Derivation of EOMs for $N_p + 1$ Connected Panels using Kane's Method

I Introduction

$\theta_i = \theta_{i,d} + \theta_{i,0}$ positive angle is in the upward direction.

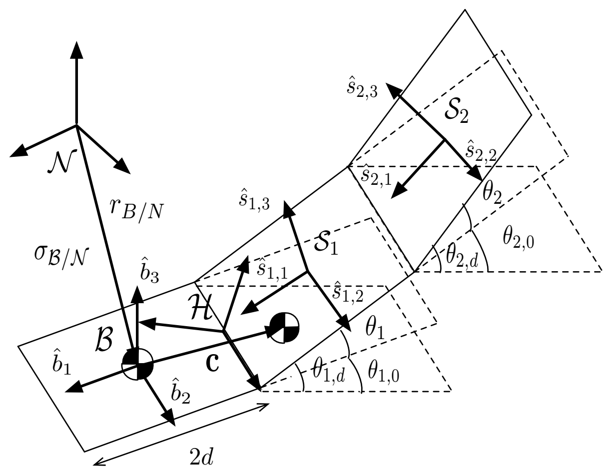

Figure 1: Frame and variable definitions for the system discussed here, where $N_p = 2$.

II Derivation of Equations of Motion - Kane's Method

The choice of state variables and their respective chosen generalized speeds are:

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{r}_{B/N} \\ \boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}} \\ \theta_1 \\ \cdot \\ \theta_{N_p} \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} \dot{\boldsymbol{r}}_{B/N} \\ \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ \dot{\theta}_1 \\ \cdot \\ \dot{\theta}_{N_p} \end{bmatrix} \tag{1}$$

To create the partial velocity table, some velocities first need to be defined

$$\dot{\boldsymbol{r}}_{B/N} = \dot{\boldsymbol{r}}_{B/N} \tag{2}$$

$$\dot{\boldsymbol{r}}_{C/N} = \dot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{c}} \tag{3}$$

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{4}$$

$$\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Big(\sum_{k=1}^{i} \dot{\theta}_k\Big)\hat{\boldsymbol{s}}_{i,2} \tag{5}$$

$$\boldsymbol{r}_{S_i/N} = \boldsymbol{r}_{B/N} + \boldsymbol{r}_{S_i/B} = \boldsymbol{r}_{B/N} + \boldsymbol{r}_{H/B} - [d\hat{\boldsymbol{s}}_{i,1} + \sum_{n=1}^{i-1} 2d\hat{\boldsymbol{s}}_{n,1}] \tag{6}$$

$$\dot{\boldsymbol{r}}_{S_i/N} = \dot{\boldsymbol{r}}_{B/N} + \boldsymbol{r}'_{S_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B} = \dot{\boldsymbol{r}}_{B/N} + d\Big[\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)\hat{\boldsymbol{s}}_{i,3} + \sum_{n=1}^{i-1} 2\hat{\boldsymbol{s}}_{n,3}\Big(\sum_{k=1}^{n}\dot{\theta}_k\Big)\Big] - [\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{7}$$

Where

$$\hat{\boldsymbol{s}}'_{i,j} = \boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{B}} \times \hat{\boldsymbol{s}}_{i,j} = \Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)\hat{\boldsymbol{s}}_{i,2} \times \hat{\boldsymbol{s}}_{i,j} \tag{8}$$

is used to get the derivative.
The summation in equation 6 and 7 can be out of bounds for certain values of i. When this occurs, the summation becomes equal to zero. $\boldsymbol{c}$ is defined as the position vector between the body frame and the COM of the system:

$$\boldsymbol{c} = \frac{1}{m_{sc}}\Big[\sum_{i=1}^{N_p} m_p \boldsymbol{r}_{S_i/B}\Big] \tag{9}$$

$$\dot{\boldsymbol{c}} = \boldsymbol{c}' - [\tilde{\boldsymbol{c}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{10}$$

$$\boldsymbol{c}' = \frac{m_p d}{m_{sc}} \sum_{i=1}^{N_p}\Big[\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)\hat{\boldsymbol{s}}_{i,3} + \sum_{n=1}^{i-1} 2\hat{\boldsymbol{s}}_{n,3}\Big(\sum_{k=1}^{n}\dot{\theta}_k\Big)\Big] \tag{11}$$

$$\boldsymbol{c}' = \frac{m_p d}{m_{sc}}\Big[\sum_{i=1}^{N_p}\dot{\theta}_i \sum_{n=i}^{N_p}(2[N_p - n] + 1)\hat{\boldsymbol{s}}_{n,3}\Big] \tag{12}$$

Now the following partial velocity table can be created (here: $j = r - 6$):

