Representations

of the

Physical Universe

Todo list

Preface

Dedicated to $\otimes$,

...

Contents

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Part 0

Prerequisite Material

Chapter 1

Old Notions Revisited

1.1 The Cartesian Coordinate System

One of the most revolutionary ideas in mathematics and physics draws its name from the seventeenth century philosopher René Descartes. The idea is simple: any point P in the Euclidean plane can be represented by a pair of real numbers (x, y) in a one-to-one correspondence. The numbers themselves represent distances along two perpendicular axes that meet at a special point, O, called the **origin** and corresponding to $(0, 0)$. This construction gives the **cartesian coordinate system**, which does something amazing: it relates the geometry of the plane to the algebra of variables and equations. Algebra could be represented geometrically, and conversely geometric problems could be solved by transforming them into algebraic ones. The two great classical branches of mathematics, the study of space and the study of quantity, have been joined.

Insert 2D Plane with Coordinates

If the above construction seems obvious to the reader, it should be the same type of obviousness that would strike one when staring at a wheel, perplexed by how such a simple concept went undiscovered for so long in so many cultures. It should speak to the nontrivial character of the Cartesian coordinate system that Descartes himself did not discover the full picture as we know it today. He introduced a variant where distances were measured from a *single* reference axis. It took many years before the idea to add a second perpendicular axis took root, and for the modern treatment to be established.

Categorical diagram of the concepts **POINT ↔ ASSIGNING SPECIFIC VALUE TO** (x, y), **CURVE ↔ EQUATION RELATING VALUES OF** (x, y), **PLANE ↔ ALL POSSIBLE VALUES OF** (x, y), **GEOMETRY ↔ ALGEBRA**

Coordinates soon extended beyond just being pairs of numbers (x, y). Their use applied just as well to describing points in 3D space in reference to a fixed origin by a triple of coordinates (x, y, z). Soon enough after, they were generalized to arbitrarily high dimensions: any point in n dimensional space could be represented by an n**-tuple** $(x_1, x_2, \ldots, x_n)$. Despite some controversy at the time of their invention, these tuples of numbers would subsequently be used to lay the foundations for modern physics and mathematics. Modern linear algebra, multivariable calculus, and the myriad of connections between algebra and geometry begin with the concept of a coordinate.

Since then, the use of coordinate systems has proven indispensable to scientists and mathematicians throughout history. Newton used Descartes' ideas to formulate his infinitesimal calculus. Maxwell used it to analyze electromagnetic fields, discovering mathematically that light is a wave in the electromagnetic field. Einstein, going further, made use of coordinate systems on a four dimensional space of three dimensions for physical space and one for time to formulate his theory of relativity and gravitation. Today, physicists and engineers base their calculations in the frames of coordinate systems. In mathematics, Descartes' idea planted the roots for what would turn into the modern field of algebraic geometry.

When studying a geometry in some n-dimensional space, we pick an origin and set of axes to form a coordinate system. Once a cartesian coordinate system has been set, can represent any point as a tuple of n real numbers. Because of this, n-dimensional space is frequently just referred to as $\mathbb{R}^n$.

For a ball falling, we could set the origin at some point on the ground, and pick one axis parallel to the ground, "distance", and one perpendicular, "height". We can decide to measure the axes in meters, or we could decide to measure in feet (nothing stops us from making bad choices of coordinates). The physical point P where the ball lies is represented by $(x, y) = (0\,\mathrm{m}, 10\,\mathrm{m})$.

Insert Ball Falling

We can now study y, free of geometry, as just a function on which we can do arithmetic and calculus. If we are given equations of motion, say

$$\frac{d^2y}{dt^2} = -g\,\mathrm{m/s^2}, \qquad \frac{d^2x}{dt^2} = 0\,\mathrm{m/s^2}$$

with initial conditions,

$$x = 0\,\mathrm{m},\ y = 10\,\mathrm{m} \qquad \frac{dy}{dt} = \frac{dx}{dt} = 0\,\mathrm{m/s}$$

then we can perform our well-known kinetic calculations for the system, and see how the system evolves in the *time* direction. **A recurring theme will be that dynamics of a system in n-dimensional space can be thought of just a special type of geometry in $n+1$ dimensional space, putting time as an added dimension**.

Because the purpose of this text is to study the ways in which geometry, algebra, and physics connect, it is worthwhile to dwell on the *philosophy* behind coordinate systems.

The ball will fall from 10 meters, according to the force of gravity. That is the way the world works. It doesn't matter what coordinate system we set up to do that calculation, we should get the *exact same result*. Plainly: nature doens't *care* what coordinate system we use. This fact, obvious as it may be, is worth thinking about: No matter what coordinate system we use, the equation of motion should give the same dynamics. The laws of physics should be *independent of any coordinate system*.

Newton's law $\mathbf{F} = m\mathbf{a}$ relates the force vector to the acceleration vector. The vector representing the force $\mathbf{F}$ that you apply on a surface is an object independent of coordinate system, and so is the resulting acceleration vector. The *components* of these vectors (F_x, F_y, F_z) and (a_x, a_y, a_z), however, depend on what you have chosen for the x, y, z axes. These components *represent* a real physical vector, but only once we pick a coordinate system. If we were to pick a different coordinate system, the numbers representing the vector would change.

When we write an equation describing a physical law, it should be valid regardless of the coordinate system we use. $\mathbf{F} = m\mathbf{a}$ will always be true whether we rotate our frame of reference or not. On

the other hand, if Newton's law of motion *only* said that the *first* 'x' component of the Force was equal to the *first* 'x' component of the acceleration, and said nothing about the other 2 components, then in different coordinate systems since 'x' means different things, we would get totally different equations of motion. No physical law will ever say something just about the first or just about the second components of two vectors: it must equate the entirety of the two vectors.

As another example if the equation for work looked like $F_x dx = dW$, then would give different results in different coordinate systems, because it puts emphasis on just one of the three components (the first 'x' coordinate) over the others. While in some coordinate system dx may point in the direction of the displacement and be nonzero, there may be a different coordinate system where $dx = 0$, making the work done zero. So the equation for work would be coordinate dependent: it would be wrong. The need for such invariance is why the true formula uses all three spacial dimensions and looks like:

$$\mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz = dW.$$

Although it isn't obvious yet that this is a quantity that is invariant regardless of the coordinate system used, at the very least it doesn't put one component above any of the others.

1.2 Linear Algebra & Coordinates

The traditional concept of a coordinate system, a series of perpendicular lines that together associate ordered tuples of numbers to each point in n-dimensional space, is not representative of all coordinate systems. For one, we do not need the requirement that the lines be perpendicular. Our coordinate system could instead look like this:

Graphic of non-perp lines and representing a point like that

In the language of linear algebra: once we choose an origin, choosing a set of coordinate axes is the same as choosing a basis for the space (a coordinate basis). For any point in space, we can relate coordinates x'_i in the new system in terms of coordinates x_i in the old system by matrix multiplication: $x'_i = \sum_{j=1}^{n} \mathbf{A}_{ij} x_j$. This is exactly what's called a change of basis in linear algebra. Transformations

between coordinate bases are exactly the invertible **linear transformations** .

As in linear algebra, we need our coordinate system to both **span** the space so that we can represent any point, and be **linearly independent** so that every point that we can represent in our coordinate system will have a unique representation. That's all that a basis is: it specifies a good coordinate system.

Definition 1.1. A set of vectors is said to span a space $\mathbb{R}^n$ if every point P can be represented as $a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$

Definition 1.2. A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called linearly independent if there is only one way to represent the zero vector $\mathbf{0}$ as a combination of them, namely as $\mathbf{0} = 0v_1 + \cdots + 0v_k$.

This second definition is the same as saying every point that we can represent in our system has a unique representation. Let's make this clear. If there were two ways to represent a point P: as

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

and

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

then subtracting these two different combinations would give a nonzero way to represent zero. Conversely, if there were a nonzero combination of vectors summing to zero, then we could add that combination to the coordinate representation of any point and get a *different* representation of the same point. So coordinate representations for all vectors are unique as long as long as there is only one representation for zero the one where each component equals zero.

Intuitively, linear independence means that there is no superfluous information in the set of vectors. We cannot linearly combine vectors in some subset to get another vector in the set; each vector is adding its own unique additional piece of information, making the set able to span in an additional direction.

Bases that don't span, or are not linearly independent, would lead to coordinate systems like these:

Show a 2-D basis in a 3-D space, and a basis of 3 vectors in 2-D space

Very often in mathematics, we ask "does a solution exist?", and "if there is a solution, is it unique?". These two questions are dual to one another. If a set of vectors spans the space, then there *exists* a way to represent any point (at least one way to represent any point). If a set of vectors is linearly independent, then *if* you can represent a point, that representation is *unique* (no more than one way to represent any point).

Now to stress the same idea again: because points in $\mathbb{R}^n$ and vectors are essentially the same thing, the idea that points in space are invariant of a coordinate system applies just as well to vectors. If we choose a basis for our vector space $\mathbf{v}_1, \dots \mathbf{v}_n$, then we can express any vector $\mathbf{u}$ by a unique combination $\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. We then say that in this basis, we can represent $\mathbf{u}$ by a list of numbers. Often, it is written:

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

But in some sense, writing this as an equality is wrong. The vector $\mathbf{u}$ is something physical: a velocity, a force, the flow of water. It doesn't depend on the coordinate system. On the other hand, the right hand side is just a list of numbers that depend entirely on the coordinate system chosen. If we change coordinate systems, the right hand side changes. Because $\mathbf{u}$ exists (say, in the real world) independently of coordinates used, it does not change.

A geometric vector like u is *not* a list of numbers. Once we pick a basis, u can be *represented by* a list of numbers, but if we change into a different basis, those numbers all have to change as well. This exact same idea will be the reason why a tensor is *not* just a multi-dimensional array (like the ones encountered in computer science). It can be *represented by* a multi-dimensional array once a coordinate system is chosen, but the numbers in each entry will differ depending on the coordinate system we pick.

This is very confusing (and will also be part of the reason why it's so hard to understand tensors as an undergraduate). In most math courses, we can freely call any list of numbers a 'vector'. After all, you can add lists and scale them so they do form a 'vector space'. This is a really unfortunate linguistic degeneracy in mathematics terminology. The type of vectors that we see in physics (acceleration, force, electric

field, etc.) are *geometric vectors* that have nothing *a priori* to do with lists of numbers until we represent them as such by using coordinate systems. On the other hand, abstract structures that we can add and multiply by scalars are *algebraic vectors*, and lists are an example of that. To avoid confusing lists of numbers with the geometric vectors in the physical world, we will call lists of numbers *tuples* rather than vectors.

So returning to the geometric vector $\mathbf{u}$, a more careful way to write it would be:

$$\mathbf{u} = \begin{pmatrix}\mathbf{v}_1 \dots \mathbf{v}_n\end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n. \tag{1.1}$$

Once we pick a basis, that column of coordinates means something. If we denote our basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ by B, then we will use the notation

$$\mathbf{u} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_B = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

Let us do a very simple example to start. In the 2-D plane, say we have our original basis $\mathbf{v}_1, \mathbf{v}_2$ and we rotate it by $\pi/4$ radians to get a new basis. Say we have a point P whose coordinate representation was $\mathbf{v}_1 + \mathbf{v}_2$, or $\begin{pmatrix}1\\1\end{pmatrix}$ in the original basis.

Now our new basis is the old one rotated by $\pi/4$ so

$$\begin{aligned}\mathbf{v}'_1 &= \frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2 \\ \mathbf{v}'_2 &= -\frac{\sqrt{2}}{2}\mathbf{v}_1 + \frac{\sqrt{2}}{2}\mathbf{v}_2.\end{aligned}$$

elementary transform with $1/\sqrt{2}$s

As a matrix transform, we can write this as[1]:

$$\begin{pmatrix}\mathbf{v}_1' & \mathbf{v}_2'\end{pmatrix} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix} \begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$$

This relates the actual basis vectors themselves. On the other hand, if we wanted to see the *coordinates* representing $\mathbf{v}_1'$ and $\mathbf{v}_1'$, then in the new basis they would simply be represented in coordinates as:

$$\mathbf{v}_1' = \begin{pmatrix}1\\0\end{pmatrix}_{\text{new}}, \ \mathbf{v}_2' = \begin{pmatrix}0\\1\end{pmatrix}_{\text{new}}$$

and in the old basis they'd be represented as:

$$\mathbf{v}_1' = \begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}}, \ \mathbf{v}_2' = \begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}}$$

If we know that we can describe a point as $\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}$ in the new basis, then we can easily get its description in the old basis as:

$$\begin{aligned}\begin{pmatrix}x\\y\end{pmatrix}_{\text{new}} &= x\mathbf{v}_1' + y\mathbf{v}_2' \\ &= x\begin{pmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} + y\begin{pmatrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}} \\ &= \left(\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}_{\text{old}\leftarrow\text{new}} \begin{pmatrix}x\\y\end{pmatrix}_{\text{new}}\right)_{\text{old}}\end{aligned} \tag{1.2}$$

This is the same matrix that related the basis vectors. We'll call it $\mathbf{A}$. $\mathbf{A}$ takes the new coordinate representations (x, y) and tells us how they'd look like in the *old* basis.

So then it is the *inverse* $\mathbf{A}^{-1}$ that tells us how our old coordinate representations of a point P will look like in our new basis.

[1]The tuple of basis vectors $\mathbf{v}_i$ is written as a row rather than a column to be consistent with Equation (1.1). Then the coordinates are represented in a column. Because of the way we do matrix multiplication, then the matrix acts on the right. It's an issue of styling and indexing, and not physically meaningful. If we were to write this coordinate transform using columns & not rows, we'd get a matrix that's the transpose of the one above, and matrix transposes would appear in subsequent equations, making them less tidy.

For our point P, represented as $(1,1)$ in our original basis, in the new basis, we would have:

$$P = \begin{pmatrix}1\\1\end{pmatrix}_{\text{old}} = \left(\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix}1\\1\end{pmatrix}_{\text{old}}\right)_{\text{new}} = \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}_{\text{new}}$$

Indeed, $\mathbf{v}_1 + \mathbf{v}_2 = \sqrt{2}\mathbf{v}'_1 + 0\mathbf{v}'_2$.

That is the central idea. If we *vary* the basis $\mathbf{v}_i$ to a different basis, $\mathbf{v}'_i$, then the coordinates a'_i will vary the *other* way, so that the geometric vector

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = a'_1\mathbf{v}'_1 + \cdots + a'_n\mathbf{v}'_n$$

is *invariant* regardless of coordinate choice.

Let's make this precise in the general case. If we start with a set of basis vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and we make the linear transformation to a new basis $\{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ so that, as before:

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n) = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A} \tag{1.3}$$

then since the vector $\mathbf{u}$ should not change when we change our basis, we must have:

$$\begin{pmatrix}a'_1\\ \vdots \\ a'_n\end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix} \tag{1.4}$$

so that

$$(\mathbf{v}'_1, \ldots, \mathbf{v}'_n)\begin{pmatrix}a'_1\\ \vdots \\ a'_n\end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\mathbf{A}\mathbf{A}^{-1}\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)\begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}$$

as desired.

EXERCISE: PROVE IT HAS TO BE A INVERSE

We say that the basis vectors $\mathbf{v}_i$ **co-vary** and the coordinates a_i **contra-vary** with the change of basis. The idea, although it sounds simple, is rather hard to get the feel of. It's worth thinking a good bit about how coordinates and bases need to vary in opposite ways

so that the physical object represented by the coordinates stays the same regardless of how we look at it.

This will be a caption for a sketch of a 3-D rotation

When you rotate your character in a video game (and in real life too, by the way), the world rotates *contrary* to the direction that you've rotated in. That's because the coordinates of what you see have *contra-varied* while your basis vectors, given by the direction you face have *co-varied*. The end result is that despite changing your coordinate system, physics stays the same: invariant. The universe did not rotate itself just because you did. This extends beyond just rotations to *all* linear transformations point.

1.3 The Notion of Length on Vector Spaces

Let us consider the property of orthogonality. It's well known that for geometric vectors, there's more that we can do than just add, scale, and transform them: we can take dot products[2] between them. When we have two geometric vectors in space, their dot product is a well defined number. If it is zero, then the vectors are orthogonal to one another. From the dot product and the magnitudes, it is possible to calculate the angle between two given vectors.

When vectors are represented in terms of tuples of numbers, the dot product was taught to us as "multiply component by component, and then sum that up". This is not, in general, what the dot product really is. Consider a basis transformation as below:

$$\mathbf{v}_1' = 2\mathbf{v}_1 + \mathbf{v}_2$$
$$\mathbf{v}_2' = \mathbf{v}_1 + 2\mathbf{v}_2.$$

It's easy to compute the inverse of this matrix and see how the new coordinates should work, but it worthwhile looking at this geometrically. It is not a rotation, but more of a "stretching". Notice that while in our original perspective, if we viewed $\mathbf{v}_1$ and $\mathbf{v}_2$ as orthogonal vectors, in the *new* perspective, they are *no longer* orthogonal. This

[2]Of course in 3-D we can also take a cross product. This will be discussed in the following chapters.

is very important: linear transformations in general do not preserve orthogonality.

Graphic of orthogonality being changed under affine transform

In particular, this linear transformation has *stretched* our vector space and changed the notion of distance. Even though rotations keep distances preserved, general linear transformations don't care about a notion of distance.

If we were to take the "dot product" $\mathbf{v}_1' \cdot \mathbf{v}_2'$ just by multiplying corresponding coordinates and summing them up, then in the new basis we'd get zero, but in the original basis we would *not*. This dot product actually changes depending on the coordinate system that we use! In some sense, this is expected: all we're doing is multiplying contravariant coordinates together and summing them up. The result should be contravariant as well (in fact doubly contravariant).

The failure of the dot product to be invariant is intimately related to the fact that transformations can change lengths. This should not be too surprising. After all, the length of a vector is defined by the square root of its dot product with itself. If the dot product we learned is not invariant under general coordinate transformations, what is the right way to measure length?

It is here that there is a big subtlety. A vector space on its own does not have a notion of length. We've just seen choosing different bases would give rise to different length scales and notions of "perpendicular" as well. Endowing a vector space with a way to universally tell what the length of a vector is, or whether two vectors are perpendicular is actually adding *extra structure* to the space. It picks a whole class of specific coordinate systems and says "these are the orthogonal reference frames; the others are skewed and stretched perspectives". This allows us to measure length using an invariant *inner product*.

I wanna really elaborate how an inner product *picks* a frame to be orthogonal, promoting it above all the others.

Euclidean space, as well as the world in which we live in, both have an natural way to measure length between two points that is invariant of the coordinate system used. A vector space on its own does not, and so it is called an *affine space*. In affine space, although there are notions like "parallel", there is not a notion of distance. Adding an

inner product to affine space gives rise to Euclidean space.

Insert graphic here, with an affine transformation of the plane

Caption something like "If Euclid's space allowed for not just rotations/translations but also general linear transformations like this, being perpendicular would be a coordinate-dependent idea. Note parallel lines, on the other hand, stay parallel, so affine space *does* have a notion of parallel"

This will be discussed in much greater detail in the following two chapters.

The takeaway from this discussion is that general linear transformations don't preserve distances. Because of this, the usual definition of the dot product gives different results depending on the choice of coordinate system. To make length an *invariant* regardless of coordinate choice, the definition of the inner product between two vectors needs to be appropriately modified. Without an *inner product* to measure distance, $\mathbb{R}^n$ is *not* Euclidean space $\mathbf{E}^n$, and is called affine space. Affine space looks the same regardless of the linear transformations we apply, while Euclidean space only looks the same when we rotate or translate our frame (because those are the two transformations that don't change lengths).

The reason we haven't encountered this problem so far in physics is simply because we always have worked in orthonormal frames given by an orthonormal basis like $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, where the inner product in fact *is* just the sum of the products of corresponding components. $\mathbb{R}^n$ without an inner product is affine, not Euclidean. Often however, when talking about Euclidean space, authors refer to it as $\mathbb{R}^n$ instead of $\mathbf{E}^n$ simply because $\mathbf{E}^n$ is modeled as $\mathbb{R}^n$ endowed with an inner product.

1.4 Nonlinear Coordinate Systems are Locally Linear

Perhaps you may be wondering why we've spent so much time on changing between coordinate systems represented by basis vectors centered at a fixed origin. Consider the change between cartesian and polar coordinates. What does this have to do with the linear changes of coordinates that we've been discussing?

We could use something like a polar system of (r, θ) or a spherical system (r, ϕ, θ). These coordinate systems are not representable in terms of axes, but instead look like this:

Graphic of polar coordinate system/spherical

This is an example of a non-linear coordinate transformation. They are more commonly referred to as **curvilinear**. Whereas linear ones map lines to lines, curvilinear ones more generally map lines to curves. The idea for making sure that the equations of physics still stay true for non-linear coordinate transformations is to note that just like a curve locally looks like a line, a *non-linear* transformation locally looks like a *linear* one. The linear transformation that it locally looks like is called the **Jacobian** J. If the laws of physics are invariant under linear transformations locally at each point, then *globally*, they will be invariant under non-linear ones as well. That is why we cared about studying covariance and contravariance for linear transformations: more complicated cases can be reduced to their local linear behavior.

As an example, consider going from a cartesian to a polar coordinate system. We have $x = r\cos\theta$ and $y = r\sin\theta$. Certainly, this is not a linear transformation of coordinates. There is sinusoidal dependence on θ in this transformation. Physics and geometry, however, do not have laws in terms of absolute coordinates (it doesn't make sense to say "That object is located at 50 meters") but only in terms of relative distances (you'd instead say "That object is located 50 meters *relative to* me"). It is the changes over relative distances between points that we care about, and these are obtained by integrating the *infinitesimal* changes at each point.

So although x, y do not depend linearly on theta, through the use of the chain rule, we have a local linear relationship in their infinitesimal changes:

$$\begin{aligned} dx &= \cos\theta\ dr - r\sin\theta\ d\theta \\ dy &= \sin\theta\ dr + r\cos\theta\ d\theta \end{aligned}$$

At any given point, this relationship can be written as a linear change of basis.

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

So every nonlinear transformation from some coordinate system $x_1 \dots x_n$ to $x' \dots x'_n$ has the local linear transformation law:

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}_{\text{old}\leftarrow\text{new}} \begin{pmatrix} dx'_1 \\ \cdots \\ dx'_n \end{pmatrix}_{\text{new}}$$

$$\Rightarrow \begin{pmatrix} dx'_1 \\ \cdots \\ dx'_n \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^{-1}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}_{\text{old}}$$

This is exactly the analogue of Equation (1.4), so indeed the changes in coordinates dx_i can be called *contravariant*, just like the coordinates were for the linear transformation case. All of this is just an extension of the principle of local linearity from calculus.

Similarly, we can express the new derivative operators in terms of the old ones by using the chain rule. For polar coordinates we have

$$\frac{\partial f}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y}$$

or more compactly we can relate just the differential operators themselves:

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

so that more generally:

$$\begin{pmatrix} \frac{\partial}{\partial x'_1} \\ \vdots \\ \frac{\partial}{\partial x'_n} \end{pmatrix}_{\text{new}} = \begin{pmatrix} \frac{\partial x_1}{\partial x'_1} & \cdots & \frac{\partial x_1}{\partial x'_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x'_1} & \cdots & \frac{\partial x_n}{\partial x'_n} \end{pmatrix}^{T}_{\text{new}\leftarrow\text{old}} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}_{\text{old}} \tag{1.5}$$

If you look carefully, you will see that this accordingly mirrors the covariant change of vectors in Equation (1.3). So while the infinitesimal changes in the coordinates themselves are contravariant, just like the linear coordinates themseles, the *differential operators* corresponding

to changes in these coordinates become *covariant*, just like the vectors $\mathbf{v}_i$ in the linear case. This is the first correspondence that will hint that our basis vectors $\mathbf{v}_i$ actually *correspond* to the differential operators $\frac{\partial}{\partial x_i}$.

Indeed, as we begin to move away from 3-D and n-D Euclidean space, we will see why the old notions of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are better viewed as the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$, and in general $\mathbf{v}_i \to \frac{\partial}{\partial x_i}$. This change of notation will allow us to easily pass onto far more general spaces than the Euclidean ones we've gotten used to.

1.5 Einstein's Summation Convention

Row tuples, column tuples, matrices representing basis transformations (old to new and new to old), co-variance and contra-variance. These ideas have constituted the entirety of this first chapter, and although hopefully they have not been too difficult conceptually, the matrix manipulation even at this early level is already a pain. We have to write everything in terms of tuples, and we need to arbitrarily decide which ones are rows and which ones are columns, and which matrices are transposed so that all the matrix multiplications make sense, as defined in linear algebra. A young physicist named Albert Einstein used a convention of writing all these equations so that we did not have to explicitly write out tuples of abstract basis vectors and coordinates.

The first step is to avoid explicitly writing out row tuples and column tuples. To do this, instead of writing out a whole tuple to represent a vector like $(v_1, \ldots, v_n)$ we will simply write v_i. v_i should be viewed as the whole vector, rather than a specific ith component of $\mathbf{v}$. The index i is *free* and can be anything.

The reason it is preferable to view v_i as the whole vector rather than a specific ith component of it is because of the same reason given at the end of Section 1.1, that we never care about just a specific component, but rather the vector as a whole.

The second step is to be able to differentiate between *covariant* quantities and contravariant quantities. The convention is this: if the quantity is covariant, like a basis vector, then write its index *downstairs*: $\mathbf{v}_i$. On the other hand, of a quantity is contravariant (like a component of a vector) then write its index *upstairs* as v^i

instead[3]. Then we can write Equation (1.1) as

$$\mathbf{u} = \sum_i a^i \mathbf{v}_i \tag{1.6}$$

For another example, the gradient operator was written $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and is now written simply as $\partial/\partial x^i$ (adopting upper indices for x^i since coordinates contra-vary). This means that Equation (1.5) can be written as

$$\frac{\partial}{\partial x'^i} = \sum_j \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j}. \tag{1.7}$$

Since i is free to be anything (while j is bound since it is being summed over), this equation holds for every i in $1, \ldots, n$ and so is indeed an equation relating two *vectors* on both sides. Note also that this means the transformation matrix can be written as $\frac{\partial x^j}{\partial x'^i}$. By writing just a summation and not any explicit matrices, we avoid having to worry about unnecessary troubles like transposes, etc. Note also that this matrix has both upper and lower indices so that the upper index in $\frac{\partial x^j}{\partial x'^i}$ gets multiplied with the lower index $\frac{\partial}{\partial x^j}$ to cancel out, and all that remains is the lower index i. All the trouble of seeing what's covariant, what's contravariant, and what's invariant is washed away into just seeing how the upper indices and lower indices cancel out in the end.

Sometimes in the mathematics literature in places where co- and contra-variance are of lesser importance, upper and lower indices are ignored and the notation describing something like a matrix-vector multiplication looks like this:

$$(\mathbf{A}\mathbf{v})_i = \sum_j \mathbf{A}_{ij} \mathbf{v}_j.$$

That is, the ith component of the product $\mathbf{A}\mathbf{v}$ is the sum over j of the ijth component of $\mathbf{A}$ with the jth component of $\mathbf{v}$. For a matrix-matrix multiplication this would look like

$$(\mathbf{A}\mathbf{B})_{ik} = \sum_j \mathbf{A}_{ij} \mathbf{B}_{jk}.$$

[3]If you are worried that this will be confused with exponentiation, don't be. In practice, such confusion almost never arises.

Notice the pattern: when we wish to multiply these objects, we sum over a common index (j in the above equations) that we make both objects share. The point is *an index is always repeated and summed over* when we do a multiplication.

Einstein took the bold, but ultimately brilliant step of making the convention that if we ever write an index twice, that *automatically means* that we are summing over it (unless we explicitly say we aren't). The above equations, in Einstein's scheme, now become

$$(\mathbf{A}\vec{v})_i = \mathbf{A}_{ij}\vec{v}_j$$
$$(\mathbf{AB})_{ik} = \mathbf{A}_{ij}\mathbf{B}_{jk}.$$

A dot product between $\mathbf{v}$ and itself would simply be $\mathbf{v}\cdot\mathbf{v} = v_i v_i$ in Einstein's convention. Note since i is an index that is summed over (and doesn't appear in the end result), we could just as well replace it by any symbol $v_i v_i = v_j v_j = v_\smile v_\smile = \mathbf{v}\cdot\mathbf{v}$.

Now returning back to co-variance and contra-variance, this scheme makes every equation in the chapter shockingly short. Equation (1.1) becomes $\mathbf{u} = a^i \mathbf{v}_i$, Equation (1.3) becomes $\mathbf{v}'_i = A^j_i \mathbf{v}_j$, Equation (1.4) becomes $a'_i = (A^{-1})^j_i a_j$. Equation (1.7) is further reduced to just

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}. \tag{1.8}$$

Similarly the transformation of infinitesimal coordinate changes is now just

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j. \tag{1.9}$$

and both of these are really just the chain rule plain and simple.

EXERCISE: Check invariance quickly using this method.

By doing exercises, this method is very quick and steady to get the hang of. We will adopt it for the rest of the book.

1.6 Exercises

Chapter 2

New Horizons Developed

2.1 The Manifold

Several thousand years ago, the first sentient human beings noticed that the landscape of the earth looked flat, and seemed to stretch out infinitely far in every direction. It is perhaps from this observation that the Euclidean plane was first conceived, and indeed it is from the fact that the earth looked like Euclid's 2D plane that geometry got its name to literally mean "measuring the earth". But the fact is that the earth is *not* a flat plane, stretching out infinitely. It turned out to be a sphere. What is true, however, is that *locally*, the geometry of the earth looks very similar to that of Euclidean space.

