Autonomous Vehicle Simulation (AVS) Laboratory, University of Colorado

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VARIABLE SPEED CONTROL MOMENT GYROSCOPE MODEL

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Scope/Contents
The VSCMG class is an instantiation of the state effector abstract class. The integrated test is validating the interaction between the VSCMG module and the rigid body hub that it is attached to. More specifically, the VSCMG module models three different cases: balanced wheels, simple jitter, and fully coupled jitter. The details of each mode is described in detail in this document. There are integrated tests that confirm that all three models are agreeing with physics and the tests use both energy and momentum conservation as validation. There are also unit tests verifying other functionality of the module

Rev	Change Description	By	Date
1.0	Initial Draft	J. Martin	20180718

Contents

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1 Model Description

1.1 Introduction

This module is modeling a VSCMG connected to a rigid body hub. The VSCMG model has three modes that can be ran: balanced wheels, simple jitter, and fully-coupled imbalanced wheels.

The balanced wheels option is modeling the VSCMG as having their principle inertia axes aligned with spin axis, $\hat{\boldsymbol{g}}_s$, and the center of mass of the wheel is coincident with $\hat{\boldsymbol{g}}_s$. This results in the reaction wheel not changing the mass properties of the spacecraft and results in simpler equations. The simple jitter option is approximating the jitter due to mass imbalances by applying an external force and torque to the spacecraft that is proportional to the wheel speeds squared. This is an approximation because in reality this is an internal force and torque. Finally, the fully-coupled mode is modeling VSCMG imbalance dynamics by modeling the static and dynamic imbalances as internal forces and torques which is physically realistic and allows for energy and momentum conservation.

Figure 1 shows the frame and variable definitions used for this problem. The formulation involves a rigid hub with its center of mass location labeled as point B_c, and N_{rw} RWs with their center of mass locations labeled as W_{c_i}. The frames being used for this formulation are the body-fixed frame, $\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, the motor frame of the i^{th} RW, $\mathcal{M}_i : \{\hat{\boldsymbol{m}}_{s_i}, \hat{\boldsymbol{m}}_{2_i}, \hat{\boldsymbol{m}}_{3_i}\}$ which is also body-fixed, and the wheel-fixed frame of the i^{th} RW, $\mathcal{W}_i : \{\hat{\boldsymbol{g}}_{s_i}, \hat{\boldsymbol{w}}_{2_i}, \hat{\boldsymbol{w}}_{3_i}\}$. The dynamics are modeled with respect to the $\mathcal{B}$ frame which can be generally oriented. The $\mathcal{W}_i$ frame is oriented such that the $\hat{\boldsymbol{g}}_{s_i}$ axis is aligned with the RW spin axis which is the same as the motor torque axis $\hat{\boldsymbol{m}}_{s_i}$, the $\hat{\boldsymbol{w}}_{2_i}$ axis is perpendicular to $\hat{\boldsymbol{g}}_{s_i}$ and points in the direction towards the RW center of mass W_{c_i}. The $\hat{\boldsymbol{w}}_{3_i}$ completes the right hand rule. The $\mathcal{M}_i$ frame is defined as being equal to the $\mathcal{W}_i$ frame at the beginning of the simulation and therefore the $\mathcal{W}_i$ and $\mathcal{M}_i$ frames are offset by an angle, θ_i, about the $\hat{\boldsymbol{m}}_{s_i} = \hat{\boldsymbol{g}}_{s_i}$ axes.

A few more key variables in Figure 1 need to be defined. The rigid spacecraft structure without the VSCMGs is called the hub. Point B is the origin of the $\mathcal{B}$ frame and is a general body-fixed point that does not have to be identical to the total spacecraft center of mass, nor the rigid hub center of mass B_c. Point W_i is the origin of the $\mathcal{W}_i$ frame and can also have any location relative to point B. Point C is the center of mass of the total spacecraft system including the rigid hub and the VSCMGs. Due to the VSCMG imbalance, the vector $\boldsymbol{c}$, which points from point B to point C, will vary as seen by a body-fixed observer. The scalar variable d_i is the center of mass offset of the VSCMG, or the distance from the spin axis, $\hat{\boldsymbol{g}}_{\text{s}_i}$ to W_{c_i}. Finally, the inertial frame orientation is defined through $\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$, while the origin of the inertial frame is labeled as N.

The key equations are highlighted below and the full deriviation can be found in.[?]

1.2 Back-Substitution

The goal of back-substitution is to manipulate the rotational and translational equations of motion to conform to the following form,

$$\begin{bmatrix}[A] & [B]\\ [C] & [D]\end{bmatrix}\begin{bmatrix}\ddot{\boldsymbol{r}}_{B/N}\\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_{\text{trans}}\\ \boldsymbol{v}_{\text{rot}}\end{bmatrix} \tag{1}$$

where $[A]$, $[B]$, $[C]$, and $[D]$ are 3x3 matrices representing $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ coefficients within the translational and rotational EOMs. $\boldsymbol{v}_{\text{trans}}$ is a 3x1 vector that represents the right-hand side (RHS) of the translational EOM, and $\boldsymbol{v}_{\text{rot}}$ is a 3x1 vector that represents the RHS of the rotational EOM. The matrices $[A]$, $[B]$, $[C]$, $[D]$ and vectors $\boldsymbol{v}_{\text{trans}}$, $\boldsymbol{v}_{\text{rot}}$ are broken down as follows.

$$[A] = [A_{\text{hub}}] + [A_{\text{contr}}] \tag{2}$$
$$[B] = [B_{\text{hub}}] + [B_{\text{contr}}] \tag{3}$$
$$[C] = [C_{\text{hub}}] + [C_{\text{contr}}] \tag{4}$$
$$[D] = [D_{\text{hub}}] + [D_{\text{contr}}] \tag{5}$$
$$\boldsymbol{v}_{\text{trans}} = \boldsymbol{v}_{\text{trans,hub}} + \boldsymbol{v}_{\text{trans,contr}} \tag{6}$$
$$\boldsymbol{v}_{\text{rot}} = \boldsymbol{v}_{\text{rot,hub}} + \boldsymbol{v}_{\text{rot,contr}} \tag{7}$$

where $[A_{\text{hub}}]$ represents the contribution to $[A]$ from the spacecraft hub and $[A_{\text{contr}}]$ represents the contribution to $[A]$ from the effectors (i.e. RWs or VSCMGs), etc. $[A_{\text{hub}}]$ etc are the same regardless

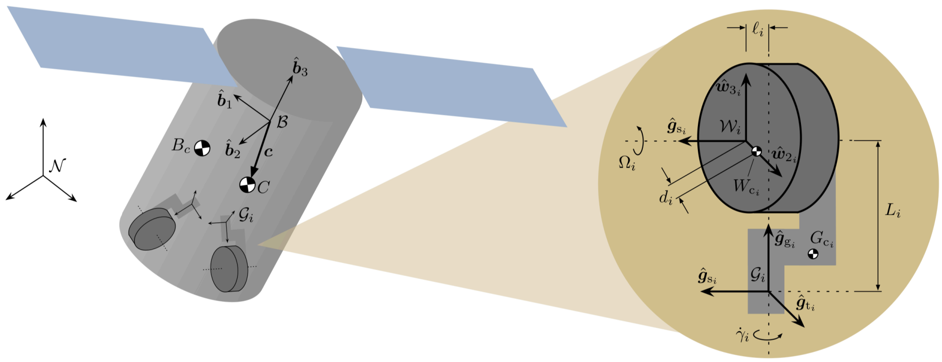

Fig. 1: VSCMG and spacecraft frame and variable definitions

of the type of effector used, and are provided in the equation below.

$$[A_{\text{hub}}] = m_{\text{sc}}[I_{3\times3}] \tag{8}$$

$$[B_{\text{hub}}] = -m_{\text{sc}}[\tilde{\boldsymbol{c}}] \tag{9}$$

$$[C_{\text{hub}}] = m_{\text{sc}}[\tilde{\boldsymbol{c}}] \tag{10}$$

$$[D_{\text{hub}}] = [I_{\text{sc},B}] \tag{11}$$

$$\boldsymbol{v}_{\text{trans,hub}} = \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c} \tag{12}$$

$$\boldsymbol{v}_{\text{rot,hub}} = \boldsymbol{L}_B - [I_{\text{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} \tag{13}$$

1.3 Balanced VSCMG

1.3.1 Equations of Motion

The balanced VSCMG equations of motion are provided here for the reader's convenience. Note that translation is not coupled with $\dot{\Omega}$ or $\ddot{\gamma}_i$.

