Autonomous Vehicle Simulation (AVS) Laboratory, University of Colorado

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LINEAR FUEL SLOSH DYNAMICS MODEL

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Scope/Contents
The fuel slosh class is an instantiation of the state effector abstract class. The integrated test is validating the interaction between the fuel slosh module and the rigid body hub that it is attached to. The fuel slosh model has a relationship with the fuel tank model because the amount of decreasing mass of the fuel slosh particle is dependent on the remaining mass in the fuel tank, and vice-versa. This fuel slosh model is being approximated as a linear spring mass damper system attached to the spacecraft, however there can be general number of fuel slosh particles attached to the spacecraft and can be oriented in any directions with the respect to the rigid body hub. This document outlines the analytical development of the model and describes the efforts completed to validate/verify the model.

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1.0	Initial Draft	C. Allard	20171221

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1 Model Description

1.1 Derivation of Equations of Motion - Newtonian Mechanics

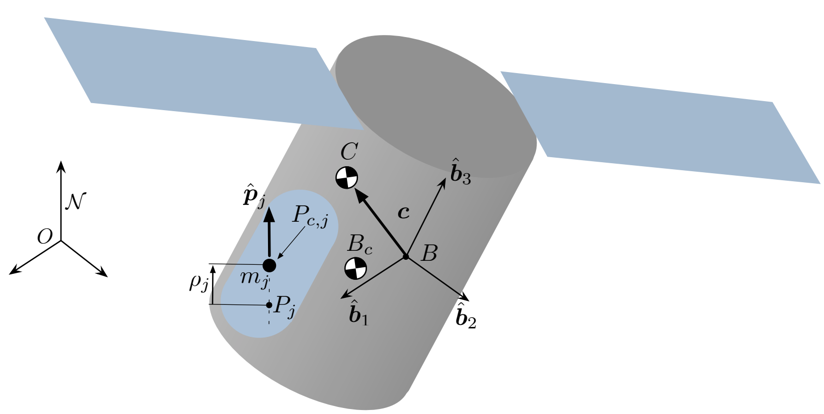

Fig. 1: Frame and variable definitions used for formulation

1.1.1 Rigid Spacecraft Hub Translational Motion

Following a similar derivation as in previous work [1], the derivation begins with Newton's second law for the center of mass of the spacecraft.

$$\ddot{\boldsymbol{r}}_{C/N} = \frac{\boldsymbol{F}}{m_{\text{sc}}} \tag{1}$$

Ultimately the acceleration of the body frame or point B is desired

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \ddot{\boldsymbol{c}} \tag{2}$$

The definition of $\boldsymbol{c}$ can be seen in Eq. (3).

$$\boldsymbol{c} = \frac{1}{m_{\text{sc}}}\Big(m_{\text{hub}}\boldsymbol{r}_{B_c/B} + \sum_{j=1}^{N_P} m_j \boldsymbol{r}_{P_{c,j}/B}\Big) \tag{3}$$

To find the inertial time derivative of $\boldsymbol{c}$, it is first necessary to find the time derivative of $\boldsymbol{c}$ with respect to the body frame. A time derivative of any vector, $\boldsymbol{v}$, with respect to the body frame is denoted by $\boldsymbol{v}'$; the inertial time derivative is labeled as $\dot{\boldsymbol{v}}$. The first and second body-relative time derivatives of $\boldsymbol{c}$ can be seen in Eqs. (7) and (8).

$\boldsymbol{r}_{P_{c,j}/B}$ is defined in the following

$$\boldsymbol{r}_{P_{c,j}/B} = \boldsymbol{r}_{P_j/B} + \rho_j \hat{\boldsymbol{p}}_j \tag{4}$$

And, the first and second body time derivatives of $\boldsymbol{r}_{P_{c,j}/B}$ are

$$\boldsymbol{r}'_{P_{c,j}/B} = \dot{\rho}_j \hat{\boldsymbol{p}}_j \tag{5}$$

$$\boldsymbol{r}''_{P_{c,j}/B} = \ddot{\rho}_j \hat{\boldsymbol{p}}_j \tag{6}$$

$\boldsymbol{c}'$ and $\boldsymbol{c}''$ are defined in the following equations

$$\boldsymbol{c}' = \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j \dot{\rho}_j \hat{\boldsymbol{p}}_j \tag{7}$$

$$\boldsymbol{c}'' = \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j \ddot{\rho}_j \hat{\boldsymbol{p}}_j \tag{8}$$

