



**Autonomous Vehicle Simulation (AVS) Laboratory,
University of Colorado**

Basilisk Technical Memorandum

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3D PENDULUM C++ MODEL TO SIMULATE FUEL SLOSH

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Status: Tested
Scope/Contents
The Basilisk fuelSloshPendulum model module is responsible for simulating the fuel slosh in spacecraft tanks. This phenomenon makes the fuel exchange energy and momentum with the spacecraft, thus creates disturbances on attitude and on precise orbit determination. This model together with the fuelSloshParticle, that is a spring mass damper model, provides an efficient and precise simulation of the real phenomenon.

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1 Introduction

Flexing appended bodies and fuel slosh are the sources of excitation of vibration for many spacecraft. This vibration impacts both the translational and rotational spacecraft motion. In order to analyze these effects, it is desirable to incorporate the dynamics in fast computer simulations. However, developing the associated equations of motion (EOMs) is not a trivial task for general three-dimensional motion and often requires a custom re-derivation of the EOMs which can be time intensive. There is a need to introduce EOMs that would be applicable to many spacecraft configurations and could be used without the need of re-derivation. The EOMs considered in this document are the translational rotational dynamics of a spacecraft with a tank with sloshing fuel dynamics.

The vibrational driver being considered is fuel slosh. Slosh occurs when there is relative motion between a tank and the liquid it contains. The main structure of a spacecraft and the liquid exchange linear momentum, angular momentum, and energy. Mathematically, the equations of motion of the structure and the liquid are tightly coupled. In space, the liquid is subjected not only to inertial forces, but also to microgravity, viscous, and surface tension forces. Furthermore, a moving liquid inside a tank produces a change in the position of the center of mass of the whole system in addition to internal torques and forces when a liquid wave hits the walls of the tank.

The most rigorous mathematical approach to sloshing phenomena is given by the Navier-Stokes equations with nonlinear boundary conditions. Several Computational Fluid Dynamics (CFD) methods have been applied to solve this problem using different formulations. Quasi-simultaneous methods can be used to solve the coupled equations of motion. From a control perspective, however, as pointing and maneuvering requirements tighten, slosh models are needed on spacecraft with large fuel tanks to

evaluate the effect of the liquid movement on the attitude control loop. The combined CFD-rigid-body model, although more exact, has some drawbacks from a simulation point of view. First, the inherent complexity of the combined model might not be feasible in early stages of the design. Second, integrating continuum and lumped models can be computationally time consuming.

To avoid these complications, often simplified slosh models are used for control loop modeling. Slosh is comprised of several different kinds of movement, many of which are highly nonlinear. Small-amplitude waves and stable nonlinear rotary slosh can be approximated using lumped mechanical multi-mode models, including either masses, springs, and dampers or pendulums. Using a lumped model may be a useful simplification in a dynamic model for control design purposes. It can be viewed as a complement of the more accurate CFD approach. In this work, a systematic approach to slosh modeling is proposed using approximated multi-mode mechanical models. The slosh model is integrated into the rigid spacecraft providing a fully-coupled model that satisfies momentum and energy conservations. The estimation of the parameters of the model, which may be either computed through CFD simulation or observed experimentally, is not considered.

Another key challenge, beyond the derivation, is the ability to create a fast numerical simulation. If the EOMs are formulated in a way where all the modes are fully cross-coupled, a large system mass matrix must be inverted which is computationally expensive. A solution to decrease the computational impact for a rigid body spacecraft configuration with N fuel slosh pendulums is the back-substitution method. The resulting system mass matrix needed to be inverted for this problem is an $(N+6) \times (N+6)$ matrix. Fast computational speed is achieved by analytically back-substituting the flexing EOM into the rigid body EOM (requires an $N \times N$ inverse), and obtaining a closed form solution for the inertial angular acceleration that can be computed with the typical 3×3 matrix inverse. The slosh modes are then solved as a second stage using the found body acceleration and angular acceleration terms.

The document is outlined as follows. First, the nonlinearly coupled EOMs for the rigid body translational and rotational degrees of freedom are developed, followed by the EOM of the fuel slosh. Next, an analytical back-substitution is presented which helps speed up the numerical evaluation. Lastly, a validation of the EOMs through energy and momentum checks is illustrated with numerical simulation of the EOMs.

2 Problem Statement

The formulation assumes that there is a rigid hub, with N_P lumped masses in the tank for the fuel. Subscript j is used to indicate the j th fuel slosh mass, m_j . Figure 1 displays the frame and variable definitions used for this formulation.

There are four coordinate frames defined for this formulation. The inertial reference frame is indicated by $\mathcal{N} : \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$. The body fixed coordinate frame, $\mathcal{B} : \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$, which is anchored to the hub and can be oriented in any direction. The initial pendulum frame, $\mathcal{P}_{0,j} : \{\hat{p}_{0,j,1}, \hat{p}_{0,j,2}, \hat{p}_{0,j,3}\}$, is a frame with its origin located at tank geometrical center, T . The $\mathcal{P}_{0,j}$ frame is a fixed frame respect to the body frame, oriented such that $\hat{p}_{0,j,1}$ points to the fuel slosh mass in its initial position, P_j . The constant distance from point T to point P_j is defined as l_j .

There are a few more key locations that need to be defined. Point B is the origin of the body frame, and can have any location with respect to the hub. Point B_c is the location of the center of mass of the rigid hub. $P_{c,j}$ is the instantaneous position of the fuel slosh mass m_j . d is vector from the center of the body reference system to the tank geometrical center. l_j is the vector from T to $P_{c,j}$.

Figure 2 provides further detail of the fuel slosh parameters and reference frames. As seen in Figure 1, an individual slosh particle is free to move in every direction while connected by rigid weightless rod to the geometrical center of the tank. A linear damper effect is considered using a damping matrix, D . The variables, φ_j and ϑ_j are state variables and quantify the angular displacement from initial position for the corresponding slosh mass.

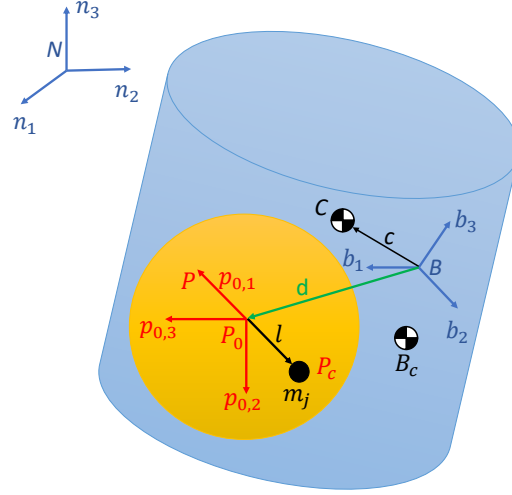


Fig. 1: Frame and variable definitions used for formulation

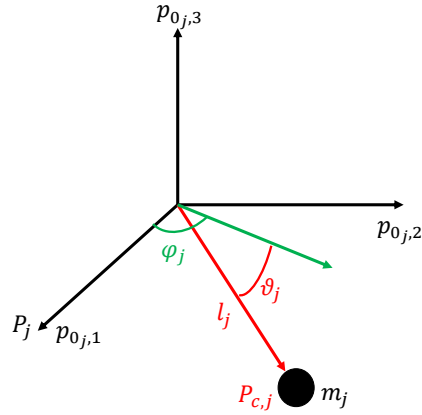


Fig. 2: Further detail of fuel slosh and reference frames

Using the variables and frames defined, the following section outlines the derivation of equations of motion for the spacecraft.