Table 1: Partial Velocity Table

r	$\boldsymbol{v}_r^C$	$\boldsymbol{v}_r^B$	$\boldsymbol{\omega}_r^{\mathcal{B}}$	$\boldsymbol{v}_r^{S_i}$	$\boldsymbol{\omega}_r^{\mathcal{S}_i}$
$1-3$	$[I_{3\times3}]$	$[I_{3\times3}]$	$[0_{3\times3}]$	$[I_{3\times3}]$	$[0_{3\times3}]$
$4-6$	$-[\tilde{\boldsymbol{c}}]$	$[0_{3\times3}]$	$[I_{3\times3}]$	$-[\tilde{\boldsymbol{r}}_{S_i/B}]$	$[I_{3\times3}]$
$7-N_p$	$\sum_{n=j}^{N_p}(2[N_p-n]+1)\hat{\boldsymbol{s}}_{n,3}$	$[0_{3\times1}]$	$[0_{3\times1}]$	if $j \leq i$: $d[\hat{\boldsymbol{s}}_{i,3} + \sum_{n=j}^{i-1} 2\hat{\boldsymbol{s}}_{n,3}]$ else: $[0_{3\times1}]$	if $j \leq i$: $\hat{\boldsymbol{s}}_{i,2}$ else: $[0_{3\times1}]$

Using these partial velocity definitions, the follow sections will step through the formulation for the translational, rotational and panel EOMs developed using Kane's method.

II.A Hub Translational Motion

Starting with the definition of a generalized force:

$$\boldsymbol{F}_r = \sum_{r}^{N} \boldsymbol{v}_r^T \boldsymbol{F} \tag{13}$$

Using this definition the external force applied on the system for the translational equations is defined as:

$$\boldsymbol{F}_{1-3} = [\boldsymbol{v}_{1-3}^C]^T \boldsymbol{F}_{\text{ext}} = \boldsymbol{F}_{\text{ext}} \tag{14}$$

Using the definition of generalized inertia forces,

$$\boldsymbol{F}_r^* = \sum_r^N \Big[\boldsymbol{\omega}_r^T \boldsymbol{T}^* + \boldsymbol{v}_r^T(-m_r \boldsymbol{a}_r)\Big] \tag{15}$$

the inertia forces for the hub translational motion are defined as

$$\boldsymbol{F}_{1-3}^* = [\boldsymbol{v}_{1-3}^B]^T(-m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N}) + \sum_{i=1}^{N_p}[\boldsymbol{v}_{1-3}^{S_i}]^T(-m_{\text{p}_i}\ddot{\boldsymbol{r}}_{S_i/N}) = -m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{i=1}^{N_p} -m_{\text{p}_i}\ddot{\boldsymbol{r}}_{S_i/N} \tag{16}$$

Finally, Kane's equation is:

$$\boldsymbol{F}_r + \boldsymbol{F}_r^* = 0 \tag{17}$$

therefore

$$\boldsymbol{F}_{\text{ext}} - m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{i=1}^{N_p} -m_{\text{p}_i}\ddot{\boldsymbol{r}}_{S_i/N} = 0 \tag{18}$$

Expanding and rearranging results in

$$m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{i=1}^{N_p} m_{\text{p}}(\ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{S_i/B}) = \boldsymbol{F}_{\text{ext}} \tag{19}$$

Where

$$\ddot{\boldsymbol{r}}_{S_i/B} = \boldsymbol{r}''_{S_i/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B}) \tag{20}$$

Plugging Eq. (20) into Eq. (19) results in

$$m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{i=1}^{N_p} m_{\text{p}}\Big[\ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{r}''_{S_i/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B})\Big] = \boldsymbol{F}_{\text{ext}} \tag{21}$$

The body frame derivative can be written explicitly using the simplification used in Eqs. 11 and 12 (this simplification only works when $\boldsymbol{r}''_{S_i/B}$ is summed over all panels)

$$m_{\text{p}}\sum_{i=1}^{N_p}\boldsymbol{r}''_{S_i/B} = \sum_{i=1}^{N_p}\Big[\ddot{\theta}_i\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3} + \Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)^2(2[N_p-i]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{i,1}\Big] \tag{22}$$

Combining like terms and rearranging results in

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}\Big]\ddot{\theta}_i = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} - \sum_{i=1}^{N_p}\Big[\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)^2(2[N_p-i]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{i,1}\Big] \tag{23}$$

II.B Hub Rotational Motion

The torque acting on the spacecraft, $\boldsymbol{L}_B$ needs to be defined as a general active force. Using Eq. (13) active forces acting on the spacecraft for the rotational equations can be defined as:

$$\boldsymbol{F}_{4-6} = [\boldsymbol{\omega}_{4-6}^{\mathcal{B}}]^T\boldsymbol{L}_B = \boldsymbol{L}_B \tag{24}$$

