Autonomous Vehicle Simulation (AVS) Laboratory

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SPHERICAL HARMONICS C++ MODEL

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Spherical harmonics model and implementation is described.

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Contents

1 Introduction

A spherical harmonics model and implementation is developed and described.

2 Document ID

`AVS-SIM-subModuleTemplate`.

3 Mathematical model

3.1 Gravity models

Gravity models are usually based on solutions of the Laplace equation $(\nabla^2 U(\bar{\mathbf{r}}) = 0)$. It is very important to state that this equation only models a gravity potential outside a body. For computing a potential inside a body the Poisson equation is used instead.
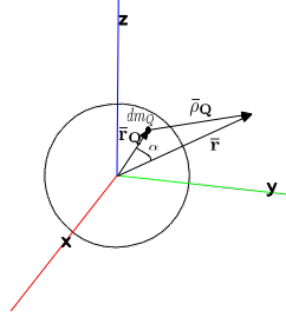
The spherical harmonic potential is a solution of the Laplace equation using orthogonal spherical harmonics. It can be derived solving the Laplace equation in spherical coordinates, using the separation of variables technique and solving a Sturm-Liouville problem. In this work, the solution will be found using another technique, which essentially follows Vallado's book.[4]

For each element of mass dm_{Q} the potential can be written as

$$dU(\bar{\mathbf{r}}) = G\frac{dm_{\mathsf{Q}}}{\rho_{\mathsf{Q}}} \tag{1}$$

where ρ_{Q} is the distance between the element of mass and the position vector $\bar{\mathbf{r}}$ where the potential is computed. This position vector is usually given in a body-fixed frame. The relation between the position vector $\bar{\mathbf{r}}$, the position of the element of mass $\bar{\mathbf{r}}_{\mathsf{Q}}$ and ρ_{Q} can be given using the cosine theorem and the angle α between the two position vectors, as can be appreciated in Figure 1.

$$\rho_{\mathsf{Q}} = \sqrt{r^2 + r_{\mathsf{Q}}^2 - 2rr_{\mathsf{Q}}\cos(\alpha)} = r\sqrt{1 - 2\frac{r_{\mathsf{Q}}}{r}\cos(\alpha) + \left(\frac{r_{\mathsf{Q}}}{r}\right)^2} = r\sqrt{1 - 2\gamma\cos(\alpha) + \gamma^2} \tag{2}$$

Fig. 1: Geometry of the Spherical Harmonics Representation.

where $\gamma = r_Q/r$.

The potential can be obtained by integrating dU through the whole body.

$$U(\bar{\mathbf{r}}) = G \int_{body} \frac{\mathrm{d}m_Q}{r\sqrt{1 - 2\gamma\cos(\alpha) + \gamma^2}} \tag{3}$$

If the potential is computed outside the body, γ will always be less than 1, and the inverse of the square root can be approximated using the Legendre polynomials $P_l[\beta]$.[4] Even though this derivation does not use the Laplace equation, it still assumes that the potential is computed outside the body.

The Legendre polynomials can be written as

$$P_l[\beta] = \frac{1}{2^l l!} \frac{d^l}{d\beta^l} (\beta^2 - 1)^l \tag{4}$$

The potential is

$$U(\bar{\mathbf{r}}) = \frac{G}{r} \int_{body} \sum_{l=0}^{\infty} \gamma^l P_l[\cos(\alpha)] \mathrm{d}m_Q \tag{5}$$