Using the transport theorem [2] yields the following definition for $\ddot{\boldsymbol{c}}$

$$\ddot{\boldsymbol{c}} = \boldsymbol{c}'' + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}\right) \tag{9}$$

Eq. (2) is updated to include Eq. (9)

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \boldsymbol{c}'' - 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' - \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}\right) \tag{10}$$

Substituting Eq.(8) into Eq.(10) results in

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j \ddot{\rho}_j \hat{\boldsymbol{p}}_j = -2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' - \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \left(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}\right) \tag{11}$$

Moving second order terms to the left hand side and introducing the tilde matrix [2] to replace the cross product operators simplifies the equation to

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{N_P} m_j \hat{\boldsymbol{p}}_j \ddot{\rho}_j = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{12}$$

Equation (12) is the translational motion equation and is the first EOM needed to describe the motion of the spacecraft. The following section develops the rotational EOM.

1.1.2 Rigid Spacecraft Hub Rotational Motion

Starting with Euler's equation when the body fixed coordinate frame origin is not coincident with the center of mass of the body [2]

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{L}_B + m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} \tag{13}$$

where $\boldsymbol{L}_B$ is the total external torque about point B. The definition of the angular momentum vector of the spacecraft about point B is

$$\boldsymbol{H}_{\text{sc},B} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times \dot{\boldsymbol{r}}_{B_c/B} + \sum_{j=1}^{N_P} m_j \boldsymbol{r}_{P_{c,j}/B} \times \dot{\boldsymbol{r}}_{P_{c,j}/B} \tag{14}$$

Now the inertial time derivative of Eq. (14) is taken and yields

$$\dot{\boldsymbol{H}}_{\text{sc},B} = [I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times \ddot{\boldsymbol{r}}_{B_c/B} + \sum_{j=1}^{N_P} m_j \boldsymbol{r}_{P_{c,j}/B} \times \ddot{\boldsymbol{r}}_{P_{c,j}/B} \tag{15}$$

$\ddot{\boldsymbol{r}}_{P_{c,j}/B}$ is

$$\ddot{\boldsymbol{r}}_{P_{c,j}/B} = \boldsymbol{r}''_{P_{c,j}/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B}) \tag{16}$$

Incorporating Eq.- (16) into Eq. (15) results in

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{sc},B} &= [I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times \ddot{\boldsymbol{r}}_{B_c/B} \\ &+ \sum_{j=1}^{N_P} m_j \boldsymbol{r}_{P_{c,j}/B} \times \Big[\boldsymbol{r}''_{P_{c,j}/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B})\Big]\end{aligned} \tag{17}$$

Applying the parallel axis theorem the following inertia tensor terms are defined as

$$[I_{\text{sc},B}] = [I_{\text{hub},B}] + m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T + \sum_{j=1}^{N_P} m_j [\tilde{\boldsymbol{r}}_{P_{c,j}/B}][\tilde{\boldsymbol{r}}_{P_{c,j}/B}]^T \tag{18}$$

Taking the body-relative time derivative of Equation (18) yields

$$[I'_{\text{sc},B}] = -\sum_{j=1}^{N_P} m_j \Big([\tilde{\boldsymbol{r}}'_{P_{c,j}/B}][\tilde{\boldsymbol{r}}_{P_{c,j}/B}] + [\tilde{\boldsymbol{r}}_{P_{c,j}/B}][\tilde{\boldsymbol{r}}'_{P_{c,j}/B}]\Big) \tag{19}$$

The Jacobi Identity, $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - \boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c})$, is used to combine terms and produce the following simplified equation

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{sc},B} &= [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ \sum_{j=1}^{N_P}\Big[m_j \boldsymbol{r}_{P_{c,j}/B} \times \boldsymbol{r}''_{P_{c,j}/B} + m_j \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \Big(\boldsymbol{r}_{P_{c,j}/B} \times \boldsymbol{r}'_{P_{c,j}/B}\Big)\Big]\end{aligned} \tag{20}$$

Eqs. (13) and (20) are equated and yield

$$\begin{aligned}\boldsymbol{L}_B + m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} &= [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ \sum_{j=1}^{N_P}\Big[m_j \boldsymbol{r}_{P_{c,j}/B} \times \boldsymbol{r}''_{P_{c,j}/B} + m_j \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \Big(\boldsymbol{r}_{P_{c,j}/B} \times \boldsymbol{r}'_{P_{c,j}/B}\Big)\Big]\end{aligned} \tag{21}$$

Finally, using tilde matrix and simplifying yields the modified Euler equation, which is the second EOM necessary to describe the motion of the spacecraft.

