# Noncommutative Algebra

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## 1 $A$-algebras

Let $A$ be a $\mathbb{K}$-algebra, with $\mathbb{K}$ a field. We will study specific cases of $A$-modules.

An $A$-mod is called *simple* if it has no nontrivial proper submodule.

**Theorem 1** (Schur's Lemma)**.** *If $M, L$ are simple $A$-modcules and $\phi \in Hom_A(M, L)$, then either $\phi$ is an isomorphism or $M \neq L$ and $\phi = 0$.*

*Proof.* Follows directly from the fact that image and kernel of $\phi$ are submodules $\square$

From this it follows that $\mathrm{End}_A(L)$ is a division algebra over $\mathbb{K}$

An $A$-mod $E$ is semisimple iff $E = \oplus_i L_i$ with $L_i$ simple.

Consider $A' = \mathrm{End}_A(E)$ with an elemetn $\phi \in A'$ so that $\phi(am) = a\phi(m)$. Then $E$ is a bimodule over $A, A'$.

$$A \times E \times A' \to E, \qquad (a, m, \phi) \mapsto \phi(am) \tag{1}$$

$A'$ is called the commutant of $A$, so $A''$ is the double commutatnt. There is a canonical map $\rho A \to A''$

How big is the image of $A$ in $A''$?

**Theorem 2** (Jacobson Density)**.** *If $E$ is semisimple over $A$, then the image of $\rho$ is dense in the sense that for any $f \in End_{A''}(E), x_1, \dots, x_n \in E$, then $\exists a \in A : ax_1 = f(x_1)$. If $x_i$ generated $E$ then this makes $\rho$ surjective.*

*Proof.* For $n = 1$ consider $x \in E$, show $f(x) = ax$. $E = Ax \oplus E'$ so take $\pi : E \to Ax \subseteq E$, which is $A$-linear. Then $f(x) = f(\pi(x)) = \pi(f(x)) = ax$

Now consider $E^n$, which is still semisimple. So $A'_n := \mathrm{End}_A(E^n) = \mathrm{Mat}(n, A')$. Then $A''_n := \mathrm{End}_{A'}(E^n)$ so define $F^{\oplus n}(x_1 \dots x_n) = (f(x_1), \dots, f(x_n)) = a(x_1, \dots x_n)$. $\square$

This is the classical Burnside-Wedderburn theorem.

Let $E$ be a simple faithful $A$-module with $A$ finite dimensional. That is, $A \to \mathrm{End}_{\mathbb{K}}(E)$ is injective. So $D = \mathrm{End}_A(E)$ is a division algebra. $E$ is finite dimensional over $D$. Then $A = \mathrm{End}_D(E) = Mat(n, D)$. So in the case of $\mathbb{K} = \bar{\mathbb{K}} : A = Mat(n, \mathbb{K})$. Let $\phi \in D$, so $\mathbb{K}(\phi) = \mathbb{K}$.

## 2 Group Algebras and Representations

A direct sum of siple modules is closed under quotients so $A$ semisimple iff the left regular representation $A \times A \to A$ is semisimple. If $G$ is a finite group and $\mathbb{K}$ is a field, consider $A = \mathbb{K}G$. It is semisimple if $|G|$ is not divisible by the characteristic of $\mathbb{K}$

Since $M$ is an $A$-module and $A$ is simple, then for any $N \subseteq M$, we can write $M = N \oplus N'$.

$$0 \quad \to \quad N \quad \to \quad M \quad \to \quad M/N \quad \to \quad 0 \tag{2}$$

We want to show it splits. Pick $\pi' : M \to N$ any $\mathbb{K}$-linear map.

$$\pi = \frac{1}{|G|} \sum_{g \in G} g\pi' g^{-1}$$

Then $h\pi h^{-1} = \pi$. Now $\pi\big|_N(x) = x$, so we have splitting.

*Second Proof (Weyl's Unitary Trick).* Let $\mathbb{K} = \mathbb{C}$, so pick a $G$-equivalent metric on $M$ so obtained from any metric $\langle v, w\rangle'$ on $M$.

$$\langle v, w\rangle = \sum_{g \in G} \langle gv, gw\rangle' \Rightarrow \langle gv, gw\rangle = \langle v, w\rangle \tag{3}$$

So we can take $N' = N^{\perp}$

$\square$

**Term.** *$\mathbb{K}G$-modules $\Leftrightarrow$ $G$-representation. A simple $G$ module is called irreducible.*

For the first few weeks we will study $G$ modules for finite groups $G$.

When the characteristic doesn't divide $|G|$, $\mathbb{K}G$ is semisimple, so we can completely decompose it as a direct sum of irreducibles $\mathbb{K}G = \oplus_i^N L_i^{r_i}$.

Observe that there are only finitely many irreducible modules since the group algebra is finite dimensional.
Define

$$E = \oplus_{i=1}^N L_i \tag{4}$$

By Schur's Lemma, the endomorphism algebra of each $L_i$ over itself is a division algebra over $\mathbb{K}$. $\mathbb{K}G$ acting on $E$ is faithful, and so by Jacobson's density theorem,

$$\mathrm{End}_G(E) = \mathbb{K} \times \cdots \times \mathbb{K} \tag{5}$$

So $\mathbb{K}G = \mathrm{End}_{\mathbb{K}\times\ldots\mathbb{K}}(E) = \prod_{i=1}^{N} \mathrm{End}_{\mathbb{K}}(L_i)$

**Theorem 3.** *If $G$ is a finite group then there are finitely many classes of irreducible $G$-representations, regardless of characteristic. For non-problematic characteristics, then $\mathbb{K}G \to \prod_{i=1}^{N} End_{\mathbb{K}}(L_i)$ is an isomorphism.*

So for two $G, H$-representation $U, V$ we get $G \times H$ acts on $U \otimes V$ by $(g,h)(u \otimes v) = gu \otimes hv$. This means so then $G \times G$ acts on $u \otimes v$ when they are both in $G$-representations. Define $G$ acting on $U \otimes V$ by $g(u \otimes v) = gu \otimes gv$. This is the tensor product representation of $U, V$.

In order to do this more generally for an algebra $A$ we need to be able to embed $A$ into $A \times A$ (in a diagonal way) $\Delta : A \to A \times A$. When we can do this, $A$ is called a bi-algebra. We have shown that the group algebra of any finite group is a bi-algebra.

The represenation ring of $G$ over $\mathbb{K}$, $\mathrm{Rep}_{\mathbb{K}}(G)$ is equal to $\oplus_{i=1}^{N}\mathbb{Z}[L_i]$. We define the product of two elements to be $[L_i][L_j] = \sum_k r_{ij}^k [L_k]$ iff $L_i \otimes L_j = \bigoplus_k L_k^{r_{ij}^k}$. This is a commutative ring since tensor proudct is commutative. The identity of the multiplication is the trivial representation. Associativity of tensor product makes this ring associative.

# 3 Matrix Coefficients

Let $U$ be a simple $G$-module and pick a basis $u_i$ for $U$. Then $\forall g \in G$, $gu_i = \sum_{i=1}^{\dim(U)} a_i^j(g) u_j$. Then $a_i^j(g)$ are the matrix coefficients of $U$.

So $\mathbb{C}G = \prod \mathrm{End}_{\mathbb{C}}(L_i) = \prod Mat(n_i, \mathbb{C})$.

Let $U, V$ be two irreps. Let $f : U \to V$ be a $\mathbb{C}$-linear map. Let the matrix representation be $F_i^j$

$$F = E_j^k \Rightarrow \sum_{g \in G} a_i^j(g) b_k^l(g^{-1}) = 0 \tag{6}$$