To define the generalized inertia forces, using Eq. (15) the definition of $\boldsymbol{T}^*$ needs to be defined for a rigid body:

$$\boldsymbol{T}^* = -[I_c]\dot{\boldsymbol{\omega}} - [\tilde{\boldsymbol{\omega}}][I_c]\boldsymbol{\omega} \tag{25}$$

$$\begin{aligned}
\boldsymbol{F}^*_{4-6} &= [\boldsymbol{\omega}^{\mathcal{B}}_{4-6}]^T\boldsymbol{T}^*_{\text{hub}} + [\boldsymbol{v}^{B}_{4-6}]^T(-m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N}) + \sum_{i=1}^{N_p}\Big([\boldsymbol{v}^{S_i}_{4-6}]^T(-m_{\text{p}}\ddot{\boldsymbol{r}}_{S_i/N}) + [\boldsymbol{\omega}^{\mathcal{S}_i}_{4\text{-}6}]^T\boldsymbol{T}^*_{\text{p}_i}\Big)\\
&= -[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{i}^{N_p}\Big(-m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\ddot{\boldsymbol{r}}_{S_i/N} - [I_{\text{p}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_{i/\mathcal{N}}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_{i/\mathcal{N}}}][I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{S}_{i/\mathcal{N}}}\Big)
\end{aligned} \tag{26}$$

Using Kane's equation, Eq. (17), the following equations of motion for the rotational dynamics are defined:

$$\begin{aligned}
&\boldsymbol{L}_B - [I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\
&\qquad + \sum_{i}^{N_p}\Big(-m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\ddot{\boldsymbol{r}}_{S_i/N} - [I_{\text{p}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_{i/\mathcal{N}}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_{i/\mathcal{N}}}][I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{S}_{i/\mathcal{N}}}\Big) = 0
\end{aligned} \tag{27}$$

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}_{i/\mathcal{N}}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{k=i}^{i}\ddot{\theta}_k\hat{\boldsymbol{s}}_{i,1} + \sum_{k=i}^{i}\dot{\theta}_k(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\hat{\boldsymbol{s}}_{i,2}) \tag{28}$$

Plugging Eq. (28) into Eq. (27)

$$\begin{aligned}
&\boldsymbol{L}_B - [I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\
&\quad + \sum_{i=1}^{N_p}\Big(-m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\ddot{\boldsymbol{r}}_{S_i/N} - [I_{\text{p}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - I_{s_{i,2}}\sum_{k=i}^{i}\ddot{\theta}_k\hat{\boldsymbol{s}}_{i,2} + [I_{\text{p}_i,S_i}]\sum_{k=i}^{i}\dot{\theta}_k(\hat{\boldsymbol{s}}_{i,2}\times\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}})\\
&\qquad - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{p}_i,S_i}]\big(\sum_{k=i}^{i}\dot{\theta}_k\big)\hat{\boldsymbol{s}}_{i,2} - \big(\sum_{k=i}^{i}\dot{\theta}_k\big)\hat{\boldsymbol{s}}_{i,2}\times[I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\Big) = 0
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\boldsymbol{L}_B - [I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}[I_{\text{p}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}I_{s_{i,2}}\sum_{k=i}^{i}\ddot{\theta}_k\hat{\boldsymbol{s}}_{i,2} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\\
&\quad + \sum_{i=1}^{N_p}\Big(-m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\Big[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{S_i/B}\Big] - I_{s_{i,2}}\sum_{k=i}^{i}\dot{\theta}_k[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\\
&\quad + \sum_{k=i}^{i}\dot{\theta}_k\Big[I_{s_{i,1}}\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T - I_{s_{i,3}}\hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T\Big]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{k=i}^{i}\dot{\theta}_k\Big[I_{s_{i,3}}\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T - I_{s_{i,1}}\hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T\Big]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\Big) = 0
\end{aligned} \tag{30}$$

$$\sum_{i=1}^{N_p}-m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{r}''_{S_i/B} = -m_{\text{p}}d\sum_{i=1}^{N_p}\Bigg(\ddot{\theta}\sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]\hat{\boldsymbol{s}}_{k,3} + \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]\hat{\boldsymbol{s}}_{i,1}\Bigg) \tag{31}$$

$$
\begin{aligned}
&\boldsymbol{L}_B - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} - [I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}[I_{\text{sp}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\ddot{\theta}_i\sum_{k=i}^{N_p} I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&+ \sum_{i=1}^{N_p}\Bigg(- \ddot{\theta}\sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{k,3} - \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{i,1} \\
&- m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\Big[2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B})\Big] - I_{s_{i,2}}\sum_{k=i}^{i}\dot{\theta}_k[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2} \\
&- \sum_{k=i}^{i}\dot{\theta}_k(I_{s_{i,3}} - I_{s_{i,1}})(\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T + \hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\Bigg) = 0 \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{L}_B - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} - [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p} I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{k,3}\Big]\ddot{\theta}_i - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&+ \sum_{i=1}^{N_p}\Bigg(- 2m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\Big[\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B}\Big] - \big(\sum_{k=1}^{i}\dot{\theta}_k\big)(I_{s_{i,3}} - I_{s_{i,1}})(\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T + \hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&- \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{i,1} - I_{s_{i,2}}\big(\sum_{k=1}^{i}\dot{\theta}_k\big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) = 0 \qquad (33)
\end{aligned}
$$