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} = \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c}$$

The rotational equation of motion includes $\dot{\Omega}_i$ and $\ddot{\gamma}_i$ terms, and is thus coupled with VSCMG motion as seen below.

$$\begin{aligned} m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N} I_{\text{V}_{g_i}}\hat{\boldsymbol{g}}_{\text{g}_i}\ddot{\gamma}_i + \sum_{i=1}^{N} I_{\text{W}_{s_i}}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i \\ = \boldsymbol{L}_B - [I_{\text{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[+ I_{\text{W}_{t_i}}\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\text{t}_i} + \Omega_i\dot{\gamma}_i(I_{\text{W}_{s_i}} - I_{\text{W}_{t_i}})\hat{\boldsymbol{g}}_{\text{t}_i} \\ + [\tilde{\boldsymbol{\omega}}][I_{\text{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\text{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\text{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}}\Big] \end{aligned}$$

The gimbal torque equation is given below.

$$I_{\text{V}_{g_i}}(\hat{\boldsymbol{g}}_{\text{g}_i}^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i) = u_{\text{g}_i} + (I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}})\omega_s\omega_t + I_{\text{W}_{s_i}}\Omega_i\omega_t$$

The wheel torque equation is given below.

$$I_{\text{W}_{s_i}}(\hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}} + \dot{\Omega}_i) = -I_{\text{W}_{s_i}}\omega_t\dot{\gamma}_i + u_{\text{s}_i}$$

1.3.2 Modified EOM for Back-Substitution

To make use of back-substitution and define the back-substitution contribution matrices for a balanced VSCMG, the EOM must be arranged into the following form:

$$
\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + \Big[[I_{\text{sc},B}] - \sum_{i=1}^{N}\big(I_{\text{V}_{g_i}}\hat{\boldsymbol{g}}_{\text{g}_i}\hat{\boldsymbol{g}}_{\text{g}_i}^T\dot{\boldsymbol{\omega}} + I_{\text{W}_{s_i}}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T\big)\Big]\dot{\boldsymbol{\omega}} \\
= \boldsymbol{L}_B - [I_{\text{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[\big(u_{\text{s}_i} - I_{\text{W}_{s_i}}\omega_t\dot{\gamma}_i\big)\hat{\boldsymbol{g}}_{\text{s}_i} + I_{\text{W}_{s_i}}\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\text{t}_i} \\
+ \big(u_{\text{g}_i} + (I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}})\omega_s\omega_t + I_{\text{W}_{s_i}}\Omega_i\omega_t\big)\hat{\boldsymbol{g}}_{\text{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\text{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\text{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\text{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}}\Big]
\end{aligned}
\tag{14}
$$

1.3.3 Back-Substitution Contribution Matrices

The balanced VSCMG back-substitution contribution matrices are given by,

$$[A_{\text{contr}}] = [0_{3\times3}] \tag{15}$$

$$[B_{\text{contr}}] = [0_{3\times3}] \tag{16}$$

$$[C_{\text{contr}}] = [0_{3\times3}] \tag{17}$$

$$[D_{\text{contr}}] = -\sum_{i=1}^{N}\left[I_{\mathsf{V}_{g_i}}\hat{\boldsymbol{g}}_{\mathsf{g}_i}\hat{\boldsymbol{g}}_{\mathsf{g}_i}^T + I_{\mathsf{W}_{s_i}}\hat{\boldsymbol{g}}_{\mathsf{s}_i}\hat{\boldsymbol{g}}_{\mathsf{s}_i}^T\right] \tag{18}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \boldsymbol{0} \tag{19}$$

$$\boldsymbol{v}_{\text{rot,contr}} = -\sum_{i=1}^{N}\Big[\big(u_{\mathsf{s}_i} - I_{\mathsf{W}_{s_i}}\omega_t\dot{\gamma}_i\big)\hat{\boldsymbol{g}}_{\mathsf{s}_i} + I_{\mathsf{W}_{s_i}}\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathsf{t}_i} + \big(u_{\mathsf{g}_i} + (I_{\mathsf{V}_{s_i}} - I_{\mathsf{V}_{t_i}})\omega_s\omega_t + I_{\mathsf{W}_{s_i}}\Omega_i\omega_t\big)\hat{\boldsymbol{g}}_{\mathsf{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\mathsf{G}_{i},G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\mathsf{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\mathsf{W}_{i},W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}}\Big] \tag{20}$$

1.4 Imbalanced VSCMG

1.4.1 Translational EOMs

The translational equation of motion for an imbalanced VSCMG is:

$$\begin{aligned}
\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\Big[m_{\text{G}_i}\Big[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}\Big]\boldsymbol{r}_{G_{c_i}/G_i} - m_{\text{W}_i}d_i c\theta_i\hat{\boldsymbol{g}}_{\text{s}_i} + m_{\text{W}_i}\ell_i\hat{\boldsymbol{g}}_{\text{t}_i}\Big]\ddot{\gamma}_i + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\left[m_{\text{W}_i}d_i\hat{\boldsymbol{w}}_{3_i}\right]\dot{\Omega}_i \\
= \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{c} - \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\Big[m_{\text{G}_i}\dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}'_{G_{c_i}/B} \\
+ m_{\text{W}_i}\Big[\left(2d_i\dot{\gamma}_i\Omega_i\text{s}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\text{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\text{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\Big]\Big]
\end{aligned} \tag{21}$$

This equation represents 3 DOF and contains all second order states ($\ddot{\boldsymbol{r}}_{B/N}$, $\dot{\boldsymbol{\omega}}$, $\ddot{\gamma}_i$, $\dot{\Omega}_i$). Removing wheel imbalance terms and assuming a symmetrical VSCMG (i.e. $\boldsymbol{r}_{G_{c_i}/G_i} = \boldsymbol{0}$, $\ell_i = 0$, $d_i = 0$) gives the following balanced VSCMG transational equation of motion.

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} = \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c} \tag{22}$$

Thus, the balanced VSCMG translational equation of motion does not contain any second-order terms relating to the wheel or gimbal, and agrees with Eq. 1.3.1.

1.4.2 Rotational EOMs

The rotational equations of motion are:

$$
\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N}\Big[[I_{\text{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\text{g}_i} + m_{\text{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + [I_{\text{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\text{g}_i} \\
+ m_{\text{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}](\ell_i\hat{\boldsymbol{g}}_{\text{t}_i} - d_i c\theta_i\hat{\boldsymbol{g}}_{\text{s}_i})\Big]\ddot{\gamma}_i + \sum_{i=1}^{N}\Big[[I_{\text{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\text{s}_i} + m_{\text{W}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big]\dot{\Omega}_i \\
= \boldsymbol{L}_B - [I_{\text{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[[I_{\text{G}_i,G_{c_i}}]'\dot{\gamma}_i\hat{\boldsymbol{g}}_{\text{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\text{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\text{g}_i} + m_{\text{G}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\boldsymbol{r}'_{G_{c_i}/B} \\
+ m_{\text{G}_i}\dot{\gamma}_i[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\text{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [I_{\text{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\text{t}_i} + [I_{\text{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} + [\tilde{\boldsymbol{\omega}}][I_{\text{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \\
+ m_{\text{W}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B} + m_{\text{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\left[\left(2d_i\dot{\gamma}_i\Omega_i\text{s}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\text{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\text{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\right]\Big]
\end{aligned} \tag{23}
$$

The rotational equation of motion for a VSCMG with balanced wheels may be found by setting imbalance terms to zero, leading to:

$$
\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N} I_{\text{V}_{g_i}}\hat{\boldsymbol{g}}_{\text{g}_i}\ddot{\gamma}_i + \sum_{i=1}^{N} I_{\text{W}_{s_i}}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i \\
= \boldsymbol{L}_B - [\tilde{\boldsymbol{\omega}}][I_{\text{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\Big[\omega_t\dot{\gamma}_i(I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}} + I_{\text{V}_{g_i}})\hat{\boldsymbol{g}}_{\text{s}_i} \\
+ \big[\omega_s\dot{\gamma}_i(I_{\text{V}_{s_i}} - I_{\text{V}_{t_i}} - I_{\text{V}_{g_i}}) + I_{\text{W}_{s_i}}\Omega_i(\dot{\gamma} + \omega_g)\big]\hat{\boldsymbol{g}}_{\text{t}_i} - \omega_t I_{\text{W}_{s_i}}\Omega_i\hat{\boldsymbol{g}}_{\text{g}_i}\Big]
\end{aligned} \tag{24}
$$

This equation agrees with that found in Eq. **??**.

