Representations
of a
Physical Universe

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Part 0

Prerequisite Material

Chapter 1

Old Notions Revisited

1.1 The Cartesian Coordinate System

One of the most important revolutions in mathematics and physics was ushered in by an idea of the seventeenth century mathematician and philosopher René Descartes. The idea was that any point P in the Euclidean plane could be represented by a pair of numbers (x, y). The numbers themselves represented distances along two perpendicular axes that met at a point $(0, 0)$, called the origin. By introducing this concept, he had done something amazing. He had related the *geometry* of the plane to the *algebra* of variables and equations. Algebra could be *represented* geometrically, and conversely geometric problems could be solved by going into the realm of algebra.

Insert 2D Plane with Coordinates

Coordinates soon became more than just pairs of numbers (x, y). Their use was extended to 3D space, and later to arbitrarily high dimensions. They would subsequently be used to lay the foundations for modern physics and mathematics. Linear algebra, multivariable calculus, and all the connections between algebra and geometry begin with the concept of a coordinate.

Since then, the use of coordinate systems has proven indispensable to physicists and mathematicians throughout history. Newton used Descarte's coordinate system to formulate his infinitesimal calculus. Maxwell used it to analyze electromagnetic fields, discovering mathematically that light is a wave in the electromagnetic field. Einstein, going further, made use of coordinate systems to formulate his theory of gravitation. Today, physicists and engineers do their calculations within the frame of coordinate systems. In mathematics, Descarte's idea planted the roots for what would turn into the modern field of algebraic geometry.

When studying a geometric phenomenon in some n-dimensional space, say $\mathbb{R}^n$, we pick an origin and axes to form our coordinate system. For a ball falling, we could set the origin at some point on the ground, and pick one axis parallel to the ground, and one perpendicular. We can decide to measure the axes in meters, or we could decide to do it in feet (nothing stops us from making bad choices). The physical point P where the ball lies is represented by $(x, y) = (0\text{ m}, 10\text{ m})$. The coordinate y is a natural choice of coordinate, as it corresponds to our intuitive notion of height.

Insert Ball Falling

We can now study y, free of geometry, as just a function which we can do arithmetic and calculus on. If we are given an equations of motion, say

$$\frac{d^2y}{dt^2} = -g, \qquad \frac{d^2x}{dt^2} = 0$$

with initial conditions,

$$\frac{dy}{dt} = 0, \qquad \frac{dx}{dt} = 0$$

then we can perform our well-known kinetic calculations for the system, and see how the system evolves in the *time* direction. **A recurring theme will be that dynamics of a system in n-dimensional space can be thought of just a special type of geometry in $n+1$ dimensional space, putting time as an added dimension**.

Because the purpose of this text is to study the ways in which geometry, algebra, and physics connect, it is worthwhile to dwell on the *philosophy* behind coordinate systems.

The ball will fall from 10 meters, according to the force of gravity. That is the way the world works. It doesn't matter what coordinate system we set up to do that calculation, we should get the *exact same result*. Plainly: nature doens't *care* what coordinate system we use. This fact, obvious as it may be, is worth thinking about: No matter what coordinate system we use, the equation of motion should give the same dynamics. The laws of physics should be *independent of any coordinate system.*

Newton's law $\mathbf{F} = m\mathbf{a}$ relates the force vector to the acceleration vector. The vector representing the force $\mathbf{F}$ that you apply on a surface is an object independent of coordinate system, and so is the resulting acceleration vector. The *components* of these vectors (F_x, F_y, F_z) and (a_x, a_y, a_z), however, depend on what you have chosen for the x, y, z axes. These components *represent* a real physical vector, but only once we pick a coordinate system. If we were to pick a different coordinate system, the numbers representing the vector would change.

When we write an equation describing a physical law, it should be valid regardless of the coordinate system we use. $\mathbf{F} = m\mathbf{a}$ will always be true whether we rotate our frame of reference or not. On the other hand, if Newton's law of motion *only* said that the *first* 'x' component of the Force was equal to the *first* 'x' component of the acceleration, and said nothing about the other 2 components, then in different coordinate systems since 'x' means different things, we would get totally different equations of motion. No physical law will ever say something just about the first or just about the second components of two vectors: it must equate the entirety of the two vectors.

As another example if the equation for work looked like $F_x dx = dW$, then would give different results in different coordinate systems, because it puts emphasis on just one of the three components (the first 'x' coordinate) over the others. While in some coordinate system dx may point in the direction of the displacement and be nonzero, there may be a different coordinate system where $dx = 0$, making the work done zero. So the equation for work would be coordinate dependent: it would be wrong. The need for such invariance is why the true formula uses all three spacial dimensions and looks like:

$$\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz = dW.$$

Although it isn't obvious yet that this is a quantity that is invariant regardless of the coordinate system used, at the very least it doesn't put one component above any of the others.

1.2 Linear Algebra & Coordinates

The traditional concept of a coordinate system, a series of perpendicular lines that together associate ordered tuples of numbers to each point in n-dimensional space, is not representative of all coordinate systems. For one, we do not need the requirement that the lines be perpendicular. Our coordinate system could instead look like this:

Graphic of non-perp lines and representing a point like that

In the language of linear algebra: once we choose an origin, choosing a set of coordinate axes is the same as choosing a basis for the space (a coordinate basis). For any point in space, we can relate coordinates x'_i in the new system in terms of coordinates x_i in the old system by matrix multiplication: $x'_i = \sum_{j=1}^{n} \mathbf{A}_{ij} x_j$. This is exactly what's called a change of basis in linear algebra. Transformations between coordinate bases are exactly the invertible **linear transformations** .

As in linear algebra, we need our coordinate system to both **span** the space so that we can represent any point, and be **linearly independent** so that every point that we can represent in our coordinate system will have a unique representation. That's all that a basis is: it specifies a good coordinate system.

Definition 1.1. *A set of vectors is said to span a space* $\mathbb{R}^n$ *if every point* P *can be represented as* $a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$

Definition 1.2. *A set of vectors* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is called linearly independent if there is only one way to represent the zero vector* $\mathbf{0}$ *as a combination of them, namely as* $\mathbf{0} = 0v_1 + \cdots + 0v_k$.

This second definition is the same as saying every point that we can represent in our system has a unique representation. Let's make this clear. If there were two ways to represent a point P: as

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

and

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

then subtracting these two different combinations would give a nonzero way to represent zero. Conversely, if there were a nonzero combination of vectors summing to zero, then we could add that combination to the coordinate representation of any point and get a *different* representation of the same point. So coordinate representations for all vectors are unique as long as long as there is only one representation for zero the one where each component equals zero.

Intuitively, linear independence means that there is no superfluous information in the set of vectors. We cannot linearly combine vectors in some subset to get another vector in the set; each vector is adding its own unique additional piece of information, making the set able to span in an additional direction.

Bases that don't span, or are not linearly independent, would lead to coordinate systems like these:

Show a 2-D basis in a 3-D space, and a basis of 3 vectors in 2-D space

Very often in mathematics, we ask "does a solution exist?", and "if there is a solution, is it unique?". These two questions are dual to one another. If a set of vectors spans the space, then there *exists* a way to represent any point (at least one way to represent any point). If a set of vectors is linearly independent, then *if* you can represent a point, that representation is *unique* (no more than one way to represent any point).

Now to stress the same idea again: because points in $\mathbb{R}^n$ and vectors are essentially the same thing, the idea that points in space are invariant of a coordinate system applies just as well to vectors. If we choose a basis for our vector space $\mathbf{v}_1, \dots \mathbf{v}_n$, then we can express any vector $\mathbf{u}$ by a unique combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. We then say that in this basis, we can represent $\mathbf{u}$ by a list of numbers. Often, it is written:

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

But in some sense, writing this as an equality is wrong. The vector $\mathbf{u}$ is something physical: a velocity, a force, the flow of water. It doesn't depend on the coordinate system. On the other hand, the right hand side is just a list of numbers that depend entirely on the coordinate system chosen. If we change coordinate systems, the right hand side changes. Because $\mathbf{u}$ exists (say, in the real world) independently of coordinates used, it does not change.

A geometric vector like u is *not* a list of numbers. Once we pick a basis, u can be *represented by* a list of numbers, but if we change into a different basis, those numbers all have to change as well. This exact same idea will be the reason why a tensor is *not* just a multi-dimensional array (like the ones encountered in computer science). It can be *represented by* a multi-dimensional array once a coordinate system is chosen, but the numbers in each entry will differ depending on the coordinate system we pick.

This is very confusing (and will also be part of the reason why it's so hard to understand tensors as an undergraduate). In most math courses, we can freely call any list of numbers a 'vector'. After all, you can add lists and scale them so they do form a 'vector space'. This is a really unfortunate linguistic degeneracy in mathematics terminology. The type of vectors that we see in physics (acceleration, force, electric field, etc.) are *geometric vectors* that have nothing *a priori* to do with lists of numbers until we represent them as such by using coordinate systems. On the other hand, abstract structures that we can add and multiply by scalars are *algebraic vectors*, and lists are an example of that. To avoid confusing lists of numbers with the geometric vectors in the physical world, we will call lists of numbers *tuples* rather than vectors.

So returning to the geometric vector $\mathbf{u}$, a more careful way to write it would be:

$$\mathbf{u} = \begin{pmatrix}\mathbf{v}_1 \dots \mathbf{v}_n\end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n. \tag{1.1}$$

Once we pick a basis, that column of coordinates means something. If we denote our basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ by B, then we will use the notation

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_B = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Let us do a very simple example to start. In the 2-D plane, say we have our original basis $\mathbf{v}_1, \mathbf{v}_2$ and we rotate it by $\pi/4$ radians to get a new basis. Say we have a point P whose coordinate representation was $\mathbf{v}_1 + \mathbf{v}_2$, or $\begin{pmatrix}1\\1\end{pmatrix}$ in the original basis.

Now our new basis is the old one rotated by $\pi/4$ so

$$\begin{aligned}\mathbf{v}_1' &= \frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2\\ \mathbf{v}_2' &= -\frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2.\end{aligned}$$

PUT GRAPHIC HERE

As a matrix transform, we can write this as[1]:

$$\begin{pmatrix}\mathbf{v}_1' & \mathbf{v}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$$

This relates the actual basis vectors themselves. On the other hand, if we wanted to see the *coordinates* representing $\mathbf{v}_1'$ and $\mathbf{v}_1'$, then in the new basis they would simply be represented in coordinates as:

$$\mathbf{v}_1' = \begin{pmatrix}1\\0\end{pmatrix}_{\text{new}},\ \mathbf{v}_2' = \begin{pmatrix}0\\1\end{pmatrix}_{\text{new}}$$

and in the old basis they'd be represented as:

$$\mathbf{v}_1' = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}},\ \mathbf{v}_2' = \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}}$$

If we know that we can describe a point as $\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}$ in the new basis, then we can easily get its description in the old basis as:

$$\begin{aligned}\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}} &= x\mathbf{v}_1' + y\mathbf{v}_2'\\ &= x\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} + y\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} \qquad (1.2)\\ &= \left(\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}\leftarrow\text{new}}\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}\right)_{\text{old}}\end{aligned}$$

[1]The tuple of basis vectors $\mathbf{v}_i$ is written as a row rather than a column to be consistent with Equation (1.1). Then the coordinates are represented in a column. Because of the way we do matrix multiplication, then the matrix acts on the right. It's an issue of styling and indexing, and not physically meaningful. If we were to write this coordinate transform using columns & not rows, we'd get a matrix that's the transpose of the one above, and matrix transposes would appear in subsequent equations, making them less tidy.

This is the same matrix that related the basis vectors. We'll call it $\mathbf{A}$. $\mathbf{A}$ takes the new coordinate representations (x, y) and tells us how they'd look like in the *old* basis.

So then it is the *inverse* $\mathbf{A}^{-1}$ that tells us how our old coordinate representations of a point P will look like in our new basis.

For our point P, represented as $(1, 1)$ in our original basis, in the new basis, we would have:

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\text{old}} = \left(\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\text{old}} \right)_{\text{new}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}_{\text{new}}$$

Indeed, $\mathbf{v}_1 + \mathbf{v}_2 = \sqrt{2}\mathbf{v}'_1 + 0\mathbf{v}'_2$.

That is the central idea. If we *vary* the <u>basis</u> $\mathbf{v}_i$ to a different basis, $\mathbf{v}'_i$, then the <u>coordinates</u> a'_i will vary the *other* way, so that the geometric vector

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = a'_1\mathbf{v}'_1 + \cdots + a'_n\mathbf{v}'_n$$

is *invariant* regardless of coordinate choice.

