Autonomous Vehicle Simulation (AVS) Laboratory, University of Colorado

Basilisk Technical Memorandum
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DERIVATION OF REACTION WHEEL JITTER BACK-SUBSTITUTION

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Scope/Contents
This document derives the back substitution method for reaction wheel jitter to conform with Basilisk's dynamics architecture.

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1 Introduction

The goal is to manipulate the reaction wheel equations of motion to conform to the following form

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{\text{trans}} \\ \boldsymbol{v}_{\text{rot}} \end{bmatrix} \tag{1}$$

Solving the system-of-equations by

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \left([D] - [C][A]^{-1}[B]\right)^{-1} (\boldsymbol{v}_{\text{rot}} - [C][A]^{-1}\boldsymbol{v}_{\text{trans}}) \tag{2}$$

$$\ddot{\boldsymbol{r}}_{B/N} = [A]^{-1}(\boldsymbol{v}_{\text{trans}} - [B]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}) \tag{3}$$

2 Balanced Reaction Wheel Back-Substitution

2.1 Equations of Motion

The translational equation of motion is not coupled with $\dot{\Omega}$ as seen in the equation below.

$$m_{\text{sc}}[I_{3\times3}]\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{4}$$

The rotational equation of motion includes $\dot{\Omega}$ terms, and is thus coupled with wheel motion as seen below.

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N} J_{\mathsf{s}_i}\hat{\boldsymbol{g}}_{\mathsf{s}_i}\dot{\Omega}_i = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\mathsf{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\mathsf{s}_i}) + \boldsymbol{L}_B \tag{5}$$

The motor torque equation can be seen below.

$$\dot{\Omega}_i = \frac{u_{\mathsf{s}_i}}{J_{\mathsf{s}_i}} - \hat{\boldsymbol{g}}_{\mathsf{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{6}$$

2.2 Back-Substitution Derivation

Plugging Eq. (6) into Eq. (5)

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + ([I_{\text{sc},B}] - \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i})$$
$$- [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \quad (7)$$

2.3 Back-Substitution Contribution Matrices

The following can be defined:

$$[A_{\text{contr}}] = [0_{3\times3}] \tag{8}$$

$$[B_{\text{contr}}] = [0_{3\times3}] \tag{9}$$

$$[C_{\text{contr}}] = [0_{3\times3}] \tag{10}$$

$$[D_{\text{contr}}] = -\sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T \tag{11}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \boldsymbol{0} \tag{12}$$

$$\boldsymbol{v}_{\text{rot,contr}} = -\sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \tag{13}$$

3 Imbalanced Reaction Wheel Back-Substitution

3.1 Equations of Motion

The translational equation of motion is

$$\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}\dot{\Omega}_i = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}$$
$$(14)$$

The rotational equation of motion is

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N}\left([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\right)\dot{\Omega}_i$$
$$= \sum_{i=1}^{N}\left[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\right)\right]$$
$$- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B$$
$$(15)$$

The motor torque equation is

$$\left[m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T\right]\ddot{\boldsymbol{r}}_{B/N} + \left[(J_{11_i} + m_{\text{rw}_i}d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\right]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \left[J_{11_i} + m_{\text{rw}_i}d_i^2\right]\dot{\Omega}_i$$
$$= -J_{13_i}\omega_{w_{2_i}}\omega_{s_i} + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i}d_i^2) - m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \quad (16)$$

3.2 Derivation of Back-Substitution

$$\dot{\Omega}_i = -\left(\frac{m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T}{J_{11_i} + m_{\text{rw}_i} d_i^2}\right) \ddot{\boldsymbol{r}}_{B/N} - \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\left[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\right]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$+\frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\left(\omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i}\omega_{w_{2_i}}\omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}\right)$$
$$\tag{17}$$

$$\boldsymbol{a}_{\Omega_i} = -\frac{m_{\text{rw}_i} d_i}{J_{11_i} + m_{\text{rw}_i} d_i^2}\hat{\boldsymbol{w}}_{3_i} \tag{18}$$

$$\boldsymbol{b}_{\Omega_i} = -\frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\left[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i} + J_{13_i}\hat{\boldsymbol{w}}_{3_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{w}}_{3_i}\right] \tag{19}$$

$$c_{\Omega_i} = \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\left(\omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i}\omega_{w_{2_i}}\omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}\right)$$
$$\tag{20}$$

$$\dot{\Omega}_i = \boldsymbol{a}_{\Omega_i}^T \ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{b}_{\Omega_i}^T \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\Omega_i} \tag{21}$$

Plugging the equation above into Eq. (14) and multiplying both sides by m_{sc},

$$\left[m_{\text{sc}}[I_{3\times3}] + \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}\boldsymbol{a}_{\Omega_i}^T\right]\ddot{\boldsymbol{r}}_{B/N} + \left[-m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}\boldsymbol{b}_{\Omega_i}^T\right]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= \boldsymbol{F} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \sum_{i=1}^{N}\left[m_{\text{rw}_i} d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - m_{\text{rw}_i} d_i c_{\Omega_i}\hat{\boldsymbol{w}}_{3_i}\right] \tag{22}$$

Moving on to rotation...

$$\left[m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\left([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\right)\boldsymbol{a}_{\Omega_i}^T\right]\ddot{\boldsymbol{r}}_{B/N}$$
$$+ \left[[I_{\text{sc},B}] + \sum_{i=1}^{N}\left([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\right)\boldsymbol{b}_{\Omega_i}^T\right]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B$$
$$+ \sum_{i=1}^{N}\left[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\right)\right.$$
$$\left. - \left([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\right)c_{\Omega_i}\right] \tag{23}$$

3.3 Back-Substitution Contribution Matrices

$$[A_{\text{contr}}] = \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}\boldsymbol{a}_{\Omega_i}^T \tag{24}$$

$$[B_{\text{contr}}] = \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{b}_{\Omega_i}^T \tag{25}$$

$$[C_{\text{contr}}] = \sum_{i=1}^{N} \left([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \right) \boldsymbol{a}_{\Omega_i}^T \tag{26}$$

$$[D_{\text{contr}}] = \sum_{i=1}^{N} \left([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \right) \boldsymbol{b}_{\Omega_i}^T \tag{27}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \sum_{i=1}^{N} \left[m_{\text{rw}_i} d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - m_{\text{rw}_i} d_i c_{\Omega_i} \hat{\boldsymbol{w}}_{3_i} \right] \tag{28}$$

$$\boldsymbol{v}_{\text{rot,contr}} = \sum_{i=1}^{N} \left[m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i, W_{c_i}}]' \Omega_i \hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \left([I_{\text{rw}_i, W_{c_i}}] \Omega_i \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \boldsymbol{r}'_{W_{c_i}/B} \right) \right.$$
$$\left. - \left([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \right) c_{\Omega_i} \right] \tag{29}$$