And now in modern times, as we look out into the cosmos and see them stretching out in every direction, our first human bias creeps in and tells us "this thing must be infinite, stretching out in every direction". Just as people thought the world was $\mathbb{R}^2$ in ancient times, in this age we entertain the thought that our universe could be three-dimensional Euclidean space $\mathbb{R}^3$. Indeed, most of the time when we do simple classical physics, we embed our system into a space that is $\mathbb{R}^3$ and work there. It is an easy space to work in.

But just as the earth's surface looked *locally* like Euclidean 2-space but in fact turned out to globally be wildly different, we should not be surprised if it turns out that the universe, despite locally looking Euclidean, has wildly different global structure.

This is exactly what a manifold intuitively is: an object that at each point locally resembles Euclidean space. The property of being locally Euclidean is similar to the property that differentiable func-

tions have of being locally linear. It allows us to use calculus on them to reduce nonlinear objects to linear ones locally.

Concept 2.1. *A manifold M is a set of points which, in the neighborhood of every point, locally looks like euclidean space*

A line is a one-dimensional manifold (in fact it *is* a Euclidean space). The circle is a one-dimensional manifold locally resembling a line, and so are ellipses, parabolas, hyperbolas, and the graph of any smooth function. A sphere is the two dimensional manifold that ancient humans mistook for the Euclidean plane itself. The Mobius strip is also a two-dimensional manifold. Although globally it is a twisted band, locally it too looks like two-dimensional Euclidean space. Every geometric object referred to as a "curve" or a "surface" has been an example of a manifold this whole time.

In this chapter, we work towards building the language necessary to formally define what we mean by a manifold. First, we reinforce the intuitive ideas through examples.

2.2 Examples of Manifolds

Example 2.2. The sphere is a two-dimensional manifold

This is the classic example of a manifold that isn't just $\mathbb{R}^n$. As we have said before, at every point on the sphere, things look locally like $\mathbb{R}^2$. Throughout this text, the sphere will often be our first go-to setting when we want to take concepts from Euclidean space and generalize them to manifolds. It is easy to visualize things on this space, so often the first question to ask when generalizing something is "well, how would this thing look on the sphere?". Of course for this reason ellipsoids, paraboloids, and hyperboloids are all also manifolds. Similarly familiar objects like cylinders, tori, or tori with multiple holes are also manifolds.

Every point on the sphere looks exactly the same as any other. This is a highly symmetric manifold, that will later be referred to as a *homogenous space* because of this property.

Example 2.3. Manifolds need not be connected. The disjoint union of two manifolds is also a manifold.

Consider the set consisting of two separate spheres. Since each sphere individually is a manifold, then any point in this disjoint union belongs to one of the spheres, and so it has a locally euclidean neighborhood.

Example 2.4. The graph of the curve $y = f(x)$ where f is a smooth function defines a one-dimensional manifold.

Because f is smooth, we know that every point of the graph of the curve will locally look like its tangent line. Since the tangent line is precisely one-dimensional Euclidean space, every point on the curve looks locally Euclidean. Indeed, we only needed f to be differentiable.

Example 2.5. The set of points forming the graph of $y = f(x_1, \ldots, x_n)$ defines an n-dimensional manifold when f is smooth.

The exact same argument as before holds, except with a tangent line generalized to a tangent plane, etc. Locally at each point the set looks like Euclidean n-space. The idea of an n-dimensional "tangent *something*" at each point p on a manifold M that generalizes the notion of a tangent line or plane to higher dimensions, will be made precise in coming sections by talking about the *tangent space* of M at p, T_pM.

Example 2.6. The figure-eight and the cone are not manifolds.

Figure 8 and cone aren't manifolds

Both of these objects have a "cusp-like" point where multiple lines intersect. Zooming in near that point will preserve these cusps, and so the space does not look like Euclidean space near that point. There is no tangent space for the figure eight at the point of intersection that looks like a line: it would look like two lines intersecting. Similarly for the cone, there would not be a tangent space at the cusp that looks like a plane. At that point, the manifold looks like a continuum of intersecting lines. Intuitively, then, manifolds cannot have sharp cusps. As a result, the cube and triangle are not examples of manifolds.

The power of manifolds lies in the fact that not only are geometric objects manifolds, but so are many of the algebraic objects that we have been working with.

Example 2.7. Consider a two-dimensional parallelogram, with opposite sides identified as the same. This is a manifold.

Parallelogram with opposite sides identified

This is like a room, where if you exit on one side, you come back on the other side. The one dimensional version of this is a circle, and we will show how this space can be thought of as a "product of circles" or a "circle of circles" in two dimensions. Clearly this parallelogram locally looks like euclidean two-space in the neighborhood of any point on the interior. The only possible problem is at the edges. Because each edge is identified with its opposite one, however, a neighborhood of a point at the edge of the parallelogram will simple wrap around to the other side, and look just as euclidean as the neighborhood around any other point.

Example 2.8. The set of $n \times n$ matrices forms a real manifold of dimension n^2.

Note that any $n \times n$ matrix can be equivalently viewed as a vector living in $\mathbb{R}^{n^2}$. Given any matrix, locally we can go in each of n^2 direction by appropriately varying one of the n^2 components of the matrix. So this manifold not only locally looks like $\mathbb{R}^{n^2}$ but can in fact be identified as being *the same* as $\mathbb{R}^{n^2}$.

Example 2.9. The set of *invertible* $n \times n$ matrices forms a manifold of dimension n^2

It is not obvious that this is a manifold. We need to know that near any point on this set M, we can go in each of n^2 linearly independent directions. The way we can see this is that invertible matrices are precisely those matrices A with *nonzero determinant*, $\det A \neq 0$. If I pick a given point A on this manifold corresponding to a matrix with nonzero determinant, then say it has determinant $d \neq 0$. Because the determinant is a *continuous* function, then changing any of the n^2 components of A by a small number δ will change the determinant by some small amount as well. We just need to pick δ small enough so that the resulting change has magnitude less than d, and therefore keeps the determinant away from 0. As long as we pick the neighborhood around A small enough, we still can vary in any of n^2 directions, and so it still *locally* looks like $\mathbb{R}^{n^2}$.

Example 2.10. The set of rotations in Euclidean 3-space is a manifold.

From mechanics and engineering, or just by playing with an object for a little bit, it is known that there are three independent ways to rotate something (about each of the three spacial axes). Any given rotation can be specified by three "Euler Angles" that describe how much to rotate about each axis, in a specific order. For a specific rotation given by these three angles, we intuitively expect that we can *perturb* this rotation in three independent ways by slightly changing one of the three angles. There would then be a three-dimensional neighborhood of the rotations *near* the original one. So locally, we look like Euclidean 3-space.

Proposition 2.11. *Manifolds are a powerful and useful idea in both engineering and applied mathematics.*

Oftentimes, we care about studying the possible states that a system can have, like the ways that an object can rotate, as in the prior example. This goes much beyond possible rotations, and can go as far as

AARON EXPLAIN WHY MANIFOLDS ARE USEFUL I DONT KNOW ANYTHING PRACTICAL

2.3 Elementary Topology

Before we can begin to do geometry on a space, we need to go even more fundamental still, into the real of topology. Geometry takes its name from the "study of the earth", while topology takes its name from the more abstract "study of places" (*topos*). Topology deals with ideas that, to a geometer or a physicist, border on the unphysical. In fact, it is likely easier to work to work with topological objects by relying on pure logical reasoning rather than geometrical intuition. Topology is in many ways an extension of set theory, and is a central tool in mathematical analysis of functions when defining and studying ideas like continuity.

Elementary topology begins with **point-set topology**, in which the central objects of study are the open and closed subsets of a main

set X. This should in fact be a familiar idea from studying the real line $\mathbb{R}$, on which there is the **Euclidean Topology**.

On the real line, intervals of the form (a, b) consisting of all x so that $a < x < b$ were called **open intervals**. One of the interesting things about open intervals is that on an interval like (a, b) there is no *greatest* number in that set.

Proposition 2.12. *There is no maximum number on the open set* (a, b)

Proof. This will be a proof by contradiction. If there were a maximal number $x \in (a, b)$ then it must necessarily be *strictly* less than b (Note since b is not strictly less than itself, $b \notin (a, b)$). Therefore, since $x < b$ we can consider $(x + b)/2$. This average is strictly greater than x and less than b and so is in (a, b), contradicting the fact that x was the maximal element in (a, b). □

By the same argument, there is no minimal element in (a, b). This set has the interesting property that for a point $x \in (a, b)$, there are points both to the right and left of x that are still in (a, b). Such subsets of $\mathbb{R}$ are called open

Definition 2.13. A subset S of $\mathbb{R}$ is called open iff for any point $x \in S$, we can pick an ε sufficiently small so that the open interval $(x - \varepsilon, x + \varepsilon) \subseteq S$ as well.

For example, any point in the interval $(-1, 3)$ satisfies this. For example 2.99 is contained in the interval $(2.99 - 0.005, 2.99 + 0.005)$ which is a subset of $(-1, 3)$. In other words, any point in an open set has that all the points sufficiently close to it are also in that set.

The notion of an "open interval" of size ε around a point x generalizes into higher dimensions by talking about an "open ball" of radius ε around a point p

$$B_\varepsilon(p) := \{q \in \mathbb{R}^n : |p - q| < \varepsilon\} \tag{2.1}$$

that is, the set of all points q within ε of p. So a subset S of $\mathbb{R}^n$ is called open iff every point $p \in S$ has a ball $B_\varepsilon(p)$ of some sufficiently small positive radius ε is contained in S.

So on the real line open intervals are open sets, and in fact any union of open intervals is still an open set (in fact an exercise will

show that all open sets on the real line can be expressed as a (possibly infinite) union of open intervals). Intersections of open intervals are also open, but only when there are *finitely many* such intersections. On the other hand, infinite unions of open sets are still open:

Proposition 2.14. *In the Euclidean topology of $\mathbb{R}^n$, any arbitrary union $V = \bigcup_\alpha U_\alpha$ of open sets U_α is open*

Proof. Let p be a point in this union V, then since it belongs to the union, it must belong to at least one of the U_α. This means that some $B_\varepsilon(p) \subseteq U_\alpha$, implying $B_\varepsilon(p) \subseteq V$ as well. □

Hopefully this proof illustrates something that is common in many topology proofs: working with simple logic works better than appealing to geometric intuition. Similarly

Proposition 2.15. *In the Euclidean topology, any* finite *intersection $V = \bigcap_\alpha U_\alpha$ of open sets is open.*

Proof. Let p be a point in the intersection V, then since it belongs to the intersection, it must belong to *all* of the U_α. For each U_α there a positive ε so that $B_\varepsilon(p) \subseteq U_\alpha$. Since the intersection is finite, pick the minimum such epsilon and it will still be positive. Moreover $B_\varepsilon(p)$ will then be contained in each U_α so will be contained in the intersection V as well. □

It's clear to see that if the intersection were *not* finite, then the set of εs that we take the minimum over may give us a minimum value of zero. This is illustrated in an example as one of the exercises. We are now in a good place to define what we mean by a **topology** on a set.

Definition 2.16. A *topology* on a set X is a family $\mathcal{T}$ of subsets of X so that

- Both the empty set $\varnothing$ and X are in $\mathcal{T}$
- Any finite/infinite union of sets in $\mathcal{T}$ is in $\mathcal{T}$
- Any *finite* intersection of sets in $\mathcal{T}$ is in $\mathcal{T}$

The elements of the topology $\mathcal{T}$ are called the ***open sets*** of X. Complements of open sets are called ***closed sets***. X is then called a ***topological space***.

Note that on $\mathbb{R}$, the closed intervals $[a, b]$ are the complements of the unions of open sets informally denoted by $(-\infty, a) \cup (b, \infty)$.

We can always arbitrarily define a topology and call "these sets, and all their arbitrary unions" open, but it is better to work with a natural topologies like the Euclidean one discussed above.

Possibly talk about the algebra of open/closed sets?

Why do we care about open sets so much? They allow us to define a notion of continuity for functions between any topological spaces:

Definition 2.17 (Continuity)**.** A function between two topological spaces

$$f : X \to T$$

is ***continuous at a point*** $p \in X$ iff for every open set $V \subseteq T$ containing $f(p)$, there is an open set $U \subseteq X$ so that $f(U) \subseteq V$.

A function is ***continuous*** if it is continuous at every point $p \in X$.

DRAW THIS

On the Euclidean topology, this is exactly equivalent to the $\varepsilon - \delta$ definitions taught in calculus class. An exercise will sketch out a simple proof of this fact. This is one of the reasons that open sets are worth studying: it allows us to turn language using numerical epsilons and deltas into a "coordinate-free" language of maps between sets. Intuitively, a continuous function is one that doesn't locally mess with the space too much around any point.

Definition 2.18 (Homeomorphism)**.** A function between two topological spaces

$$f : X \to T$$

is a ***homeomorphism*** iff it is bijective, with inverse

$$f^{-1} : T \to X$$

so that both f and f^{-1} are continuous.

An example of a homeomorphism that is one of the most common ideas associated with topology is that something like a coffee cup is homeomorphic to a doughnut as a topological space.

ELABORATE HERE: coffee cup = doughnut

In order to talk about manifolds *locally* looking like Euclidean space, we need to be able to have a rigorous way of talking about things *locally*. We want a good way of defining a *neighborhood* around a point p. Certainly an open ball containing p is a neighborhood of p because it contains "All the points near p". In the Euclidean topology, any open set U containing p also contains a ball of some radius around p, and thus contains a neighborhood of p. We could say that a neighborhood is any open set containing p, or more generally:

Definition 2.19 (Neighborhood)**.** A ***neighborhood*** V of a point p is a subset of X that contains an open set containing p. That is $p \in U \subseteq V$.

This way, there is no need for V to be open, but it does *contain* an open set containing all the points close to p.

Draw a sphere, with a patch.

Definition 2.20 (The Manifold)**.** A ***manifold*** M of dimension n is a topological space such that for every point $p \in M$, there is a neighborhood U of p that is homeomorphic to an open subset of $\mathbb{R}^n$.

This is exactly what we have been speaking about intuitively the whole time: for any point, a neighborhood around that point looks like Euclidean space. On a manifold, then, these neighborhoods are open sets that can be mapped in a one-to-one manner to open sets in $\mathbb{R}^n$. We use this notion to define coordinate charts:

Definition 2.21 (Coordinate Charts)**.** A ***coordinate chart*** (U, φ) is an open set $U \subseteq M$ together with a "coordinate" map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism of U to and open subset of $\mathbb{R}^n$.

The revolutionary idea of Descartes has been translated beyond Euclidean space, onto manifolds. Coordinate charts are vital to going from geometric data into algebraic calculations. They allow physicists to lay down a coordinate system that is valid for at least a *part* of the manifold, on which numerical calculations can be done. Because φ is one-to-one, we can parameterize the part of the manifold on U by n parameters.

Example 2.22 (Charts on the Sphere)**.** The sphere has a ***stereographic projection*** to the plane that covers every point of the sphere except the north pole.

Charts on the sphere

We would like to be able to patch up the whole manifold with coordinate charts, so that we can work globally. Such a patch that covers the entire manifold is called an *atlas*.

Definition 2.23 (Atlas)**.** An ***atlas*** on a manifold is a set of coordinate charts $(U_\alpha, \varphi_\alpha)$ whose union $\bigcup_\alpha U_\alpha = M$.

Note that because each φ_α is a homeomorphism, meaning its inverse is also continuous, then if $U_\alpha \cap U_\beta \neq \varnothing$, we have $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a continuous function as well, from $\mathbb{R}^n$ to $\mathbb{R}^n$. These are the *transition maps* between coordinate patches.

A **smooth manifold** is one where the φ_α are not only continuous but in fact infinitely differentiable (i.e. **smooth**). This makes the transition maps smooth functions on $\mathbb{R}^n$. Essentially all the examples we deal with for the remainder of this book will be smooth manifolds.

2.4 Embeddings vs. Intrinsic Geometry

The whole embeddings section. Talk about how we always picture manifolds as embedded into Euclidean space, but nothing about them *intrinsically demands it*

Mention any n-dimensional manifold can be embedded in $\mathbb{R}^{2n}$

Talk about how the torus is exactly the space in Example 2.7

2.5 Vectors Reimagined

Let us go back to $\mathbb{R}^3$. Studying the point-set topology of $\mathbb{R}^3$ as we have been doing for manifolds is a very easy thing to do, and isn't so rewarding. The reason we do it is so we can apply it to studying the *functions* on $\mathbb{R}^3$. A function on $\mathbb{R}^3$ takes three a tuple of three real inputs (x, y, z) and outputs a real number $f(x, y, z)$.

As you should be aware of by now, (x, y, z) is not a point, but instead a *coordinate representation* of some point. The manifold of Euclidean 3-space $\mathbf{E}^3$ can be modelled by $\mathbb{R}^3$ once a coordinate representation is chosen. This distinction is so slight that it is barely noted, but it is worth noting. A function $f : \mathbf{E}^3 \to \mathbb{R}$ in fact takes a point $p \in \mathbf{E}^3$ and outputs a real number $f(p)$.

We also study vector fields on Euclidean space. Before thinking too hard about coordinate co-variance and contra-variance, we would just pick an orthogonal reference frame, and label an orthonormal basis by three vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Then, the form of a vector field looked like

$$\mathbf{F} = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

and again, (x, y, z) is in fact a coordinate representation of the invariant point $p = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

In higher dimensions, a function (i.e. a scalar field) could be specified in terms of a coordinate representation by $f(x_1, \ldots, x_n)$. Usually, it is customary to replace $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ by the notation $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n$ to denote an orthonormal frame in n dimensions, just so that we do not run out of alphabet characters.

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}^i(x^1, \ldots, x^n)\ \hat{\mathbf{e}}_i = \mathbf{F}^i(x^j)\ \hat{\mathbf{e}}_i$$

where we are now beginning to use Einstein summation convention. Of course, we do not *need* an orthonormal frame to describe a vector. We can use any basis $\mathbf{e}_i$ and write $p = x^i\mathbf{e}_i$, $\mathbf{F}(p) = \mathbf{F}^i(p)\mathbf{e}_i$.

The first important thing to note is that a scalar field is an invariant, *physical* thing on $\mathbb{R}^n$ (and on manifolds in general). When we talk about the temperature at each point in space, or classically when we talk about the energy density of the electric field at each point in space, that means there is a specific invariant number at each point. It doesn't matter whether we set up coordinates x^i on the space or not. The temperature at a point does not depend on what coordinate system we represent that point in.

Similarly, a *vector field* is also an invariant physical thing! The components $\mathbf{F}^i$ of the vector field, of course, *will* depend on the coordinate system we use, but the field $\mathbf{F}$ itself will *not*. Just like the wind

doesn't change its motion across the earth when we change the coordinate system we use for measuring it, $\mathbf{F} = \mathbf{F}^i \mathbf{e}_i$ will remain invariant at every point. Only the components $\mathbf{F}^i$ will contra-vary against our co-variant $\mathbf{e}_i$.

Using these ideas from $\mathbb{R}^n$, consider a manifold M. A scalar function on M is no difficult thing to define: it associates to each point $p \in M$ a value $f(p)$. Because we can form an atlas of charts $(U_\alpha, \varphi_\alpha)$, we can locally form a function $f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ on Euclidean $\mathbb{R}^n$. Then we can say f is continuous/differentiable/smooth if every $f \circ \varphi_\alpha^{-1}$ is. In physics, we will frequently be concerned with smooth functions in particular. They are natural objects of study. The set of smooth, infinitely differentiable functions on a manifold M will be denoted by $C^\infty(M)$, while the space of continuous functions will be denoted by $C^0(M)$.

The φ_α^{-1} allow us to take a tuple of n coordinates $q^1, \ldots, q^n$ on $\mathbb{R}^n$ and associate that to a unique point on M that is part of U. So we see how a scalar field $f : M \to R$ can descend to a function on the local coordinates.

Show a graph of the curves for coordinates q_i

The question now is what about a *vector field* $\mathbf{F}$ on our manifold M? What does this thing mean? We want to associate to each point on M a vector at that point. Note that this doesn't mean we want to associate to each point on M just some tuple in $\mathbb{R}^n$, because a vector is *not* just a tuple of numbers (that's coordinate dependent). We want to associate a physical vector at that point.

Intuitively it should make sense what we mean by this:

Ball with a vector field on it

In Section 1.4, we showed how when adopting polar coordinates, the corresponding vectors $\hat{\mathbf{r}}, \hat{\theta}$ change direction depending on the point p we're describing. Each of them points in the direction of an increase in the corresponding coordinate. So for a general curvilinear coordinate system on a manifold q^i, the associated vectors $\mathbf{e}_i$ would be tangent to the paths traced out by varying q_i.

The vectors e_i are tangents to the curves traced out by letting each q^i vary (i.e. $\phi^{-1}(q^i)$).

At each point $p \in M$, the local tangent vector $\mathbf{e}_i$ corresponding to coordinate q^i is associated with increasing q^i while holding all other coordinates fixed. This already gives us enough information to know

one thing: the rate of change of a scalar function f along $\mathbf{e}_i$. It is exactly the partial derivative with respect to q^i

$$D_{\mathbf{e}_i} = \frac{\partial f}{\partial q^i} \tag{2.2}$$

We already know from Section 1.5 that partial derivative operators with respect to coordinates are *co*-variant, just like the vectors $\mathbf{e}_i$ themselves. So for any coordinate system q^i, we have a set of differential operators $\partial/\partial q^i$ corresponding to directional derivatives along the basis vectors $\mathbf{e}_i$. So we have a correspondence between coordinate frames $\mathbf{e}_i$ at a point and the differential operators associated with their coordinate system q^i.

$$\text{Covariant Quantities:} \qquad \mathbf{e}_i \longleftrightarrow \frac{\partial}{\partial q^i}$$

The set of vectors at p correspond to all the possible directions that paths on M can pass through p. If we have a path $\gamma \in M$ parameterized by t so that $\gamma(t_0) = p$ then we can compute derivative along the direction $\mathbf{v}$ tangent to γ as

$$D_{\mathbf{v}} f = \frac{d}{dt}\left[f\left(\gamma(t)\right)\right] \tag{2.3}$$

So although even though we don't have an idea of how to represent $\mathbf{v}$ in terms of our coordinate system q^i, we know how to represent its directional derivative operator. That is just:

$$D_{\mathbf{v}} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} \tag{2.4}$$

where dq^i/dt is the change in $q^i = \varphi^i(\gamma(t))$.

$$\frac{dq^i}{dt} = \frac{d}{dt}\big[\varphi^i(\gamma(t))\big] \tag{2.5}$$

This gives us the "component" v^i associated with $\partial/\partial q^i$ for our directional derivative in our coordinate basis:

$$D_{\mathbf{v}} = \frac{d}{dt}\big[\varphi^i(\gamma(t))\big]\frac{\partial}{\partial q^i} = v^i\frac{\partial}{\partial q^i} \tag{2.6}$$

But this is a correspondence between two physical, invariant objects associated with a direction:

$$\text{Tangent Vectors at } p \rightleftharpoons \text{Directional Derivatives at } p$$

$$\mathbf{v} = v^i \mathbf{e}_i \longleftrightarrow D_{\mathbf{v}} = v^i \frac{\partial}{\partial q^i}$$

Independent of the coordinates q^i that we use, the differential operator $D_{\mathbf{v}}$ will give the same invariant value when acting on a physical scalar field f, namely $v^i \frac{\partial f}{\partial q^i}$. So this differential operator is an invariant just like its corresponding vector $\mathbf{v}$. Moreover, in the case of Euclidean space, the components v^i of the directional derivative operator are exactly the same as the components of $\mathbf{v}$ itself:

$$D_{\mathbf{v}} = \mathbf{v} \cdot \nabla = v^i \frac{\partial}{\partial x^i} \tag{2.7}$$

Just like with vector addition, the sum $\mathbf{u} + \mathbf{v}$ corresponds to the operator $D_{\mathbf{u}+\mathbf{v}} = D_{\mathbf{u}} + D_{\mathbf{v}}$, and a multiple $c\mathbf{u}$ gives the multiple of the original derivative $D_{c\mathbf{u}} = cD_{\mathbf{u}}$. Directional derivatives form a vector space in one-to-one correspondence with the vectors at p. For this reason we say that these directional derivatives *are* the vectors at p. Vectors, at their core, represent flows along direction, which is no different from what a directional derivative operator along that point represents. We will sometimes abbreviate it as $v^i \partial_i$.

This leads us to define the **tangent space** of the vectors at a point p of a manifold M.

Definition 2.24 (Tangent Space)**.** The tangent space at a point p of a manifold M, T_pM is the set of first order derivative operators at p.

Going back to $\mathbb{R}^n$, this means that this whole time we could have treated $\hat{\mathbf{i}}$ as $\partial/\partial x$, $\hat{\mathbf{j}}$ as $\partial/\partial y$, etc. Using this idea is powerful, because now just from having a coordinate patch, q^i on a manifold, we obtain the full set of tangent vectors $\partial/\partial q^i$ at each point on the manifold.

The fundamental observation is that vectors are in one-to-one correspondence with the *first order behaviour* of curves passing through p (i.e. the instantaneous velocity along that curve). This in turn corresponds to first-order derivative operators at p (corresponding to moving along the curve's direction instantaneously). Vectors can be viewed as equivalence classes of curves through p, where two curves

γ_1, γ_2 are equivalent if they share the same tangent at p.

The guiding philosophy of this section is that everything vectors represent: namely direction and magnitude, can be derived from the way that we compute changes in a scalar function via directional derivatives. From multivariable calculus, it is easy to see that the first order change in a function along a curve is the sum of the changes due to each coordinate:

$$\frac{df}{dt} = \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} \tag{2.8}$$

Given a function f and given a specific direction of first order changes dq^i/dt, this gives a *real, invariant* quantity. Note that this quantity is dependent on two objects: the *function* that we are differentiating and the *direction* along which we differentiate.

When we focus a specific direction along a curve: $dq^i/dt = v^i$, and allow the function to be arbitrary we get a vector corresponding to change along our path:

$$\mathbf{v} = v^i \partial_i \tag{2.9}$$

This needs to be *fed* a function f to give a real number value.

On the other hand, if we pick a specific *function* with first order behaviour given by $\partial_i f = \omega_i$ and allow for the *direction* to be arbitrary, we get something else

$$df = \omega_i dq^i \tag{2.10}$$

This needs to be *fed* a direction dq^i to give an associated real value. We will call this new object a **covector** or a **1-form**, and it is often denoted by the greek letter ω. It is an entirely different object from a vector. Where a vector represents a direction that we can differentiate functions alone, this represents a *function differential* which we can take changes of, along a direction. It is the dual to the notion of a vector, but behaves in very much the same way.

The sum of two one-forms is still a one form, as are scalar multiples, so the one forms at a point p form a vector space called the **cotangent space** at p.

Definition 2.25 (Cotangent Space)**.** The cotangent space at a point p of a manifold M, T_p^*M is the set of all first order differentials at p.

Vectors were defined in terms of curves: they represent the first order approximations to curves (namely their tangents, and the derivatives along them). 1-Forms are defined in terms of *functions*: they represent the first order approximations of functions (namely their differentials). At first glance, you may think that for a dimension n manifold, there are only n directions to go in, whereas we have seemingly infinite freedom in picking the functions passing through p so there are many more 1-forms than there are vectors, but this isn't true.

Just as there are a huge number of functions that we can define at p, there are a huge number of curves that go through p, but because we only care about first order behaviour at p for the curves, many curves give the *same* tangent vector. Similarly, many functions will give the *same* 1-form, if they have the same first order behaviour. It should start to become clear that there is duality between these two ideas

Concept 2.26. The first-order behaviour of curves at a point p (i.e. vectors) is dual (in a sense that will be discussed in the next chapter) to the first-order behaviour of the functions at p. On a manifold of dimension n, both of these are vector spaces of dimension n.

Vectors	1-Forms
First-order approximations to curves	First-order approximations to functions
To differentiate functions along	To be integrated along curves
Act on 1-Forms	Act on Vectors
Contravariant components	Covariant components
Space of dimension n	Space of dimension n

This is all well and good, but how can we *visualize* these two concepts. Visualizing vectors is easy, we've been doing it since calculus class, if not before. They are "arrows", tangent to curves, showing the first-order behaviour of that curve. 1-forms on the other hand are a different story. If we pick a point p in 2D, then the first order behavior of a function can be visualized by adding in a dimension and showing the local tangent plane approximating that function. Alternatively,

we can stay in 2-D and just show the local *level curves* (these are lines for tangent plane).

GRAPHIC: Vector on the left, form on the right

In 3D vectors would look the same, and forms could be visualized by the local level-planes of the function at p.

Exercise on constructing a dual basis of forms to a vector

If we are given a coordinate system q^i then on a given vector $\mathbf{v} = v^i\partial_i$ representing a direction, a form $\omega_i dx^i$ representing the first order behavior of a function can *act* on $\mathbf{v}$ to give the first order change along $\mathbf{v}$.