Let's make this precise in the general case. If we start with a set of basis vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and we make the linear transformation to a new basis $\{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ so that, as before:

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A} \tag{1.3}$$

then since the vector $\mathbf{u}$ should not change when we change our basis, we must have:

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \tag{1.4}$$

so that

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A}\mathbf{A}^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

as desired. **THIS WOULD BE A GOOD EXERCISE: PROVE IT HAS TO BE A INVERSE**

We say that the basis vectors $\mathbf{v}_i$ **co-vary** and the coordinates a_i **contra-vary** with the change of basis. The idea, although it sounds simple, is rather hard to get the feel of. It's worth thinking a good bit about how coordinates and bases need to vary in opposite ways so that the physical object represented by the coordinates stays the same regardless of how we look at it.

This will be a caption for a sketch of a 3-D rotation

When you rotate your character in a video game (and in real life too, by the way), the world rotates *contrary* to the direction that you've rotated in. That's because the coordinates of what you see have *contra-varied* while your basis vectors, given by the direction you face have *co-varied.* The end result is that despite changing your coordinate system, physics stays the same: invariant. The universe did not rotate itself just because you did. This extends beyond just rotations to *all* linear transformations point.

1.3 The Notion of Length on Vector Spaces

Let us consider the property of orthogonality. It's well known that for geometric vectors, there's more that we can do than just add, scale, and transform them: we can take dot products[2] between them. When we have two geometric vectors in space, their dot product is a well defined number. If it is zero, then the vectors are orthogonal to one another. From the dot product and the magnitudes, it is possible to calculate the angle between two given vectors.

When vectors are represented in terms of tuples of numbers, the dot product was taught to us as "multiply component by component, and then sum that up". This is not, in general, what the dot product really is. Consider a basis transformation as below:

$$\begin{aligned}\mathbf{v}_1' &= 2\mathbf{v}_1 + \mathbf{v}_2\\ \mathbf{v}_2' &= \mathbf{v}_1 + 2\mathbf{v}_2.\end{aligned}$$

It's easy to compute the inverse of this matrix and see how the new coordinates should work, but it worthwhile looking at this geometrically. It is not a rotation, but more of a "stretching". Notice that

[2]Of course in 3-D we can also take a cross product. This will be discussed in the following chapters.

while in our original perspective, if we viewed $\mathbf{v}_1$ and $\mathbf{v}_2$ as orthogonal vectors, in the *new* perspective, they are *no longer* orthogonal. This is very important: linear transformations in general do not preserve orthogonality.

Include graphic here

In particular, this linear transformation has *stretched* our vector space and changed the notion of distance. Even though rotations keep distances preserved, general linear transformations don't care about a notion of distance.

If we were to take the "dot product" $\mathbf{v}_1' \cdot \mathbf{v}_2'$ just by multiplying corresponding coordinates and summing them up, then in the new basis we'd get zero, but in the original basis we would *not*. This dot product actually changes depending on the coordinate system that we use! In some sense, this is expected: all we're doing is multiplying contravariant coordinates together and summing them up. The result should be contravariant as well (in fact doubly contravariant).

The failure of the dot product to be invariant is intimately related to the fact that transformations can change lengths. This should not be too surprising. After all, the length of a vector is defined by the square root of its dot product with itself. If the dot product we learned is not invariant under general coordinate transformations, what is the right way to measure length?

It is here that there is a big subtlety. A vector space on its own does not have a notion of length. We've just seen choosing different bases would give rise to different length scales and notions of "perpendicular" as well. Endowing a vector space with a way to universally tell what the length of a vector is, or whether two vectors are perpendicular is actually adding *extra structure* to the space. It picks a whole class of specific coordinate systems and says "these are the orthogonal reference frames; the others are skewed and stretched perspectives". This allows us to measure length using an invariant *inner product*.

Euclidean space, as well as the world in which we live in, both have an natural way to measure length between two points that is invariant of the coordinate system used. A vector space on its own does not, and so it is called an *affine space*. In affine space, although there are notions like "parallel", there is not a notion of distance. Adding an inner product to affine space gives rise to Euclidean space.

Insert graphic here, with an affine transformation of the

plane Caption something like "If Euclid's space allowed for not just rotations/translations but also general linear transformations like this, being perpendicular would be a coordinate-dependent idea. Note parallel lines, on the other hand, stay parallel, so affine space *does* have a notion of parallel"

This will be discussed in much greater detail in the following two chapters.

The takeaway from this discussion is that general linear transformations don't preserve distances. Because of this, the usual definition of the dot product gives different results depending on the choice of coordinate system. To make length an *invariant* regardless of coordinate choice, the definition of the inner product between two vectors needs to be appropriately modified. Without an *inner product* to measure distance, $\mathbb{R}^n$ is *not* Euclidean space $\mathbf{E}^n$, and is called affine space. Affine space looks the same regardless of the linear transformations we apply, while Euclidean space only looks the same when we rotate or translate our frame (because those are the two transformations that don't change lengths).

The reason we haven't encountered this problem so far in physics is simply because we always have worked in orthonormal frames given by an orthonormal basis like $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, where the inner product in fact *is* just the sum of the products of corresponding components. $\mathbb{R}^n$ without an inner product is affine, not Euclidean. Often however, when talking about Euclidean space, authors refer to it as $\mathbb{R}^n$ instead of $\mathbf{E}^n$ simply because $\mathbf{E}^n$ is modeled as $\mathbb{R}^n$ endowed with an inner product.

1.4 Nonlinear Coordinate Systems are Locally Linear

Perhaps you may be wondering why we've spent so much time on changing between coordinate systems represented by basis vectors centered at a fixed origin. Consider the change between cartesian and polar coordinates. What does this have to do with the linear changes of coordinates that we've been discussing?

We could use something like a polar system of (r, θ) or a spherical system (r, ϕ, θ). These coordinate systems are not representable in terms of axes, but instead look like this:

Graphic of polar coordinate system/spherical

This is an example of a non-linear coordinate transformation. They are more commonly referred to as **curvilinear**. Whereas linear ones map lines to lines, curvilinear ones more generally map lines to curves. The idea for making sure that the equations of physics still stay true for non-linear coordinate transformations is to note that just like a curve locally looks like a line, a *non-linear* transformation locally looks like a *linear* one. The linear transformation that it locally looks like is called the **Jacobian** J. If the laws of physics are invariant under linear transformations locally at each point, then *globally*, they will be invariant under non-linear ones as well. That is why we cared about studying covariance and contravariance for linear transformations: more complicated cases can be reduced to their local linear behavior.

As an example, consider going from a cartesian to a polar coordinate system. We have $x = r\cos\theta$ and $y = r\sin\theta$. Certainly, this is not a linear transformation of coordinates. There is sinusoidal dependence on θ in this transformation. Physics and geometry, however, do not have laws in terms of absolute coordinates (it doesn't make sense to say "That object is located at 50 meters") but only in terms of relative distances (you'd instead say "That object is located 50 meters *relative to* me"). It is the changes over relative distances between points that we care about, and these are obtained by integrating the *infinitesimal* changes at each point.

So although x, y do not depend linearly on theta, through the use of the chain rule, we have a local linear relationship in their infinitesimal changes:

$$dx = \cos\theta \ dr - r\sin\theta \ d\theta$$
$$dy = \sin\theta \ dr + r\cos\theta \ d\theta$$

At any given point, this relationship can be written as a linear change of basis.

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

So every nonlinear transformation from some coordinate system $x_1 \dots x_n$

to $x' \dots x'_n$ has the local linear transformation law:

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}_{\text{old}\leftarrow\text{new}} \begin{pmatrix} dx'_1 \\ \dots \\ dx'_n \end{pmatrix}_{\text{new}}$$

$$\Rightarrow \begin{pmatrix} dx'_1 \\ \dots \\ dx'_n \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}}$$

This is exactly the analogue of Equation (1.4), so indeed the changes in coordinates dx_i can be called *contravariant*, just like the coordinates were for the linear transformation case. All of this is just an extension of the principle of local linearity from calculus.

Similarly, we can express the new derivative operators in terms of the old ones by using the chain rule. For polar coordinates we have

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y} \end{aligned}$$

or more compactly we can relate just the differential operators themselves:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

so that more generally:

$$\begin{pmatrix} \frac{\partial}{\partial x'_1} \\ \vdots \\ \frac{\partial}{\partial x'_n} \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^{T}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}_{\text{old}} \tag{1.5}$$

If you look carefully, you will see that this accordingly mirrors the covariant change of vectors in Equation (1.3). So while the infinitesimal changes in the coordinates themselves are contravariant, just like the linear coordinates themseles, the *differential operators* corresponding to changes in these coordinates become *covariant*, just like the vectors $\mathbf{v}_i$ in the linear case. This is the first correspondence that will hint that our basis vectors $\mathbf{v}_i$ actually *correspond* to the differential operators $\frac{\partial}{\partial x_i}$.

Indeed, as we begin to move away from 3-D and n-D Euclidean space, we will see why the old notions of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are better viewed as the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$, and in general $\mathbf{v}_i \to \frac{\partial}{\partial x_i}$. This change of notation will allow us to easily pass onto far more general spaces than the Euclidean ones we've gotten used to.

1.5 Einstein's Summation Convention

Row tuples, column tuples, matrices representing basis transformations (old to new and new to old), co-variance and contra-variance. These ideas have constituted the entirety of this first chapter, and although hopefully they have not been too difficult conceptually, the matrix manipulation even at this early level is already a pain. We have to write everything in terms of tuples, and we need to arbitrarily decide which ones are rows and which ones are columns, and which matrices are transposed so that all the matrix multiplications make sense, as defined in linear algebra. A young physicist named Albert Einstein used a convention of writing all these equations so that we did not have to explicitly write out tuples of abstract basis vectors and coordinates.

The first step is to avoid explicitly writing out row tuples and column tuples. To do this, instead of writing out a whole tuple to represent a vector like $(v_1, \ldots, v_n)$ we will simply write v_i. v_i should be viewed as the whole vector, rather than a specific *ith* component of $\mathbf{v}$. The index i is *free* and can be anything.

The reason it is preferable to view v_i as the whole vector rather than a specific ith component of it is because of the same reason given at the end of Section 1.1, that we never care about just a specific component, but rather the vector as a whole.

The second step is to be able to differentiate between *covariant* quantities and contravariant quantities. The convention is this: if the quantity is covariant, like a basis vector, then write its index *downstairs*: $\mathbf{v}_i$. On the other hand, of a quantity is contravariant (like a component of a vector) then write its index *upstairs* as v^i

instead[3]. Then we can write Equation (1.1) as

$$\mathbf{u} = \sum_i a^i \mathbf{v}_i \tag{1.6}$$

For another example, the gradient operator was written $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and is now written simply as $\partial/\partial x^i$ (adopting upper indices for x^i since coordinates contra-vary). This means that Equation (1.5) can be written as

$$\frac{\partial}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}. \tag{1.7}$$

Since i is free to be anything (while j is bound since it is being summed over), this equation holds for every i in $1, \ldots, n$ and so is indeed an equation relating two *vectors* on both sides. Note also that this means the transformation matrix can be written as $\frac{\partial x^j}{\partial x'^i}$. By writing just a summation and not any explicit matrices, we avoid having to worry about unnecessary troubles like transposes, etc. Note also that this matrix has both upper and lower indices so that the upper index in $\frac{\partial x^j}{\partial x'^i}$ gets multiplied with the lower index $\frac{\partial}{\partial x^j}$ to cancel out, and all that remains is the lower index i. All the trouble of seeing what's covariant, what's contravariant, and what's invariant is washed away into just seeing how the upper indices and lower indices cancel out in the end.

Sometimes in the mathematics literature in places where co- and contra-variance are of lesser importance, upper and lower indices are ignored and the notation describing something like a matrix-vector multiplication looks like this:

$$(\mathbf{A}\mathbf{v})_i = \sum_j \mathbf{A}_{ij}\mathbf{v}_j.$$

That is, the ith component of the product $\mathbf{A}\mathbf{v}$ is the sum over j of the ijth component of $\mathbf{A}$ with the jth component of $\mathbf{v}$. For a matrix-matrix multiplication this would look like

$$(\mathbf{A}\mathbf{B})_{ik} = \sum_j \mathbf{A}_{ij}\mathbf{B}_{jk}.$$

[3]If you are worried that this will be confused with exponentiation, don't be. In practice, such confusion almost never arises.

Notice the pattern: when we wish to multiply these objects, we sum over a common index (j in the above equations) that we make both objects share. The point is *an index is always repeated and summed over* when we do a multiplication.

Einstein took the bold, but ultimately brilliant step of making the convention that if we ever write an index twice, that *automatically means* that we are summing over it (unless we explicitly say we aren't). The above equations, in Einstein's scheme, now become

$$\begin{aligned}(\mathbf{A}\vec{v})_i &= \mathbf{A}_{ij}\vec{v}_j \\ (\mathbf{AB})_{ik} &= \mathbf{A}_{ij}\mathbf{B}_{jk}.\end{aligned}$$

A dot product between $\mathbf{v}$ and itself would simply be $\mathbf{v}\cdot\mathbf{v} = v_i v_i$ in Einstein's convention. Note since i is an index that is summed over (and doesn't appear in the end result), we could just as well replace it by any symbol $v_i v_i = v_j v_j = v_{\ddot\smile} v_{\ddot\smile} = \mathbf{v}\cdot\mathbf{v}$.

