (\text{Hom}(V, U))_c$  and  $W_1 \rightarrow \text{Hom}(V_c, U)$  are also invertible. If the direct sum on  $W$  is given by commuting CSOs  $J_W, J'_W$ , and  $H$  satisfies both  $\text{Hom}(J_1, Id_U) \circ H = H \circ J_W$  and  $\text{Hom}(J_2, Id_U) \circ H = H \circ J'_W$ , then  $H$  respects the direct sums and the induced maps are  $c$ -linear.*

PROOF. The claims follow from Lemmas 1.53, 1.61, and 5.51. If the projection and inclusion operators induced on  $W = W_c \oplus W_a$  by  $J_W, J'_W$  are  $P'_i, Q'_i$ , then the induced maps  $\text{Hom}(Q_c, Id_U) \circ H \circ Q'_c : W_c \rightarrow \text{Hom}(V_c, U)$  and  $P'_c \circ H \circ Q'_c : W_c \rightarrow (\text{Hom}(V, U))_c$  are related by composition with the  $c$ -linear invertible map from the above Example:

$$\text{Hom}(Q_c, Id_U) \circ H \circ Q'_c = (\text{Hom}(Q_c, Id_U) \circ Q''_c) \circ (P'_c \circ H \circ Q'_c).$$

■

EXERCISE 5.74. The results of the previous Lemma have analogues for a map  $\text{Hom}(V, U) \rightarrow W$ . ■

EXAMPLE 5.75. For  $U$  and commuting CSOs  $J_1, J_2$  on  $V$  as in Example 5.71, suppose there are also commuting CSOs  $J'_1, J'_2$  on  $V'$ . If  $B : V \rightarrow V'$  respects the direct sums  $V_c \oplus V_a \rightarrow V'_c \oplus V'_a$  (equivalently,  $B \circ J_1 \circ J_2 = J'_1 \circ J'_2 \circ B$  by Lemmas 1.83 and 5.43), then for  $c = i, a$ , there are induced maps  $P'_i \circ B \circ Q_i : V_i \rightarrow V'_i$ . By Lemma 1.55, for any map  $A : W \rightarrow U$ , the map  $\text{Hom}(A, B) : \text{Hom}(U, V) \rightarrow \text{Hom}(W, V')$  respects the direct sums

$$\text{Hom}(U, V_c) \oplus \text{Hom}(U, V_a) \rightarrow \text{Hom}(W, V'_c) \oplus \text{Hom}(W, V'_a)$$

from Example 5.71, and for  $i = c, a$ , the induced map

$$\text{Hom}(Id_W, P'_i) \circ \text{Hom}(A, B) \circ \text{Hom}(Id_U, Q_i)$$

is equal to  $\text{Hom}(A, P'_i \circ B \circ Q_i)$ .

EXAMPLE 5.76. For  $U, V, V', A$  as in Example 5.75, if  $B : V \rightarrow V'$  is  $c$ -linear with respect to both pairs  $J_1, J'_1$  and  $J_2, J'_2$ , then  $B$  satisfies the hypothesis from Example 5.75, and by Lemma 5.22,  $\text{Hom}(A, B)$  is also  $c$ -linear with respect to the pairs  $\text{Hom}(Id_U, J_1), \text{Hom}(Id_W, J'_1)$  and  $\text{Hom}(Id_U, J_2), \text{Hom}(Id_W, J'_2)$ . Lemma 5.51 applies, so the induced maps from Example 5.75,  $P'_c \circ B \circ Q_c : V_c \rightarrow V'_c$ , and also  $\text{Hom}(A, P'_c \circ B \circ Q_c)$ , are both  $c$ -linear.

EXAMPLE 5.77. Given  $U$  and  $V$ , suppose  $\mathbf{U} = (U, J_U)$  and there are commuting CSOs  $J_V, J'_V$  on  $V$ . Then  $J_1 = \text{Hom}(J_U, Id_V), J_2 = \text{Hom}(Id_U, J_V), J_3 = \text{Hom}(Id_U, J'_V)$  are three commuting CSOs on  $\text{Hom}(U, V)$ , and Example 5.46 applies. There are three direct sums:  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(12)} \oplus (\text{Hom}(U, V))_{a(12)}$ , where  $(\text{Hom}(U, V))_{c(12)} = \text{Hom}_c(\mathbf{U}, (V, J_V))$  as in Example 5.65,  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(13)} \oplus (\text{Hom}(U, V))_{a(13)}$ , where  $(\text{Hom}(U, V))_{c(13)} = \text{Hom}_c(\mathbf{U}, (V, J'_V))$ , and  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(23)} \oplus (\text{Hom}(U, V))_{a(23)}$ , where  $(\text{Hom}(U, V))_{c(23)} = \text{Hom}(U, V_c)$  as in Example 5.71.

Each  $(\text{Hom}(U, V))_{c(i_1 i_2)}$  admits a direct sum with projection onto

$$((\text{Hom}(U, V))_{c(i_1 i_2)})_c = (\text{Hom}(U, V))_{c(123)} = \text{Hom}_c(\mathbf{U}, V_c),$$

as follows:

There are two ways to construct the projection

$$P^1 : (\text{Hom}(U, V))_{c(23)} = \text{Hom}(U, V_c) \rightarrow (\text{Hom}(U, V))_{c(123)} = \text{Hom}_c(\mathbf{U}, V_c),$$

which will turn out to give the same map. Denote the projection  $P'_c : V \rightarrow V_c$  as in Example 5.42 with corresponding inclusion  $Q'_c$ ; then  $P_{c(23)} = \text{Hom}(Id_U, P'_c) : \text{Hom}(U, V) \rightarrow \text{Hom}(U, V_c)$  is the projection from (5.6) in Example 5.71, with corresponding inclusion  $Q_{c(23)} = \text{Hom}(Id_U, Q'_c)$ .

The first construction of  $P^1$  is to consider  $\text{Hom}(U, V_c)$  as a space with commuting CSOs  $\text{Hom}(J_U, Id_{V_c}), \text{Hom}(Id_U, J_{V_c})$  and directly apply Example 5.65 to get a projection

$$\begin{aligned} P^1 &= \frac{1}{2} \cdot (Id_{\text{Hom}(U, V_c)} - \text{Hom}(J_U, Id_{V_c}) \circ \text{Hom}(Id_U, J_{V_c})) \\ (5.7) \quad &= \frac{1}{2} \cdot (Id_{\text{Hom}(U, V_c)} - \text{Hom}(J_U, P'_c \circ J_V \circ Q'_c)). \end{aligned}$$

Second, consider the subspace  $(\text{Hom}(U, V))_{c(23)}$ , with two induced CSOs that commute, as in Example 5.46:

$$(P_{c(23)} \circ \text{Hom}(J_U, Id_V) \circ Q_{c(23)}), \quad (P_{c(23)} \circ \text{Hom}(Id_U, J_V) \circ Q_{c(23)}).$$

Then, using  $P_{c(23)} = \text{Hom}(Id_U, P'_c)$  and  $Q_{c(23)} = \text{Hom}(Id_U, Q'_c)$ , the projection  $P_{c((23)1)}$  defined by these two CSOs as in Example 5.46 is the same as (5.7).

The projection  $P_{c((12)3)} : (\text{Hom}(U, V))_{c(12)} \rightarrow (\text{Hom}(U, V))_{c(123)}$  can also be defined by two methods with the same result (and similarly for  $P_{c((13)2)}$ ). The commuting induced CSOs:

$$(P_{c(12)} \circ \text{Hom}(J_U, Id_V) \circ Q_{c(12)}), \quad (P_{c(12)} \circ \text{Hom}(Id_U, J'_V) \circ Q_{c(12)})$$

define  $P_{c((12)3)}$  as in Example 5.46. The other way to define the projection is to consider the map  $\text{Hom}(Id_U, P'_c) : \text{Hom}(U, V) \rightarrow \text{Hom}(U, V_c)$ , which is  $c$ -linear in



two different ways: with respect to the pair  $\text{Hom}(J_U, Id_V)$ ,  $\text{Hom}(J_U, Id_{V_c})$  and also the pair  $\text{Hom}(Id_U, J_V)$ ,  $\text{Hom}(Id_U, J_{V_c})$ , as in Lemma 5.68. The induced map  $P^2 = P^1 \circ \text{Hom}(Id_U, P'_c) \circ Q_{c(12)} : (\text{Hom}(U, V))_{c(12)} \rightarrow \text{Hom}_c(U, V_c)$  is  $c$ -linear; it can be denoted  $\text{Hom}_c(Id_U, P'_c)$  as in Notation 5.69. By the equality of the composite projections from Example 5.46,

$$(5.8) \quad P^2 = P^1 \circ \text{Hom}(Id_U, P'_c) \circ Q_{c(12)}$$

$$(5.9) \quad \begin{aligned} &= P_{c((23)1)} \circ P_{c(23)} \circ Q_{c(12)} \\ &= P_{c((12)3)} \circ P_{c(12)} \circ Q_{c(12)} = P_{c((12)3)}. \end{aligned}$$

The expression (5.9) is an example of the construction (1.6) from Theorem 1.86.

