Representations
of a
Physical Universe

Alex Atanasov & Aaron Hillman

August 12, 2016

Contents

Part 0

Old Notions and New Horizons

Chapter 1

Old Notions Revisited

1.1 The Cartesian Coordinate System

One of the most important revolutions in mathematics and physics was ushered in by an idea of the seventeenth century mathematician and philosopher René Descartes. The idea was that any point P in the Euclidean plane could be represented by a pair of numbers (x, y). The numbers themselves represented distances along two perpendicular axes that met at a point $(0, 0)$, called the origin. By introducing this concept, he had done something amazing. He had related the *geometry* of the plane to the *algebra* of variables and equations. Algebra could be *represented* geometrically, and conversely geometric problems could be solved by going into the realm of algebra.

Insert 2D Plane with Coordinates

Coordinates soon became more than just pairs of numbers (x, y). Their use was extended to 3D space, and later to arbitrarily high dimensions. They would subsequently be used to lay the foundations for modern physics and mathematics. Linear algebra, multivariable calculus, and all the connections between algebra and geometry begin with the concept of a coordinate.

Since then, the use of coordinate systems has proven indispensable to physicists and mathematicians throughout history. Newton used Descarte's coordinate system to formulate his infinitesimal calculus. Maxwell used it to analyze electromagnetic fields, discovering mathematically that light is a wave in the electromagnetic field. Einstein, going further, made use of coordinate systems to formulate his theory of gravitation. Today, physicists and engineers do their calculations within the frame of coordinate systems. In mathematics, Descarte's idea planted the roots for what would turn into the modern field of algebraic geometry.

When studying a geometric phenomenon in some n-dimensional space, say $\mathbb{R}^n$, we pick an origin and axes to form our coordinate system. For a ball falling, we could set the origin at some point on the ground, and pick one axis parallel to the ground, and one perpendicular. We can decide to measure the axes in meters, or we could decide to do it in feet (nothing stops us from making bad choices). The physical point P where the ball lies is represented by $(x, y) = (0 \text{ m}, 10 \text{ m})$. The coordinate y is a natural choice of coordinate, as it corresponds to our intuitive notion of height.

Insert Ball Falling

We can now study y, free of geometry, as just a function which we can do arithmetic and calculus on. If we are given an equations of motion, say

$$\frac{d^2y}{dt^2} = -g, \quad \frac{d^2x}{dt^2} = 0$$

with initial conditions,

$$\frac{dy}{dt} = 0, \quad \frac{dx}{dt} = 0$$

then we can perform our well-known kinetic calculations for the system, and see how the system evolves in the *time* direction. **A recurring theme will be that dynamics of a system in n-dimensional space can be thought of just a special type of geometry in $n+1$ dimensional space, putting time as an added dimension**.

Because the purpose of this text is to study the ways in which geometry, algebra, and physics connect, it is worthwhile to dwell on the *philosophy* behind coordinate systems.

The ball will fall from 10 meters, according to the force of gravity. That is the way the world works. It doesn't matter what coordinate system we set up to do that calculation, we should get the *exact same result*. Plainly: nature doens't *care* what coordinate system we use. This fact, obvious as it may be, is worth thinking about: No matter what coordinate system we use, the equation of motion should give the same dynamics. The laws of physics should be *independent of any coordinate system.*

Newton's law $\mathbf{F} = m\mathbf{a}$ relates the force vector to the acceleration vector. The vector representing the force $\mathbf{F}$ that you apply on a surface is an object independent of coordinate system, and so is the resulting acceleration vector. The *components* of these vectors (F_x, F_y, F_z) and (a_x, a_y, a_z), however, depend on what you have chosen for the x, y, z axes. These components *represent* a real physical vector, but only once we pick a coordinate system. If we were to pick a different coordinate system, the numbers representing the vector would change.

When we write an equation describing a physical law, it should be valid regardless of the coordinate system we use. $\mathbf{F} = m\mathbf{a}$ will always be true whether we rotate our frame of reference or not. On the other hand, if Newton's law of motion *only* said that the *first* 'x' component of the Force was equal to the *first* 'x' component of the acceleration, and said nothing about the other 2 components, then in different coordinate systems since 'x' means different things, we would get totally different equations of motion. No physical law will ever say something just about the first or just about the second components of two vectors: it must equate the entirety of the two vectors.

As another example if the equation for work looked like $F_x dx = dW$, then would give different results in different coordinate systems, because it puts emphasis on just one of the three components (the first 'x' coordinate) over the others. While in some coordinate system dx may point in the direction of the displacement and be nonzero, there may be a different coordinate system where $dx = 0$, making the work done zero. So the equation for work would be coordinate dependent: it would be wrong. The need for such invariance is why the true formula uses all three spacial dimensions and looks like:

$$\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz = dW.$$

Although it isn't obvious yet that this is a quantity that is invariant regardless of the coordinate system used, at the very least it doesn't put one component above any of the others.

1.2 Linear Algebra & Coordinates

The traditional concept of a coordinate system, a series of perpendicular lines that together associate ordered tuples of numbers to each point in n-dimensional space, is not representative of all coordinate systems. For one, we do not need the requirement that the lines be perpendicular. Our coordinate system could instead look like this:

Graphic of non-perp lines and representing a point like that

In the language of linear algebra: once we choose an origin, choosing a set of coordinate axes is the same as choosing a basis for the space (a coordinate basis). For any point in space, we can relate coordinates x'_i in the new system in terms of coordinates x_i in the old system by matrix multiplication: $x'_i = \sum_{j=1}^{n} \mathbf{A}_{ij} x_j$. This is exactly what's called a change of basis in linear algebra. Transformations between coordinate bases are exactly the invertible **linear transformations** .

As in linear algebra, we need our coordinate system to both **span** the space so that we can represent any point, and be **linearly independent** so that every point that we can represent in our coordinate system will have a unique representation. That's all that a basis is: it specifies a good coordinate system.

Definition 1.1. *A set of vectors is said to span a space $\mathbb{R}^n$ if every point P can be represented as $a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$*

Definition 1.2. *A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called linearly independent if there is only one way to represent the zero vector $\mathbf{0}$ as a combination of them, namely as $\mathbf{0} = 0v_1 + \cdots + 0v_k$.*

This second definition is the same as saying every point that we can represent in our system has a unique representation. Let's make this clear. If there were two ways to represent a point P: as

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

and

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

then subtracting these two different combinations would give a nonzero way to represent zero. Conversely, if there were a nonzero combination of vectors summing to zero, then we could add that combination to the coordinate representation of any point and get a *different* representation of the same point. So coordinate representations for all vectors are unique as long as long as there is only one representation for zero the one where each component equals zero.

Intuitively, linear independence means that there is no superfluous information in the set of vectors. We cannot linearly combine vectors in some subset to get another vector in the set; each vector is adding its own unique additional piece of information, making the set able to span in an additional direction.

