Autonomous Vehicle Simulation (AVS) Laboratory

AVS-Sim Technical Memorandum

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MRP STEERING ADCS CONTROL MODULE

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Status: Initial Documentation
Scope/Contents
This module uses the MRP Steering control logic to determine the ADCS control torque vector $\boldsymbol{L}_r$.

Rev:	Change Description	By
Draft	Initial Documentation Draft	H. Schaub

Contents

1 Initialization

Simply call the module reset function prior to using this control module. This will reset the prior function call time variable, and reset the attitude error integral measure. The control update period Δt is evaluated automatically.

2 Steering Law Goals

This technical note develops a new MRP based steering law that drives a body frame $\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ towards a time varying reference frame $\mathcal{R} : \{\hat{\boldsymbol{r}}_1, \hat{\boldsymbol{r}}_2, \hat{\boldsymbol{r}}_3\}$. The inertial frame is given by $\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$. The RW coordinate frame is given by $\mathcal{W}_\rangle : \{\hat{\boldsymbol{g}}_{s_i}, \hat{\boldsymbol{g}}_{t_i}, \hat{\boldsymbol{g}}_{g_i}\}$. The Using MRPs, the overall control goal is

$$\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}} \rightarrow 0 \tag{1}$$

The reference frame orientation $\boldsymbol{\sigma}_{\mathcal{R}/\mathcal{N}}$, angular velocity $\boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$ and inertial angular acceleration $\dot{\boldsymbol{\omega}}_{\mathcal{R}/\mathcal{N}}$ are assumed to be known.

The rotational equations of motion of a rigid spacecraft with N Reaction Wheels (RWs) attached are given by[?]

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}]\left([I_{RW}]\boldsymbol{\omega} + [G_s]\boldsymbol{h}_s\right) - [G_s]\boldsymbol{u}_s + \boldsymbol{L} \tag{2}$$

where the inertia tensor $[I_{RW}]$ is defined as

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N}\left(J_{t_i}\hat{\boldsymbol{g}}_{t_i}\hat{\boldsymbol{g}}_{t_i}^T + J_{g_i}\hat{\boldsymbol{g}}_{g_i}\hat{\boldsymbol{g}}_{g_i}^T\right) \tag{3}$$

The spacecraft inertial without the N RWs is $[I_s]$, while J_{s_i}, J_{t_i} and J_{g_i} are the RW inertias about the body fixed RW axis $\hat{\boldsymbol{g}}_{s_i}$ (RW spin axis), $\hat{\boldsymbol{g}}_{t_i}$ and $\hat{\boldsymbol{g}}_{g_i}$. The $3 \times N$ projection matrix $[G_s]$ is then defined as

$$[G_s] = \left[\cdots {}^{\mathcal{B}}\hat{\boldsymbol{g}}_{s_i} \cdots\right] \tag{4}$$

The RW inertial angular momentum vector $\boldsymbol{h}_s$ is defined as

$$h_{s_i} = J_{s_i}(\omega_{s_i} + \Omega_i) \tag{5}$$

Here Ω_i is the i^{th} RW spin relative to the spacecraft, and the body angular velocity is written in terms of body and RW frame components as

$$\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3 = \omega_{s_i}\hat{\boldsymbol{g}}_{s_i} + \omega_{t_i}\hat{\boldsymbol{g}}_{t_i} + \omega_{g_i}\hat{\boldsymbol{g}}_{g_i} \tag{6}$$

3 MRP Steering Law

3.1 Steering Law Stability Requirement

As is commonly done in robotic applications where the steering laws are of the form $\dot{\boldsymbol{x}} = \boldsymbol{u}$, this section derives a kinematic based attitude steering law. Let us consider the simple Lyapunov candidate function[?,?]