Similarly, since the composite inclusions are equal:

$$(5.10) \quad Q_{c(12)} \circ Q_{c((12)3)} = \text{Hom}(Id_U, Q'_c) \circ Q_{c((23)1)},$$

the inclusion  $Q_{c((12)3)}$  is equal the induced map:

$$Q_{c((12)3)} = P_{c(12)} \circ \text{Hom}(Id_U, Q'_c) \circ Q_{c((23)1)} = \text{Hom}_c(Id_U, Q'_c).$$

EXAMPLE 5.78. Given  $U$  and  $V$ , suppose  $\mathbf{V} = (V, J_V)$  and there are commuting CSOs  $J_U, J'_U$  on  $U$ . Then  $J_1 = \text{Hom}(J_U, Id_V)$ ,  $J_2 = \text{Hom}(J'_U, Id_V)$ ,  $J_3 = \text{Hom}(Id_U, J_V)$  are three commuting CSOs on  $\text{Hom}(U, V)$ , and Example 5.46 applies. There are three direct sums:  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(13)} \oplus (\text{Hom}(U, V))_{a(13)}$ , where  $(\text{Hom}(U, V))_{c(13)} = \text{Hom}_c((U, J_U), \mathbf{V})$  as in Example 5.65,  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(23)} \oplus (\text{Hom}(U, V))_{a(23)}$ , where  $(\text{Hom}(U, V))_{c(23)} = \text{Hom}_c((U, J'_U), \mathbf{V})$ , and  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(12)} \oplus (\text{Hom}(U, V))_{a(12)}$ , as in Example 5.72. Each  $(\text{Hom}(U, V))_{c(i_1 i_2)}$  admits a direct sum with projection onto

$$((\text{Hom}(U, V))_{c(i_1 i_2)})_c = (\text{Hom}(U, V))_{c(123)} = \{A : U \rightarrow V : A \circ J_U = A \circ J'_U = J_V \circ A\}.$$

The invertible map  $P''_c \circ \text{Hom}(P_c, Id_U) : \text{Hom}(U_c, V) \rightarrow (\text{Hom}(U, V))_{c(12)}$  from Example 5.72 (where in this case,  $P''_c = P_{c(12)}$ ) is  $c$ -linear with respect to  $\text{Hom}(Id_{U_c}, J_V)$  and  $P_{c(12)} \circ J_3 \circ Q_{c(12)}$ , and is also  $c$ -linear with respect to  $\text{Hom}(P_c \circ J_U \circ Q_c, Id_V)$  and  $P_{c(12)} \circ J_1 \circ Q_{c(12)}$ , so by Lemma 5.51, it respects the direct sums and induces an invertible,  $c$ -linear map  $\text{Hom}_c(U_c, \mathbf{V}) \rightarrow (\text{Hom}(U, V))_{c(123)}$ , as indicated in the following diagram.

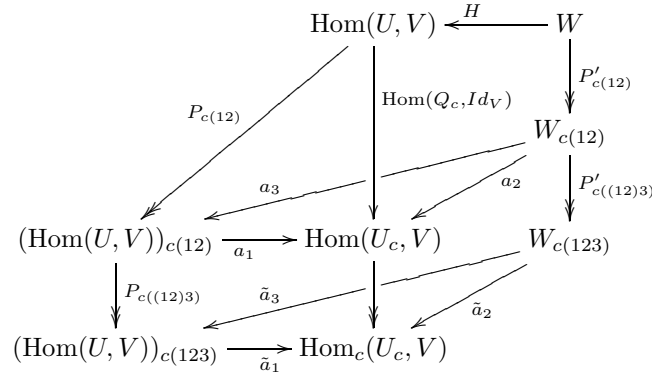
$$\begin{array}{ccccc} & & \text{Hom}(U, V) & & \\ & \swarrow \text{Hom}(Q_c, Id_V) & \downarrow P_{c(13)} & \searrow P_{c(23)} & \\ & \text{Hom}(U_c, V) & \downarrow P_{c(12)} & & (\text{Hom}(U, V))_{c(23)} \\ & \swarrow \text{Hom}(P_c, Id_V) & & & \\ & (\text{Hom}(U, V))_{c(12)} & & & \\ \downarrow & \swarrow & \downarrow & \swarrow & \\ \text{Hom}_c(U_c, \mathbf{V}) & \xrightarrow{\quad} & (\text{Hom}(U, V))_{c(123)} & \xleftarrow{\quad} & \end{array}$$

LEMMA 5.79. Given  $\mathbf{V} = (V, J_V)$  and  $U$  with commuting CSOs  $J_U^1, J_U^2$ , let  $W$  be a space with three commuting CSOs  $J_W^1, J_W^2, J_W^3$ . If  $H : W \rightarrow \text{Hom}(U, V)$  satisfies  $\text{Hom}(J_U^1, Id_V) \circ H = H \circ J_W^1$  and  $\text{Hom}(J_U^2, Id_V) \circ H = H \circ J_W^2$  and  $\text{Hom}(Id_U, J_V) \circ H = H \circ J_W^3$ , then  $H$  respects the corresponding direct sums from Example 5.78 and the induced maps are  $c$ -linear. If also  $H$  is invertible, then the induced maps are invertible.

PROOF. That  $H$  respects the direct sums produced by the three corresponding pairs of CSOs, and that the induced maps

$$P_{c(i_1 i_2)} \circ H \circ Q'_{c(i_1 i_2)} : W_{c(i_1 i_2)} \rightarrow (\text{Hom}(U, V))_{c(i_1 i_2)}$$

(for example, the arrow labeled  $a_3$  in the diagram below) are  $c$ -linear, follow from Lemma 5.55, which also showed the induced map  $\tilde{a}_3 : W_{c(123)} \rightarrow (\text{Hom}(U, V))_{c(123)}$  is  $c$ -linear, and invertible if  $H$  is. In the diagram,  $a_1$  and  $\tilde{a}_1$  are the canonical invertible maps which appeared as horizontal arrows in the diagram from Example 5.78; the adjacent projection arrows are also copied from that diagram. Lemma 5.73 showed that  $H$  also respects the direct sum  $W_{c(12)} \oplus W_{a(12)} \rightarrow \text{Hom}(U_c, V) \oplus \text{Hom}(U_a, V)$  and that the induced map  $\text{Hom}(Q_c, Id_V) \circ H \circ Q'_{c(12)}$ , denoted  $a_2$  in the diagram below, is  $c$ -linear with respect to the CSO induced by  $J_W^1$  and the CSO induced by  $\text{Hom}(J_U^1, Id_V)$ . In fact,  $a_2$  is also  $c$ -linear with respect to the other pair of CSOs, induced by  $J_W^3$  and  $\text{Hom}(Id_U, J_V)$ . By Lemma 5.51,  $a_2$  respects the direct sums produced by the commuting CSOs and induces a  $c$ -linear map  $\tilde{a}_2 : W_{c(123)} \rightarrow \text{Hom}_c(U_c, V)$ ; it satisfies the identity  $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$ .



EXAMPLE 5.80. Given  $U$  and  $V$ , suppose there are three commuting CSOs  $J_U$ ,  $J'_U$ ,  $J''_U$  on  $U$ . Then  $J_1 = \text{Hom}(J_U, Id_V)$ ,  $J_2 = \text{Hom}(J'_U, Id_V)$ ,  $J_3 = \text{Hom}(J''_U, Id_V)$  are three commuting CSOs on  $\text{Hom}(U, V)$ , and Example 5.46 applies. There are three direct sums:  $\text{Hom}(U, V) = (\text{Hom}(U, V))_{c(i_1 i_2)} \oplus (\text{Hom}(U, V))_{a(i_2 i_2)}$ , each of which is equivalent to a direct sum  $\text{Hom}(U_{c(i_1 i_2)}, V) \oplus \text{Hom}(U_{a(i_1 i_2)}, V)$  as in Example 5.72, with projection  $\text{Hom}(Q_{c(i_2, i_2)}, Id_V) : \text{Hom}(U, V) \rightarrow \text{Hom}(U_{c(i_1 i_2)}, V)$  and inclusion  $\text{Hom}(P_{c(i_2, i_2)}, Id_V)$ . Each  $(\text{Hom}(U, V))_{c(i_1 i_2)}$  admits a direct sum with projection  $P'_{c((i_1 i_2) i_3)}$  onto

$$((\text{Hom}(U, V))_{c(i_1 i_2)})_c = (\text{Hom}(U, V))_{c(123)} = \{A : U \rightarrow V : A \circ J_U = A \circ J'_U = A \circ J''_U\}.$$

Each subspace  $\text{Hom}(U_{c(i_2 i_2)}, V)$  has two commuting CSOs, and Example 5.72 applies again; there are equivalent direct sums:  $\text{Hom}(U_{c(i_1 i_2)}, V) = (\text{Hom}(U_{c(i_1 i_2)}, V))_c \oplus (\text{Hom}(U_{c(i_1 i_2)}, V))_a$ , and  $\text{Hom}(U_{c(i_1 i_2)}, V) = \text{Hom}(U_{c(123)}, V) \oplus \text{Hom}((U_{c(i_1 i_2)})_a, V)$ . The following diagram shows the  $(i_1 i_2) = (12)$  case, the other two cases being similar.

$$\begin{array}{ccccc}
& & & \text{Hom}(U, V) & \\
& & \text{Hom}(Q_{c(12)}, Id_V) & \nearrow & \\
& & & \downarrow P'_{c(12)} & \\
& \text{Hom}(U_{c(12)}, V) & \xrightarrow{a_1} & (\text{Hom}(U, V))_{c(12)} & \\
& \nearrow \text{Hom}(Q_{c((12)3)}, Id_V) & & \downarrow P'_{c((12)3)} & \\
& \text{Hom}(U_{c(123)}, V) & \xrightarrow{a_2} & (\text{Hom}(U_{c(12)}, V))_c & \xrightarrow{a_3} & (\text{Hom}(U, V))_{c(123)} \\
& & \downarrow P''_c & & & \\
& & & & & 
\end{array}$$

The horizontal arrows are

$$\begin{aligned}
a_1 &= P'_{c(12)} \circ \text{Hom}(P_{c(12)}, Id_V) \\
a_2 &= P''_c \circ \text{Hom}(P_{c((12)3)}, Id_V) \\
a_3 &= P'_{c((12)3)} \circ a_1 \circ Q''_c.
\end{aligned}$$

Both  $a_1$  and  $a_2$  are  $c$ -linear and invertible, canonically induced from the equivalent direct sums as in Example 5.72. The  $a_1$  map is also  $c$ -linear with respect to the CSOs induced by  $J''_U$ , so the induced map  $a_3$  is  $c$ -linear and invertible by Lemma 5.51. The  $c$ -linear invertible composite

$$a_3 \circ a_2 = P'_{c((12)3)} \circ P'_{c(12)} \circ \text{Hom}(P_{c((12)3)} \circ P_{c(12)}, Id_V)$$

is canonical, not depending on the choice of  $(i_1 i_2)$ , as in Example 5.46.