Moving things to the correct sides

$$
\begin{aligned}
&m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p} I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{k,3}\Big]\ddot{\theta}_i \\
&= -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Bigg(2m_{\text{p}}[\tilde{\boldsymbol{r}}_{S_i/B}]\Big[\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B}\Big] + \big(\sum_{k=1}^{i}\dot{\theta}_k\big)(I_{s_{i,3}} - I_{s_{i,1}})(\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T + \hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&+ \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{i,1} + I_{s_{i,2}}\big(\sum_{k=1}^{i}\dot{\theta}_k\big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) + \boldsymbol{L}_B \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
&m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p} I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{k,3}\Big]\ddot{\theta}_i \\
&= -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Bigg(-m_{\text{p}}\Big[[\tilde{\boldsymbol{r}}'_{S_i/B}][\tilde{\boldsymbol{r}}_{S_i/B}] + [\tilde{\boldsymbol{r}}_{S_i/B}][\tilde{\boldsymbol{r}}'_{S_i/B}]\Big]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \big(\sum_{k=1}^{i}\dot{\theta}_k\big)(I_{s_{i,3}} - I_{s_{i,1}})(\hat{\boldsymbol{s}}_{i,1}\hat{\boldsymbol{s}}_{i,3}^T + \hat{\boldsymbol{s}}_{i,3}\hat{\boldsymbol{s}}_{i,1}^T)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&+ m_{\text{p}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{r}'_{S_i/B} + \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{i,1} + I_{s_{i,2}}\big(\sum_{k=1}^{i}\dot{\theta}_k\big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) + \boldsymbol{L}_B \qquad (35)
\end{aligned}
$$

End

$$
\begin{aligned}
&m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p} I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \sum_{k=i}^{N_p}\Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{k,3}\Big]\ddot{\theta}_i = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\
&-[I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Bigg(m_{\text{p}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{r}'_{S_i/B} + \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big] m_{\text{p}} d\hat{\boldsymbol{s}}_{i,1} + I_{s_{i,2}}\big(\sum_{k=1}^{i}\dot{\theta}_k\big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) + \boldsymbol{L}_B \qquad (36)
\end{aligned}
$$

II.C Panel Motions

Following the similar pattern for translational and rotational equations the generalized active forces are defined, where $j = r - 6$:

$$\boldsymbol{F}_r = \boldsymbol{\omega}_r^{\mathcal{S}_j} \cdot (-k_j(\theta_j - \theta_{j,0})\hat{\boldsymbol{s}}_{j,2} - c_j\dot{\theta}_j\hat{\boldsymbol{s}}_{j,2} + k_{j+1}(\theta_{j+1} - \theta_{j+1,0})\hat{\boldsymbol{s}}_{j,2} + c_{j+1}\dot{\theta}_{j+1}\hat{\boldsymbol{s}}_{j+1,2}) \\ + 2d\hat{\boldsymbol{s}}_{j,3} \cdot (\boldsymbol{F}_{ext,j+1} - m_{conn,j}\ddot{\boldsymbol{r}}_{\boldsymbol{conn,j}}) = -k_j(\theta_j - \theta_{j,0}) - c_j\dot{\theta}_j + k_{j+1}(\theta_{j+1} - \theta_{j+1,0}) + c_{j+1}\dot{\theta}_{j+1} \\ + 2d\hat{\boldsymbol{s}}_{j,3} \cdot (\boldsymbol{F}_{ext,j+1} - m_{conn,j}\ddot{\boldsymbol{r}}_{\boldsymbol{conn,j}}) = K + 2d\hat{\boldsymbol{s}}_{j,3} \cdot (\boldsymbol{F}_{ext,j+1} - m_{conn,j}\ddot{\boldsymbol{r}}_{\boldsymbol{conn,j}}) \quad (37)$$

Where $m_{conn,j}$ and $\ddot{\boldsymbol{r}}_{\boldsymbol{conn,j}}$ are the mass and acceleration of the connected panels after the jth panel, and:

$$K = -k_j(\theta_j - \theta_{j,0}) - c_j\dot{\theta}_j + k_{j+1}(\theta_{j+1} - \theta_{j+1,0}) + c_{j+1}\dot{\theta}_{j+1} \quad (38)$$

The generalized inertia forces are defined as:

$$\boldsymbol{F}_r^* = \boldsymbol{\omega}_r^{\mathcal{S}_j} \cdot \boldsymbol{T}_{\mathrm{p}_j}^* + \boldsymbol{v}_r^{S_j} \cdot (-m_{\mathrm{p}}\ddot{\boldsymbol{r}}_{S_j/N}) = \\ \boldsymbol{\omega}_r^{\mathcal{S}_j} \cdot \Big[-[I_{\mathrm{p}_j,S_j}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{N}}][I_{\mathrm{p}_j,S_j}]\boldsymbol{\omega}_{\mathcal{S}_j/\mathcal{N}} \Big] + \boldsymbol{v}_r^{S_j} \cdot (-m_{\mathrm{p}}\ddot{\boldsymbol{r}}_{S_j/N}) \quad (39)$$