Now returning back to co-variance and contra-variance, this scheme makes every equation in the chapter shockingly short. Equation (1.1) becomes $\mathbf{u} = a^i\mathbf{v}_i$, Equation (1.3) becomes $\mathbf{v}'_i = A^j_i\mathbf{v}_j$, Equation (1.4) becomes $a'_i = (A^{-1})^j_i a_j$. Equation (1.7) is further reduced to just

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}.$$

Similarly the transformation of infinitesimal coordinate changes is now just

$$dx'^i = \frac{\partial x'^i}{\partial x^j}dx^j.$$

and both of these are really just the chain rule plain and simple.

EXERCISE: Check invariance quickly using this method.

By doing exercises, this method is very quick and steady to get the hang of. We will adopt it for the rest of the book.

1.6 Exercises

Chapter 2

New Horizons Developed

2.1 The Manifold

Several thousand years ago, the first sentient human beings noticed that the landscape of the earth looked flat, and seemed to stretch out infinitely far in every direction. It is perhaps from this observation that the Euclidean plane was first conceived, and indeed it is from the fact that the earth looked like Euclid's 2D plane that geometry got its name to literally mean "measuring the earth". But the fact is that the earth is *not* a flat plane, stretching out infinitely. It turned out to be a sphere. What is true, however, is that *locally*, the geometry of the earth looks very similar to that of Euclidean space.

And now in modern times, as we look out into the cosmos and see them stretching out in every direction, our first human bias creeps in and tells us "this thing must be infinite, stretching out in every direction". Just as people thought the world was $\mathbb{R}^2$ in ancient times, in this age we entertain the thought that our universe could be three-dimensional Euclidean space $\mathbb{R}^3$. Indeed, most of the time when we do simple classical physics, we embed our system into a space that is $\mathbb{R}^3$ and work there. It is an easy space to work in.

But just as the earth's surface looked *locally* like Euclidean 2-space but in fact turned out to globally be wildly different, we should not be surprised if it turns out that the universe, despite locally looking Euclidean, has wildly different global structure.

This is exactly what a manifold intuitively is: an object that at each point locally resembles Euclidean space. The property of being locally Euclidean is similar to the property that differentiable functions have of being locally linear. It allows us to use calculus on them to reduce nonlinear objects to linear ones locally.

Concept 2.1. *A manifold M is a set of points which, in the neighborhood of every point, locally looks like euclidean space*

A line is a one-dimensional manifold (in fact it *is* a Euclidean space). The circle is a one-dimensional manifold locally resembling a line, and so are ellipses, parabolas, hyperbolas, and the graph of any smooth function. A sphere is the two dimensional manifold that ancient humans mistook for the Euclidean plane itself. The Mobius strip is also a two-dimensional manifold. Although globally it is a twisted band, locally it too looks like two-dimensional Euclidean space. Every geometric object referred to as a "curve" or a "surface" has been an example of a manifold this whole time.

In this chapter, we work towards building the language necessary to formally define what we mean by a manifold. First, we reinforce the intuitive ideas through examples.

2.2 Examples of Manifolds

Example 2.2. *The sphere is a two-dimensional manifold*

This is the classic example of a manifold that isn't just $\mathbb{R}^n$. As we have said before, at every point on the sphere, things look locally like $\mathbb{R}^2$. Throughout this text, the sphere will often be our first go-to setting when we want to take concepts from Euclidean space and generalize them to manifolds. It is easy to visualize things on this space, so often the first question to ask when generalizing something is "well, how would this thing look on the sphere?". Of course for this reason ellipsoids, paraboloids, and hyperboloids are all also manifolds. Similarly familiar objects like cylinders, tori, or tori with multiple holes are also manifolds.

Every point on the sphere looks exactly the same as any other. This is a highly symmetric manifold, that will later be referred to as a *homogenous space* because of this property.

Example 2.3. *Manifolds need not be connected. The disjoint union of two manifolds is also a manifold.*

Consider the set consisting of two separate spheres. Since each sphere individually is a manifold, then any point in this disjoint union belongs to one of the spheres, and so it has a locally euclidean neighborhood.

Example 2.4. *The graph of the curve $y = f(x)$ where f is a smooth function defines a one-dimensional manifold.*

Because f is smooth, we know that every point of the graph of the curve will locally look like its tangent line. Since the tangent line is precisely one-dimensional Euclidean space, every point on the curve looks locally Euclidean. Indeed, we only needed f to be differentiable.

Example 2.5. *The set of points forming the graph of $y = f(x_1, \ldots, x_n)$ defines an n-dimensional manifold when f is smooth.*

The exact same argument as before holds, except with a tangent line generalized to a tangent plane, etc. Locally at each point the set looks like Euclidean n-space. The idea of an n-dimensional "tangent *something*" at each point p on a manifold M that generalizes the notion of a tangent line or plane to higher dimensions, will be made precise in coming sections by talking about the *tangent space* of M at p, T_pM.

Example 2.6. *The figure-eight and the cone are not manifolds.*

DRAW IT

Both of these objects have a "cusp-like" point where multiple lines intersect. Zooming in near that point will preserve these cusps, and so the space does not look like Euclidean space near that point. There is no tangent space for the figure eight at the point of intersection that looks like a line: it would look like two lines intersecting. Similarly for the cone, there would not be a tangent space at the cusp that looks like a plane. At that point, the manifold looks like a continuum of intersecting lines. Intuitively, then, manifolds cannot have sharp cusps. As a result, the cube and triangle are not examples of manifolds.

The power of manifolds lies in the fact that not only are geometric objects manifolds, but so are many of the algebraic objects that we have been working with.

Example 2.7. *Consider a two-dimensional parallelogram, with opposite sides identified as the same. This is a manifold.*

DRAW THIS

This is like a room, where if you exit on one side, you come back on the other side. The one dimensional version of this is a circle, and we will show how this space can be thought of as a "product of circles" or a "circle of circles" in two dimensions. Clearly this parallelogram locally looks like euclidean two-space in the neighborhood of any point on the interior. The only possible problem is at the edges. Because each edge is identified with its opposite one, however, a neighborhood of a point at the edge of the parallelogram will simple wrap around to the other side, and look just as euclidean as the neighborhood around any other point.

Example 2.8. *The set of $n \times n$ matrices forms a real manifold of dimension n^2.*

Note that any $n \times n$ matrix can be equivalently viewed as a vector living in $\mathbb{R}^{n^2}$. Given any matrix, locally we can go in each of n^2 direction by appropriately varying one of the n^2 components of the matrix. So this manifold not only locally looks like $\mathbb{R}^{n^2}$ but can in fact be identified as being *the same* as $\mathbb{R}^{n^2}$.

Example 2.9. *The set of* invertible $n \times n$ *matrices forms a manifold of dimension* n^2

It is not obvious that this is a manifold. We need to know that near any point on this set M, we can go in each of n^2 linearly independent directions. The way we can see this is that invertible matrices are precisely those matrices A with *nonzero determinant*, $\det A \neq 0$. If I pick a given point A on this manifold corresponding to a matrix with nonzero determinant, then say it has determinant $d \neq 0$. Because the determinant is a *continuous* function, then changing any of the n^2 components of A by a small number δ will change the determinant by some small amount as well. We just need to pick δ small enough so that the resulting change has magnitude less than d, and therefore keeps the determinant away from 0. As long as we pick the neighborhood around A small enough, we still can vary in any of n^2 directions, and so it still *locally* looks like $\mathbb{R}^{n^2}$.

Example 2.10. *The set of rotations in Euclidean* 3*-space is a manifold.*

From mechanics and engineering, or just by playing with an object for a little bit, it is known that there are three independent ways to rotate something (about each of the three spacial axes). Any given rotation can be specified by three "Euler Angles" that describe how much to rotate about each axis, in a specific order. For a specific rotation given by these three angles, we intuitively expect that we can *perturb* this rotation in three independent ways by slightly changing one of the three angles. There would then be a three-dimensional neighborhood of the rotations *near* the original one. So locally, we look like Euclidean 3-space.

Proposition 2.11. *Manifolds are a powerful and useful idea in both engineering and applied mathematics.*

Oftentimes, we care about studying the possible states that a system can have, like the ways that an object can rotate, as in the prior example. This goes much beyond possible rotations, and can go as far as

AARON EXPLAIN WHY ITS USEFUL I DONT KNOW ANYTHING PRACTICAL

2.3 Elementary Topology

Before we can begin to do geometry on a space, we need to go even more fundamental still, into the real of topology. Geometry takes its name from the "study of the earth", while topology takes its name from the more abstract "study of places" (*topos*). Topology deals with ideas that, to a geometer or a physicist, border on the unphysical. In fact, it is likely easier to work to work with topological objects by relying on pure logical reasoning rather than geometrical intuition. Topology is in many ways an extension of set theory, and is a central tool in mathematical analysis of functions when defining and studying ideas like continuity.

Elementary topology begins with **point-set topology**, in which the central objects of study are the open and closed subsets of a main set X. This should in fact be a familiar idea from studying the real line $\mathbb{R}$, on which there is the **Euclidean Topology**.

On the real line, intervals of the form (a, b) consisting of all x so that $a < x < b$ were called **open intervals**. One of the interesting things about open intervals is that on an interval like (a, b) there is no *greatest* number in that set.

Proposition 2.12. *There is no maximum number on the open set* (a, b)

Proof. This will be a proof by contradiction. If there were a maximal number $x \in (a, b)$ then it must necessarily be *strictly* less than b (Note since b is not strictly less than itself, $b \notin (a, b)$). Therefore, since $x < b$ we can consider $(x + b)/2$. This average is strictly greater than x and less than b and so is in (a, b), contradicting the fact that x was the maximal element in (a, b). $\square$

By the same argument, there is no minimal element in (a, b). This set has the interesting property that for a point $x \in (a, b)$, there are points both to the right and left of x that are still in (a, b). Such subsets of $\mathbb{R}$ are called open

Definition 2.13. *A subset S of $\mathbb{R}$ is called open iff for any point $x \in S$, we can pick an ε sufficiently small so that the open interval $(x - \varepsilon, x + \varepsilon) \subseteq S$ as well.*

For example, any point in the interval $(-1, 3)$ satisfies this. For example 2.99 is contained in the interval $(2.99 - 0.005, 2.99 + 0.005)$ which is a subset of $(-1, 3)$. In other words, any point in an open set has that all the points sufficiently close to it are also in that set.

The notion of an "open interval" of size ε around a point x generalizes into higher dimensions by talking about an "open ball" of radius ε around a point p

$$B_\varepsilon(p) := \{q \in \mathbb{R}^n : |p - q| < \varepsilon\} \tag{2.1}$$

that is, the set of all points q within ε of p. So a subset S of $\mathbb{R}^n$ is called open iff every point $p \in S$ has a ball $B_\varepsilon(p)$ of some sufficiently small positive radius ε is contained in S.

So on the real line open intervals are open sets, and in fact any union of open intervals is still an open set (in fact an exercise will show that all open sets on the real line can be expressed as a (possibly infinite) union of open intervals). Intersections of open intervals are also open, but only when there are *finitely many* such intersections. On the other hand, infinite unions of open sets are still open:

Proposition 2.14. *In the Euclidean topology of $\mathbb{R}^n$, any arbitrary union $V = \bigcup_\alpha U_\alpha$ of open sets U_α is open*

Proof. Let p be a point in this union V, then since it belongs to the union, it must belong to at least one of the U_α. This means that some $B_\varepsilon(p) \subseteq U_\alpha$, implying $B_\varepsilon(p) \subseteq V$ as well. $\square$

Hopefully this proof illustrates something that is common in many topology proofs: working with simple logic works better than appealing to geometric intuition. Similarly

Proposition 2.15. *In the Euclidean topology, any* finite *intersection $V = \bigcap_\alpha U_\alpha$ of open sets is open.*

Proof. Let p be a point in the intersection V, then since it belongs to the intersection, it must belong to *all* of the U_α. For each U_α there a positive ε so that $B_\varepsilon(p) \subseteq U_\alpha$. Since the intersection is finite, pick the minimum such epsilon and it will still be positive. Moreover $B_\varepsilon(p)$ will then be contained in each U_α so will be contained in the intersection V as well. $\square$

It's clear to see that if the intersection were *not* finite, then the set of εs that we take the minimum over may give us a minimum value of zero. This is illustrated in an example as one of the exercises. We are now in a good place to define what we mean by a **topology** on a set.

