In calculus class you were taught the fundamental theorem, that the total difference of a function's value at the end of an interval from its value at the beginning is the sum of the infinitesimal changes in the function over the points of the interval:

$$\int_a^b f'(x)dx = f|_a^b \tag{1}$$

And later, in multivariable calculus, you encountered more elaborate integral formulae, such as the divergence theorem of Gauss:

$$\int_\Omega \nabla \cdot \mathbf{F}\ dV = \int_S \mathbf{F} \cdot d\mathbf{S} \tag{2}$$

where Ω is the volume of a 3D region we are integrating over, with infinitesimal volume element dV and S is the surface that forms the boundary of Ω. dS then represents an infinitesimal parallelogram through which $\mathbf{F}$ is flowing out, giving the flux integral on the right. Read in english, Gauss' divergence theorem says "Summing up the infinitesimal flux over every volume element of the region is the same as calculating the total flux coming out of the region". The total flux coming out of a region is the sum of its parts over the region. You might see that in english, this reads very similar to the description of the fundamental theorem of calculus.

Alongside this, there is Stokes' theorem for a 2D region. In english: summing up the infinitesimal amount of circulation of a vector field $\mathbf{F}$ over every infinitesimal area is equal to calculating the total circulation of $\mathbf{F}$ around the boundary of the region. In mathematical language:

$$\int_R \nabla \times \mathbf{F}\ dA = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{3}$$

where R is our region and C is its boundary.

Perhaps now, the pattern is more evident. In all the above cases, summing up some *differential* of the function on the interior of some region is the same as summing up the function itself at the *boundary* of the region. All these theorems, that on their own look so strange to a first-year calculus student, are part of a much more general statement, the **General Stokes' Theorem**:

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{4}$$

Above, ω is an object that will generalize both the "functions" and "vector fields" that you've seen in multivariable calculus, and d will generalize of all the differential operators (gradient, divergence, curl) that you've dealt with. Lastly, when Ω is the region in question $\partial\Omega$ represents the *boundary* of the region Ω. The fact that it looks like a derivative symbol is no coincidence, as we'll see that the natural way to define the "derivative" of a region is as its boundary.

Through abstraction, we can reach results like this that not only look elegant and beautiful, but also provide us with insight into the natural way to view the objects that we've been working with for centuries. This gives us not only understanding of what language to use when studying mathematics, but also what is the natural language in which to describe the natural world. The general Stokes' theorem is one of the first examples of this beautiful phenomenon, and this book will work to illustrate many more.

For the first half of this chapter, we will work towards giving the intuition behind this result. On our way, we will begin to slowly move into a much more general setting, beyond the 3-dimensional world in which most of multivariable calculus was taught. That doesn't just mean we'll be going into n-dimensional space. We'll move outside of euclidean spaces that look like $\mathbb{R}^n$, into non-euclidean geometries. This will put into question what we really mean by the familiar concepts of "vector", "derivative", and "distance" as the bias towards Euclidean geometry no longer remains central in our minds. At its worst, the introduction of new concepts and notation will seem confusing and even unnecessary. At its best, it will open your mind away from the biases you've gained from growing up in a euclidean-looking world, and give you a glimpse of how modern mathematics *actually* looks.

Modern mathematics is learning that the earth isn't flat. To someone who's never had those thoughts, it is difficult to get used to, tiring, and sometimes even rage inducing, but to someone who has spent months thinking and reflecting on it, it quickly becomes second nature. Far from being the study of numbers or circles, it is a systematic climb towards abstraction. It is a struggle towards creating a language free from your all-encompassing human bias, to try and describe a world that language, for so many centuries, has failed to grasp. It is humbling, and in the strangest of ways, it is profoundly beautiful.

-1.1 The Notion of a Form

-1.2 The Derivative and the Boundary

-1.3 Stokes' Theorem

-1.4 Stokes' Theorem

-1.5 Movement, Lie's Ideas

-1.6 Distance

-1.7 Tensor