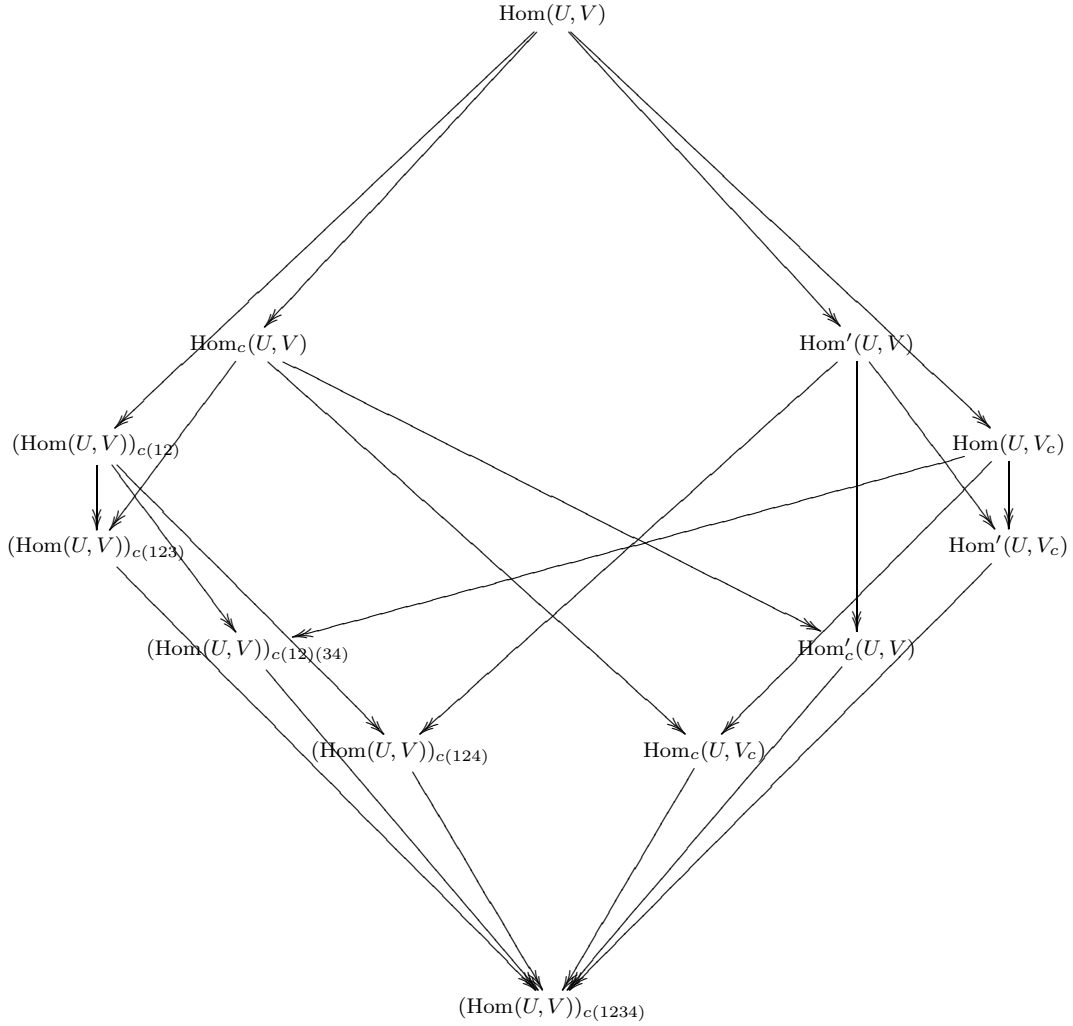
EXAMPLE 5.81. For  $V$  with commuting CSOs  $J_V$ ,  $J'_V$ , and  $U$  with commuting CSOs  $J_U$ ,  $J'_U$ , the space  $\text{Hom}(U, V)$  has four commuting CSOs:

$$J_1 = \text{Hom}(J_U, Id_V), \quad J_2 = \text{Hom}(J'_U, Id_V),$$

$$J_3 = \text{Hom}(Id_U, J_V), \quad J_4 = \text{Hom}(Id_U, J'_V).$$

Theorem 5.49 applies, to give a collection of subspaces of  $\text{Hom}(U, V)$ . The subspace  $(\text{Hom}(U, V))_{c(12)}$  was considered in Example 5.78. As in Example 5.65, denote  $(\text{Hom}(U, V))_{c(13)} = \text{Hom}_c(U, V)$ , and (temporarily) denote a similar construction  $(\text{Hom}(U, V))_{c(24)} = \text{Hom}'(U, V)$ . From Example 5.71, denote  $(\text{Hom}(U, V))_{c(34)} = \text{Hom}(U, V_c)$ . The subspaces  $(\text{Hom}(U, V))_{c(14)}$  and  $(\text{Hom}(U, V))_{c(23)}$  are omitted from the following commutative diagram, but otherwise the positions of the objects match the corresponding positions in the diagram from Theorem 5.49 and Example

5.64.



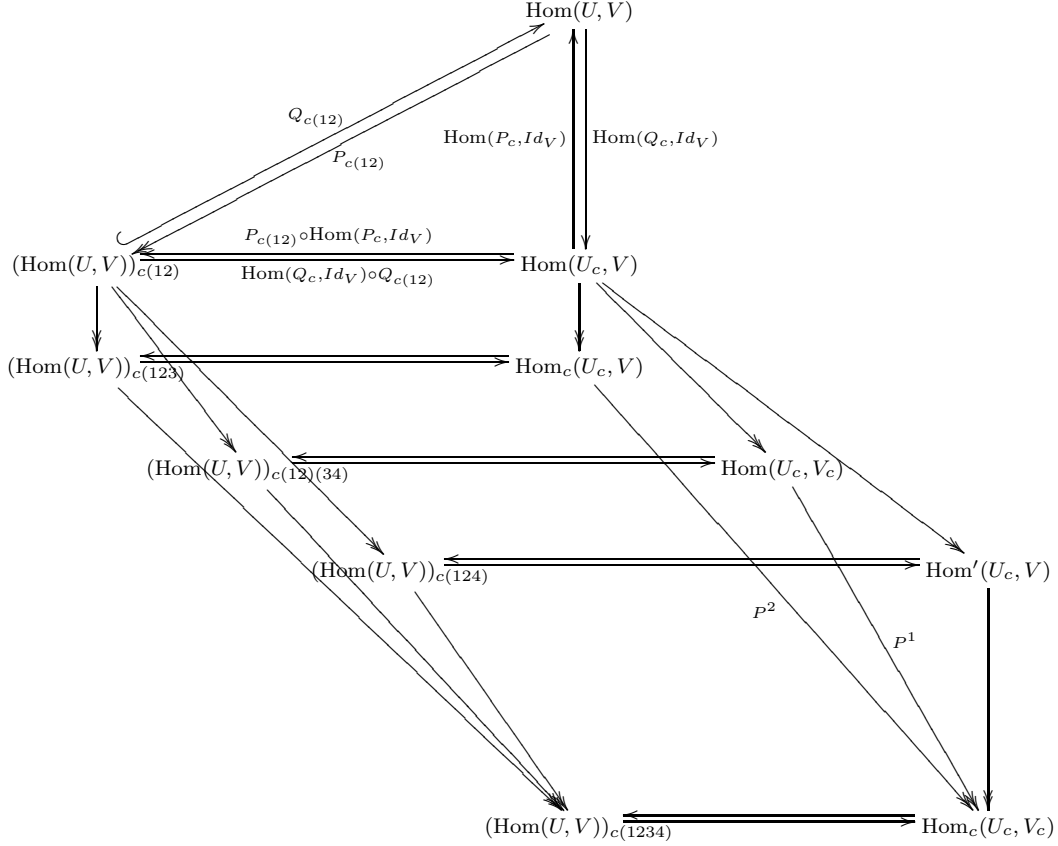
For example,  $\text{Hom}'_c(U, V)$  denotes the subspace

$$(\text{Hom}(U, V))_{c(13)(24)} = \{A : U \rightarrow V : A \circ J_U = J_V \circ A \text{ and } A \circ J'_U = J'_V \circ A\}.$$

If we ignore  $J'_U$ , then the two projections onto  $\text{Hom}_c(U, V_c)$  in the above diagram are as in Example 5.77. Similarly ignoring  $J_U$ , the two projections onto  $\text{Hom}'(U, V_c)$  are also as in Example 5.77.

EXAMPLE 5.82. The space  $(\text{Hom}(U, V))_{c(12)}$  from Example 5.81 is related to  $\text{Hom}(U_c, V)$  as in Example 5.78, by an invertible map  $P_{c(12)} \circ \text{Hom}(P_c, Id_V)$ . Both  $(\text{Hom}(U, V))_{c(12)}$  and  $\text{Hom}(U_c, V)$  admit three induced CSOs, and  $\text{Hom}(U_c, V)$  admits three direct sums as in Example 5.77. The map  $P_{c(12)} \circ \text{Hom}(P_c, Id_V)$  is c-linear with respect to the three corresponding pairs of CSOs, so by Lemma 5.55, it respects the direct sums and induces c-linear invertible maps as indicated by the unlabeled horizontal arrows in the following diagram. The left part is copied from the diagram in Example 5.81, and the top triangle and top square appeared already

in the diagram for Example 5.78.



The projections  $P^1$ ,  $P^2$  are labeled to match (5.7), (5.8) from Example 5.77, and the lower right vertical arrow is also from (5.8).