Definition 2.16. *A* topology *on a set X is a family $\mathcal{T}$ of subsets of X so that*

- *Both the empty set $\emptyset$ and X are in $\mathcal{T}$*
- *Any finite/infinite union of sets in $\mathcal{T}$ is in $\mathcal{T}$*
- *Any* finite *intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$*

The elements of the topology $\mathcal{T}$ are called the **open sets** *of X. Complements of open sets are called* **closed sets***. X is then called a* **topological space***.*

Note that on $\mathbb{R}$, the closed intervals $[a, b]$ are the complements of the unions of open sets informally denoted by $(-\infty, a) \cup (b, \infty)$.

We can always arbitrarily define a topology and call "these sets, and all their arbitrary unions" open, but it is better to work with a natural topologies like the Euclidean one discussed above.

Talk about the algebra of open/closed sets?

Why do we care about open sets so much? They allow us to define a notion of continuity for functions between any topological spaces:

Definition 2.17 (Continuity)**.** *A function between two topological spaces*

$$f : X \to T$$

is **continuous at a point** $p \in X$ *iff for every open set* $V \subseteq T$ *containing* $f(p)$*, there is an open set* $U \subseteq X$ *so that* $f(U) \subseteq V$*.*

A function is **continuous** *if it is continuous at every point* $p \in X$*.*

DRAW THIS

On the Euclidean topology, this is exactly equivalent to the $\varepsilon - \delta$ definitions taught in calculus class. An exercise will sketch out a simple proof of this fact. This is one of the reasons that open sets are worth studying: it allows us to turn language using numerical epsilons and deltas into a "coordinate-free" language of maps between sets. Intuitively, a continuous function is one that doesn't locally mess with the space too much around any point.

Definition 2.18 (Homeomorphism)**.** *A function between two topological spaces*

$$f : X \to T$$

is a **homeomorphism** *iff it is bijective, with inverse*

$$f^{-1} : T \to X$$

so that both f *and* f^{-1} *are continuous.*

An example of a homeomorphism that is one of the most common ideas associated with topology is that something like a coffee cup is homeomorphic to a doughnut as a topological space.

ELABORATE HERE: COFFEE CUP = DOUGHNUT

In order to talk about manifolds *locally* looking like Euclidean space, we need to be able to have a rigorous way of talking about things *locally*. We want a good way of defining a *neighborhood* around a point p. Certainly an open ball containing p is a neighborhood of p because it contains "All the points near p". In the Euclidean topology, any open set U containing p also contains a ball of some radius around p, and thus contains a neighborhood of p. We could say that a neighborhood is any open set containing p, or more generally:

Definition 2.19 (Neighborhood)**.** *A* **neighborhood** *V of a point p is a subset of X that contains an open set containing p. That is $p \in U \subseteq V$.*

This way, there is no need for V to be open, but it does *contain* an open set containing all the points close to p.

Draw a sphere, with a patch.

Definition 2.20 (The Manifold)**.** *A* **manifold** *M of dimension n is a topological space such that for every point $p \in M$, there is a neighborhood U of p that is homeomorphic to an open subset of $\mathbb{R}^n$.*

This is exactly what we have been speaking about intuitively the whole time: for any point, a neighborhood around that point looks like Euclidean space. On a manifold, then, these neighborhoods are open sets that can be mapped in a one-to-one manner to open sets in $\mathbb{R}^n$. We use this notion to define coordinate charts:

Definition 2.21 (Coordinate Charts)**.** *A* **coordinate chart** (U, φ) *is an open set $U \subseteq M$ together with a "coordinate" map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism of U to and open subset of $\mathbb{R}^n$.*

The revolutionary idea of Descartes has been translated beyond Euclidean space, onto manifolds. Coordinate charts are vital to going from geometric data into algebraic calculations. They allow physicists to lay down a coordinate system that is valid for at least a *part* of the manifold, on which numerical calculations can be done. Because φ is one-to-one, we can parameterize the part of the manifold on U by n parameters.

Example 2.22 (Charts on the Sphere)**.** *The sphere has a* **stereographic projection** *to the plane that covers every point of the sphere except the north pole.*

FINISH THIS

We would like to be able to patch up the whole manifold with coordinate charts, so that we can work globally. Such a patch that covers the entire manifold is called an *atlas*.

Definition 2.23 (Atlas)**.** *An* **atlas** *on a manifold is a set of coordinate charts* $(U_\alpha, \varphi_\alpha)$ *whose union* $\bigcup_\alpha U_\alpha = M$*.*

Note that because each φ_α is a homeomorphism, meaning its inverse is also continuous, then if $U_\alpha \cap U_\beta \neq \emptyset$, we have $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a continuous function as well, from $\mathbb{R}^n$ to $\mathbb{R}^n$. These are the *transition maps* between coordinate patches.

A **smooth manifold** is one where the φ_α are not only continuous but in fact infinitely differentiable (i.e. **smooth**). This makes the transition maps smooth functions on $\mathbb{R}^n$. Essentially all the examples we deal with for the remainder of this book will be smooth manifolds.

2.4 Embeddings vs. Intrinsic Geometry

Talk about how we always picture manifolds as embedded into Euclidean space, but nothing about them *intrinsically demands it*

Mention any n-dimensional manifold can be embedded in $\mathbb{R}^{2n}$

Talk about how the torus is exactly the space in Example 2.7

2.5 Vectors Reimagined

Let us go back to $\mathbb{R}^3$. Studying the point-set topology of $\mathbb{R}^3$ as we have been doing for manifolds is a very easy thing to do, and isn't so rewarding. The reason we do it is so we can apply it to studying the *functions* on $\mathbb{R}^3$. A function on $\mathbb{R}^3$ takes three a tuple of three real inputs (x, y, z) and outputs a real number $f(x, y, z)$.

As you should be aware of by now, (x, y, z) is not a point, but instead a *coordinate representation* of some point. The manifold of

Euclidean 3-space $\mathbf{E}^3$ can be modelled by $\mathbb{R}^3$ once a coordinate representation is chosen. This distinction is so slight that it is barely noted, but it is worth noting. A function $f : \mathbf{E}^3 \to \mathbb{R}$ in fact takes a point $p \in \mathbf{E}^3$ and outputs a real number $f(p)$.

We also study vector fields on Euclidean space. Before thinking too hard about coordinate co-variance and contra-variance, we would just pick an orthogonal reference frame, and label an orthonormal basis by three vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Then, the form of a vector field looked like

$$\mathbf{F} = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

and again, (x, y, z) is in fact a coordinate representation of the invariant point $p = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

In higher dimensions, a function (i.e. a scalar field) could be specified in terms of a coordinate representation by $f(x_1, \ldots, x_n)$. Usually, it is customary to replace $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ by the notation $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n$ to denote an orthonormal frame in n dimensions, just so that we do not run out of alphabet characters.

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}^i(x^1, \ldots, x^n)\ \hat{\mathbf{e}}_i = \mathbf{F}^i(x^j)\ \hat{\mathbf{e}}_i$$

where we are now beginning to use Einstein summation convention. Of course, we do not *need* an orthonormal frame to describe a vector. We can use any basis $\mathbf{e}_i$ and write $p = x^i\mathbf{e}_i$, $\mathbf{F}(p) = \mathbf{F}^i(p)\mathbf{e}_i$.

The first important thing to note is that a scalar field is an invariant, *physical* thing on $\mathbb{R}^n$ (and on manifolds in general). When we talk about the temperature at each point in space, or classically when we talk about the energy density of the electric field at each point in space, that means there is a specific invariant number at each point. It doesn't matter whether we set up coordinates x^i on the space or not. The temperature at a point does not depend on what coordinate system we represent that point in.

Similarly, a *vector field* is also an invariant physical thing! The components $\mathbf{F}^i$ of the vector field, of course, *will* depend on the coordinate system we use, but the field $\mathbf{F}$ itself will *not*. Just like the wind doesn't change its motion across the earth when we change the coordinate system we use for measuring it, $\mathbf{F} = \mathbf{F}^i\mathbf{e}_i$ will remain invariant at every point. Only the components $\mathbf{F}^i$ will contra-vary against our co-variant $\mathbf{e}_i$.

Using these ideas from $\mathbb{R}^n$, consider a manifold M. A scalar function on M is no difficult thing to define: it associates to each point $p \in M$ a value $f(p)$. Because we can form an atlas of charts $(U_\alpha, \varphi_\alpha)$, we can locally form a function $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ on Euclidean $\mathbb{R}^n$. Then we can say f is continuous/differentiable/smooth if every $f \circ \varphi_\alpha^{-1}$ is.

The φ_α^{-1} allow us to take a tuple of n coordinates $q^1, \ldots, q^n$ on $\mathbb{R}^n$ and associate that to a unique point on M that is part of U. So we see how a scalar field $f : M \to R$ can descend to a function on the local coordinates.

Show a graph of the curves for coordinates q_i

The question now is what about a *vector field* $\mathbf{F}$ on our manifold M? What does this thing mean? We want to associate to each point on M a vector at that point. Note that this doesn't mean we want to associate to each point on M just some tuple in $\mathbb{R}^n$, because a vector is *not* just a tuple of numbers (that's coordinate dependent). We want to associate a physical vector at that point.

Intuitively it should make sense what we mean by this:

PICTURE OF A BALL WITH A VECTOR FIELD ON IT

In Section 1.4, we showed how when adopting polar coordinates, the corresponding vectors $\hat{\mathbf{r}}, \hat{\theta}$ change direction depending on the point p we're describing. Each of them points in the direction of an increase in the corresponding coordinate. So for a general curvilinear coordinate system on a manifold q^i, the associated vectors $\mathbf{e}_i$ would be tangent to the paths traced out by varying q_i.

The vectors e_i are tangents to the curves traced out by letting each q^i vary (i.e. $\phi^{-1}(q^i)$).

At each point $p \in M$, the local tangent vector $\mathbf{e}_i$ corresponding to coordinate q^i is associated with increasing q^i while holding all other coordinates fixed. This already gives us enough information to know one thing: the rate of change of a scalar function f along $\mathbf{e}_i$. It is exactly the partial derivative with respect to q^i

$$D_{\mathbf{e}_i} = \frac{\partial f}{\partial q^i} \tag{2.2}$$

We already know from Section 1.5 that partial derivative operators with respect to coordinates are *co*-variant, just like the vectors $\mathbf{e}_i$ themselves. So for any coordinate system q^i, we have a set of differential operators $\partial/\partial q^i$ corresponding to directional derivatives along the basis vectors $\mathbf{e}_i$. So we have a correspondence between coordinate frames $\mathbf{e}_i$ at a point and the differential operators associated with their coordinate system q^i.

$$\text{Covariant Quantities:} \qquad\qquad \mathbf{e}_i \longleftrightarrow \frac{\partial}{\partial q^i}$$

The set of vectors at p correspond to all the possible directions that paths on M can pass through p. If we have a path $\gamma \in M$ parameterized by t so that $\gamma(t_0) = p$ then we can compute derivative along the direction $\mathbf{v}$ tangent to γ as

$$D_{\mathbf{v}} f = \frac{d}{dt}\left[f\left(\gamma(t)\right)\right] \tag{2.3}$$

So although even though we don't have an idea of how to represent $\mathbf{v}$ in terms of our coordinate system q^i, we know how to represent its directional derivative operator. That is just:

$$D_{\mathbf{v}} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} \tag{2.4}$$

where dq^i/dt is the change in $q^i = \varphi^i(\gamma(t))$.

$$\frac{dq^i}{dt} = \frac{d}{dt}\left[\varphi^i(\gamma(t))\right] \tag{2.5}$$

This gives us the "component" v^i associated with $\partial/\partial q^i$ for our directional derivative in our coordinate basis:

$$D_{\mathbf{v}} = \frac{d}{dt}\left[\varphi^i(\gamma(t))\right]\frac{\partial}{\partial q^i} = v^i\frac{\partial}{\partial q^i} \tag{2.6}$$

But this is a correspondence between two physical, invariant objects associated with a direction:

$$\text{Tangent Vectors at } p \rightleftharpoons \text{Directional Derivatives at } p$$

$$\mathbf{v} = v^i\mathbf{e}_i \longleftrightarrow D_{\mathbf{v}} = v^i\frac{\partial}{\partial q^i}$$

Independent of the coordinates q^i that we use, the differential operator $D_{\mathbf{v}}$ will give the same invariant value when acting on a physical scalar field f, namely $v^i \frac{\partial f}{\partial q^i}$. So this differential operator is an invariant just like its corresponding vector $\mathbf{v}$. Moreover, in the case of Euclidean space, the components v^i of the directional derivative operator are exactly the same as the components of $\mathbf{v}$ itself:

$$D_{\mathbf{v}} = \mathbf{v} \cdot \nabla = v^i \frac{\partial}{\partial x^i} \tag{2.7}$$

Just like with vector addition, the sum $\mathbf{u} + \mathbf{v}$ corresponds to the operator $D_{\mathbf{u}+\mathbf{v}} = D_{\mathbf{u}} + D_{\mathbf{v}}$, and a multiple $c\mathbf{u}$ gives the multiple of the original derivative $D_{c\mathbf{u}} = cD_{\mathbf{u}}$. Directional derivatives form a vector space in one-to-one correspondence with the vectors at p. For this reason we say that these directional derivatives *are* the vectors at p. Vectors, at their core, represent flows along direction, which is no different from what a directional derivative operator along that point represents. We will sometimes abbreviate it as $v^i \partial_i$.