Using Kane's equation the following equations of motion are defined:

$$K + 2d\hat{\boldsymbol{s}}_{j,3} \cdot (\boldsymbol{F}_{ext,j+1} - m_{conn,j}\ddot{\boldsymbol{r}}_{\boldsymbol{conn,j}}) + \hat{\boldsymbol{s}}_{i,2} \cdot \Big[-[I_{\mathrm{p}_i,S_i}]\dot{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{S}_i/\mathcal{N}}][I_{\mathrm{p}_i,S_i}]\boldsymbol{\omega}_{\mathcal{S}_i/\mathcal{N}} \Big] \\ + d\hat{\boldsymbol{s}}_{j,3} \cdot (-m_{\mathrm{p}}\ddot{\boldsymbol{r}}_{S_j/N}) = 0 \quad (40)$$

Defining the inertial derivative:

$$\dot{\boldsymbol{\omega}}_{\mathcal{S}_j/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p} \ddot{\theta}_i\hat{\boldsymbol{s}}_{j,2} + \sum_{i=1}^{N_p} \dot{\theta}_i\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \hat{\boldsymbol{s}}_{j,2} \quad (41)$$

Which can be plugged into Eq. (40):

$$K - I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - I_{s_j,2}\sum_{i=1}^{N_p}\ddot{\theta}_i - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} \\ + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} - 2d\hat{\boldsymbol{s}}_{j,3}^T\sum_{i=j+1}^{N_p} m_{\mathrm{p}}\ddot{\boldsymbol{r}}_{S_i/N} - d\hat{\boldsymbol{s}}_{j,3}^T m_{\mathrm{p}}\ddot{\boldsymbol{r}}_{S_j/N} = 0 \quad (42)$$

$$K - I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - I_{s_j,2}\sum_{i=1}^{N_p}\ddot{\theta}_i - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} \\ + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} - 2d\hat{\boldsymbol{s}}_{j,3}^T\sum_{i=j+1}^{N_p} m_{\mathrm{p}}[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{S_i/B}] - d\hat{\boldsymbol{s}}_{j,3}^T m_{\mathrm{p}}[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{S_j/B}] = 0 \quad (43)$$

$$K - [d\hat{\boldsymbol{s}}_{j,3}^T + \sum_{i=j+1}^{N_p} 2d\hat{\boldsymbol{s}}_{j,3}^T]\ddot{\boldsymbol{r}}_{B/N} - I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - I_{s_j,2}\sum_{i=1}^{N_p}\ddot{\theta}_i - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} \\ + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} - 2d\hat{\boldsymbol{s}}_{j,3}^T\sum_{i=j+1}^{N_p} m_{\mathrm{p}}[\boldsymbol{r}''_{S_i/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_i/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_i/B})\Big] \\ - d\hat{\boldsymbol{s}}_{j,3}^T m_{\mathrm{p}}[\boldsymbol{r}''_{S_j/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{S_j/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_j/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{S_j/B})\Big] = 0 \quad (44)$$

$$
\begin{aligned}
K - \Big[dm_p\hat{\boldsymbol{s}}_{j,3}^T + \sum_{i=j+1}^{N_p} 2dm_p\hat{\boldsymbol{s}}_{j,3}^T\Big]\ddot{\boldsymbol{r}}_{B/N} - \Big[I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T - m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}] - \sum_{i=j+1}^{N_p} 2m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \\
I_{s_j,2}\sum_{i=1}^{N_p}\ddot{\theta}_i - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} - m_p d\hat{s}_{j,3}^T[\boldsymbol{r}''_{S_j/B} + \sum_{i=1}^{N_p} 2\boldsymbol{r}''_{S_i/B}] \\
- m_p d\hat{s}_{j,3}^T\Big[2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B} + \sum_{i=j+1}^{N_p}\Big(4[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}\Big)\Big] = 0
\end{aligned}
\tag{45}
$$