**THEOREM 5.83.** *For  $V$  with commuting CSOs  $J_V$ ,  $J'_V$ , and  $U$  with commuting CSOs  $J_U$ ,  $J'_U$ , let  $W$  be a space with four commuting CSOs  $J_W^1, J_W^2, J_W^3, J_W^4$ . If  $H : W \rightarrow \text{Hom}(U, V)$  satisfies  $\text{Hom}(J_U, Id_V) \circ H = H \circ J_W^1$  and  $\text{Hom}(J'_U, Id_V) \circ H = H \circ J_W^2$  and  $\text{Hom}(Id_U, J_V) \circ H = H \circ J_W^3$  and  $\text{Hom}(Id_U, J'_V) \circ H = H \circ J_W^4$ , then  $H$  respects the corresponding direct sums from Examples 5.81 and 5.82, and the induced maps are  $c$ -linear. If also  $H$  is invertible, then the induced maps are invertible.*

**PROOF.** The claims for the induced maps

$$\begin{aligned}
 W_{c(i_1 i_2)} &\rightarrow (\text{Hom}(U, V))_{c(i_1 i_2)} \\
 W_{c(i_1 i_2 i_3)} &\rightarrow (\text{Hom}(U, V))_{c(i_1 i_2 i_3)} \\
 W_{c(i_1 i_2)(i_3 i_4)} &\rightarrow (\text{Hom}(U, V))_{c(i_1 i_2)(i_3 i_4)} \\
 \tilde{a}_3 : W_{c(1234)} &\rightarrow (\text{Hom}(U, V))_{c(1234)}
 \end{aligned}$$

follow from Lemma 5.56. The target spaces are as in the diagram from Example 5.81, for example,  $H$  induces a map  $W_{c(13)(24)} \rightarrow \text{Hom}'_c(U, V)$ , labeled  $a_3$  in the diagram below, and  $a_3$  induces  $\tilde{a}_3$ . The claims for the induced maps

$$a_2 = \text{Hom}(Q_c, Id_V) \circ H \circ Q'_{c(12)} : W_{c(12)} \rightarrow \text{Hom}(U_c, V),$$

and  $W_{c(123)} \rightarrow \text{Hom}_c(U_c, (V, J_V))$  ( $= \text{Hom}_c(U_c, V)$  in the diagram from Example 5.82), and  $W_{c(124)} \rightarrow \text{Hom}_c(U_c, (V, J'_V)) = \text{Hom}'(U_c, V)$  follow from Lemma 5.79. The map  $a_2$  is  $c$ -linear with respect to the pair of CSOs induced by  $J_W^3$  and  $\text{Hom}(Id_U, J_V)$ , as mentioned in the Proof of Lemma 5.79, and is also  $c$ -linear with respect to the pair of CSOs induced by  $J_W^4$  and  $\text{Hom}(Id_U, J'_V)$ , so it satisfies the hypotheses of Lemma 5.55, and respects the corresponding direct sums in the diagram from Example 5.82. The maps induced by  $a_2$  are  $c$ -linear:

$$\begin{aligned} W_{c(12)(34)} &\rightarrow \text{Hom}(U_c, V_c) \\ \tilde{a}_2 : W_{c(1234)} &\rightarrow \text{Hom}_c(U_c, V_c). \end{aligned}$$

In the following diagram, the left half is copied from the diagram from Example 5.82, where  $a_1 = \text{Hom}(Q_c, Id_V) \circ Q_{c(12)}$  induces  $\tilde{a}_1$ , and they are both  $c$ -linear and invertible. The space  $\text{Hom}'_c(U, V)$  and projections  $P_{c(13)(24)}$ ,  $P_3$  are copied from Example 5.81, and the right half of the diagram is part of the diagram from Theorem 5.49.

$$\begin{array}{ccccc} & & \text{Hom}(U, V) & \xleftarrow{H} & W \\ & \swarrow P_{c(12)} & \downarrow \text{Hom}(Q_c, Id_V) & \searrow P'_{c(12)} & \downarrow \\ (\text{Hom}(U, V))_{c(12)} & \xrightarrow{a_1} & \text{Hom}(U_c, V) & \xleftarrow{a_2} & W_{c(12)} \\ & \searrow P_1 & \downarrow P_{c(13)(24)} & \searrow P'_{c(13)(24)} & \downarrow P'_1 \\ & & \text{Hom}'_c(U, V) & \xleftarrow{a_3} & W_{c(13)(24)} \\ & & \downarrow P_2 & \searrow P'_2 & \downarrow \\ & & & & W_{c(1234)} \\ & \swarrow P_3 & \searrow \tilde{a}_3 & \searrow \tilde{a}_2 & \\ & & (\text{Hom}(U, V))_{c(1234)} & \xrightarrow{\tilde{a}_1} & \text{Hom}_c(U_c, V_c) \end{array}$$

Finally, we remark that  $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$ ; an analogous property was observed in the Proof of Lemma 5.79. The identity can be checked directly, using the  $c$ -linearity of  $H$ . The map  $\tilde{a}_2$  acts on  $w \in W_{c(1234)}$  as:  $\tilde{a}_2 : w \mapsto P'_c \circ H(w) \circ Q_c$ , where  $P'_c$  is the projection  $V \twoheadrightarrow V_c$ . ■

EXAMPLE 5.84. Given  $U_1, U_2, V_1, V_2$ , and  $\mathbf{U}_1 = (U_1, J_{U_1})$ , the canonical map (Definition 1.22)

$$j : \text{Hom}(\mathbf{U}_1, V_1) \otimes \text{Hom}(U_2, V_2) \rightarrow \text{Hom}(\mathbf{U}_1 \otimes U_2, V_1 \otimes V_2)$$

is  $c$ -linear with respect to the induced CSOs. A similar statement holds if any one of the four spaces has a CSO.

If every one of the above four spaces has a CSO,  $\mathbf{U}_1 = (U_1, J_{U_1})$ ,  $\mathbf{U}_2 = (U_2, J_{U_2})$ ,  $\mathbf{V}_1 = (V_1, J_{V_1})$ ,  $\mathbf{V}_2 = (V_2, J_{V_2})$ , then  $\text{Hom}(U_1, V_1) \otimes \text{Hom}(U_2, V_2)$  admits four commuting CSOs:

$$\begin{aligned} J'_1 &= [\text{Hom}(J_{U_1}, Id_{V_1}) \otimes Id_{\text{Hom}(U_2, V_2)}] \\ J'_2 &= [Id_{\text{Hom}(U_1, V_1)} \otimes \text{Hom}(J_{U_2}, Id_{V_2})] \\ J'_3 &= [\text{Hom}(Id_{U_1}, J_{V_1}) \otimes Id_{\text{Hom}(U_2, V_2)}] \\ J'_4 &= [Id_{\text{Hom}(U_1, V_1)} \otimes \text{Hom}(Id_{U_2}, J_{V_2})], \end{aligned}$$

and  $\text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2)$  also admits four commuting CSOs:

$$\begin{aligned} J_1 &= \text{Hom}([J_{U_1} \otimes Id_{U_2}], Id_{V_1 \otimes V_2}) \\ J_2 &= \text{Hom}([Id_{U_1} \otimes J_{U_2}], Id_{V_1 \otimes V_2}) \\ J_3 &= \text{Hom}(Id_{U_1 \otimes U_2}, [J_{V_1} \otimes Id_{V_2}]) \\ J_4 &= \text{Hom}(Id_{U_1 \otimes U_2}, [Id_{V_1} \otimes J_{V_2}]). \end{aligned}$$

Since  $j$  is c-linear with respect to each pair  $J'_i, J_i$ , Theorem 5.83 applies;  $j$  induces maps on corresponding subspaces, which are c-linear with respect to (possibly several pairs of) corresponding CSOs, and which are invertible if  $j$  is. From the diagrams in Theorem 5.49 and Examples 5.64, 5.81, some induced maps from Lemma 5.56 are evident:

$$\begin{aligned} \text{Hom}(\mathbf{U}_1, V_1) \otimes_c \text{Hom}(U_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes U_2, V_1 \otimes \mathbf{V}_2) \\ \text{Hom}(U_1, \mathbf{V}_1) \otimes_c \text{Hom}(\mathbf{U}_2, V_2) &\rightarrow \text{Hom}_c(U_1 \otimes \mathbf{U}_2, \mathbf{V}_1 \otimes V_2) \\ \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes \text{Hom}(U_2, V_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes U_2, \mathbf{V}_1 \otimes V_2) \\ \text{Hom}(U_1, V_1) \otimes \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(U_1 \otimes \mathbf{U}_2, V_1 \otimes \mathbf{V}_2) \\ \text{Hom}(U_1, \mathbf{V}_1) \otimes_c \text{Hom}(U_2, \mathbf{V}_2) &\rightarrow \text{Hom}(U_1 \otimes U_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2) \\ \text{Hom}(U_1, \mathbf{V}_1) \otimes_c \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(U_1 \otimes \mathbf{U}_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2) \\ a_3 : \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow \text{Hom}'_c(U_1 \otimes U_2, V_1 \otimes V_2) \\ \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes_c \text{Hom}(U_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes U_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2) \\ \tilde{a}_3 : \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes_c \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow (\text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(1234)}. \end{aligned}$$

For example, the seventh map, labeled  $a_3$  as in the diagram from Theorem 5.83, is c-linear with respect to both corresponding pairs of induced CSOs, and for c-linear maps  $A$  and  $B$ , takes  $A \otimes B$  to the map  $j(A \otimes B) : U_1 \otimes U_2 \rightarrow V_1 \otimes V_2$ , which is c-linear with respect to the pair  $[J_{U_1} \otimes Id_{U_2}], [J_{V_1} \otimes Id_{V_2}]$ , and also c-linear with respect to the pair  $[Id_{U_1} \otimes J_{U_2}], [Id_{V_1} \otimes J_{V_2}]$ .

Also, if some but not all of the four spaces have CSOs, then there may still be some induced maps, for example, the first one in the above list makes sense if only  $U_1$  and  $V_2$  have CSOs.