1.4.3 Gimbal Torque Equation

The VSCMG gimbal torque equation of motion is given by:

$$
\begin{aligned}
&\hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\Bigg[m_{\mathsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\Bigg]\ddot{\boldsymbol{r}}_{B/N} + \hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\Bigg[[I_{\mathsf{V}_i,V_{c_i}}] + m_{\mathsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}][\tilde{\boldsymbol{r}}_{V_{c_i}/B}]^T\Bigg]\dot{\boldsymbol{\omega}} + \hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\Bigg[[I_{\mathsf{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\mathsf{g}_i} \\
&\quad + [I_{\mathsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathsf{g}_i} + [P_i]\big(\ell_i\hat{\boldsymbol{g}}_{\mathsf{t}_i} - d_i c\theta_i\hat{\boldsymbol{g}}_{\mathsf{s}_i}\big) + [Q_i][\tilde{\hat{\boldsymbol{g}}}_{\mathsf{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i}\Bigg]\ddot{\gamma}_i + \hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\Bigg[[I_{\mathsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\mathsf{s}_i} + [P_i]d_i\hat{\boldsymbol{w}}_{3_i}\Bigg]\dot{\Omega}_i \\
&\qquad = -\hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\Bigg[\dot{\gamma}_i[Q_i][\tilde{\hat{\boldsymbol{g}}}_{\mathsf{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [P_i]\big[\big(2d_i\dot{\gamma}_i\Omega_i\mathsf{s}\theta_i - \ell_i\dot{\gamma}_i^2\big)\hat{\boldsymbol{g}}_{\mathsf{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\mathsf{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\big] \\
&\qquad\qquad + [I_{\mathsf{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathsf{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [I_{\mathsf{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\mathsf{t}_i} + [I_{\mathsf{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \\
&\qquad\qquad + [\tilde{\boldsymbol{\omega}}][I_{\mathsf{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathsf{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{G_{c_i}/V_{c_i}} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{G_{c_i}/V_{c_i}}\big) \\
&\quad + m_{\mathsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{W_{c_i}/V_{c_i}} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{W_{c_i}/V_{c_i}}\big) + m_{\mathsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{V_{c_i}/B} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{V_{c_i}/B}\big)\Bigg] + u_{\mathsf{g}_i}
\end{aligned}
\tag{25}
$$

Where,

$$[I_{\mathsf{V}_i,V_{c_i}}] = [I_{\mathsf{G}_i,V_{c_i}}] + [I_{\mathsf{W}_i,V_{c_i}}] \tag{26}$$

$$[I_{\mathsf{G}_i,V_{c_i}}] = [I_{\mathsf{G}_i,G_{c_i}}] + m_{\mathsf{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]^T \tag{27}$$

$$[I_{\mathsf{W}_i,V_{c_i}}] = [I_{\mathsf{W}_i,W_{c_i}}] + m_{\mathsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]^T \tag{28}$$

$$[P_i] = m_{\mathsf{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}] - m_{\mathsf{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}] + m_{\mathsf{W}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{29}$$

$$[Q_i] = m_{\mathsf{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}] - m_{\mathsf{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}] + m_{\mathsf{G}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{30}$$

$$[\tilde{\boldsymbol{\omega}}]^2 = [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \tag{31}$$

Removing all imbalance terms, (45) simplifies to the equation found in (1.3.1)

$$I_{\mathsf{V}_{g_i}}(\hat{\boldsymbol{g}}_{\mathsf{g}_i}^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}_i) = u_{\mathsf{g}_i} + (I_{\mathsf{V}_{s_i}} - I_{\mathsf{V}_{t_i}})\omega_s\omega_t + I_{\mathsf{W}_{s_i}}\Omega_i\omega_t \tag{32}$$

1.4.4 Wheel Torque Equation

The wheel torque equation is given by:

$$\begin{aligned}
&\left[m_{\mathsf{W}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\right] \ddot{\boldsymbol{r}}_{B/N} + \left[\hat{\boldsymbol{g}}_{\mathsf{s}_i}^T [I_{\mathsf{W}_i, W_{c_i}}] + m_{\mathsf{W}_i} d_i \hat{\boldsymbol{g}}_{\mathsf{s}_i}^T [\tilde{\hat{\boldsymbol{w}}}_{2_i}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T\right] \dot{\boldsymbol{\omega}} \\
&\qquad + \left[J_{12_i} s\theta_i + J_{13_i} c\theta_i - m_{\mathsf{W}_i} d_i \ell_i s\theta_i\right] \ddot{\gamma}_i + \left[J_{11_i} + m_{\mathsf{W}_i} d_i^2\right] \dot{\Omega}_i \\
&= -\hat{\boldsymbol{g}}_{\mathsf{s}_i}^T \Bigg[[I_{\mathsf{W}_i, W_{c_i}}]' \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathsf{W}_i, W_{c_i}}] \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathsf{W}_i} d_i [\tilde{\hat{\boldsymbol{w}}}_{2_i}] \Big[2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \boldsymbol{r}_{W_{c_i}/B} \Big] \Bigg] \\
&\qquad + (J_{13_i} s\theta_i - J_{12_i} c\theta_i) \Omega \dot{\gamma} - m_{\mathsf{W}_i} d_i^2 \dot{\gamma}_i^2 c\theta_i s\theta_i + u_{\mathsf{s}_i}
\end{aligned} \tag{33}$$

Removing imbalance terms gives (recall that for the simplified case $\theta_i = 0$),

$$I_{\mathsf{W}_{s_i}} (\hat{\boldsymbol{g}}_{\mathsf{s}_i}^T \dot{\boldsymbol{\omega}} + \dot{\Omega}_i) = -I_{\mathsf{W}_{s_i}} \omega_t \dot{\gamma}_i + u_{\mathsf{s}_i} \tag{34}$$

which agrees with (1.3.1).

1.4.5 Modified EOM for Back-Substitution

Translational EOM This aforementioned translation equation of motion may be rewritten to confirm with back-substitution requirements in the following way:

$$\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\boldsymbol{u}_{r_i}\ddot{\gamma}_i + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\boldsymbol{v}_{r_i}\dot{\Omega}_i = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c} - \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N}\boldsymbol{k}_{r_i} \tag{35}$$

where,

$$\boldsymbol{u}_{r_i} = m_{\mathsf{G}_i}[\tilde{\hat{\boldsymbol{g}}}_{\mathsf{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} - m_{\mathsf{W}_i}d_i c\theta_i\hat{\boldsymbol{g}}_{\mathsf{s}_i} + m_{\mathsf{W}_i}\ell_i\hat{\boldsymbol{g}}_{\mathsf{t}_i} \tag{36}$$

$$\boldsymbol{v}_{r_i} = m_{\mathsf{W}_i}d_i\hat{\boldsymbol{w}}_{3_i} \tag{37}$$

$$\boldsymbol{k}_{r_i} = m_{\mathsf{G}_i}\dot{\gamma}_i[\tilde{\hat{\boldsymbol{g}}}_{\mathsf{g}_i}]\boldsymbol{r}'_{G_{c_i}/B} + m_{\mathsf{W}_i}\left[\left(2d_i\dot{\gamma}_i\Omega_i\mathsf{s}\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\mathsf{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\mathsf{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\right] \tag{38}$$

$$\begin{aligned}\Bigg[m_{\text{sc}}[I_{3\times3}] + \sum_{i=1}^{N}\Bigg(\boldsymbol{u}_{r_i}\boldsymbol{a}_{\gamma_i}^T + \frac{\left(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\right)\left(\boldsymbol{a}_{\Omega_i}^T + c_{\Omega_i}\boldsymbol{a}_{\gamma_i}^T\right)}{1 - c_{\Omega_i}c_{\gamma_i}}\Bigg)\Bigg]\ddot{\boldsymbol{r}}_{B/N} \\ + \Bigg[-m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\Bigg(\boldsymbol{u}_{r_i}\boldsymbol{b}_{\gamma_i}^T + \frac{\left(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\right)\left(\boldsymbol{b}_{\Omega_i}^T + c_{\Omega_i}\boldsymbol{b}_{\gamma_i}^T\right)}{1 - c_{\Omega_i}c_{\gamma_i}}\Bigg)\Bigg]\dot{\boldsymbol{\omega}} \\ = \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}]^2\boldsymbol{c} - \sum_{i=1}^{N}\Bigg(\boldsymbol{k}_{r_i} + \boldsymbol{u}_{r_i}d_{\gamma_i} + \frac{\left(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\right)\left(c_{\Omega_i}d_{\gamma_i} + d_{\Omega_i}\right)}{1 - c_{\Omega_i}c_{\gamma_i}}\Bigg)\end{aligned} \tag{39}$$