$$[I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ - \sum_{j=1}^{N_P}\Big(m_j[\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\boldsymbol{r}''_{P_{c,j}/B} + m_j[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\boldsymbol{r}'_{P_{c,j}/B}\Big) + \boldsymbol{L}_B - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} \quad (22)$$

Rearranging Eq. (22) to be in the same form as the previous sections results in

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{N_P} m_j[\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\hat{\boldsymbol{p}}_j\ddot{\rho}_j = \\ -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{N_P} m_j[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\boldsymbol{r}'_{P_{c,j}/B} + \boldsymbol{L}_B \quad (23)$$

1.1.3 Fuel Slosh Motion

Figure 1 shows that a single fuel slosh particle is free to move along its corresponding $\hat{\boldsymbol{p}}_j$ direction and this formulation is generalized to include N_P number of fuel slosh particles. The derivation begins with Newton's law for each fuel slosh particle:

$$m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N} = \boldsymbol{F}_C - k_j\rho_j\hat{\boldsymbol{p}}_j - c_j\dot{\rho}_j\hat{\boldsymbol{p}}_j \quad (24)$$

Where $\boldsymbol{F}_C$ is the constraint force that maintains the fuel slosh mass to travel along the direction $\hat{\boldsymbol{p}}_j$. The forces due to the spring and damper are explicitly included in Eq. (24) and result in a restoring force and damping force. $\ddot{\boldsymbol{r}}_{P_{c,j}/N}$ is defined in the following equation.

$$\ddot{\boldsymbol{r}}_{P_{c,j}/N} = \ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{P_{c,j}/B} \quad (25)$$

The inertial acceleration vector $\ddot{\boldsymbol{r}}_{P_{c,j}/B}$ is defined in Eq. (16). Plugging this definition into Eq. (24) results in

$$m_j\Big[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\rho}_j\hat{\boldsymbol{p}}_j + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}_{P_{c,j}/B})\Big] \\ = \boldsymbol{F}_C - k_j\rho_j\hat{\boldsymbol{p}}_j - c_j\dot{\rho}_j\hat{\boldsymbol{p}}_j \quad (26)$$

Equation (26) is the dynamical equation for a fuel slosh particle, however, the constraint force, $\boldsymbol{F}_C$, is undefined. Since the fuel slosh particle is free to move in the $\hat{\boldsymbol{p}}_j$ direction, the component of $\boldsymbol{F}_C$ along the $\hat{\boldsymbol{p}}_j$ direction is zero. Leveraging this insight, Eq. (26) is projected into the $\hat{\boldsymbol{p}}_j$ direction by multiplying both sides of the equation by $\hat{\boldsymbol{p}}_j{}^T$.

$$m_j\Big(\hat{\boldsymbol{p}}_j{}^T\ddot{\boldsymbol{r}}_{B/N} + \ddot{\rho}_j + 2\hat{\boldsymbol{p}}_j{}^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}'_{P_{c,j}/B} + \hat{\boldsymbol{p}}_j{}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}_{P_{c,j}/B} + \hat{\boldsymbol{p}}_j{}^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\times\boldsymbol{r}_{P_{c,j}/B})\Big) \\ = -k_j\rho_j - c_j\dot{\rho}_j \quad (27)$$

Moving the second order terms to the left hand side and introducing the tilde matrix notation yields the final equation needed to describe the motion of the spacecraft.

$$m_j\hat{\boldsymbol{p}}_j{}^T\ddot{\boldsymbol{r}}_{B/N} - m_j\hat{\boldsymbol{p}}_j{}^T[\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_j\ddot{\rho}_j \\ = -k_j\rho_j - c_j\dot{\rho}_j - 2m_j\hat{\boldsymbol{p}}_j{}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P_{c,j}/B} - m_j\hat{\boldsymbol{p}}_j{}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{P_{c,j}/B} \quad (28)$$

1.2 Derivation of Equations of Motion - Kane's Method

The choice of generalized coordinates and their respective generalized speeds are:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{r}_{B/N} \\ \boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}} \\ \rho_1 \\ \cdot \\ \rho_{N_P} \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} \dot{\boldsymbol{r}}_{B/N} \\ \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ \dot{\rho}_1 \\ \cdot \\ \dot{\rho}_{N_P} \end{bmatrix} \tag{29}$$

The necessary velocities needed to be defined are as follows

$$\dot{\boldsymbol{r}}_{B/N} = \dot{\boldsymbol{r}}_{B/N} \tag{30}$$

$$\dot{\boldsymbol{r}}_{C/N} = \dot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{c}} \tag{31}$$