Let  $Q'_{c(12)}$ ,  $P'_{c(12)}$ , and  $Q_{c(12)}$ ,  $P_{c(12)}$  denote the inclusion and projection operators for the direct sums produced by  $J'_1, J'_2$ , and  $J_1, J_2$ , respectively, as appearing in the diagram from the Proof of Theorem 5.83. By Lemma 5.73, the map  $j$  also respects the direct sum

$$\text{Hom}(\mathbf{U}_1 \otimes_c \mathbf{U}_2, V_1 \otimes V_2) \oplus \text{Hom}(U_1 \otimes_a U_2, V_1 \otimes V_2),$$

and induces a c-linear map, labeled  $a_2$  in Theorem 5.83,

$$\begin{aligned} a_2 &= \text{Hom}(Q_c, Id_{V_1 \otimes V_2}) \circ j \circ Q'_{c(12)} : \\ \text{Hom}(\mathbf{U}_1, V_1) \otimes_c \text{Hom}(\mathbf{U}_2, V_2) &\rightarrow \text{Hom}(\mathbf{U}_1 \otimes_c \mathbf{U}_2, V_1 \otimes V_2) \end{aligned}$$

which is equal to the composite of the induced map

$$P_{c(12)} \circ j \circ Q'_{c(12)} : \text{Hom}(\mathbf{U}_1, V_1) \otimes_c \text{Hom}(\mathbf{U}_2, V_2) \rightarrow (\text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(12)}$$

with the invertible c-linear map from Example 5.82, labeled  $a_1$  in Theorem 5.83,

$$\text{Hom}(Q_c, Id_{V_1 \otimes V_2}) \circ Q_{c(12)} : (\text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(12)} \rightarrow \text{Hom}(\mathbf{U}_1 \otimes_c \mathbf{U}_2, V_1 \otimes V_2).$$

The map  $a_2$  is c-linear with respect to all three corresponding pairs of induced CSOs, so as in Example 5.81 and Theorem 5.83, it respects the corresponding direct sums, to induce c-linear maps:

$$\begin{aligned} \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes_c \text{Hom}(\mathbf{U}_2, V_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes_c \mathbf{U}_2, \mathbf{V}_1 \otimes V_2) \\ \text{Hom}(U_1, V_1) \otimes'_c \text{Hom}(U_2, V_2) &\rightarrow \text{Hom}(\mathbf{U}_1 \otimes_c \mathbf{U}_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2) \\ \text{Hom}(\mathbf{U}_1, V_1) \otimes_c \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes_c \mathbf{U}_2, V_1 \otimes \mathbf{V}_2) \\ \tilde{a}_2 : \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes_c \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2) &\rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes_c \mathbf{U}_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2). \end{aligned}$$

As remarked in the Proof of Theorem 5.83, let  $\tilde{a}_1$  denote the invertible map induced by  $a_1$ ,

$$(\text{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(1234)} \rightarrow \text{Hom}_c(\mathbf{U}_1 \otimes_c \mathbf{U}_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2),$$

so that then  $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$ . The map  $\tilde{a}_2$  is invertible if  $j$  is; it acts on  $w \in \text{Hom}_c(\mathbf{U}_1, \mathbf{V}_1) \otimes_c \text{Hom}_c(\mathbf{U}_2, \mathbf{V}_2)$  as:  $\tilde{a}_2 : w \mapsto P'_c \circ j(w) \circ Q_c$ , where  $P'_c$  is the projection  $V_1 \otimes V_2 \rightarrow \mathbf{V}_1 \otimes_c \mathbf{V}_2$ .

EXERCISE 5.85. Given  $V$  and  $\mathbf{W} = (W, J_W)$ ,  $\text{Hom}(\text{Hom}(V, W), W)$  admits two commuting CSOs,  $\text{Hom}(\text{Hom}(Id_V, J_W), Id_W)$  and  $\text{Hom}(Id_{\text{Hom}(V, W)}, J_W)$ , so as in Example 5.65 there is a direct sum

$$\text{Hom}_c(\text{Hom}(V, \mathbf{W}), \mathbf{W}) \oplus \text{Hom}_a(\text{Hom}(V, \mathbf{W}), \mathbf{W}).$$

The image of the canonical map (Definition 1.9)  $d_{VW} : V \rightarrow \text{Hom}(\text{Hom}(V, W), W)$  is contained in  $\text{Hom}_c(\text{Hom}(V, \mathbf{W}), \mathbf{W})$ , i.e, for each  $v \in V$ ,  $d_{VW}(v) : \text{Hom}(V, \mathbf{W}) \rightarrow \mathbf{W}$  is c-linear. ■

EXERCISE 5.86. Given  $\mathbf{V} = (V, J_V)$  and  $\mathbf{W} = (W, J_W)$ ,  $\text{Hom}(\text{Hom}(V, W), W)$  admits three commuting CSOs,

$$\begin{aligned} J_1 &= \text{Hom}(\text{Hom}(J_V, Id_W), Id_W), \\ J_2 &= \text{Hom}(\text{Hom}(Id_V, J_W), Id_W), \\ J_3 &= \text{Hom}(Id_{\text{Hom}(V, W)}, J_W). \end{aligned}$$

As in Example 5.78, there are three direct sums;  $(\text{Hom}(\text{Hom}(V, W), W))_{c(23)}$  was considered in the previous Exercise. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ d_{VW} : \mathbf{V} \rightarrow (\text{Hom}(\text{Hom}(V, W), W))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let  $Q_c : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \hookrightarrow \text{Hom}(V, W)$  denote the inclusion from Example 5.65. Then the image of

$$\text{Hom}(Q_c, Id_W) \circ d_{VW} : V \rightarrow \text{Hom}(\text{Hom}_c(\mathbf{V}, \mathbf{W}), W)$$



is contained in  $\text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$ , i.e., for any  $v \in V$ ,

$$d_{VW}(v) \circ Q_c : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \rightarrow \mathbf{W} : H \mapsto (Q_c(H))(v) = H(v)$$

is a c-linear map. From the commutativity of the diagram from Example 5.78, considering  $\text{Hom}(Q_c, Id_W) \circ d_{VW}$  as a map  $\mathbf{V} \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$ , it is identical to the composite of the above map  $P_{c((23)1)} \circ P_{c(23)} \circ d_{VW}$  with the canonical map  $(\text{Hom}(\text{Hom}(V, W), W))_{c(123)} \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$ , so it is c-linear. ■

EXERCISE 5.87. Given  $U, V, W$ , with  $\mathbf{U} = (U, J_U)$  and  $\mathbf{V} = (V, J_V)$ , the canonical map (Definition 1.6)  $t_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}(V, W), \text{Hom}(U, W))$  respects the direct sums and the induced map

$$\text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\text{Hom}(\mathbf{V}, W), \text{Hom}(\mathbf{U}, W))$$

is c-linear.

HINT. Exercises 5.33 and 5.34, and then Lemma 5.51, apply. ■

EXAMPLE 5.88. For  $W = \mathbb{R}$  in the previous Exercise,  $t_{UV}^{\mathbb{R}} = t_{UV} : \text{Hom}(U, V) \rightarrow \text{Hom}(V^*, U^*)$ .  $V^*$  has a CSO  $J_V^*$  as in Example 5.5, and similarly  $J_U^*$  is a CSO for  $U^*$ .  $t_{UV}$  respects the direct sums

$$\text{Hom}_c(\mathbf{U}, \mathbf{V}) \oplus \text{Hom}_a(U, V) \rightarrow \text{Hom}_c(V^*, U^*) \oplus \text{Hom}_a(V^*, U^*)$$

and the induced map  $\text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(V^*, U^*)$ ,  $A \mapsto t_{UV}(A) = \text{Hom}(A, Id_{\mathbb{R}}) = A^*$ , is c-linear.

EXERCISE 5.89. Given  $U, V, W$ , with  $\mathbf{W} = (W, J)$ , the image of

$$t_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}(V, W), \text{Hom}(U, W))$$

is contained in  $\text{Hom}_c(\text{Hom}(V, \mathbf{W}), \text{Hom}(U, \mathbf{W}))$ , i.e., for any  $A \in \text{Hom}(U, V)$ , the map  $t_{UV}^W(A) : \text{Hom}(V, \mathbf{W}) \rightarrow \text{Hom}(U, \mathbf{W})$  is c-linear. ■

EXERCISE 5.90. Given  $U, V, W$ , with  $\mathbf{U} = (U, J_U)$ ,  $\mathbf{V} = (V, J_V)$ , and  $\mathbf{W} = (W, J)$ , for any  $A \in \text{Hom}_c(\mathbf{U}, \mathbf{V})$ , the map  $t_{UV}^W(A) : \text{Hom}(V, W) \rightarrow \text{Hom}(U, W)$  (or, more precisely,  $t_{UV}^W(Q'_c(A))$ , where  $Q'_c$  is the inclusion of  $\text{Hom}_c(\mathbf{U}, \mathbf{V})$  in  $\text{Hom}(U, V)$ ) is c-linear with respect to both pairs  $\text{Hom}(Id_V, J_W)$ ,  $\text{Hom}(Id_U, J_W)$  and  $\text{Hom}(J_U, Id_W)$ ,  $\text{Hom}(J_V, Id_W)$ , so  $t_{UV}^W(A)$  respects the direct sums and induces a c-linear map  $\text{Hom}_c(\mathbf{V}, \mathbf{W}) \rightarrow \text{Hom}_c(\mathbf{U}, \mathbf{W})$ . The resulting map, denoted

$$t_{\mathbf{U}\mathbf{V}}^{\mathbf{W}} : \text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \text{Hom}_c(\mathbf{U}, \mathbf{W})),$$

is c-linear.