Rotational EOM This equation may be abbreviated as,

$$\begin{aligned} m_{\textsf{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\textsf{sc},B}]\dot{\boldsymbol{\omega}} + \sum_{i=1}^{N}\boldsymbol{u}_{\omega_i}\ddot{\gamma}_i + \sum_{i=1}^{N}\boldsymbol{v}_{\omega_i}\dot{\Omega}_i \\ = \boldsymbol{L}_B - [I_{\textsf{sc},B}]'\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}][I_{\textsf{sc},B}]\boldsymbol{\omega} - \sum_{i=1}^{N}\boldsymbol{k}_{\omega_i} \end{aligned} \tag{40}$$

where,

$$\boldsymbol{u}_{\omega_i} = [I_{\textsf{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{g}_i} + m_{\textsf{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\textsf{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i} + [I_{\textsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{g}_i} + m_{\textsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}](\ell_i\hat{\boldsymbol{g}}_{\textsf{t}_i} - d_i c\theta_i\hat{\boldsymbol{g}}_{\textsf{s}_i}) \tag{41}$$

$$\boldsymbol{v}_{\omega_i} = [I_{\textsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{s}_i} + m_{\textsf{W}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i} \tag{42}$$

$$\begin{aligned} \boldsymbol{k}_{\omega_i} = {}& [I_{\textsf{G}_i,G_{c_i}}]'\dot{\gamma}_i\hat{\boldsymbol{g}}_{\textsf{g}_i} + [\tilde{\boldsymbol{\omega}}][I_{\textsf{G}_i,G_{c_i}}]\dot{\gamma}_i\hat{\boldsymbol{g}}_{\textsf{g}_i} + m_{\textsf{G}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{G_{c_i}/B}]\boldsymbol{r}'_{G_{c_i}/B} + m_{\textsf{G}_i}\dot{\gamma}_i[\tilde{\boldsymbol{r}}_{G_{c_i}/B}][\tilde{\hat{\boldsymbol{g}}}_{\textsf{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} \\ & + [I_{\textsf{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\textsf{t}_i} + [I_{\textsf{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} + [\tilde{\boldsymbol{\omega}}][I_{\textsf{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} + m_{\textsf{W}_i}[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B} \\ & + m_{\textsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\left[\left(2d_i\dot{\gamma}_i\Omega_i s\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\textsf{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\textsf{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\right] \end{aligned} \tag{43}$$

$$\begin{aligned} & \left[m_{\textsf{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\Big([I_{\textsf{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\textsf{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{a}_{\Omega_i}^T\right]\ddot{\boldsymbol{r}}_{B/N} \\ & \quad + \left[[I_{\textsf{sc},B}] + \sum_{i=1}^{N}\Big([I_{\textsf{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\textsf{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{b}_{\Omega_i}^T\right]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ & = \sum_{i=1}^{N}\Big[m_{\textsf{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\textsf{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\textsf{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\textsf{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big) \\ & \qquad - \Big([I_{\textsf{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\textsf{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)c_{\Omega_i}\Big] \\ & \qquad - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\textsf{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\textsf{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \end{aligned} \tag{44}$$

Gimbal Torque Equation The gimbal torque equation is,

$$\begin{aligned} & \hat{\boldsymbol{g}}_{\textsf{g}_i}^T\Big[m_{\textsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\Big]\ddot{\boldsymbol{r}}_{B/N} + \hat{\boldsymbol{g}}_{\textsf{g}_i}^T\Big[[I_{\textsf{V}_i,V_{c_i}}] + m_{\textsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}][\tilde{\boldsymbol{r}}_{V_{c_i}/B}]^T\Big]\dot{\boldsymbol{\omega}} + \hat{\boldsymbol{g}}_{\textsf{g}_i}^T\Big[[I_{\textsf{G}_i,G_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{g}_i} \\ & + [I_{\textsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{g}_i} + [P_i]\big(\ell_i\hat{\boldsymbol{g}}_{\textsf{t}_i} - d_i c\theta_i\hat{\boldsymbol{g}}_{\textsf{s}_i}\big) + [Q_i][\tilde{\hat{\boldsymbol{g}}}_{\textsf{g}_i}]\boldsymbol{r}_{G_{c_i}/G_i}\Big]\ddot{\gamma}_i + \hat{\boldsymbol{g}}_{\textsf{g}_i}^T\Big[[I_{\textsf{W}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{\textsf{s}_i} + [P_i]d_i\hat{\boldsymbol{w}}_{3_i}\Big]\dot{\Omega}_i \\ & = -\hat{\boldsymbol{g}}_{\textsf{g}_i}^T\Big[\dot{\gamma}_i[Q_i][\tilde{\hat{\boldsymbol{g}}}_{\textsf{g}_i}]\boldsymbol{r}'_{G_{c_i}/G_i} + [P_i]\big[\left(2d_i\dot{\gamma}_i\Omega_i s\theta_i - \ell_i\dot{\gamma}_i^2\right)\hat{\boldsymbol{g}}_{\textsf{s}_i} - d_i\dot{\gamma}_i^2 c\theta_i\hat{\boldsymbol{g}}_{\textsf{t}_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}\big] \\ & \quad + [I_{\textsf{G}_i,G_{c_i}}]'\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\textsf{G}_i,G_{c_i}}]\boldsymbol{\omega}_{\mathcal{G}_i/\mathcal{N}} + [I_{\textsf{W}_i,W_{c_i}}]\Omega\dot{\gamma}\hat{\boldsymbol{g}}_{\textsf{t}_i} + [I_{\textsf{W}_i,W_{c_i}}]'\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} \\ & \quad + [\tilde{\boldsymbol{\omega}}][I_{\textsf{W}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\textsf{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{G_{c_i}/V_{c_i}} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{G_{c_i}/V_{c_i}}\big) \\ & + m_{\textsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{W_{c_i}/V_{c_i}} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{W_{c_i}/V_{c_i}}\big) + m_{\textsf{V}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}]\big(2[\tilde{\boldsymbol{\omega}}]\boldsymbol{r}'_{V_{c_i}/B} + [\tilde{\boldsymbol{\omega}}]^2\boldsymbol{r}_{V_{c_i}/B}\big)\Big] + u_{\textsf{g}_i} \end{aligned} \tag{45}$$

where,

$$[I_{\mathsf{V}_i,V_{c_i}}] = [I_{\mathsf{G}_i,V_{c_i}}] + [I_{\mathsf{W}_i,V_{c_i}}] \tag{46}$$

$$[I_{\mathsf{G}_i,V_{c_i}}] = [I_{\mathsf{G}_i,G_{c_i}}] + m_{\mathsf{G}_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}]^T \tag{47}$$

$$[I_{\mathsf{W}_i,V_{c_i}}] = [I_{\mathsf{W}_i,W_{c_i}}] + m_{\mathsf{W}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}][\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}]^T \tag{48}$$

$$[P_i] = m_{\mathsf{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}] - m_{\mathsf{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}] + m_{\mathsf{W}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{49}$$

$$[Q_i] = m_{\mathsf{G}_i}\rho_{W_i}[\tilde{\boldsymbol{r}}_{G_{c_i}/V_{c_i}}] - m_{\mathsf{W}_i}\rho_{G_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/V_{c_i}}] + m_{\mathsf{G}_i}[\tilde{\boldsymbol{r}}_{V_{c_i}/G_i}] \tag{50}$$

$$[\tilde{\boldsymbol{\omega}}]^2 = [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \tag{51}$$

Wheel Torque Equation The wheel torque equation is,

$$
\begin{aligned}
&\left[m_{\mathrm{W}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\right] \ddot{\boldsymbol{r}}_{B/N} + \left[\hat{\boldsymbol{g}}_{\mathrm{s}_i}^T [I_{\mathrm{W}_i,W_{c_i}}] + m_{\mathrm{W}_i} d_i \hat{\boldsymbol{g}}_{\mathrm{s}_i}^T [\tilde{\hat{\boldsymbol{w}}}_{2_i}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T\right] \dot{\boldsymbol{\omega}} \\
&\qquad + \left[J_{12_i} s\theta_i + J_{13_i} c\theta_i - m_{\mathrm{W}_i} d_i \ell_i s\theta_i\right] \ddot{\gamma}_i + \left[J_{11_i} + m_{\mathrm{W}_i} d_i^2\right] \dot{\Omega}_i \\
&= -\hat{\boldsymbol{g}}_{\mathrm{s}_i}^T \left[[I_{\mathrm{W}_i,W_{c_i}}]' \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}][I_{\mathrm{W}_i,W_{c_i}}] \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} + m_{\mathrm{W}_i} d_i [\tilde{\hat{\boldsymbol{w}}}_{2_i}] \left[2[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}] \boldsymbol{r}_{W_{c_i}/B} \right] \right] \\
&\qquad + (J_{13_i} s\theta_i - J_{12_i} c\theta_i) \Omega \dot{\gamma} - m_{\mathrm{W}_i} d_i^2 \dot{\gamma}_i^2 c\theta_i s\theta_i + u_{\mathrm{s}_i} \qquad (52)
\end{aligned}
$$