$$V(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) = 2\ln\left(1 + \boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}^T\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}\right) \tag{7}$$

in terms of the MRP attitude tracking error $\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}$. Using the MRP differential kinematic equations

$$\dot{\boldsymbol{\sigma}}_{\mathcal{B}/\mathcal{R}} = \frac{1}{4}[B(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}})]^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{R}} = \frac{1}{4}\left[(1-\sigma^2_{\mathcal{B}/\mathcal{R}})[I_{3\times3} + 2[\tilde{\boldsymbol{\sigma}}_{\mathcal{B}/\mathcal{R}}] + 2\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}\boldsymbol{\sigma}^T_{\mathcal{B}/\mathcal{R}}\right]{}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{R}} \tag{8}$$

where $\sigma^2_{\mathcal{B}/\mathcal{R}} = \boldsymbol{\sigma}^T_{\mathcal{B}/\mathcal{R}}\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}$, the time derivative of V is

$$\dot{V} = \boldsymbol{\sigma}^T_{\mathcal{B}/\mathcal{R}}\left({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{R}}\right) \tag{9}$$

To create a kinematic steering law, let $\mathcal{B}^*$ be the desired body orientation, and $\boldsymbol{\omega}_{\mathcal{B}^*/\mathcal{R}}$ be the desired angular velocity vector of this body orientation relative to the reference frame $\mathcal{R}$. The steering law requires an algorithm for the desired body rates $\boldsymbol{\omega}_{\mathcal{B}^*/\mathcal{R}}$ relative to the reference frame make $\dot{V}$ in Eq. (**??**) negative definite. For this purpose, let us select

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}^*/\mathcal{R}} = -\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) \tag{10}$$

where $\boldsymbol{f}(\boldsymbol{\sigma})$ is an even function such that

$$\boldsymbol{\sigma}^T\boldsymbol{f}(\boldsymbol{\sigma}) > 0 \tag{11}$$

The Lyapunov rate simplifies to the negative definite expression:

$$\dot{V} = -\boldsymbol{\sigma}^T_{\mathcal{B}/\mathcal{R}}\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) < 0 \tag{12}$$

3.2 Saturated MRP Steering Law

A very simple example would be to set

$$\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) = K_1\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}} \tag{13}$$

where $K_1 > 0$. This yields a kinematic control where the desired body rates are proportional to the MRP attitude error measure. If the rate should saturate, then $\boldsymbol{f}()$ could be defined as

$$\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) = \begin{cases} K_1\sigma_i & \text{if } |K_1\sigma_i| \leqslant \omega_{\text{max}} \\ \omega_{\text{max}}\text{sgn}(\sigma_i) & \text{if } |K_1\sigma_i| > \omega_{\text{max}} \end{cases} \tag{14}$$

where

$$\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}} = (\sigma_1, \sigma_2, \sigma_3)^T$$

A smoothly saturating function is given by

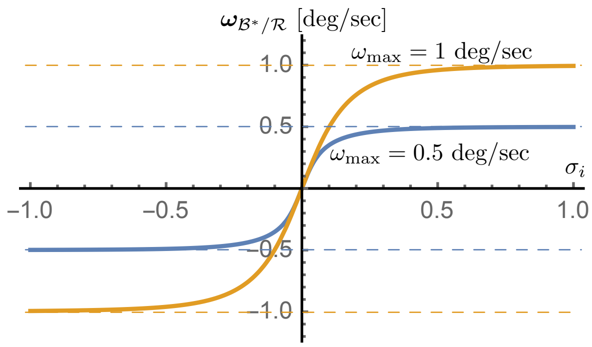
$$\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) = \arctan\left(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}\frac{K_1\pi}{2\omega_{\text{max}}}\right)\frac{2\omega_{\text{max}}}{\pi} \tag{15}$$