Bases that don't span, or are not linearly independent, would lead to coordinate systems like these:

Show a 2-D basis in a 3-D space, and a basis of 3 vectors in 2-D space

Very often in mathematics, we ask "does a solution exist?", and "if there is a solution, is it unique?". These two questions are dual to one another. If a set of vectors spans the space, then there *exists* a way to represent any point (at least one way to represent any point). If a set of vectors is linearly independent, then *if* you can represent a point, that representation is *unique* (no more than one way to represent any point).

Now to stress the same idea again: because points in $\mathbb{R}^n$ and vectors are essentially the same thing, the idea that points in space are invariant of a coordinate system applies just as well to vectors. If we choose a basis for our vector space $\mathbf{v}_1, \ldots \mathbf{v}_n$, then we can express any vector $\mathbf{u}$ by a unique combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. We then say that in this basis, we can represent $\mathbf{u}$ by a list of numbers. Often, it is written:

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

But in some sense, writing this as an equality is wrong. The vector $\mathbf{u}$ is something physical: a velocity, a force, the flow of water. It doesn't depend on the coordinate system. On the other hand, the right hand side is just a list of numbers that depend entirely on the coordinate system chosen. If we change coordinate systems, the right hand side changes. Because $\mathbf{u}$ exists (say, in the real world) independently of coordinates used, it does not change.

A geometric vector like u is *not* a list of numbers. Once we pick a basis, u can be *represented by* a list of numbers, but if we change into a different basis, those numbers all have to change as well. This exact same idea will be the reason why a tensor is *not* just a multi-dimensional array (like the ones encountered in computer science). It can be *represented by* a multi-dimensional array once a coordinate system is chosen, but the numbers in each entry will differ depending on the coordinate system we pick.

This is very confusing (and will also be part of the reason why it's so hard to understand tensors as an undergraduate). In most math courses, we can freely call any list of numbers a 'vector'. After all, you can add lists and scale them so they do form a 'vector space'. This is a really unfortunate linguistic degeneracy in mathematics terminology. The type of vectors that we see in physics (acceleration, force, electric field, etc.) are *geometric vectors* that have nothing *a priori* to do with lists of numbers until we represent them as such by using coordinate systems. On the other hand, abstract structures that we can add and multiply by scalars are *algebraic vectors*, and lists are an example of that. To avoid confusing lists of numbers with the geometric vectors in the physical world, we will call lists of numbers *tuples* rather than vectors.

So returning to the geometric vector $\mathbf{u}$, a more careful way to write it would be:

$$\mathbf{u} = \begin{pmatrix}\mathbf{v}_1 \dots \mathbf{v}_n\end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n. \tag{1.1}$$

Once we pick a basis, that column of coordinates means something. If we denote our basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ by B, then we will use the notation

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_B = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Let us do a very simple example to start. In the 2-D plane, say we have our original basis $\mathbf{v}_1, \mathbf{v}_2$ and we rotate it by $\pi/4$ radians to get a new basis. Say we have a point P whose coordinate representation was $\mathbf{v}_1 + \mathbf{v}_2$, or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the original basis.

Now our new basis is the old one rotated by $\pi/4$ so

$$\begin{aligned}\mathbf{v}_1' &= \frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2\\ \mathbf{v}_2' &= -\frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2.\end{aligned}$$

PUT GRAPHIC HERE

As a matrix transform, we can write this as[1]:

$$\begin{pmatrix}\mathbf{v}_1' & \mathbf{v}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$$

This relates the actual basis vectors themselves. On the other hand, if we wanted to see the *coordinates* representing $\mathbf{v}_1'$ and $\mathbf{v}_1'$, then in the new basis they would simply be represented in coordinates as:

$$\mathbf{v}_1' = \begin{pmatrix}1\\0\end{pmatrix}_{\text{new}}, \ \mathbf{v}_2' = \begin{pmatrix}0\\1\end{pmatrix}_{\text{new}}$$

and in the old basis they'd be represented as:

$$\mathbf{v}_1' = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}}, \ \mathbf{v}_2' = \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}}$$

If we know that we can describe a point as $\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}$ in the new basis, then we can easily get its description in the old basis as:

$$\begin{aligned}\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}} &= x\mathbf{v}_1' + y\mathbf{v}_2'\\ &= x\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} + y\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} \qquad (1.2)\\ &= \left(\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}\leftarrow\text{new}}\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}\right)_{\text{old}}\end{aligned}$$

[1]The tuple of basis vectors $\mathbf{v}_i$ is written as a row rather than a column to be consistent with Equation (1.1). Then the coordinates are represented in a column. Because of the way we do matrix multiplication, then the matrix acts on the right. It's an issue of styling and indexing, and not physically meaningful. If we were to write this coordinate transform using columns & not rows, we'd get a matrix that's the transpose of the one above, and matrix transposes would appear in subsequent equations, making them less tidy.

This is the same matrix that related the basis vectors. We'll call it $\mathbf{A}$. $\mathbf{A}$ takes the new coordinate representations (x, y) and tells us how they'd look like in the *old* basis.

So then it is the *inverse* $\mathbf{A}^{-1}$ that tells us how our old coordinate representations of a point P will look like in our new basis.

For our point P, represented as $(1, 1)$ in our original basis, in the new basis, we would have:

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\text{old}} = \left(\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\text{old}} \right)_{\text{new}} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}_{\text{new}}$$

Indeed, $\mathbf{v}_1 + \mathbf{v}_2 = \sqrt{2}\mathbf{v}'_1 + 0\mathbf{v}'_2$.

That is the central idea. If we *vary* the <u>basis</u> $\mathbf{v}_i$ to a different basis, $\mathbf{v}'_i$, then the <u>coordinates</u> a'_i will vary the *other* way, so that the geometric vector

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = a'_1\mathbf{v}'_1 + \cdots + a'_n\mathbf{v}'_n$$

is *invariant* regardless of coordinate choice.

Let's make this precise in the general case. If we start with a set of basis vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and we make the linear transformation to a new basis $\{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ so that, as before:

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A} \tag{1.3}$$

then since the vector $\mathbf{u}$ should not change when we change our basis, we must have:

$$\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \tag{1.4}$$

so that

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A}\mathbf{A}^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

as desired. **THIS WOULD BE A GOOD EXERCISE: PROVE IT HAS TO BE A INVERSE**

We say that the basis vectors $\mathbf{v}_i$ **co-vary** and the coordinates a_i **contra-vary** with the change of basis. The idea, although it sounds simple, is rather hard to get the feel of. It's worth thinking a good bit about how coordinates and bases need to vary in opposite ways so that the physical object represented by the coordinates stays the same regardless of how we look at it.

This will be a caption for a sketch of a 3-D rotation

When you rotate your character in a video game (and in real life too, by the way), the world rotates *contrary* to the direction that you've rotated in. That's because the coordinates of what you see have *contra-varied* while your basis vectors, given by the direction you face have *co-varied*. The end result is that despite changing your coordinate system, physics stays the same: invariant. The universe did not rotate itself just because you did. This extends beyond just rotations to *all* linear transformations point.