3 Derivation of Equations of Motion

3.1 Rigid Spacecraft Hub Translational Motion

The derivation begins with Newton's first law for the center of mass of the spacecraft.

$$\ddot{\mathbf{r}}_{C/N} = \frac{\mathbf{F}}{m_{sc}} \quad (1)$$

Ultimately the acceleration of the body frame or point B is desired

$$\ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \ddot{\mathbf{c}} \quad (2)$$

The definition of \mathbf{c} can be seen in Eq. (3).

$$\mathbf{c} = \frac{1}{m_{\text{sc}}} \left(m_{\text{hub}} \mathbf{r}_{Bc/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \right) \quad (3)$$

To find the inertial time derivative of \mathbf{c} , it is first necessary to find the time derivative of \mathbf{c} with respect to the body frame. A time derivative of any vector, \mathbf{v} , with respect to the body frame is denoted by \mathbf{v}' ; the inertial time derivative is labeled as $\dot{\mathbf{v}}$. The first and second body-relative time derivatives of \mathbf{c} can be seen in Eqs. (4) and (5).

$$\mathbf{c}' = \frac{1}{m_{\text{sc}}} \left(\sum_{j=1}^{N_P} m_j \mathbf{r}'_{P_{c,j}/B} \right) \quad (4)$$

$$\mathbf{c}'' = \frac{1}{m_{\text{sc}}} \left(\sum_{j=1}^{N_P} m_j \mathbf{r}''_{P_{c,j}/B} \right) \quad (5)$$

Remembering that the derivative of d is null respect to the body frame, the first and second body time derivatives of $\mathbf{r}_{P_{c,j}/B}$ are

$$\mathbf{r}_{P_{c,j}/B} = l_j \begin{bmatrix} \cos(\varphi_j) \cos(\vartheta_j) \\ \sin(\varphi_j) \cos(\vartheta_j) \\ -\sin(\vartheta_j) \end{bmatrix} + d \quad (6)$$

$$\mathbf{r}'_{P_{c,j}/B} = l_j \begin{bmatrix} -\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ \dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ -\dot{\vartheta}_j \cos(\vartheta_j) \end{bmatrix} \quad (7)$$

$$\mathbf{r}''_{P_{c,j}/B} = l_j \begin{bmatrix} -\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ \ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ -\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \end{bmatrix} \quad (8)$$

Eqs. (4) and (5) are next reformulated to include these new definitions:

$$\mathbf{c}' = \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(\dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} - \dot{\vartheta}_j \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right] \quad (9)$$

$$\mathbf{c}'' = \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \right. \right. \\ \left. \left. + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} + \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \right. \right. \\ \left. \left. - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \left(-\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right] \quad (10)$$

Using the transport theorem yields the following definition for $\ddot{\mathbf{c}}$

$$\ddot{\mathbf{c}} = \mathbf{c}'' + 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (11)$$

Eq. (2) is updated to include Eq. (11)

$$\ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \mathbf{c}'' - 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' - \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} - \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (12)$$

Substituting Eq.(10) into Eq.(12) results in

$$\begin{aligned} \ddot{\mathbf{r}}_{B/N} = \ddot{\mathbf{r}}_{C/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \Bigg[& \left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \right. \\ & - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \Big) \hat{\mathbf{p}}_{0,j,1} + \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right. \\ & - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \Big) \hat{\mathbf{p}}_{0,j,2} + \left(-\ddot{\vartheta}_j \cos(\vartheta_j) \right. \\ & \left. \left. + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right] - 2\boldsymbol{\omega}_{B/N} \times \mathbf{c}' - \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{c} - \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{c}) \quad (13) \end{aligned}$$

Moving second order terms to the left hand side and introducing the tilde matrix to replace the cross product operators simplifies the equation to

$$\begin{aligned} \ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}] \dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \Bigg[& \left(\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) + \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} \\ & + \left(-\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) + \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \ddot{\vartheta}_j \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \Big] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' \\ - [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \Bigg[& \left(-\dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,1} \\ & + \left(-\dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + \dot{\vartheta}_j^2 \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \Big] \quad (14) \end{aligned}$$

Rearranging the terms:

$$\begin{aligned} \ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}] \dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \Bigg[& \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \\ & + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \Big] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' \\ - [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \Bigg[& \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \\ & + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \\ & + \left(2\sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \Big] \quad (15) \end{aligned}$$

Equation (15) is the translational motion equation and is the first EOM needed to describe the motion of the spacecraft. The following section develops the rotational EOM.

3.2 Rigid Spacecraft Hub Rotational Motion

Starting with Euler's equation when the body fixed coordinate frame origin is not coincident with the center of mass of the body

$$\dot{\mathbf{H}}_{sc,B} = \mathbf{L}_B + m_{sc}\ddot{\mathbf{r}}_{B/N} \times \mathbf{c} \quad (16)$$

where \mathbf{L}_B is the total external torque about point B . The definition of the angular momentum vector of the spacecraft about point B is

$$\mathbf{H}_{sc,B} = [\mathbf{I}_{hub,B_c}]\boldsymbol{\omega}_{B/N} + m_{hub}\mathbf{r}_{B_c/B} \times \dot{\mathbf{r}}_{B_c/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} \quad (17)$$

Now the inertial time derivative of Eq. (17) is taken and yields

$$\dot{\mathbf{H}}_{sc,B} = [\mathbf{I}_{hub,B_c}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [\mathbf{I}_{hub,B_c}]\boldsymbol{\omega}_{B/N} + m_{hub}\mathbf{r}_{B_c/B} \times \ddot{\mathbf{r}}_{B_c/B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \ddot{\mathbf{r}}_{P_{c,j}/B} \quad (18)$$

The terms $\ddot{\mathbf{r}}_{B_c/B}$ and $\ddot{\mathbf{r}}_{P_{c,j}/B}$ are found using the transport theorem and knowing that $\mathbf{r}_{B_c/B}$ is fixed with respect to the body frame.

$$\ddot{\mathbf{r}}_{B_c/B} = \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{B_c/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{B_c/B}) \quad (19)$$

$$\ddot{\mathbf{r}}_{P_{c,j}/B} = \mathbf{r}_{P_{c,j}/B}'' + 2\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}' + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}) \quad (20)$$

Incorporating Eqs. (19) - (20) into Eq. (18) results in

$$\begin{aligned} \dot{\mathbf{H}}_{sc,B} = & [\mathbf{I}_{hub,B_c}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [\mathbf{I}_{hub,B_c}]\boldsymbol{\omega}_{B/N} + m_{hub}\mathbf{r}_{B_c/B} \times (\dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{B_c/B}) + \\ & + m_{hub}\mathbf{r}_{B_c/B} \times [\boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{B_c/B})] + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times [\mathbf{r}_{P_{c,j}/B}'' + 2\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B}' + \\ & + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{r}_{P_{c,j}/B} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B})] \end{aligned} \quad (21)$$