The angle α must be integrated. However, the cosine of the angle α can be decomposed using the geocentric latitude and the longitude associated to vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}_Q$. These angles will be called (ϕ, λ) and (ϕ_Q, λ_Q) respectively. Using the addition theorem it is possible to write.[4]

$$P_l[\cos(\alpha)] = P_l[\sin(\phi_Q)]P_l[\sin(\phi)] + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!}(a_{l,m}a'_{l,m} + b_{l,m}b'_{l,m}) \tag{6}$$

where

$$a_{l,m} = P_{l,m}[\sin(\phi_Q)]\cos(m\lambda_Q) \tag{7}$$
$$b_{l,m} = P_{l,m}[\sin(\phi_Q)]\sin(m\lambda_Q) \tag{8}$$
$$a'_{l,m} = P_{l,m}[\sin(\phi)]\cos(m\lambda) \tag{9}$$
$$b'_{l,m} = P_{l,m}[\sin(\phi)]\sin(m\lambda) \tag{10}$$

where $P_{l,m}[x]$ are the associated Legendre functions. "l" is called degree and "m", order. The polynomials can be computed as

$$P_{l,m}[\beta] = (1 - \beta^2)^{\frac{m}{2}} \frac{d^m}{d\beta^m} P_l[\beta] \tag{11}$$

As can be seen, $a_{l,m}$ and $b_{l,m}$ must be integrated, but $a'_{l,m}$ and $a'_{l,m}$ can be taken outside the integral. Therefore, it is possible to define

$$C'_{l,m} = \int_{body} (2 - \delta_m) r_Q^l \frac{(l-m)!}{(l+m)!} a_{l,m} dm_Q \tag{12}$$

$$S'_{l,m} = \int_{body} (2 - \delta_m) r_Q^l \frac{(l-m)!}{(l+m)!} b_{l,m} dm_Q \tag{13}$$

where δ_m is the Kronecker delta.

Then

$$U(\bar{\mathbf{r}}) = \frac{G}{r} \sum_{l=0}^{\infty} C'_{l,0} \frac{P_l[\sin(\phi)]}{r^l} + \frac{G}{r} \sum_{l=0}^{\infty} \sum_{m=1}^{l} \frac{P_{l,m}[\sin(\phi)]}{r^l} [C'_{l,m} \cos(m\lambda) + S'_{l,m} \sin(m\lambda)] \tag{14}$$

Non-dimensional coefficients $C_{l,m}$ and $S_{l,m}$ are usually used

$$C'_{l,m} = C_{l,m} R_{\text{ref}}^l m_Q \tag{15}$$

$$S'_{l,m} =_{\text{CoM}} S_{l,m} R_{\text{ref}}^l m_Q \tag{16}$$

where m_Q is the total mass of the body and R_{ref} is a reference radius. If the coefficients $C_{l,m}$ and $S_{l,m}$ are given, the reference radius must be specified. Usually, the reference is chosen as the maximum radius or the mean radius.[3]

The potential is then

$$U(\bar{\mathbf{r}}) = \frac{\mu}{r} \sum_{l=0}^{\infty} C_{l,0} \left(\frac{R_{\text{ref}}}{r}\right)^l P_l[\sin(\phi)] + \frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=1}^{l} \left(\frac{R_{\text{ref}}}{r}\right)^l P_{l,m}[\sin(\phi)][C_{l,m} \cos(m\lambda) + S_{l,m} \sin(m\lambda)] \tag{17}$$

Since $P_l[x] = P_{l,0}[x]$ the potential can be written in a more compact way

$$U(\bar{\mathbf{r}}) = \frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\text{ref}}}{r}\right)^l P_{l,m}[\sin(\phi)][C_{l,m} \cos(m\lambda) + S_{l,m} \sin(m\lambda)] \tag{18}$$

Some coefficients have a very interesting interpretation.

$$C_{0,0} = 1 \tag{19}$$

$$S_{l,0} = 0 \quad \forall l \geqslant 0 \tag{20}$$

$$C_{1,0} = \frac{Z_{\text{CoM}}}{R_{\text{ref}}} \tag{21}$$

$$C_{1,1} = \frac{X_{\text{CoM}}}{R_{\text{ref}}} \tag{22}$$

$$S_{1,1} = \frac{Y_{\text{CoM}}}{R_{\text{ref}}} \tag{23}$$

where $[X_{\text{CoM}}, Y_{\text{CoM}}, Z_{\text{CoM}}]$ represents the center of mass of the celestial body. Therefore, if the origin of the coordinate system coincides with the center of mass, all these coefficients are identically zero. Similarly, the second order coefficients are related to the second order moments (moments of inertia).