1.4.6 Back-Substitution Matrices for Imbalanced VSCMG

The contributions are,

$$[A_{\text{contr}}] = \sum_{i=1}^{N} \Big[\boldsymbol{u}_{r_i}\boldsymbol{a}_{\gamma_i}^T + \big(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\big)\boldsymbol{p}_i^T\Big] \tag{53}$$

$$[B_{\text{contr}}] = \sum_{i=1}^{N} \Big[\boldsymbol{u}_{r_i}\boldsymbol{b}_{\gamma_i}^T + \big(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\big)\boldsymbol{q}_i^T\Big] \tag{54}$$

$$[C_{\text{contr}}] = \sum_{i=1}^{N} \Big[\boldsymbol{u}_{\omega_i}\boldsymbol{a}_{\gamma_i}^T + \big(\boldsymbol{v}_{\omega_i} + \boldsymbol{u}_{\omega_i}c_{\gamma_i}\big)\boldsymbol{p}_i^T\Big] \tag{55}$$

$$[D_{\text{contr}}] = \sum_{i=1}^{N} \Big[\boldsymbol{u}_{\omega_i}\boldsymbol{b}_{\gamma_i}^T + \big(\boldsymbol{v}_{\omega_i} + \boldsymbol{u}_{\omega_i}c_{\gamma_i}\big)\boldsymbol{q}_i^T\Big] \tag{56}$$

$$\boldsymbol{v}_{\text{trans,contr}} = -\sum_{i=1}^{N} \Big[\boldsymbol{k}_{r_i} + \boldsymbol{u}_{r_i}d_{\gamma_i} + \big(\boldsymbol{v}_{r_i} + \boldsymbol{u}_{r_i}c_{\gamma_i}\big)s_i\Big] \tag{57}$$

$$\boldsymbol{v}_{\text{rot,contr}} = -\sum_{i=1}^{N} \Big[\boldsymbol{k}_{\omega_i} + \boldsymbol{u}_{\omega_i}d_{\gamma_i} + \big(\boldsymbol{v}_{\omega_i} + \boldsymbol{u}_{\omega_i}c_{\gamma_i}\big)s_i\Big] \tag{58}$$

This concludes the derivation of the back-substitution of imbalanced VSCMGs.

This concludes the equations that are necessary to define the three different modes of the VSCMG. Reference[1] explains in further detail the EOMs for the simple-jitter and fully-coupled modes. Reference[3] gives more details on the derivation for balanced VSCMGs.

2 Model Functions

This model is used to approximate the behavior of a VSCMG. Below is a list of functions that this model performs:

- Compute it's contributions to the mass properties of the spacecraft
- Provides matrix contributions for the back substitution method
- Compute it's derivatives for θ and Ω
- Adds energy and momentum contributions to the spacecraft
- Convert commanded torque to applied torque. This takes into account friction, and minimum and maximum torque, and speed saturation
- Write output messages for states like Ω and applied torque

3 Model Assumptions and Limitations

Below is a summary of the assumptions/limitations:

- The reaction wheel is considered a rigid body
- The spin axis is body fixed, therefore does not take into account bearing flexing
- There is no error placed on the torque when converting from the commanded torque to the applied torque
- For balanced wheels and simple jitter mode the mass properties of the reaction wheels are assumed to be included in the mass and inertia of the rigid body hub, therefore there is zero contributions to the mass properties from the reaction wheels in the dynamics call.
- For fully-coupled imbalanced wheels mode the mass properties of the reaction wheels are assumed to not be included in the mass and inertia of the rigid body hub.
- For balanced wheels and simple jitter mode the inertia matrix is assumed to be diagonal with one of it's principle inertia axis equal to the spin axis, and the center of mass of the reaction wheel is coincident with the spin axis.
- For simple jitter, the parameters that define the static and dynamic imbalances are U_s and U_d.
- For fully-coupled imbalanced wheels the inertia off-diagonal terms, J_{12} and J_{23} are equal to zero and the remaining inertia off-diagonal term J_{13} is found through the setting the dynamic imbalance parameter U_d: $J_{13} = U_d$. The center of mass offset, d, is found using the static imbalance parameter U_s: $d = \frac{U_s}{m_{\text{rw}}}$
- The friction model is modeling static, Coulomb, and viscous friction. Other higher order effects of friction are not included.

- The speed saturation model only has one boundary, whereas in some reaction wheels once the speed boundary has been passed, the torque is turned off and won't turn back on until it spins down to another boundary. This model only can turn off and turn on the torque and the same boundary

4 Test Description and Success Criteria

The tests are located in `simulation/dynamics/VSCMGs/_UnitTest/test_VSCMGStateEffector_integrated.py` **and** `simulation/dynamics/VSCMGs/_UnitTest/ test_VSCMGStateEffector_ConfigureVSCMGRequests.py`. Depending on the test, there are different success criteria. These are outlined in the following subsections:

4.1 Balanced Wheels Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 VSCMGs attached to the spacecraft, and they are in "Balanced Wheels" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy
- Achieving the expected final attitude
- Achieving the expected final position

4.2 Simple Jitter Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 VSCMGs attached to the spacecraft, and they are in "Simple Jitter" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Achieving the expected final attitude
- Achieving the expected final position

4.3 Fully Coupled Jitter Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 VSCMGs attached to the spacecraft, and they are in "Fully Coupled Jitter" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy
- Achieving the expected final attitude
- Achieving the expected final position

4.4 Fully Coupled Jitter Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 VSCMGs attached to the spacecraft, and they are in "Fully Coupled Jitter" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Achieving the expected final attitude
- Achieving the expected final position

5 Test Parameters

Since this is an integrated test, the inputs to the test are the physical parameters of the spacecraft along with the initial conditions of the states. These parameters are outlined in Tables 2- 8. Additionally, the error tolerances can be seen in Table 9. The error tolerances are different depending on the test. The energy-momentum conservation values will normally have an agreement down to 1e-14, but to ensure cross-platform agreement the tolerance was chose to be 1e-10. The position and attitude checks have a tolerance set to 1e-7 and is because 8 significant digits were chosen as the values being compared to. The BOE tests depend on the integration time step but as the time step gets smaller the accuracy gets better. So 1e-8 tolerance was chosen so that a larger time step could be used but still show agreement. The Friction tests give the same numerical outputs down to 1e-15 between python and Basilisk, but 1e-10 was chosen to ensure cross platform agreement. Finally, the saturation and minimum torque tests have 1e-10 to ensure cross-platform success, but these values will typically agree to machine precision.

Table 2: Spacecraft Hub Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
mHub	mass	750.0	kg
IHubPntBc_B	Inertia in $\mathcal{B}$ frame	$\begin{bmatrix} 900.0 & 0.0 & 0.0 \\ 0.0 & 800.0 & 0.0 \\ 0.0 & 0.0 & 600.0 \end{bmatrix}$	kg-m^2
r_BcB_B	CoM Location in $\mathcal{B}$ frame	$\begin{bmatrix} -0.0002 & 0.0001 & 0.1 \end{bmatrix}^T$	m

Table 3: VSCMG 1 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\begin{bmatrix} 1.0 & 0.0 & 0.0 \end{bmatrix}^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$\begin{bmatrix} 0.1 & 0.0 & 0.0 \end{bmatrix}^T$	m

Table 4: VSCMG 2 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$[0.0 \quad 1.0 \quad 0.0]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.0 \quad 0.1 \quad 0.0]^T$	m

Table 5: VSCMG 3 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 1.0]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 0.1]^T$	m

Table 6: VSCMG 1 parameters for friction tests

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\left[\frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}\right]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.5 \quad -0.5 \quad 0.5]^T$	m

Table 7: VSCMG 2 parameters for friction tests

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\left[\frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}\right]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[-0.5 \quad 0.5 \quad -0.5]^T$	m

Table 8: Initial Conditions for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
(RW 1) OmegaInit	(RW 1) Initial Ω	500	RPM
(RW 2) OmegaInit	(RW 2) Initial Ω	200	RPM
(RW 3) OmegaInit	(RW 3) Initial Ω	-150	RPM
r_CN_NInit	Initial Position of S/C	$[-4020339 \quad 7490567 \quad 5248299]^T$	m
v_CN_NInit	Initial Velocity of S/C	$[-5199.78 \quad -3436.68 \quad 1041.58]^T$	m/s
sigma_BNInit	Initial MRP of $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 0.0]^T$	-
omega_BN_BInit	Initial Angular Velocity of $\mathcal{B}$ frame	$[0.08 \quad 0.01 \quad 0.0]^T$	rad/s

Table 9: Error Tolerance - Note: Relative Tolerance is abs($\frac{\text{truth}-\text{value}}{\text{truth}}$)