Applying the parallel axis theorem the following inertia tensor terms are defined as

$$[\mathbf{I}_{hub,B}] = [\mathbf{I}_{hub,B_c}] + m_{hub}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T \quad (22)$$

$$[\mathbf{I}_{sc,B}] = [\mathbf{I}_{hub,B}] + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \quad (23)$$

Taking the body-relative time derivative of Equation (23) yields

$$[\mathbf{I}_{sc,B}]' = - \sum_{j=1}^{N_P} m_j \left([\tilde{\mathbf{r}}_{P_{c,j}/B}]' [\tilde{\mathbf{r}}_{P_{c,j}/B}] + [\tilde{\mathbf{r}}_{P_{c,j}/B}] [\tilde{\mathbf{r}}_{P_{c,j}/B}]' \right) \quad (24)$$

The Jacobi Identity, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$, is applied to the cross products of Eq. (21):

$$\begin{aligned} \dot{\mathbf{H}}_{sc,B} = & [\mathbf{I}_{hub,B_c}]\dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [\mathbf{I}_{hub,B_c}]\boldsymbol{\omega}_{B/N} - m_{hub}\mathbf{r}_{B_c/B} \times (\mathbf{r}_{B_c/B} \times \dot{\boldsymbol{\omega}}_{B/N}) + \\ & + m_{hub}\boldsymbol{\omega}_{B/N} \times [\mathbf{r}_{B_c/B} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{B_c/B})] + \sum_{j=1}^{N_P} m_j \left\{ \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}'' - \mathbf{r}_{P_{c,j}/B} \times (\mathbf{r}_{P_{c,j}/B}' \times \boldsymbol{\omega}_{B/N}) \right. \\ & - \mathbf{r}_{P_{c,j}/B}' \times (\mathbf{r}_{P_{c,j}/B} \times \boldsymbol{\omega}_{B/N}) + \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}') \\ & \left. - \mathbf{r}_{P_{c,j}/B} \times (\mathbf{r}_{P_{c,j}/B} \times \dot{\boldsymbol{\omega}}_{B/N}) + \boldsymbol{\omega}_{B/N} \times [\mathbf{r}_{P_{c,j}/B} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{r}_{P_{c,j}/B})] \right\} \end{aligned} \quad (25)$$

Rearranging the terms in Eq. (25) yields:

$$\begin{aligned} \dot{\mathbf{H}}_{\text{sc},B} = & \left\{ [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \right\} \dot{\boldsymbol{\omega}}_{B/N} \\ & + \boldsymbol{\omega}_{B/N} \times \left\{ [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\mathbf{r}}_{B_c/B}][\tilde{\mathbf{r}}_{B_c/B}]^T + \sum_{j=1}^{N_P} m_j[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]^T \right\} \boldsymbol{\omega}_{B/N} \\ & + \sum_{j=1}^{N_P} m_j \left\{ \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}'' - \left[[\tilde{\mathbf{r}}_{P_{c,j}/B}][\tilde{\mathbf{r}}_{P_{c,j}/B}]' + [\tilde{\mathbf{r}}_{P_{c,j}/B}]'[\tilde{\mathbf{r}}_{P_{c,j}/B}] \right] \boldsymbol{\omega}_{B/N} \right. \\ & \left. + \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}') \right\} \quad (26) \end{aligned}$$

Using Eqs. (23) and (24) to simplify results in Eq. (26), the following simplified equation is obtained:

$$\begin{aligned} \dot{\mathbf{H}}_{\text{sc},B} = & [I_{\text{sc},B}] \dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} + [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} + \sum_{j=1}^{N_P} \left[m_j \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}'' \right. \\ & \left. + m_j \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}') \right] \quad (27) \end{aligned}$$

Eqs. (16) and (27) are equated and yield

$$\begin{aligned} \mathbf{L}_B + m_{\text{sc}} \ddot{\mathbf{r}}_{B/N} \times \mathbf{c} = & [I_{\text{sc},B}] \dot{\boldsymbol{\omega}}_{B/N} + \boldsymbol{\omega}_{B/N} \times [I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} + [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} + \sum_{j=1}^{N_P} \left[m_j \mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}'' \right. \\ & \left. + m_j \boldsymbol{\omega}_{B/N} \times (\mathbf{r}_{P_{c,j}/B} \times \mathbf{r}_{P_{c,j}/B}') \right] \quad (28) \end{aligned}$$

Finally, using tilde matrix and simplifying yields the modified Euler equation, which is the second EOM necessary to describe the motion of the spacecraft.

$$\begin{aligned} [I_{\text{sc},B}] \dot{\boldsymbol{\omega}}_{B/N} = & -[\tilde{\boldsymbol{\omega}}_{B/N}][I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} \left(m_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}_{P_{c,j}/B}'' \right. \\ & \left. + m_j [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}_{P_{c,j}/B}' \right) + \mathbf{L}_B - m_{\text{sc}}[\tilde{\mathbf{c}}] \ddot{\mathbf{r}}_{B/N} \quad (29) \end{aligned}$$

Rearranging Eq. (29) to be in the same form as the previous sections results in

$$\begin{aligned} m_{\text{sc}}[\tilde{\mathbf{c}}] \ddot{\mathbf{r}}_{B/N} + [I_{\text{sc},B}] \dot{\boldsymbol{\omega}}_{B/N} - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] & \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \right. \\ & \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \right] = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} \\ & - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}_{P_{c,j}/B}' + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \right. \\ & \left. \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \right. \right. \\ & \left. \left. \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \right\} \quad (30) \end{aligned}$$

3.3 Fuel Slosh Motion

The fuel slosh motion is being approximated by a lumped mechanical multi-mode model. Figure 1 shows that a single fuel slosh particle is free to move around the geometrical center of the tank at a fixed distance l_j and this formulation is generalized to include N_P number of fuel slosh particles. The derivation begins with Euler's law for each fuel slosh particle:

$$\dot{\mathbf{H}}_{T,j} = \mathbf{L}_{T,j} + m_j \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (31)$$

Where the $\mathbf{L}_{T,j}$ represents the external torques. It contains the damping term that is modeled as \mathbf{D} damping matrix multiplied by the relative velocity between the tank and the fuel, \mathbf{l}'_j . It takes into account the torque due to gravity and any other external torque.

$$\mathbf{L}_{T,j} = \mathbf{l}_j \times \mathbf{F}_g - \mathbf{D} \mathbf{l}'_j + \boldsymbol{\tau}_{ext,j} \quad (32)$$

It is necessary to express \mathbf{H}_T of the fuel slosh particle as:

$$\mathbf{H}_{T,j} = m_j \mathbf{l}_j \times \dot{\mathbf{l}}_j \quad (33)$$

Deriving this equation we obtain:

$$\dot{\mathbf{H}}_{T,j} = m_j \mathbf{l}_j \times \ddot{\mathbf{l}}_j \quad (34)$$

Using the transport theorem once again:

$$\ddot{\mathbf{l}}_j = \mathbf{l}''_j + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \quad (35)$$