Finally, the coefficients and Legendre polynomials are usually normalized to avoid computational issues. The factor $N_{l,m}$ is called the normalization factor

$$N_{l,m} = \sqrt{\frac{(l-m)!(2-\delta_m)(2l+1)}{(l+m)!}} \tag{24}$$

The normalized coefficients are

$$\bar{C}_{l,m} = \frac{C_{l,m}}{N_{l,m}} \tag{25}$$

$$\bar{S}_{l,m} = \frac{S_{l,m}}{N_{l,m}} \tag{26}$$

The normalized associated Legendre functions are

$$\bar{P}_{l,m}[x] = P_{l,m}[x]N_{l,m} \tag{27}$$

The potential may be written as

$$U(\bar{\mathbf{r}}) = \frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\text{ref}}}{r}\right)^l \bar{P}_{l,m}[\sin(\phi)]\big[\bar{C}_{l,m}\cos(m\lambda) + \bar{S}_{l,m}\sin(m\lambda)\big] \tag{28}$$

3.2 Pines' Representation of Spherical Harmonics Gravity

There are many ways to algorithmically compute the potential and its first and secondary derivatives. One of such algorithms is the one proposed by Pines.[2]

The spherical harmonics representation as it was presented has a singularity at the poles for the gravity field. The Pines' formulation avoids this problem and is more numerically stable for high degree and high order terms.

Unfortunately, this formulation does not contain the normalization factor which is necessary if the coefficients are normalized. In a paper written by Lundberg and Schutz,[1] a normalized representation of the Pines' formulation is given, but it contains an approximation.

For this work, and in order to code the spherical harmonics formulation, a formulation similar to Pines' using the Lundberg-Schutz paper will be derived. However, no approximations will be used. Therefore, the algorithm will be developed here without using the exact formulations given in those papers. For the sake of brevity, not every single derivation will be carried out, but it is possible to get the results following the expressions obtained in this section.

In the Pines' formulation the radius and the director cosines are used as coordinates. The potential will be given as $U[r,s,t,u]$, where

$$r = \sqrt{x^2 + y^2 + z^2} \tag{29}$$

$$s = \frac{x}{r} \tag{30}$$

$$t = \frac{y}{r} \tag{31}$$

$$u = \frac{z}{r} \tag{32}$$

For a function of these coordinates, the dependance will be given using square brackets (e.g. $f[r,s,t,u]$).

Since $u = \sin(\phi) = \cos(90° - \phi)$, it is possible to write

$$P_{l,m}[\sin(\phi)] = P_{l,m}[u] \tag{33}$$

The derived Legendre functions $A_{l,m}[u]$ are defined such that

$$P_{l,m}[u] = (1 - u^2)^{\frac{m}{2}} A_{l,m}[u] \tag{34}$$

From the definition of $P_{l,m}$ (Equation 11), it is possible to write

$$A_{l,m}[u] = \frac{d^m}{du^m} P_l[u] = \frac{1}{2^l l!} \frac{d^{l+m}}{du^{l+m}} (u^2 - 1)^l \tag{35}$$

The term $(1 - u^2)^{\frac{m}{2}}$ can be written as $(1 - \sin^2(\phi))^{\frac{m}{2}} = |\cos(\phi)|^m = \cos^m(\phi)$.
If the complex number ξ is defined such that (j is the imaginary unit)

$$\xi = \cos(\phi)\cos(\lambda) + j\cos(\phi)\sin(\lambda) = \frac{x}{r} + j\frac{y}{r} = s + jt \tag{36}$$

it is possible to write

$$\xi^m = \cos^m(\phi)e^{jm\lambda} = (s + jt)^m \tag{37}$$

The following sequences may be defined

$$R_m[s,t] = Re\{\xi^m\} \tag{38}$$

$$I_m[s,t] = Im\{\xi^m\} \tag{39}$$

Putting all together, it is possible to write

$$U(\bar{\mathbf{r}}) = \frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\text{ref}}}{r}\right)^l A_{l,m}[u]\{C_{l,m}R_m[s,t] + S_{l,m}I_m[s,t]\} \tag{40}$$