1.3 The Notion of Length on Vector Spaces

Let us consider the property of orthogonality. It's well known that for geometric vectors, there's more that we can do than just add, scale, and transform them: we can take dot products[2] between them. When we have two geometric vectors in space, their dot product is a well defined number. If it is zero, then the vectors are orthogonal to one another. From the dot product and the magnitudes, it is possible to calculate the angle between two given vectors.

When vectors are represented in terms of tuples of numbers, the dot product was taught to us as "multiply component by component, and then sum that up". This is not, in general, what the dot product really is. Consider a basis transformation as below:

$$\begin{aligned} \mathbf{v}_1' &= 2\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{v}_2' &= \mathbf{v}_1 + 2\mathbf{v}_2. \end{aligned}$$

It's easy to compute the inverse of this matrix and see how the new coordinates should work, but it worthwhile looking at this geometrically. It is not a rotation, but more of a "stretching". Notice that

[2]Of course in 3-D we can also take a cross product. This will be discussed in the following chapters.

while in our original perspective, if we viewed $\mathbf{v}_1$ and $\mathbf{v}_2$ as orthogonal vectors, in the *new* perspective, they are *no longer* orthogonal. This is very important: linear transformations in general do not preserve orthogonality.

Include graphic here

In particular, this linear transformation has *stretched* our vector space and changed the notion of distance. Even though rotations keep distances preserved, general linear transformations don't care about a notion of distance.

If we were to take the "dot product" $\mathbf{v}_1' \cdot \mathbf{v}_2'$ just by multiplying corresponding coordinates and summing them up, then in the new basis we'd get zero, but in the original basis we would *not*. This dot product actually changes depending on the coordinate system that we use! In some sense, this is expected: all we're doing is multiplying contravariant coordinates together and summing them up. The result should be contravariant as well (in fact doubly contravariant).

The failure of the dot product to be invariant is intimately related to the fact that transformations can change lengths. This should not be too surprising. After all, the length of a vector is defined by the square root of its dot product with itself. If the dot product we learned is not invariant under general coordinate transformations, what is the right way to measure length?

It is here that there is a big subtlety. A vector space on its own does not have a notion of length. We've just seen choosing different bases would give rise to different length scales and notions of "perpendicular" as well. Endowing a vector space with a way to universally tell what the length of a vector is, or whether two vectors are perpendicular is actually adding *extra structure* to the space. It picks a whole class of specific coordinate systems and says "these are the orthogonal reference frames; the others are skewed and stretched perspectives". This allows us to measure length using an invariant *inner product*.

Euclidean space, as well as the world in which we live in, both have an natural way to measure length between two points that is invariant of the coordinate system used. A vector space on its own does not, and so it is called an *affine space*. In affine space, although there are notions like "parallel", there is not a notion of distance. Adding an inner product to affine space gives rise to Euclidean space. This will be discussed in much greater detail in the following two chapters.

1.4 Nonlinear Coordinate Systems are Locally Linear

Perhaps you may be wondering why we've spent so much time on changing between coordinate systems represented by basis vectors centered at a fixed origin. Consider the change between cartesian and polar coordinates. What does this have to do with the linear changes of coordinates that we've been discussing?

We could use something like a polar system of (r, θ) or a spherical system (r, ϕ, θ). These coordinate systems are not representable in terms of axes, but instead look like this:

Graphic of polar coordinate system/spherical

This is an example of a non-linear coordinate transformation. They are more commonly referred to as **curvilinear**. Whereas linear ones map lines to lines, curvilinear ones more generally map lines to curves. The idea for making sure that the equations of physics still stay true for non-linear coordinate transformations is to note that just like a curve locally looks like a line, a *non-linear* transformation locally looks like a *linear* one. The linear transformation that it locally looks like is called the **Jacobian** J. If the laws of physics are invariant under linear transformations locally at each point, then *globally*, they will be invariant under non-linear ones as well. That is why we cared about studying covariance and contravariance for linear transformations: more complicated cases can be reduced to their local linear behavior.

As an example, consider going from a cartesian to a polar coordinate system. We have $x = r\cos\theta$ and $y = r\sin\theta$. Certainly, this is not a linear transformation of coordinates. There is sinusoidal dependence on θ in this transformation. Physics and geometry, however, do not have laws in terms of absolute coordinates (it doesn't make sense to say "That object is located at 50 meters") but only in terms of relative distances (you'd instead say "That object is located 50 meters *relative to* me"). It is the changes over relative distances between points that we care about, and these are obtained by integrating the *infinitesimal* changes at each point.

So although x, y do not depend linearly on theta, through the use of the chain rule, we have a local linear relationship in their infinitesimal changes:

$$dx = \cos\theta \; dr - r\sin\theta \; d\theta$$
$$dy = \sin\theta \; dr + r\cos\theta \; d\theta$$

At any given point, this relationship can be written as a linear change of basis.

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

So every nonlinear transformation from some coordinate system $x_1 \ldots x_n$ to $x' \ldots x'_n$ has the local linear transformation law:

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}_{\text{old}\leftarrow\text{new}} \begin{pmatrix} dx'_1 \\ \cdots \\ dx'_n \end{pmatrix}_{\text{new}}$$

$$\Rightarrow \begin{pmatrix} dx'_1 \\ \cdots \\ dx'_n \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}}$$

This is exactly the analogue of Equation (1.4), so indeed the changes in coordinates dx_i can be called *contravariant*, just like the coordinates were for the linear transformation case. All of this is just an extension of the principle of local linearity from calculus.

Similarly, we can express the new derivative operators in terms of the old ones by using the chain rule. For polar coordinates we have

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y}$$

or more compactly we can relate just the differential operators themselves:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

so that more generally:

$$\begin{pmatrix} \frac{\partial}{\partial x'_1} \\ \vdots \\ \frac{\partial}{\partial x'_n} \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^T_{\text{new}\leftarrow\text{old}} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}_{\text{old}} \tag{1.5}$$

If you look carefully, you will see that this accordingly mirrors the co-variant change of vectors in Equation (1.3). So while the infinitesimal changes in the coordinates themselves are contravariant, just like the linear coordinates themseles, the *differential operators* corresponding to changes in these coordinates become *covariant*, just like the vectors $\mathbf{v}_i$ in the linear case. This is the first correspondence that will hint that our basis vectors $\mathbf{v}_i$ actually *correspond* to the differential operators $\frac{\partial}{\partial x_i}$.

Indeed, as we begin to move away from 3-D and n-D Euclidean space, we will see why the old notions of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are better viewed as the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$, and in general $\mathbf{v}_i \to \frac{\partial}{\partial x_i}$. This change of notation will allow us to easily pass onto far more general spaces than the Euclidean ones we've gotten used to.