Lemma 2.27. *For a basis $\frac{\partial}{\partial q^i}$ of the tangent space T_pM at a point p, the associated first order behavior corresponding to first order change only along one coordinate dq^i forms a basis for the cotangent space* and *we have $\partial_i dq^j = \delta_i^j$*

Proof. The fact that dq^i form a basis for the set of differentials of functions at p is clear since every function has the first order approximation $df = \partial_i f dq^i = \omega_i dq^i$, exactly in the span of the dq^i.
For the second part, for a specific coordinate q^i, ∂_i corresponds to finding a change along q^i, holding all other q^j fixed. If the local behavior of a function was just dq^j for a specific j different from i, then since this does not result in a change by varying i, we get that $\partial_i dq^j = 0$ if $i \neq j$. On the other hand for dq^i, representing the local behavior of a function that goes up by 1 unit for a unit change in dq^i, we'd get ∂_i of such a function is exactly 1, so we write $\partial_i dq^i = 1$ (no summation) for any specific i, so that in general $\partial_i dq^j = \delta_i^j$ exactly. □

From this, we know how forms act on vectors $\omega(\mathbf{v})$:

$$\omega(\mathbf{v}) = \omega_i dq^i(v^j\partial_j) = \omega_i v^j \; dq^i\partial_j = \omega_i v^j \delta_i^j = \omega_i v^i \tag{2.11}$$

For a real vector space V (in this case our tangent space), linear maps $\alpha : V \to \mathbb{R}$ that send vectors in $\mathbf{v} \in V$ to real numbers $\alpha(\mathbf{v})$ are called **linear functionals** on the vector space. Note that a linear combination of two linear functionals $a\alpha + b\beta$ is still a linear functional that acts by $(a\alpha + b\beta)(\mathbf{v}) = a\alpha(\mathbf{v}) + b\beta(\mathbf{v}) \in \mathbb{R}$. This is exactly what one forms are, they act on vectors and send them to real numbers corresponding to the first order change along $\mathbf{v}$ of ω. In general, for

a vector space V, the vector space of linear functionals on V is called the **dual space**, denoted V^*.

Definition 2.28. For a finite-dimensional vector space V, the set of all linear maps $\alpha : V \to \mathbb{R}$ forms a vector space V^* called the dual space of V.

EXERCISE: show $(V^*)^* = V$

In linear algebra, perhaps this idea is *too* simple. If our vector space is represented as column vectors, the linear functionals are the row vectors.

Proposition 2.29. *An element in V^* is determined entirely by its action on a basis $\mathbf{v}_i$ of V. In particular this means that* $\dim V^* = \dim V$.

Proof. Say two functionals α, β act exactly the same on the basis $\mathbf{v}_i$, then by their linearity they will act exactly the same everywhere:

$$\alpha(\mathbf{v}) = \alpha(a^i \mathbf{v}_i) = a^i \alpha(\mathbf{v}_i) = a^i \beta(\mathbf{v}_i) = \beta(a^i \mathbf{v}_i) = \beta(\mathbf{v})$$
$$\Rightarrow \alpha = \beta$$

This means a basis for the dual space can be built by the $(\mathbf{v}^*)^i \in V^*$ defined so that $(\mathbf{v}^*)^i$ acts as zero on all basis vectors $\mathbf{v}_j$ except $\mathbf{v}_i$, on which it's action gives 1, i.e. $(\mathbf{v}^*)^i \mathbf{v}_j = \delta^i_j$. □

2.6 What Follows

Fix the "What Follows" section according to the course of the book

The rest of this book will expand both on the geometry of fields and manifolds, and also on the larger ideas of groups, homogenous spaces, and representations.

In Chapter 3, we will continue studying the fields that live on manifolds. We'll prove the General Stokes' theorem, an elegant generalization of the divergence, curl, and line integral theorems that have been taught in multivariable calculus. From there, we will study

more thoroughly the concept of distance on a vector space and on a manifold using a metric, and how this relates vector fields to differential forms. The notion of a derivative will be extended to manifolds, and will take the form of a "Lie Derivative".

In Chapter 4, we will introduce Fourier Analysis as a powerful tool for studying functions on the real line and in Euclidean space. We'll see how the set of functions on a manifold naturally forms a vector space (of infinite dimension) and consider the Fourier Transform as a change of basis.

In Chapter 5, we will shift to looking at the representation theory of *finite* groups and illustrate the parallels. We will then return to the study of continuous group actions on especially symmetric "homogenous" spaces, and show how Fourier analysis is related to their representation theory. Towards the end, we will expand on the idea behind a group actions on manifolds and look at the representation theory, giving a small glimpse into harmonic analysis: the Fourier transform on manifolds. Just as in the first chapter, we'll recognize the importance of the underlying differential geometry of the group action. The underlying differential structure is known as the "Lie Algebra" of the group, and we will discuss that.

In Chapter 6, we introduce some background behind Lie Algebras. We put almost all of our focus on one special case: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The relationship between this algebra and the symmetries of the sphere are explored, as well as its applications in quantum physics for studying angular momentum. The representation theory of a variant of this algebra gives rise to the concept of spin.

In Chapter 7, we move further into physics, going over classical Lagrangian and Hamiltonian Mechanics. We discuss Noether's theorem in both the Lagrangian and Hamiltonian Pictures, and then we move to study Hamiltonian mechanics using the language of differential geometry that we have developed. This will give rise to *symplectic geometry*. In chapter 7, combining this with representation theory gives rise to *quantum mechanics*.

In Chapter 8, we apply differential geometry first to the study of

electromagnetism, and then to gravitation. We shall arrive at Einstein's theory of gravity. Along the way, we study in even greater detail the notion of a metric, a connection, and curvature.

In Chapter 9, we use the representation theory and differential geometry that we have developed so far to study how quantum mechanics can arise from quantizing a symplectic manifold.

Finally, Chapter 10 studies Lie algebras in greater detail, working towards the *classification of complex semisimple Lie algebras*. Along the way, we will look at the relationship between representation theory of Lie algebras and modern physics.

2.7 Exercises

Exercises: topology, connectedness, holonomy? definitions etc.

EXERCISE: Define fiber bundle here and in particular the principal bundle... Extend to gauge?

Part 1

A Language for Nature

Chapter 3

Differential Geometry

In calculus class, the fundamental theorem of calculus is introduced: that the total difference of a function's value at the end of an interval from its value at the beginning is the sum of the infinitesimal changes in the function over the points of the interval:

$$\int_a^b f'(x)dx = f\Big|_a^b \tag{3.1}$$

And later, in multivariable calculus, more elaborate integral formulae are encountered, such as the divergence theorem of Gauss:

$$\int_\Omega \nabla \cdot \mathbf{F}\ dV = \int_S \mathbf{F} \cdot d\mathbf{S} \tag{3.2}$$

where Ω is the volume of a 3D region we are integrating over, with infinitesimal volume element dV and S is the surface that forms the boundary of Ω. dS then represents an infinitesimal parallelogram through which $\mathbf{F}$ is flowing out, giving the flux integral on the right. Read in English, Gauss' divergence theorem says "Summing up the infinitesimal flux over every volume element of the region is the same as calculating the total flux coming out of the region". The total flux coming out of a region is the sum of its parts over the region. You might see that in English, this reads very similar to the description of the fundamental theorem of calculus.

Alongside this, there is Stokes' theorem for a 2D region. In English: summing up the infinitesimal amount of circulation of a vector field $\mathbf{F}$ over every infinitesimal area is equal to calculating the total circulation of $\mathbf{F}$ around the boundary of the region. In mathematical language:

$$\int_R \nabla \times \mathbf{F}\ dA = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{3.3}$$

where R is our region and C is its boundary.

Perhaps now, the pattern is more evident. In all the above cases, summing up some *differential* of the function on the interior of some region is the same as summing up the function itself at the *boundary* of the region. All these theorems, that on their own look so strange to a first-year calculus student, are part of a much more general statement, the **General Stokes' Theorem**:

Theorem 3.1 (General Stokes' Theorem)**.**

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega. \tag{3.4}$$

Above, ω is an object that will generalize both the "functions" and "vector fields" that you've seen in multivariable calculus, and d will generalize of all the differential operators (gradient, divergence, curl) that you've dealt with. Lastly, when Ω is the region in question $\partial\Omega$ represents the *boundary* of the region Ω. The fact that it looks like a derivative symbol is no coincidence, as we'll see that the natural way to define the "derivative" of a region is as its boundary.

Through abstraction, we can reach results like this that not only look elegant and beautiful, but also provide us with insight into the natural way to view the objects that we've been working with for centuries. This gives us not only understanding of what language to use when studying mathematics, but also what is the natural language in which to describe the natural world. The general Stokes' theorem is one of the first examples of this beautiful phenomenon, and this book will work to illustrate many more.

For the first half of this chapter, we will work towards giving the intuition behind this result. On our way, we will begin to slowly move into a much more general setting, beyond the 3-dimensional world in which most of multivariable calculus was taught. That doesn't just mean we'll be going into n-dimensional space. We'll move outside of euclidean spaces that look like $\mathbb{R}^n$, into non-euclidean geometries. This will put into question what we really mean by the familiar concepts of "vector", "derivative", and "distance" as the bias towards Euclidean geometry no longer remains central in our minds. At its worst,

the introduction of new concepts and notation will seem confusing and even unnecessary. At its best, it will open your mind away from the biases you've gained from growing up in a euclidean-looking world, and give you a glimpse of how modern mathematics *actually* looks.

Modern mathematics is learning that the earth isn't flat. To someone who's never had those thoughts, it is difficult to get used to, tiring, and sometimes even rage inducing, but to someone who has spent months thinking and reflecting on it, it quickly becomes second nature. Far from being the study of numbers or circles, it is a systematic climb towards abstraction. It is a struggle towards creating one language, free from all-encompassing human bias, in order to try and describe a world that all other human languages, for so many centuries, have failed to grasp. It is humbling, and in the strangest of ways, it is profoundly beautiful.

3.1 The Derivative and the Boundary

Let's start working towards understanding Equation (3.4). First, let's work with what we've already seen to try and explore the relation between integrating within a region and integrating on the boundary.

If we are in one dimension, we have a function f defined on the interval $x \in [a, b]$. Proving Equation (3.1) is much easier than you'd think. Let's take a bunch of steps: $x_i = a + (b-a)i/N$, so that $x_0 = a, x_N = b$. Then all we need is to form the telescoping sum:

$$\begin{aligned} f|_a^b &= f(x_N) - f(x_0) \\ &= \sum_{i=1}^{N} f(x_i) - f(x_{i-1}). \end{aligned}$$

If we make the number of steps N large enough, the stepsize shrinks so that in the limit, we get

$$\begin{aligned} \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i) - f(x_{i-1}) &= \lim_{N\to\infty} \sum_{i=1}^{N} \Delta f \\ &= \int_a^b df. \end{aligned}$$

Of course, the way its written more often is:

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{\Delta f}{\Delta x}\Delta x=\int_a^b\frac{df}{dx}dx.$$

What is the idea of what we've done? At each point we've taken a difference of f at that point with f at the preceding one. Because we're summing over all points, the sum of differences between neighboring points will lead to cancellation everywhere *except* at the boundary, where there will not be further neighbors to cancel out the $f(b)$ and $f(a)$. From this, we get Equation (3.1).

Note: Now for a distinction which may seem like it isn't important. We haven't integrated from point a to point b. We have integrated from where the coordinate x take *value* a, to the where coordinate x takes *value* b. a and b are NOT points. They are numbers, values for our coordinate x. As we have said in the preceding chapter, the idea that numbers form a *representation* for points is ingenius, but numbers are *not* points. Although we could write this interval as $[a, b]$ in terms of some variable x, it would be a completely different interval should we have chosen a different coordinate u. This is why, when doing u-substitution, we change the bounds. In coordinate free, language, then:

Theorem 3.2 (Fundamental Theorem of Calculus)**.** *For a given interval I with endpoints p_0, p_1 and a smooth function f, we have*

$$\int_{p_0}^{p_1} df = f\Big|_{p_0}^{p_1} \tag{3.5}$$

Notice something: the end result doesn't depend on the partition x_i at all, so long as it becomes infinitesimal as $N \to \infty$. That is to say: we are summing up the change of f over some interval, but it doesn't matter what coordinate system we use to describe this interval. The integral is *coordinate independent*. We chose to use x as our coordinate, describing the interval as going from $x = a$ to $x = b$, but we didn't *have* to make this specific choice. This makes perfect physical sense. For example, if we had a temperature at each point in space, the temperature difference between two fixed points some shouldn't depend on whether we use meters or feet to measure their distance apart.

Written mathematically:

$$\int_I df = \int_I \frac{df}{dx} dx = \int_I \frac{df}{du} du$$

If we chose an I that's very small around some point, essentially an infinitesimal line segment, we get:

$$\frac{df}{dx} dx = \frac{df}{du} du \Rightarrow \frac{df}{dx} = \frac{df}{du} \frac{du}{dx}$$

this is the u-substitution rule from calculus.

Now what if f was a function defined not on the real line $\mathbb{R}$, but on 2-dimensional space $\mathbb{R}^2$, or more generally n-dimensional space $\mathbb{R}^n$. To each point $p = (p_1, \ldots, p_n)$ we associate $f(p)$. Now again, consider $f(p_f) - f(p_i)$ for two points in this space.

For any curve C going between p_i and p_f, say defined by $\mathbf{r}(t)$ for t a real number going from a to b, we can make the same partition $t_i = a + (b-a)i/N$ and let N get large. Again, it becomes a telescoping sum:

$$\begin{aligned} f(p_f) - f(p_i) =& f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \\ =& \sum_{i=1}^{N} f(\mathbf{r}(t_i)) - f(\mathbf{r}(t_{i-1})) \\ =& \sum_{i=1}^{N} \Delta f_i \to \int_C df. \end{aligned}$$

Now if we cared about coordinates, we could ask "how can we write df in terms of dt or dx_i?".

We know from the multivariable chain rule that the infinitesimal change of f is the sum of the change in f due to every individual variable, so:

$$df = \sum_i \frac{df}{dx_i} dx_i \tag{3.6}$$

We know that dx_i together must lie along C. In terms of t since $x_i = r_i(t)$, we have $dx_i = \frac{dr_i}{dt} dt$ giving:

Theorem 3.3 (Fundamental Theorem of Line Integrals)**.** *For a smooth function f defined on a piecewise-smooth curve C parameterized by* $\mathbf{r}(t)$

$$f|_{p_i}^{p_f} = \int_C \sum_i \frac{df}{dx_i}\frac{dr_i}{dt}dt = \int_C \nabla f \cdot \frac{d\mathbf{r}}{dt}dt = \int_C \nabla f \cdot d\mathbf{r} \tag{3.7}$$

The proof of this was no different from the 1-D case.

Let's go further. Consider a region in three dimensions and a vector field

$$\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

We want to relate the total flux coming out of the region to the infinitesimal flux at each point inside the region. To do this, as before, we will subdivide the region. This time, it will not be into a series of intervals, but instead into a mesh of increasingly small *cubes*, as below.

Flux Graphic

See that the flux out a side of each cube is cancelled out by the corresponding side on its neighboring cube. That means that the only sides that do not cancel are for the cubes at the boundary[11], giving us the desired flux out.

So if we sum the fluxes over all infinitesimal cubes, we will get the total flux out of the boundary. For a single cube of sides dx, dy, dz, drawn below, the total flux will be the sum over each side.

$$\begin{aligned}\text{Flux} = \quad & F_z(x, y, z + dz/2)dxdy - F_z(x, y, z - dz/2)dxdy \\ & + F_y(x, y + dy/2, z)dxdz - F_y(x, y - dy/2, z)dxdz \\ & + F_x(x + dx/2, y, z)dydz - F_x(x - dx/2, y, z)dydz\end{aligned}$$

[1]You may be worried that the cubes do not perfectly fit into the boundary when it is not rectangular. As the mesh gets smaller and smaller, it approximates the region better so this does not pose a problem. This idea can be made rigorous (c.f.

Stokes' Theorem Rigor

)

Figure of infinitesimal flux on a cube: Stokes' proof

But we can write this as:

$$\left(\frac{\partial F_x(x,y,z)}{\partial x} + \frac{\partial F_y(x,y,z)}{\partial y} + \frac{\partial F_z(x,y,z)}{\partial z}\right) dxdydz = \nabla \cdot \mathbf{F}\ dV$$

So the total flux will be the sum over all these cubes of each of their total fluxes. But then this becomes exactly the divergence theorem:

Theorem 3.4 (Divergence Theorem, Gauss)**.** *For a smooth vector field* $\mathbf{F}$ *defined on a piecewise-smooth region* Ω*, then we can relate*

$$\int_\Omega \nabla \cdot \mathbf{F}\ dV = \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{S}$$

It is an easy **exercise** to show that this exact same argument holds for an n-cube.

What did we do? In the fundamental theorem of calculus/line integrals, we had a function f evaluated on the 1-D boundary, and we chopped the curve into little pieces that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece, and found that it was $df = f'(x_i)dx$, meaning that the evaluation on the boundary could be expressed as an integral of this differential quantity over the curve. That is Equation (3.1).

For the divergence theorem, we had a vector field $\mathbf{F}$, again *evaluated on the boundary*, this time in the form of a surface integral. We chopped the region into little pieces (cubes now) that cancelled on their neighboring boundaries, making a telescoping sum. Then we evaluated the contribution at each individual piece and found that it was $\nabla \cdot \mathbf{F}\ dV$, meaning that the integration on the boundary could be expressed as an integral of this differential quantity over the region. That is Equation (3.2).

Through abstraction, we see that there is really no difference. Perhaps now Equation (3.4) does not look so mysterious and far-off.

For Equation (3.3), we have a vector field $\mathbf{F}$ evaluated on the boundary in the form of a contour integral around a region. This is the total circulation of $\mathbf{F}$ around the region. Let us chop the region into little pieces.

Figure of infinitesimal circulation on a sqauare: Stokes' proof

On an infinitesimal square, we get that the circulation is:

$$\begin{aligned}\text{Circulation} = \quad & F_y(x + dx/2, y)dy - F_y(x - dx/2, y)dy \\ & + F_x(x, y - dy/2)dx - F_x(x, y + dy/2)dx\end{aligned}$$

This can be written as:

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy = (\nabla \times \mathbf{F})\ dA$$

so that

Theorem 3.5 (Stokes' Theorem in 2D)**.** *For a smooth vector field on a piecewise smooth region S*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F}\ dA \tag{3.8}$$

Exercise **(MAKE AN EXERCISE)**

Make an exercise to generalize classical Stokes' to 3D

generalizes this to a surface in 3D, to get the 3D version of Stokes' theorem :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \tag{3.9}$$

The philosophy behind these proofs is always the same. It is the manipulation of the differentials that seems wildly different every time. The curl looks nothing like a divergence, and a divergence is distinct from a gradient. Moreover, its not clear in what way each one generalizes the one dimensional derivative $df = f'(x)dx$. This is the problem that the symbol 'd' in Equation (3.4) was made to solve.

We must stop thinking of the 1D derivative, the gradient, the divergence, and the curl, as unrelated operations. They are in fact, the same operation, applied in different circumstances. Infinitesimal change, flux, and circulation are all the same differential action applied to different types of objects.

Perhaps part of this was clear from multivariable calculus: the gradient is nothing more than a generalization of the derivative to functions on higher dimensions. Then why are there seemingly two different, unrelated types of "derivative" on vector fields? Instead of a regular, gradient-like object, we have two: the divergence and the curl.

It will turn out that the reason that there are two is this: the vector fields that we take curls of are a different type of object from the vector fields we take divergences of. Looking forward, we'll see that we only take the curl on a vector field that is "meant to be integrated along curves" (**1-form**), and the curl gives us another vector field "meant to be integrated over surfaces" (**2-form**). On the other hand, the divergence takes a vector field "meant to be integrated over surfaces" (**2-form**) and gives us a scalar field "meant to be integrated over volumes" (**3-form**). Every object that we've encountered when integrating: from functions in 1-D or 3-D, to vector fields in n-D, have been examples of these **forms**.

To get to this idea, we first need to stop thinking of functions and vector fields as totally separate objects. A function is an object that is "meant to be evaluated at a point" (**0-form**). The derivative takes us from a function to a 1-form, meant to be integrated along a curve. It is the exact same object as the one in Section 2.5. The gradient, properly speaking, is not a vector describing the local behavior of a *curve* but is the opposite: a 1-form describing the local behavior of a *function*. So the correct way of writing this, is to go from the old $\mathbb{R}^3$ notation

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial k}\hat{\mathbf{k}}$$

to the modern language

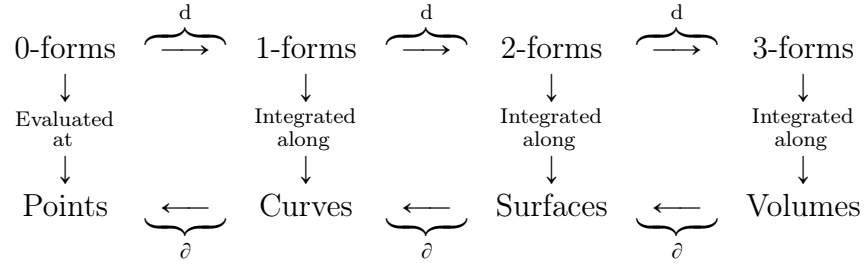
$$\mathrm{d}f = \frac{\partial f}{\partial x^i}dx^i \ \Big(= \nabla f \cdot d\mathbf{r}\Big) \tag{3.10}$$

This is a 1-form, as we already know.

What we would like is to have the old multivariable calculus chain

$$\text{functions} \overbrace{\longrightarrow}^{\text{grad}} \text{vector fields} \overbrace{\longrightarrow}^{\text{curl}} \text{vector fields} \overbrace{\longrightarrow}^{\text{div}} \text{functions}$$

be converted to



where ∂ is the **boundary operator** that takes us from an n dimensional manifold to its $n-1$ dimensional boundary. At this point, we won't be afraid to use our new word: manifold, when referring collectively to curves, surfaces, volumes, or any of their higher dimensional generalizations (points too, as the degenerate 0 dimensional case).

As a final note of this section, let us try to give a sketch for why on a region Ω, we denote its boundary with the partial derivative symbol as $\partial\Omega$. Picture in your mind a ball (interior of a sphere) of radius r, B_r. If we increase the radius by a tiny amount h then we have a slightly larger radius B_{r+h}. If we took the difference $B_{r+h} - B_r$, by which we mean all the points of B_{r+h} that are not in B_r, we would be left with a thin shell. In the limit as $h \to 0$, this becomes a sphere of radius r, precisely the boundary of B_r (note that a sphere is always the two-dimensional boundary of the ball). See how similar this is to taking derivatives. This is why ∂B_r is what we use to denote the sphere boundary of the ball.

You may ask "but what about dividing by h at the end, like we do for a regular derivative?". This also has an interpretation. The 3D volume of a sphere is zero, since it is a 2-D boundary. Dividing by h as h goes to zero puts increasing "weight" on the shell so that as the shell shrinks to becoming absolutely thin, 3-D integrals on it become 2-D [2].

3.2 The Notion of a Form

A differential form ω, in short, is an object that is meant to be integrated. We've seen the example of 1-forms in the preceding chapter.

[2]For those familiar with the terminology: dividing by h corresponds to multiplying by a dirac delta that spikes exactly on the sphere. This turns integrals over 3-D space into 2-D integrals on the sphere

At a point p on a manifold, one-forms ω are exactly all the first-order behaviors of the functions at p. Just as we can have a vector field on $\mathbb{R}^3$, or manifolds in general, we have 1-form fields. You have seen this before: the gradient is in fact a one-form field

$$(\nabla f \cdot d\mathbf{r})(p) = \frac{\partial f}{\partial q^i}(p)\ dq^i \tag{3.11}$$

The components of the gradient are *covariant* with our change of coordinate system, just like those of a 1-form (and unlike those of a vector field).

A general 1-form associates a first-order function behavior $\omega_i(p)\ dq^i$ to every point p in space. Just because ω is some differential behavior of a function at a point p doesn't mean that ω actually *is* the differential of a function. That is, it doesn't mean there exists a function so that $\partial f/\partial q^i = \omega_i$ at every point p so that $\omega = \mathrm{d}f$.

It rarely true that $\omega = \mathrm{d}f$. In fact, this happens *exactly* when ω actually *does* correspond to a gradient. This is exactly what we called a conservative vector field in introductory physics and in multivariable calculus. In this language, we call ω an **exact form** when it is the differential of a function.

This is what we care about when integrating. It is more fundamental than $\mathbf{F}$, but what does it mean *physically*? If $\mathbf{F}$ was a force field, then since we know $\mathbf{F} \cdot d\mathbf{r} = dW$, this form ω represents all possible infinitesimal changes in work dW at a given point, depending on what changes dx, dy, dz we do.

If we were actually *given* the changes in each of the coordinates dx, dy, dz, we could plug them in to ω and get the first-order approximation of the amount of work done over that distance. This is a point that has been said before: ω does not represent a specific change in work, but rather the *relationship* between the changes in coordinate and the change in work. If you *give it* an infinitesimal displacement, it will tell you the associated work. When integrating along a curve, the displacement is simply the tangent vector to the curve.

Even simpler than one-forms are the **zero forms**, with no differentials appearing. A zero-form precisely a scalar function at $f(p)$ each point p. Regardless of how we change our coordinate system, the value of the *function* at point p is the same.

We are now in a good place to define d, at least for going from functions (zero-forms) to one-forms. Given a function f, $\mathrm{d}f$ will produce

a form representing the local change in f depending on the displacement. We call d the **exterior derivative** operator.

For example, for a potential energy function ϕ, $\mathrm{d}\phi$ can be written as

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i} dx^i \tag{3.12}$$

because of d, we will no longer have to use the gradient at all. This is more important than simply meaning that we'll grow to stop using $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. It is something much deeper. In in two-dimensional motion, if you have some potential ϕ at a point p, then of course the value of ϕ at p is independent of any coordinate system you use. If you have two cartesian coordinates, say x, y, then you can define the x, y components of force by

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = F_x \; dx + F_y \; dy$$

If our coordinates were r, θ, then the analogous force would be the covariant components of the same *form*, in a different coordinate system:

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial \theta} d\theta + \frac{\partial \phi}{\partial r} dr = F_\theta \; d\theta + F_r \; dr$$

Note that the first component has units not of force, but of force times distance. It is precisely the torque that the potential induces. In this sense, quantities like torque are precisely just generalizations of force to non-cartesian coordinate systems (polar, in this case). The second component is just radial force, plain and simple.

To hammer the point across: these two "forces" have components that mean completely different things, and cannot easily be compared. On the other hand, since $\mathrm{d}\phi$ is independent of coordinate system, we get:

$$\mathrm{d}\phi = F_x \; dx + F_y \; dy = F_\theta \; d\theta + F_r \; dr \tag{3.13}$$

Because we know how to go from x, y to r, θ and because this nonlinear change of coordinates is *linear* at every point on the differentials, this would allow us to go between the language of "x-y force" and the language of "torque + radial force about the origin" at any point.

All forces (including the generalized forces, like torque) are covariant coefficients of the invariant differential form for work. If you're working in a coordinate system q^i, whether it be cartesian x, y, z or polar r, θ, then the coefficient corresponding to dq^i is precisely the generalized force associated with that coordinate in your system.

3.3 The Exterior Algebra and the Wedge

If a 1-form must be fed a vector of some associated change of coordinates dq^i, then what about a 2 form? A 2-form, ω, should associate a "flux" out of a plane, so we need ω to be given a plane associated with *two* directions v^i, u^i, and then ω acting on these two directions would give the associated "flux" out of the infinitesimal parallelogram gained by varying q^i in both directions. So if ω were a 1-form, it would act on one vector as $\omega(\mathbf{v}) = \omega_i v^i$, but now ω as a 2-form acts on two vectors:

$$\omega(\mathbf{u}, \mathbf{v}) \longleftrightarrow \text{Flux Coming out of } \mathbf{u} \text{ and } \mathbf{v}\text{s Parallelogram}$$

This is an intuitive *geometric idea*. This is what we want to be true, and the following observations will be *algebraic properties of* ω based on our geometric notions of flux.

1-form giving flux out of a parallelogram.. how is this even drawn?

Observation 3.6. $\omega(\mathbf{v}, \mathbf{v}) = 0$

The parallelogram generated by $\mathbf{v}$ and itself has no second dimension, so it has no area. Therefore there isn't room for any flux to come out of it.

Observation 3.7. $\omega(2\mathbf{u}, \mathbf{v}) = 2\omega(\mathbf{u}, \mathbf{v})$ *and* $\omega(\mathbf{u}, 2\mathbf{v}) = 2\omega(\mathbf{u}, \mathbf{v})$. *Moreover, in general* $\omega(a\mathbf{u}, \mathbf{v}) = \omega(\mathbf{u}, a\mathbf{v}) = a\omega(\mathbf{u}, \mathbf{v})$ *for* $a > 0$.

Of course, if we scaled the parallelogram by some positive amount a along one of the sides, then its total area scales by a, so that ω gives a times as much "stuff" coming out of the scaled parallelogram. This observation, together with our linear algebraic ideas, suggest that this should naturally extend beyond positive a so that $\omega(-\mathbf{u}, \mathbf{v}) =$

$-\omega(\mathbf{u}, \mathbf{v})$, giving us negative flux through the parallelogram. But what does that mean? This means that if one of the vectors gets scaled negatively, the *orientation* of the new parallelogram reverses.

Oriented Parallelograms

So now even though the pair $-\mathbf{u}, \mathbf{v}$ are on the same plane, the notion of "out" through their parallelogram has reversed. This makes physical sense,

Concept 3.8. What matters, when finding the flux associated with ω along two directions $\mathbf{u}, \mathbf{v}$

1. The *plane* that $\mathbf{u}, \mathbf{v}$ span, through which the flux is going
2. That scaling $\mathbf{u}$ or $\mathbf{v}$ also scales the flux.
3. The *orientation* for which direction is in and which is out.