The $\boldsymbol{r}''_{S_j/B}$ terms contain $\ddot{\theta}$ terms and thus need to be rewritten to a usable form. This is done by writing it out for several panels and finding a pattern, the result of this is shown next:

$$
\begin{aligned}
K - \Big[dm_p\hat{\boldsymbol{s}}_{j,3}^T + \sum_{i=j+1}^{N_p} 2dm_p\hat{\boldsymbol{s}}_{j,3}^T\Big]\ddot{\boldsymbol{r}}_{B/N} - \Big[I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T - m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}] - \sum_{i=j+1}^{N_p} 2m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \\
I_{s_j,2}\sum_{i=1}^{N_p}\ddot{\theta}_i - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} \\
- m_p d\hat{s}_{j,3}^T\Big[2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B} + \sum_{i=j+1}^{N_p}\Big(4[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}\Big)\Big] \\
- m_p d^2\hat{s}_{j,3}^T\sum_{i=1}^{N_p}\Big[\ddot{\theta}_i\sum_{k=i}^{N_p}(2\hat{s}_{k,3} + 4\hat{s}_{k,3}(N_p - j) - H[k-j]4\hat{s}_{k,3}(k-j)) - H[j-i]\hat{s}_{j,3} + \\
\Big(\sum_{n=1}^{i}\dot{\theta}\Big)^2(2\hat{s}_{i,1} + 4\hat{s}_{i,1}(N_p - j) - H[i-j]4\hat{s}_{i,1}(i-j) - \Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2\hat{s}_{j,1})\Big] = 0
\end{aligned}
\tag{46}
$$

This finally leads to:

$$
\begin{aligned}
\Big[dm_p\hat{\boldsymbol{s}}_{j,3}^T + \sum_{i=j+1}^{N_p} 2dm_p\hat{\boldsymbol{s}}_{j,3}^T\Big]\ddot{\boldsymbol{r}}_{B/N} + \Big[I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T - m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}] - \sum_{i=j+1}^{N_p} 2m_p d\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \\
\sum_{i=1}^{N_p}\Big[I_{s_j,2}H[j-i] + m_p d^2\hat{s}_{j,3}^T\sum_{k=i}^{N_p}(2\hat{s}_{k,3} + 4\hat{s}_{k,3}(N_p - j) - H[k-j]4\hat{s}_{k,3}(k-j)) - H[j-i]\hat{s}_{j,3}\Big]\ddot{\theta}_i \\
= K + 2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1} - \left(I_{s_{j,1}} - I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}} - m_p d\hat{s}_{j,3}^T\Big[2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B} + \\
\sum_{i=j+1}^{N_p}\Big(4[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B} + 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}\Big) + \Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2(2\hat{s}_{i,1} + 4\hat{s}_{i,1}(N_p - j) - H[i-j]4\hat{s}_{i,1}(i-j) - \Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2\hat{s}_{j,1})\Big]
\end{aligned}
\tag{47}
$$

Where $H[x]$ is the Heaviside function.

III Back Substitution Method

The Back substitution method is used to gain a simpler expression that combines the three equations of motion. First, Eq. (47) is rearranged so that the second order state variables for the panel motions are isolated on the left hand side:

$$\sum_{i=1}^{N_p}\Big[I_{s_j,2}H[j-i]+m_pd^2\hat{s}_{j,3}^T\sum_{k=i}^{N_p}(2\hat{s}_{k,3}+4\hat{s}_{k,3}(N_p-j)-H[k-j]4\hat{s}_{k,3}(k-j))-H[j-i]\hat{s}_{j,3}\Big]\ddot{\theta}_i =$$
$$-\Big[dm_p\hat{\boldsymbol{s}}_{j,3}^T+\sum_{i=j+1}^{N_p}2dm_p\hat{\boldsymbol{s}}_{j,3}^T\Big]\ddot{\boldsymbol{r}}_{B/N}-\Big[I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T-m_pd\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]-\sum_{i=j+1}^{N_p}2m_pd\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}+$$
$$K+2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1}-\left(I_{s_{j,1}}-I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}}-m_pd\hat{s}_{j,3}^T\Big[2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B}+[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}+$$
$$\sum_{i=j+1}^{N_p}\Big(4[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B}+2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}\Big)+\Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2(2\hat{s}_{i,1}+4\hat{s}_{i,1}(N_p-j)-H[i-j]4\hat{s}_{i,1}(i-j)-\Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2\hat{s}_{j,1})\Big] \tag{48}$$

Now, defining the elements of a matrix $[A]$ as:

$$a_{j,i}=I_{s_j,2}H[j-i]+m_pd^2\hat{s}_{j,3}^T\sum_{k=i}^{N_p}\Big(2\hat{s}_{k,3}+4\hat{s}_{k,3}(N_p-j)-H[k-j]4\hat{s}_{k,3}(k-j)\Big)-H[j-i]\hat{s}_{j,3} \tag{49}$$

And defining the row elements of a matrix $[F]$ as:

$$\boldsymbol{f}_{j,1}=-[dm_p\hat{\boldsymbol{s}}_{j,3}^T+\sum_{i=j+1}^{N_p}2dm_p\hat{\boldsymbol{s}}_{j,3}^T] \tag{50}$$

With a matrix $[G]$ which has row elements defined as:

$$\boldsymbol{g}_{j,1}=-[I_{s_j,2}\hat{\boldsymbol{s}}_{j,2}^T-m_pd\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}]-\sum_{i=j+1}^{N_p}2m_pd\hat{s}_{j,3}^T[\tilde{\boldsymbol{r}}_{S_i/B}] \tag{51}$$