1.5 Einstein's Summation Convention

Row tuples, column tuples, matrices representing basis transformations (old to new and new to old), co-variance and contra-variance. These ideas have constituted the entirety of this first chapter, and although hopefully they have not been too difficult conceptually, the matrix manipulation even at this early level is already a pain. We have to write everything in terms of tuples, and we need to arbitrarily decide which ones are rows and which ones are columns, and which matrices are transposed so that all the matrix multiplications make sense, as defined in linear algebra. A young physicist named Albert Einstein used a convention of writing all these equations so that we did not have to explicitly write out tuples of abstract basis vectors and coordinates.

The first step is to avoid explicitly writing out row tuples and column tuples. To do this, instead of writing out a whole tuple to represent a vector like $(v_1, \ldots, v_n)$ we will simply write $\mathbf{v}_i$. v_i should be viewed as the whole vector, rather than an *ith* component. The index i is *free* and can be anything.

The reason it is preferable to view v_i as the whole vector rather than a specific ith component of it is because of the same reason given at the end of Section 1.1, that we never care about just a specific component, but rather the vector as a whole.

The second step is to be able to differentiate between *covariant* quantities and contravariant quantities. The convention is this: if the quantity is covariant, like a basis vector, then write its index *downstairs*: $\mathbf{v}_i$. On the other hand, of a quantity is contravariant (like a coordinate) then write its index *upstairs* as a^i instead[3]. Then we can write Equation (1.1) as

$$\mathbf{u} = \sum_i a^i \mathbf{v}_i \tag{1.6}$$

For another example, the gradient operator was written $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and is now written simply as $\partial/\partial x^i$ (adopting upper indices for x^i since coordinates contra-vary). This means that Equation (1.5) can be written as

$$\frac{\partial}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j}. \tag{1.7}$$

Since i is free to be anything (while j is bound since it is being summed over), this equation holds for every i in $1, \ldots, n$ and so is indeed an equation relating two *vectors* on both sides. Note also that this means the transformation matrix can be written as $\frac{\partial x^j}{\partial x'^i}$. By writing just a summation and not any explicit matrices, we avoid having to worry about unnecessary troubles like transposes, etc. Note also that this matrix has both upper and lower indices so that the upper index in $\frac{\partial x^j}{\partial x'^i}$ gets multiplied with the lower index $\frac{\partial}{\partial x^j}$ to cancel out, and all that remains is the lower index i. All the trouble of seeing what's covariant, what's contravariant, and what's invariant is washed away into just seeing how the upper indices and lower indices cancel out in the end.

Sometimes in the mathematics literature in places where co- and contra-variance are of lesser importance, upper and lower indices are

[3]If you are worried that this will be confused with exponentiation, don't be. In practice, such confusion rarely arises.

ignored and the notation describing something like a matrix-vector multiplication looks like this:

$$(\mathbf{A}\vec{v})_i = \sum_j \mathbf{A}_{ij}\vec{v}_j.$$

That is, the ith component of the product $\mathbf{A}\vec{v}$ is the sum over j of the ijth component of $\mathbf{A}$ with the jth component of $\vec{v}$. For a matrix-matrix multiplication this would look like

$$(\mathbf{AB})_{ik} = \sum_j \mathbf{A}_{ij}\mathbf{B}_{jk}.$$

Notice the pattern: when we wish to multiply these objects, we sum over a common index (j in the above equations) that we make both objects share. The point is *an index is always repeated and summed over* when we do a multiplication.

Einstein took the bold, but ultimately brilliant step of making the convention that if we ever write an index twice, that *automatically means* that we are summing over it (unless we explicitly say we aren't). The above equations, in Einstein's scheme, now become

$$(\mathbf{A}\vec{v})_i = \mathbf{A}_{ij}\vec{v}_j$$
$$(\mathbf{AB})_{ik} = \mathbf{A}_{ij}\mathbf{B}_{jk}.$$

A dot product between v and itself would simply be $\vec{v}\cdot\vec{v} = \vec{v}_i\vec{v}_i$ in Einstein's convention.

Now returning back to caring about co-variance and contra-variance, this scheme makes every equation in the chapter shockingly short. Equation (1.1) becomes $\mathbf{u} = a^i\mathbf{v}_i$, Equation (1.3) becomes $\mathbf{v}'_i = A_i^j\mathbf{v}_j$, Equation (1.4) becomes $a'_i = (A^{-1})_i^j a_j$. Equation (1.7) is further reduced to just

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}.$$

Similarly the transformation of infinitesimal coordinate changes is now just

$$dx'^i = \frac{\partial x'^i}{\partial x^j}dx^j.$$

EXERCISE: Check invariance quickly using this method.

By doing exercises, this method is very quick and steady to get the hang of. We will adopt it for the rest of the book.

Chapter 2

New Horizons Developed

2.1 The Manifold

Several thousand years ago, the first sentient human beings noticed that the landscape of the earth looked flat, and seemed to stretch out infinitely far in every direction. It is perhaps from this observation that the Euclidean plane was first conceived, and indeed it is from the fact that the earth looked like Euclid's 2D plane that geometry got its name to literally mean "measuring the earth". But the fact is that the earth is *not* a flat plane, stretching out infinitely. It turned out to be a sphere. What is true, however, is that *locally*, the geometry of the earth looks very similar to that of Euclidean space.

And now in modern times, as we look out into the cosmos and see them stretching out in every direction, our first human bias creeps in and tells us "this thing must be infinite, stretching out in every direction". Just as people thought the world was $\mathbb{R}^2$ in ancient times, in this age we entertain the thought that our universe could be three-dimensional Euclidean space $\mathbb{R}^3$. Indeed, most of the time when we do simple classical physics, we embed our system into a space that is $\mathbb{R}^3$ and work there. It is an easy space to work in.

But just as the earth's surface looked *locally* like Euclidean 2-space but in fact turned out to globally be wildly different, we should not be surprised if it turns out that the universe, despite locally looking Euclidean, has wildly different global structure.

This is exactly what a manifold intuitively is: an object that at each point locally resembles Euclidean space. The property of being locally Euclidean is similar to the property that differentiable functions have of being locally linear. It allows us to use calculus on them to reduce nonlinear objects to linear ones locally.

A line is a one-dimensional manifold (in fact it *is* a Euclidean space). circle is a one-dimensional manifold as it locally resembles a line, and so are ellipses, parabolas, hyperbolas, and the graph of any smooth function. A sphere is the two dimensional manifold that ancient humans mistook for the plane itself. The Mobius strip is also a two-dimensional manifold. Although globally it is a twisted band, locally it, too looks like flat two-dimensional space. All the geometric objects that you've encountered that before were classified into sets like "curves" or "surfaces" are bundled together into one category: manifolds.

