Engel's Theorem

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Theorem 1 (Engel)**.** *If $L \subseteq \mathfrak{gl}(V)$ consists only of nilpotent elements, then $\exists v \in V$ so that $xv = 0 \forall \in L$*

Consequence 1. *If L is such a Lie algebra, then we can continue to find v_2 in $V/\mathbb{K}v_1$ so that v_2 is again such an eigenvector. So*

$$\begin{aligned} xv_1 = &\quad 0 \\ xv_2 = &\quad a_1(x)v_1 \\ xv_3 = &\quad a_2(x)v_2 + a_1(x)v_1 \end{aligned} \tag{1}$$

Thus we can represent any x by a strictly *upper triangular matrix.*

Proof. By induction on dimension of L. For dimension 1, this means that $L = \mathbb{K}x$, so x is nilpotent of degree n so pick a nonzero vector in the image of x^{n-1}, then x acting on this gives zero.

Now assume it holds true for all L' of dimension $< \dim L$. If $K \subseteq L$ is a proper Lie subalgebra, then $\mathrm{ad} : K \to \mathrm{End}(L/K)$. By the induction hypothesis there is a vector in L/K killed by $\mathrm{ad}K$, so $\exists z : [K, z] \in K$, so $K \subsetneq N_L(K)$. If we take K as maximal, this argument makes its normalizer universal, so it is an ideal. In fact (since it's easy to find one-dimensional sub-algebras, and by fourth isomorphism theorem), K maximal means it has codimension one.

So $L = K \oplus \mathbb{F}z$ for some z in $L \backslash K$. Now let $W = \{v \in V : Kv = 0\}$. Since K is an ideal, W is stable under L. Now consider the nilpotent z acting on W and note it has an eigenvector of eigenvalue 0. This shows that both parts of the direct sum satisfy the conclusion, so the conclusion is proven. ☐

Note further that central quotients and central extensions preserve nilpotency. Since every Lie algebra is a central extension of the linear ad of that lie algebra, we have shown nilpotency for all Lie algebras.

Theorem 2 (Lie)**.** *If $L \subseteq \mathfrak{gl}(V)$ is solvable over $\mathbb{F}$ closed and characteristic 0 (basically $\mathbb{C}$), then $\exists v \in V$ so that $xv = \lambda(x)v$, $\lambda : L \to \mathbb{C}$.*

Consequence 2. *A solvable lie algebra is conjugate to a subalgebra of upper triangular matrices. In particular the maximal solvable Lie subalgebra in $\mathfrak{gl}(V)$ is the upper triangular matrices, up to conjugation.*

Proof. Again induct on $\dim L$. If $\dim L = 1$ then $L = \mathbb{C}x$ so we can find an eigenvector for x.

Assume it holds true for all L' of dimension $< \dim L$. Let $K' = [L, L]$, so L/K' is abelian so any subalgebra is an ideal (so subalgebras containing K' are ideals by 4th isomorphism theorem). Pick an ideal of codimension 1, so there's a $v \in V$ such that $\forall x \in K$. $\square$