This leads us to define the **tangent space** of the vectors at a point p of a manifold M.

Definition 2.24 (Tangent Space)**.** *The tangent space at a point p of a manifold M, T_pM is the set of first order derivative operators at p.*

Going back to $\mathbb{R}^n$, this means that this whole time we could have treated $\hat{\mathbf{i}}$ as $\partial/\partial x$, $\hat{\mathbf{j}}$ as $\partial/\partial y$, etc. Using this idea is powerful, because now just from having a coordinate patch, q^i on a manifold, we obtain the full set of tangent vectors $\partial/\partial q^i$ at each point on the manifold.

The fundamental observation is that vectors are in one-to-one correspondence with the *first order behaviour* of curves passing through p (i.e. the instantaneous velocity along that curve). This in turn corresponds to first-order derivative operators at p (corresponding to moving along the curve's direction instantaneously). Vectors can be viewed as equivalence classes of curves through p, where two curves γ_1, γ_2 are equivalent if they share the same tangent at p.

The guiding philosophy of this section is that everything vectors represent: namely direction and magnitude, can be derived from the way that we compute changes in a scalar function via directional derivatives. From multivariable calculus, it is easy to see that the first order change in a function along a curve is the sum of the changes due to each coordinate:

$$\frac{df}{dt} = \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} \tag{2.8}$$

Given a function f and given a specific direction of first order changes dq^i/dt, this gives a *real, invariant* quantity. Note that this quantity is dependent on two objects: the *function* that we are differentiating and the *direction* along which we differentiate.

When we focus a specific direction along a curve: $dq^i/dt = v^i$, and allow the function to be arbitrary we get a vector corresponding to change along our path:

$$\mathbf{v} = v^i\partial_i \tag{2.9}$$

This needs to be *fed* a <u>function</u> f to give a real number value.

On the other hand, if we pick a specific *function* with first order behaviour given by $\partial_i f = \omega_i$ and allow for the *direction* to be arbitrary, we get something else

$$df = \omega_i dq^i \tag{2.10}$$

This needs to be *fed* a <u>direction</u> dq^i to give an associated real value. We will call this new object a **covector** or a **1-form**, and it is often denoted by the greek letter ω. It is an entirely different object from a vector. Where a vector represents a direction that we can differentiate functions alone, this represents a *function differential* which we can take changes of, along a direction. It is the dual to the notion of a vector, but behaves in very much the same way.

The sum of two one-forms is still a one form, as are scalar multiples, so the one forms at a point p form a vector space called the **cotangent space** at p.

Definition 2.25 (Cotangent Space)**.** *The cotangent space at a point p of a manifold M, T_p^*M is the set of all first order differentials at p.*

Vectors were defined in terms of curves: they represent the first order approximations to curves (namely their tangents, and the derivatives along them). 1-Forms are defined in terms of *functions*: they represent the first order approximations of functions (namely their differentials). At first glance, you may think that for a dimension n

manifold, there are only n directions to go in, whereas we have seemingly infinite freedom in picking the functions passing through p so there are many more 1-forms than there are vectors, but this isn't true.

Just as there are a huge number of functions that we can define at p, there are a huge number of curves that go through p, but because we only care about <u>first order behaviour</u> at p for the curves, many curves give the *same* tangent vector. Similarly, many functions will give the *same* 1-form, if they have the same <u>first order behaviour</u>. It should start to become clear that there is duality between these two ideas

Concept 2.26. *The first-order behaviour of curves at a point p (i.e. vectors) is dual (in a sense that will be discussed in the next chapter) to the first-order behaviour of the functions at p. On a manifold of dimension n, both of these are vector spaces of dimension n.*

Vectors	1-Forms
First-order approximations to <u>curves</u>	First-order approximations to <u>functions</u>
To differentiate <u>functions</u> along	To be integrated along <u>curves</u>
Contravariant components	Covariant components
Space of dimension n	Space of dimension n

This is all well and good, but how can we *visualize* these two concepts. Visualizing vectors is easy, we've been doing it since calculus class, if not before. They are "arrows", tangent to curves, showing the first-order behaviour of that curve. 1-forms on the other hand are a different story. If we pick a point p in 2D, then the first order behavior of a function can be visualized by adding in a dimension and showing the local tangent plane approximating that function. Alternatively, we can stay in 2-D and just show the local *level curves* (these are lines for tangent plane).

GRAPHIC HERE, VECTOR ON THE LEFT, FORM ON THE RIGHT

In 3D vectors would look the same, and forms could be visualized by the local level-planes of the function at p.

Exercise on constructing a dual basis of forms to a vector

2.6 What Follows

Tentative:

The rest of this book will expand both on the geometry of fields and manifolds, and also on the larger ideas of groups, homogenous spaces, and representations.

In Chapter 3, we will continue studying the fields that live on manifolds. We'll prove the General Stokes' theorem, an elegant generalization of the divergence, curl, and line integral theorems that have been taught in multivariable calculus. From there, we will study more thoroughly the concept of distance on a vector space and on a manifold using a metric, and how this relates vector fields to differential forms. The notion of a derivative will be extended to manifolds, and will take the form of a "Lie Derivative".

In Chapter 4, we will introduce Fourier Analysis as a powerful tool for studying functions on the real line and in Euclidean space. We'll see how the set of functions on a manifold naturally forms a vector space (of infinite dimension) and consider the Fourier Transform as a change of basis.

In Chapter 5, we will shift to looking at the representation theory of *finite* groups and illustrate the parallels. We will then return to the study of continuous group actions on especially symmetric "homogenous" spaces, and show how Fourier analysis is related to their representation theory. Towards the end, we will expand on the idea behind a group actions on manifolds and look at the representation theory, giving a small glimpse into harmonic analysis: the Fourier transform on manifolds. Just as in the first chapter, we'll recognize the importance of the underlying differential geometry of the group action. The underlying differential structure is known as the "Lie Algebra" of the group, and we will discuss that.

In Chapter 6, we introduce some background behind Lie Algebras. We put almost all of our focus on one special case: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The relationship between this algebra and the symmetries of the sphere are explored, as well as its applications in quantum physics for studying angular momentum. The representation theory of a variant of this algebra gives rise to the concept of spin.

In Chapter 7, we move further into physics, going over classical Lagrangian and Hamiltonian Mechanics. We discuss Noether's theorem in both the Lagrangian and Hamiltonian Pictures, and then we move to study Hamiltonian mechanics using the language of differential geometry that we have developed. This will give rise to *symplectic geometry.* In chapter 7, combining this with representation theory gives rise to *quantum mechanics.*

In Chapter 8, we apply differential geometry first to the study of electromagnetism, and then to gravitation. We shall arrive at Einstein's theory of gravity. Along the way, we study in even greater detail the notion of a metric, a connection, and curvature.

In Chapter 9, we use the representation theory and differential geometry that we have developed so far to study how quantum mechanics can arise from quantizing a symplectic manifold.

Finally, Chapter 10 studies Lie algebras in greater detail, working towards the *classification of complex semisimple Lie algebras.* Along the way, we will look at the relationship between representation theory of Lie algebras and modern physics.

2.7 Exercises

Part 1

A Better Language

Chapter 3

Differential Geometry

In calculus class, the fundamental theorem of calculus is introduced: that the total difference of a function's value at the end of an interval from its value at the beginning is the sum of the infinitesimal changes in the function over the points of the interval:

$$\int_a^b f'(x)dx = f\Big|_a^b \tag{3.1}$$

And later, in multivariable calculus, more elaborate integral formulae are encountered, such as the divergence theorem of Gauss:

$$\int_\Omega \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S} \tag{3.2}$$

where Ω is the volume of a 3D region we are integrating over, with infinitesimal volume element dV and S is the surface that forms the boundary of Ω. dS then represents an infinitesimal parallelogram through which $\mathbf{F}$ is flowing out, giving the flux integral on the right. Read in English, Gauss' divergence theorem says "Summing up the infinitesimal flux over every volume element of the region is the same as calculating the total flux coming out of the region". The total flux coming out of a region is the sum of its parts over the region. You might see that in English, this reads very similar to the description of the fundamental theorem of calculus.

Alongside this, there is Stokes' theorem for a 2D region. In English: summing up the infinitesimal amount of circulation of a vector field $\mathbf{F}$ over every infinitesimal area is equal to calculating the total circulation of $\mathbf{F}$ around the boundary of the region. In mathematical language:

$$\int_R \nabla \times \mathbf{F}\, dA = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{3.3}$$

where R is our region and C is its boundary.

Perhaps now, the pattern is more evident. In all the above cases, summing up some *differential* of the function on the interior of some region is the same as summing up the function itself at the *boundary* of the region. All these theorems, that on their own look so strange to a first-year calculus student, are part of a much more general statement, the **General Stokes' Theorem**:

Theorem 3.1 (General Stokes' Theorem)**.**

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{3.4}$$

Above, ω is an object that will generalize both the "functions" and "vector fields" that you've seen in multivariable calculus, and d will generalize of all the differential operators (gradient, divergence, curl) that you've dealt with. Lastly, when Ω is the region in question $\partial\Omega$ represents the *boundary* of the region Ω. The fact that it looks like a derivative symbol is no coincidence, as we'll see that the natural way to define the "derivative" of a region is as its boundary.

Through abstraction, we can reach results like this that not only look elegant and beautiful, but also provide us with insight into the natural way to view the objects that we've been working with for centuries. This gives us not only understanding of what language to use when studying mathematics, but also what is the natural language in which to describe the natural world. The general Stokes' theorem is one of the first examples of this beautiful phenomenon, and this book will work to illustrate many more.

For the first half of this chapter, we will work towards giving the intuition behind this result. On our way, we will begin to slowly move into a much more general setting, beyond the 3-dimensional world in which most of multivariable calculus was taught. That doesn't just mean we'll be going into n-dimensional space. We'll move outside of euclidean spaces that look like $\mathbb{R}^n$, into non-euclidean geometries. This will put into question what we really mean by the familiar concepts of "vector", "derivative", and "distance" as the bias towards Euclidean geometry no longer remains central in our minds. At its worst, the introduction of new concepts and notation will seem confusing and even unnecessary. At its best, it will open your mind away from the biases you've gained from growing up in a euclidean-looking world, and give you a glimpse of how modern mathematics *actually* looks.

Modern mathematics is learning that the earth isn't flat. To someone who's never had those thoughts, it is difficult to get used to, tiring, and sometimes even rage inducing, but to someone who has spent months thinking and reflecting on it, it quickly becomes second nature. Far from being the study of numbers or circles, it is a systematic climb towards abstraction. It is a struggle towards creating one language, free from all-encompassing human bias, in order to try and describe a world that all other human languages, for so many centuries, have failed to grasp. It is humbling, and in the strangest of ways, it is profoundly beautiful.

3.1 The Derivative and the Boundary

Let's start working towards understanding Equation (3.4). First, let's work with what we've already seen to try and explore the relation between integrating within a region and integrating on the boundary.

If we are in one dimension, we have a function f defined on the interval $x \in [a, b]$. Proving Equation (3.1) is much easier than you'd think. Let's take a bunch of steps: $x_i = a + (b-a)i/N$, so that $x_0 = a, x_N = b$. Then all we need is to form the telescoping sum:

$$\begin{aligned} f|_a^b &= f(x_N) - f(x_0) \\ &= \sum_{i=1}^{N} f(x_i) - f(x_{i-1}). \end{aligned}$$

If we make the number of steps N large enough, the stepsize shrinks so that in the limit, we get

$$\begin{aligned} \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i) - f(x_{i-1}) &= \lim_{N\to\infty} \sum_{i=1}^{N} \Delta f \\ &= \int_a^b df. \end{aligned}$$

Of course, the way its written more often is:

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{\Delta f}{\Delta x}\Delta x=\int_a^b\frac{df}{dx}dx.$$

What is the idea of what we've done? At each point we've taken a difference of f at that point with f at the preceding one. Because we're summing over all points, the sum of differences between neighboring points will lead to cancellation everywhere *except* at the boundary, where there will not be further neighbors to cancel out the $f(b)$ and $f(a)$. From this, we get Equation (3.1).