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{32}$$

$$\dot{\boldsymbol{r}}_{P_{c,j}/N} = \dot{\boldsymbol{r}}_{B/N} + \boldsymbol{r}'_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} = \dot{\boldsymbol{r}}_{B/N} + \dot{\rho}_j \hat{\boldsymbol{p}}_j - [\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{33}$$

Now the following partial velocity table can be created:

Table 2: Partial Velocity Table

r	$\boldsymbol{v}_r^B$	$\boldsymbol{\omega}_r^{\mathcal{B}}$	$\boldsymbol{v}_r^{P_c}$
$1-3$	$[I_{3\times3}]$	$[0_{3\times3}]$	$[I_{3\times3}]$
$4-6$	$[0_{3\times3}]$	$[I_{3\times3}]$	$-[\tilde{\boldsymbol{r}}_{P_{c,j}/B}]$
$7-N_P$	$[0_{3\times1}]$	$[0_{3\times1}]$	$\hat{\boldsymbol{p}}_j$

An additional partial velocity that is needed is $[\boldsymbol{v}_{1-3}^C]$ for the external force applied on the spacecraft, $\boldsymbol{F}_{\text{ext}}$. Using Eq.(31) the following is defined:

$$[\boldsymbol{v}_{1-3}^C] = [I_{3\times3}] \tag{34}$$

Using these partial velocity definitions, the follow sections will step through the formulation for the translational, rotational and slosh EOMs developed using Kane's method.

1.2.1 Rigid Spacecraft Hub Translational Motion

Starting with the definition of a generalized force:

$$\boldsymbol{F}_r = \boldsymbol{v}_r^T \boldsymbol{F} \tag{35}$$

Using this definition the external force applied on the spacecraft for the translational equations is defined as:

$$\boldsymbol{F}_{1-3} = [\boldsymbol{v}_{1-3}^C]^T \boldsymbol{F}_{\text{ext}} = \boldsymbol{F}_{\text{ext}} \tag{36}$$

Using the definition of generalized inertia forces,

$$\boldsymbol{F}_r^* = \sum_r^N \Big[\boldsymbol{\omega}_r^T \boldsymbol{T}^* + \boldsymbol{v}_r^T(-m_r \boldsymbol{a}_r)\Big] \tag{37}$$

the inertia forces for the hub translational motion are defined as

$$\boldsymbol{F}^*_{1-3} = [\boldsymbol{v}^B_{1-3}]^T(-m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N}) + \sum_{j}^{N_P}[\boldsymbol{v}^{P_c}_{1-3}]^T(-m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N}) = -m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{j}^{N_P} -m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N} \tag{38}$$

Finally, Kane's equation is:

$$\boldsymbol{F}_r + \boldsymbol{F}^*_r = 0 \tag{39}$$

therefore

$$\boldsymbol{F}_{\text{ext}} - m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{j}^{N_P} -m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N} = 0 \tag{40}$$

Expanding and rearranging results in

$$m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{j}^{N_P} m_j(\ddot{\boldsymbol{r}}_{B/N} + \ddot{\boldsymbol{r}}_{P_{c,j}/B}) = \boldsymbol{F}_{\text{ext}} \tag{41}$$

Plugging Eq. (16) into Eq. (41) results in

$$\begin{split} m_{\text{hub}}\ddot{\boldsymbol{r}}_{B/N} + \sum_{j}^{N_P} m_j\Big[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\rho}_j\hat{\boldsymbol{p}}_j + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} \\ + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B})\Big] = \boldsymbol{F}_{\text{ext}} \end{split} \tag{42}$$

Combining like terms and rearranging results in

$$m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{N_P} m_j\hat{\boldsymbol{p}}_j\ddot{\rho}_j = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{43}$$

Which is identical to Eq. (12) found using Newtonian mechanics.