Test	Relative Tolerance
Energy and Momentum Conservation	1e-8
Position, Attitude Check	1e-8

6 Test Results

6.1 Balanced Wheels Scenario - Integrated Test Results

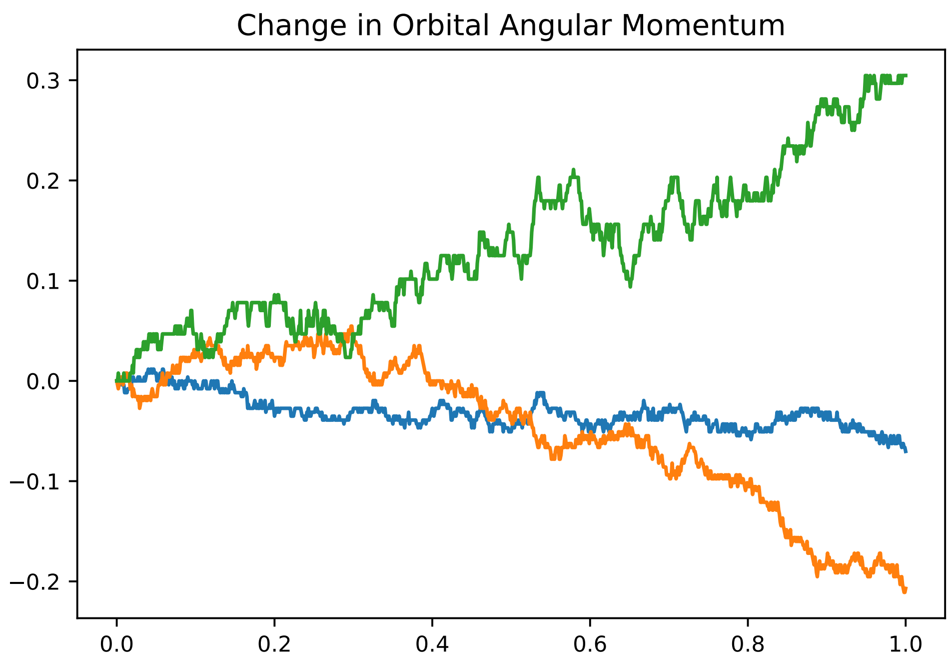

Fig. 2: Change in Orbital Angular Momentum BalancedWheels

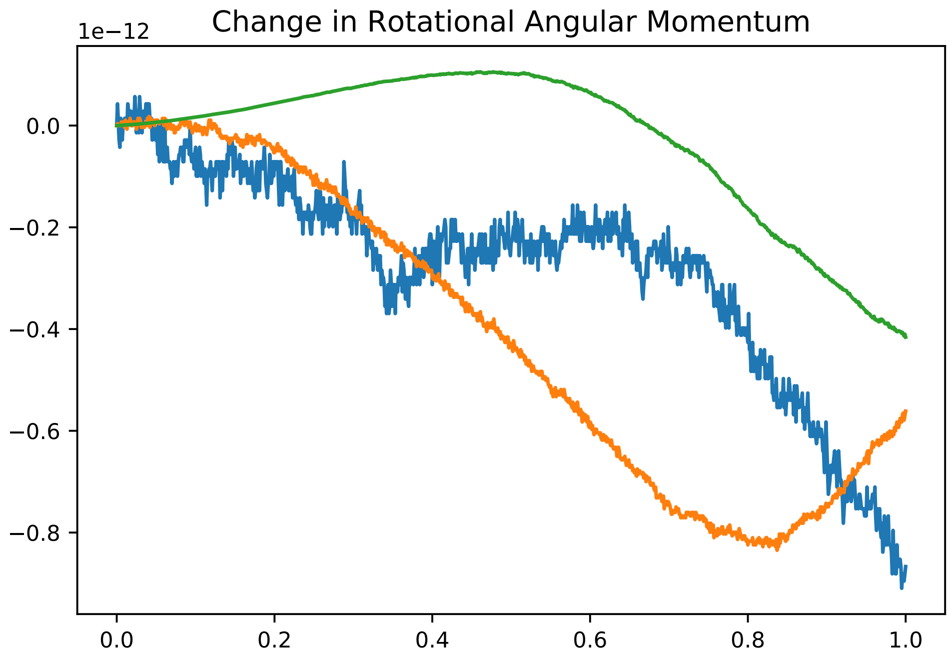

Fig. 3: Change in Orbital Energy BalancedWheels

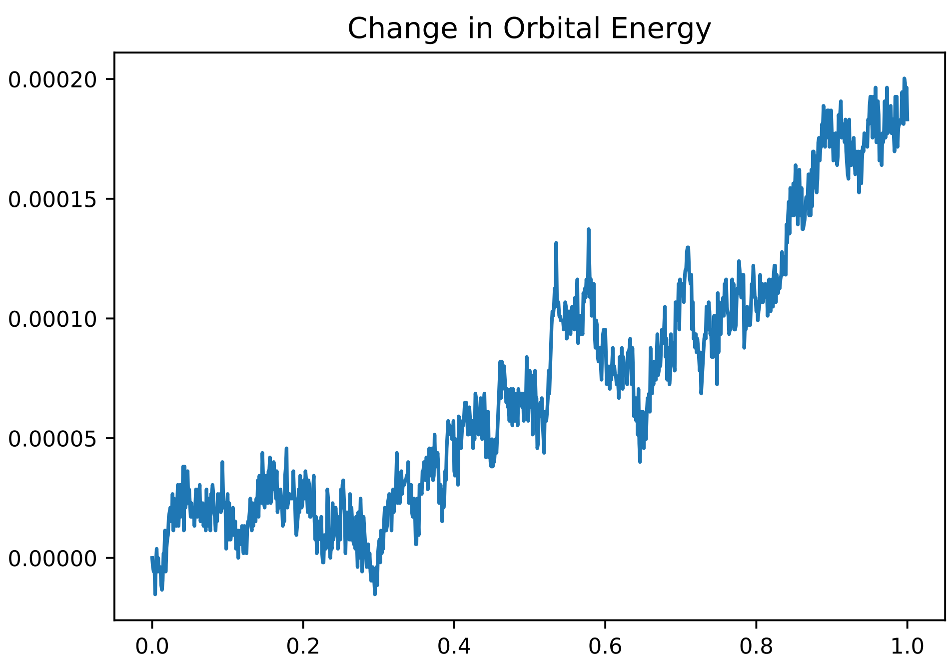

Fig. 4: Change in Rotational Angular Momentum BalancedWheels

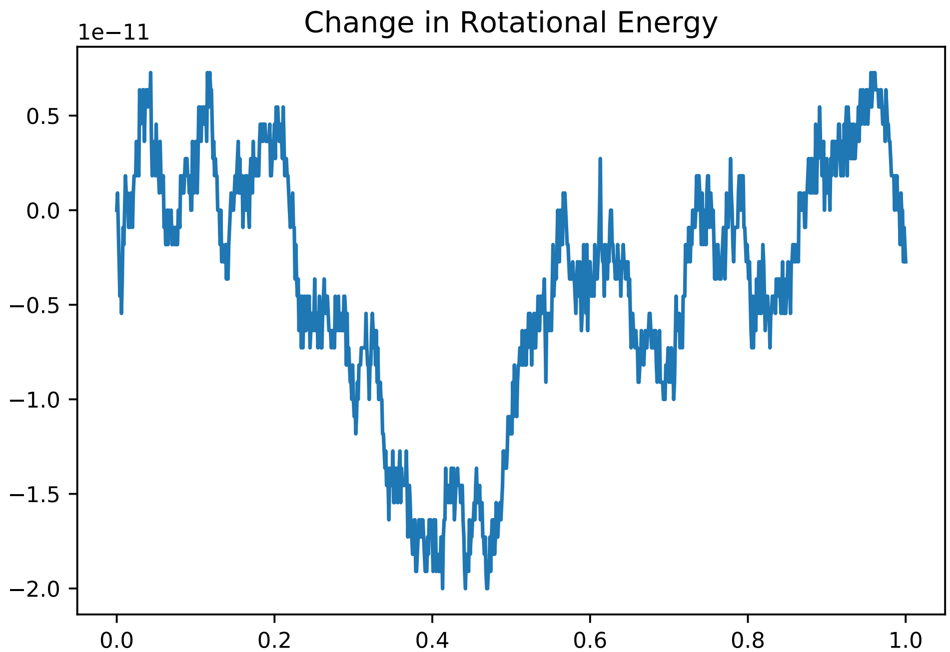

Fig. 5: Change in Rotational Energy BalancedWheels

6.2 Fully Coupled Jitter Scenario - Integrated Test Results

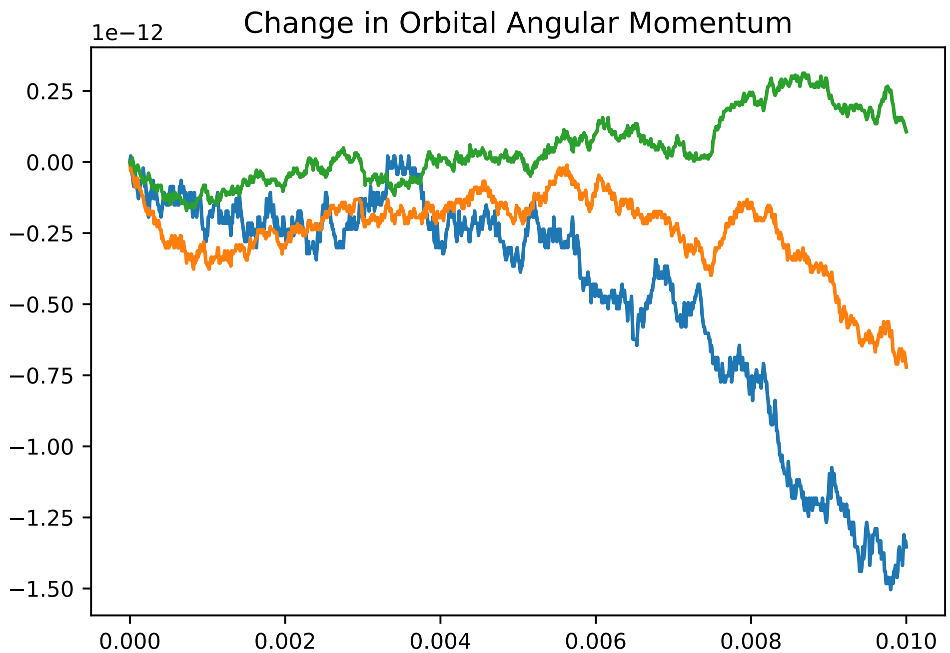

Fig. 6: Change in Orbital Angular Momentum JitterFullyCoupled

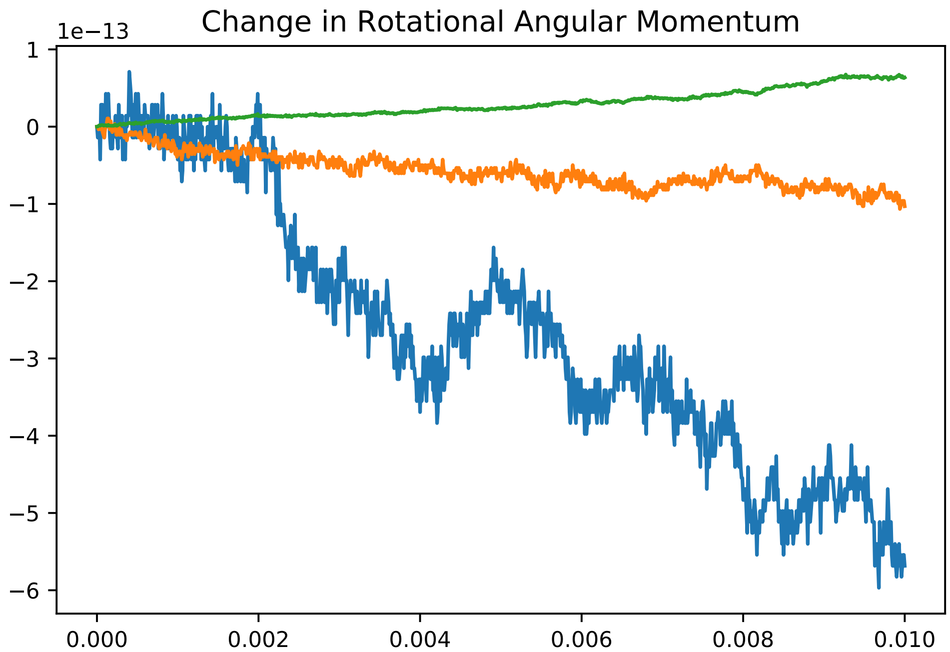

Fig. 7: Change in Orbital Energy JitterFullyCoupled

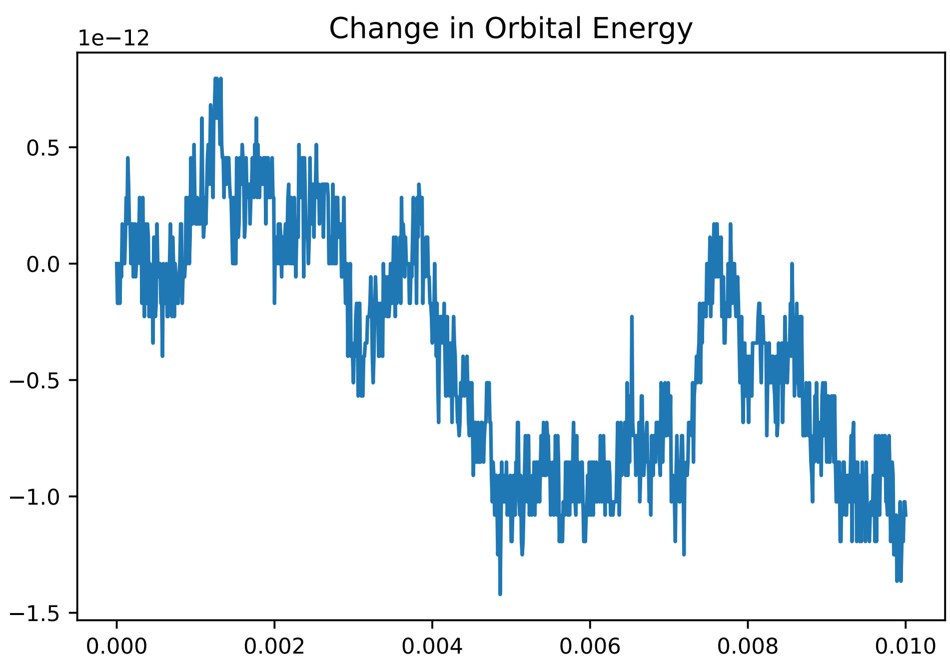

Fig. 8: Change in Rotational Angular Momentum JitterFullyCoupled

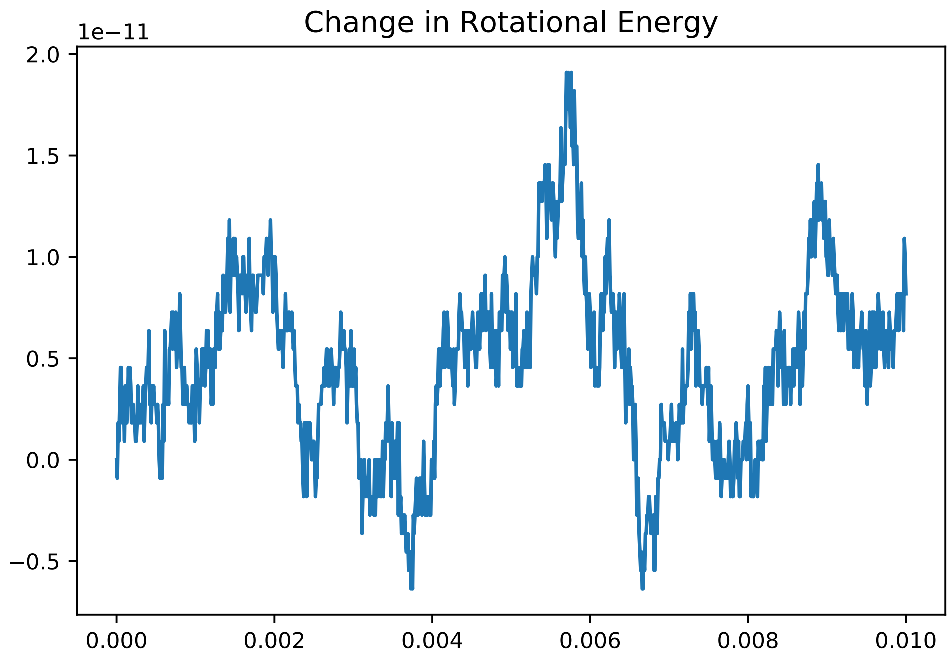

Fig. 9: Change in Rotational Energy JitterFullyCoupled

6.3 Fully Coupled Jitter with Gravity Scenario - Integrated Test Results