where

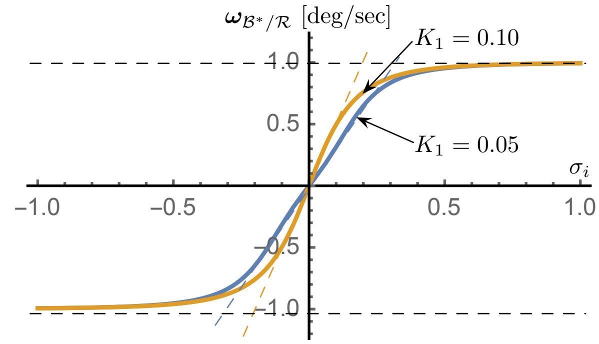
$$\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) = \begin{pmatrix} f(\sigma_1) \\ f(\sigma_2) \\ f(\sigma_3) \end{pmatrix} \tag{16}$$

Here as $\sigma_i \rightarrow \infty$ then the function f smoothly converges to the maximum speed rate $\pm\omega_{\text{max}}$. For small $|\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}|$, this function linearizes to

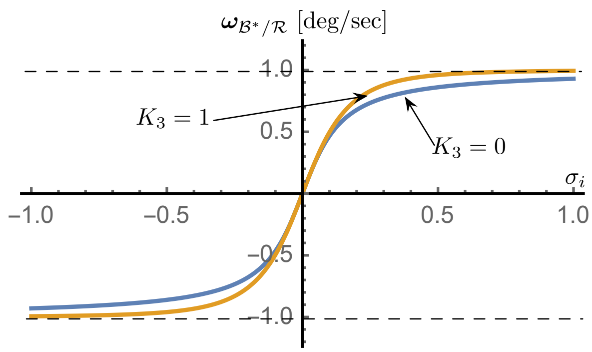
$$\boldsymbol{f}(\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}) \approx K_1\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}} + \text{ H.O.T} \tag{17}$$



(a) ω_{max} dependency with $K_1 = 0.1$, $K_3 = 1$



(b) K_1 dependency with $\omega_{\text{max}} = 1^\circ/s$, $K_3 = 1$



(c) K_3 dependency with $\omega_{\text{max}} = 1^\circ/s$, $K_1 = 0.1$

Fig. 1: Illustrations of MRP Steering Parameters Influence.

If the MRP shadow set parameters are used to avoid the MRP singularity at 360°, then $|\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{R}}|$ is upper limited by 1. To control how rapidly the rate commands approach the ω_{max} limit, Eq. (**??**) is modified to include a cubic term:

$$f(\sigma_i) = \arctan\left((K_1\sigma_i + K_3\sigma_i^3)\frac{\pi}{2\omega_{\text{max}}} \right)\frac{2\omega_{\text{max}}}{\pi} \tag{18}$$

The order of the polynomial must be odd to keep $\boldsymbol{f}()$ an even function. A nice feature of Eq. (**??**) is that the control rate is saturated individually about each axis. If the smoothing component is removed to reduce this to a bang-band rate control, then this would yield a Lyapunov optimal control which minimizes $\dot{V}$ subject to the allowable rate constraint ω_{max}.

Figure **??** illustrates how the parameters ω_{max}, K_1 and K_3 impact the steering law behavior. The maximum steering law rate commands are easily set through the ω_{max} parameters. The gain K_1 controls the linear stiffness when the attitude errors have become small, while K_3 controls how rapidly the steering law approaches the speed command limit.

The required velocity servo loop design is aided by knowing the body-frame derivative of ${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}*/\mathcal{R}}$ to implement a feed-forward components. Using the $\boldsymbol{f}()$ function definition in Eq. (**??**), this requires the time derivatives of $f(\sigma_i)$.