In order to normalize the coefficients ($\bar{C}_{l,m}$ and $\bar{S}_{l,m}$) and the derived Legendre functions ($\bar{A}_{l,m} = N_{l,m}A_{l,m}$), each term is divided an multiplied by the normalization factor $N_{l,m}$. Then

$$U(\bar{\mathbf{r}}) = \frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\text{ref}}}{r}\right)^l \bar{A}_{l,m}[u]\{\bar{C}_{l,m}R_m[s,t] + \bar{S}_{l,m}I_m[s,t]\} \tag{41}$$

The sets $D_{l,m}[s,t]$, $E_{l,m}[s,t]$, and $F_{l,m}[s,t]$, are defined as

$$D_{l,m}[s,t] = \bar{C}_{l,m}R_m[s,t] + \bar{S}_{l,m}I_m[s,t] \tag{42}$$

$$E_{l,m}[s,t] = \bar{C}_{l,m}R_{m-1}[s,t] + \bar{S}_{l,m}I_{m-1}[s,t] \tag{43}$$

$$F_{l,m}[s,t] = \bar{S}_{l,m}R_{m-1}[s,t] - \bar{C}_{l,m}I_{m-1}[s,t] \tag{44}$$

The value $\rho_l[r]$ is also defined as

$$\rho_l[r] = \frac{\mu}{r}\left(\frac{R_{\text{ref}}}{r}\right)^l \tag{45}$$

The gravity potential may be finally computed as

$$U(\bar{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \rho_l[r]\bar{A}_{l,m}[u]D_{l,m}[s,t] \tag{46}$$

This is the final expression that will be used to compute the gravity potential.

3.3 Recursion Formulas

Several recursion formulas are needed in order to algorithmically implement the Pines' formulation. They will be given without proof, but they are easily derived using the definitions above.

3.3.1 Recursion formula for $\rho_l[r]$

Initial condition: $\rho_0[r] = \frac{\mu}{r}$

$$\rho_l[r] = \rho \cdot \rho_{l-1}[r] \tag{47}$$

where $\rho = R_{\text{ref}}/r$.

3.3.2 Recursion formula for $R_m[s,t]$

Initial condition: $R_0[s,t] = 1$

$$R_m[s,t] = sR_{m-1}[s,t] - tI_{m-1}[s,t] \tag{48}$$

3.3.3 Recursion formula for $I_m[s,t]$

Initial condition: $I_0[s,t] = 0$

$$I_m[s,t] = sI_{m-1}[s,t] + tR_{m-1}[s,t] \tag{49}$$

3.3.4 Recursion formula for $\bar{A}_{l,m}[u]$

From Equation (35), it is possible to see that

$$A_{l,l}[u] = (2l-1)A_{l-1,l-1}[u] \tag{50}$$
$$A_{l,l-1}[u] = uA_{l,l}[u] \tag{51}$$

There are several recursion formulas for computing Legendre polynomials $A_{l,m}[u]$, for $m < l - 1$. The following formula, which is stable for high degrees,[1] will be used:

$$A_{l,m}[u] = \frac{1}{l-m}((2l-1)uA_{l-1,m}[u] - (l+m-1)A_{l-2,m}[u]) \tag{52}$$

Using Equations (50), (51), and (52), and the definition $\bar{A}_{l,m}[u] = N_{l,m}A_{l,m}[u]$, the following recursion formulas can be derived.