HINT. Lemma 5.51 and the previous Exercises apply. The last claim can be checked directly. However, by following the diagrams from Examples 5.81 and 5.82, a little more can be obtained. Let  $Q_i$  and  $P_i$  denote the operators for the direct sum  $\text{Hom}(V, W) = \text{Hom}_c(\mathbf{V}, \mathbf{W}) \oplus \text{Hom}_a(V, W)$ , so that

$$\text{Hom}(Q_c, Id_{\text{Hom}(U, W)}) : \text{Hom}(\text{Hom}(V, W), \text{Hom}(U, W)) \rightarrow \text{Hom}(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \text{Hom}(U, W))$$

is as in Example 5.82. Then

$$\text{Hom}(Q_c, Id_{\text{Hom}(U, W)}) \circ t_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \text{Hom}(U, W))$$

is c-linear with respect to  $\text{Hom}(J_U, Id_V)$  and the CSO induced by

$$J^3 = \text{Hom}(Id_{\text{Hom}(V, W)}, \text{Hom}(J_U, Id_W)),$$

and is also c-linear with respect to  $\text{Hom}(Id_U, J_V)$  and the CSO induced by

$$J^1 = \text{Hom}(\text{Hom}(J_V, Id_W), Id_{\text{Hom}(U, W)}).$$

So by Lemma 5.51, it induces a c-linear map

$$\text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \text{Hom}(\mathbf{U}, W)).$$

As claimed above, the image of this induced map is contained in the subspace  $\text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}), \text{Hom}_c(\mathbf{U}, \mathbf{W}))$ . For  $A \in \text{Hom}_c(\mathbf{U}, \mathbf{V})$  and  $K \in \text{Hom}_c(\mathbf{V}, \mathbf{W})$ ,

$$(t_{UV}^W(Q'_c(A))) \circ Q_c : K \mapsto (Q_c(K)) \circ (Q'_c(A)) = K \circ A \in \text{Hom}_c(\mathbf{U}, \mathbf{W}).$$

■

EXERCISE 5.91. Given  $W$  and  $\mathbf{V} = (V, J_V)$ , the involution

$$T_{V;W} = \text{Hom}(d_{VW}, Id_{\text{Hom}(V, W)}) \circ t_{V, \text{Hom}(V, W)}^W$$

on  $\text{Hom}(V, \text{Hom}(V, W))$  from Definition 4.2 is c-linear with respect to both induced CSOs as follows:

$$\begin{aligned} \text{Hom}(\mathbf{V}, \text{Hom}(V, W)) &\rightarrow \text{Hom}(V, \text{Hom}(\mathbf{V}, W)) \\ \text{Hom}(V, \text{Hom}(\mathbf{V}, W)) &\rightarrow \text{Hom}(\mathbf{V}, \text{Hom}(V, W)) \end{aligned}$$

and induces a c-linear involution on  $\text{Hom}_c(\mathbf{V}, \text{Hom}(\mathbf{V}, W))$ .

HINT.

$$t_{V, \text{Hom}(V, W)}^W : \text{Hom}(\mathbf{V}, \text{Hom}(V, W)) \rightarrow \text{Hom}(\text{Hom}(\text{Hom}(V, W), W), \text{Hom}(\mathbf{V}, W))$$

is c-linear by Exercise 5.33, and  $\text{Hom}(d_{VW}, Id_{\text{Hom}(V, W)}) :$

$$\text{Hom}(\text{Hom}(\text{Hom}(V, W), W), \text{Hom}(\mathbf{V}, W)) \rightarrow \text{Hom}(V, \text{Hom}(\mathbf{V}, W))$$

is c-linear by Lemma 5.22. Similarly,

$$t_{V, \text{Hom}(V, W)}^W : \text{Hom}(V, \text{Hom}(\mathbf{V}, W)) \rightarrow \text{Hom}(\text{Hom}(\text{Hom}(\mathbf{V}, W), W), \text{Hom}(V, W))$$

is c-linear by Exercise 5.34, and  $\text{Hom}(d_{VW}, Id_{\text{Hom}(V, W)}) :$

$$\text{Hom}(\text{Hom}(\text{Hom}(\mathbf{V}, W), W), \text{Hom}(V, W)) \rightarrow \text{Hom}(\mathbf{V}, \text{Hom}(V, W))$$

is c-linear by Exercise 5.32. The composite  $T_{V;W}$  is an involution to which Lemma 5.54 applies. ■

EXERCISE 5.92. Given  $\mathbf{V} = (V, J_V)$  and  $\mathbf{W} = (W, J_W)$ , the map  $T_{V;W}$  on  $\text{Hom}(V, \text{Hom}(V, W))$  is c-linear with respect to three corresponding pairs of commuting CSOs and induces c-linear maps as follows:

$$\begin{aligned} \text{Hom}_c(\mathbf{V}, \text{Hom}(V, \mathbf{W})) &\rightarrow \text{Hom}(V, \text{Hom}_c(\mathbf{V}, \mathbf{W})) \\ \text{Hom}(V, \text{Hom}_c(\mathbf{V}, \mathbf{W})) &\rightarrow \text{Hom}_c(\mathbf{V}, \text{Hom}(V, \mathbf{W})) \end{aligned}$$

and induces a c-linear involution on  $\text{Hom}_c(\mathbf{V}, \text{Hom}_c(\mathbf{V}, \mathbf{W}))$ .

HINT. Exercises 5.37 and 5.91, and then Examples 5.57 and 5.77 apply. ■

EXERCISE 5.93. Given  $U, V, W$ , with  $\mathbf{U} = (U, J_U)$  and  $\mathbf{V} = (V, J_V)$ , the canonical map (Definition 1.30)  $e_{UV}^W : \text{Hom}(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}(\text{Hom}(\mathbf{V}, W) \otimes \mathbf{U}, W)$  respects the direct sums and the induced map

$$\text{Hom}_c(\mathbf{U}, \mathbf{V}) \rightarrow \text{Hom}(\text{Hom}(\mathbf{V}, W) \otimes_c \mathbf{U}, W)$$

is c-linear.

HINT. Exercises 5.35 and 5.36, and then Lemma 5.73, apply. ■

EXERCISE 5.94. Given  $U, V, W$ , with  $\mathbf{W} = (W, J)$ , the image of

$$e_{UV}^W : \text{Hom}(U, V) \rightarrow \text{Hom}(\text{Hom}(V, W) \otimes U, W)$$

is contained in  $\text{Hom}_c(\text{Hom}(V, \mathbf{W}) \otimes U, \mathbf{W})$ , i.e., for any  $A \in \text{Hom}(U, V)$ , the map  $e_{UV}^W(A) : \text{Hom}(V, \mathbf{W}) \otimes U \rightarrow \mathbf{W}$  is c-linear. ■

EXERCISE 5.95. Given  $\mathbf{U} = (U, J_U)$  and  $\mathbf{W} = (W, J_W)$ ,  $\text{Hom}(\text{Hom}(V, W) \otimes U, W)$  admits three commuting CSOs, as in Example 5.78, so there are three direct sums;  $(\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(23)}$  was considered in the previous Exercise. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W : \text{Hom}(\mathbf{U}, V) \rightarrow (\text{Hom}(\text{Hom}(V, \mathbf{W}) \otimes \mathbf{U}, \mathbf{W}))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let

$$Q_c : \text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U} \hookrightarrow \text{Hom}(V, W) \otimes U$$

denote the inclusion from Example 5.59. Then the image of

$$\text{Hom}(Q_c, Id_W) \circ e_{UV}^W : \text{Hom}(\mathbf{U}, V) \rightarrow \text{Hom}(\text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U}, W)$$

is contained in  $\text{Hom}_c(\text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$ , i.e., for any  $A \in \text{Hom}(U, V)$ ,

$$e_{UV}^W(A) \circ Q_c : \text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U} \rightarrow \mathbf{W} : B \otimes u \mapsto (e_{UV}^W(A))(Q_c(B \otimes u)) = B(A(u))$$

is a c-linear map. From the commutativity of the diagram from Example 5.78, considering  $\text{Hom}(Q_c, Id_W) \circ e_{UV}^W$  as a map  $\text{Hom}(\mathbf{U}, V) \rightarrow \text{Hom}_c(\text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$ , it is identical to the composite of the above map  $P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W$  with the canonical map  $(\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(123)} \rightarrow \text{Hom}_c(\text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$ , so it is c-linear. ■

EXERCISE 5.96. Given  $\mathbf{V} = (V, J_V)$  and  $\mathbf{W} = (W, J_W)$ ,  $\text{Hom}(\text{Hom}(V, W) \otimes U, W)$  admits three commuting CSOs, as in Example 5.78, so there are three direct sums;  $(\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(23)}$  was considered in Exercises 5.94, 5.95. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W : \text{Hom}(U, \mathbf{V}) \rightarrow (\text{Hom}(\text{Hom}(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W}))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let  $Q_c : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U \rightarrow \text{Hom}(V, W) \otimes U$  denote the inclusion from Examples 5.58 and 5.65. Then the image of

$$\text{Hom}(Q_c, Id_W) \circ e_{UV}^W : \text{Hom}(U, \mathbf{V}) \rightarrow \text{Hom}(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, W)$$

is contained in  $\text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$ , i.e., for any  $A \in \text{Hom}(U, V)$ ,

$$e_{UV}^W(A) \circ Q_c : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U \rightarrow \mathbf{W} : B \otimes u \mapsto (e_{UV}^W(A))(Q_c(B \otimes u)) = B(A(u))$$

is a c-linear map. From the commutativity of the diagram from Example 5.78, considering  $\text{Hom}(Q_c, Id_W) \circ e_{UV}^W$  as a map  $\text{Hom}(U, \mathbf{V}) \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$ , it is identical to the composite of the above map  $P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W$  with the canonical map  $(\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(123)} \rightarrow \text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$ , so it is c-linear. ■

EXERCISE 5.97. Given  $U, V, W$ , with  $\mathbf{U} = (U, J_U)$ ,  $\mathbf{V} = (V, J_V)$ , and  $\mathbf{W} = (W, J_W)$ , we consider the three commuting CSOs on  $\text{Hom}(V, W) \otimes U$ , so Example 5.80 applies to  $\text{Hom}(\text{Hom}(V, W) \otimes U, W)$ . The lower square in the following commutative diagram is a specific case of a square from the diagram in Example 5.80, with the same labeling of arrows, and (12) referring to the CSOs induced by  $J_U$  and  $J_V$ . The  $e_1, e_2$  arrows are the maps induced by  $e_{UV}^W$  on the two equivalent direct sums from Exercise 5.93.

$$\begin{array}{ccc}
 & & \text{Hom}_c(\mathbf{U}, \mathbf{V}) \\
 & \swarrow e_1 & \downarrow e_2 \\
 \text{Hom}(\text{Hom}(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U}, W) & \xrightarrow{a_1} & (\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(12)} \\
 \downarrow \text{Hom}(Q_{c((12)3)}, Id_W) & & \downarrow P'_{c((12)3)} \\
 \text{Hom}(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U}, W) & \xrightarrow{a_3 \circ a_2} & (\text{Hom}(\text{Hom}(V, W) \otimes U, W))_{c(123)}
 \end{array}$$