This is exactly what we've seen before with the **Right-Hand Rule**. Because of this, and from what we've seen before with objects like cross products, we would *expect* that $\omega(\mathbf{u}, \mathbf{v})$ represents one orientation, and $\omega(\mathbf{v}, \mathbf{u})$ represents the opposite one so that

$$\omega(\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{v}, \mathbf{u})$$

we can add this as another property *but* we can instead actually prove it if we make just one more geometric observation

Observation 3.9. $\omega(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{w}) + \omega(\mathbf{v}, \mathbf{w})$

If $\mathbf{u} = a\mathbf{v}$ are linearly dependent then this is just the scaling observation. If $\mathbf{v}$ is linearly dependent with $\mathbf{w}, \mathbf{u} = a\mathbf{w}$, then this is just the observation that the parallelogram of $(\mathbf{u}, \mathbf{w})$ would have the same amount of associated area if you were to add vectors in the direction of $\mathbf{w}$ to $\mathbf{u}$ so $\omega(\mathbf{u} + a\mathbf{w}, \mathbf{w}) = \omega(\mathbf{u}, \mathbf{w})$. It's the same idea if $\mathbf{u}$ is linearly dependent with $\mathbf{w}$. Now let's assume all vectors are linearly independent. Geometrically, the two planes associated with $(\mathbf{u}, \mathbf{w})$ and $(\mathbf{v}, \mathbf{w})$, and the plane associated with the sum $(\mathbf{u} + \mathbf{v}, \mathbf{w})$ can look like:

Proving additivity for forms, sum of planes. Need to discuss about this argument

Now ω represents a constant flux in space. Enclosing a region by these three planes should mean that the flux that goes through the first two is the flux that comes out of the third.

The same exact argument can be applied to show this holds for the second slot in $\omega(-,-)$.

Corollary 3.10. *The above observations imply that ω is linear and* **antisymmetric** *in its arguments so that $\omega(\mathbf{u},\mathbf{v}) = -\omega(\mathbf{v},\mathbf{u})$. That is, as we expected, reversing the order of $\mathbf{u}$ and $\mathbf{v}$ reverses the orientation of the plane.*

Proof. We have shown that ω is compatible with scaling and addition in each argument, so it is linear in both.

Now consider $\omega(\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v})$. By our first observation, such a parallelogram has no area, so this is zero. On the other hand, by the linearity of ω in both arguments:

$$\begin{aligned} 0 &= \omega(\mathbf{u},\mathbf{u}) + \omega(\mathbf{u},\mathbf{v}) + \omega(\mathbf{v},\mathbf{u}) + \omega(\mathbf{v},\mathbf{v}) \\ &= \omega(\mathbf{u},\mathbf{v}) + \omega(\mathbf{v},\mathbf{u}) \\ &\Rightarrow \omega(\mathbf{u},\mathbf{v}) = -\omega(\mathbf{v},\mathbf{u}) \end{aligned}$$

□

2-forms are bilinear operators that act on pairs of vectors that represent coordinate changes $(\mathbf{u},\mathbf{v})$. They associate a flux to a given plane defined by such coordinate changes. In $\mathbb{R}^3$, the space of 1-forms is spanned by dx, dy, dz. We can also know the flux through any plane if we knew the flux on the xy, yz, and zx planes, so our basis for our set of 2-forms should also be three dimensional. It is spanned by three elements that we will write as

$$dx \wedge dy,\ dy \wedge dz,\ dx \wedge dz$$

The first element represents a flux of magnitude $|dx\, dy|$ through the xy plane an no flux through the other two. On the other hand $dy \wedge dx$ would represent a flux in the OTHER direction so that $dx \wedge dy = -dy \wedge dx$. We have invented something called the **wedge product**.

$$\begin{aligned} &\text{1-form in } \mathbb{R}^3 = \omega_x dx + \omega_y dy + \omega_z dz \\ &\text{2-form in } \mathbb{R}^3 = \omega_{xy}(dx \wedge dy) + \omega_{yz}(dy \wedge dz) + \omega_{xz}(dx \wedge dz) \end{aligned}$$

In multivariable calculus, we would define planes, together with orientations, by specifying a normal vector to those planes. In a general dimension, we aren't able to associate a normal vector to a plane, so we should talk about the plane *itself*. For this reason we define the **wedge product**.

Definition 3.11 (The Wedge Product)**.** The product $\wedge$ on a real vector space V defined so as to satisfy the following properties

- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V,\ (a\mathbf{a} + b\mathbf{b}) \wedge \mathbf{c} = a(\mathbf{a} \wedge \mathbf{c}) + b(\mathbf{b} \wedge \mathbf{c})$
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in V,\ \mathbf{c} \wedge (a\mathbf{a} + b\mathbf{b}) = a\mathbf{c} \wedge \mathbf{b} + b\mathbf{c} \wedge \mathbf{a}$
- $\forall \mathbf{a} \in V,\ \mathbf{a} \wedge \mathbf{a} = 0$

As before, this last condition, together with bi-linearity, implies anti-symmetry of $\wedge$.

In our case, when we have a basis dq^i for our cotangent space of 1-forms, the basis for our space of 2-forms can be written as $dq^i \wedge dq^j$ for $i < j$. The 2-form $dq^3 \wedge dq^4$, for example, represents a flux of magnitude $|dq^3 dq^4|$ out of the q^3q^4 plane and no flux out of q^iq^j planes for any other i, j. On the other hand $dq^4 \wedge dq^3$ would be the same flux in the opposite direction (and again, no flux out of any of the other q^iq^j planes).

It may be frustrating to see this new product without any previous background. Let's do an example of wedging two 1-forms in 3D: α and β, and see what happens.

$$\alpha \wedge \beta = (\alpha_x dx + \alpha_y dy + \alpha_z dz) \wedge (\beta_x dx + \beta_y dy + \beta_z dz)$$

$$\begin{aligned} = \quad & \alpha_x\beta_x(dx \wedge dx) + \alpha_x\beta_y(dx \wedge dy) + \alpha_x\beta_z(dx \wedge dz) \\ & + \alpha_y\beta_x(dy \wedge dx) + \alpha_y\beta_y(dy \wedge dy) + \alpha_y\beta_z(dy \wedge dz) \\ & + \alpha_z\beta_x(dy \wedge dx) + \alpha_z\beta_y(dy \wedge dy) + \alpha_z\beta_z(dy \wedge dz) \end{aligned}$$

$$\begin{aligned} = & (\alpha_x\beta_y - \alpha_y\beta_x)(dx \wedge dy) + (\alpha_y\beta_z - \alpha_z\beta_y)(dy \wedge dz) \\ & + (\alpha_z\beta_x - \alpha_x\beta_z)(dz \wedge dx) \end{aligned}$$

Very bad equation formatting here, Aaron probs knows how to fix

but if we were back in multivariable calculus world, not caring about vectors and forms and writing everything in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, and we identified

$$\hat{\mathbf{i}} \leftrightarrow dx, \hat{\mathbf{j}} \leftrightarrow dy, \hat{\mathbf{k}} \leftrightarrow dz$$

as well as

$$\hat{\mathbf{i}} \leftrightarrow dy \wedge dz, \hat{\mathbf{j}} \leftrightarrow dz \wedge dx, \hat{\mathbf{k}} \leftrightarrow dx \wedge dy$$

then the wedge product becomes *exactly* the cross product. Why wedge then, when we already know the cross product? Because the cross product, going from vectors to vectors, only works in three dimensions. The wedge product taking us from 1-forms to 2-forms, is universally valid.

Moreover, now we can go beyond just 2-forms and form higher wedges, k-forms. The wedge of two forms $\alpha \wedge \beta$ will always be linear in both arguments and antisymmetric. For example, we can make 3-forms. In 3-D, the space of 3-forms is one dimensional spanned by $dx \wedge dy \wedge dz$. Any other wedge of these differentials will either be the same, or a negative of this one. It corresponds to the one true infinitesimal 3D volume. What about the sign? It is the distinction between flow "into" the volume.

GRAPHIC of 2-form wedge a 1-form giving "inward" orientation, and then wedge the negative of that 1-form to give the "outward" orientation

This is more general, in n dimensions the space of n-forms is one dimensional, spanned by the form $\bigwedge_{i=1}^{n} dq^i$. For some terminology:

Definition 3.12 (**Exterior Power**)**.** For a given vector space V, the vector space of of k forms spanned by wedging elements of V with themselves k times is called the kth exterior power of V, and is denoted by $\Lambda^k V$.

Our V in this case is the cotangent space at a point $T_p^* M$: the space of 1-forms and $V = \Lambda^1 V$ always. The space of zero forms $\Lambda^0(T_p^* M)$ at a point is the set of possible function values so is just $\mathbb{R}$ in our case, since we are working over the reals. The space of k forms at p is the kth exterior power of the cotangent space at p: $\Lambda^k T_p^* M$. If we consider all k-forms at p, then we get the *exterior algebra* of the cotangent space at p.

Definition 3.13 (**Exterior Algebra**)**.** The vector space of *all* k-forms is called the exterior algebra of V and is denoted ΛV.

What about the tangent space of vectors? What does the exterior algebra mean there? If $dq^1 \wedge dq^2$ is the form that associates an oriented flux to the q^1q^2 plane, then $\partial_1 \wedge \partial_2$ is the oriented plane *itself.*

In this way

$$(\text{Flux } dq^1 \wedge dq^2)(\text{Coordinate Area } \partial_1 \wedge \partial_2) = (\text{Flux})(\text{Coordinate Area}).$$

This is the invariant *total flux* coming out from varying dq^1 and dq^2 together to sweep out a coordinate area. In general, k-wedges of vectors $v^i\partial_i$ represent the oriented k-volumes *themselves*, on which k-forms act. In Einstein's convention, you can show that in for a general 2-forms and 2-vectors whose coordinates are doubly covariant and contravariant, respectively, we get the invariant value:

$$\begin{aligned}(\omega_{ab}dx^a \wedge dx^b)(v^{cd}\partial_c \wedge \partial_d) &= \omega_{ab}v^{cd}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) \\ &= \omega_{ij}v^{ij} - \omega_{ij}v^{ji}.\end{aligned}$$

Exercise to generalize n forms acting on n vectors, and also to check that it makes sense in 3D when our wedges are cross products and areas are normal vectors

You may have a question, concerning the fact that for 1-forms, we expect them to act on vectors representing infinitesimal displacements. Therefore, k-forms should act on something as well! Do they act on wedges of vectors, representing infinitesimal k-volumes? Yes, and in addition to this, we can act on a k form by a vector (or you can say act on a vector by a k form, no difference here) in the form of an **interior product**.

Interior Product

As a last note, philosophically, *where does this antisymmetry come from?* We've already seen it in the cross product, and now we have it in this wedge. Geometrically, what is happening? That the wedge product of a form with itself is zero is easy to understand: you cannot geometrically extend an object to higher dimensions without introducing new directions. Antisymmetry, on the other hand, is less obvious.

When we extend a geometric k-volume to a $k+1$-volume, there is a notion of orientation. Going from the line to the plane, we need to know "which direction is out for flux?" Similarly, for the plane to the volume, we have basically the same question "which direction is in/out for flux?", and the antisymmetry of the wedge reflects that orientation will always exist for higher k volumes.

3.4 Stokes' Theorem

To go from a 0-form f to a 1-form ω, we applied the exterior derivative operator, which could just be written as:

$$\mathrm{d}f = dq^i \frac{\partial f}{\partial q^i}. \tag{3.14}$$

Going further, perhaps we could write the exterior derivative operator explicitly asa the invariant:

$$\mathrm{d} = dq^i \frac{\partial}{\partial q^i}. \tag{3.15}$$

We can view this d operator as a 1-form whose coefficients on each dq^i, rather than being numbers, are derivative operators $\partial/\partial q^i$ with respect to the corresponding coordinates.

Now for a 1-form $\omega = \omega_i dq^i$, we want the exterior derivative to take us to a 2-form. If this d operator can be thought of as a 1-form, the obvious way to go to a form one step higher is by wedging, meaning that we would define:

$$\mathrm{d}\omega := (dq^i \frac{\partial}{\partial q^i}) \wedge \omega = dq^i \wedge \frac{\partial \omega}{\partial q^i}. \tag{3.16}$$

Note that the only reason we did this is because of what the *algebra* seemed to tell us to do, independent of any geometric intuition beforehand. This is powerful, but is this right? Is this the derivative operator that will generalize the gradient, divergence, curl, and *everything else*?

Let us first check that for a 1-form ω $\mathrm{d}\omega$ gives us Stokes' theorem, as we want:

Theorem 3.14 (Stokes' Theorem for 1-forms)**.** *If ω is a 1-form, then with ∂ defined as the boundary operator of a manifold and* d *defined as in Equation* (3.16)*, we have*

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega.$$

Proof. Take Ω, and as in all of our proofs in the section Section 3.1, let us cut Ω into a mesh of infinitesimal parallelograms. If we integrate ω over the boundary $\partial\Omega$, this is the same as integrating ω over every single individual parallelogram on the interior, as *BECAUSE OF ORIENTATION*, the integrals over the boundaries of these parallelograms will cancel between neighbors, leaving us with only the boundary, as always.

It remains to show that for an arbitrary small parallelogram, Stokes' theorem holds. This parallelogram is obtained by varying q^i along two vectors $u^i\partial_i$ and $v^i\partial_i$. After a suitable linear transformation of coordinates, we can assume WLOG that this parallelogram is obtained by changing q^1 by some fixed small amount dq^1 and q^2 by dq^2. Let's integrate ω on the boundary:

Because the parallelogram is small, we can approximate these integrals as:

$$(\partial_1\omega_2 - \partial_2\omega_1)dq^1dq^2$$

On the other hand the exterior derivative is:

$$\begin{aligned}\mathrm{d}\omega &= dq^i \wedge (\partial_i\omega)\\ &= (dq^1 \wedge dq^2)\partial_1\omega_2 + (dq^2 \wedge dq^1)\partial_2\omega_1 + \text{other}\\ &= (\partial_1\omega_2 - \partial_2\omega_1)dq^1 \wedge dq^2 + \text{other}\end{aligned}$$

where the other terms involve wedges that aren't of q^1, q^2 and will vanish along integration of this specific parallelogram. Since integrating $(\partial_1\omega_2 - \partial_2\omega_1)dq^1 \wedge dq^2$ gives exactly $(\partial_1\omega_2 - \partial_2\omega_1)dq^1dq^2$, we have proven it for this parallelogram, and by linearity and coordinate change for *any* paralellogram. Because adding these parallelograms together forms the bulk of Ω and they cancel when integrated on neighboring boundaries (if one boundary is associated with dq^i, the neighboring one is associated with $-dq^i$), this gives the desired result. □

Fix Stokes' Theorem Proof

WE DONT NEED ANY OTHER COORDINATES

Note that if $q^i = (x, y, z)$ then this is exactly Stokes' theorem in 3-D for the curl! More than this, it generalizes Stokes' theorem in $\mathbb{R}^3$: for any 2-D surface, the circulation of ω over the boundary is exactly the sum total of the curl $\mathrm{d}\omega$ over the interior.

From this let us prove what we set out to prove in the most general case:

Theorem 3.1 (General Stokes' Theorem)**.** *With ∂ defined as the boundary operator of a manifold and* d *defined as in Equation* (3.16)*, we have for a general differential k-form ω that*

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega.$$

First, a lemma:

Lemma 3.15 (Restriction of a Form)**.** *If ω is a k form of n variables that is being integrated over some k-dimensional manifold associated with changing only the first k variables, then for the integration, we can work with the restricted ω_{res} associated with setting the last $n-k$ dq^i equal to zero, and eliminating any wedge terms holding those dq^i.*

Proof. This follows from □

IM NOT SURE WE NEED THIS NOW

Now to prove the General Stokes' Theorem:

Proof. For a given manifold Ω, divide it's bulk into k-volumes that are generalizations of parallelograms to k-dimensions. Just like a line's boundary has two points, a parallelogram's has 4 lines, a parallelepiped's has 6 parallelograms, a k-volume's boundary is $2k$ $k-1$ volumes. Again, we can set up local coordinates so that this k volume has each $k-1$ obtained by holding some q^i constant and letting the others vary. For each q^i there are exactly two opposing $k-1$ volumes obtained by holding that coordinate constant, and they have opposite orientation.

Now let us perform the integration of ω over the $k-1$ boundaries. In terms of these coordinates, ω can be written as a linear combination of wedges of $k-1$ of the dq^i, meaning that each such wedge misses exactly one dq^i. □

THE PROBLEM WITH THE ABOVE IS THERE COULD BE n q^i and

Proposition 3.16. $\mathrm{d}^2 = 0$

Proof. For a general form ω, consider

$$\begin{aligned}\mathrm{d}(\mathrm{d}\omega) &= (dq^j\partial_j) \wedge (dq^i\partial_i\omega) \\ &= dq^j \wedge dq^i(\partial_j\partial_i\omega)\end{aligned}$$

This is a summation over i and j running from 1 to n. Now pick any specific term in the sum with *specific* indices a, b. This corresponds to a term:

$$dq^a \wedge dq^b(\partial_a\partial_b\omega)$$

in the sum. We can assume $a \neq b$ as otherwise that'd mean $dq^a \wedge dq^b = 0$. But that means we will have another distinct term with those indices reversed (b, a) equal to

$$dq^b \wedge dq^a(\partial_b\partial_a\omega)$$

Since partials commute but the wedge products anti-commute, this (b, a) term is equal to the *negative* of that first (a, b) term, meaning they will cancel. The whole double-sum will then become a sum of cancelling terms, giving zero. □

Corollary 3.17. $\partial^2 = 0$*: The boundary of a boundary is nothing.*

Proof. For any k-form ω, assume we are integrating a form $\mathrm{d}\omega$ on a $(k+1)$-boundary. By Stokes' Theorem (twice):

$$\int_{\partial^2\Omega} \omega = \int_{\partial\Omega} \mathrm{d}\omega = \int_{\Omega} \mathrm{d}^2\omega = \int_{\Omega} 0 = 0.$$

Since ω was any arbitrary form, we must have $\partial^2\Omega = 0$. □

This is an amazing geometric fact that we have gotten, via Stokes' theorem, from the *purely algebraically derived* fact that $\mathrm{d}^2 = 0$. The duality between forms and the manifolds we integrate them over is a gorgeous duality between algebra and geometry that extends very deeply and profoundly (c.f. Hodge Theory **INSERT TEXTS HERE**).

We have already seen that differential forms that are the exterior derivatives $\mathrm{d}\omega$ of some other form ω are called exact. We know exact 1-forms correspond to conservative vector fields. **It will be an exercise to show** exact 2-forms in $\mathbb{R}^3$ correspond to solenoidal vector fields.

Exact 2-forms on $\mathbb{R}^3$ are correspond to solenoidal vector fields

On the other hand, forms ω that have $\mathrm{d}\omega = 0$ are called **closed**. Why this language? It is taken for the corresponding geometric language for the boundary operator. If a region has no boundary, it is called closed, so if a form has zero exterior derivative, it will also be called closed. Clearly exact forms are closed, since $\mathrm{d}^2 = 0$, but are *all* the closed forms exact? In $\mathbf{R}^3$, the answer is yes, but consider this:

Example 3.18. $d\theta$ is a closed form defined on the punctured plane that is not exact.

We know $\mathrm{d}(d\theta) = 0$, so it is indeed closed. Although locally, a function θ (that this form represents the change of) can be defined just by calculating the angle from the x axis, if you go around counterclockwise in a circle containing the origin, then θ continuously increases. At the end of the revolution, even though you are at the same point, θ has increased by 2π. So although $d\theta$ makes sense locally as a differential form everywhere in the plane minus the origin, we cannot define a global smooth function representing θ without a discontinuity. The existence of a closed form that is not exact happens because the manifold on which $d\theta$ is defined is *not* $\mathbb{R}^2$, but is instead defined on $\mathbb{R}^2$ without the origin (where $d\theta$ would not be well-defined). This change in global geometric structure gives rise to these interesting closed, inexact forms.

The study of the closed forms that are not exact on a manifold is called the **De-Rham cohomology** of a manifold.

Punctured Plane for De-Rham Cohomology

3.5 Distance, a Metric

Why have we been making such a difference between vectors and forms?! In multivariable calculus I could have done all the 3-D calculation working only with my regular vector fields using $\hat{\mathbf{i}}$ *and the rest. Why*

can't I just turn these forms into vector fields?.

These are the types of questions you might ask, were this book written in dialogue. It's a fair point. For some reason, in multivariable calculus, there seemed to be no problem just using the language of vector fields for gradients, curls, and integration. What are all these forms doing here? The reason, as we've said before, is because we've no longer assumed that we were working in an "orthogonal reference frame". In fact, up until now we have been working in vector spaces of tangent vectors and forms that had *no notion of distance whatsoever.* Accordingly, we made no assumptions that our manifolds themselves had any notion of distance and length either. Even though we talked about wedge products as associated with area, it was not true area of euclidean space but merely a (contravariant) number obtained by multiplying the changes of coordinates dq^i together.

But we know that in the physical world there *is* a notion of length. It is not just affine space. There is in fact a notion of *being perpendicular.* If we had this notion, something powerful happens. Every 1-form, as we know from the previous chapter, can be viewed of as local function behavior, and therefore visualized in terms of level curves of that function. If we had a notion of being perpendicular, then we have a unique direction *perpendicular to the level curves of the first order behavior* along which ω increases. We can then associate a specific, invariant vector $\mathbf{v} = v^i\partial_i$ that is perpendicular to the level curves of ω, and along which omega increases by exactly $|\mathbf{v}|^2$.

Conversely, if we have a notion of being perpendicular, then to a specific vector, we can pick a unique 1-form $\omega = \omega_i dx^i$ whose level curves are perpendicular to the direction of $\mathbf{v}$ and such that the increase of ω along $\mathbf{v}$ is $|\mathbf{v}|^2$.

What we want is to associate an invariant *physical length element* to a small local change of coordinates v^i around any point. In 3-dimensional Euclidean space $\mathbf{E}^3$, with an orthogonal coordinate system x, y, z we know how to obtain a physical change in length from the changes of the coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3.17}$$

We are let by Pythogoras: In euclidean space $\mathbb{E}^n$, for an orthogonal

frame, the length associated with a change of coordinates is

$$|x| = \sqrt{\sum x_i^2} \Rightarrow |x|^2 = x_i x_i \tag{3.18}$$

where we've used lower indices for coordinates because in an orthogonal frame there's no big deal between covariance and contravariance. More generally, in an orthogonal frame, $\mathbf{E}^n$ has a dot product producing a scalar $\mathbf{x} \cdot \mathbf{y} = x_i y_i$. In general coordinates, though, we've already seen $v^i v^i$ is not invariant just by noting the two upper indices, and also by analyzing how it trasforms directly in Section 1.3.

We want to define an *invariant* **inner product** $\langle \mathbf{v} | \mathbf{w} \rangle$ on our tangent space, bilinear in both arguments, just like the one in $\mathbf{E}^n$. Then we will write:

$$\begin{aligned} \langle \mathbf{v} | \mathbf{w} \rangle &= \left\langle v^i \partial_i | w^j \partial_j \right\rangle \\ &= v^i w^i \left\langle \partial_i | \partial_j \right\rangle \end{aligned} \tag{3.19}$$

This object, $\langle \partial_i | \partial_j \rangle$, is the inner product between just the basis vectors themselves. Its double co-variance cancels out the double contravariance of the coordinates themselves to give an invariant quantity to this inner product. In an orthonormal frame, since the vectors are orthonormal we would have $\langle \partial_i | \partial_j \rangle = \delta_{ij}$ so that the inner product $\delta_{ij} v^i w^j$ is exactly the same as the dot product we are used to.

Now we can get the length of a vector:

$$|\mathbf{v}|^2 = \langle \mathbf{v} | \mathbf{v} \rangle = v^i v^j \left\langle \partial_i | \partial_j \right\rangle \tag{3.20}$$

in general, for an *arbitrary* change of coordinates dq^i, we can finally associate a *physical length* in space by:

$$ds^2 = dq^i dq^j \left\langle \partial_i | \partial_j \right\rangle \tag{3.21}$$

Because of the significance of this object $\langle \partial_i | \partial_j \rangle$, we will them by g_{ij}. All together, this invariant inner product that takes in two coordinate changes:

$$ds^2 = g_{ij} (dq^1)^i (dq^2)^j \tag{3.22}$$

will be called the **metric** on our tangent space. A metric turns the tangent space into an **inner product space**

At a very far glance, you might ask if this thing is some sort of 2-form, but it is not an anti-symmetric two-form associated with integrating over areas. There are no wedges between the differentials, and

in fact since the dot product (which is the inner product in orthonormal basis) is symmetric, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, this holds true here, so in fact $g_{ij} = g_{ji}$ is symmetric as a matrix. This is a different type of product of 1-forms, that will be discussed in the next section. For now, it is the tool by which we can associate a length with a coordinate change dq^i, and how we can take scalar products of two vectors together.

To avoid abstraction without geometric concreteness, let's do a simple but nontrivial example. In cartesian coordinates on Euclidean space, $g_{ij} = \delta_{ij}$, but in polar coordinates, we now have $\partial_r, \partial_\theta$. At each point, these vectors are orthogonal, and ∂_r corresponds to a unit increase in r, meaning that it has magnitude 1. On the other hand, ∂_θ corresponds to a unit increase in theta. Increasing θ by $d\theta$ on a circle of radius r gives an infinitesimal length of $rd\theta$. The total line element then is

$$ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{3.23}$$

At every point, this gives the *physical* length associated with any local infinitesimal change of coordinates $(dr, d\theta) = v^i$ that form the components of a vector.

Immediately, being able to associate an invariant length to a tangent vector on our space allows us to be able to define the length of any curve on our manifold. For a curve $\gamma(t) \in M$ parameterized by t. For a coordinate patch, we get $q^i = \varphi^i(\gamma)$. On each tangent space we have the tangent vector to γ:

$$\mathbf{v}_{\gamma,t} = \frac{d}{dt}\left[\varphi^i(\gamma(t))\right] \frac{\partial}{\partial q^i} \tag{3.24}$$

for which we can now calculate an associated length:

$$|\mathbf{v}_{\gamma,t}|^2 = g_{ij}\ \dot{\varphi}^i(\gamma)\dot{\varphi}^j(\gamma). \tag{3.25}$$

Proposition 3.19 (The Metric Defines the Length of Curves)**.** *The length of a curve γ parameterized by t on a manifold M with metric g is*

$$\int_{t_i}^{t_f} \sqrt{g_{ij}\ \dot{\varphi}^i(\gamma(t))\dot{\varphi}^j(\gamma(t))}\ dt \tag{3.26}$$

and can be more clearly seen as

$$\int_\gamma \sqrt{g_{ij}\frac{dq^i}{dt}\frac{dq^j}{dt}}dt = \int_\gamma \sqrt{g_{ij}dq^i dq^j} \tag{3.27}$$

where dq^i is the infinitesimal coordinate change along the curve at that point.

We said that the integrals over curves should be one-forms, but the integrand doesn't look like it is. Note, however, that $\sqrt{g_{ij}dq^i dq^j}$ is exactly the "square root of a 1-form product", which could be viewed as the "norm" of a 1-form in some sense. This object is not a 1-form, it is a **measure**. The difference between the two is that while a 1-form will give a negative of its original value if integrated in the opposite direction of the curve, a measure will *always* associate a positive value to any region it is integrated over. These are centrally important to calculating positive geometric invariants like length, volume, etc.

In the polar example, this automatically gives us the equation for arc-length:

$$\int_\gamma \sqrt{dr^2 + r^2 d\theta^2} = \int_{t_i}^{t_f} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2}\, dt \tag{3.28}$$

and indeed, once g is known, we can find the equation for arc length in *any* coordinate system.

Arc Length in worthwhile coordinate systems

Now note, the metric tensor was not something that we had before on a general manifold M. It is added information. A manifold doesn't need a metric or any notion of distance to be well-defined. After all, its definition is just topological. When M is endowed with with a metric g at every tangent space T_pM, we say it is a **Riemannian manifold**.

Definition 3.20 (Riemannian Manifold)**.** A Riemannian manifold is a smooth manifold M with a positive definite bilinear form $g(\mathbf{u}, \mathbf{v})$ on each tangent space. This form is called a ***metric*** or a ***Riemannian metric***.

g is positive definite in the sense that $g(\mathbf{u}, \mathbf{u}) \geqslant 0$ always, and equals zero iff $\mathbf{u} = 0$. This defines our positive-definite inner product.

This condition ensures that every nonzero vector has positive length. Now, endowed with a metric, we can measure lengths of curves, and much more.

Proposition 3.21 (The Metric Defines Angles)**.** *If two curves γ_1, γ_2 intersect at a point p on a Riemannian manifold, the angle between them can be calculated in terms of their respective tangent vectors,* $\mathbf{u}, \mathbf{v}$ *by*

$$\cos(\theta) = \frac{g_{ij}u^i v^i}{|u||v|} \tag{3.29}$$

This is the obvious analogue of the Euclidean definition, now we've just extended the dot product of vectors to general coordinates. This is how we will *define* the notion of angle in a manifold. Note under general linear transformations, angles between two vectors can appear sheared and stretched. It is the *metric* that holds us steady and tells us "this frame is orthogonal, this one is not, it's distorted and so the dot product here should be g_{ij} instead to recover the right notion of angle".