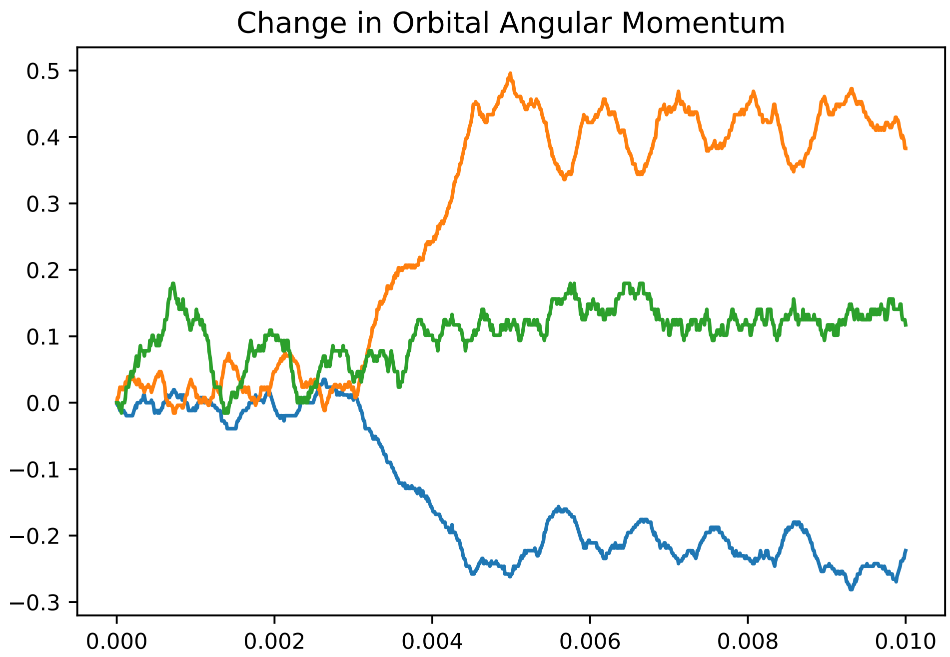

Fig. 10: Change in Orbital Angular Momentum JitterFullyCoupledGravity

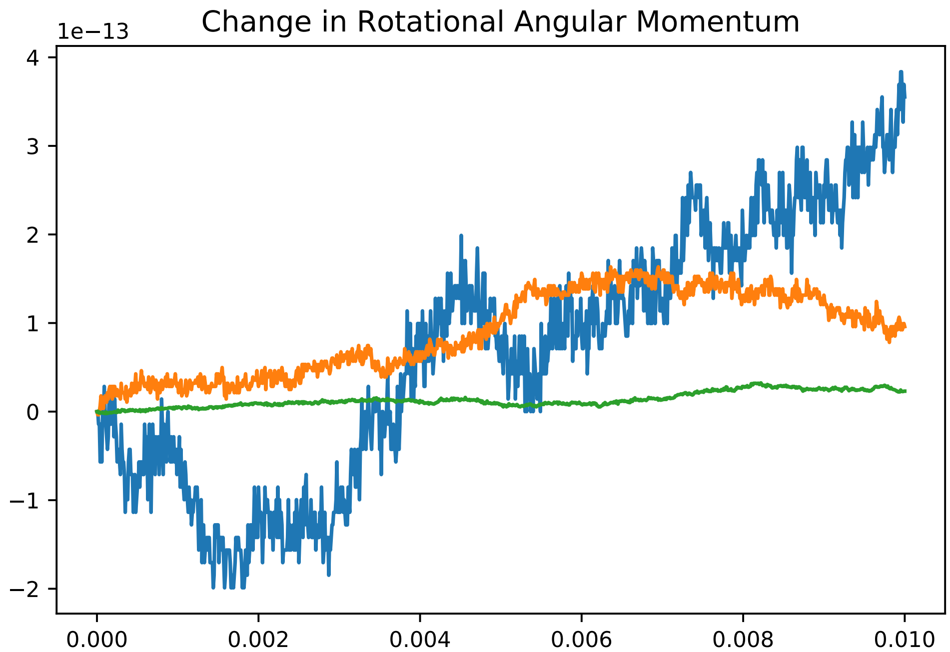

Fig. 11: Change in Orbital Energy JitterFullyCoupledGravity

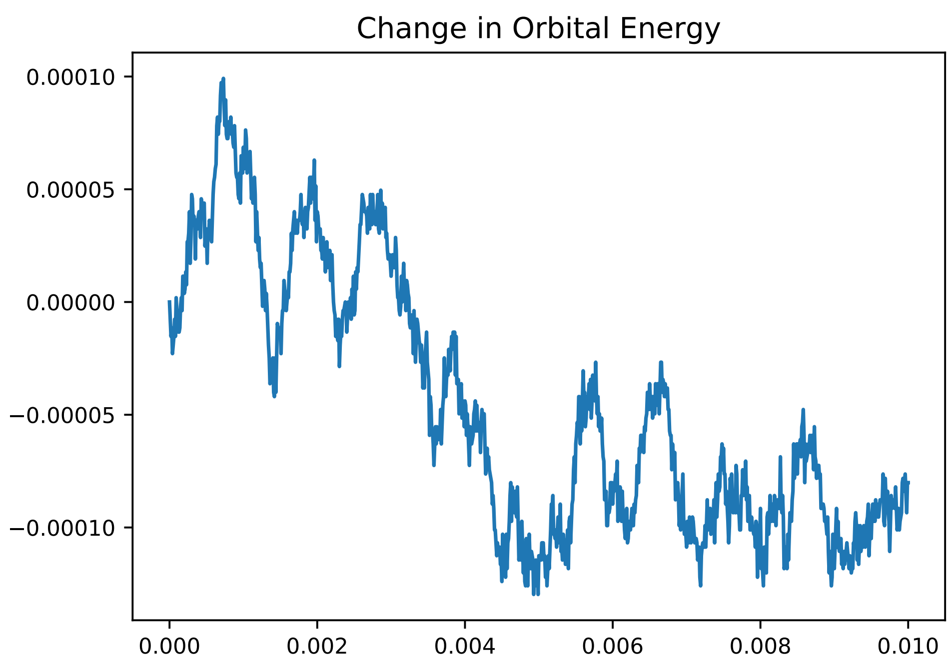

Fig. 12: Change in Rotational Angular Momentum JitterFullyCoupledGravity

6.4 Balanced Wheels, Simple Jitter, Fully Coupled Jitter and Fully Coupled Jitter with Gravity Tests Results

Table 10: Test results.

Test	Pass/Fail
Balanced Wheels	PASSED
Simple Jitter	PASSED
Fully Coupled Jitter	PASSED
Fully Coupled Jitter + Gravity	PASSED

7 User Guide

This section is to outline the steps needed to setup a reaction wheel state effector in python using Basilisk.

1. Import the vscmgStateEffector class and the spacecraftPlus class:
   import vscmgStateEffector and *import spacecraftPlus*
2. Define an instantiation of a vscmgObject:
   vscmgObject = vscmgStateEffector.vscmgStateEffector()
3. Set parameters for vscmgObject. A common set up might include:
   VSCMG.rGB_B = [[0.],[0.],[0.]]
   VSCMG.gsHat0_B = [[1.],[0.],[0.]]
   VSCMG.gtHat0_B = [[0.],[1.],[0.]]