$$\frac{{}^{\mathcal{B}}\mathrm{d}({}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}*/\mathcal{R}})}{\mathrm{d}t} = \boldsymbol{\omega}'_{\mathcal{B}*/\mathcal{R}} = -\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\sigma}_{\mathcal{B}*/\mathcal{R}}}\dot{\boldsymbol{\sigma}}_{\mathcal{B}*/\mathcal{R}} = -\begin{pmatrix} \frac{\partial f}{\partial \sigma_1}\dot{\sigma}_1 \\ \frac{\partial f}{\partial \sigma_2}\dot{\sigma}_2 \\ \frac{\partial f}{\partial \sigma_3}\dot{\sigma}_3 \end{pmatrix} \tag{19}$$

where

$$\dot{\boldsymbol{\sigma}}_{\mathcal{B}*/\mathcal{R}} = \begin{pmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{pmatrix} = \frac{1}{4}[B(\boldsymbol{\sigma}_{\mathcal{B}*/\mathcal{R}})]^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}*/\mathcal{R}} \tag{20}$$

Using the general $f()$ definition in Eq. (**??**), its sensitivity with respect to σ_i is

$$\frac{\partial f}{\partial \sigma_i} = \frac{(K_1 + 3K_3\sigma_i^2)}{1 + (K_1\sigma_i + K_3\sigma_i^3)^2\left(\frac{\pi}{2\omega_{\text{max}}}\right)^2} \tag{21}$$

4 Angular Velocity Servo Sub-System

To implement the kinematic steering control, a servo sub-system must be included which will produce the required torques to make the actual body rates track the desired body rates. The angular velocity tracking error vector is defined as

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{B}*} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{B}*/\mathcal{N}} \tag{22}$$

where the $\mathcal{B}^*$ frame is the desired body frame from the kinematic steering law. Note that

$$\boldsymbol{\omega}_{\mathcal{B}*/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}*/\mathcal{R}} + \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}} \tag{23}$$

where $\boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$ is obtained from the attitude navigation solution, and $\boldsymbol{\omega}_{\mathcal{B}*/\mathcal{R}}$ is the kinematic steering rate command. To create a rate-servo system that is robust to unmodeld torque biases, the state $\boldsymbol{z}$ is defined as:

$$\boldsymbol{z} = \int_{t_0}^{t_f} {}^{\mathcal{B}}\delta\boldsymbol{\omega}\ \mathrm{d}t \tag{24}$$

The rate servo Lyapunov function is defined as

$$V_{\boldsymbol{\omega}}(\delta\boldsymbol{\omega}, \boldsymbol{z}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I_{\text{RW}}]\delta\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{z}^T[K_I]\boldsymbol{z} \tag{25}$$

where the vector $\delta\omega$ and tensor $[I_{\text{RW}}]$ are assumed to be given in body frame components, $[K_i]$ is a symmetric positive definite matrix. The time derivative of this Lyapunov function is

$$\dot{V}_{\boldsymbol{\omega}} = \delta\boldsymbol{\omega}^T\left([I_{\text{RW}}]\delta\boldsymbol{\omega}' + [K_I]\boldsymbol{z}\right) \tag{26}$$

Using the identities $\boldsymbol{\omega}'_{\mathcal{B}/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and $\boldsymbol{\omega}'_{\mathcal{R}/\mathcal{N}} = \dot{\boldsymbol{\omega}}_{\mathcal{R}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}$,[?] the body frame derivative of $\delta\boldsymbol{\omega}$ is

$$\delta\boldsymbol{\omega}' = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \boldsymbol{\omega}'_{\mathcal{B}*/\mathcal{R}} - \dot{\boldsymbol{\omega}}_{\mathcal{R}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}} \tag{27}$$

Substituting Eqs. (**??**) and (**??**) into the $\dot{V}_{\boldsymbol{\omega}}$ expression in Eq. (**??**) yields

$$\begin{aligned}\dot{V}_{\boldsymbol{\omega}} = \delta\boldsymbol{\omega}^T\Big(&- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{RW}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [G_s]\boldsymbol{h}_s\right) - [G_s]\boldsymbol{u}_s + \boldsymbol{L} + [K_I]\boldsymbol{z} \\ &- [I_{\text{RW}}](\boldsymbol{\omega}'_{\mathcal{B}*/\mathcal{R}} + \dot{\boldsymbol{\omega}}_{\mathcal{R}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}})\Big)\end{aligned} \tag{28}$$