In order to formally define a manifold, we must first define a **topological space**. A topological space X is a set with a family of open subsets T satisfying

1) $\emptyset$ (empty set) and X are in T

2) A finite union of members of T is in T

3) A finite intersection of members of T is in T

and T is called the **topology** on X. We can talk about points x in the set X and we say that a collection of open sets $\{O_\alpha\}$ **covers** X if X is contained in their union. Intuitively, the information contained in such a collection of sets should be enough to talk about all of X. The last thing we need is a **chart**, which is an open set O_a paired with a continuous function $\varphi_\alpha : O_\alpha \to \mathbb{R}^n$ with continuous inverse. This map is identifying each point in the subset O_α with a corresponding point in $\mathbb{R}^n$. We can now define a manifold

DRAW A PICTURE OF A SPHERE COVERED BY CHARTS

We need to be able to use **Talk about the sphere here (not the metric part)**

Just like in $\mathbb{R}^n$ where had different coordinate systems around an origin, on a manifold M, we will *locally* at each point have coordinate systems that look exactly like the ones we used for $\mathbb{R}^n$ in section 1.2.

Just because we have a coordinate system to describe the manifold doens't mean we have everything. It may seem strange, but up until now we have missed talking about a vital part of geometry: the notion of *distance*. Even though we've talked in all the way points can be represented by coordinates, none of these numbers representing coordinates have any *inherent* notion of distance to them.

Concept 2.1. *A smooth manifold is a continuous collection of points which can be invertibly mapped into Euclidean space in a neighborhood of any point.*

Now that we prepared the stage, we are ready to study the fields which can exist on it.

2.2 Intrinsic Geometry

2.3 The Field

One of the most important aspects of physics is studying the *fields* that live on manifolds. Just like in multivariable calculus, this means the study of scalar fields ϕ that associate a number to every point P. Examples are voltage, potential energy, mass/charge density, etc. This also includes the study of vector fields $\mathbf{v}$ associating a vector to each point. These can be wind speeds, electric fields.

While scalar fields associate a number $\phi(P)$ to each point P, which looks the same regardless of the local coordinate system used at P, vector fields will associate a specific vector $\mathbf{u}$ to a point P whose coordinate representation will, of course, change depending on the local coordinate system at P.

[I want to somehow motivate why the vectors should looks like $\partial/\partial q_i$]

Show a graph of the curves for coordinates q_i

2.4 What Follows

The rest of this book will expand both on the geometry of fields and manifolds, and also on the larger ideas of groups, homogenous spaces, and representations.

In chapter 1, we will continue studying the fields that live on manifolds. We'll prove the General Stokes' theorem, an elegant generalization of the divergence, curl, and line integral theorems that have been taught in multivariable calculus. From there, we will study more thoroughly the concept of a metric, and how this relates vector fields to differential forms. The notion of a derivative will be extended to manifolds, and will take the form of a “Lie Derivative”.

In chapter 2, we will introduce Fourier Analysis as a powerful tool for studying functions on the real line an Euclidean space. Then we will shift to looking at the representation theory of *finite* groups and illustrate the parallels. We will then return to the study of continuous group actions on especially symmetric “homogenous” spaces, and show how Fourier analysis is related to their representation theory. Towards the end, we will expand on the idea behind a group actions on manifolds and look at the representation theory, giving a small glimpse into harmonic analysis: the Fourier transform on manifolds. Just as in the first chapter, we'll recognize the importance of the underlying differential geometry of the group action. The underlying differential structure is known as the “Lie Algebra” of the group, and we will discuss that.

In chapter 3, we introduce some background behind Lie Algebras. We put almost all of our focus on one special case: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The relationship between this algebra and the symmetries of the sphere are explored, as well as its applications in quantum physics for studying angular momentum. The representation theory of a variant of this algebra gives rise to the concept of spin.

In chapter 4, we move further into physics, going over classical Lagrangian and Hamiltonian Mechanics. We discuss Noether's theorem in both the Lagrangian and Hamiltonian Pictures, and then we move to study Hamiltonian mechanics using the language of differential geometry that we have developed. This will give rise to *symplectic geometry.* In chapter 7, combining this with representation theory gives rise to *quantum mechanics.*

In chapter 5, we apply differential geometry first to the study of electromagnetism, and then to gravitation. We shall arrive at Einstein's theory of gravity. Along the way, we study in even greater detail the notion of a metric, a connection, and curvature.

In chapter 7, we use the representation theory and differential geometry that we have developed so far to study how quantum mechanics can arise from quantizing a symplectic manifold.

Finally, chapter 8 studies Lie algebras in greater detail, working towards the *classification of complex semisimple Lie algebras*. Along the way, we will look at the relationship between representation theory of Lie algebras and modern physics.

Part 1

A Better Language

Chapter 3

Differential Geometry

In calculus class you were taught the fundamental theorem, that the total difference of a function's value at the end of an interval from its value at the beginning is the sum of the infinitesimal changes in the function over the points of the interval:

$$\int_a^b f'(x)dx = f\Big|_a^b \tag{3.1}$$

And later, in multivariable calculus, you encountered more elaborate integral formulae, such as the divergence theorem of Gauss:

$$\int_\Omega \nabla \cdot \mathbf{F}\, dV = \int_S \mathbf{F} \cdot d\mathbf{S} \tag{3.2}$$

where Ω is the volume of a 3D region we are integrating over, with infinitesimal volume element dV and S is the surface that forms the boundary of Ω. dS then represents an infinitesimal parallelogram through which $\mathbf{F}$ is flowing out, giving the flux integral on the right. Read in english, Gauss' divergence theorem says "Summing up the infinitesimal flux over every volume element of the region is the same as calculating the total flux coming out of the region". The total flux coming out of a region is the sum of its parts over the region. You might see that in english, this reads very similar to the description of the fundamental theorem of calculus.

Alongside this, there is Stokes' theorem for a 2D region. In english: summing up the infinitesimal amount of circulation of a vector field $\mathbf{F}$ over every infinitesimal area is equal to calculating the total circulation of $\mathbf{F}$ around the boundary of the region. In mathematical language:

$$\int_R \nabla \times \mathbf{F}\, dA = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{3.3}$$

where R is our region and C is its boundary.

Perhaps now, the pattern is more evident. In all the above cases, summing up some *differential* of the function on the interior of some region is the same as summing up the function itself at the *boundary* of the region. All these theorems, that on their own look so strange to a first-year calculus student, are part of a much more general statement, the **General Stokes' Theorem**:

Theorem 3.1 (General Stokes' Theorem)**.**

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{3.4}$$

Above, ω is an object that will generalize both the "functions" and "vector fields" that you've seen in multivariable calculus, and d will generalize of all the differential operators (gradient, divergence, curl) that you've dealt with. Lastly, when Ω is the region in question $\partial\Omega$ represents the *boundary* of the region Ω. The fact that it looks like a derivative symbol is no coincidence, as we'll see that the natural way to define the "derivative" of a region is as its boundary.

Through abstraction, we can reach results like this that not only look elegant and beautiful, but also provide us with insight into the natural way to view the objects that we've been working with for centuries. This gives us not only understanding of what language to use when studying mathematics, but also what is the natural language in which to describe the natural world. The general Stokes' theorem is one of the first examples of this beautiful phenomenon, and this book will work to illustrate many more.