Also defining the vector $\boldsymbol{v}$

$$\boldsymbol{v}_{j,1}=K+2d\hat{\boldsymbol{s}}_{j,3}^T\boldsymbol{F}_{ext,j+1}-\left(I_{s_{j,1}}-I_{s_{j,3}}\right)\omega_{s_{j,3}}\omega_{s_{j,1}}-m_pd\hat{s}_{j,3}^T\Big[2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B}+[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}+$$
$$\sum_{i=j+1}^{N_p}\Big(4[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{S_j/B}+2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{S_j/B}\Big)+\Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2(2\hat{s}_{i,1}+4\hat{s}_{i,1}(N_p-j)-H[i-j]4\hat{s}_{i,1}(i-j)-\Big(\sum_{n=1}^{i}\dot{\theta}_i\Big)^2\hat{s}_{j,1})\Big] \tag{52}$$

Eq. (48) can then be written in matrix form to utilize some linear algebra techniques.

$$[A]\begin{bmatrix}\ddot{\theta}_1\\ \vdots\\ \ddot{\theta}_{N_p}\end{bmatrix}=[F]\ddot{\boldsymbol{r}}_{B/N}+[G]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}+\boldsymbol{v} \tag{53}$$

Eq. (53) can now be solved by inverting matrix $[A]$. Note the definition $[E]=[A]^{-1}$.

$$\begin{bmatrix}\ddot{\theta}_1\\ \vdots\\ \ddot{\theta}_{N_p}\end{bmatrix}=[E][F]\ddot{\boldsymbol{r}}_{B/N}+[E][G]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}+[E]\boldsymbol{v} \tag{54}$$

And the subcomponents of $[E]$ are defined as

$$[E]=\begin{bmatrix}\boldsymbol{e}_1^T\\ \vdots\\ \boldsymbol{e}_{N_p}^T\end{bmatrix} \tag{55}$$

Since the modified Euler's equation, Eq. (36), has $\ddot{\theta}_i$ terms, it is more convenient to use the expression for $\ddot{\theta}_i$ as

$$\ddot{\theta}_i = e_i^T[F]\ddot{\boldsymbol{r}}_{B/N} + e_i^T[G]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + e_i^T\boldsymbol{v} \tag{56}$$

The next step in the back substitution method is to analytically substitute Eq. (56) into the translational and rotational EOMs. Performing this substitution for translation yields:

$$\begin{aligned} m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}[e_i^T[F]\ddot{\boldsymbol{r}}_{B/N} + e_i^T[G]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + e_i^T\boldsymbol{v}]\Big] = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' \\ - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} - \sum_{i=1}^{N_p}\Big[\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)^2(2[N_p-i]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{i,1}\Big] \end{aligned} \tag{57}$$

Combining like terms yields:

$$\begin{aligned} \Bigg\{m_{\text{sc}}[I_{3\times3}] + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}\Big]e_i^T[F]\Bigg\}\ddot{\boldsymbol{r}}_{B/N} + \Bigg\{-m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}\Big]e_i^T[G]\Bigg\}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} - \sum_{i=1}^{N_p}\Big[\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)^2(2[N_p-i]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{i,1}\Big] \\ - \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}e_i^T\boldsymbol{v}\Big] \end{aligned} \tag{58}$$

Substitution into the rotational equation of motion:

$$\begin{aligned} m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{k,3})[e_i^T[F]\ddot{\boldsymbol{r}}_{B/N} + e_i^T[G]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + e_i^T\boldsymbol{v}]\Big] \\ = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Bigg(m_{\text{p}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{r}'_{S_i/B} + \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{i,1} \\ + I_{s_{i,2}}\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) + \boldsymbol{L}_B \end{aligned} \tag{59}$$

And combining like terms yields:

$$\begin{aligned} \Bigg\{m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{k,3})e_i^T[F]\Big]\Bigg\}\ddot{\boldsymbol{r}}_{B/N} \\ + \Bigg\{[I_{\text{sc},B}] + \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{k,3})e_i^T[G]\Big]\Bigg\}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N_p}\Bigg(m_{\text{p}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{S_i/B}]\boldsymbol{r}'_{S_i/B} + \Big(\sum_{k=1}^{i}\dot{\theta}\Big)^2\Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{i,1} \\ + I_{s_{i,2}}\Big(\sum_{k=1}^{i}\dot{\theta}_k\Big)[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\hat{\boldsymbol{s}}_{i,2}\Bigg) + \boldsymbol{L}_B - \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(I_{s_{k,2}}\hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p}2[\tilde{\boldsymbol{r}}_{S_n/B}]\Big]m_{\text{p}}d\hat{\boldsymbol{s}}_{k,3})e_i^T\boldsymbol{v}\Big] \end{aligned} \tag{60}$$