Let $[P]^T = [P] >$ be a symmetric positive definite rate feedback gain matrix. The servo rate feedback control is defined as

$$\begin{aligned}[G_s]\boldsymbol{u}_s = [P]\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}*/\mathcal{N}}]\left([I_{\text{RW}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [G_s]\boldsymbol{h}_s\right) \\ - [I_{\text{RW}}](\boldsymbol{\omega}'_{\mathcal{B}*/\mathcal{R}} + \dot{\boldsymbol{\omega}}_{\mathcal{R}/\mathcal{N}} - \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{R}/\mathcal{N}}) + \boldsymbol{L}\end{aligned} \tag{29}$$

Defining the right-hand-side as $\boldsymbol{L}_r$, this is rewritten in compact form as

$$[G_s]\boldsymbol{u}_s = \boldsymbol{L}_r \tag{30}$$

The array of RW motor torques can be solved with the typical minimum norm inverse

$$\boldsymbol{u}_s = [G_s]^T \left([G_s][G_s]^T\right)^{-1} \boldsymbol{L}_r \tag{31}$$

To analyze the stability of this rate servo control, the $[G_s]\boldsymbol{u}_s$ expression in Eq. (**??**) is substituted into the Lyapunov rate expression in Eq. (**??**).

$$\begin{aligned}
\dot{V}_\omega &= \delta\boldsymbol{\omega}^T\Big(- [P]\delta\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\left([I_{RW}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [G_s]\boldsymbol{h}_s\right) + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}*/\mathcal{N}}]\left([I_{\text{RW}}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [G_s]\boldsymbol{h}_s\right)\Big) \\
&= \delta\boldsymbol{\omega}^T\Big(- [P]\delta\boldsymbol{\omega} - [\widetilde{\delta\boldsymbol{\omega}}]\left([I_{RW}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [G_s]\boldsymbol{h}_s\right)\Big) \\
&= -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} < 0
\end{aligned} \tag{32}$$

Thus, in the absence of unmodeled torques, the servo control in Eq. (**??**) is asymptotically stabilizing in rate tracking error $\delta\boldsymbol{\omega}$.

Next, the servo robustness to unmodeled external torques is investigated. Let us assume that the external torque vector $\boldsymbol{L}$ in Eq. (**??**) only approximates the true external torque, and the unmodeled component is given by $\Delta\boldsymbol{L}$. Substituting the true equations of motion and the same servo control in Eq. (**??**) into the Lyapunov rate expression in Eq. (**??**) leads to

$$\dot{V}_\omega = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}^T\Delta\boldsymbol{L} \tag{33}$$

This $\dot{V}_\omega$ is no longer negative definite due to the underdetermined sign of the $\delta\boldsymbol{\omega}^T\Delta\boldsymbol{L}$ components. Equating the Lyapunov rates in Eqs. (**??**) and (**??**) yields the following servo closed loop dynamics:

$$[I_{\text{RW}}]\delta\boldsymbol{\omega}' + [P]\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} = \Delta\boldsymbol{L} \tag{34}$$

Assuming that $\Delta\boldsymbol{L}$ is either constant as seen by the body frame, or at least varies slowly, then taking a body-frame time derivative of Eq. (**??**) is

$$[I_{\text{RW}}]\delta\boldsymbol{\omega}'' + [P]\delta\boldsymbol{\omega}' + [K_I]\delta\boldsymbol{\omega} = \Delta\boldsymbol{L}' \approx 0 \tag{35}$$

As $[I_{\text{RW}}]$, $[P]$ and $[K_I]$ are all symmetric positive definite matrices, these linear differential equations are stable, and $\delta\boldsymbol{\omega} \to 0$ given that assumption that $\Delta\boldsymbol{L}' \approx 0$.

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