For the first half of this chapter, we will work towards giving the intuition behind this result. On our way, we will begin to slowly move into a much more general setting, beyond the 3-dimensional world in which most of multivariable calculus was taught. That doesn't just mean we'll be going into n-dimensional space. We'll move outside of euclidean spaces that look like $\mathbb{R}^n$, into non-euclidean geometries. This will put into question what we really mean by the familiar concepts of "vector", "derivative", and "distance" as the bias towards Euclidean geometry no longer remains central in our minds. At its worst, the introduction of new concepts and notation will seem confusing and even unnecessary. At its best, it will open your mind away from the biases you've gained from growing up in a euclidean-looking world, and give you a glimpse of how modern mathematics *actually* looks.

Modern mathematics is learning that the earth isn't flat. To someone who's never had those thoughts, it is difficult to get used to, tiring, and sometimes even rage inducing, but to someone who has spent months thinking and reflecting on it, it quickly becomes second nature. Far from being the study of numbers or circles, it is a systematic climb towards abstraction. It is a struggle towards creating one language, free from all-encompassing human bias, in order to try and describe a world that all other human languages, for so many centuries, have failed to grasp. It is humbling, and in the strangest of ways, it is profoundly beautiful.

3.1 The Derivative and the Boundary

Let's start working towards understanding Equation (3.4). First, let's work with what we've already seen to try and explore the relation between integrating within a region and integrating on the boundary.

If we are in one dimension, we have a function f defined on the interval $x \in [a, b]$. Proving Equation (3.1) is much easier than you'd think. Let's take a bunch of steps: $x_i = a + (b - a)i/N$, so that $x_0 = a, x_N = b$. Then all we need is to form the telescoping sum:

$$\begin{aligned} f|_a^b &= f(x_N) - f(x_0) \\ &= \sum_{i=1}^{N} f(x_i) - f(x_{i-1}). \end{aligned}$$

If we make the number of steps N large enough, the stepsize shrinks so that in the limit, we get

$$\begin{aligned} \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i) - f(x_{i-1}) &= \lim_{N\to\infty} \sum_{i=1}^{N} \Delta f \\ &= \int_a^b df. \end{aligned}$$

Of course, the way its written more often is:

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{\Delta f}{\Delta x}\Delta x = \int_a^b \frac{df}{dx}dx.$$

What is the idea of what we've done? At each point we've taken a difference of f at that point with f at the preceding one. Because we're summing over all points, the sum of differences between neighboring points will lead to cancellation everywhere *except* at the boundary, where there will not be further neighbors to cancel out the $f(b)$ and $f(a)$. From this, we get Equation (3.1).
Note: Now for a distinction which may seem like it isn't important. We haven't integrated from point a to point b. We have integrated from where the coordinate x take *value* a, to the where coordinate x takes *value* b. a and b are *NOT* points. They are numbers, values for our coordinate x. As we have said in the preceding chapter, the idea that numbers form a *representation* for points is ingenius, but numbers are *not* points. Although we could write this interval as $[a, b]$ in terms of some variable x, it would be a completely different interval should we have chosen a different coordinate u. This is why, when doing u-substitution, we change the bounds. In coordinate free, language, then:

Theorem 3.2 (Fundamental Theorem of Calculus)**.** *For a given interval I with endpoints p_0, p_1 and a smooth function f, we have*

$$\int_{p_0}^{p_1} df = f\Big|_{p_0}^{p_1} \tag{3.5}$$

Notice something: the end result doesn't depend on the partition x_i at all, so long as it becomes infinitesimal as $N \to \infty$. That is to say: we are summing up the change of f over some interval, but it doesn't matter what coordinate system we use to describe this interval. The integral is *coordinate independent.* We chose to use x as our coordinate, describing the interval as going from $x = a$ to $x = b$, but we didn't *have* to make this specific choice. This makes perfect physical sense. For example, if we had a temperature at each point in space, the temperature difference between two fixed points some shouldn't depend on whether we use meters or feet to measure their distance apart.

Written mathematically:

$$\int_I df = \int_I \frac{df}{dx}dx = \int_I \frac{df}{du}du$$

If we chose an I that's very small around some point, essentially an infinitesimal line segment, we get:

$$\frac{df}{dx}dx = \frac{df}{du}du \Rightarrow \frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

this is the u-substitution rule from calculus.

Now what if f was a function defined not on the real line $\mathbb{R}$, but on 2-dimensional space $\mathbb{R}^2$, or more generally n-dimensional space $\mathbb{R}^n$. To each point $p = (p_1, \ldots, p_n)$ we associate $f(p)$. Now again, consider $f(p_f) - f(p_i)$ for two points in this space.

For any curve C going between p_i and p_f, say defined by $\mathbf{r}(t)$ for t a real number going from a to b, we can make the same partition $t_i = a+(b-a)i/N$ and let N get large. Again, it becomes a telescoping sum:

$$\begin{aligned} f(p_f) - f(p_i) =& f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ =& \sum_{i=1}^{N} f(\mathbf{r}(t_i)) - f(\mathbf{r}(t_{i-1})) \\ =& \sum_{i=1}^{N} \Delta f_i \to \int_C df. \end{aligned}$$

Now if we cared about coordinates, we could ask "how can we write df in terms of dt or dx_i?".

We know from the multivariable chain rule that the infinitesimal change of f is the sum of the change in f due to every individual variable, so:

$$df = \sum_i \frac{df}{dx_i}dx_i \tag{3.6}$$

We know that dx_i together must lie along C. In terms of t since $x_i = r_i(t)$, we have $dx_i = \frac{dr_i}{dt}dt$ giving:

Theorem 3.3 (Fundamental Theorem of Line Integrals)**.** *For a smooth function f defined on a piecewise-smooth curve C parameterized by* $\mathbf{r}(t)$

$$f|_{p_i}^{p_f} = \int_C \sum_i \frac{df}{dx_i}\frac{dr_i}{dt}dt = \int_C \nabla f \cdot \frac{d\mathbf{r}}{dt}dt = \int_C \nabla f \cdot d\mathbf{r} \tag{3.7}$$

The proof of this was no different from the 1-D case.

Let's go further. Consider a region in three dimensions. We want to relate the total flux coming out of the region to the infinitesimal flux at each point inside the region. To do this, as before, we will subdivide the region. This time, it will not be into a series of intervals, but instead into a mesh of increasingly small *cubes*, as below.

PUT A GRAPHIC HERE

See that the flux out a side of each cube is cancelled out by the corresponding side on its neighboring cube. That means that the only sides that do not cancel are for the cubes at the boundary[11], giving us the desired flux out.