The composite  $P'_{c((12)3)} \circ e_2$  is  $c$ -linear, so the composite  $\text{Hom}(Q_{c((12)3)}, Id_W) \circ e_1$  is also  $c$ -linear. As in Exercises 5.94, 5.95, 5.96, the image of  $\text{Hom}(Q_{c((12)3)}, Id_W) \circ e_1$  is contained in  $\text{Hom}_c(\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes \mathbf{U}, \mathbf{W})$ , i.e., if  $Q'_c$  denotes the inclusion of  $\text{Hom}_c(\mathbf{U}, \mathbf{V})$  in  $\text{Hom}(U, V)$  and  $Q_c = Q_{c((12)3)} \circ Q_{c(12)}$  denotes the inclusion of  $\text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U}$  in  $\text{Hom}(V, W) \otimes U$ , then for any  $A \in \text{Hom}_c(\mathbf{U}, \mathbf{V})$ , the map

$$e_{UV}^W(Q'_c(A)) \circ Q_c : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U} \rightarrow \mathbf{W}$$

is  $c$ -linear. ■

#### 5.4. Real trace with complex vector values

In this section we develop the notion of vector-valued trace of  $\mathbb{R}$ -linear maps, where the value spaces have complex structure operators. The approach will be to refer to Chapter 2, while avoiding scalar multiplication.

Recall from Definition 2.47 and Notation 2.52 the canonical map  $n' : V \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(U, V \otimes W) : (n'(v \otimes E)) : u \mapsto v \otimes (E(u))$ . The map  $n'$  is invertible if  $U$  or  $V$  is finite-dimensional; the following Theorem drops the prime notation.

THEOREM 5.98. *If  $\mathbf{U} = (U, J_U)$  and  $\mathbf{V} = (V, J_V)$  and  $\mathbf{W} = (W, J_W)$ , then  $n : V \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(U, V \otimes W)$  is  $c$ -linear with respect to corresponding pairs of the three commuting CSOs induced on each space, so it respects the direct sums and induces maps*

$$\begin{aligned}
 n^1 : V \otimes \text{Hom}_c(\mathbf{U}, \mathbf{W}) &\rightarrow \text{Hom}_c(\mathbf{U}, V \otimes \mathbf{W}) \\
 n^2 : \mathbf{V} \otimes_c \text{Hom}(\mathbf{U}, W) &\rightarrow \text{Hom}_c(\mathbf{U}, \mathbf{V} \otimes W) \\
 n^3 : \mathbf{V} \otimes_c \text{Hom}(U, \mathbf{W}) &\rightarrow \text{Hom}(U, \mathbf{V} \otimes_c \mathbf{W}) \\
 \mathbf{n} : \mathbf{V} \otimes_c \text{Hom}_c(\mathbf{U}, \mathbf{W}) &\rightarrow \text{Hom}_c(\mathbf{U}, \mathbf{V} \otimes_c \mathbf{W}),
 \end{aligned}$$

which are invertible if  $n$  is.

PROOF. The  $c$ -linearity is straightforward to check, and then this is a special case of Lemma 5.55. The direct sums for the domain  $V \otimes \text{Hom}(U, W)$  are as in Example 5.61. The projection onto the subspace  $V \otimes \text{Hom}_c(\mathbf{V}, \mathbf{W})$  is equal to  $[Id_V \otimes P_H]$  as in Example 5.58, where  $P_H$  is the projection  $\text{Hom}(U, W) \rightarrow$

$\text{Hom}_c(\mathbf{U}, \mathbf{W})$ . The direct sums for the target space  $\text{Hom}(U, V \otimes W)$  are as in Example 5.77. The following diagram shows some of the canonical projections, including  $P^2$  as in (5.8).

$$\begin{array}{ccc}
 V \otimes \text{Hom}(U, W) & \xrightarrow{n} & \text{Hom}(U, V \otimes W) \\
 \downarrow [Id_V \otimes P_H] & & \downarrow \\
 V \otimes \text{Hom}_c(\mathbf{U}, \mathbf{W}) & \xrightarrow{n^1} & \text{Hom}_c(\mathbf{U}, V \otimes \mathbf{W}) \\
 \downarrow & & \downarrow P^2 = \text{Hom}_c(Id_U, P'_c) \\
 \mathbf{V} \otimes_c \text{Hom}_c(\mathbf{U}, \mathbf{W}) & \xrightarrow{\mathbf{n}} & \text{Hom}_c(\mathbf{U}, \mathbf{V} \otimes_c \mathbf{W})
 \end{array}$$

■

Recall from Theorem 2.71 the special case, where the canonical map

$$n' : \text{Hom}(V, W) \otimes V \rightarrow \text{Hom}(V, V \otimes W) : A \otimes v \mapsto (u \mapsto v \otimes (A(u))),$$

is invertible if  $V$  is finite-dimensional. The following result is a minor modification of Theorem 5.98.

**COROLLARY 5.99.** *If  $\mathbf{V} = (V, J_V)$  and  $\mathbf{W} = (W, J_W)$ , then  $n' : \text{Hom}(V, W) \otimes V \rightarrow \text{Hom}(V, V \otimes W)$  is  $c$ -linear with respect to corresponding pairs of the three commuting CSOs induced on each space, so it respects the direct sums and induces maps*

$$\begin{aligned}
 n'_1 : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes V &\rightarrow \text{Hom}_c(\mathbf{V}, V \otimes \mathbf{W}) \\
 n'_2 : \text{Hom}(\mathbf{V}, W) \otimes_c \mathbf{V} &\rightarrow \text{Hom}_c(\mathbf{V}, \mathbf{V} \otimes W) \\
 n'_3 : \text{Hom}(V, \mathbf{W}) \otimes_c \mathbf{V} &\rightarrow \text{Hom}(V, \mathbf{V} \otimes_c \mathbf{W}) \\
 \mathbf{n}' : \text{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} &\rightarrow \text{Hom}_c(\mathbf{V}, \mathbf{V} \otimes_c \mathbf{W}),
 \end{aligned}$$

which are invertible if  $n'$  is. ■

Recall from Definition 2.69 that  $\text{Hom}(\text{Hom}(V, W) \otimes V, W)$  contains a distinguished element  $Ev_{VW} : A \otimes v \mapsto A(v)$ .

**LEMMA 5.100.** *If  $\mathbf{W} = (W, J_W)$ , then  $Ev_{VW}$  is  $c$ -linear with respect to the induced CSO, i.e., it is an element of the subspace  $\text{Hom}_c(\text{Hom}(V, \mathbf{W}) \otimes V, \mathbf{W})$ . ■*

**THEOREM 5.101.** *If  $V$  is finite-dimensional and  $\mathbf{W} = (W, J_W)$ , then the map*

$$Tr_{V;W} = Ev_{VW} \circ (n')^{-1} : \text{Hom}(V, V \otimes \mathbf{W}) \rightarrow \mathbf{W}$$

*is  $c$ -linear.*

**PROOF.** The map  $n'$  is from Corollary 5.99: it is  $c$ -linear with respect to  $[\text{Hom}(Id_V, J_W) \otimes Id_V]$  and  $\text{Hom}(Id_V, [Id_V \otimes J_W])$  (without assuming any CSO on  $V$ ). The equality  $Tr_{V;W} \circ n' = Ev_{VW}$  is from Theorem 2.71. The result could also be proved by applying Corollary 2.59 with  $B = J_W$ . ■

**THEOREM 5.102.** *If  $V$  is finite-dimensional and  $W$  admits commuting CSOs  $J_1, J_2$ , then the map  $Tr_{V;W}$  respects the direct sums*

$$\text{Hom}(V, V \otimes W_c) \oplus \text{Hom}(V, V \otimes W_a) \rightarrow W_c \oplus W_a,$$

*and the induced  $c$ -linear map  $\text{Hom}(V, V \otimes W_c) \rightarrow W_c$  is equal to  $Tr_{V;W_c}$ .*

PROOF. This is a special case of Lemma 2.60. The direct sums on  $V \otimes W$  and  $\text{Hom}(V, V \otimes W)$  are as in Examples 5.58, 5.71, with canonical projections as indicated in the diagram.  $Tr_{V;W}$  is c-linear with respect to both corresponding pairs of CSOs by Theorem 5.101, and the c-linearity of the induced map follows from Lemma 5.51.

$$\begin{array}{ccc} \text{Hom}(V, V \otimes W) & \xrightarrow{Tr_{V;W}} & W \\ \text{Hom}(Id_V, [Id_V \otimes P_c]) \downarrow & & \downarrow P_c \\ \text{Hom}(V, V \otimes W_c) & \xrightarrow{Tr_{V;W_c}} & W_c \end{array}$$

■

EXAMPLE 5.103. Given  $\mathbf{U} = (U, J_U)$ ,  $\mathbf{V} = (V, J_V)$ ,  $\mathbf{W} = (W, J_W)$ , the canonical invertible map

$$q : \text{Hom}(V, \text{Hom}(U, W)) \rightarrow \text{Hom}(V \otimes U, W)$$

from Definition 1.39 is c-linear with respect to the three corresponding pairs of induced CSOs. Lemma 5.79 applies, so that the following induced maps are c-linear and invertible:

$$\begin{aligned} \text{Hom}_c(\mathbf{V}, \text{Hom}(\mathbf{U}, W)) &\rightarrow \text{Hom}(\mathbf{V} \otimes_c \mathbf{U}, W) \\ \text{Hom}(V, \text{Hom}_c(\mathbf{U}, \mathbf{W})) &\rightarrow \text{Hom}_c(V \otimes \mathbf{U}, \mathbf{W}) \\ \text{Hom}_c(\mathbf{V}, \text{Hom}(U, \mathbf{W})) &\rightarrow \text{Hom}_c(\mathbf{V} \otimes U, \mathbf{W}) \\ \text{Hom}_c(\mathbf{V}, \text{Hom}_c(\mathbf{U}, \mathbf{W})) &\rightarrow \text{Hom}_c(\mathbf{V} \otimes_c \mathbf{U}, \mathbf{W}). \end{aligned}$$

The direct sums for the domain  $\text{Hom}(V, \text{Hom}(U, W))$  are as in Example 5.77. The direct sums for the target space  $\text{Hom}(V \otimes U, W)$  are as in Example 5.78.