With the following definitions:

$$[A_{\text{contr}}] = \sum_{i=1}^{N_p}\Big[\sum_{k=i}^{N_p}(2[N_p-k]+1)dm_{\text{p}}\hat{\boldsymbol{s}}_{k,3}\Big]e_i^T[F] \tag{61}$$

$$[B_{\text{contr}}] = \sum_{i=1}^{N_p} \Big[\sum_{k=i}^{N_p} (2[N_p - k] + 1) d m_{\text{p}} \hat{\boldsymbol{s}}_{k,3} \Big] e_i^T [G] \tag{62}$$

$$\boldsymbol{v}_{\text{trans,contr}} = -\sum_{i=1}^{N_p} \Big[\Big(\sum_{k=1}^{i} \dot{\theta}_k \Big)^2 (2[N_p - i] + 1) d m_{\text{p}} \hat{\boldsymbol{s}}_{i,1} \Big] - \sum_{i=1}^{N_p} \Big[\sum_{k=i}^{N_p} (2[N_p - k] + 1) d m_{\text{p}} \hat{\boldsymbol{s}}_{k,3} e_i^T \boldsymbol{v} \Big] \tag{63}$$

$$[C_{\text{contr}}] = \sum_{i=1}^{N_p} \Big[\sum_{k=i}^{N_p} (I_{s_{k,2}} \hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2 [\tilde{\boldsymbol{r}}_{S_n/B}] \Big] m_{\text{p}} d \hat{\boldsymbol{s}}_{k,3}) \Big] e_i^T [F] \tag{64}$$

$$[D_{\text{contr}}] = \sum_{i=1}^{N_p} \Big[\sum_{k=i}^{N_p} (I_{s_{k,2}} \hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2 [\tilde{\boldsymbol{r}}_{S_n/B}] \Big] m_{\text{p}} d \hat{\boldsymbol{s}}_{k,3}) \Big] e_i^T [G] \tag{65}$$

$$\begin{aligned}[v_{\text{rot,contr}}] = -\sum_{i=1}^{N_p} \Bigg(m_{\text{p}} [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] [\tilde{\boldsymbol{r}}_{S_i/B}] \boldsymbol{r}'_{S_i/B} + \Big(\sum_{k=1}^{i} \dot{\theta} \Big)^2 \Big[[\tilde{\boldsymbol{r}}_{S_i/B}] + \sum_{n=i+1}^{N_p} 2 [\tilde{\boldsymbol{r}}_{S_n/B}] \Big] m_{\text{p}} d \hat{\boldsymbol{s}}_{i,1} + I_{s_{i,2}} \Big(\sum_{k=1}^{i} \dot{\theta}_k \Big) [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \hat{\boldsymbol{s}}_{i,2} \Bigg) \\ - \sum_{i=1}^{N_p} \Big[\sum_{k=i}^{N_p} (I_{s_{k,2}} \hat{\boldsymbol{s}}_{k,2} + \Big[[\tilde{\boldsymbol{r}}_{S_k/B}] + \sum_{n=k+1}^{N_p} 2 [\tilde{\boldsymbol{r}}_{S_n/B}] \Big] m_{\text{p}} d \hat{\boldsymbol{s}}_{k,3}) \Big] e_i^T \boldsymbol{v} \end{aligned} \tag{66}$$

The full back substitution matrices then become:

$$[A] = m_{\text{sc}} [I_{3\times 3}] + [A_{\text{contr}}] \tag{67}$$

$$[B] = -m_{\text{sc}} [\tilde{\boldsymbol{c}}] + [B_{\text{contr}}] \tag{68}$$

$$\boldsymbol{v}_{\text{trans}} = \boldsymbol{F} - 2 m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \boldsymbol{c}' - m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \boldsymbol{c} + \boldsymbol{v}_{\text{trans,contr}} \tag{69}$$

$$[C] = m_{\text{sc}} + [C_{\text{contr}}] \tag{70}$$

$$[D] = [I_{\text{sc},B}] + [D_{\text{contr}}] \tag{71}$$

$$\boldsymbol{v}_{\text{rot}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] [I_{\text{sc},B}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B + \boldsymbol{v}_{\text{rot,contr}} \tag{72}$$

This produces the following simplified equations:

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_{\text{trans}} \\ \boldsymbol{v}_{\text{rot}} \end{bmatrix} \tag{73}$$

Solving the system-of-equations by

$$\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \Big([D] - [C]][A]^{-1}[B] \Big)^{-1} (\boldsymbol{v}_{\text{rot}} - [C][A]^{-1} \boldsymbol{v}_{\text{trans}}) \tag{74}$$

$$\ddot{\boldsymbol{r}}_{B/N} = [A]^{-1} (\boldsymbol{v}_{\text{trans}} - [B] \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}) \tag{75}$$

Now Eq. (74) and (75) can be used to solve for $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and $\ddot{\boldsymbol{r}}_{B/N}$. Once these second order state variables are solved for, Eq. (56) can be used to directly solve for $\ddot{\theta}_i$. This shows that the back substitution method can work seamlessly for interconnected bodies.