It is also worth noting a simple fact

Lemma 3.22. *The metric tensor g_{ij} is invertible as a matrix.*

Proof. Assume g isn't invertible. Then there is a nontrivial null space for g. Then there exists a nonzero vector $\mathbf{u}$ with $g\mathbf{u} = 0$, but that means $g(\mathbf{u}, \mathbf{u}) = 0$ without $u = 0$, contradicting the positive definiteness of the metric. □

Now since a metric gives us an inner product, to each vector $\mathbf{v}$ is associated a corresponding $\langle \mathbf{v}| - \rangle$ operation that wants to take in a vector. But this is *exactly* what a 1-form is. We then have:

Proposition 3.23 (Metric Defines an Isomorphism Between Vectors and Covectors)**.** *A given metric g associates to each vector $\mathbf{u}$ a corresponding 1-form $\omega_{\mathbf{u}}$ in the sense that for all vectors, $\mathbf{v}$, $\omega_{\mathbf{u}}$ acts according to inner product*

$$\omega_{\mathbf{u}}\mathbf{v} = \langle \mathbf{u}|\mathbf{v}\rangle, \omega = g\mathbf{u} \tag{3.30}$$

In coordinate notation, we have that for $\mathbf{u} = u^i\partial_i$ give rise to

$$(\omega_{\mathbf{u}})_i dx^i = (g_{ij}dx^i dx^j)(u^j\partial_j) = (g_{ij}u^j)dx^i \tag{3.31}$$

so the tuple of coefficients for the form corresponding is the matrix g_{ij} multiplying the tuple u^j for the vector (note we could have had it multiplying u^i since g_{ij} is symmetric so we can switch i and j above).

Of course, in an orthonormal frame, where $g_{ij} = \delta_{ij}$ is just the identity, this means that the form corresponding to ∂_i is just dx^i and things are simple. It is for this reason that forms and vectors have been interchangeable in our orthonormal frames with trivial metric. In any interesting coordinate system, however, this breaks immediately.

Note, then, that if $g\mathbf{v} = \omega$ gives a form associated to a vector, and this is just a matrix-vector equation then we would expect that

$$g^{-1}\omega = \mathbf{v} \tag{3.32}$$

gives a *vector* associated to every 1-form. In index notation

$$(g^{-1})^{ij}\omega_j = v^i \tag{3.33}$$

For this reason, g and its inverse[3] are said to "raise and lower indices". This inverse g then gives us an inner product on the cotangent space:

$$\langle \alpha | \beta \rangle = g^{ij}\alpha_i\beta_j \tag{3.34}$$

its easy (almost immediate) to check that the inner product of 1-forms gives the same number as the inner product of their corresponding vectors.

Mathematicians, who often write vectors and forms just as $\mathbf{v}$ and ω, without ever resorting to index notation have invented a very creative notation for the way that g associates vectors to their 1-forms.

Proposition 3.24 (Musical Isomorphisms)**.** *At each point p on a Riemannian manifold M, the metric g induces an isomorphism between the tangent space T_pM and the cotangent space T_p^*M:*

$$\flat : T_pM \to T_p^*M \tag{3.35}$$

so that for each vector $\mathbf{v} \in T_pM$, $\omega = \mathbf{v}^\flat$ is the associated 1-form. The inverse of the metric gives the opposite direction

$$\sharp : T_p^*M \to T_pM \tag{3.36}$$

so that $\mathbf{v} = \omega^\sharp$ is the associated vector to the 1-form ω.

[3]Note that often $(g^{-1})^{ij}$ is often just written as g^{ij} in the literature simply because upper indices automatically mean that we have inverted the covariant g_{ij}, and so no confusion ensues.

Diagram of Musical Isomorphisms

As mentioned in the beginning of this section, since the metric gives the notion of "orthogonality" to the tangent space, geometrically the form corresponding to a vector $\mathbf{v}$ is the local linear behavior of a function with level curves perpendicular to $\mathbf{v}$, and increasing in the direction of $\mathbf{v}$ so that $\omega_{\mathbf{v}}(\mathbf{v}) = |\mathbf{v}|^2$. $\mathbf{v}$ would then be exactly the *gradient vector* to this local behavior, pointing in the direction of greatest ascent.

This means we can define the gradient vector to a function, finally.

Definition 3.25 (The Gradient Vector)**.** For a function f, we define ∇f as the vector field

$$\nabla f = (\mathrm{d}f)^\sharp. \tag{3.37}$$

In a coordinate system q^i this is:

$$(\nabla f)^k \partial_k = \left(g^{ij} \frac{\partial f}{\partial q^j} \right) \partial_i \tag{3.38}$$

That is, we take the form corresponding to the first-order behavior of f and because of the metric we now have the vector field corresponding to the *direction of greatest ascent*. Comparing components, we have that

$$\nabla^i f = g^{ij}(\partial_j f) \Rightarrow \nabla^i = g^{ij}\partial_j \tag{3.39}$$

So the gradient operator is the "contravariant" version of the partial derivative. Often it is just written as the "raised partial" ∂^i to avoid confusion with what ∇ will denote in later sections. The powerful thing about knowing the metric is that it allows us to define the gradient operator in *any* coordinate system. In the polar example, this is

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \Rightarrow \nabla f = \frac{\partial f}{\partial r}\partial_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\partial_\theta.$$

Often, engineers want to define this in terms of the "normalized basis vectors" of length 1: $\hat{\mathbf{r}} = \partial_r$, $\hat{\boldsymbol{\theta}} = \partial_\theta / r$ so that

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}.$$

3.6 Multilinear Algebra: $\oplus, \otimes$ and Tensors

INSERT SOME QUOTE ABOUT HOW LIKE 97% OF MATH IS JUST TERMINOLOGY

Before moving on to completing the study of differential geometry, it is worth reflecting on the mathematical machinery that we have developed.

We began differential geometry seriously in Section 2.5 by noting that there were two first order behaviors that were dual: vectors, which represented the first order behavior of curves and gave rise to direction, and forms, which represented the first order behavior of functions, and represented an operator that took a direction and gave us an associated change.

Both sets of first order behavior had the property of being vector spaces: their elements could be added together and scaled. From this we defined the tangent and cotangent spaces. If we changed our coordinate system q^i, on both spaces this induced a linear transformation (since nonlinear coordinate transforms are locally linear, c.f. Section 1.4) on the *components* of the vectors. The tangent space would have its components transform one way and the cotangent space would have the components transform the other way. As a result, these two spaces were not naturally isomorphic.

With the introduction of the metric g_{ij}, whose components are "doubly" covariant, we get a linear map that lets us go from the contravariant components v^i of a vector to the covariant ω_i of a form. This also gives us an inner product.

While v_i and ω_i could both be represented by a tuple, a list of numbers that changed either with or against the basis ∂_i when we changed coordinates, g_{ij} is represented by a *matrix* of numbers. In fact this matrix is symmetric. The metric, g, is and object that's neither a vector nor a covector but something "higher".

We will work towards making intuitive and rigorous the concept of a **tensor**, which generalizes that of a vector and covector from before:

Concept 3.26 (Tensor)**.** A tensor is a physical quantity that can be represented by a multidimensional array of k indices. Because the numbers within the tensor depend on the coordinate system, they will

change (either co-variantly or contra-variantly). Components associated with some indices can be covariant, coordinates associated with other indices can be contravariant.

We've already seen before: a k-form, the wedge product of k 1-forms is a tensor with k components. For a set of indices i_k (which we will call a **multi-index**), we associate the flux that is going through the k-volume $dq^{i_1} \wedge dq^{i_2} \wedge \cdots \wedge dq^{i_k}$. This tensor has the property that if we flip any two neighboring i_k, say i_1 with i_2, then the value at that new multi-index is the negative of the value of the old, because $\wedge$ is anti-symmetric. This tensor is thus called anti-symmetric: flipping neighboring indices flips the sign.

In order to appreciate this all better, however, we need to go back and better our understanding of the language of vector spaces.

Consider an n-dimensional vector space, V with a basis $\{\mathbf{v}_i\}$ and an m-dimensional vector space W with basis $\{\mathbf{w}_i\}$.

Definition 3.27. The ***direct sum*** $V \oplus W$ is the $n+m$-dimensional vector space of ordered pairs $(\mathbf{v}, \mathbf{w})$ with $\mathbf{v} \in V$, $\mathbf{w} \in W$, with addition defined component-wise.

GRAPHIC: DIRECT SUM, R with R, and R with $\mathbf{R}^2$

The direct sum is one of the easiest concepts to grasp. It is exactly what it seems, at first glance, a addition of new directions to V. The only possible uncertainty is "what if the two bases overlap?". But when taking the direct sum, V and W are assumed to be entirely different vector spaces from the start, and we assume they share no elements. For this reason, we can take $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$ even though the two vector spaces are the same.

The direct sum is simply the addition of new possibilities, new directions to V by W. It is clearly commutative $V \oplus W = W \oplus V$. We can write a direct sum of the individual elements as:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \oplus \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} \tag{3.40}$$

An element in the direct sum $V \oplus W$ corresponds uniquely to a vector in V paired with a vector in W. If a given vector space can be written as a direct sum of two nontrivial sub-spaces, we say it can be **decomposed**.

Example 3.28. A plane in $\mathbb{R}^3$ can be **decomposed** as a direct sum of any two distinct lines going through it. On the other hand $\mathbb{R}^3$ clearly cannot be decomposed into a direct sum of that plane and a line going through that same plane. Even though the plane and the line are isomorphic to $\mathbb{R}^2$ and $\mathbb{R}$, in context, since their embedding in this 3-space is linearly dependent, their combined span is not $\mathbb{R}^3$.

On the other hand, there *is* a way to pick a plane and a line so that their combined span is $\mathbb{R}^3$ so $\mathbb{R}^3$ *is* decomposable into $\mathbb{R}^2$ and $\mathbb{R}$. It depends on *which* $\mathbb{R}^2$ and *which* $\mathbb{R}$ we choose inside $\mathbb{R}^3$.

The notion of a direct sum is so broad and general that it will be seen for the remainder of this book. It is the idea that V and W are two entirely different worlds, and $V \oplus W$ is their combination.

A very intuitive way to understand the direct product further comes, oddly enough, from quantum mechanical language. In quantum mechanics, the various states that something can have are represented by vectors. Say something can be one of three shapes

$$\{\square,\ \triangle,\ \bigcirc\}$$

Now in quantum mechanics we can mix states so something could look like $\frac{1}{\sqrt{2}}\square + \frac{1}{\sqrt{2}}\triangle$, so these form a vector space V with vectors $a\square + b\triangle + c\bigcirc$. If we had another two states, say ☺ and ☹ that formed a space of vectors $a☺ + b☹$ labelled W then the direct sum $V \oplus W$ would be spanned by the basis:

$$\{\square,\ \triangle,\ \bigcirc,\ ☺, ☹\}$$

The idea of the direct sum should resemble exactly how you learned addition as a toddler.

Within *this* scheme, it is easy to explain the **direct product** also known more commonly as the **tensor product**.

View 1 (3-Year-Old's Perspective)**.** The tensor product of a space spanned by $\{\square,\ \triangle,\ \bigcirc\}$ with a space spanned by $\{☺, ☹\}$ gives rise to a new space, spanned by the *composite pairs*

$$\begin{Bmatrix} \square\otimes☺, & \triangle\otimes☺, & \bigcirc\otimes☺, \\ \square\otimes☹, & \triangle\otimes☹, & \bigcirc\otimes☹ \end{Bmatrix}$$

This is a space of dimension $\dim V \times \dim W$.

The tensor product $V \otimes W$ represents all the states of a *composite system* of one object that has a state in V and another object that has a state in W. It's especially important to stress is that $V \otimes W$ is a *completely different space* from either V or W, and unlike the direct sum, neither of the original vector spaces are a part of this new space.

So in general, for a space V with basis $\mathbf{v}_i$ and a space W with basis $\mathbf{w}_i$, the tensor product space is simply a "product" of V and W in the sense that it is spanned by the abstract new vectors $\mathbf{v}_i \otimes \mathbf{w}_j$. There is no more structure to the tensor product other than requiring that $\otimes$ is distributive, just like multiplication is: $\mathbf{a}\otimes(\mathbf{b}+\mathbf{c}) = \mathbf{a}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{c}$, and vice versa for $(\mathbf{a}+\mathbf{b})\otimes\mathbf{c}$. Also, we say $\mathbf{a}\otimes(c\mathbf{b}) = c(\mathbf{a}\otimes\mathbf{b})$. So tensoring with c times a vector gives c times the tensor with that vector. This is the exact same distributivity that ordinary multiplication has $x(cy) = c(xy)$. The tensor product is no more than a new space of "formal multiplication of vectors, satisfying distributivity".

Definition 3.29 (Tensor Product)**.** For two vector spaces V, W, we define $V \otimes W$ as the vector space of all products $\mathbf{v} \otimes \mathbf{w}$ satisfying the following bi-linearity conditions:

1. $(a\mathbf{v}) \otimes \mathbf{w} = a(\mathbf{v} \otimes \mathbf{w})$ and $\mathbf{v} \otimes (a\mathbf{w}) = a(\mathbf{v} \otimes \mathbf{w})$
2. $(\mathbf{v}_1 + \mathbf{v}_2) \otimes \mathbf{w} = \mathbf{v}_1 \otimes \mathbf{w} + \mathbf{v}_2 \otimes \mathbf{w}$ and $\mathbf{v} \otimes (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \otimes \mathbf{w}_1 + \mathbf{v} \otimes \mathbf{w}_2$

Now we can relate pairs of vectors $\mathbf{v} \in V$ and $\mathbf{w} \in W$ to an element in the tensor product basis by writing $\mathbf{v}, \mathbf{w}$ in terms of the $\mathbf{v}_i$ and $\mathbf{w}_i$ bases and using distributivity of $\otimes$:

$$\mathbf{v} \otimes \mathbf{w} = (v^i \mathbf{v}_i) \otimes (w^i \mathbf{w}_i) = v^i w^i \mathbf{v}_i \otimes \mathbf{w}_i$$

The second view is the easy answer given when a student asks "what is a tensor?". A professor may reply "you can just think of it as a matrix". This is also exactly the right way for computer scientist to view tensors, but their language refers to vectors as "arrays" and tensors, consequently, as "double arrays".

View 2 (Computer Scientist's Perspective)**.** For elements in V represented as lists of n numbers, and elements in W represented as lists of m numbers, the elements of $V \otimes W$ are represented as $n \times m$ tables A_{ij} so that the entry of A_{ij} is the coefficient of basis element $\mathbf{v}_i \otimes \mathbf{w}_j$.

By distributivity, for individual lists v^i and w^j their associated tensor product $\mathbf{v} \otimes \mathbf{w}$ has entries $v^i w^j$.

The third view is similar. If we start with a vector space V, $V \oplus W$ is similar to saying that "we now have more vectors" that we can add into V. That means the operation '+' an act not just between vectors in V but also between vectors in V and vectors in W. We have "extended the set of vectors we can add together" to include W, giving the resulting larger space of $V \oplus W$.

In this vein, the tensor product can be viewed as an extension of *scalar multiplication*

View 3 (Extension of Scalars)**.** We view $V \otimes W$ as extending the vectors in V from having *scalars* as their components to having *vectors* in W as their components. That is, we "extend scalars" from the one-dimensional $\mathbb{R}$ to the higher space W.

Then a general vector $\mathbf{a}$ in $V \otimes W$ can be viewed as a vector in $\mathbf{V}$ whose components are not scalars, but are themselves vectors in $\mathbf{W}$. This is really the same perspective as the computer scientist's.

The last view is probably the least intuitive, but is the one that every mathematician will call the "correct" way to define the tensor product. What it lacks in intuition, it makes up for in motivating why this tensor product operation is worth studying *at all*.

The tensor product can first be motivated by the action of a bilinear forms. We've seen the metric as an example of a (symmetric) bilinear form on the tangent space. Moreover two any two vectors we can also associate an area $A(\mathbf{u}, \mathbf{v})$ to be the signed area of the parallelogram that they span. This is an antisymmetric bilinear form.

Consider now a general bilinear form $B(\mathbf{v}, \mathbf{w})$ acting on a vectors $\mathbf{v}, \mathbf{w}$ in a space V. In fact, to go to greater generality, consider $B(\mathbf{v}, \mathbf{w})$ acting on vectors $\mathbf{v} \in V$ for its first argument, and vectors $\mathbf{w}$ in another space W for the second argument. B takes a vector from V and a vector to W and sends them to a scalar. B need not be positive definite, symmetric, define a dot product, and can have a null space. It can be anything.

If $\{\mathbf{v}_i\}$ is a basis for V and $\{\mathbf{w}_i\}$ for W then we will know how B acts on any pair of vectors in (V, W) iff we know $B(\mathbf{v}_i, \mathbf{w}_j)$ for every pair $(\mathbf{v}_i, \mathbf{w}_j)$. We need $\dim V \times \dim W$ dimensions of information know the action of B. In full notation:

$$B(\mathbf{a}, \mathbf{b}) = B(a^i \mathbf{v}_i, b^j \mathbf{w}_j) = a^i b^j B(\mathbf{v}_i, \mathbf{w}_j) \tag{3.41}$$

Just as in differential geometry, a 1-form's coefficient ω_i was defined by its action on *each* $\partial_i = \mathbf{v}_i$, this *bilinear* form's coefficient $B(\mathbf{v}_i, \mathbf{w}_j)$ is defined by its action on each *pair* of basis vectors $\mathbf{v}_i, \mathbf{w}_j$. Let us form a vector space whose basis is not $\mathbf{v}_i$ or $\mathbf{w}_j$ but instead all possible *pairs* of the form $\mathbf{v}_i \otimes \mathbf{w}_j$. B is acting exactly on these pairs.

So vectors in this space look like linear combinations of the basis vectors $c^{ij}\mathbf{v}_i \otimes \mathbf{w}_j$ and B acts as a *linear operator* on this space: $B(c^{ij}\mathbf{v}_i \otimes \mathbf{w}_j) = c^{ij}B(\mathbf{v}_i \otimes \mathbf{w}_j)$. But besides looking at the pairs $\mathbf{v}_i \otimes \mathbf{w}_j$, can we take such a product between any two vectors, $\mathbf{a} \otimes \mathbf{b}$? We'll define $\otimes$ exactly so that it gives us what we want:

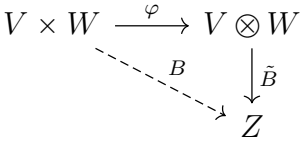
$$\begin{aligned} B(a^i\mathbf{v}_i, b^j\mathbf{w}_j) &= a^i b^j B(\mathbf{v}_i, \mathbf{w}_j) \\ \Rightarrow (a^i\mathbf{v}_i) \otimes (b^j\mathbf{w}_j) &= a^i b^j \mathbf{v}_i \otimes \mathbf{w}_j \end{aligned}$$

That is, the tensor product is *defined* to be bilinear so that $B(\mathbf{u}, \mathbf{v})$ corresponds exactly to $B(\mathbf{u} \otimes \mathbf{v})$ on the tensor product space.

Bilinear operators are important, however they seem less compatible with the tools of linear algebra. We don't want to generalize "linear" algebra to "quadratic" or "cubic" algebra because that'd be wildly more complicated and less elegant. Therefore, we view all bilinear operators as *linear* operators acting on *tensor spaces*, and the only bilinear operator we have to worry about from now on is $\otimes$ itself.

View 4 (Category Theorist's Perspective)**.** The tensor product is defined so that the following definition holds.

Definition 3.30 (Universal Property of the Tensor Product)**.** For two vector spaces V, W, we define a bilinear map φ from[4] $V \times W$ into their tensor product space $V \otimes W$. The tensor product space satisfies the *universal property* that for any bilinear map $B(-,-) : V \times W \to Z$ into a space Z, there is a *unique* linear map $\tilde{B}(-)$ induced on $V \otimes W$ so that the following diagram commutes:



[4]In general, for a function with 2 arguments: the first in a set A, the second in a set B, we say the function's domain is $A \times B$. This is just notation.

Important Note. *Although* $\mathbf{a}\otimes\mathbf{b}$ *is an element in the tensor product space, not* all *elements in this space are tensor products of two vectors. The ones that are are called* **simple tensors**. *It's easy to see that not all tensors are simple, as a tensor of the form* $\mathbf{a}\otimes\mathbf{b}$ *really holds only* $\dim V + \dim W$ *pieces of information, less than the dimension of the whole space.*

So we can view the metric g as acting on the tensor product space $T_pM \otimes T_pM$ spanned by tensors of tangent vectors $\partial_i \otimes \partial_j$. For each such basis element we have a corresponding element $dq^i \otimes dq^j$ in our dual basis. This means that the proper way to write the metric in this tensor language is

$$g = g_{ij}\ dq^i \otimes dq^j \tag{3.42}$$

For this reason, g is referred to as not just a metric but a **metric tensor**.

Observation 3.31. *A metric* g *is a linear functional on* $T_pM \otimes T_pM$, *making it an element of the dual space* $(T_pM \otimes T_pM)^*$.

We see how given a tensor product basis $\partial_i \otimes \partial_j$, the dual basis is just tensors of the dual vectors of the original space $dq^i \otimes dq^j$. This means that:

Observation 3.32. *For finite dimensional vector spaces,* $(V\otimes W)^* = V^* \otimes W^*$.

An element in a tensor space will have components that are described by two indices i and j, each of which can be either co/contravariant.

Example 3.33. In $T_pM \otimes T_pM$ a general element will be written as $c^{ij}\ \partial_i \otimes \partial_j$. On the other hand in $T_p^*M \otimes T_p^*M$, an element is written as $c_{ij}\ dq^i \otimes dq^j$ (and a metric tensor is such and element). We could abstractly define $T_pM \otimes T_p * M$ as a vector space, and there each element would be written as $c^i_j\ \partial_i \otimes dq^j$, with one index co-variant, one index contra-variant.

Important Note. *Tensor spaces are still vector spaces, as we can add and scale their elements. This is a weird thing to get used to, as it doesn't feel right to say "tensors are vectors", but this is due to the multiple meanings associated with the word "vector", some of which are not compatible with the notion of a tensor.*

In this vein of covariant and contravariant indices, let us go back to transformation laws. If we change from q^i to q'^i, then the basis vectors/covectors at a point p change according to Equations (1.8),(1.9). Their components change in the respectively opposite ways depending on the Jacobian matrix $J = \frac{\partial q'^j}{\partial q^i}$ and its inverse, $J^{-1} = \frac{\partial q^i}{\partial q'^j}$ at p:

$$v'^i = \frac{\partial q'^i}{\partial q^j} v^j, \quad \omega'_i = \frac{\partial q^j}{\partial q'^i} \omega_j. \tag{3.43}$$

On the other hand, the metric tensor would change like:

$$g'_{ij} = \frac{\partial q^k}{\partial q'^i} \frac{\partial q^l}{\partial q'^j} g_{kl} \tag{3.44}$$

That is, because the invariant length of a vector $g_{ij} v^i v^j$ involves the contra-variant components twice, g must co-variantly change twice (once for each index) to keep that inner product invariant. We'd say that g is a tensor with two covariant indices, or a $(0, 2)$-tensor.

Of course, we can take tensor products of tensor products and form higher rank tensors. For a tensor living in $V_1 \otimes \cdots \otimes V_r$ we call its **rank** r. Vectors and covectors are rank 1, scalars are rank 0, and the metric tensor is rank 2. We use the notation $V^{\otimes r}$ to denote $V \otimes \cdots \otimes V$ r times, and save space.

Higher tensor spaces also satisfy a similar universal property on **multilinear maps**.

Definition 3.34 (Multilinear Map)**.** A multilinear map M into some vector space Z

$$M : V_1 \times \cdots \times V_r \to Z \tag{3.45}$$

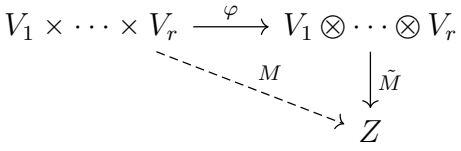
is a map with r arguments, with the ith argument taking a vector from V_i that is *linear in each argument.*

Example 3.35. The function giving the signed k-volume enclosed by a set of k vectors is a multilinear map.

Proposition 3.36 (Universal Property of Higher Tensor Spaces)**.** *For a multilinear map $M : V_1 \times \cdots \times V_r \to Z$, there is a multilinear map into their tensor product space*

$$\varphi : V_1 \times \cdots \times V_r \to V_1 \otimes \cdots \otimes V_r \tag{3.46}$$

so that any multilinear map M induces a unique *linear map $\tilde{M}$ on the tensor space so that the following diagram commutes.*



Definition 3.37 (An (a, b)-Tensor)**.** A tensor of a contra-variant indices and b covariant indices is called an (a, b)-tensor, and can be written in local coordinates at a point p on M as:

$$T^{i_1 \ldots i_a}_{j_1 \ldots j_b} \partial_{i_1} \otimes \cdots \otimes \partial_{i_a} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_b} \in (T_p M)^{\otimes a} \otimes (T_p^* M)^{\otimes b}$$

and we know how $T^{j_1,\ldots,j_b}_{i_1,\ldots,i_a}$ linearly transforms on each index. Moments like this are when we should be grateful to Einstein's summation convention for saving us $a + b$ summation signs out front.

If we have a vector space V and we want to consider ALL the tensors we can get from applying $V \otimes V$ to itself any number of times, then we get the **tensor algebra** $T^{\cdot}V$ of V, defined as

$$\bigoplus_{r=1}^{\infty} V^{\otimes r} \tag{3.47}$$

In this algebra, we have our addition operation $+$ as before, and $\otimes$ becomes a multiplication operation.

EXERCISE ON A GRADED ALGEBRA

Some examples of tensors

We gave the example of k-forms are antisymmetric tensors of rank k, and the wedge product $\wedge$ does have all of the properties of $\otimes$, but it includes the extra property that $\omega \wedge \omega = 0$, which gives it its anti-symmetry. If we exchange any two indices in the component $\omega_{j_1,\ldots,j_k}$ of a tensor representing a k-form, the new component is the opposite sign. The tensors that satisfy this property are called **anti-symmetric**.

Alternatively, we can define the space of anti-symmetric tensors of rank r to be the space of tensors of rank r with the additional

constraint that $T \otimes T = 0$ for any tensor T in that space. This gives us an antisymmetric space.

Performing this anti-symmetrization on $V^{\otimes k}$ gives the kth exterior power $\Lambda^k V$. Anti-symmetrizing the whole tensor algebra gives the whole *exterior algebra* ΛV from before.

On the other hand, the metric tensor is *symmetric*, in that exchanging the indices of a component g_{ij} gives a new component gij with the same value. Such tensors whose components *don't* change if you permute the indices are called **symmetric**. Alternatively for the space of rank r tensors over V, $V^{\otimes r}$, we can define the subspace space of symmetric tensors of rank r by taking $V^{\otimes r}$ and identifying $\mathbf{v}_1 \otimes \mathbf{v}_2 = \mathbf{v}_2 \otimes \mathbf{v}_1$ for any vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$. Doing this on the whole tensor algebra gives the **Symmetric Algebra**.

3.7 The Hodge Star and the Laplacian (*Optional*)

We've seen how we can get length of curve from the metric associating an infinitesimal length to a coordinate change dq^i. We should expect, then, that we can get volumes in our space as well.

For a point p on a manifold, let us choose coordinates x^i so that we have an *orthonormal basis* $\hat{\partial}_i$ at p, so that $g_{ij} = \delta_{ij}$. That means that the differential form for this orthonormal basis:

$$\omega = dx^1 \wedge \cdots \wedge dx^n \tag{3.48}$$

gives exactly the area of a generalized parallelogram (in fact generalized rectangle) associated with n changes in the x^i. Of course.. in an orthonormal basis we didn't need to use the language of forms. The magnitude of the volume is just $|dx^1 \dots dx^n|$.

Now say we *change* our $\hat{\partial}_i$ to a new arbitrary coordinate system q'^i with local basis ∂'_i by $\partial'_i = A^j_i \hat{\partial}_j$. Now, from linear algebra, we know that for a transformation $\mathbf{A}$ acting on an orthonormal basis $\mathbf{e}_i$, the determinant of $\mathbf{A}$ is the volume enclosed by the new vectors $\mathbf{e}'_i = \mathbf{A}\mathbf{e}_i$ the absolute value of the volume enclosed by our original choice of unit vectors must have been scaled by $|\det \mathbf{A}|$.

For an arbitrary basis, we don't keep track of "how to go back to an orthonormal basis" but we *do* always keep track of the metric.

An if originally $g_{ij} = \left\langle \hat{\partial}_i | \hat{\partial}_j \right\rangle = \delta_{ij}$ then now $g'_{ij} = A^k_i A^l_j g_{ij}$. The determinant of these matrices, on the other hand is:

$$\begin{aligned} \det g_{ij} &= \det \delta_i j = 1 \\ \Rightarrow \det g'_{ij} &= \det(A^k_i A^l_j g_{ij}) \\ &= \det A^j_i \det A^l_j \det g_{ij} \\ &= (\det A)^2 \end{aligned}$$

This means that the absolute value of the volume enclosed by the new vectors ∂'_i is *in general* equal to $\sqrt{\det g_{ij}}$. Then we have a corresponding **volume form**.

Definition 3.38 (Volume Form)**.** For an n-dimensional Riemannian manifold M with metric g, the invariant volume form Vol_n is given by

$$\mathrm{Vol}_n = \sqrt{\det g_{ij}}\; dq^1 \wedge \cdots \wedge dq^n \tag{3.49}$$

This associates to each infinitesimal n volume given by n changes in the coordinates dq^i, an invariant physical volume.

With a metric, we can take an inner product not only on the tangent space at a point p but also on the cotangent space as well. Can we extend our inner product to act on exterior powers of the cotangent space? That is, can we take an inner product of k-forms if we have an inner product on the 1-forms?