So if we sum the fluxes over all infinitesimal cubes, we will get the total flux out of the boundary. For a single cube of sides dx, dy, dz, drawn below, the total flux will be the sum over each side.

$$\begin{aligned}\text{Flux} = \quad & \mathbf{F}(x, y, z + dz/2)dxdy - \mathbf{F}(x, y, z - dz/2)dxdy \\ & + \mathbf{F}(x, y + dy/2, z)dxdz - \mathbf{F}(x, y - dy/2, z)dxdz \\ & + \mathbf{F}(x + dx/2, y, z)dydz - \mathbf{F}(x - dx/2, y, z)dydz\end{aligned}$$

SHOW GRAPHIC HERE

But we can write this as:

$$\left(\frac{\partial \mathbf{F}(x,y,z)}{\partial x} + \frac{\partial \mathbf{F}(x,y,z)}{\partial y} + \frac{\partial \mathbf{F}(x,y,z)}{\partial z}\right) dxdydz = \nabla \cdot F\ dV$$

So the total flux will be the sum over all these cubes of each of their total fluxes. But then this becomes exactly the divergence theorem:

[1]You may be worried that the cubes do not perfectly fit into the boundary when it is not rectangular. As the mesh gets smaller and smaller, this does not pose a problem. This can be made more rigorous (c.f. **GIVE A REFERNCE HERE**)

Theorem 3.4 (Divergence Theorem, Gauss)**.** *For a smooth vector field* $\mathbf{F}$ *defined on a piecewise-smooth region* Ω*, then we can relate*

$$\int_{\Omega} \nabla \cdot \mathbf{F}\ dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

It is an easy **exercise** to show that this exact same argument holds for an n-cube.

What did we do? In the fundamental theorem of calculus/line integrals, we had a function f evaluated on the 1-D boundary, and we chopped the curve into little pieces that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece, and found that it was $df = f'(x_i)dx$, meaning that the evaluation on the boundary could be expressed as an integral of this differential quantity over the curve. That is Equation (3.1).

For the divergence theorem, we had a vector field $\mathbf{F}$, again *evaluated on the boundary*, this time in the form of a surface integral. We chopped the region into little pieces (cubes now) that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece and found that it was $\nabla \cdot \mathbf{F}\ dV$, meaning that the integration on the boundary could be expressed as an integral of this differential quantity over the region. That is Equation (3.2).

Through abstraction, we see that there is really no difference. Perhaps now Equation (3.4) does not look so mysterious and far-off.

For Equation (3.3), we have a vector field $\mathbf{F}$ evaluated on the boundary in the form of a contour integral around a region. This is the total circulation of $\mathbf{F}$ around the region. Let us chop the region into little pieces.

INSERT GRAPHIC HERE

On an infinitesimal square, we get that the circulation is:

$$\begin{aligned} \text{Circulation} = \quad & \mathbf{F}(x + dx/2, y)dy - \mathbf{F}(x - dx/2, y)dy \\ & + \mathbf{F}(x, y - dy/2)dx - \mathbf{F}(x, y + dy/2)dx \end{aligned}$$

This can be written as:

$$\left(\frac{\partial \mathbf{F}}{\partial x} - \frac{\partial \mathbf{F}}{\partial y}\right) dxdy = \nabla \times \mathbf{F}\ dA$$

so that

Theorem 3.5. *For a smooth vector field on a piecewise smooth region* S

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) dA \tag{3.8}$$

Exercise **(MAKE AN EXERCISE)** generalizes this to a surface in 3D, to get the 3D version of Stokes' theorem :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \tag{3.9}$$

The philosophy behind these proofs is always the same. It is the manipulation of the differentials that seems wildly different every time. The curl looks nothing like a divergence, and a divergence is distinct from a gradient. Moreover, its not clear in what way each one generalizes the one dimensional derivative $df = f'(x)dx$. This is the problem that the symbol 'd' in Equation (3.4) was made to solve.

We need to stop thinking of the 1-d derivative, the gradient, the divergence, and the curl, as unrelated operations. They are in fact, the same operation, applied in different circumstances. Infinitesimal change, flux, and circulation are all the same type of derivative, acting on different types of objects.

Perhaps part of this was clear from multivariable calculus: the gradient is nothing more than a generalization of the derivative to functions of multiple variables. But then why are there seemingly two different, unrelated types of "derivative" on vector fields? Instead of a regular, gradient-like object, we have two: the divergence and the curl.

It will turn out that the reason that there are two is this: the vector fields that we take curls of are a different type of object from the vector fields we take the divergence of. To see this more clearly, we need to stop thinking of functions and vector fields as totally separate objects. Every object that we've encountered when integrating: from functions in 1-D or 3-D, to vector fields in n-D, have been examples of **forms**.

As a final note of this section, let us try to give a sketch for why on a region Ω, we denote its boundary with the partial derivative symbol as $\partial\Omega$. Picture in your mind a ball (interior of a sphere) of radius

r, B_r. If we increase the radius by a tiny amount h then we have a slightly larger radius B_{r+h}. If we took the difference $B_{r+h} - B_r$, by which we mean all the points of B_{r+h} that are not in B_r, we would be left with a thin shell. In the limit as $h \to 0$, this becomes a sphere of radius r, precisely the boundary of B_r (note that a sphere is always the two-dimensional boundary of the ball). See how similar this is to taking derivatives. This is why ∂B_r is what we use to denote the sphere boundary of the ball.

You may ask "but what about dividing by h at the end, like we do for a regular derivative?". This also has an interpretation. The 3D volume of a sphere is zero, since it is a 2-D boundary. Dividing by h as h goes to zero puts increasing "weight" on the shell so that as the shell shrinks to becoming absolutely thinness, 3-D integrals on it become 2-D [2].

3.2 The Notion of a Form

A differential form ω, in short, is an object that is meant to be integrated. The simplest example of a differential form is something you have often dealt with: $\omega = g(x)dx$. At every point p in space, ω represents the infinitesimal change $g(p)dx$. If we were using another coordinate system u instead of x, to measure length, then at point p, if we want to write ω in terms of the coordinate change du, we would have

$$\omega = g(p)dx = \left(g(p)\frac{dx}{du} \right) du = \tilde{g}(p)du \tag{3.10}$$

So if we change our coordinate system, because dx changes to du, g must change to $\tilde{g} = g\frac{dx}{du}$ to counteract this, so that the total change is the same. Because at each point, ω represents a one-dimensional differential line segment, it is meant to be integrated along a one dimensional *curve*.

So more generally than the real line, on a curve, you want to integrate some vector field that perhaps you would write in cartesian

[2] For those familiar with the terminology: dividing by h corresponds to multiplying by a dirac delta that spikes exactly on the sphere. This turns integrals over 3-D space into 2-D integrals on the sphere

coordinates like:

$$\mathbf{F} = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}$$

But you are not actually integrating this field $\mathbf{F}$. You're integrating $\mathbf{F} \cdot d\mathbf{r}$, appropriately multiplying $\mathbf{F}$ by an infinitesimal change in distance along the curve. This gives the *form* that you would integrate:

$$\omega = Pdx + Qdy + Rdz$$

This is what we care about when integrating. It is more fundamental than $\mathbf{F}$, but what does it mean *physically*? If $\mathbf{F}$ was a force field, then since we know $\mathbf{F} \cdot d\mathbf{r} = dW$, this form ω represents all possible infinitesimal changes in work dW at a given point, depending on what changes dx, dy, dz we do.