\mathbf{l}_j prime and second derivative are equal to (7) and (8) because the \mathbf{d} vector is constant in body frame. The previous results are reported here:

$$\mathbf{l}_j = l_j \begin{bmatrix} \cos(\varphi_j) \cos(\vartheta_j) \\ \sin(\varphi_j) \cos(\vartheta_j) \\ -\sin(\vartheta_j) \end{bmatrix} \quad (36)$$

$$\mathbf{l}'_j = l_j \begin{bmatrix} -\dot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ \dot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ -\dot{\vartheta}_j \cos(\vartheta_j) \end{bmatrix} \quad (37)$$

$$\mathbf{l}''_j = l_j \begin{bmatrix} -\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \\ \ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \\ -\dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \\ -\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \end{bmatrix} \quad (38)$$

Equating eqs. (31) and (34) and using eq. (35), we obtain:

$$m_j \mathbf{l}_j \times [\mathbf{l}''_j + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j)] = \mathbf{L}_{T,j} + m_j \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (39)$$

Substituting eq. (38) in eq. (39)

$$\begin{aligned}
m_j \mathbf{l}_j \times & \left[l_j \left(-\ddot{\varphi}_j \sin(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \cos(\varphi_j) \cos(\vartheta_j) - \dot{\vartheta}_j^2 \cos(\varphi_j) \cos(\vartheta_j) \right. \right. \\
& + 2\dot{\varphi}_j \dot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) \Big) \hat{\mathbf{p}}_{0,j,1} + l_j \left(\ddot{\varphi}_j \cos(\varphi_j) \cos(\vartheta_j) - \ddot{\vartheta}_j \sin(\varphi_j) \sin(\vartheta_j) - \dot{\varphi}_j^2 \sin(\varphi_j) \cos(\vartheta_j) \right. \\
& \left. \left. - \dot{\vartheta}_j^2 \sin(\varphi_j) \cos(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j \cos(\varphi_j) \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,2} + l_j \left(-\ddot{\vartheta}_j \cos(\vartheta_j) + \dot{\vartheta}_j^2 \sin(\vartheta_j) \right) \hat{\mathbf{p}}_{0,j,3} \right. \\
& \left. + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] = \mathbf{L}_{T,j} + m_j \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (40)
\end{aligned}$$

Execute separately the following vectorial product, using C and S to indicate \cos and \sin , respectively.

$$\begin{aligned}
& l_j^2 \begin{bmatrix} 0 & S(\vartheta_j) & S(\varphi_j)C(\vartheta_j) \\ -S(\vartheta_j) & 0 & -C(\varphi_j)C(\vartheta_j) \\ -S(\varphi_j)C(\vartheta_j) & C(\varphi_j)C(\vartheta_j) & 0 \end{bmatrix} \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{aligned} & -\ddot{\varphi}_j S(\varphi_j)C(\vartheta_j) - \ddot{\vartheta}_j C(\varphi_j)S(\vartheta_j) \\ & -\dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j) - \dot{\vartheta}_j^2 C(\varphi_j)C(\vartheta_j) \\ & + 2\dot{\varphi}_j \dot{\vartheta}_j S(\varphi_j)S(\vartheta_j) \\ & \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)S(\vartheta_j) \\ & -\dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j) - \dot{\vartheta}_j^2 S(\varphi_j)C(\vartheta_j) \\ & - 2\dot{\varphi}_j \dot{\vartheta}_j C(\varphi_j)S(\vartheta_j) \\ & -\ddot{\vartheta}_j C(\vartheta_j) + \dot{\vartheta}_j^2 S(\vartheta_j) \end{aligned} \right] \end{matrix} \\
& = l_j^2 \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{aligned} & \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) - \dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \dot{\vartheta}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ & - 2\dot{\varphi}_j \dot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j)C^2(\vartheta_j) + \dot{\vartheta}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ & \ddot{\varphi}_j S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) + \dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \dot{\vartheta}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ & - 2\dot{\varphi}_j \dot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)C^2(\vartheta_j) - \dot{\vartheta}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) \\ & \ddot{\varphi}_j S^2(\varphi_j)C^2(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j)S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \dot{\varphi}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) \\ & + \dot{\vartheta}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j S^2(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\varphi}_j C^2(\varphi_j)C^2(\vartheta_j) \\ & - \ddot{\vartheta}_j C(\varphi_j)S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \dot{\varphi}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) - \dot{\vartheta}_j^2 C(\varphi_j)S(\varphi_j)C^2(\vartheta_j) \\ & - 2\dot{\varphi}_j \dot{\vartheta}_j C^2(\varphi_j)C(\vartheta_j)S(\vartheta_j) \end{aligned} \right] \end{matrix} \\
& = l_j^2 \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{aligned} & \ddot{\varphi}_j C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - \ddot{\vartheta}_j S(\varphi_j) - \dot{\varphi}_j^2 S(\varphi_j)C(\vartheta_j)S(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j C(\varphi_j)S^2(\vartheta_j) \\ & \ddot{\varphi}_j S(\varphi_j)C(\vartheta_j)S(\vartheta_j) + \ddot{\vartheta}_j C(\varphi_j) + \dot{\varphi}_j^2 C(\varphi_j)C(\vartheta_j)S(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j S(\varphi_j)S^2(\vartheta_j) \\ & \ddot{\varphi}_j C^2(\vartheta_j) - 2\dot{\varphi}_j \dot{\vartheta}_j C(\vartheta_j)S(\vartheta_j) \end{aligned} \right] \end{matrix} \quad (41)
\end{aligned}$$

Using the results obtained in eq. (41), project eq. (40) on the axis of rotation of φ and ϑ . The φ rotation axis is $\hat{\mathbf{p}}_{0,j,3}$, while we will call $\hat{\mathbf{p}}'_{0,j,2}$ the ϑ rotation axis.

$$\hat{\mathbf{p}}_{0,j,3} = \begin{matrix} \mathcal{P}_{0,j} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix} \quad (42)$$

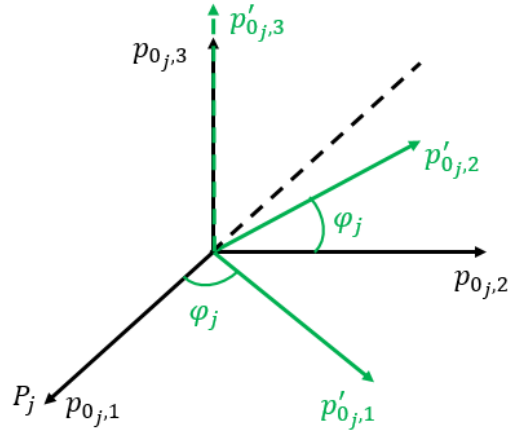


Fig. 3: Rotation Axes

$$m_j l_j^2 \left[\ddot{\vartheta}_j \cos^2(\vartheta_j) - 2\dot{\vartheta}_j \dot{\vartheta}_j \cos(\vartheta_j) \sin(\vartheta_j) \right] + m_j \hat{\mathbf{p}}_{0,j,3}^T \mathbf{l}_j \times \left[\dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] \\ = \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + m_j \hat{\mathbf{p}}_{0,j,3}^T \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (43)$$

$$\hat{\mathbf{p}}'_{0,j,2} = \begin{matrix} \mathcal{P}_{0,j} \\ \left[\begin{array}{c} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{array} \right] \end{matrix} \quad (44)$$

$$m_j l_j^2 \left[\ddot{\vartheta}_j + \dot{\vartheta}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) \right] + m_j \hat{\mathbf{p}}_{0,j,2}^T \mathbf{l}_j \times \left[\dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{l}_j + 2\boldsymbol{\omega}_{B/N} \times \mathbf{l}'_j + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{l}_j) \right] \\ = \hat{\mathbf{p}}_{0,j,2}^T \mathbf{L}_{T,j} + m_j \hat{\mathbf{p}}_{0,j,2}^T \ddot{\mathbf{r}}_{T/N} \times \mathbf{l}_j \quad (45)$$