1.2.2 Rigid Spacecraft Hub Rotational Motion

The torque acting on the spacecraft, $\boldsymbol{L}_B$ needs to be defined as a general active force. Using Eq. (35) active forces acting on the spacecraft for the rotational equations can be defined as:

$$\boldsymbol{F}_{4-6} = [\boldsymbol{\omega}^{\mathcal{B}}_{4-6}]^T\boldsymbol{L}_B = \boldsymbol{L}_B \tag{44}$$

To define the generalized inertia forces, using Eq. (37) the definition of $\boldsymbol{T}^*$ needs to be defined for a rigid body and applying it to the hub:

$$\boldsymbol{T}^* = -[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{45}$$

$$\begin{split} \boldsymbol{F}^*_{4-6} &= [\boldsymbol{\omega}^{\mathcal{B}}_{4-6}]^T\boldsymbol{T}^* + \sum_{j}^{N_P}[\boldsymbol{v}^{P_c}_{4-6}]^T(-m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N}) \\ &= -[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{j}^{N_P} -[\tilde{\boldsymbol{r}}_{P_{c,j}/B}]^T(-m_j\ddot{\boldsymbol{r}}_{P_{c,j}/N}) \end{split} \tag{46}$$

Using Kane's equation, Eq. (39), the following equations of motion for the rotational dynamics are defined:

$$\boldsymbol{L}_B + -[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{j}^{N_P} -[\tilde{\boldsymbol{r}}_{P_{c,j}/B}](m_j \ddot{\boldsymbol{r}}_{P_{c,j}/N}) = 0 \tag{47}$$

Plugging in some definitions

$$[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j}^{N_P} m_j [\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\Big[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\rho}_j \hat{\boldsymbol{p}}_j + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B})\Big] = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \tag{48}$$

Combining like terms results in the same equation seen in Eq. (23) found using Newtonian mechanics.

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{j=1}^{N_P} m_j [\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\hat{\boldsymbol{p}}_j \ddot{\rho}_j = - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{j=1}^{N_P} m_j [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\boldsymbol{r}'_{P_{c,j}/B} + \boldsymbol{L}_B \tag{49}$$

1.2.3 Fuel Slosh Motion

Following the similar pattern for translational and rotational equations the generalized active forces are defined:

$$\boldsymbol{F}_7 = \boldsymbol{v}_r^{P_c} \cdot (-k\rho_j \hat{\boldsymbol{p}}_j - c\dot{\rho}\hat{\boldsymbol{p}}_j) = -k\rho_j - c\dot{\rho} \tag{50}$$

The generalized inertia forces are defined as:

$$\boldsymbol{F}_7^* = \boldsymbol{v}_r^{P_c} \cdot (-m_j \ddot{\boldsymbol{r}}_{P_{c,j}/N}) = \hat{\boldsymbol{p}}_j^T(-m_j \ddot{\boldsymbol{r}}_{P_{c,j}/N}) \tag{51}$$

Using Kane's equation the following equations of motion are defined:

$$- k\rho_j - c\dot{\rho} - m_j \hat{\boldsymbol{p}}_j^T \Big[\ddot{\boldsymbol{r}}_{B/N} + \ddot{\rho}_j \hat{\boldsymbol{p}}_j + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}'_{P_{c,j}/B} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{P_{c,j}/B})\Big] = 0 \tag{52}$$

Rearranging and combining like terms results in:

$$m_j \hat{\boldsymbol{p}}_j{}^T \ddot{\boldsymbol{r}}_{B/N} - m_j \hat{\boldsymbol{p}}_j{}^T [\tilde{\boldsymbol{r}}_{P_{c,j}/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_j \ddot{\rho}_j = -k_j \rho_j - c_j \dot{\rho}_j - 2m_j \hat{\boldsymbol{p}}_j{}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}'_{P_{c,j}/B} - m_j \hat{\boldsymbol{p}}_j{}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{P_{c,j}/B} \tag{53}$$

Which is identical to Eq. (28)!

2 Model Functions

This module is intended to be used an approximation to a flexing body attached to the spacecraft. Examples include solar arrays, antennas, and other appended bodies that would exhibit flexing behavior. Below is a list of functions that this model performs:

- Compute it's contributions to the mass properties of the spacecraft
- Provides matrix contributions for the back substitution method
- Compute it's derivatives for θ and $\dot{\theta}$
- Adds energy and momentum contributions to the spacecraft

3 Model Assumptions and Limitations

Below is a summary of the assumptions/limitations:

- Is an approximation to a flexing body
- Is developed in such a way that does not require constraints to be met
- The hinged rigid bodies must have a diagonal inertia tensor with respect the $\mathcal{S}_i$ frame as seen in Figure 1
- Only linear spring and damping terms
- The dual hinged rigid body will always stay attached to the hub (the hinge does not have torque limits)
- The hinge does not have travel limits, therefore if the spring is not stiff enough it will unrealistically travel through bounds such as running into the spacecraft hub
- The EOMs are nonlinear equations of motion, therefore there can be inaccuracies (and divergence) that result from integration. Having a time step of $<= 0.10$ sec is recommended, but this also depends on the natural frequency of the system

4 Test Description and Success Criteria

This test is located in `simulation/dynamics/dualHingedRigidBodies/UnitTest/test_dualHingedRigidBodyStateEffector.py`. In this integrated test there are two dual hinged rigid bodies connected to the spacecraft hub. Depending on the scenario, there are different success criteria. These are outlined in the following subsections:

4.1 Gravity integrated test

In this test the simulation is placed into orbit around Earth with point gravity and has no damping in the hinged rigid bodies. The following parameters are being tested.