   VSCMG.ggHat_B = [[0.],[0.],[1.]]
   *VSCMG.Omega_max = 6000. * macros.RPM*
   VSCMG.IW1 = 100./VSCMG.Omega_max
   *VSCMG.IW2 = 0.5*VSCMG.IW1*
   *VSCMG.IW3 = 0.5*VSCMG.IW1*
   VSCMG.IG1 = 0.1
   VSCMG.IG2 = 0.2
   VSCMG.IG3 = 0.3
   *VSCMG.U_s = 4.8e-06 * 1e4*
   *VSCMG.U_d = 1.54e-06 * 1e4*
   VSCMG.l = 0.01
   VSCMG.L = 0.1
   VSCMG.rGcG_G = [[0.0001],[-0.02],[0.1]]
   VSCMG.massW = 6.
   'VSCMG.massG = 6.
   VSCMG.VSCMGModel = 0
4. Create an instantiation of a spacecraftPlus:
   scObject = spacecraftPlus.SpacecraftPlus()
5. Finally, add the VSCMG object to your spacecraftPlus:
   rwFactory.addToSpacecraft("VSCMG", vscmgStateEffector, scObject). See spacecraftPlus documentation on how to set up a spacecraftPlus object.

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[2] H. Olsson, K.J. Å ström, C. Canudas de Wit, M. Gäfvert, and P. Lischinsky. Friction models and friction compensation. *European Journal of Control*, 4(3):176 – 195, 1998.

[3] Hanspeter Schaub and John L. Junkins. *Analytical Mechanics of Space Systems*. AIAA Education Series, Reston, VA, 3rd edition, 2014.