If we were actually *given* the changes in each of the coordinates dx, dy, dz, we could plug them in to ω and get the first-order approximation of the amount of work done over that distance. This is very important to understand! ω does not represent a specific change in work, but rather the *relationship* between the changes in coordinate and the change in work. If you *give it* an infinitesimal displacement, it will tell you the associated work. When integrating along a curve, the displacement is simply the tangent vector to the curve.

Because there is only one differential multiplying each term (be it dx or dy), we call such ω **one-forms**. It is easy to show **MAKE AN EXERCISE** that the sum of one-forms is still a one-form, and that multiplying a one-form by a function keeps it as a one-form.

Even simpler than one-forms are the **zero forms**, with no differentials appearing. A zero-form precisely a scalar function at $f(p)$ each point p. Regardless of how we change our coordinate system, the value of the *function* at point p is the same.

We are now in a good place to define d, at least for going from functions (zero-forms) to one-forms. Given a function f, $\mathrm{d}f$ will produce a form representing the local change in f depending on the displacement. We call d the **exterior derivative** operator.

For example, for a potential energy function ϕ, $\mathrm{d}\phi$ can be written as

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i} dx^i \tag{3.11}$$

because of d, we will no longer have to use the gradient at all. This is more important than simply meaning that we'll grow to stop using $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. It is something much deeper. In in two-dimensional motion, if you have some potential ϕ at a point p, then of course the value of ϕ at p is independent of any coordinate system you use. If you have two cartesian coordinates, say x, y, then you can define the x, y components of force by

$$\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}}$$

If our coordinates were r, θ, then the analogous force would be

$$\mathbf{G} = G_\theta\hat{\theta} + G_r\hat{\mathbf{r}} = \frac{\partial \phi}{\partial \theta}\hat{\theta} + \frac{\partial \phi}{\partial r}\hat{\mathbf{r}}$$

Note that the first component has units not of force, but of force times distance. It is precisely the torque that the potential induces. In this sense, quantities like torque are precisely just generalizations of force to non-cartesian coordinate systems (polar, in this case). The second component is just radial force, plain and simple.

These two "forces" have components that mean completely different things, and cannot easily be compared. On the other hand, since $\mathrm{d}\phi$ is independent of coordinate system, we get:

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = \frac{\partial \phi}{\partial \theta}d\theta + \frac{\partial \phi}{\partial r}dr \tag{3.12}$$

All forces (including the generalized forces, like torque) come from the differential form. If you're working in a coordinate system x^i, whether it be cartesian x, y, z or polar r, θ, then the coefficient corresponding to dx^i is precisely the generalized force associated with that coordinate.

So if we have a 1-form ω we can write it in terms of its components as $\omega = \sum_{i=1}^n \omega_i dx^i$. In general, only a special set of ω are exterior derivatives of functions. In multivariable calculus, we studied conservative vector fields as arising from gradients of functions. The language we'll use here for the same phenomenon is that ω is **exact** if it is the exterior derivative of a 0-form.

Concept 3.6 (One-Forms Relate Change to Direction)**.** *For a function ϕ, the one-form $\omega = \mathrm{d}\phi$ gives the first-order change in ϕ along* *a given direction* $(dx^1, \ldots, dx^n)$. *In general, for a one-form* ω *that is not exact,* ω *along a given direction* $(dx^1, \ldots, dx^n)$ *gives the change in a quantity that cannot be represented by a function of the coordinates. This occurs, for example, with non-conservative forces such as friction or when calculating heat added to a system.*

A classic example is $d\theta$. Although locally, θ can be defined just by calculating the angle from the x axis, if you go around counterclockwise in a circle containing the origin, then θ continuously increases. At the end of the revolution, even though you are at the same point, θ has increased by 2π. So although $d\theta$ makes sense locally as a differential form everywhere in the plane minus the origin, we cannot define a global smooth function representing θ without a discontinuity.

We've now shown how the fundamental theorem of line integrals deals with the exterior derivative. The next step is to go into 2D and show how we can define the exterior derivative in just the right way to get Stokes' theorem for curl in 2D (also known as Green's theorem). Because the language is suggestive, you would expect that there are two-forms for integrating over two-dimensional regions.

Now instead of having individual quantities like dx to represent an infinitesimal-length line segment, we will want quantities to represent a *infinitesimal areas* that will cover the surface that we integrate on. These areas need to be defined by two directions, dx^i and dx^j.

This is different from the vectors and forms that we've encountered before. Forms probably seem very similar to vectors. There are components associated with each dx^i. Even though philosophically they are deeply tied with infinitesimal quantities and integration, together with vectors they both correspond to some object that deals with 1-D lengths.

A 2-form, on the other hand, is a "product" of one forms in a similar way to how area is the product of lengths:

Insert Graphic Here

We denote the 2-form representing the infinitesimal area formed by one-forms dx^i and dx^j by $dx^i \wedge dx^j$. This is called the wedge product between dx^i and dx^j.

Concept 3.7. *The wedge product of coordinate one-forms* dx^i, dx^j *is geometrically defined to be the infinitesimal parallelogram with one* *side along the increment of dx^i and the other side along the increment of dx^j*

Note that it is not as easy as just defining the area to be $dxdy$, like a simple scalar. This two-form is a vector-like object. Indeed, the set of all two forms in some dimension form a vector space: we can add them, we can scale them by functions, and we have 0 to be a trivial two form of no area.

What properties does this wedge product have?

Proposition 3.8 (Properties of $\wedge$). *The wedge product satisfies:*

1. $dx^i \wedge dx^i = 0$
2. $(\alpha dx^i) \wedge dx^k = \alpha(dx^i \wedge dx^j)$
3. $(dx^i + dx^j) \wedge dx^k = dx^i \wedge dx^k + dx^j \wedge dx^k$

Three forms? Infinitesimal parallelepipeds. Past that, it gets difficult to visualize, but you get the idea. Moreover, the formalism does not change.

Talk about coordinate independence of the FTOC and now how we get it for the proof in the divergence theorem

Example in 1-D, 2-D, and 3-D

3.3 Stokes' Theorem

3.4 To Manifolds, Coordinate Freedom

3.5 Vectors, Forms, and Tensors

3.6 Distance, a Metric

3.7 Movement, Lie's Ideas

First, something cool. Euler's identity $\rightarrow e^{a\frac{\partial}{\partial x}}$

3.8 Exercises

Chapter 4

Beyond Harmonics: Representation Theory

Part 2

Physics

Chapter 5

Symmetries of the sphere: $\mathrm{SU}(2)$ and friends

Chapter 6

Classical Mechanics and Symplectic Geometry

Chapter 7

Einstein's Gravity

Part 3

More Advanced Topics

Chapter 8

An Introduction to Quantization

Chapter 9

Classification of Simple Lie Algebras over $\mathbb{C}$

Index