Note: Now for a distinction which may seem like it isn't important. We haven't integrated from point a to point b. We have integrated from where the coordinate x take *value* a, to the where coordinate x takes *value* b. a and b are *NOT* points. They are numbers, values for our coordinate x. As we have said in the preceding chapter, the idea that numbers form a *representation* for points is ingenius, but numbers are *not* points. Although we could write this interval as $[a, b]$ in terms of some variable x, it would be a completely different interval should we have chosen a different coordinate u. This is why, when doing u-substitution, we change the bounds. In coordinate free, language, then:

Theorem 3.2 (Fundamental Theorem of Calculus)**.** *For a given interval I with endpoints p_0, p_1 and a smooth function f, we have*

$$\int_{p_0}^{p_1} df = f\Big|_{p_0}^{p_1} \tag{3.5}$$

Notice something: the end result doesn't depend on the partition x_i at all, so long as it becomes infinitesimal as $N\to\infty$. That is to say: we are summing up the change of f over some interval, but it doesn't matter what coordinate system we use to describe this interval. The integral is *coordinate independent.* We chose to use x as our coordinate, describing the interval as going from $x = a$ to $x = b$, but we didn't *have* to make this specific choice. This makes perfect physical sense. For example, if we had a temperature at each point in space, the temperature difference between two fixed points some shouldn't depend on whether we use meters or feet to measure their distance apart.

Written mathematically:

$$\int_I df = \int_I \frac{df}{dx} dx = \int_I \frac{df}{du} du$$

If we chose an I that's very small around some point, essentially an infinitesimal line segment, we get:

$$\frac{df}{dx} dx = \frac{df}{du} du \Rightarrow \frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

this is the u-substitution rule from calculus.

Now what if f was a function defined not on the real line $\mathbb{R}$, but on 2-dimensional space $\mathbb{R}^2$, or more generally n-dimensional space $\mathbb{R}^n$. To each point $p = (p_1, \ldots, p_n)$ we associate $f(p)$. Now again, consider $f(p_f) - f(p_i)$ for two points in this space.

For any curve C going between p_i and p_f, say defined by $\mathbf{r}(t)$ for t a real number going from a to b, we can make the same partition $t_i = a + (b-a)i/N$ and let N get large. Again, it becomes a telescoping sum:

$$\begin{aligned} f(p_f) - f(p_i) =& f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ =& \sum_{i=1}^{N} f(\mathbf{r}(t_i)) - f(\mathbf{r}(t_{i-1})) \\ =& \sum_{i=1}^{N} \Delta f_i \to \int_C df. \end{aligned}$$

Now if we cared about coordinates, we could ask "how can we write df in terms of dt or dx_i?".

We know from the multivariable chain rule that the infinitesimal change of f is the sum of the change in f due to every individual variable, so:

$$df = \sum_i \frac{df}{dx_i} dx_i \tag{3.6}$$

We know that dx_i together must lie along C. In terms of t since $x_i = r_i(t)$, we have $dx_i = \frac{dr_i}{dt} dt$ giving:

Theorem 3.3 (Fundamental Theorem of Line Integrals)**.** *For a smooth function f defined on a piecewise-smooth curve C parameterized by* $\mathbf{r}(t)$

$$f|_{p_i}^{p_f} = \int_C \sum_i \frac{df}{dx_i}\frac{dr_i}{dt}dt = \int_C \nabla f \cdot \frac{d\mathbf{r}}{dt}dt = \int_C \nabla f \cdot d\mathbf{r} \tag{3.7}$$

The proof of this was no different from the 1-D case.

Let's go further. Consider a region in three dimensions and a vector field

$$\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

We want to relate the total flux coming out of the region to the infinitesimal flux at each point inside the region. To do this, as before, we will subdivide the region. This time, it will not be into a series of intervals, but instead into a mesh of increasingly small *cubes*, as below.

PUT A GRAPHIC HERE

See that the flux out a side of each cube is cancelled out by the corresponding side on its neighboring cube. That means that the only sides that do not cancel are for the cubes at the boundary[11], giving us the desired flux out.

So if we sum the fluxes over all infinitesimal cubes, we will get the total flux out of the boundary. For a single cube of sides dx, dy, dz, drawn below, the total flux will be the sum over each side.

$$\begin{aligned}\text{Flux} = \quad & F_z(x,y,z+dz/2)dxdy - F_z(x,y,z-dz/2)dxdy \\ & + F_y(x,y+dy/2,z)dxdz - F_y(x,y-dy/2,z)dxdz \\ & + F_x(x+dx/2,y,z)dydz - F_x(x-dx/2,y,z)dydz\end{aligned}$$

SHOW GRAPHIC HERE

But we can write this as:

$$\left(\frac{\partial F_x(x,y,z)}{\partial x} + \frac{\partial F_y(x,y,z)}{\partial y} + \frac{\partial F_z(x,y,z)}{\partial z}\right) dxdydz = \nabla \cdot \mathbf{F}\ dV$$

[1]You may be worried that the cubes do not perfectly fit into the boundary when it is not rectangular. As the mesh gets smaller and smaller, it approximates the region better so this does not pose a problem. This idea can be made rigorous (c.f. **GIVE A REFERNCE HERE**)

So the total flux will be the sum over all these cubes of each of their total fluxes. But then this becomes exactly the divergence theorem:

Theorem 3.4 (Divergence Theorem, Gauss)**.** *For a smooth vector field* $\mathbf{F}$ *defined on a piecewise-smooth region* Ω*, then we can relate*

$$\int_\Omega \nabla \cdot \mathbf{F}\, dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

It is an easy **exercise** to show that this exact same argument holds for an n-cube.

What did we do? In the fundamental theorem of calculus/line integrals, we had a function f evaluated on the 1-D boundary, and we chopped the curve into little pieces that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece, and found that it was $df = f'(x_i)dx$, meaning that the evaluation on the boundary could be expressed as an integral of this differential quantity over the curve. That is Equation (3.1).

For the divergence theorem, we had a vector field $\mathbf{F}$, again *evaluated on the boundary*, this time in the form of a surface integral. We chopped the region into little pieces (cubes now) that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece and found that it was $\nabla \cdot \mathbf{F}\, dV$, meaning that the integration on the boundary could be expressed as an integral of this differential quantity over the region. That is Equation (3.2).

Through abstraction, we see that there is really no difference. Perhaps now Equation (3.4) does not look so mysterious and far-off.

For Equation (3.3), we have a vector field $\mathbf{F}$ evaluated on the boundary in the form of a contour integral around a region. This is the total circulation of $\mathbf{F}$ around the region. Let us chop the region into little pieces.

INSERT GRAPHIC HERE

On an infinitesimal square, we get that the circulation is:

$$\begin{aligned} \text{Circulation} = \quad & F_y(x + dx/2, y)dy - F_y(x - dx/2, y)dy \\ & + F_x(x, y - dy/2)dx - F_x(x, y + dy/2)dx \end{aligned}$$

This can be written as:

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy = (\nabla \times \mathbf{F})\ dA$$

so that

Theorem 3.5 (Stokes' Theorem in 2D)**.** *For a smooth vector field on a piecewise smooth region S*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F}\ dA \tag{3.8}$$

Exercise **(MAKE AN EXERCISE)** generalizes this to a surface in 3D, to get the 3D version of Stokes' theorem :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \tag{3.9}$$

The philosophy behind these proofs is always the same. It is the manipulation of the differentials that seems wildly different every time. The curl looks nothing like a divergence, and a divergence is distinct from a gradient. Moreover, its not clear in what way each one generalizes the one dimensional derivative $df = f'(x)dx$. This is the problem that the symbol 'd' in Equation (3.4) was made to solve.

We must stop thinking of the 1D derivative, the gradient, the divergence, and the curl, as unrelated operations. They are in fact, the same operation, applied in different circumstances. Infinitesimal change, flux, and circulation are all the same differential action applied to different types of objects.

Perhaps part of this was clear from multivariable calculus: the gradient is nothing more than a generalization of the derivative to functions on higher dimensions. Then why are there seemingly two different, unrelated types of "derivative" on vector fields? Instead of a regular, gradient-like object, we have two: the divergence and the curl.

It will turn out that the reason that there are two is this: the vector fields that we take curls of are a different type of object from the vector fields we take divergences of. Looking forward, we'll see that we only take the curl on a vector field that is "meant to be integrated along curves" (**1-form**), and the curl gives us another vector field "meant to be integrated over surfaces" (**2-form**). On the other hand, the divergence takes a vector field "meant to be integrated over surfaces" (**2-form**) and gives us a scalar field "meant to be integrated over volumes" (**3-form**). Every object that we've encountered when integrating: from functions in 1-D or 3-D, to vector fields in n-D, have been examples of these **forms**.

To get to this idea, we first need to stop thinking of functions and vector fields as totally separate objects. A function is an object that is "meant to be evaluated at a point" (**0-form**). The derivative takes us from a function to a 1-form, meant to be integrated along a curve. It is the exact same object as the one in Section 2.5. The gradient, properly speaking, is not a vector describing the local behavior of a *curve* but is the opposite: a 1-form describing the local behavior of a *function.* So the correct way of writing this, is to go from the old $\mathbb{R}^3$ notation

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial k}\hat{\mathbf{k}}$$

to the modern language

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}dx^i \left(= \nabla f \cdot d\mathbf{r}\right) \tag{3.10}$$

This is a 1-form, as we already know.

What we would like is to have the old multivariable calculus chain

$$\text{functions} \overbrace{\longrightarrow}^{\text{grad}} \text{vector fields} \overbrace{\longrightarrow}^{\text{curl}} \text{vector fields} \overbrace{\longrightarrow}^{\text{div}} \text{functions}$$

be converted to

$$\begin{array}{ccccccc}
\text{0-forms} & \overbrace{\longrightarrow}^{\mathrm{d}} & \text{1-forms} & \overbrace{\longrightarrow}^{\mathrm{d}} & \text{2-forms} & \overbrace{\longrightarrow}^{\mathrm{d}} & \text{3-forms} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\substack{\text{Evaluated}\\\text{at}} & & \substack{\text{Integrated}\\\text{along}} & & \substack{\text{Integrated}\\\text{along}} & & \substack{\text{Integrated}\\\text{along}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Points} & \underbrace{\longleftarrow}_{\partial} & \text{Curves} & \underbrace{\longleftarrow}_{\partial} & \text{Surfaces} & \underbrace{\longleftarrow}_{\partial} & \text{Volumes}
\end{array}$$

where ∂ is the **boundary operator** that takes us from an n dimensional manifold to its $n-1$ dimensional boundary. At this point, we won't be afraid to use our new word: manifold, when referring collectively to curves, surfaces, volumes, or any of their higher dimensional generalizations (points too, as the degenerate 0 dimensional case).

As a final note of this section, let us try to give a sketch for why on a region Ω, we denote its boundary with the partial derivative symbol as $\partial\Omega$. Picture in your mind a ball (interior of a sphere) of radius r, B_r. If we increase the radius by a tiny amount h then we have a slightly larger radius B_{r+h}. If we took the difference $B_{r+h} - B_r$, by which we mean all the points of B_{r+h} that are not in B_r, we would be left with a thin shell. In the limit as $h \to 0$, this becomes a sphere of radius r, precisely the boundary of B_r (note that a sphere is always the two-dimensional boundary of the ball). See how similar this is to taking derivatives. This is why ∂B_r is what we use to denote the sphere boundary of the ball.

You may ask "but what about dividing by h at the end, like we do for a regular derivative?". This also has an interpretation. The 3D volume of a sphere is zero, since it is a 2-D boundary. Dividing by h as h goes to zero puts increasing "weight" on the shell so that as the shell shrinks to becoming absolutely thin, 3-D integrals on it become 2-D [2].

3.2 The Notion of a Form

A differential form ω, in short, is an object that is meant to be integrated. We've seen the example of 1-forms in the preceding chapter. At a point p on a manifold, one-forms ω are exactly all the first-order behaviors of the functions at p. Just as we can have a vector field on $\mathbb{R}^3$, or manifolds in general, we have 1-form fields. You have seen this before: the gradient is in fact a one-form field

$$(\nabla f \cdot d\mathbf{r})(p) = \frac{\partial f}{\partial q^i}(p)\ dq^i \tag{3.11}$$

The components of the gradient are *covariant* with our change of coordinate system, just like those of a 1-form (and unlike those of a vector field).

[2] For those familiar with the terminology: dividing by h corresponds to multiplying by a dirac delta that spikes exactly on the sphere. This turns integrals over 3-D space into 2-D integrals on the sphere

A general 1-form associates a first-order function behavior $\omega_i(p)\, dq^i$ to every point p in space. Just because ω is some differential behavior of a function at a point p doesn't mean that ω actually *is* the differential of a function. That is, it doesn't mean there exists a function so that $\partial f/\partial q^i = \omega_i$ at every point p so that $\omega = \mathrm{d}f$.

It rarely true that $\omega = \mathrm{d}f$. In fact, this happens *exactly* when ω actually *does* correspond to a gradient. This is exactly what we called a conservative vector field in introductory physics and in multivariable calculus. In this language, we call ω an **exact form** when it is the differential of a function.

