Notes on Lie Theory

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Let $\mathfrak{g}$ be a Lie algebra of dimension n. Note for two solvable ideals $\mathfrak{h}, \mathfrak{l}$ of $\mathfrak{g}$, their sum $\mathfrak{h} + \mathfrak{l}$ is also solvable:

$$(\mathfrak{h} + \mathfrak{l})/\mathfrak{h} = \mathfrak{l}/(\mathfrak{h} \cap \mathfrak{l})$$

This is the second isomorphism theorem. Since $\mathfrak{l}$ is solvable, its quotients will be too, since commutators of this quotient are just quotients of commutators of $\mathfrak{l}$ (this is the fourth isomorphism theorem).

Assume that a maximal solvable algebra is not unique, so there is another: then take those two and sum them to make a bigger one, condtradicting their maximality. Therefore there is a special maximal solvable subalgebra in $\mathfrak{g}$, and let us call it $\mathrm{rad}\mathfrak{g}$. Now $\mathfrak{g}/\mathrm{rad}\mathfrak{g}$ has no nontrivial solvable ideals, so is semisimple. This means that any Lie algebra can be expressed as an extension of a semisimple Lie algebra by a solvable one. In fact, this extension is a semidirect sum.

We now prove Engel's Theorem:

Theorem 1 (Engel's Theorem)**.** *If every element in a Lie algebra is nilpotent, then the algebra is nilpotent.*

Proof. □

If every element of $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is nilpotent.

If $\mathfrak{g} \subset \mathfrak{gl}_n$ is solvable then there is a basis in which every element is upper triangular.

The killing form is nondegenerate for a simple Lie algebra (hint, we can assume that it is concrete WLOG)

Cartan's Criterion: $\mathfrak{g}$ is solvable iff it is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$