Initial condition: $\bar{A}_{0,0}[u] = 1$

The diagonal terms are computed as

$$\bar{A}_{l,l}[u] = \sqrt{\frac{(2l-1)(2-\delta_l)}{(2l)(2-\delta_{l-1})}}\bar{A}_{l-1,l-1}[u] \tag{53}$$

The low diagonal terms are then calculated as

$$\bar{A}_{l,l-1}[u] = u\sqrt{\frac{(2l)(2-\delta_{l-1})}{2-\delta_l}}\bar{A}_{l,l}[u] \tag{54}$$

Finally, for $l \geqslant (m+2)$, $N1_{l,m}$ and $N2_{l,m}$ are defined such that

$$N1_{l,m} = \sqrt{\frac{(2l+1)(2l-1)}{(l-m)(l+m)}} \tag{55}$$

$$N2_{l,m} = \sqrt{\frac{(l+m-1)(2l+1)(l-m-1)}{(l-m)(l+m)(2l-3)}} \tag{56}$$

and $\bar{A}_{l,m}[u]$ computed using

$$\bar{A}_{l,m}[u] = uN1_{l,m}\bar{A}_{l-1,m}[u] - N2_{l,m}\bar{A}_{l-2,m}[u] \tag{57}$$

3.4 Derivatives

The first order derivatives of many of the values given are necessary to compute the gravity field (second order derivatives are needed if the Hessian is to be computed).

It is easy to show that

$$\frac{\partial D_{l,m}}{\partial s}[s,t] = mE_{l,m}[s,t] \tag{58}$$

$$\frac{\partial D_{l,m}}{\partial t}[s,t] = mF_{l,m}[s,t] \tag{59}$$

$$\frac{d\rho_l}{dr}[r] = -\frac{(l+1)}{R_{\text{ref}}}\rho_{l+1}[r] \tag{60}$$

$$\frac{\partial R_m}{\partial s}[s,t] = mR_{m-1}[s,t] \tag{61}$$

$$\frac{\partial R_m}{\partial t}[s,t] = -mI_{m-1}[s,t] \tag{62}$$

$$\frac{\partial I_m}{\partial s}[s,t] = mI_{m-1}[s,t] \tag{63}$$

$$\frac{\partial I_m}{\partial t}[s,t] = mR_{m-1}[s,t] \tag{64}$$

$$\frac{d\bar{A}_{l,m}}{du}[u] = \frac{N_{l,m}}{N_{l,m+1}}\bar{A}_{l,m+1}[u] \tag{65}$$

The gravity field can be computed using all the equations given. However, the gradient of the potential is needed. As a change of variables was realized, the chain rule must be applied. In order to avoid filling up pages with math derivations, the results will be given. With patience, the following results can be obtained applying the chain rule and using all the derivatives given.

The gravity field can be computed as

$$\bar{\mathbf{g}} = (a_1[r,s,t,u] + s \cdot a_4[r,s,t,u])\hat{\mathbf{i}} + (a_2[r,s,t,u] + t \cdot a_4[r,s,t,u])\hat{\mathbf{j}} + (a_3[r,s,t,u] + u \cdot a_4[r,s,t,u])\hat{\mathbf{k}} \tag{66}$$

where

$$a_1[r,s,t,u] = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{\rho_{l+1}[r]}{R_{\text{ref}}}m\bar{A}_{l,m}[u]E_{l,m}[s,t] \tag{67}$$

$$a_2[r,s,t,u] = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{\rho_{l+1}[r]}{R_{\text{ref}}}m\bar{A}_{l,m}[u]F_{l,m}[s,t] \tag{68}$$

$$a_3[r,s,t,u] = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{\rho_{l+1}[r]}{R_{\text{ref}}}m\frac{N_{l,m}}{N_{l,m+1}}\bar{A}_{l,m+1}[u]D_{l,m}[s,t] \tag{69}$$

$$a_4[r,s,t,u] = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{\rho_{l+1}[r]}{R_{\text{ref}}}m\frac{N_{l,m}}{N_{l+1,m+1}}\bar{A}_{l+1,m+1}[u]D_{l,m}[s,t] \tag{70}$$