This is what we care about when integrating. It is more fundamental than $\mathbf{F}$, but what does it mean *physically*? If $\mathbf{F}$ was a force field, then since we know $\mathbf{F} \cdot d\mathbf{r} = dW$, this form ω represents all possible infinitesimal changes in work dW at a given point, depending on what changes dx, dy, dz we do.

If we were actually *given* the changes in each of the coordinates dx, dy, dz, we could plug them in to ω and get the first-order approximation of the amount of work done over that distance. This is a point that has been said before: ω does not represent a specific change in work, but rather the *relationship* between the changes in coordinate and the change in work. If you *give it* an infinitesimal displacement, it will tell you the associated work. When integrating along a curve, the displacement is simply the tangent vector to the curve.

Even simpler than one-forms are the **zero forms**, with no differentials appearing. A zero-form precisely a scalar function at $f(p)$ each point p. Regardless of how we change our coordinate system, the value of the *function* at point p is the same.

We are now in a good place to define d, at least for going from functions (zero-forms) to one-forms. Given a function f, $\mathrm{d}f$ will produce a form representing the local change in f depending on the displacement. We call d the **exterior derivative** operator.

For example, for a potential energy function ϕ, $\mathrm{d}\phi$ can be written as

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i} dx^i \tag{3.12}$$

because of d, we will no longer have to use the gradient at all. This is more important than simply meaning that we'll grow to stop using

$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. It is something much deeper. In in two-dimensional motion, if you have some potential ϕ at a point p, then of course the value of ϕ at p is independent of any coordinate system you use. If you have two cartesian coordinates, say x, y, then you can define the x, y components of force by

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = F_x\ dx + F_y\ dy$$

If our coordinates were r, θ, then the analogous force would be the covariant components of the same *form*, in a different coordinate system:

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial \theta} d\theta + \frac{\partial \phi}{\partial r} dr = F_\theta\ d\theta + F_r\ dr$$

Note that the first component has units not of force, but of force times distance. It is precisely the torque that the potential induces. In this sense, quantities like torque are precisely just generalizations of force to non-cartesian coordinate systems (polar, in this case). The second component is just radial force, plain and simple.

To hammer the point across: these two "forces" have components that mean completely different things, and cannot easily be compared. On the other hand, since $\mathrm{d}\phi$ is independent of coordinate system, we get:

$$\mathrm{d}\phi = F_x\ dx + F_y\ dy = F_\theta\ d\theta + F_r\ dr \tag{3.13}$$

Because we know how to go from x, y to r, θ and because this nonlinear change of coordinates is *linear* at every point on the differentials, this would allow us to go between the language of "x-y force" and the language of "torque + radial force about the origin" at any point.

All forces (including the generalized forces, like torque) are covariant coefficients of the invariant differential form for work. If you're working in a coordinate system q^i, whether it be cartesian x, y, z or polar r, θ, then the coefficient corresponding to dq^i is precisely the generalized force associated with that coordinate in your system.

3.3 The Exterior Algebra and the Wedge

If a 1-form must be fed a vector of some associated change of coordinates dq^i, then what about a 2 form? A 2-form, ω, should associate a "flux" out of a plane, so we need ω to be given a plane associated with *two* directions v^i, u^i, and then ω acting on these two directions would give the associated "flux" out of the infinitesimal parallelogram gained by varying q^i in both directions. So if ω were a 1-form, it would act on one vector as $\omega(\mathbf{v}) = \omega_i v^i$, but now ω as a 2-form acts on two vectors:

$$\omega(\mathbf{u}, \mathbf{v}) \longleftrightarrow \text{Flux Coming out of } \mathbf{u} \text{ and } \mathbf{v}\text{s Parallelogram}$$

This is an intuitive *geometric idea.* This is what we want to be true, and the following observations will be *algebraic properties of* ω based on our geometric notions of flux.
DRAW SOMETHING HERE

Observation 3.6. $\omega(\mathbf{v}, \mathbf{v}) = 0$

The parallelogram generated by $\mathbf{v}$ and itself has no second dimension, so it has no area. Therefore there isn't room for any flux to come out of it.

Observation 3.7. $\omega(2\mathbf{u}, \mathbf{v}) = 2\omega(\mathbf{u}, \mathbf{v})$ *and* $\omega(\mathbf{u}, 2\mathbf{v}) = 2\omega(\mathbf{u}, \mathbf{v})$. *Moreover, in general* $\omega(a\mathbf{u}, \mathbf{v}) = \omega(\mathbf{u}, a\mathbf{v}) = a\omega(\mathbf{u}, \mathbf{v})$ *for* $a > 0$.

Of course, if we scaled the parallelogram by some positive amount a along one of the sides, then its total area scales by a, so that ω gives a times as much "stuff" coming out of the scaled parallelogram. This observation, together with our linear algebraic ideas, suggest that this should naturally extend beyond positive a so that $\omega(-\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{u}, \mathbf{v})$, giving us negative flux through the parallelogram. But what does that mean? This means that if one of the vectors gets scaled negatively, the *orientation* of the new parallelogram reverses.
DRAW THIS HERE

So now even though the pair $-\mathbf{u}, \mathbf{v}$ are on the same plane, the notion of "out" through their parallelogram has reversed. This makes physical sense,

Concept 3.8. *What matters, when finding the flux associated with* ω *along two directions* $\mathbf{u}, \mathbf{v}$

1. *The* plane *that* $\mathbf{u}, \mathbf{v}$ *span, through which the flux is going*

2. *That scaling* $\mathbf{u}$ *or* $\mathbf{v}$ *also scales the flux.*

3. *The* orientation *for which direction is in and which is out.*

This is exactly what we've seen before with the **Right-Hand Rule**. Because of this, and from what we've seen before with objects like cross products, we would *expect* that $\omega(\mathbf{u}, \mathbf{v})$ represents one orientation, and $\omega(\mathbf{v}, \mathbf{u})$ represents the opposite one so that

$$\omega(\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{v}, \mathbf{u})$$

we can add this as another property *but* we can instead actually prove it if we make just one more geometric observation

Observation 3.9. $\omega(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{w}) + \omega(\mathbf{v}, \mathbf{w})$

If $\mathbf{u} = a\mathbf{v}$ are linearly dependent then this is just the scaling observation. If $\mathbf{v}$ is linearly dependent with $\mathbf{w}, \mathbf{u} = a\mathbf{w}$, then this is just the observation that the parallelogram of $(\mathbf{u}, \mathbf{w})$ would have the same amount of associated area if you were to add vectors in the direction of $\mathbf{w}$ to $\mathbf{u}$ so $\omega(\mathbf{u} + a\mathbf{w}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{w})$. It's the same idea if $\mathbf{u}$ is linearly dependent with $\mathbf{w}$. Now let's assume all vectors are linearly independent. Geometrically, the two planes associated with $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{v}, \mathbf{w})$, and the plane associated with the sum $(\mathbf{u} + \mathbf{v}, \mathbf{w})$ can look like:

FIGURES TO ILLUSTRATE MY GEOMETRIC IDEA, DISCUSS WITH AARON

Now ω represents a constant flux in space. Enclosing a region by these three planes should mean that the flux that goes through the first two is the flux that comes out of the third.

The same exact argument can be applied to show this holds for the second slot in $\omega(-, -)$.

Corollary 3.10. *The above observations imply that* ω *is linear and* **antisymmetric** *in its arguments so that* $\omega(\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{v}, \mathbf{u})$*. That is, as we expected, reversing the order of* $\mathbf{u}$ *and* $\mathbf{v}$ *reverses the orientation of the plane.*

Proof. We have shown that ω is compatible with scaling and addition in each argument, so it is linear in both.

Now consider $\omega(\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v})$. By our first observation, such a parallelogram has no area, so this is zero. On the other hand, by the linearity of ω in both arguments:

$$\begin{aligned}0 &= \omega(\mathbf{u},\mathbf{u}) + \omega(\mathbf{u},\mathbf{v}) + \omega(\mathbf{v},\mathbf{u}) + \omega(\mathbf{v},\mathbf{v})\\ &= \omega(\mathbf{u},\mathbf{v}) + \omega(\mathbf{v},\mathbf{u})\\ &\Rightarrow \omega(\mathbf{u},\mathbf{v}) = -\omega(\mathbf{v},\mathbf{u})\end{aligned}$$

□

2-forms are bilinear operators that act on pairs of vectors that represent coordinate changes $(\mathbf{u},\mathbf{v})$. They associate a flux to a given plane defined by such coordinate changes. In $\mathbb{R}^3$, the space of 1-forms is spanned by dx, dy, dz. We can also know the flux through any plane if we knew the flux on the xy, yz, and zx planes, so our basis for our set of 2-forms should also be three dimensional. It is spanned by three elements that we will write as

$$dx \wedge dy,\ \ dy \wedge dz,\ \ dx \wedge dz$$

The first element represents a flux of magnitude $|dx\ dy|$ through the xy plane an no flux through the other two. On the other hand $dy \wedge dx$ would represent a flux in the OTHER direction so that $dx \wedge dy = -dy \wedge dx$. We have invented something called the **wedge product**.

$$\begin{aligned}&\text{1-form in } \mathbb{R}^3 = \omega_x dx + \omega_y dy + \omega_z dz\\ &\text{2-form in } \mathbb{R}^3 = \omega_{xy}(dx \wedge dy) + \omega_{yz}(dy \wedge dz) + \omega_{xz}(dx \wedge dz)\end{aligned}$$

In multivariable calculus, we would define planes, together with orientations, by specifying a normal vector to those planes. In a general dimension, we aren't able to associate a normal vector to a plane, so we should talk about the plane *itself*. For this reason we define the **wedge product**

Definition 3.11 (The Wedge Product)**.** *The product $\wedge$ on a real vector space V defined so as to satisfy the following properties*

- $\forall \mathbf{a},\mathbf{b},\mathbf{c} \in V,\ \ (a\mathbf{a}+b\mathbf{b}) \wedge \mathbf{c} = a(\mathbf{a}\wedge\mathbf{c}) + b(\mathbf{b}\wedge\mathbf{c})$
- $\forall \mathbf{a},\mathbf{b},\mathbf{c} \in V,\ \ \mathbf{c} \wedge (a\mathbf{a}+b\mathbf{b}) = a\mathbf{c}\wedge\mathbf{b} + b\mathbf{c}\wedge\mathbf{a}$
- $\forall \mathbf{a} \in V,\ \ \mathbf{a}\wedge\mathbf{a} = 0$

As before, this last condition, together with bi-linearity, implies anti-symmetry of $\wedge$.

In our case, when we have a basis dq^i for our cotangent space of 1-forms, the basis for our space of 2-forms can be written as $dq^i \wedge dq^j$ for $i < j$. The 2-form $dq^3 \wedge dq^4$, for example, represents a flux of magnitude $|dq^3 dq^4|$ out of the $q^3 q^4$ plane and no flux out of $q^i q^j$ planes for any other i, j. On the other hand $dq^4 \wedge dq^3$ would be the same flux in the opposite direction (and again, no flux out of any of the other $q^i q^j$ planes).

It may be frustrating to see this new product without any previous background. Let's do an example of wedging two 1-forms in 3D: α and β, and see what happens.

$$\alpha \wedge \beta = (\alpha_x dx + \alpha_y dy + \alpha_z dz) \wedge (\beta_x dx + \beta_y dy + \beta_z dz)$$

$$\begin{aligned} = \quad & \alpha_x\beta_x(dx \wedge dx) + \alpha_x\beta_y(dx \wedge dy) + \alpha_x\beta_z(dx \wedge dz) \\ & + \alpha_y\beta_x(dy \wedge dx) + \alpha_y\beta_y(dy \wedge dy) + \alpha_y\beta_z(dy \wedge dz) \\ & + \alpha_z\beta_x(dy \wedge dx) + \alpha_z\beta_y(dy \wedge dy) + \alpha_z\beta_z(dy \wedge dz) \end{aligned}$$

$$\begin{aligned} &= (\alpha_x\beta_y - \alpha_y\beta_x)(dx \wedge dy) + (\alpha_y\beta_z - \alpha_z\beta_y)(dy \wedge dz) \\ &\quad + (\alpha_z\beta_x - \alpha_x\beta_z)(dz \wedge dx) \end{aligned}$$

OK AARON FORMAT THIS IDK HOW but if we were back in multivariable calculus world, not caring about vectors and forms and writing everything in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, and we identified

$$\hat{\mathbf{i}} \leftrightarrow dx, \hat{\mathbf{j}} \leftrightarrow dy, \hat{\mathbf{k}} \leftrightarrow dz$$

as well as

$$\hat{\mathbf{i}} \leftrightarrow dy \wedge dz, \hat{\mathbf{j}} \leftrightarrow dz \wedge dx, \hat{\mathbf{k}} \leftrightarrow dx \wedge dy$$

then the wedge product becomes *exactly* the cross product. Why wedge then, when we already know the cross product? Because the cross product, going from vectors to vectors, only works in three dimensions. The wedge product taking us from 1-forms to 2-forms, is universally valid.