Remembering that $\mathbf{r}_{T/N} = \mathbf{r}_{B/N} + \mathbf{d}$, and that \mathbf{d} is constant in the body frame, we can use once again the transport theorem to write:

$$\ddot{\mathbf{d}} = \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{d} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{d}) \quad (46)$$

$$\ddot{\mathbf{r}}_{T/N} = \ddot{\mathbf{r}}_{B/N} + \dot{\boldsymbol{\omega}}_{B/N} \times \mathbf{d} + \boldsymbol{\omega}_{B/N} \times (\boldsymbol{\omega}_{B/N} \times \mathbf{d}) \quad (47)$$

Rearranging the terms as done in the previous sections

$$m_j l_j^2 \ddot{\vartheta}_j \cos^2(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} + m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} = -m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} \\ + \hat{\mathbf{p}}_{0,j,3}^T \mathbf{L}_{T,j} + 2m_j l_j^2 \dot{\vartheta}_j \dot{\vartheta}_j \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \quad (48)$$

$$m_j l_j^2 \ddot{\vartheta}_j - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} + m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} = -m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} \\ + \hat{\mathbf{p}}_{0,j,2}^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\vartheta}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0,j,2}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \quad (49)$$

Eqs. (48) and (49) are the Fuel Slosh Particle equations.

4 Back-substitution Method

The equations presented in the previous sections result in $2N_P + 6$ coupled differential equations. Therefore, if the EOMs were placed into state space form, a system mass matrix of size $2N_P + 6$ would need to be inverted to numerically integrate the EOMs. This can result in a computationally expensive simulation. The computation effort to numerically invert an $N \times N$ matrix scales with N^3 . In the following section, the EOMs are manipulated using a back-substitution method to increase the computational efficiency.

This manipulation involves inverting twice a (3×3) matrix, the A^{-1} and $(D - CA^{-1}B)^{-1}$ matrices as it is shown in eqs. (75) and (76). Then the system is completely solved back substituting for fuel slosh and translational motions. The derivation of the back-substitution method can be seen in the following sections.

4.1 Fuel Slosh Motion

Starting from eq. (48)

$$\ddot{\varphi}_j = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left\{ m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} - m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} \right. \\ \left. + \hat{\mathbf{p}}_{0j,3}^T \mathbf{L}_{T,j} + 2m_j l_j^2 \dot{\varphi}_j \dot{\vartheta} \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (50)$$

$$\ddot{\varphi}_j = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left(m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} + a_{\varphi_j} \right) \quad (51)$$

Writing it this way instead

$$\ddot{\varphi}_j = \mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j} \quad (52)$$

Where

$$\mathbf{a}_{\varphi_j}^T = -\frac{\hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j]}{l_j^2 \cos^2(\vartheta_j)} \quad (53)$$

$$\mathbf{b}_{\varphi_j}^T = \frac{\hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}])}{l_j^2 \cos^2(\vartheta_j)} \quad (54)$$

$$c_{\varphi_j} = \frac{1}{m_j l_j^2 \cos^2(\vartheta_j)} \left\{ -m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0j,3}^T \mathbf{L}_{T,j} \right. \\ \left. + 2m_j l_j^2 \dot{\varphi}_j \dot{\vartheta} \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0j,3}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (55)$$

Doing the same for eq. (49)

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2} \left\{ m_j \hat{\mathbf{p}}_{0j,2}^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0j,2}^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} - m_j \hat{\mathbf{p}}_{0j,2}^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} \right. \\ \left. + \hat{\mathbf{p}}_{0j,2}^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\varphi}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) - m_j \hat{\mathbf{p}}_{0j,2}^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}'_j + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (56)$$

$$\ddot{\vartheta}_j = \frac{1}{m_j l_j^2} \left(m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}]) \dot{\boldsymbol{\omega}}_{B/N} - m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] \ddot{\mathbf{r}}_{B/N} + a_{\vartheta_j} \right) \quad (57)$$

Writing this a different way

$$\ddot{\vartheta}_j = \mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j} \quad (58)$$

Where

$$\mathbf{a}_{\vartheta_j}^T = -\frac{\hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j]}{l_j^2} \quad (59)$$

$$\mathbf{b}_{\vartheta_j}^T = \frac{\hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] ([\tilde{\mathbf{l}}_j] + [\tilde{\mathbf{d}}])}{l_j^2} \quad (60)$$

$$c_{\vartheta_j} = \frac{1}{m_j l_j^2} \left\{ -m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{d} + \hat{\mathbf{p}}_{0,j,2}'^T \mathbf{L}_{T,j} - m_j l_j^2 \dot{\varphi}_j^2 \cos(\vartheta_j) \sin(\vartheta_j) \right. \\ \left. - m_j \hat{\mathbf{p}}_{0,j,2}'^T [\tilde{\mathbf{l}}_j] \left[2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j' + [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{l}_j \right] \right\} \quad (61)$$

4.2 Translation

Plugging these definitions into the translation equation

$$\ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}] \dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \ddot{\varphi}_j \right. \\ \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \ddot{\vartheta}_j \right] = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' \\ - [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{sc}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \\ \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\ \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \quad (62)$$

Results in

$$\begin{aligned}
& \ddot{\mathbf{r}}_{B/N} - [\tilde{\mathbf{c}}] \dot{\boldsymbol{\omega}}_{B/N} - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) (\mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j}) \right. \\
& \quad \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) (\mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j}) \right] \\
& = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \\
& \quad \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
& \quad \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \quad (63)
\end{aligned}$$

Simplifying

$$\begin{aligned}
& \left\{ [I_{3 \times 3}] - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \quad \left. \left. + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \ddot{\mathbf{r}}_{B/N} + \left\{ -[\tilde{\mathbf{c}}] - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \quad \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = \ddot{\mathbf{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \frac{1}{m_{\text{sc}}} \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
& \quad \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (64)
\end{aligned}$$

Multiply both sides by m_{sc} .

$$\begin{aligned}
& \left\{ m_{\text{sc}}[I_{3 \times 3}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \ddot{\mathbf{r}}_{B/N} + \left\{ -m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \quad \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = m_{\text{sc}} \ddot{\mathbf{r}}_{C/N} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} - \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 \right. \\
& \quad \left. + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (65)
\end{aligned}$$

4.3 Rotation

Same thing for rotation:

$$\begin{aligned}
& m_{\text{sc}}[\tilde{\mathbf{c}}] \ddot{\mathbf{r}}_{B/N} + [I_{\text{sc},B}] \dot{\boldsymbol{\omega}}_{B/N} - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) (\mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j}) + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right. \right. \\
& \quad \left. \left. + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) (\mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j}) \right] = \\
& \quad \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}][I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}][\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} \right. \\
& \quad \left. + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \quad \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j \right] \right\} \quad (66)
\end{aligned}$$