THEOREM 5.104. For finite-dimensional  $V$ , and  $U, W$  with CSOs  $J_U, J_W$ , the generalized trace

$$Tr_{V;U,W} : \text{Hom}(V \otimes U, V \otimes W) \rightarrow \text{Hom}(U, W)$$

is c-linear with respect to both pairs of corresponding commuting CSOs, and respects the direct sums, inducing a c-linear map, denoted

$$Tr_{V;\mathbf{U},\mathbf{W}} : \text{Hom}_c(V \otimes \mathbf{U}, V \otimes \mathbf{W}) \rightarrow \text{Hom}_c(\mathbf{U}, \mathbf{W}).$$

PROOF. The c-linearity claims follow from Theorem 2.26, and then Lemma 5.51 applies. That is enough for the Proof, but to see how the generalized trace is related to two different vector-valued traces, consider the following diagram,

$$\begin{array}{ccccccc} \text{Hom}(V \otimes U, V \otimes W) & & & & & & \\ \downarrow & \swarrow q & \xrightarrow{q^{-1}} & & \xrightarrow{Tr_{V;U,W}} & & \\ & M_{11} & \xrightleftharpoons[\text{Hom}(Id_V, n)]{\text{Hom}(Id_V, n^{-1})} & M_{12} & \xrightarrow{Tr_{V;\text{Hom}(U,W)}} & \text{Hom}(U, W) & \\ & \downarrow & & \downarrow & & \downarrow P_H & \\ \text{Hom}_c(V \otimes \mathbf{U}, V \otimes \mathbf{W}) & \xrightleftharpoons[q_1]{} & M_{21} & \xrightleftharpoons{} & M_{22} & \xrightarrow{Tr_{V;\text{Hom}_c(\mathbf{U},\mathbf{W})}} & \text{Hom}_c(\mathbf{U}, \mathbf{W}) \end{array}$$

where

$$\begin{aligned} M_{11} &= \text{Hom}(V, \text{Hom}(U, V \otimes W)) \\ M_{12} &= \text{Hom}(V, V \otimes \text{Hom}(U, W)) \\ M_{21} &= \text{Hom}(V, \text{Hom}_c(\mathbf{U}, V \otimes \mathbf{W})) \\ M_{22} &= \text{Hom}(V, V \otimes \text{Hom}_c(\mathbf{U}, \mathbf{W})). \end{aligned}$$

All the vertical arrows are the canonical projections of the direct sums produced by the commuting CSOs induced by  $J_U$  and  $J_W$ . The right square is commutative by Lemma 2.60; this is an example of Theorem 5.102, where the projection  $M_{12} \rightarrow M_{22}$  is equal to  $\text{Hom}(Id_V, [Id_V \otimes P_H])$ . The map

$$n : V \otimes \text{Hom}(U, W) \rightarrow \text{Hom}(U, V \otimes W),$$

is as in Notation 2.52 and Theorem 5.98;  $n$  is invertible and  $c$ -linear with respect to the commuting corresponding pairs of CSOs induced by  $J_U$  and  $J_W$ . Example 5.76 applies to  $\text{Hom}(Id_V, n)$  and the middle square in the diagram: the induced map (lower middle arrow) is invertible,  $c$ -linear, and equal to  $\text{Hom}(Id_V, n^1)$ , where  $n^1$  is the induced map from Theorem 5.98. The map  $q$  is as in Theorem 2.53, which asserts the commutativity of the diagram's top triangle. By Example 5.103, the map  $q$  similarly induces an invertible  $c$ -linear map,  $q_1$ . We can conclude, for  $K : V \rightarrow V \otimes \text{Hom}_c(\mathbf{U}, \mathbf{W})$ ,

$$Tr_{V; \text{Hom}_c(\mathbf{U}, \mathbf{W})}(K) = Tr_{V; \mathbf{U}, \mathbf{W}}(q_1(n^1 \circ K)).$$

■





## Bibliography

- [AF] F. ANDERSON and K. FULLER, *Rings and Categories of Modules*, GTM **13**, Springer-Verlag, New York, 1974. MR0417223 (54 #5281)
- [Bhatia] R. BHATIA, *Matrix Analysis*, GTM **169**, Springer-Verlag, New York, 1997. MR1477662 (98i:15003)
- [B] N. BOURBAKI, *Algebra I, Chapters 1–3*, Hermann, Paris, 1974. MR0354207 (50 #6689)
- [C] A. COFFMAN, *Real linear maps preserving some complex subspaces*, Beiträge zur Algebra und Geometrie (1) **56** (2015), 159–173. MR3305441
- [EPW] D. EISENBUD, S. POPESCU, and C. WALTER, *Lagrangian subbundles and codimension 3 subcanonical subschemes*, Duke Math. J. (3) **107** (2001), 427–467. MR1828297 (2002d:14075)
- [EHM] P. ELLIA, A. HIRSCHOWITZ, and L. MANIVEL, *Problème de Brill-Noether pour les fibrés de Steiner. Applications aux courbes gauches*, Ann. Sci. École Norm. Sup. (4) (no. 6) **32** (1999), 835–857. MR1717579 (2001g:14068)
- [FH] W. FULTON and J. HARRIS, *Representation Theory*, GTM **129**, Springer-Verlag, New York, 1991. MR1153249 (93a:20069)
- [GM] T. GARRITY and R. MIZNER, *Invariants of vector-valued bilinear and sesquilinear forms*, Linear Algebra Appl. **218** (1995), 225–237. MR1324060 (96d:15044)
- [Geroch] R. GEROCH, *Mathematical Physics*, University of Chicago, 1985. MR0824481 (87d:00002)
- [Graham] A. GRAHAM, *Kronecker Products and Matrix Calculus: with Applications*, Ellis Horwood Series, Halsted Press, New York, 1981. MR0640865 (83g:15001)
- [G<sub>1</sub>] W. GREUB, *Linear Algebra*, 3rd ed., GMW **97**, Springer-Verlag, New York, 1967. MR0224622 (37 #221)
- [G<sub>2</sub>] W. GREUB, *Multilinear Algebra*, GMW **136**, Springer-Verlag, New York, 1967. MR0224623 (37 #222)
- [H] L. HÖRMANDER, *An Introduction to Complex Analysis in Several Variables*, USHM Van Nostrand, Princeton, 1966. MR0203075 (34 #2933)
- [HJ] R. HORN and C. JOHNSON, *Topics in Matrix Analysis*, Cambridge, 1991. MR1091716 (92e:15003)
- [J] N. JACOBSON, *Basic Algebra I*, Freeman & Co., San Francisco, 1974. MR0356989 (50 #9457)
- [JSV] A. JOYAL, R. STREET, and D. VERITY, *Traced monoidal categories*, Math. Proc. Camb. Phil. Soc. (3) **119** (1996), 447–468. MR1357057 (96m:18014)
- [K] C. KASSEL, *Quantum Groups*, GTM **155**, Springer-Verlag, New York, 1995. MR1321145 (96e:17041)
- [LP] A. LASCOUX and P. PRAGACZ, *Schur  $Q$ -functions and degeneracy locus formulas for morphisms with symmetries*, in Recent Progress in Intersection Theory (Bologna, 1997), Trends Math., Birkhäuser, Boston, 2000, 239–263. MR1849297 (2002g:14004)
- [L] D. W. LEWIS, *Trace forms, Kronecker sums, and the shuffle matrix*, Linear and Multilinear Algebra **40** (1996), 221–227. MR1382078 (97h:15038)
- [MB] S. MAC LANE and G. BIRKHOFF, *Algebra*, 3rd ed, Chelsea, New York, 1993. MR0941522 (89i:00009)
- [Magnus] J. MAGNUS, *Linear Structures*, Griffin’s Statistical Monographs and Courses **42**, Oxford University Press, New York, 1988. MR0947343 (89j:15003)

- [Maltsiniotis] G. MALTSINIOTIS, *Traces dans les catégories monoïdales, dualité et catégories monoïdales fibrées*, Cahiers Topologie Géom. Différentielle Catég. (3) **36** (1995), 195–288. MR1352535 (97d:18007)
- [Neudecker] H. NEUDECKER, *Some theorems on matrix differentiation with special reference to Kronecker matrix products*, Journal of the American Statistical Association **64** (1969), 953–963.
- [Nissen] D. H. NISSEN, *A note on the variance of a matrix*, Econometrica (3 – 4) **36** (1968), 603–604.
- [O] I. OJEDA, *Kronecker square roots and the block vec matrix*, The American Mathematical Monthly (1) **122** (2015), 60–64. MR3324956
- [P] T. PATE, *Tensor products, symmetric products, and permanents of positive semidefinite Hermitian matrices*, Linear and Multilinear Algebra (1 – 4) **31** (1992), 27–36. MR1199038 (94a:15048)
- [PS] K. PONTO and M. SHULMAN, *Shadows and traces in bicategories*, J. Homotopy Relat. Struct. (2) **8** (2013), 151–200. MR3095324
- [Stolz-Teichner] S. STOLZ and P. TEICHNER, *Traces in monoidal categories*, Trans. Amer. Math. Soc. (8) **364** (2012), 4425–4464. MR2912459
- [Sukhov-Tumanov] A. SUKHOV and A. TUMANOV, *Regularization of almost complex structures and gluing holomorphic discs to tori*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) (no. 2) **10** (2011), 389–411. MR2856153 (2012m:32031)
- [VZ] C. VAFA, E. ZASLOW, et al, eds., *Mirror Symmetry*, Clay Math. Monog. **1**, AMS, Providence, 2003. MR2003030 (2004g:14042)