Moreover, now we can go beyond just 2-forms and form higher wedges, k-forms. The wedge of two forms $\alpha \wedge \beta$ will always be linear in both arguments and antisymmetric. For example, we can make 3-forms. In 3-D, the space of 3-forms is one dimensional spanned by $dx \wedge dy \wedge dz$. Any other wedge of these differentials will either be the same, or a negative of this one. It corresponds to the one true infinitesimal 3D volume. What about the sign? It is the distinction between flow "into" the volume.

GRAPHIC of 2-form wedge a 1-form giving "inward" orientation, and then wedge the negative of that 1-form to give the "outward" orientation

This is more general, in n dimensions the space of n-forms is one dimensional, spanned by the form $\bigwedge_{i=1}^{n} dq^i$. For some terminology:

Definition 3.12 (**Exterior Power**). *For a given vector space V, the vector space of of k forms spanned by wedging elements of V with themselves k times is called the kth exterior power of V, and is denoted by $\Lambda^k V$.*

Our V in this case is the cotangent space at a point T_p^*M: the space of 1-forms and $V = \Lambda^1 V$ always. The space of zero forms $\Lambda^0(T_p^*M)$ at a point is the set of possible function values so is just $\mathbb{R}$ in our case, since we are working over the reals. The space of k forms at p is the kth exterior power of the cotangent space at p: $\Lambda^k T_p^* M$. If we consider all k-forms at p, then we get the *exterior algebra* of the cotangent space at p.

Definition 3.13 (**Exterior Algebra**). *The vector space of* all *k-forms is called the exterior algebra of V and is denoted ΛV.*

What about the tangent space of vectors? What does the exterior algebra mean there? If $dq^1 \wedge dq^2$ is the form that associates an oriented flux to the $q^1 q^2$ plane, then $\partial_1 \wedge \partial_2$ is the oriented plane *itself*.

In this way

$$(\text{Flux } dq^1 \wedge dq^2)(\text{Coordinate Area } \partial_1 \wedge \partial_2) = (\text{Flux})(\text{Coordinate Area}).$$

This is the invariant *total flux* coming out from varying dq^1 and dq^2 together to sweep out a coordinate area. In general, k-wedges of vectors $v^i \partial_i$ represent the oriented k-volumes *themselves*, on which k-forms act. In Einstein's convention, you can show that in for a general 2-forms and 2-vectors whose coordinates are doubly covariant and contravariant, respectively, we get the invariant value:

$$\begin{aligned}(\omega_{ab}dx^a \wedge dx^b)(v^{cd}\partial_c \wedge \partial_d) &= \omega_{ab}v^{cd}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) \\ &= \omega_{ij}v^{ij} - \omega_{ij}v^{ji}.\end{aligned}$$

It is an exercise to generalize this, and also to check that it makes sense in 3D when our wedges are cross products and areas are normal vectors

As a last note, philosophically, *where does this antisymmetry come from?* We've already seen it in the cross product, and now we have it in this wedge. Geometrically, what is happening? That the wedge product of a form with itself is zero is easy to understand: you cannot geometrically extend an object to higher dimensions without introducing new directions. Antisymmetry, on the other hand, is less obvious.

When we extend a geometric k-volume to a $k+1$-volume, there is a notion of orientation. Going from the line to the plane, we need to know "which direction is out for flux?" Similarly, for the plane to the volume, we have basically the same question "which direction is in/out for flux?", and the antisymmetry of the wedge reflects that orientation will always exist for higher k volumes.

3.4 Stokes' Theorem

To go from a 0-form f to a 1-form ω, we applied the exterior derivative operator, which could just be written as:

$$\mathrm{d}f = dq^i \frac{\partial f}{\partial q^i}. \tag{3.14}$$

Going further, perhaps we could write the exterior derivative operator explicitly asa the invariant:

$$\mathrm{d} = dq^i \frac{\partial}{\partial q^i}. \tag{3.15}$$

We can view this d operator as a 1-form whose coefficients on each dq^i, rather than being numbers, are derivative operators $\partial/\partial q^i$ with respect to the corresponding coordinates.

Now for a 1-form $\omega = \omega_i dq^i$, we want the exterior derivative to take us to a 2-form. If this d operator can be thought of as a 1-form, the obvious way to go to a form one step higher is by wedging, meaning that we would define:

$$\mathrm{d}\omega := (dq^i \frac{\partial}{\partial q^i}) \wedge \omega = dq^i \wedge \frac{\partial \omega}{\partial q^i}. \tag{3.16}$$

Note that the only reason we did this is because of what the *algebra* seemed to tell us to do, independent of any geometric intuition beforehand. This is powerful, but is this right? Is this the derivative operator that will generalize the gradient, divergence, curl, and *everything else*?

Let us first check that for a 1-form ω $\mathrm{d}\omega$ gives us Stokes' theorem, as we want:

Theorem 3.14 (Stokes' Theorem for 1-forms)**.** *If ω is a 1-form, then with ∂ defined as the boundary operator of a manifold and* d *defined as in Equation* (3.16)*, we have*

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega.$$

Proof. Take Ω, and as in all of our proofs in the section Section 3.1, let us cut Ω into a mesh of infinitesimal parallelograms. If we integrate ω over the boundary $\partial\Omega$, this is the same as integrating ω over every single individual parallelogram on the interior, as *BECAUSE OF ORIENTATION*, the integrals over the boundaries of these parallelograms will cancel between neighbors, leaving us with only the boundary, as always.

It remains to show that for an arbitrary small parallelogram, Stokes' theorem holds. This parallelogram is obtained by varying q^i along two vectors $u^i\partial_i$ and $v^i\partial_i$. After a suitable linear transformation of coordinates, we can assume WLOG that this parallelogram is obtained by changing q^1 by some fixed small amount dq^1 and q^2 by dq^2. Let's integrate ω on the boundary:

Because the parallelogram is small, we can approximate these integrals as:

$$(\partial_1\omega_2 - \partial_2\omega_1)dq^1 dq^2$$

On the other hand the exterior derivative is:

$$\begin{aligned}
\mathrm{d}\omega &= dq^i \wedge (\partial_i \omega) \\
&= (dq^1 \wedge dq^2)\partial_1\omega_2 + (dq^2 \wedge dq^1)\partial_2\omega_1 + \text{other} \\
&= (\partial_1\omega_2 - \partial_2\omega_1) dq^1 \wedge dq^2 + \text{other}
\end{aligned}$$

where the other terms involve wedges that aren't of q^1, q^2 and will vanish along integration of this specific parallelogram. Since integrating $(\partial_1\omega_2 - \partial_2\omega_1)dq^1 \wedge dq^2$ gives exactly $(\partial_1\omega_2 - \partial_2\omega_1)dq^1 dq^2$, we have proven it for this parallelogram, and by linearity and coordinate change for *any* paralellogram. Because adding these parallelograms together forms the bulk of Ω and they cancel when integrated on neighboring boundaries (if one boundary is associated with dq^i, the neighboring one is associated with $-dq^i$), this gives the desired result. □

WE DONT NEED ANY OTHER COORDINATES

Note that if $q^i = (x, y, z)$ then this is exactly Stokes' theorem in 3-D for the curl! More than this, it generalizes Stokes' theorem in $\mathbb{R}^3$: for any 2-D surface, the circulation of ω over the boundary is exactly the sum total of the curl $\mathrm{d}\omega$ over the interior.

From this let us prove what we set out to prove in the most general case:

Theorem 3.1 (General Stokes' Theorem)**.** *With ∂ defined as the boundary operator of a manifold and* d *defined as in Equation* (3.16)*, we have for a general differential k-form ω that*

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega.$$

First, a lemma:

Lemma 3.15 (Restriction of a Form)**.** *If ω is a k form of n variables that is being integrated over some k-dimensional manifold associated with changing only the first k variables, then for the integration, we can work with the restricted ω_{res} associated with setting the last $n-k$ dq^i equal to zero, and eliminating any wedge terms holding those dq^i.*

Proof. This follows from □

IM NOT SURE WE NEED THIS NOW

Now to prove the General Stokes' Theorem:

Proof. For a given manifold Ω, divide it's bulk into k-volumes that are generalizations of parallelograms to k-dimensions. Just like a line's boundary has two points, a parallelogram's has 4 lines, a parallelepiped's has 6 parallelograms, a k-volume's boundary is $2k$ $k-1$ volumes. Again, we can set up local coordinates so that this k volume has each $k-1$ obtained by holding some q^i constant and letting the others vary. For each q^i there are exactly two opposing $k-1$ volumes obtained by holding that coordinate constant, and they have opposite orientation.

Now let us perform the integration of ω over the $k-1$ boundaries. In terms of these coordinates, ω can be written as a linear combination of wedges of $k-1$ of the dq^i, meaning that each such wedge misses exactly one dq^i. □

THE PROBLEM WITH THE ABOVE IS THERE COULD BE n q^i and

Proposition 3.16. $\mathrm{d}^2 = 0$

Proof. For a general form ω, consider

$$\begin{aligned} \mathrm{d}(\mathrm{d}\omega) &= (dq^j\partial_j) \wedge (dq^i\partial_i\omega) \\ &= dq^j \wedge dq^i(\partial_j\partial_i\omega) \end{aligned}$$

This is a summation over i and j running from 1 to n. Now pick any specific term in the sum with *specific* indices a, b. This corresponds to a term:

$$dq^a \wedge dq^b(\partial_a\partial_b\omega)$$

in the sum. We can assume $a \neq b$ as otherwise that'd mean $dq^a \wedge dq^b = 0$. But that means we will have another distinct term with those indices reversed (b, a) equal to

$$dq^b \wedge dq^a(\partial_b\partial_a\omega)$$

Since partials commute but the wedge products anti-commute, this (b, a) term is equal to the *negative* of that first (a, b) term, meaning they will cancel. The whole double-sum will then become a sum of cancelling terms, giving zero. □

Corollary 3.17. $\partial^2 = 0$*: The boundary of a boundary is nothing.*

Proof. Assume we are integrating a form $\mathrm{d}\omega$ on a boundary. By Stokes' Theorem (twice):

$$\int_{\partial^2\Omega} \omega = \int_{\partial\Omega} \mathrm{d}\omega = \int_{\Omega} \mathrm{d}^2\omega = \int_{\Omega} 0 = 0$$

□

This is an amazing geometric fact that we have gotten, via Stokes' theorem, from the *purely algebraically derived* fact that $\mathrm{d}^2 = 0$. The duality between forms and the manifolds we integrate them over is a gorgeous duality between algebra and geometry that extends very deeply and profoundlys (c.f. Hodge Theory **INSERT TEXTS HERE**).

We have already seen that differential forms that are the exterior derivatives $\mathrm{d}\omega$ of some other form ω are called exact. We know exact 1-forms correspond to conservative vector fields. **It will be an exercise to show** exact 2-forms in $\mathbb{R}^3$ correspond to solenoidal vector fields. On the other hand, forms ω that have $\mathrm{d}\omega = 0$ are called **closed**. Why this language? It is taken for the corresponding geometric language for the boundary operator. If a region has no boundary, it is called closed, so if a form has zero exterior derivative, it will also be called closed. Clearly exact forms are closed, since $\mathrm{d}^2 = 0$, but are *all* the closed forms exact? In $\mathbf{R}^3$, the answer is yes, but consider this:

Example 3.18. *$d\theta$ is a closed form defined on the punctured plane that is not exact.*

We know $\mathrm{d}(d\theta) = 0$, so it is indeed closed. Although locally, a function θ (that this form represents the change of) can be defined just by calculating the angle from the x axis, if you go around counterclockwise in a circle containing the origin, then θ continuously increases. At the end of the revolution, even though you are at the same point, θ has increased by 2π. So although $d\theta$ makes sense locally as a differential form everywhere in the plane minus the origin, we cannot define a global smooth function representing θ without a discontinuity. The existence of a closed form that is not exact happens because the manifold on which $d\theta$ is defined is *not* $\mathbb{R}^2$, but is instead defined on $\mathbb{R}^2$ without the origin (where $d\theta$ would not be well-defined). This change in global geometric structure gives rise to these interesting closed, inexact forms.

The study of the closed forms that are not exact on a manifold is called the **De-Rham cohomology** of a manifold.

Insert Graphic Here

3.5 Distance, a Metric

3.6 Multilinear Algebra: $\oplus, \otimes$ and Tensors

3.7 Movement, Lie's Ideas

First, something cool. Euler's identity $\rightarrow e^{a\frac{\partial}{\partial x}}$

3.8 Exercises

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