Next

$$\begin{aligned}
& \left\{ m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& + \left. \left. \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \Big\} \ddot{\mathbf{r}}_{B/N} + \left\{ [I_{\text{sc},B}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& - \left. \left. \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \Big\} \dot{\boldsymbol{\omega}}_{B/N} \\
& = \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}] [I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} \right. \\
& + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& - \left. \left. \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& - \left. \left. 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\
& \left. \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \right\} \quad (67)
\end{aligned}$$

4.4 Remaining Back-substitution Steps

The following definitions can be defined:

$$\begin{aligned}
[A] = & \left\{ m_{\text{sc}}[I_{3 \times 3}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T \right. \right. \\
& \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \quad (68)
\end{aligned}$$

$$\begin{aligned}
[B] = & \left\{ -m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T \right. \right. \\
& \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \quad (69)
\end{aligned}$$

$$\begin{aligned}
[C] = & \left\{ m_{\text{sc}}[\tilde{\mathbf{c}}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{a}_{\varphi_j}^T \right. \right. \\
& \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{a}_{\vartheta_j}^T \right] \right\} \quad (70)
\end{aligned}$$

$$\begin{aligned}
[D] = & \left\{ [I_{\text{sc},B}] - \sum_{j=1}^{N_P} m_j l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \mathbf{b}_{\varphi_j}^T \right. \right. \\
& \left. \left. + \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \mathbf{b}_{\vartheta_j}^T \right] \right\} \quad (71)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{\text{trans}} = & m_{\text{sc}} \ddot{\mathbf{r}}_{C/N} - 2m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c}' - m_{\text{sc}} [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\boldsymbol{\omega}}_{B/N}] \mathbf{c} \\
& - \sum_{j=1}^{N_P} m_j l_j \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \\
& \quad \left. \left. - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\
& \quad \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \quad (72)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{\text{rot}} = & \mathbf{L}_B - [\tilde{\boldsymbol{\omega}}_{B/N}] [I_{\text{sc},B}] \boldsymbol{\omega}_{B/N} - [I'_{\text{sc},B}] \boldsymbol{\omega}_{B/N} \\
& - \sum_{j=1}^{N_P} m_j \left\{ [\tilde{\boldsymbol{\omega}}_{B/N}] [\tilde{\mathbf{r}}_{P_{c,j}/B}] \mathbf{r}'_{P_{c,j}/B} + l_j [\tilde{\mathbf{r}}_{P_{c,j}/B}] \left[\left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j^2 \right. \right. \\
& \quad \left. \left. + \left(-\cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) \dot{\vartheta}_j^2 + \left(2 \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} \right. \right. \right. \\
& \quad \left. \left. - 2 \cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) \dot{\varphi}_j \dot{\vartheta}_j - \left(\sin(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} - \cos(\varphi_j) \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} \right) c_{\varphi_j} \right. \\
& \quad \left. \left. - \left(\cos(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,1} + \sin(\varphi_j) \sin(\vartheta_j) \hat{\mathbf{p}}_{0,j,2} + \cos(\vartheta_j) \hat{\mathbf{p}}_{0,j,3} \right) c_{\vartheta_j} \right] \right\} \quad (73)
\end{aligned}$$

Therefore the translation and rotation EOMs are written in the following form

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_{B/N} \\ \dot{\boldsymbol{\omega}}_{B/N} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{\text{trans}} \\ \mathbf{v}_{\text{rot}} \end{bmatrix} \quad (74)$$

Solving the system-of-equations by

$$\dot{\boldsymbol{\omega}}_{B/N} = \left([D] - [C][A]^{-1}[B] \right)^{-1} (\mathbf{v}_{\text{rot}} - [C][A]^{-1} \mathbf{v}_{\text{trans}}) \quad (75)$$

$$\ddot{\mathbf{r}}_{B/N} = [A]^{-1} (\mathbf{v}_{\text{trans}} - [B] \dot{\boldsymbol{\omega}}_{B/N}) \quad (76)$$

Now the other state variables can be solved using Eqs. (52) and (58):

$$\ddot{\varphi}_j = \mathbf{a}_{\varphi_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\varphi_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\varphi_j} \quad (77)$$

$$\ddot{\vartheta}_j = \mathbf{a}_{\vartheta_j}^T \ddot{\mathbf{r}}_{B/N} + \mathbf{b}_{\vartheta_j}^T \dot{\boldsymbol{\omega}}_{B/N} + c_{\vartheta_j} \quad (78)$$

5 Rotational Kinetic Energy

The total rotational kinetic energy (i.e. kinetic energy about the center of mass) of the spacecraft is:

$$T_{rot} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, C} \cdot \dot{\mathbf{r}}_{B_c, C} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/C} \cdot \dot{\mathbf{r}}_{P_{c,j}/C} \quad (79)$$

Expanding these terms results in

$$T_{rot} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} (\dot{\mathbf{r}}_{B_c, B} - \dot{\mathbf{c}}) \cdot (\dot{\mathbf{r}}_{B_c, B} - \dot{\mathbf{c}}) + \sum_{j=1}^{N_P} \frac{1}{2} m_j (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \cdot (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \quad (80)$$

Expanding further

$$\begin{aligned} T_{rot} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} (\dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} - 2 \dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{c}} + \dot{\mathbf{c}} \cdot \dot{\mathbf{c}}) \\ + \sum_{j=1}^{N_P} \frac{1}{2} m_j (\dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} - 2 \dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{c}} + \dot{\mathbf{c}} \cdot \dot{\mathbf{c}}) \end{aligned} \quad (81)$$

Combining like terms results in

$$\begin{aligned} T_{rot} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} \\ - \left[m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \right] \cdot \dot{\mathbf{c}} + \frac{1}{2} \left[m_{\text{hub}} + \sum_{j=1}^{N_P} m_j \right] \dot{\mathbf{c}} \cdot \dot{\mathbf{c}} \end{aligned} \quad (82)$$

Performing a final simplification yields

$$T_{rot} = \frac{1}{2} \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}^T [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \frac{1}{2} m_{\text{hub}} \dot{\mathbf{r}}_{B_c, B} \cdot \dot{\mathbf{r}}_{B_c, B} + \sum_{j=1}^{N_P} \frac{1}{2} m_j \dot{\mathbf{r}}_{P_{c,j}/B} \cdot \dot{\mathbf{r}}_{P_{c,j}/B} - \frac{1}{2} m_{SC} \dot{\mathbf{c}} \cdot \dot{\mathbf{c}} \quad (83)$$

6 Rotational Angular Momentum

The total rotational angular momentum of the spacecraft about point C is

$$\mathbf{H}_{rot, C} = [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}} \mathbf{r}_{B_c, C} \times \dot{\mathbf{r}}_{B_c, C} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/C} \times \dot{\mathbf{r}}_{P_{c,j}/C} \quad (84)$$

Expanding these terms yields

$$\mathbf{H}_{rot, C} = [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}} (\mathbf{r}_{B_c, B} - \mathbf{c}) \times (\dot{\mathbf{r}}_{B_c, B} - \dot{\mathbf{c}}) + \sum_{j=1}^{N_P} m_j (\mathbf{r}_{P_{c,j}/B} - \mathbf{c}) \times (\dot{\mathbf{r}}_{P_{c,j}/B} - \dot{\mathbf{c}}) \quad (85)$$