Answering this question is a good example of when actually going to *higher* abstraction helps guide us along and give us the right answer. It's what we've been doing so far, to be able to formulate differential geometry universally and independent of an orthogonal frame. Let us then ask the more general question: given an inner-product space V and an inner product space W is there a natural inner product structure on $V \otimes W$?

Say our basis for $V \otimes W$, as before is $\mathbf{v}_i \otimes \mathbf{w}_j$. We want to imagine a way to take:

$$\langle \mathbf{v}_i \otimes \mathbf{w}_j | \mathbf{v}_k \otimes \mathbf{w}_l \rangle$$

The naive guess would be to just let this become

$$\langle \mathbf{v}_i \otimes \mathbf{w}_j | \mathbf{v}_k \otimes \mathbf{w}_l \rangle = \langle \mathbf{v}_i | \mathbf{v}_k \rangle \langle \mathbf{w}_j | \mathbf{w}_l \rangle \tag{3.50}$$

so that for general tensors $\mathbf{T}, \mathbf{S} \in V \otimes W$ we would want to extend it bilinearly so that:

$$\langle \mathbf{T} | \mathbf{S} \rangle = \left\langle T^{ij} \mathbf{v}_i \otimes \mathbf{w}_j | S^{kl} \mathbf{v}_k \otimes \mathbf{w}_l \right\rangle = T^{ij} S^{kl} \langle \mathbf{v}_i | \mathbf{v}_k \rangle \langle \mathbf{w}_j | \mathbf{w}_l \rangle \quad (3.51)$$

Is this bilinear and positive definite? If so, it's a valid inner product on the tensor space, uniquely defined.

Proposition 3.39 (Inner Product on $V \otimes W$)**.** *The above construction defines an inner product on the tensor space.*

Proof. We see that the inner product must be bilinear for Equation (3.51) to hold. In fact, bilinearity follows just by definition because

$$\begin{aligned}
\langle \mathbf{T} | \mathbf{S} \rangle &= \left\langle T^{ij} \mathbf{v}_i \otimes \mathbf{w}_j | S^{kl} \mathbf{v}_k \otimes \mathbf{w}_l \right\rangle \\
&= \left\langle (T^{ij} \mathbf{v}_i) \otimes \mathbf{w}_j | (S^{kl} \mathbf{v}_k) \otimes \mathbf{w}_l \right\rangle && \text{(Property of } \otimes\text{)} \\
&= \left\langle T^{ij} \mathbf{v}_i | S^{kl} \mathbf{v}_k \right\rangle \langle \mathbf{w}_j | \mathbf{w}_l \rangle && \text{Construction, Eq. (3.50)} \\
&= T^{ij} S^{kl} \langle \mathbf{v}_i | \mathbf{v}_k \rangle \langle \mathbf{w}_j | \mathbf{w}_l \rangle && \text{By Inner Product on } V
\end{aligned}$$

note that on something like the tangent space this would become:

$$\langle \mathbf{T} | \mathbf{S} \rangle = g_{ik} g_{ij} T^{ij} S^{kl} \quad (3.52)$$

Positive definiteness is simple to show as well. Pick the bases $\mathbf{v}_i, \mathbf{w}_i$ on each space to be orthonormal.

$$\begin{aligned}
\langle \mathbf{T} | \mathbf{T} \rangle &= T^{ij} T^{kl} \langle \mathbf{v}_i | \mathbf{v}_k \rangle \langle \mathbf{w}_j | \mathbf{w}_l \rangle \\
&= T^{ij} T^{kl} \delta_{ik} \delta_{jl} \\
&= \sum_{i,j} (T^{ij})^2 \geqslant 0
\end{aligned}$$

and is only equal to zero when $T^{ij} = 0$ always, i.e. $\mathbf{T} = 0$. Since this metric assigns positive value to any vector, it is positive definite. The fact that we used an orthonormal basis doesn't change this invariant property, it just makes the proof easier.

Do this proof without going into an orthonormal basis maybe?

□

An example of an inner product on V lifting to a tensor space is when we define an inner product on the set of matrices:

Example 3.40 (Inner Product of Matrices)**.** On a tensors space $V \otimes V$ we can always interpret the tensor coefficients themselves as matrices and considering $\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_{ij}$ corresponding to the i, j th entry. If we have an orthogonal basis $\mathbf{e}_i$ for both, then an inner product on V gives rise to an inner product on these matrices of $V \otimes V$ by

$$\langle \mathbf{e}_{ij} | \mathbf{e}_{kl} \rangle = \delta_{ik}\delta_{jl} \tag{3.53}$$

this means each of the n^2 entries $\mathbf{e}_{ij}$ is orthogonal from the others and we have an orthogonal basis for our matrix space.

Moreover this gives rise to the *matrix norm* in an orthonormal basis

$$|A|^2 = \sum_{ij} (A^{ij})^2 \tag{3.54}$$

or in a general basis, with a metric g this is

$$|A|^2 = g_{ik} g_{jl} A^{ij} A^{kl} \tag{3.55}$$

which collapses to what we would get using Equation (3.53) when $g_{ij} = \delta_{ij}$.

Proposition 3.41 (Inner Product on $V^{\otimes r}$)**.** *V has an inner product structure, then so does $V^{\otimes r}$.*

Proof. By induction, the base case $r = 1$ our assumption and for the inductive step, we have just shown that if V, W have inner product structure, so does $V \otimes W$. Let $W = V^{\otimes(r-1)}$ and we are done. □

But then if we can take inner products of arbitrary rank tensors, then we can *restrict* our inner product to just the antisymmetric tensors and get

Definition 3.42 (Inner Product on k-forms)**.** The inner product on the kth exterior power is the restriction of the inner product defined above on $\mathbf{V}^{\otimes k}$ to the subspace of antisymmetric tensors $\Lambda^k V$.

In particular, since the cotangent space $T_p^* M$ has inner product given by g^{ij}, then the space of k forms $\Lambda^k T_p^* M$ inherits an inner product:

$$\langle \alpha | \beta \rangle = g^{i_1 j_1} \dots g^{i_k j_k} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_k} \tag{3.56}$$

So now we can find norms of differential forms, which should correspond in some way to areas and k-volumes. There is one differential form in particular that is worth seeing the norm of: the top one, the n-form of our manifold:

$$\left\langle dq^1 \wedge \cdots \wedge dq^n | dq^1 \wedge \cdots \wedge dq^n \right\rangle \tag{3.57}$$

Now this is n times contravariant. As a specific anti-symmetric tensor in the space, it should be written

$$\omega_{i_1 \ldots i_n} dq^{i_1} \wedge \cdots \wedge dq^{i_n} \tag{3.58}$$

where ω is antisymmetric so that $\omega_{12\ldots n} = +1/n!$ and any permutation that flips the indices an *even* number of times is $+1/n!$ while any permutation that flips an *odd* number of times (associated with the volume oriented the other way) is $-1/n!$. Because there are $n!$ permutations, this will add up to produce a coefficient of 1 in front of the oriented volume $dq^1 \wedge \cdots \wedge dq^n$, as desired. Then the inner product of this form with itself is:

$$g^{i_1 j_1} \ldots g^{i_n j_n} \omega_{i_1 \ldots i_n} \omega_{j_1 \ldots j_n} \tag{3.59}$$

Since $i_1 \ldots i_n$ need to be some permutation of $1 \ldots n$,this can be written as a sum over *all* permutations of the n numbers:

$$\sum_{\sigma,\pi} g^{\sigma(1)\pi(1)} \ldots g^{\sigma(n)\pi(n)} \omega_{\sigma(1)\ldots\sigma(n)} \omega_{\pi(1)\ldots\pi(n)} \tag{3.60}$$

where σ, π denote permutations on $1 \ldots n$ numbers. If we're careful with the combinatorics, though, and write $i_1 = \sigma(1) \ldots i_n = \sigma(n)$ this is the same sum as:

$$\sum_{i_1,\ldots i_n,\rho} g^{i_1 \rho(i_1)} \ldots g^{i_n \rho(i_n)} \omega_{i_1 \ldots i_n} \omega_{\rho(i_1)\ldots\rho(i_n)} \tag{3.61}$$

where $\rho = \sigma^{-1}\pi$ is the relative permutation from the first one. Since $\omega_{i_1 \ldots i_n}$ is $\pm 1/n!$ depending on the order, $\omega_{i_1 \ldots i_n}$ *times* $\omega_{\rho(i_1)\ldots\rho(i_n)}$ will be $+ 1/n!$ if ρ is an *even* number of interchanges and $-1/n!$ if ρ is an odd number of changes so this becomes:

$$\frac{1}{n!^2} \sum_{i_1 \ldots i_n,\rho} \operatorname{sgn}(\rho)\; g^{i_1 \rho(i_1)} \ldots g^{i_n \rho(i_n)} \tag{3.62}$$

Recognize that this is exactly

$$\frac{1}{n!^2}\det(g^{ij}) \tag{3.63}$$

So that the norm of this 1-form is

$$|dq^1 \wedge \cdots \wedge dq^n| = \frac{\sqrt{\det g^{ij}}}{k!} = \frac{1}{k!}\frac{1}{\sqrt{\det g_{ij}}} \tag{3.64}$$

Again, the determinant has appeared. This is another way to get that our volume form $\sqrt{\det g_{ij}}\ dq^1 \wedge \cdots \wedge dq^n$ is an invariant form with norm $1/n!$. In general inner products between k-forms involve these k determinants that make the calculation difficult. The important thing is to know that such inner products do exist.

Consider a k form and an $(n-k)$-form (note these spaces have the same dimension, as shown in **an exercise**). Necessarily, their wedge is an n form. All 0-forms (functions) are invariant, and they are a 1-dimensional space. On the other hand, now that we have a notion of metric, we have an invariant object "on the other side" of the tower of forms: at $\Lambda^n(T_p^*M)$, namely the volume form Vol_n.

On one hand, for something like a 1-form we would say vectors are functionals bringing us to 1-dimensional space of invariant real numbers representing function values at p: $\Lambda^0 T_p^*M$. On the *other* hand we could say that $(n-1)$-forms bring us to $\Lambda^n T_p^*M$ which is *another* one-dimensional space. And for a given basis dq^i we have a dual basis in the $(n-1)$ forms of all $(n-1)$ wedges missing a dq^i, appropriately oriented, ready to send us to $\Lambda^n T_p^*M$. The same happens for k-forms and their sister $(n-k)$-forms.

Just like our metric lets us go between the 1-forms and their dual space of vectors because we have a notion of associating "vector $\mathbf{v}_\omega$ perpendicular to the first order behavior of the form" in such a way so that $\mathbf{v}_\omega(\omega) = \langle\omega|\omega\rangle$ or more generally, using the notation $\omega^\sharp = \mathbf{v}_\omega$ then the dual to β, $\beta^\sharp$ is such that:

$$\alpha(\beta^\sharp) = \left\langle \alpha, \beta^\sharp \right\rangle. \tag{3.65}$$

for all 1-forms α.

Similarly, now for a given k-form α we define its **Hodge dual** or **Hodge star** to be an $n-k$ form $\star\alpha$ such that:

Definition 3.43 (Hodge Star)**.** For a given k-form β, we define the operator $\star : \Lambda^k T_p^* M \to \Lambda^{n-k} T_p^* M$ so that for any other k-form α

$$\alpha \wedge (\star\beta) = \langle \alpha | \beta \rangle \operatorname{Vol}_n \tag{3.66}$$

The hodge star fundamentally stems from the same idea that says "If we have a notion of being orthogonal from a metric, then as soon as we have an area element for an oriented plane in $\mathbb{R}^{\not\vdash}$, we can associate a normal vector of that area" which allowed us to integrate vector fields. That idea is exactly how a 2-form ω representing flux, in Euclidean space can just be represented by a vector field $(\star\omega)^\sharp$. This is more general, though: to any k-volume there is (upon choosing orientation) and $n-k$ that is orthogonal to it. Correspondingly, to a k form there is an associated $n-k$ form.

It is worth doing a calculation:

Observation 3.44 (Hodge Star of dq^i)**.** *We have*

$$\star dq^i = \omega \text{ s.t. } (dq^j \wedge \omega = \left\langle dq^j | dq^i \right\rangle \operatorname{Vol}_n) \tag{3.67}$$

and since

$$\left\langle dq^j | dq^i \right\rangle \operatorname{Vol}_n = g^{ij} \sqrt{\det g_{ij}}\; dq^1 \wedge \cdots \wedge dq^n \tag{3.68}$$

and

$$dq^1 \wedge \cdots \wedge dq^n = (-1)^{j+1} dq^j \wedge dq^1 \wedge \cdots \wedge dq^{j-1} \wedge dq^{j+1} \wedge \cdots \wedge dq^n \tag{3.69}$$

then

$$\star dq^i = (-1)^{j+1} g^{ij} \sqrt{\det g_{ij}} dq^1 \wedge \cdots \wedge dq^{j-1} \wedge dq^{j+1} \wedge \cdots \wedge dq^n \tag{3.70}$$

It is not too difficult to see that the hodge star applied twice, since it brings us back to k forms, gives the same form *up to a sign* that depends on n and k.

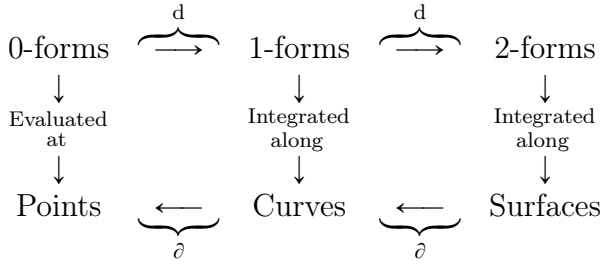
Now the divergence operator should tell us how much is going into some infinitesimal volume. It is associated with applying the exterior derivative to a $n-1$ form representing the oriented "flux" through each side of this volume. However, with the metric, we now have a way of taking a 1-form ω and getting an $n-1$ form by using our Hodge star to get $\star\omega$.

More intuitively, why this works is that for an $n-1$ plane in n dimensions, we can associate to it just a *unit normal* (because we can define the notion of "perpendicular" using the metric). The flux going through this plane as an $n-1$ form can just as well be described by a 1 form representing the component of flow along that normal vector. With all this, for any vector field, we can form an associated 1-form using the metric, and from *that* another associated $n-1$ form. Then, we can take an exterior derivative to get:

Definition 3.45 (The Divergence)**.** The **divergence** of a vector field $\mathbf{v}$ is defined as

$$\text{div } \mathbf{v} = \star \mathrm{d} \star (\mathbf{v}^\flat) \tag{3.71}$$

This answers a question from before: in two dimensions, the exterior derivative gave us a chain like:



The first d stood for the gradient's analogue on forms, and the second for the curl's, but we know there is also a divergence in 2 dimensions. Where is it's corresponding analogue with forms? We needed a metric to define it. That gives an operation $\star \mathrm{d} \star$ between 1-forms and 0-forms. For the interest of the reader, in general the operator $\star \mathrm{d} \star$ is called the **codifferential**[5], δ. It is dual to d in the sense that it is a differential operator taking us from k-forms to $k-1$-forms now.

Aside from this, having a definition for the divergence operator is incredibly useful because it gives rise to the central differential operator in all of physics:

Definition 3.46 (The Laplacian)**.** The **Laplacian** operator on a function (0-form) f is defined as

$$\triangle f = \text{div grad } f = \star \mathrm{d} \star \mathrm{d} f = (\star \mathrm{d})^2 f \tag{3.72}$$

[5]This is up to an annoying factor of a sign, depending on n and k.

Again, purely for the interest of the reader: the Laplacian on 0-forms can be written as $\delta\mathrm{d}$, and on general k-forms the natural definition can be shown to be

$$\triangle = \delta\mathrm{d} + \mathrm{d}\delta \tag{3.73}$$

this allows for the definition of the wave equation (and others) on k-forms.

For calculations, it is important to have the Laplacian of a function, f, explicitly:

$$\begin{aligned}
\triangle f &= \star\mathrm{d}((\star dq^i)\ \partial_i f) \\
&= \star\mathrm{d}\left[(-1)^{i+1} g^{ij}\sqrt{\det g}\ \partial_i f\ dq^1 \wedge \cdots \wedge dq^{j-1} \wedge dq^{j+1} \wedge \cdots \wedge dq^n\right] \\
&= \star\left[\partial_j(g^{ij}\sqrt{\det g}\ \partial_i f)\ dq^1 \wedge \cdots \wedge dq^n\right] \\
&= \partial_j(g^{ij}\sqrt{\det g}\ \partial_i f)\ \star\left[dq^1 \wedge \cdots \wedge dq^n\right] \\
&= \frac{1}{\sqrt{\det g}}\partial_j(g^{ij}\sqrt{\det g}\ \partial_i f) \\
&= \frac{1}{\sqrt{\det g}}\partial_j\left(\sqrt{\det g}\ \partial^j f\right)
\end{aligned}$$

Of course with a simple metric δ_{ij} in an orthonormal frame this becomes just $\partial_j\partial^j f$, the sum of all second derivatives.

We can even understand what this is saying intuitively. There is a function f. There is an associated vector field $\partial^i f$ along the direction of increase of this function. Say something (like energy) moves according to this field, wanting to go in the direction of maximum increase. For a given point p, we calculate the associated $n-1$ form of the flux of this thing and turn that back into a scalar that tells us the "influx" per unit volume of this thing. If there is just as much coming in as coming out, then $\triangle f = 0$ (Laplace's equation) and we are in equilibrium. Otherwise $\triangle f$ tells us there is a source/sink ρ within this volume (Poisson's equation).

3.8 Movement, Lie's Ideas

To end this chapter, lets begin with something beautiful.

$$e^{i\pi} + 1 = 0$$

Hopefully this equation is familiar to the reader. More generally, though manipulating Taylor series in calculus class, it is known that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{3.74}$$

The exponential function gives uniquely fascinating results on imaginary numbers.

But now, we will consider something even stranger than exponentiating an imaginary number. Imagine we are working on the simplest manifold that we have been used to: the 1-dimensional real line $\mathbb{R}$. Consider

$$e^{\frac{\partial}{\partial x}}$$

What could this nonsensical looking thing possibly mean? As before, by working through using the known Taylor series of e^x we get:

$$e^{\partial_x} = 1 + \frac{\partial}{\partial x} + \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{3!}\left(\frac{\partial}{\partial x}\right)^3 + \dots \tag{3.75}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!}\frac{\partial^k}{\partial x^k} \tag{3.76}$$

This is a formal sum of powers of the derivative operator. This object, when acting on a function gives:

$$e^{\partial_x} f = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{\partial^k f}{\partial x^k} \tag{3.77}$$

This takes f and operates on the function giving a *new* one. But if we *recognize this as a Taylor series* then evaluating this at a point x_0 gives:

$$(e^{\partial_x} f)(x_0) = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{\partial^k f(x_0)}{\partial x^k} = f(x_0 + 1) \tag{3.78}$$

This thing, e^{∂_x} *evaluates* f *to the right of* x_0 *by* 1. It is the **translation** operator for functions on $\mathbb{R}$. But then more generally:

$$(e^{a\partial_x} f)(x_0) = f(x_0 + a) \tag{3.79}$$

so that $e^{a\partial_x}$ translates the argument of f over by a. This is a remarkably simple and beautiful (not to mention unexpected) result about the exponential function and its action on derivatives.

By the exact same logic, in much fuller generality, say we are on a manifold M and we have set up coordinates q^i with associated tangent vector fields ∂_i on each point p on the coordinate patch. Then the *exact* same argument, using Taylor series says that for a multivariable function f on M

$$(e^{X^i \partial_i} f)(q^i) = f(q^i + X^i) \tag{3.80}$$

It evaluates f over by a coordinate change X^i from the starting point q^i.

The exponential map is then fundamental! It takes us from a vector field X defined locally on the tangent spaces of M and *moves us* along the flow of the vector field on M. The exponential map allows vector fields to becomes actual *flows* along the manifold.

Definition 3.47. For a manifold M, a flow φ is a mapping

$$\varphi : M \times \mathbb{R} \to M \tag{3.81}$$

such that for all points p on the manifold and all real numbers s, t:

- $\varphi(p, 0) = p$
- $\varphi(p, s + t) = \varphi(\varphi(p, s), t)$

and such that the mapping is *continuous* (with respect to the topology of M).

This means that starting at a point p, a flow gives rise to a curve $\gamma = \varphi(p, t)$. The vector field of the tangent vectors to γ at the points that it passes through can be **exponentiated** to give the flow along γ.

Flow of exponentiated vector field

Starting at p and given a vector field X on M, getting a curve γ associated with flowing along X means solving the differential equation:

$$\frac{d}{dt} q^i = X^i. \tag{3.82}$$

The solutions of many dynamical systems in physics and engineering can be seen as curves found by solving differential equations that end up amounting to Equation (3.82). Much more on this will be discussed in Chapter 7.

Important Note. *Certainly there are pathological vector fields that don't generate flows, and not all vector fields are guaranteed to generate global flows on the manifold. The study of when vector fields generate global flows, and the subtleties arising therein will be glossed over in this text.*

Now since vector fields on a manifold form a vector space, what does it mean to exponentiate cX? It gives rise to the same curve, but parameterized by a parameter t that traverses it c times as fast. For negative c this would mean reversing direction.

What about a sum of two vector fields? Naively we'd say that moving along the flow of X and then moving along the flow of Y should be the same as moving along the flow of $X+Y$, which should in turn be the same as moving along the flow of Y and *then* along the flow of X. For translations ∂_x, ∂_y in $\mathbb{R}^2$, this turns out to be true. Consider however the sphere, with X and Y defined so as to generate circles like in the graphic below. Then moving along the flows of X or Y corresponds to rotating the sphere around the z or x axis, respectively.

MAKE A GRAPHIC: Two vectors at different points on a sphere.. what's their difference? Two paths getting two different results.

For these vector fields on the sphere, $e^X e^Y \neq e^{X+Y} \neq e^Y e^X$.

What happens algebraically? Just exponentiating two **coordinate vector fields** ∂_i, ∂_j associated with coordinates q^i, q^j

$$\begin{aligned} e^{\partial_i+\partial_j} &= 1 + (\partial_i + \partial_j) + \frac{1}{2}(\partial_i + \partial_j)^2 + \dots \\ &= 1 + (\partial_i + \partial_j) + \frac{1}{2}(\partial_i^2 + \partial_j^2 + \partial_i\partial_j + \partial_j\partial_i) + \dots \\ &= (1 + \partial_i + \frac{1}{2}\partial_i^2 + \dots)(1 + \partial_j + \frac{1}{2}\partial_j^2 + \dots) \\ &= e^{\partial_i} e^{\partial_j} \end{aligned}$$

Since in this case partials commute and behave just like real numbers, the same algebra giving $e^{a+b} = e^a e^b$ still holds. This was for *coordinate vector fields*. This would still hold, of course, if we put *constants* in front of the vectors $X = a\partial_i, Y = \partial_j$, then $e^{X+Y} = e^X e^Y = e^Y e^X$ because constants commute as well.

On the other hand, if we took an exponential of a sum of *two arbitrary vector fields* $X = X^i\partial_i, Y = Y^i\partial_i$ whose components *them-*

selves were functions of $p \in M$ so the $\partial_i X^j \neq X^j \partial_i$, then we would see something else. Just to second order:

$$\begin{aligned} e^{X^i\partial_i} e^{Y^i\partial_i} &= (1 + X^i\partial_i + \frac{1}{2}(X^i\partial_i)^2)(1 + Y^i\partial_i + \frac{1}{2}Y^i\partial_i^2) \\ &= 1 + (X^i + Y^i)\partial_i + \frac{1}{2}(X^i\partial_i)^2 + \frac{1}{2}(Y^i\partial_i)^2 + X^i\partial_i(Y^j\partial_j) \end{aligned}$$

on the other hand

$$\begin{aligned} e^{Y^i\partial_i} e^{X^i\partial_i} &= (1 + Y^i\partial_i + \frac{1}{2}(Y^i\partial_i)^2)(1 + X^i\partial_i + \frac{1}{2}X^i\partial_i^2) \\ &= 1 + (Y^i + X^i)\partial_i + \frac{1}{2}(Y^i\partial_i)^2 + \frac{1}{2}(X^i\partial_i)^2 + Y^i\partial_i(X^j\partial_j) \end{aligned}$$

These share all the same terms in common except their last:

$$X^i\partial_i(Y^j\partial_j) \quad \text{vs.} \quad Y^i\partial_i(X^j\partial_j)$$

These terms correspond exactly to the products XY and YX of the derivatives. The lowest order difference between $e^X e^Y$ and $e^Y e^X$ lies in the difference $XY - YX$.

The product rule can be applied to expand both, giving

$$\Rightarrow (X^i\partial_i Y^j)\partial_j + X^i Y^j \partial_i\partial_j \quad \text{vs.} \quad (Y^i\partial_i X^j)\partial_j + Y^i X^j \partial_i\partial_j$$

The second terms on both expressions are equal because $\partial_i\partial_j$ is symmetric as a tensor in $T_pM \otimes T_pM$, since partials commute. This means that the lowest order difference between $e^X e^Y$ and $e^Y e^X$ lies just in:

$$XY - YX = \left[(X^i\partial_i Y^j) - (Y^i\partial_i X^j)\right]\partial_j \tag{3.83}$$

From the fact that the second order terms in the taylor series expansions of these two flows differ, you can see that there is a difference that depends on the *order* that we flow in, just like the example of the sphere illustrated. For general vector fields, we shouldn't expect that the exponential of the sum is the product of the exponentials either. This failure to commute further means that general vector fields can't be represented as coordinate fields $\frac{\partial}{\partial q_i}$ in some coordinate system q^i.

Something even more interesting has been revealed: we took two vector fields X and Y on M, and were able to get a *third* vector field: $XY - YX$. Indeed, note that Equation (3.83) is a vector field. It is called the **commutator** vector field of X and Y.

Definition 3.48 (Commutator)**.** The commutator (also called the ***Lie bracket***) of two vector fields X and Y, denoted by $[X, Y]$, is a bilinear anti-symmetric operation defined as

$$[X, Y] := XY - YX \tag{3.84}$$

This seems to be telling us to what extent two vector fields are *incompatible*, but how can we see that this more intuitively and geometrically?

To do this we need to develop two concepts associated with flow: the **pullback** and the **pushforward**. Consider two manifolds, M and N and a smooth map $\varphi : M \to N$. In general φ will not have an inverse.

Definition 3.49 (Pullback of a Function)**.** If we have a function f defined on N, we can *pull it back* to M using ϕ by defining the pullback of f as

$$\varphi^* f = f \circ \varphi$$

Because φ doesn't have an inverse in general, we can't in general make a construction to do the opposite: taking a function on M to a function on N. The pullback illustrates a **duality** that was first hinted at by vectors and 1-forms. There, the duality was between the first order behavior of curves on the manifold against the first order behavior of functions. Here, φ allows us to take *curves* on M forward into curves on N and its pullback allows us to take *functions* on N back to functions on M.

A special case of all this is when $U \subseteq M$ and $\varphi : U_\alpha \to \mathbb{R}^n$ is a coordinate patch on M. Coordinate patches allow us to take functions on $\mathbb{R}^n$ (in particular the coordinates x^i) and associate them to functions on U (in particular coordinates q^i) but in general function on U will be pushed forward to functions on the range of $\varphi \in \mathbb{R}^n$, that don't in general extend to functions on all of $\mathbb{R}^n$.

So we cant push forward functions from M to N in general (although if φ^{-1} existed, we could). On the other hand, for a vector field X on M it *is* in general possible to associate a vector field on N. The reason is that because vector fields are determined by their actions $Xf = X^i \partial_i f$ on functions, then we can act by X on a function $f \in C^\infty(N)$ as follows:

Definition 3.50 (Pushforward of a Vector Field)**.** For a vector field $X \in \mathrm{Vec}(M)$ we define its pushforward, denoted by $\varphi_*(X) \in TN$ or $d\varphi X$ by its action on $f \in C^\infty(N)$

$$\varphi_*(X)(f) = X(\varphi^*(f))$$

Or in general $\varphi_* X = X\varphi^*$

To understand this better let's do a small general example. Say we have a manifold M a map φ into N. If we have coordinates q^i on M and q'^i on N, then φ maps a point $p \in M$ specified by q^i to a $p' \in N$ specified by $q'^i = \varphi^i(p) = \varphi^i(q_1 \dots q_n)$ If we have a field written in local coordinates on M as $X = X^i \frac{\partial}{\partial q^i}$, how do we push this forward? For a function f on N, we get

$$\begin{aligned}
\varphi_*\Big(X^i \frac{\partial}{\partial q^i}\Big)(f) &= X^i \frac{\partial}{\partial q^i}[f \circ \varphi] \\
&= X^i \frac{\partial \varphi^j}{\partial q^i} \frac{\partial f}{\partial q'^j} && \text{(Chain Rule)} \\
&= \Big(X^i \frac{\partial q'^j}{\partial q^i} \frac{\partial}{\partial q'^j}\Big)(f)
\end{aligned}$$

So we get our coordinate expression of the pushforward in terms of $\partial/\partial q'^i$. The Jacobian $\partial q'^i/\partial q^j$ is a pushforward if we view a coordinate change $\varphi : M \to M$ as an active transformation on the manifold itself.