In order to avoid computing factorials, it is easy to see that

$$\frac{N_{l,m}}{N_{l,m+1}} = \sqrt{\frac{(l-m)(2-\delta_m)(l+m+1)}{2-\delta_{m+1}}} \tag{71}$$

$$\frac{N_{l,m}}{N_{l+1,m+1}} = \sqrt{\frac{(l+m+2)(l+m+1)(2l+1)(2-\delta_m)}{(2l+3)(2-\delta_{m+1})}} \tag{72}$$

Using all these expressions, the potential and the gravity field can be computed.

sphericalHarmonics

-_loader: coeffLoader
-_Clm: double**
+_Slm: double**
+_mu: double
+_maxDegree: unsigned int

+new(loader: coeffLoader*, filename: string&, max_degree: const unsigned int, mu: double, reference_radius: double)
+computeField(pos: double*, degree: unsigned int, acc: double*, include_zero_degree: bool = False)
+getters()
+getLastError()

coeffLoader

-_errorMsg

*+load(filename: string, C_bar: double**, S_bar: double**, max_degree: unsigned int*)*
+getLastErrorMsg()
+replaceExpDesignator(str: string)

csvCoeffLoader

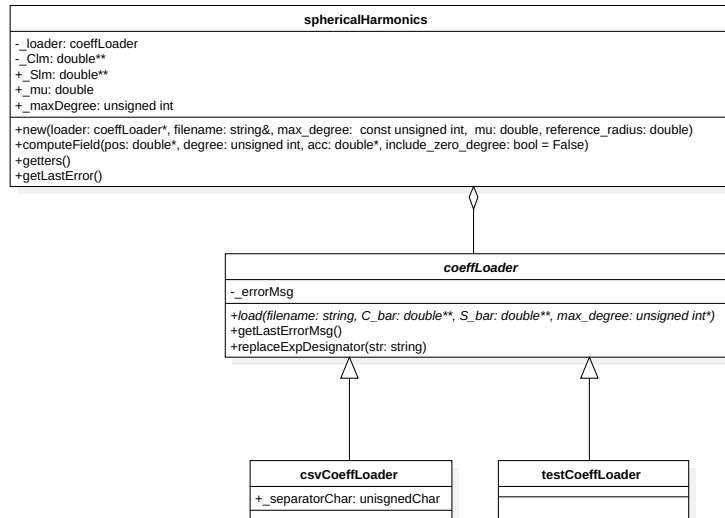+_separatorChar: unisgnedChar

testCoeffLoader

Fig. 2: UML class diagram.

4 Library

The library is structured around the class *sphericalHarmonics*, which computes the gravity field, and a family of classes which inherit from *coeffLoader* that load the coefficients from any source. The abstract class *coeffLoader* can be inherited if coefficient files with new formats are found.

A class diagram of the library can be seen in Figure 2. Figure 3 shows a use case diagram. In the typical use case, the user creates a *coeffLoader* object, and assigns it to the *sphericalHarmonics* object when its constructor is called. The class automatically loads the coefficients using *coeffLoader*'s *load()* interface and pre-computes all those values that are independent of the coordinates. The user then calls *computeField()* method to calculate the gravity field at a given planet-fixed position.
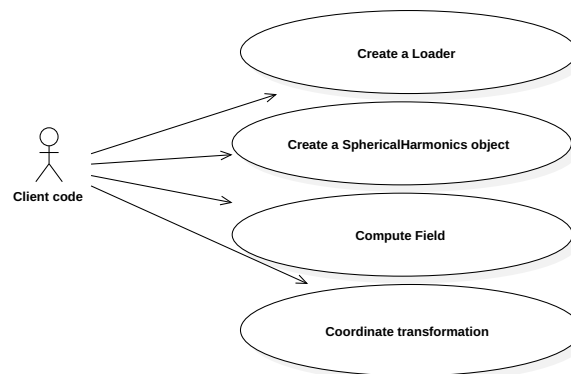
Create a Loader

Create a SphericalHarmonics object

Client code

Compute Field

Coordinate transformation

Fig. 3: UML use case diagram.

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