Distributing this result

$$\begin{aligned} \mathbf{H}_{rot, C} = [I_{\text{hub}, B_c}] \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}} (\mathbf{r}_{B_c, B} \times \dot{\mathbf{r}}_{B_c, B} - \mathbf{r}_{B_c, B} \times \dot{\mathbf{c}} - \mathbf{c} \times \dot{\mathbf{r}}_{B_c, B} + \mathbf{c} \times \dot{\mathbf{c}}) \\ + \sum_{j=1}^{N_P} m_j (\mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} - \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{c}} - \mathbf{c} \times \dot{\mathbf{r}}_{P_{c,j}/B} + \mathbf{c} \times \dot{\mathbf{c}}) \end{aligned} \quad (86)$$

Simplifying this result yields the final equation

$$\mathbf{H}_{rot,C} = [\mathbf{I}_{hub,B_c}] \boldsymbol{\omega}_{B/\mathcal{N}} + m_{hub} \mathbf{r}_{B_c,B} \times \dot{\mathbf{r}}_{B_c,B} + \sum_{j=1}^{N_P} m_j \mathbf{r}_{P_{c,j}/B} \times \dot{\mathbf{r}}_{P_{c,j}/B} - m_{SC} \mathbf{c} \times \dot{\mathbf{c}} \quad (87)$$

7 Reference System Change

As we can see in Eqs. (53), (54) and (55), the problem is singular for $\vartheta = \pi/2 + k\pi$, ($k = 0, 1, 2, \dots$). In order to solve this issue, a rotation of the pendulum frame, $\mathcal{P}_{0,j}$, is necessary. This rotation is performed when ϑ_j reaches a value multiple of $\pi/4$, to remain always far enough from the singularity. The new pendulum frame $\mathcal{P}_{0,j}^{new}$ is obtained with a rotation of the actual value of φ around the $\hat{\mathbf{p}}_{3,j}$ axes, and of ϑ around $\hat{\mathbf{p}}_{2,j}^{new}$ axes. This would lead to a new pendulum reference frame with $\hat{\mathbf{p}}_{1,j}^{new}$ aligned along the pendulum direction. The Fig. 4 shows the $\mathcal{P}_{0,j}^{new}$ respect to $\mathcal{P}_{0,j}$. At this point is easy to see that the

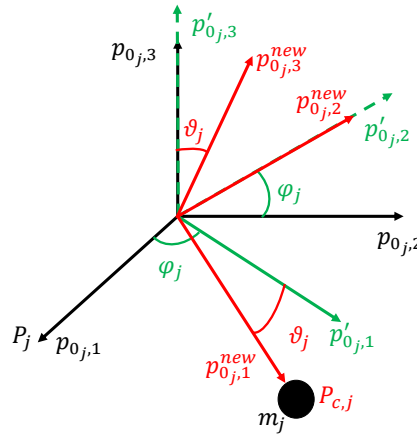


Fig. 4: $\mathcal{P}_{0,j}^{new}$ frame definition

new value of φ_j and ϑ_j are equal to 0. To compute the new value of $\dot{\varphi}_j$ and $\dot{\vartheta}_j$, Eq. (37) is reversed and it yields:

$$\dot{\varphi} = \frac{l_j[2]}{l_j} \quad (88)$$

$$\dot{\vartheta} = -\frac{l_j[3]}{l_j} \quad (89)$$

The integration can continue using these new values and the new reference systems. This would lead to discontinuities on φ_j , θ_j , φ_j and ϑ_j but not on the vectors \mathbf{l}_j and \mathbf{l}'_j .

8 Test

To validate and test the code a simulation with two slosh pendulums has been run using the following input parameters:

$$\begin{aligned} l_1 &= 0.3m, & m_1 &= 20kg, & \dot{\varphi}_{1,0} &= 0.01rad/s, & \dot{\vartheta}_{1,0} &= 0.05rad/s \\ \mathbf{d}_1 &= \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} m & \hat{\mathbf{p}}_{0,1} &= \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} & \hat{\mathbf{p}}_{0,2} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \hat{\mathbf{p}}_{0,3} &= \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix} \end{aligned} \quad (90)$$

$$l_2 = 0.4m, \quad m_2 = 40kg, \quad \dot{\varphi}_{2,0} = 0.1rad/s, \quad \dot{\vartheta}_{2,0} = 0.5rad/s$$

$$\mathbf{d}_1 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} m \quad \hat{\mathbf{p}}_{0_2,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{\mathbf{p}}_{0_2,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{\mathbf{p}}_{0_2,3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (91)$$

The following figures show how without a damping term the rotational and orbital angular momentum and energy are conserved. This means is an important results that proves the right derivation and implementation of the model. It has also be proven that reducing the time step the conservation is more precise. This is a further prove that the model is correct.