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy

4.2 No gravity integrated test

In this test, the spacecraft is placed in free space (no gravity) and has no damping in the hinged rigid bodies. The following parameters describe the success criteria.

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy

5 Test Parameters

Since this is an integrated test, the inputs to the test are the physical parameters of the spacecraft along with the initial conditions of the states. These parameters can be seen in the test file. Additionally, the error tolerances can be seen in Table 3.

Table 3: Error Tolerance - Note: Relative Tolerance is abs($\frac{\text{truth}-\text{value}}{\text{truth}}$)

Test	Relative Tolerance
Energy and Momentum Conservation	1e-10

6 Test Results

The following figures show the conservation of the quantities described in the success criteria for each scenario. The conservation plots are all relative difference plots. All conservation plots show integration error which is the desired result. In the python test these values are automatically checked therefore when the tests pass, these values have all been confirmed to be conserved.

6.1 Gravity with no damping scenario

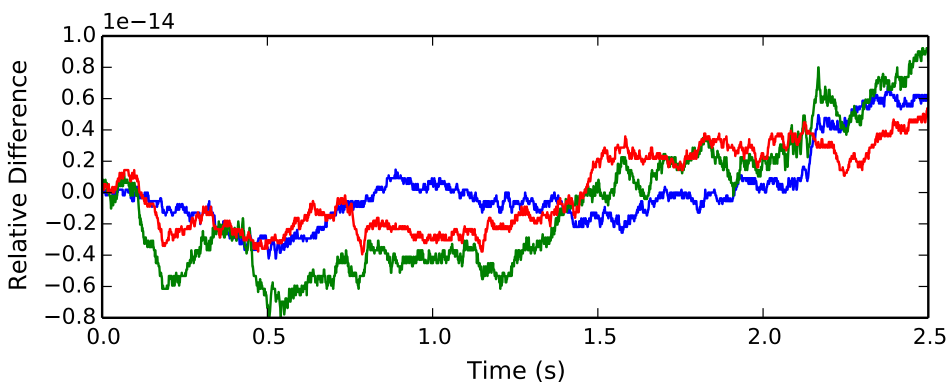

Fig. 2: Change in Orbital Angular Momentum Gravity

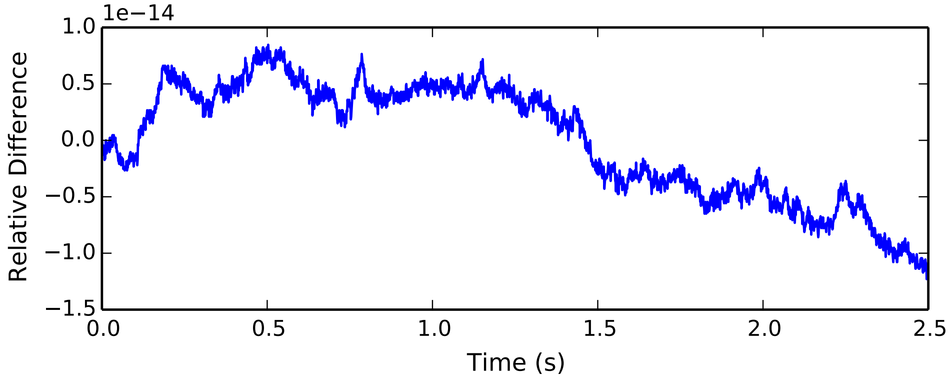

Fig. 3: Change in Orbital Energy Gravity

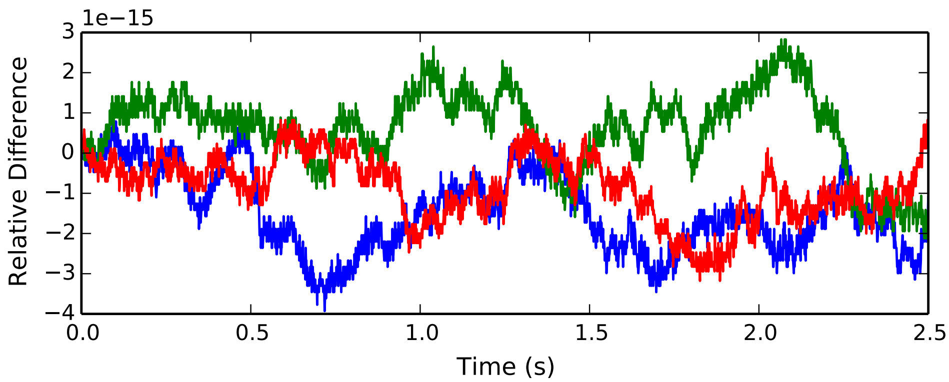

Fig. 4: Change In Rotational Angular Momentum Gravity

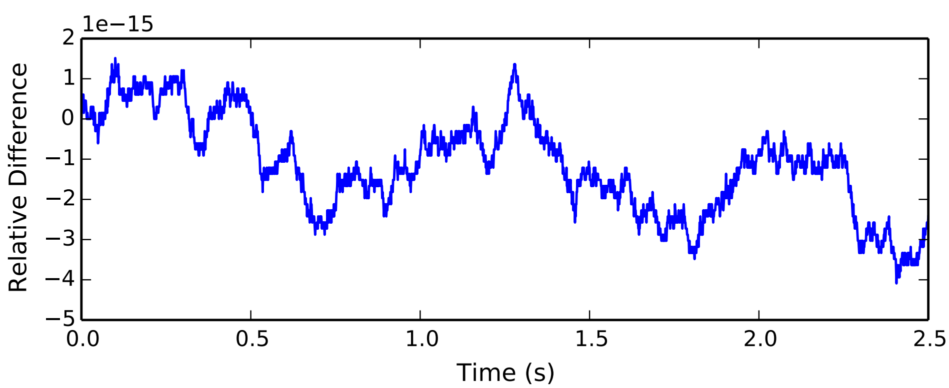

Fig. 5: Change In Rotational Energy Gravity

6.2 No Gravity with no damping scenario

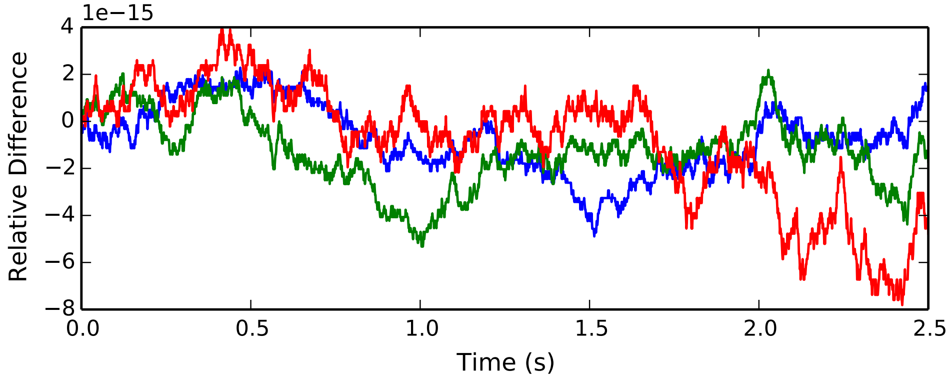

Fig. 6: Change In Orbital Angular Momentum No Gravity

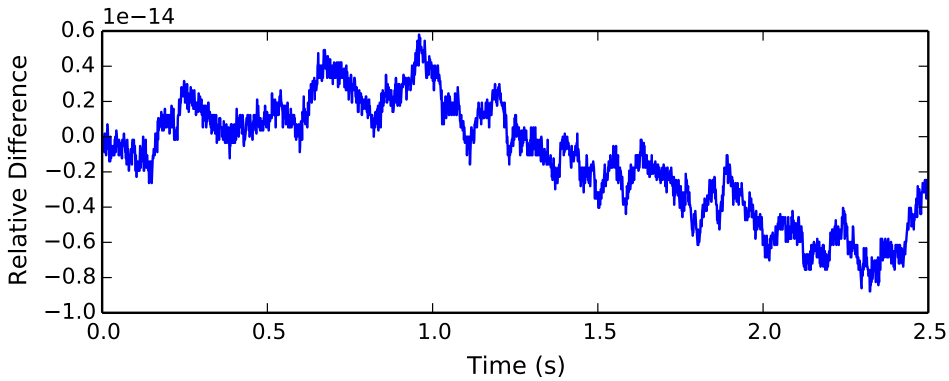

Fig. 7: Change In Orbital Energy No Gravity

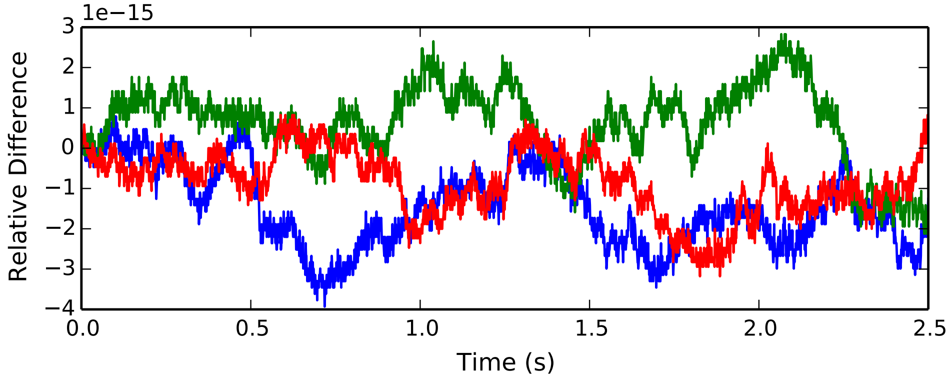

Fig. 8: Change In Rotational Angular Momentum No Gravity

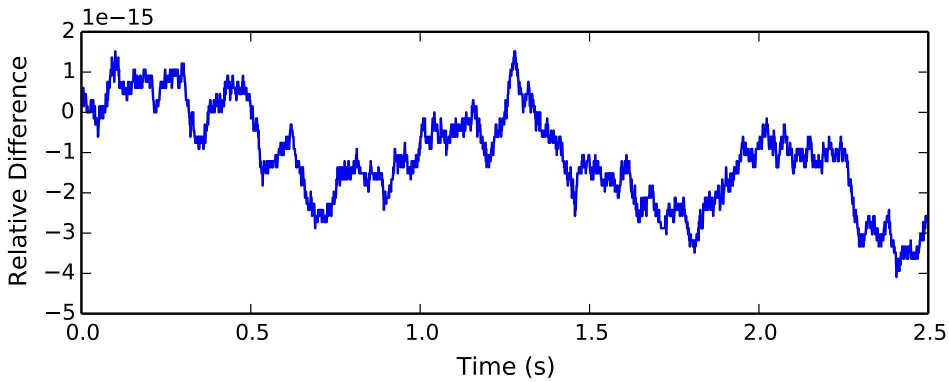

Fig. 9: Change In Rotational Energy No Gravity

7 User Guide

This section is to outline the steps needed to setup a Dual Hinged Rigid Body State Effector in python using Basilisk.

1. Import the dualHingedRigidBodyStateEffector class:
   from Basilisk.simulation import dualHingedRigidBodyStateEffector

2. Create an instantiation of a Dual Hinged Rigid body:
   panel1 = dualHingedRigidBodyStateEffector.DualHingedRigidBodyStateEffector()

3. Define all physical parameters for a Dual Hinged Rigid Body. For example:
   IPntS1_S1 = [[100.0, 0.0, 0.0], [0.0, 50.0, 0.0], [0.0, 0.0, 50.0]] Do this for all of the parameters for a Dual Hinged Rigid Body seen in the public variables in the .h file.

4. Define the initial conditions of the states:
   *panel1.theta1Init = 5*numpy.pi/180.0* *panel1.theta1DotInit = 0.0*
   *panel1.theta2Init = 5*numpy.pi/180.0* *panel1.theta2DotInit = 0.0*

5. Define a unique name for each state:
   panel1.nameOfTheta1State = "dualHingedRigidBodyTheta1" *panel1.nameOfTheta1DotState = "dualHingedRigidBodyThetaDot1"*
   panel1.nameOfTheta2State = "dualHingedRigidBodyTheta2" *panel1.nameOfTheta2DotState = "dualHingedRigidBodyThetaDot2"*

6. Finally, add the panel to your spacecraftPlus:
   scObject.addStateEffector(unitTestSim.panel1). See spacecraftPlus documentation on how to set up a spacecraftPlus object.

REFERENCES

[1] C. Allard, Hanspeter Schaub, and Scott Piggott. General hinged solar panel dynamics approximating first-order spacecraft flexing. In *AAS Guidance and Control Conference*, Breckenridge, CO, Feb. 5–10 2016. Paper No. AAS-16-156.

[2] Hanspeter Schaub and John L. Junkins. *Analytical Mechanics of Space Systems*. AIAA Education Series, Reston, VA, 3rd edition, 2014.