Being able to push forward vector fields along φ will be very useful. Now we will specialize to when $\varphi = \varphi_X(t) : M \to M$ is a flow associated with a vector field X. Sowing together the language of pushforwards and pullbacks with the language of flows gives:

Observation 3.51. *A flow along a field X gives rise to a family of invertible maps, one for each $\varphi_X(t) : M \to M$ over $t \in \mathbb{R}$. At a point p we can pull back $f(\varphi_X(t)(p))$ and push forward another field $Y(p)$ to act at $Y(\varphi_X(t)(p))$ by*

$$[\varphi_X(t)]^*(f) = e^{tX} f \tag{3.85}$$

$$[\varphi_X(t)]_*(Y) = Y[\varphi_X(t)]^* = Y e^{tX} \tag{3.86}$$

The pullback of a function along a flow is obvious. It answers "what is the value f takes on later along the flow?". The pushforward

of a field Y along X's flow is more interesting. The pushed forward Y in the new tangent space $T_{p'}M$ will act on functions f at p' just like Y would act on them if you were to pull them back to p. Now with a good language of flows, we can try to generalize the notion of a derivative along a field by taking a limit of pushforwards/pullbacks.

We know that if we have a function f we can find its change along a vector field $X = X^i\partial_i$ immediately: Xf. We'll denote the "change of an object along X" by the symbol $\mathcal{L}_X$. So the change of f along X is denoted $\mathcal{L}_X f := Xf$. Using flows, we could write this to get a limit definition of the derivative:

$$\begin{aligned}\mathcal{L}_X f(p) &= \lim_{t\to 0} \frac{f(\varphi_X(p,t)) - f(p)}{t} \\ \Rightarrow \mathcal{L}_X f &= \lim_{t\to 0} \frac{[\varphi_X(t)]^* f - f}{t}\end{aligned}$$

The pullback here means that we are *pulling back* the value of f from a point later along the flow and comparing it to the value of f at the current point.

But now, it is natural to go more general and ask *how does a vector field Y change along X?*. That is, what is $\mathcal{L}_X Y$? We expect it to be a vector that point in the direction that Y will change in the next instant if we flow along X.

Consider the radial vector field ∂_r and the circulatory field ∂_θ. If we are flowing along ∂_θ, we do *not* see ∂_r change at all! If ∂_r was the only vector field that we cared about, then flowing along ∂_θ would be like changing *nothing*. How can we make idea this more mathematical? The flow curves associated with ∂_r do not change as we flow along ∂_θ

The naive guess of $\mathcal{L}_X Y = XY$ doesn't produce a vector. It would produce something like

$$XY = (X^i\partial_i)(Y^j\partial_j) = X^i(\partial_i Y^j)\partial_j + (X^i Y^j)\partial_i\partial_j \tag{3.87}$$

which is some strange mix of a vector and a higher derivative. What did we do wrong? When we act on an object by the derivative operator X at a point p, it corresponds to a limiting process of changing the coordinates q^i in the direction $X^i dt$ and taking the difference between the value at that new point p' and the old one p, then dividing that by dt as dt goes to zero. In terms of flows it compares $[\varphi_X(t)]^* f$, which

pulls back the value of f from the point t along the flow of X against the value of f now:

$$Xf(p) = \lim_{dt\to 0} \frac{f(\varphi_X(p, dt)) - f(p)}{dt} \tag{3.88}$$

For functions, there's no problem in doing this. On the other hand, for a vector field Y^j we would get:

$$XY(p) = \lim_{h\to 0} \frac{Y(\varphi_X(p, dt)) - Y(p)}{dt} \tag{3.89}$$

In the numerator, there is a problem: $Y(\varphi_X(p, dt))$ is an element of the tangent space at $\varphi_X(p, dt)$ while $Y(p)$ is an element of the tangent space at p. These are two entirely different vector spaces, so we can't just pretend the vectors are in the same space. We can't *a-priori* compare vectors on different tangent spaces. Intuitively, though, in the real world we locally *do* compare vectors at different places on our manifold M. We will see in chapter 8 how the metric allows us to do this by using a *connection*.

Although we can't compare vectors at different points, we *can* flow the other way by $\varphi_X(-t)$ and *push forward* $Y(\varphi_X(p, t))$ along the *backwards* flow to get a vector back at T_p:
GRAPHIC HERE

$$[\varphi_X(-t)]_*\big(Y(\varphi_X(p, t))\big) \in T_pM \tag{3.90}$$

and as $t \to 0$ this is just Y, so it is worth looking at

$$\lim_{t\to 0} \frac{[\varphi_X(-t)]_*\big(Y(\varphi_X(p, t))\big) - Y(p)}{t}$$

This gives us a notion of derivative, which we will call the **Lie derivative** of Y along the flow of X. Note that at no point did we ever need to use a metric, or any notion of distance/orthogonality in defining this derivative.

If we wrote in terms of exponentials of the vector fields, then it would be:

$$[\varphi_X(-t)]_*(e^{tX}Y(p))$$

by the definition of pullback, we must have

$$\Big([\varphi_X(-t)]_*(e^{tX}Y(p))\Big)(f) = \big(e^{tX}Y(p)\big)(f(\varphi_X(-t))) = e^{tX}Ye^{-tX}f$$

Note how interesting that this initially complex thing became just a "conjugation" of Y by e^{tX}. The limit definition becomes:

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{e^{tX} Y e^{-tX} - Y}{t} \tag{3.91}$$

But expanding the exponentials to first order in t gives something remarkable:

$$\mathcal{L}_X Y = \lim_{t\to 0} \frac{(1+tX)Y(1-tX) - Y}{t} = XY - YX \tag{3.92}$$

The Lie derivative along X of Y is the commutator $[X, Y]$.

Intuitively why is it a difference of each vector field acting on the other, in both orders? There were two steps that we took when taking the Lie derivative:

1. We moved from p along the flow of X, $e^{tX}Y$, getting Y at a neighboring point p'.

2. We pushed that vector $Y(p')$ *back* along the flow of X, by defining it acting on translated functions $[\varphi_X(-t)]_* Y(p')f = Y(p')e^{-tX} f$ flowed back along the field of X

Note the first step resulted in a change in Y: the $Y(p')$ at p' is a change from our original $Y(p)$ at p because of how the coordinates Y^i are functions that change in space. We cared about how Y^i changed as we moved along that flow, giving us XY, but we didn't care about taking changes in X.

In the second step, we now *had* our changed $Y(p')$ at the neighboring point, and we wanted to bring it back, using X's flow, and compare it to our starting $Y(p)$. We did that by defining its action on functions that have been "flown" backwards long X. These functions distorted by the flow-lines of X so we wanted to see how the flow back along X varied in the Y direction, giving the $-YX$.

CAN DO BETTER ON THIS LAST ONE

Another identity that can be derived from the notion of this Lie derivative is that since $e^X e^Y - e^Y e^X = [X, Y]$ to first order, the *infinitesimal* flow along Y followed by the flow along X of a vector Z minus the infinitesimal flow along X followed by the flow along Y of that same vector should equal the flow along $[X, Y]$ of that vector. In terms of commutators:

$$\mathcal{L}_X(\mathcal{L}_Y Z) - \mathcal{L}_Y(\mathcal{L}_X Z) = \mathcal{L}_{[X,Y]}Z \tag{3.93}$$
$$\Rightarrow [X,[Y,Z]] - [Y,[X,Z]] = [[X,Y],Z] \tag{3.94}$$

This is called the **Jacobi identity**. It can easily be rewritten, using the anti-symmetry of $[-,-]$ in the elegant form:

$$[X,[Y,Z]] + [Y,[X,Z]] + [Z,[X,Y]] = 0.$$

The fact that X and Y can be joined together through a bilinear operation to produce another vector means that the space of vector field on a manifold $\mathrm{Vect}(M)$ is endowed with a structure that is bilinear and antisymmetric. This Lie bracket turns the vector space of vector fields on M into a **Lie Algebra**.

Definition 3.52 (Lie Algebra)**.** A Lie Algebra is a vector space V together with a bilinear operation called the **commutator** of the Lie algebra

$$[-,-] : V \times V \to V$$

That satisfies $[X,X] = 0$ for all $X \in V$ (so is anti-symmetric), as well as the Jacobi identity.

The Jacobi identity needs to be added for the definition of a Lie algebra because it gives all the information the failure of the commutator to have associativity $[X,[Y,Z]] \neq [[X,Y],Z]$, and Jacobi's identity shows that the difference between these two can be expressed as another commutator ($[Y,[X,Z]]$).

Lie algebras are of central importance to studying manifolds and their geometry as it relates to symmetry, and we will return to them in later chapters.

To end this entire chapter, it is worth looking back at the start of the section, to Euler's identity. Can this all be interpreted within the ideas of flows?

Relate this all to the complex exponential from the beginning of the section

3.9 Exercises

WILL FORMAT THIS LATER

Alternative definition for commutator

Nothing stops us from going from our starting point p, moving along X by dt and at least seeing how the flow of Y looks there, namely:

$$\varphi_Y(\varphi_X(p, dt), s) \tag{3.95}$$

As s varies, this defines a curve $\gamma_{Y,p'}$ passing through the point p' associated with $q^i + X^i dt$. Similarly, nothing stops us from then *pulling BACK* $\gamma_{Y,p'}$ along X^i by acting *inversely* by φ_X at each point of $\gamma_{Y,p'}$.

$$\varphi_X^{-1}(\varphi_Y(\varphi_X(p, dt), s), dt) = \varphi_X(\varphi_Y(\varphi_X(p, dt), s), -dt) \tag{3.96}$$

Alternative intuition for commutator

Denote the pulled back curve by $\gamma'_{Y,p}$. Note this at this curve also passes through p at $s = 0$. The corresponding tangent vector to γ'_Y at p which we can call $Y'(p)$ will approach $Y(p)$ as $dt \to 0$. We can define this difference:

$$\mathcal{L}_X Y = \lim_{dt \to 0} \frac{Y'(p) - Y(p)}{dt} \tag{3.97}$$

But there's also a symmetry to the drawing above. At a specific dt, if we start at p and flow along the Y' for that dt by $\varphi_{Y'}(p, t)$ to p'', then the curve $\gamma_{X,p''}$, when pulled back along Y becomes *exactly* the flow $\gamma_{X,p}$ of X at p.

And dual diagram to that

$$\varphi_Y(\varphi_X(\varphi_{Y'}(p, dt), s), -dt) = \varphi_X(s) \tag{3.98}$$

Intuitively, it seems that if we flowed for dt along Y instead of Y', then pulling that back would produce a curve who's tangent vector is off, to first order, by $Y - Y' = -\mathcal{L}_X Y dt$. But flowing along Y and comparing the tangent vector to the pulled back level curve for X is exactly $\mathcal{L}_Y X$. This gives some intuition for us to expect a striking symmetry:

$$\mathcal{L}_X Y = -\mathcal{L}_Y X \tag{3.99}$$

Chapter 4

Harmonics: Fourier Analysis

This chapter develops the ideas of Fourier analysis, a framework whose reach now extends across the science and engineering disciplines. There is no arguing the fact that it is one of the most powerful tools that mathematics has ever developed, not just for its wide practical application, and not just for its use in formulating physical theories, but for going further and influencing the way that mathematics itself is thought about and done.

Another mathematical idea that has had similar impact across the scientific fields is linear algebra: the in-depth study of the phenomenon of linearity. Aside from allowing us to go between points in physical space and tuples of numbers, linear algebra gives another powerful tool that, as of yet, has not been touched upon in this book: the eigenvector. Eigenvectors give a natural way of handling an object, once it is represented by a linear operator on a vector space. When that object is the stress tensor of a material, they describe the principal axes, when it is a high dimensional dataset, they allow for the principal-component analysis, and for markov chains and stochastic processes, they give information about long-term dynamics.

In this section, we will not just reflect on the uses and power of linear algebra and Fourier analysis, but we will go further and show that they are one and the same: Fourier analysis, viewed through the right lense, is plain linear algebra with eigenvalues. This will set the stage on which the next chapter will give us the real performance: generalizing the ideas of Fourier analysis to get something much more

unified and powerful, which it can be said to constitute a cornerstone of modern mathematics: representation theory.

The chapter is organized in stages that will form stepping stones between elementary linear algebra and full-blown analysis. We begin by performing Fourier analysis on N points connected in a discrete ring. Then, taking the limit as $N \to \infty$, we obtain the Fourier series on the circle S^1, i.e. the interval $[0, 1]$ with periodic boundary conditions. Generalizing this to arbitrary intervals $[-k, k]$, we take another limit as $k \to \infty$ and obtain the Fourier transform on the line. From here, we study higher-dimensional generalizations of these ideas: to 2-D lattices of points and continuous n-D Euclidean space. We conclude by translating these ideas to more general arrangements of N connected points: Fourier analysis on graphs, leading to spectral graph theory.

4.1 Discrete, Bounded: Eigenvalues

We begin with the definition of a **finite graph**

Definition 4.1. A finite graph G is a finite set of **vertices** $V = \{v_1, \ldots, v_n\}$ together with a set of **edges** $E = \{e_1, \ldots, e_m\}$ such that each edge is an unordered pair of vertices (v_i, v_j) indicating a connection between two vertices.

Occasionally, we will call the vertices "points" on the graph.

For this section, we consider the special case of N vertices connected in a ring by having each vertex joined to two nearest neighbors as below. Denote this graph by G.

Illustrate N points connected in a ring

We have considered functions on a manifold M, that associate a real or complex number to each point $p \in M$. Here we will consider the simpler, more finite case of functions f that associate to each *vertex* $v \in G$ a (real or complex) number $f(v)$ called the value of f at v. The ideas of functions on graphs are in many ways analogous to the notions of functions on manifolds, as we shall see.

For this ring with N points, a function can be described entirely

in terms of N numbers: the values it takes at each point.

$$f \approx \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_N) \end{pmatrix}$$

Here we write '$\approx$' instead of '$=$' because of the same point that was made in chapter 1. This collection of N numbers certainly *represents* the function f, but it is *not* f. That is, we can identify the space of real-valued functions on these N-points with $\mathbb{R}^N$, and the space of complex-valued functions with $\mathbb{C}^N$, but this choice requires that we pick a basis.

But what basis have we picked on the function space? Our basis $\{b_i\}_{i=1}^N$ consists of functions $b_i(v_j)$ that are zero everywhere except for when $i = j$, where $b_i(v_i) = 1$. Then any function f can clearly be written as

$$f = \sum_i f(v_i) b_i \tag{4.1}$$

it is easy to see that evaluating this at a vertex v_j makes every b_i vanish except for $i = j$, giving total value $f(v_j)$ as required. This set of basis functions seems to be the obvious one. We will call it the **position basis**, taking vernacular from physics.

Although the space of real-valued functions on G is isomorphic *as a vector space* to $\mathbb{R}^N$, it has much more structure than just an ordinary vector space. We know that we can multiply two functions f and g to obtain a new one $h = fg$ simply by multiplying values pointwise $h(v_i) = f(v_i)g(v_i)$. Since the functions on G are a vector space, we thus have a way to multiply two vectors in the space to get another.

Moreover, we can define an inner product on this space of functions

Definition 4.2 (Inner Product of Functions on a G)**.** Let f, g be to functions defined on G. We define an inner product $(\cdot, \cdot)$ on this space by

$$(f, g) = \sum_{i=1}^{N} f(v_i) g(v_i) \tag{4.2}$$

4.2 Continuous, Bounded: Fourier Series

4.3 Continuous, Unbounded: Fourier Transforms

4.4 Fourier Analysis on Euclidean Spaces

4.5 Fourier Analysis on Graphs: Spectral Graph Theory

4.6 Vector Spaces of Functions

We will work with functions just over $\mathbb{R}$ at first, the setting that we are most used to. It is no great surprise that functions $f : \mathbb{R} \to \mathbb{R}$ form a vector space, since we can add and scale any two functions $af + bg$ and get another function as their linear combination. We can therefore try to apply ideas from linear algebra to this vector space. Note first that it is infinite dimensional.

There is an important subspace worth looking at: $C^\infty(\mathbb{R})$ of smooth functions. Since finite linear combinations of smooth functions are still smooth, $C^\infty(\mathbb{R})$ is a subspace. We only allow for finite linear combinations of functions for now, as otherwise we need to figure out what we mean by infinite sequences and series of functions to "converge" to a limit in this vector space.

Example 4.3. The set of polynomials up to degree n on $\mathbb{R}$, $P^n(\mathbb{R})$, is a vector space. The set of polynomials up to degree n restricted to an interval $[a, b]$, $P^n[a, b]$, is also a vector space.

Note we need to include all the polynomials of degree $\leqslant n$ as well, not just degree n. Otherwise something like $(x^5 + x^3) - x^5$ would not be in the space.

In particular, now, let's consider the interval $[0, 1]$ to be of interest. We care about the space of *all functions* on $[0, 1]$. Again, this is still infinite dimensional, but say we restricted *further* and only cared about the function at the points $0, 1/4, 1/2, 3/4$ and 1. These are 5

points of interest. A function restricted to these 5 points can be represented just as a vector of 5 numbers: $[f(0)\ f(1/4)\ f(1/2)\ f(3/4)\ f(1)]$.

The linear combinations of functions on $[0,1]$ when restricted to these 5 points are then just linear combinations of these vectors in $\mathbb{R}^5$. In numerics, this would be a *low resolution* way to view the function, equivalent to something like the graphic.

5 box functions interpolating f

If we increased the number of points we sampled at, say $f(k/n)$ as k runs from 0 to n, then we represent functions by vectors in $\mathbb{R}^{n+1}$. As n gets big our resolution gets better and better. In general, though, infinitely many different functions will get sent to the same representation as a vector in $\mathbb{R}^{n+1}$ (so this discrete approximation from the set of all functions on $[0,1]$ to just $n+1$ evenly-spaced points has a huge kernel).

This type of idea is what lies behind Riemann summation for defining the integral operator. In fact, the discrete approximation of the integral operator on functions, when viewed as vectors in $\mathbb{R}^{n+1}$

Example 4.4 (Riemann Sums)**.**

$$\int_{approx} f dx = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \begin{pmatrix}\frac{1}{n} & \dots & \frac{1}{n}\end{pmatrix}\begin{pmatrix} f(0) \\ \vdots \\ f(1)\end{pmatrix}$$

The Riemann sum in this discrete space is just the linear functional $\begin{pmatrix}\frac{1}{n} & \dots & \frac{1}{n}\end{pmatrix}$ acting on the vector representing the function to give the desired value. If the full function f is Riemann-integrable, then as $n \to \infty$ this will approach the true value $\int_0^1 f dx$.

This approximation in $\mathbb{R}^{n+1}$ is not just any vector space. It has an notion of *length* for a vector

$$||f_n||^2 = \frac{1}{n}\sum_{i=0}^{n+1} f(k/n)^2 \tag{4.3}$$

As the sum of the squares of the function values at every point. The factor of $1/n$ out front is so that in the limit of $n \to \infty$ this gives a **norm**, a notion of size, on the space of functions whose squares are Riemann-integrable (such functions are called **square-integrable**)

on $[0, 1]$:

$$||f||^2 = \int_0^1 f(x)^2 dx \tag{4.4}$$

On $[0, 1]$, the norm of a function is its root mean square value. In fact this notion of length also gives a natural notion of inner product (both on the discrete space and the full space):

$$\langle f|g\rangle = \int_0^1 f(x)g(x)dx \tag{4.5}$$

Looking at it this way, it is *exactly* the same as how we defined a dot product: multiply the corresponding components together and sum that up. The difference is that now "components" have become "functional values" and the sum has become an integral.

Naturally, we can define an inner product on any interval $[a, b]$ between square integrable functions on that interval:

$$\langle f|g\rangle = \int_a^b f(x)g(x)dx \tag{4.6}$$

It is similarly possible to define an inner product for square-integrable functions on the *whole* real line:

$$\langle f|g\rangle = \lim_{L\to\infty} \int_{-L}^{L} f(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx \tag{4.7}$$

As in Chapter 3, an inner product on a real vector space is as follows.

Definition 4.5 (Real Inner Product)**.** For a real vector space V, an inner product $\langle -|-\rangle$ on V is a bilinear map into $\mathbb{R}$ so that:

1. (Linear) $\langle a\mathbf{x} + b\mathbf{y}|\mathbf{z}\rangle = a\langle \mathbf{x}|\mathbf{z}\rangle + b\langle \mathbf{y}|\mathbf{z}\rangle$
2. (Symmetric) $\langle \mathbf{x}|\mathbf{y}\rangle = \langle \mathbf{y}|\mathbf{x}\rangle$
3. (Positive-definite) $\langle \mathbf{x}|\mathbf{x}\rangle \geqslant 0$ and equality holds iff $\mathbf{x} = 0$. A vector space that possesses an inner product is called an ***inner product space***.

Inner product spaces are specific examples of **Metric Spaces**.

Definition 4.6 (Metric Space)**.** A set X is a metric space if there is a non-negative function $d : X \times X \to \mathbb{R}$ that satisfies

1. (Symmetry) $d(x, y) = d(y, x)$
2. (Positive Definite) $d(x, y)$=0 iff $x = y$
3. (Triangle Inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$

Just as on the real line and on $\mathbb{R}^n$, having a notion of distance allowed us to define closed and open sets, the same is true on metric spaces. In particular, inner product spaces have a natural topology defined by open and closed balls defining neighborhoods, and so as vector spaces that are also topological spaces, they are called **Topological Vector Spaces**.

Returning back to the inner product space of functions on $\mathbb{R}$ this inner product is far from useless. If both $|f| = 1$ and $|g| = 1$ have length then the inner product, in some well-defined sense, measures how *correlated* f is with g. If the integral of f against g is zero then there's no way to predict the behavior of g based on the behavior of f. If it is positive, then a positive f means a positive g *more often than not*, while if the integral is negative then a positive f means a *negative* g more often than not. Such a quantity in general:

$$\frac{\langle f|g\rangle}{||f||\ ||g||}$$

is geometrically what the cosine of the angle between two vectors was in $\mathbb{R}^{n+1}$. Functions that are "in the same direction" would have a positive cosine, while functions "pointing in opposite directions" would have a negative one. Uncorrelated, orthogonal functions would have a cosine of zero.

Exercise on correlation of functions and the inner product

So the vector space of square-integrable functions on $[0, 1]$ is an inner product space. It is denoted[1] by $L^2[0, 1]$. Similarly, for a general

[1] **HERE SAY SOMETHING ABOUT LEBESGUE VS RIEMANN INTEGRATION**

Riemann vs Lebesgue in a footnote

interval $[a, b]$ we denote the square integrable functions as $L^2[a, b]$, and for functions square-integrable on the *entire real line*, we write $L^2(\mathbb{R})$

The space $L^2[0, 1]$ is more than just an inner product space. It is a **Hilbert space**, an object that is of central importance in quantum mechanics, signal processing, and mathematical analysis. A Hilbert space is a vector space that allows us to take limits of vectors, too, not just finite sums, so that we can do analysis on it.

If we have a sequence of vectors x_n on our Hilbert space that *accumulates*, in the sense that for any ε we can pick an N large enough so that all the x_i after x_N are within ε of x_N

$$|x_N - x_i| < \varepsilon \tag{4.8}$$

Ex: Show this is the same as forming a Cauchy sequence

Figure: All vectors being within a neighborhood of x_N, accumulating.

Then, if we were working in $\mathbb{R}^n$, this would mean that the sequence should converge within some closed ball $B_\varepsilon(x_N)$. In fact, convergent sequences are *exactly* the ones that accumulate like this.

We want a Hilbert space H, infinite-dimensional or otherwise, to have this property of being **complete** in the inner product. That is, whenever a sequence of vectors accumulate, that sequence will in fact **converge** to a **limit** vector *still in* H.

Definition 4.7 (Real Hilbert Space)**.** A real Hilbert space H is a real inner product space that is **complete** with respect to its inner product.

Example 4.8. The rationals $\mathbb{Q}$ are not a complete metric space. The reals are (they are called the **completion** of the rationals).

Example 4.9. The space of continuous functions is not complete. Consider the sequence of functions:

$$f_n(x) = \begin{cases} -1, & \text{if } x \leqslant -1/n \\ nx, & \text{if } -1/n < x < 1/n \\ 1, & \text{otherwise} \end{cases} \tag{4.9}$$

While each f_n is continuous, the limit is the sign function of x, which is *not* continuous. Similar arguments show that the space of smooth

functions is also not complete, by considering $f_n(x) = e^{-n\pi x^2}$ approaching the zero function with a jump discontinuity at zero.

Ex: Being complete is with respect to a metric, not a topology (reals are homeomorphic to incomplete $[0, 1]$)

Theorem 4.10. *$L^2[a, b]$ is complete, as is $L^2(\mathbb{R})$.*

Proof. Any introductory text in real analysis and measure theory proves this result after developing the necessary background in proper Lebesgue integration, c.f.

Provide a measure theory reference for L^2 being complete, like Stein, but probably not Stein tb$\hbar$.

□

This is an extraordinary property, which places L^2 spaces above all others in their importance to classical analysis. This property will allow us to talk about sequences on these spaces, and hence do analysis on these spaces by talking about sequences of functions converging to a limit point "in the L^2 norm".

The two properties of having an inner product as well as having this notion of convergence in a norm are beautifully useful in formulating the original ideas of Fourier.

4.7 Trigonometric Series

Let us investigate the phenomenon of a wave in a 1-dimensional medium, like a string vibrating in one dimension. At a point x_0, the string has some displacement $u(x_0)$. Its neighbors have displacements $u(x_0 + h)$ and $u(x_0 - h)$.

Draw A String for the Wave equation

Because of **Hooke's Law** $\mathbf{F} = k\mathbf{x}$, the rightward neighbor will exert a force $\mathbf{F}$ on the string at x_0 proportional to the difference in displacement $u(x_0 + h) - u(x_0)$. Similarly, the leftward neighbor will exert a force proportional to $u(x_0 - h) - u(x_0)$. The total force at x_0 then is some scalar multiple times the sum of both these contributions:

$$\mathbf{F}_{x_0} = k[u(x_0 + h) - u(x_0) + u(x_0 - h) - u(x_0)]. \tag{4.10}$$

The mass constant k between successive neighbors can be related to the total spring constant K by $k = KL/h$. This is a simple exercise.

Make an exercise for the total spring constant

.

We also know from **Newton's Second Law** that the force is equal to the mass times the acceleration of the displacement of the string at x_0

$$\mathbf{F}_{x_0} = m\frac{\partial^2 u}{\partial t^2}. \tag{4.11}$$

The mass of the particle at x_0, because it is only a small section of the string, becomes infinitesimal as $h \to 0$ and can be expressed as ρh where ρ is the mass density of the string. Together, then:

$$\rho h\frac{\partial^2 u}{\partial t^2} = \frac{KL}{h}[u(x_0 + h) + u(x_0 - h) - 2u(x_0)] \tag{4.12}$$

This gives rise to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{KL}{\rho}\ \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{h^2} \tag{4.13}$$

The constant KL/ρ on the right hand side has units of length squared per second squared, and can be interpreted as a velocity squared c^2. The more interesting quantity on the right hand side is what succeeds that. Note that it can be re-written as:

$$\frac{1}{h}\left[\frac{u(x_0 + h) - u(x_0)}{h} - \frac{u(x_0) - u(x_0 - h)}{h}\right]$$

As $h \to 0$ the inside approaches a difference of the derivatives $\frac{\partial u}{\partial x}$ at neighboring points $x_0 + h/2, x_0 - h/2$, so that the whole term becomes the *second derivative* $\partial^2 u/\partial x^2$. The equation for the wave on the string is then:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{4.14}$$

This is the **wave equation** of central importance across the subfields of physics.

The wave equation in one dimension is important for studying simple vibrations of strings, metal beams, and waves of current in

trasmission lines and cavity resonators. However, the higher dimensional generalizations of the wave equation find even more use, especially in studying the propagation of wave energy through physical space. It is worth deriving this as well. The waves propagate through a medium with speed c depending on the properties of the medium.

One way is just to note that now we have $u(x, y)$, and as before we will have that $u(x \pm h, y \pm h)$ will contribute a term that now looks like

$$\frac{u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)}{h^2}$$

$$\begin{aligned}=&\frac{u(x+h,y)+u(x-h,y)-2u(x,y)}{h^2}\\&+\frac{u(x,y+h)+u(x,y-h)-2u(x,y)}{h^2}\end{aligned}$$

$$\to \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

so that the wave equation in 2 dimensions is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{4.15}$$

This quantity, $\frac{\partial^2 u}{\partial x^2}$ in one dimension and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ in two, and more generally

$$\sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$$

in n dimensions, is called the **Laplacian** of u. It is the divergence of the gradient, and so is often denoted $\nabla^2 u$, or otherwise just $\triangle u$.

Now for the 2-dimensional argument, you could ask why we did not take into account diagonal neighbors like $u(x + h, y + h)$. One argument could be that they are not "directly connected" to (x, y), but a better answer is that we *can* account for the diagonal neighbors (and associated $\sqrt{2}$s pop up because their distance away is not exactly h), and in the limit as $h \to 0$, the finite differences obtained would still converge to the same Laplacian.

Concept 4.11 (Laplacian, Intuitively)**.** There are many different finite-difference approximations for the Laplacian, all proportional to the idea:

$$\triangle u \propto \frac{\text{sum over } n \text{ neighbors of } u(\text{neighbors}) - n * u(x,y)}{\text{Area}}$$

In this way, if the average value of u on a small circle around x is greater than the value of u at x, the Laplacian is positive, and vice versa gives that the Laplacian is negative.

It is natural for the average behavior of the neighbors at point to "drive" the behavior of the point in the next instant. For waves, the mean value of the neighbors of that point drives that point to *accelerate* in that direction.

Similarly, for heat flow, a similar derivation using Newton's law of *cooling*, instead of motion, gives that this Laplacian: the imbalance between the point and its environment, drives the point to simply *move* to stabilize that imbalance:

Make a heat equation derivation exercise, from Newton's law of COOLING this time

Concept 4.12 (Wave and Heat Equations)**.** The motion of waves is governed by the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \triangle u$$

for some constant speed c depending on the properties of the medium.