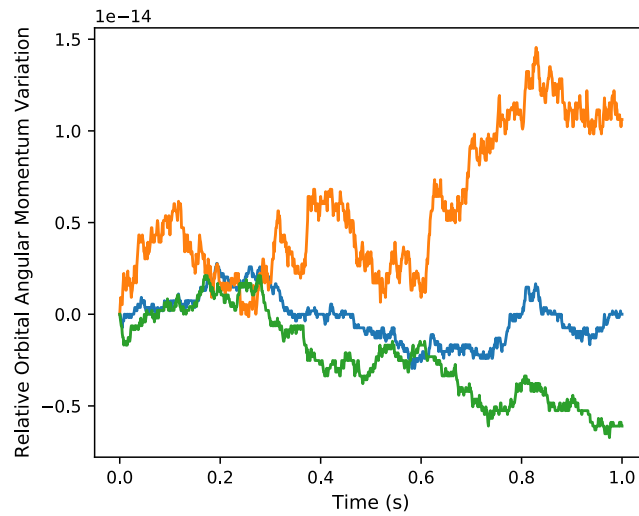


Fig. 5: Relative Orbital Angular Momentum Variation

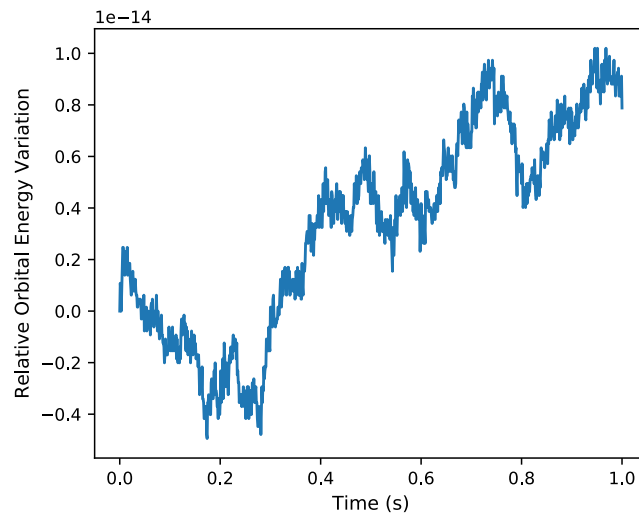
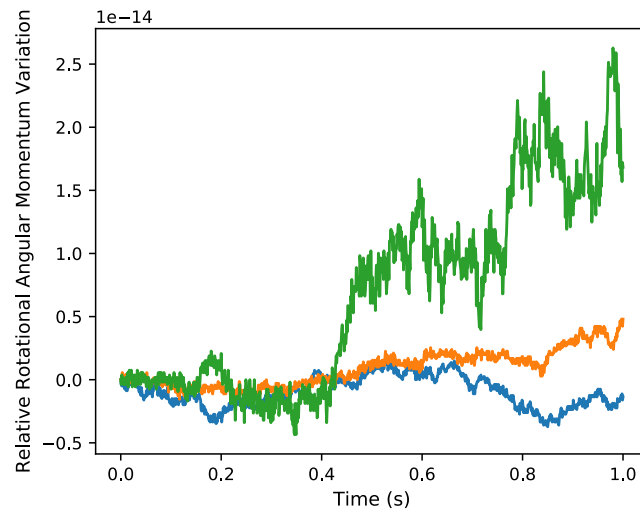
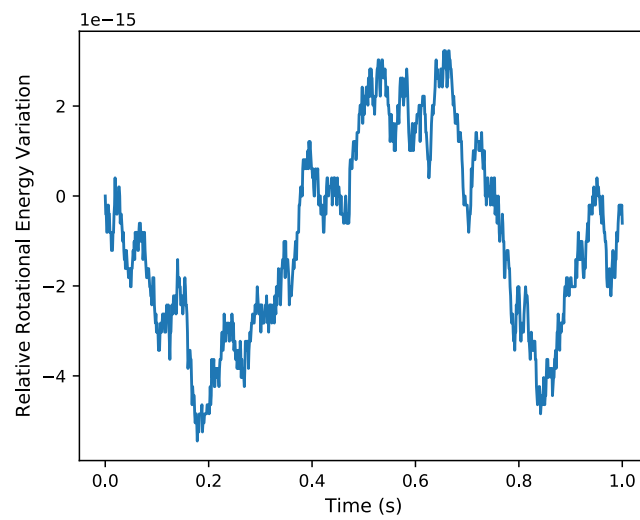


Fig. 6: Relative Orbital Energy Variation

**Fig. 7:** Relative Rotational Angular Momentum Variation**Fig. 8:** Relative Rotational Energy Variation

A second simulation has been run with the same parameters used before but introducing a damping matrix. The value of the damping are so high in order to show a significant rotational energy variation in short interval.

$$D_1 = \begin{bmatrix} 100000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{bmatrix} \frac{Ns}{m} \quad D_2 = \begin{bmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{bmatrix} \frac{Ns}{m} \quad (92)$$

The next figure shows a conservation of orbital momentum and energy. While, only the rotational angular momentum is conserved. The energy is dissipated by the damping term. This is a further prove that the model implemented is correct.

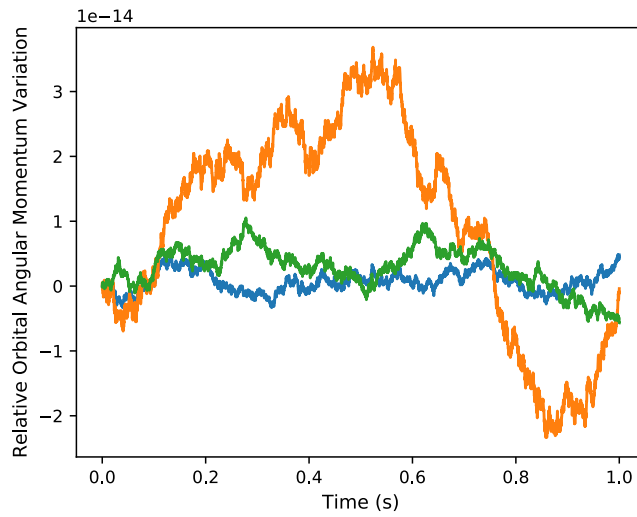


Fig. 9: Relative Rotational Energy Variation

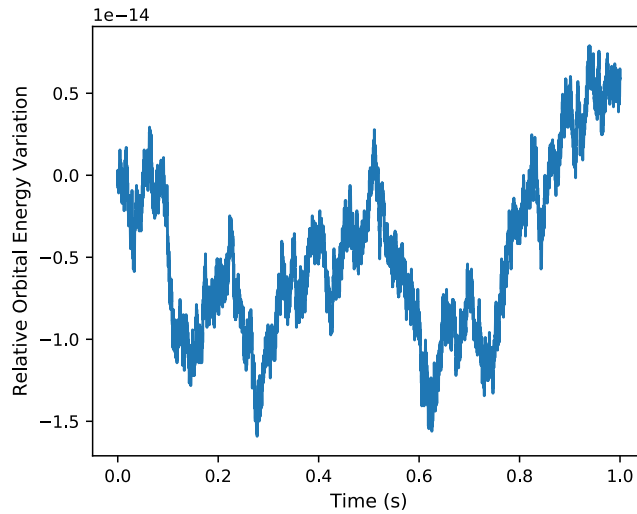
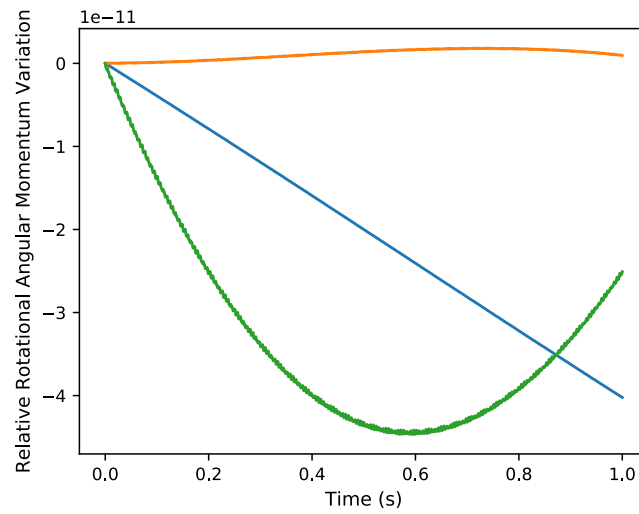
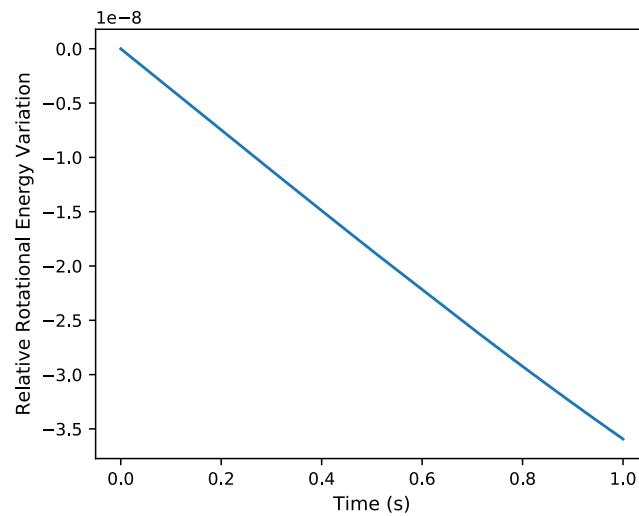


Fig. 10: Relative Rotational Energy Variation

**Fig. 11:** Relative Rotational Energy Variation**Fig. 12:** Relative Rotational Energy Variation