Further, the flow of heat is governed by the equation

$$\tau \frac{\partial u}{\partial t} = \triangle u$$

for some constant τ depending on the properties of the medium.

Fourier arrived at the idea of trigonometric series while studying heat flow. We will look at the example of the vibrating string, pinned down at its edges.

We now begin with a string defined on the interval $[0, 1]$. u satisfies the wave equation, as well as the **boundary conditions** $u(0) = u(1) = 0$. If we have an initial shape of the string $u_0(x)$ as well as an initial velocity $\partial u/\partial t$ at $t = 0$, given by $v_0(x)$, then we'd like to see what dynamics emerge.

4.8 Spectral Theory on Graphs

We begin with the definition of a **finite graph**

Definition 4.13. A finite graph G is a finite set of vertices $V = \{v_1, \ldots, v_n\}$ together with a set of edges $E = \{e_1, \ldots, e_m\}$ such that each edge is an unordered pair of vertices (v_i, v_j) indicating that they are connected.

Draw a figure of a graph

All of this information can be captured in terms of an n by n matrix called the **adjacency matrix**.

Definition 4.14 (Adjacency Matrix)**.** The adjacency matrix A for a graph G is a matrix whose i, jth entry is 1 if (v_i, v_j) is an edge, and 0 otherwise.

The adjacency matrix acting on a vector with 0s everywhere except for the ith slot will give a vector with 0s everywhere except for the slots of the **neighbors** of v_i. That is, all vertices v_j such that (v_i, v_j) is an edge. We denote the set of neighbor vertices to v_i by $N(v_i)$.

Note that this matrix is symmetric, and consists of only zeroes and ones. Similarly we define the **degree matrix** as

Definition 4.15 (Degree Matrix)**.** The degree matrix D for a graph is a diagonal matrix whose (i, i)th entry is the *number of neighbors* of v_i, that is $|N(v_i)|$

Graphs appear throughout modern mathematics and computer science, to the point that we expect that the reader is familiar with them at least at a rudimentary level. The reason we put a focus on graphs in a section that is about Fourier analysis is the following:

Concept 4.16 (Discrete Approximation)**.** In almost all practical matters of engineering and applied math, manifolds are approximated by graphs.

The reason for why we care about this is that it makes everything finite-dimensional, turning all analysis into algebra and allowing us to not worry about notions of convergence and Hilbert spaces. Because of the immense use of such finite methods in practical engineering,

physics, and computer science, we have this topic as a section of its own, not just a mere exercise of example. We will illustrate its applicability to science through a series of examples.

Example 4.17 (Signal Processing)**.** A signal $f(t)$ on a time interval $t \in [0,1]$ has the interval approximated as a graph G_n of vertices $v_0, \ldots, v_n$. v_k represents the point $k/n \in [0,1]$.

v_0 is connected only to v_1, v_n is connected only to v_{n-1} and otherwise v_i is connected to its immediate neighbors $v_{i\pm1}$. The signal is then a function on the set of $n+1$ vertices.

Consider the derivation of the wave and heat equations in the preceding chapter.

Example 4.18 (Image Processing)**.** A $2D$ image represented by $f(x,y)$ on the interval $[0,1] \times [0,1] = [0,1]^2$ is approximated as a graph G_n of vertices $v_{i,j}$ for $0 \leqslant i,j \leqslant n$. The vertex $v_{i,j}$ represents the point $(i/n, j/n) \in [0,1]^2$.

As before, each vertex v_ij on the interior is connected to its four nearest neighbors $v_{i\pm1,j\pm1}$, vertices on the corners are only connected to two such neighbors, and otherwise vertices on the edges are connected to the three such neighbors. Images are then functions on this graph of $(n+1)$ vertices.

In these examples, we can form a **discrete derivative** operator.

Example 4.19 (Discrete Derivative)**.** In the first example, we can form a discrete derivative of a function on these $(n+1)$ vertices as:

$$\Delta f(v_i) = \begin{cases} \frac{f(v_{i+1})-f(v_i)}{(1/n)}, & \text{if } i = 0 \\ \frac{f(v_i)-f(v_{i-1})}{(1/n)}, & \text{if } i = n \\ \frac{f(v_{i+1})-f(v_{i-1})}{2(1/n)}, & \text{otherwise} \end{cases} \tag{4.16}$$

This derivative is called a **central difference**. Of course we could have also picked just a **forward difference**

$$\Delta f(v_i) = \begin{cases} \frac{f(v_{i+1})-f(v_i)}{(1/n)}, & \text{if } i < n \\ \frac{f(v_i)-f(v_{i-1})}{(1/n)}, & \text{if } i = n \end{cases} \tag{4.17}$$

or the other way, as a **backward difference**. All of these operators are linear and representable as $(n+1)$ by $(n+1)$ matrices acting on

the set of functions on the graph, equivalent to[2] $\mathbb{C}^{n+1}$. Their kernel, just like the regular derivative operator, is one-dimensional: the space of constant functions.

Similarly, in the second example we can form discrete derivatives corresponding to the "x" and "y" directions in the same way as before, operating on either the first or the second index of the $v_{i,j}$, respectively.

Much like picking a derivative $\partial/\partial q^i$ is coordinate covariant, so is picking a derivative on a graph not a natural thing. We need to pick a sort of "flow" on the graph along which our derivative vector field is pointing. On a graph given at random, its not at all obvious if there is any natural way to define "derivative". This is further evidence that the coordinate-dependent notion of "derivative" is not of central importance in physics.

Random scrambled graph "where's your derivative now, fam?"

On the other hand, there *is* an operator independent of coordinate system that we *can* form: the **graph Laplacian**. In the previous section, for 1-dimension we defined the Laplacian to be the sum of the function values at the two neighboring points $x_{i\pm1}$ minus 2 times the function value at the point x_0 itself, all divided by h, the infinitesimal length associated to the point x_0. For a 2-dimensional plane, we defined the Laplacian to be the sum of the function values at the 4 neighboring points minus 4 times the value of the function at (x_0, y_0), all divided by h^2, the infinitesimal *area* associated to the point (x_0, y_0).

We now generalize this to a graph. A given point may have k neighbors. We take the sum of the function f at all k of these neighbors and subtract k times the function value at f. As a matrix, this can be seen as:

$$\sum_{v_i \in N(v_0)} [f(v_i) - f(v_0)] = \sum_{v_i \in N(v_0)} f(v_i) - D(v_0) f(v_0) \tag{4.18}$$

$$= (A - D) f \tag{4.19}$$

This last equality recasts it in terms of the linear operators of the adjacency and degree matrix acting on the vector f.

[2]Note we work over complex vector spaces for the same reason as in the previous section: we can find all the eigenvalues of an operator when working over $\mathbb{C}$.

This thing should be proportional (up to the factor of that infinitesimal area associated to each vertex, assuming each vertex has equal area) to the Laplacian as we've defined it before. For this reason, in graph theory we don't care about an overall constant that multiplies our Laplacian, and so there are multiple conventions. Note that the operators A and D are *invariants*, associated to the graph, and so the Laplacian is invariant as well. On any graph, no matter how scrambled-looking it is, there is a good definition for a Laplacian.

Definition 4.20 (Laplacian Matrix of a Graph)**.** For a graph $G = (V, E)$ represented by adjacency matrix A giving degree matrix D, the Laplacian can be written as

$$\triangle = A - D \tag{4.20}$$

Because we care about the matrix itself and not the constant multiplying it, different parts of literature define it differently as:

- (reversed) $\triangle = D - A$
- (normalized) $\triangle = 1 - D^{-1}A$
- (symmetric normalized) $\triangle = 1 - D^{-1/2}AD^{-1/2}$

We will use the symmetric normalized, as it is common in mathematical literature.

Applications to Molecular Orbital Theory

Regardless of what Laplacian definition we use, the matrix is symmetric. By the spectral theorem, this guarantees us that the eigenvalues are all real and that the eigenvectors span the space of functions on the graph, and are in fact *orthogonal*. This is as powerful a result as we could have ever hoped for, because now, if we write the heat equation as

$$-\triangle f = \frac{\partial f}{\partial t}. \tag{4.21}$$

We can expand f in an eigenbasis h_i of eigenvalues λ_i determined from the matrix for $\triangle$:

$$f = \sum_{i=1}^{n} c_i h_i \tag{4.22}$$

where c_i is obtained from orthogonality of the eigenbasis, $c_i = \langle h_i | f \rangle$ so that:

$$-\triangle f = \sum_{i=1}^{n} c_i(-\lambda_i)h_i = \sum_{i=1}^{n} c_i \lambda_i h_i = \frac{\partial f}{\partial t} = \sum_{i=1}^{n} c_i \dot{h}_i. \tag{4.23}$$

As a vector equation, this implies that components must agree:

$$-\lambda_i h_i = \dot{h}_i \Rightarrow h_i(t) = h_i(0)e^{-\lambda t} \tag{4.24}$$

Thus, we get the full solution to the heat equation:

$$f(t) = \sum_{i=0}^{n} c_i e^{-\lambda t} h_i \tag{4.25}$$

In shorthand then, the diffusion of heat operator can be written as the exponential:

$$e^{-t\triangle}$$

And for this reason we can call the Laplacian the *infinitesimal generator* of diffusion.

One important subset of graphs that is worth focusing on is the set of all **k-regular** graphs. These are graphs such that each vertex v has exactly k neighbors: $\forall v \ |N(v)| = k$.

Observation 4.21. *The vector* $(1, \ldots, 1)$ *is an eigenvector of* Δ *with eigenvalue* 0*.*

The matrix A acting on $(1, 0 \ldots, 0)$ gives the vector of 1s on all neighbors of v_1 and 0s elsewhere. It's easy to see then that A acting on $(1, \ldots, 1)$ will give a vector with $|N(v_i)|$ on its ith entry. But this is exactly the same as the action of D on $(1, \ldots, 1)$, so $A - D$ gives zero on this vector.

The equivalent on manifolds is that the constant function has vanishing Laplacian.

4.9 Spectral Theory on $\mathbb{R}$

Chapter 5

Beyond Harmonics: Representation Theory

5.1 The Representations of Finite Groups

5.2 Character Theory

5.3 The Representations of Topological Groups

5.4 Representations Induced

Part 2

Physics

Chapter 6

Symmetries of the sphere: $\mathrm{SU}(2)$ and friends

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Chapter 10

Classification of Simple Lie Algebras over $\mathbb{C}$

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Appendices

Chapter A

Eigenvalues and the Jordan Normal Form

Should this just be an appendix section?

IDK how to make chapter -> appendix which is a big problem here

Eigenvalues of a linear tranformation and their associated eigenvectors hold great value, The eigenvectors determine subspaces of V where the action of the transformation becomes as *simple as possible*: literally scalar multiplication. If we can decompose V, by using eigenvectors, into a series of spaces along which the transformation is simply scaling, we can **diagonalize** the matrix, and all the information and dynamics that it represents become extraordinarily simple to work with. The difficult-to-work-with process of matrix multiplication collapses down to multiplication by scalars.

When we studied a linear transformation $T : V \to V$ on a real vector space V of dimension n. T takes a vector $\mathbf{v}_i$ and maps it to another vector $T\mathbf{v}_i \in V$. If we pick a specific basis $B\{\mathbf{e}_i\}$ for our space, then T can be represented by a **matrix**, A^i_j whose ijth entries in the basis B can be explicitly found:

$$A^i_j = \begin{pmatrix} T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \end{pmatrix}_B = \begin{pmatrix} T(\mathbf{e}_1)^1 & \dots & T(\mathbf{e}_n)^1 \\ \vdots & \dots & \vdots \\ T(\mathbf{e}_1)^n & \dots & T(\mathbf{e}_n)^n \end{pmatrix}_B \tag{A.1}$$

Note this same transformation can be viewed through different bases

$\mathbf{e}'_i$, producing different matrix representations depending on the basis. Therefore we should realize, just like tuples of numbers only represented physical vectors once a coordinate system was chosen, the same is true for how matrices only *represent* the linear operators we care about. In particular, since we can write matrix multiplication in a basis as

$$(Tv)^i = A^i_j v^j \tag{A.2}$$

Now if we change basis from B to B' and we have a matrix Q^j_i that takes the components v^i of a vector in the B basis and gives us the B' basis $(v')^i = Q^j_i v^j$ then we can transform A^i_j according to:

$$(A^i_j)_{B'} = Q^i_k (A^k_l)_B (Q^{-1})^l_j \tag{A.3}$$

Accordingly, $A_{B'} = QA_BQ^{-1}$. This should make perfect sense: if we're given a vector's components v'^i in the B' basis, first we act by Q^{-1} to get back into the B basis, then we act by A, since we know how T acts in this basis, and then we map that vector back into the B' basis with Q.

One operator can be represented by many different matrices. Here, we will attempt to *classify* all the possible linear operators on a space. This would be a great thing, since linear operators can be so diverse, ranging from stretching to shearing to rotation, and much more.

We start with the **trivial** linear operator, 0. Nothing can be more basic than the action of sending everything to zero. The simplest linear transformation that isn't just 0 is multiplication by a scalar, λ. This is basic multiplication, and as a matrix is represented just as

$$\lambda = \lambda I = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{pmatrix} = \lambda \delta^i_j$$

Note that if we change basis, the matrix representation for scalar multiplication *doesn't* change: regardless of how you look at it, multiplication by λ is multiplication by λ. The set of linear operators that are just simple scalar multiplication form a subspace of all linear operators, and are called **simple**. Simple linear operators are literally just numbers, acting on the space by scaling everything the same way. Note the trivial 0 operator is an example of a simple operator.

Of course, the set of linear operators that are simple is a very small subset of the set of all linear operators. A linear operator on V does not in general act simply on V, but the next best thing we could hope for is that T becomes simple when its action is restricted to a *subspace* of V.

What would this mean, mathematically? We want to find vectors $\mathbf{v}$ so that

$$T\mathbf{v} = \lambda\mathbf{v} \tag{A.4}$$

Note that such a set of $\mathbf{v}$ is *also* a vector space.

Proposition A.1. *The set of vectors where $T\mathbf{v} = \lambda\mathbf{v}$, for some specific λ, is a vector space, denoted V_λ.*

Proof. If $\mathbf{v}, \mathbf{w}$ have

$$T\mathbf{v} = \lambda\mathbf{v}, T\mathbf{w} = \lambda\mathbf{w}$$

then

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w}) = a\lambda\mathbf{v} + b\lambda\mathbf{w} = \lambda(a\mathbf{v} + b\mathbf{w})$$

□

We then have that V_λ is exactly the subspace where T acts simply as λ. When $\lambda = 0$ this is exactly what we referred to as $\ker T$. How can we find this subspace? The trick all lies in noting that the number λ is *itself* just a linear transformation. Then finding when Equation (A.4) is satisfied is the same as finding when:

$$(T - \lambda)\mathbf{v} = 0 \tag{A.5}$$

So this is no different from calculating a kernel of the shifted operator $T - \lambda$. If we know how to find the kernel, the subspace where $T = 0$, we also know how to find the subspace where $T = \lambda$ by finding the kernel where $T - \lambda = 0$. It's a really simple idea.

But it's worth asking the question "does $T - \lambda$ even *have* a nontrivial kernel?". That is, for which λ is V_λ nontrivial? We know a quick way to answer this question. $T - \lambda$ will have a nontrivial kernel exactly when:

$$\det(T - \lambda) = 0 \tag{A.6}$$

In a specific basis, T can be represented by a matrix T_{ij} and λ as a operator is always the matrix $\lambda I = \lambda\delta_{ij}$ so that we can directly

compute this in a basis by computing the matrix determinant $\det(T_{ij} - \lambda\delta_{ij})$. It is not hard to see that this is a polynomial

$$\begin{pmatrix} T_1^1 - \lambda & \dots & T_1^n \\ \vdots & & \vdots \\ T_1^n & \dots & T_n^n - \lambda \end{pmatrix} \tag{A.7}$$

It's not difficult to see that the resulting determinant becomes a *polynomial of degree n* in λ. It is called the **characteristic polynomial** $p_T(\lambda)$ of the transformation T.

The determinant of a transformation is itself independent of the coordinate system we use, so the characteristic polynomial is an invariant, not dependent on the basis that gives us our T_{ij}. The roots of this polynomial are special numbers, characteristic of the transformation. These "characteristic values" or, from German, **eigenvalues**, of T are thus also invariants independent of coordinate system. They are exactly the values for which V_λ is nontrivial, so that T acts simply as one of these λ_i on each V_{λ_i}.

But over the reals, there are transformations that don't even have eigenvalues:

Example A.2. The transformation representing counterclockwise rotation of the basis, which can be represented by the real matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues.

We see this immediately since its characteristic polynomial is λ^2+1 which has no real roots. It does, however, have complex roots $\lambda = \pm i$ and we shouldn't ignore this fact. If we want to find a vector subspace on which this transformation acts simply as $\pm i$, we need to *extend* the number system that we are working over, from the reals to the complex numbers. In fact, working over the complex numbers is completely *ideal* if we want $p_T(\lambda)$ to have roots, giving us eigenvalues. The reason is because of this foundational theorem:

Theorem A.3 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial with coefficients in* $\mathbb{C}$ *has a root in* $\mathbb{C}$*.*

Proof. (Optional, for those familia with complex analysis). The easiest way to do this is by use of complex analysis. Any introductory text

in complex analysis will give the tools necessary to form this proof. If the reader is familiar with Cauchy's integral formula (**which can be derived in one of the exercises in the preceding chapter**)

Make an exercise in the preceding chapter to derive all of complex analysis' integral formulae

, then he can understand the following:

For a polynomial p of a complex variable z, assume $p(z)$ did not have a root in $\mathbb{C}$. Then $f(z) := 1/p(z)$ is bounded on $\mathbb{C}$ so that $|f(z)| \leqslant B$, and is also holomorphic in fact entire on the complex plane. This means it admits a Taylor series about $z = 0$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{A.8}$$

with a_k obtained from the Cauchy integral formula by:

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{\partial D_R(0)} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta \tag{A.9}$$

We are integrating over the boundary of the disk of radius R enclosing zero. Since this can be any disk, we let R becomes large so that $|\zeta| = |Re^{i\theta}| = |R|$ also becomes large, and so

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \oint_{\partial D_R(0)} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta \right| \\ &\leqslant \frac{1}{2\pi} \oint_{D_R(0)} \frac{|f(\zeta)|}{R^{k+1}} |d\zeta| \\ &\leqslant \frac{1}{2\pi} (2\pi R) \max_{\partial D_R(0)} \frac{f}{R^{k+1}} \\ &\leqslant \frac{B}{R^k} \end{aligned}$$

since R can be anything, we let it go to infinity and obtain $|a_k| \leqslant 0$ for all $k \geqslant 1$. This gives us that $f(z)$ is constant, and also a powerful theorem in general:

Theorem A.4 (Liouville)**.** *Any bounded entire function on $\mathbb{C}$ is constant.*

In particular $1/p(z)$ is a constant, so $p(z)$ must be a constant as well. If a polynomial does not have roots in $\mathbb{C}$, it must be a constant function. □

Since every polynomial p has a root r, we can write it as $p(z) = (z - r_1)p'(z)$, with p' a polynomial of degree $n - 1$. Applying the fundamental theorem to p', now, and continuing inductively immediately shows that.

Corollary A.5. *Every non-constant polynomial with coefficients in $\mathbb{C}$ has exactly n roots, counted with multiplicity, in $\mathbb{C}$.*

That means that a polynomial over $\mathbb{C}$, $p(x) = \sum_{i=0}^{n} a_i x^i$ splits entirely into a product of linear factors $c(x - r_1) \dots (x - r_n)$ over $\mathbb{C}$. In particular, this means that our characteristic polynomial will have n roots, counted with multiplicity. Although the complex numbers are less intuitive to be introduced to than the real numbers were, they have actually made things *easier* and much more interesting. So instead of disregarding these numbers because we think they are are unphysical, let us accept them, and later try to physically interpret what they mean!

The key word of Corollary A.5 is "with multiplicity". Our polynomial will always have exactly n roots, but they need not all be distinct. Let's say that k of them are distinct $\lambda_1 \dots \lambda_k$, each with multiplicity m_i (so $\sum_{i=1}^{k} m_i = n$). We can write the polynomial as

$$p_A(t) = (-1)^n \prod_{i=1}^{k} (t - \lambda_i)^{m_i} \tag{A.10}$$

First let's assume no root is repeated for this polynomial. Then we have $\lambda_1 \dots \lambda_n$ distinct. Each λ_i gives rise to a corresponding eigenvector $\mathbf{v}_i$.

Lemma A.6. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Proof. Let $\mathbf{v}_1$ correspond to λ_1, $\mathbf{v}_2$ to λ_2. If they are linearly dependent then

$$\exists a, b \neq 0 \text{ s.t. } a\mathbf{v}_1 + b\mathbf{v}_2 = 0 \tag{A.11}$$

But then

$$T(0) = T(a\mathbf{v}_1 + b\mathbf{v}_2) = a\lambda_1\mathbf{v}_1 + b\mathbf{v}_2 = 0 \tag{A.12}$$

Subtracting from this λ_1 times the above equation, we get that $b(\lambda_2 - \lambda_1)\mathbf{v}_2 = 0$, and since $b \neq 0, \mathbf{v}_1 \neq 0$, we get $\lambda_1 = \lambda_2$. □

So when all the λ_i are distinct, this gives rise to n pairwise linearly independent eigenvectors $\mathbf{v}_i$ of eigenvalue λ_i. But this forms a *basis* for our vector space! In this basis, we get that

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \tag{A.13}$$

This means on *each* 1-dimensional subspace $V_{\lambda_i} = \text{span}\{\mathbf{v}_i\}$, T acts simply, and the entire space V can be written as a *direct sum* (c.f. Section 3.6) of the V_{λ_i}

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i} \tag{A.14}$$

When the space V can be decomposed as a direct sum of subspaces V_i on which the operator T acts simply, T is called **semisimple**. There is another word you may have heard in place of semisimple in the past, which means the same thing: **diagonalizable**.

Definition A.7 (Semisimplicity)**.** A linear operator $T \in \text{End}(V)$ is called *semisimple* or *diagonalizable* if there exists a decomposition

$$V = \bigoplus_{i} V_i \tag{A.15}$$

such that T acts simply, as λ_i on each V_i. We write that the operator T itself can be written as a direct sum of the simple linear operators:

$$T = \bigoplus_{i} T_i \tag{A.16}$$

Where T_i is the restriction of T to V_i, $T|_{V_i}$.

Graphic of restriction of an operator to subspaces, and now its just a constant scalar

Note Equation (A.13) was a special case of semisimplicity, when all V_i had dimension 1. In general, a semisimple matrix when viewed

in a basis $\mathbf{v}_j$ so that each $\mathbf{v}_j$ lies in one of the V_i will look just like Equation (A.13), *allowing* for repeated λ_i.

But from our previous analysis, this implies that for a linear operator whose characteristic polynomial *doesn't* have repeated roots (i.e all roots are **simple**) that there the eigenvectors form a basis for the space, in which the operator looks diagonal.

Theorem A.8 (Semisimplicity for T Lacking Repeated Eigenvalues)**.** *If the characteristic polynomial of a linear operator T does not have repeated roots, then T is semisimple, and V decomposes as.*

$$V = \bigoplus_{i=1}^{n} \ker(T - \lambda_i) \tag{A.17}$$

Proof. To reiterate the idea, this follows immediately from Lemma A.6, which means the eigenvectors form a basis for V. So V can be decomposed as a direct sum of the eigenvector spaces, on each of which T acts simply by definition, so T is semisimple. □

Now note that almost all polynomials do not have repeated roots. An intuitive sketch of this is that if you pick a polynomial at random, the roots will have some random distribution on $\mathbb{C}$. The chance that two roots will happen to be at the exact same point and form a double root is infinitesimal: the same as the chance of picking a random real number and getting an integer. So a linear operator picked at random, we'd expect, would not have any double roots. That is *almost all linear operators are semisimple.* There's a way to view almost any operator so that the operator looks diagonal.

The converse of Theorem A.8 is not true. The characteristic polynomial can have repeated roots and the operator can still be semisimple. For example simple operators λ are trivially semisimple and their characteristic polynomials are $(t-\lambda)^n$. Are all operators just semisimple, diagonal when viewed in the right eigenbasis?

The classic example of an operator that is not semisimple is the one represented by the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The characteristic polynomial for this is $p_A(t) = t^2$, the same as the characteristic polynomial for the zero matrix, however the eigenspace

$V_0 = \ker A$ is only one-dimensional, and the characteristic polynomial tells us there are no other eigenspaces. So although the eigenvalue 0 has **algebraic multiplicity** 2, we say the associated eigenspace only has **geometric multiplicity** 1, since $\ker T - \lambda = 1$.

This operator is on the other side of the coin from semisimple: it is **nilpotent**. A nilpotent operator T is one so that $T^k = 0$ for some number k. The minimum such k for which $T^k = 0$ is called the **nilpotence degree** of T.

Theorem A.9 (Primary Decomposition of a Linear Operator)**.** *For a general $T \in \mathrm{End}(V)$, with its distinct eigenvalue roots to its characteristic polynomial labelled by $\lambda_1, \ldots, \lambda_k$, and each λ_i with multiplicity denoted m_i, we can decompose V as a direct sum:*

$$V = \bigoplus_{i=1}^{k} \ker\left[(T - \lambda_i)^{m_i}\right] \tag{A.18}$$

There is in fact a connection between this statement and how an integer n can be decomposed into a product of power of "simple" integers: primes, as $n = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$.

Chapter B

Category Theory

We begin by going back to elementary linear algebra. Linear transformations between vector spaces $T : V \to W$ satisfy

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w}) \tag{B.1}$$

That is, they respect linear combinations of vectors.

In more abstract language, transformations that respect algebraic structure (like linear transformations respect vector space structure) generally called **homomorphisms**. We denote the set of linear transformations from V to W by $\mathrm{Hom}(V, W)$. Linear transformations from V to itself, $\mathrm{Hom}(V, V)$ are called **endomorphisms** and are denoted by $\mathrm{End}(V) := \mathrm{Hom}(V, V)$. A particularly special endomorphism on V is the *identity* map 1_V that acts trivially on V, sending every vector to itself.

Each linear transformation in $T \in \mathrm{Hom}(V, W)$ has two important subspaces associated with it: its kernel and its image. We say the kernel, $\ker T$ is the subspace of V that T sends to zero. The image $\mathrm{im}\, T$ is the subspace of W that T maps into.

Now certain linear transformations have the property that they map every vector $\mathbf{v} \in V$ to a *unique* vector in W. Section 1.2 has shown that this is equivalent to $\ker T = 0$. Such linear transformations are called one-to-one, and they are also called **injective**.

On the other side, linear transformations can also have the property that they map *to* every vector $w \in W$. That is, $\mathrm{im}\, T = W$. Such linear transformations are called onto, or **surjective**.

When a linear transformation is both injective and surjective, it is **bijective**, meaning there is an inverse $T^{-1} : V \to W$ so that

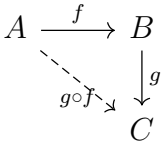
$T \circ T^{-1} = 1_V$ and $T^{-1} \circ T = 1_W$. These are called **bijections** and also **isomorphisms** between V and W. Endomorphisms from V to itself that are isomorphisms are called **automorphisms** and are denoted by $\mathrm{Aut}(V)$ or $\mathrm{GL}(V)$.

This is the inspiration for the much more general setting: Category Theory.

A **category** is simply something that consists of **objects** that are linked together by arrows called **morphisms**, also just called **maps**. A morphism f takes the **source object** A to a **target object** B. Such morphisms can be drawn using **diagrams** as

$$A \xrightarrow{f} B$$

We denote the class of morphisms from A to B by $\hom(A, B)$. A morphism from A to itself is called an **endomorphism** and the class of endomorphisms is denoted by $\mathrm{end}(A) = \hom(A, A)$. If f is a morphism from object A to B and g is a morphism from object B to C then we can **compose** g with f to form another morphism $g \circ f$ according to the following diagram.



We say "the diagram commutes" to mean that we can go either direction in the diagram, along f then g or just along $g \circ f$ and get the same result.

Further, there is a special morphism to every object A, the **identity morphism** denoted by 1_A such that for any morphism $f : B \to A$ into A, $1_A \circ f = f$ and for any morphism $g : A \to B$ out of A, we have $g \circ 1_A = g$. Note this implies that the identity is unique, as if there were another $1'_A$, then by these properties, $1'_A = 1_A \circ 1'_A = 1_A$.

The class of all vector spaces forms a category, denoted **Vect**, where morphisms are linear maps. This was our motivating example. Further along the same line of thought, groups and rings are categories, whose morphisms are exactly the algebraic homomorphisms associated with those algebraic structures. Abelian groups also form a

category, that is a **subcategory** of the category of groups. A subcategory S of C is a category whose objects and morphisms are objects and morphisms in C, respectively.

Even more simply, the class[1] of all sets forms a category, **Set**, where morphisms are exactly functions between sets.

More interestingly, the class of all *topological spaces* forms a category, with morphisms being the *continuous functions*. A sub-category of this is the category of manifolds **Man**. If we instead want our morphisms to be smooth maps, we can form the subcategory of smooth manifolds, denoted $\mathbf{Man}^{\infty}$.

[1]We keep using the word class instead of set exactly because we don't want to run into the paradox of saying "the set of all sets". The word "class" refers to a mathematical construction designed to avoid this, which is not focused on here.

Notation

References

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