Autonomous Vehicle Simulation (AVS) Laboratory,
University of Colorado

Basilisk Technical Memorandum

REACTIONWHEELSTATEEFFECTOR

Rev	Change Description	By	Date
1.0	Initial Draft	C. Allard	20170816
2.0	Updated with new friction model	C. Allard	20171120

Contents

1 Model Description

1.1 Introduction

This module is modeling a reaction wheel connected to a rigid body hub. The reaction wheel model has three modes that can be ran: balanced wheels, simple jitter, and fully-coupled imbalanced wheels. The balanced wheels option is modeling the reaction wheels as having their principle inertia axes aligned with spin axis, $\hat{\boldsymbol{g}}_s$, and the center of mass of the wheel is coincident with $\hat{\boldsymbol{g}}_s$. This results in the reaction wheel not changing the mass properties of the spacecraft and results in simpler equations. The simple jitter option is approximating the jitter due to mass imbalances by applying an external force and torque to the spacecraft that is proportional to the wheel speeds squared. This is an approximation because in reality this is an internal force and torque. Finally, the fully-coupled mode is modeling reaction wheel imbalance dynamics by modeling the static and dynamic imbalances as internal forces and torques which is physically realistic and allows for energy and momentum conservation.

Figure 1 shows the frame and variable definitions used for this problem. The formulation involves a rigid hub with its center of mass location labeled as point B_c, and N_{rw} RWs with their center of mass locations labeled as W_{c_i}. The frames being used for this formulation are the body-fixed frame, $\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, the motor frame of the i^{th} RW, $\mathcal{M}_i : \{\hat{\boldsymbol{m}}_{s_i}, \hat{\boldsymbol{m}}_{2_i}, \hat{\boldsymbol{m}}_{3_i}\}$ which is also body-fixed, and the wheel-fixed frame of the i^{th} RW, $\mathcal{W}_i : \{\hat{\boldsymbol{g}}_{s_i}, \hat{\boldsymbol{w}}_{2_i}, \hat{\boldsymbol{w}}_{3_i}\}$. The dynamics are modeled with respect to the $\mathcal{B}$ frame which can be generally oriented. The $\mathcal{W}_i$ frame is oriented such that the $\hat{\boldsymbol{g}}_{s_i}$ axis is aligned with the RW spin axis which is the same as the motor torque axis $\hat{\boldsymbol{m}}_{s_i}$, the $\hat{\boldsymbol{w}}_{2_i}$ axis is perpendicular to $\hat{\boldsymbol{g}}_{s_i}$ and points in the direction towards the RW center of mass W_{c_i}. The $\hat{\boldsymbol{w}}_{3_i}$ completes the right hand rule. The $\mathcal{M}_i$ frame is defined as being equal to the $\mathcal{W}_i$ frame at the beginning of the simulation and therefore the $\mathcal{W}_i$ and $\mathcal{M}_i$ frames are offset by an angle, θ_i, about the $\hat{\boldsymbol{m}}_{s_i} = \hat{\boldsymbol{g}}_{s_i}$ axes.

A few more key variables in Figure 1 need to be defined. The rigid spacecraft structure without the RWs is called the hub. Point B is the origin of the $\mathcal{B}$ frame and is a general body-fixed point that does not have to be identical to the total spacecraft center of mass, nor the rigid hub center of mass B_c. Point W_i is the origin of the $\mathcal{W}_i$ frame and can also have any location relative to point B. Point C is the center of mass of the total spacecraft system including the rigid hub and the RWs. Due to the RW imbalance, the vector $\boldsymbol{c}$, which points from point B to point C, will vary as seen by a body-fixed observer. The scalar variable d_i is the center of mass offset of the RW, or the distance from the spin axis, $\hat{\boldsymbol{g}}_{\mathsf{s}_i}$ to W_{c_i}. Finally, the inertial frame orientation is defined through $\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$, while the origin of the inertial frame is labeled as N.

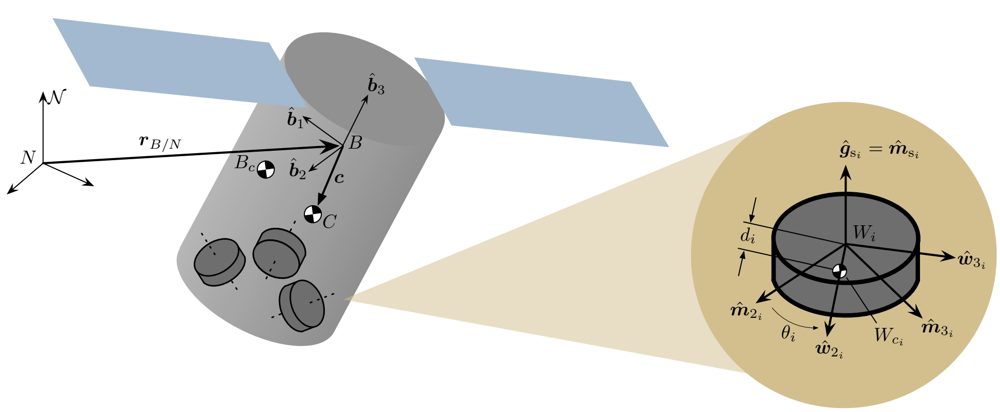


Fig. 1: Reaction wheel and spacecraft frame and variable definitions

1.2 Equations of Motion

The main introduction that is needed for this model is the equations of motion. Depending on the mode, the equations of motion are different. Each mode's equations of motion are discussed in the following sub sections.

1.2.1 Balanced Wheels

For balanced wheels, translational equation of motion is not coupled with $\dot{\Omega}$ as seen in the equation below.

$$m_{\text{sc}}[I_{3\times3}]\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} \tag{1}$$

The rotational equation of motion includes $\dot{\Omega}$ terms, and is thus coupled with wheel motion as seen below.

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) + \boldsymbol{L}_B \tag{2}$$

The motor torque equation can be seen below.

$$\dot{\Omega}_i = \frac{u_{\text{s}_i}}{J_{\text{s}_i}} - \hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{3}$$

Plugging Eq. (13) into Eq. (12)

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + ([I_{\text{sc},B}] - \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \tag{4}$$

The following can be defined:

$$[A_{\text{contr}}] = [0_{3\times3}] \tag{5}$$

$$[B_{\text{contr}}] = [0_{3\times3}] \tag{6}$$

$$[C_{\text{contr}}] = [0_{3\times3}] \tag{7}$$

$$[D_{\text{contr}}] = -\sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T \tag{8}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \boldsymbol{0} \tag{9}$$

$$\boldsymbol{v}_{\text{rot,contr}} = -\sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \tag{10}$$

These are the contributions needed for the back-substitution method used in spacecraft plus.

1.2.2 Simple Jitter

For simple jitter, like balanced wheels, the translational equation of motion is not coupled with $\dot{\Omega}$ as seen in the equation below, however the jitter does apply a force on the spacecraft.

$$m_{\text{sc}}[I_{3\times3}]\ddot{\boldsymbol{r}}_{B/N} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{F}_{\text{ext}} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + U_{s_i}\Omega_i^2\hat{\boldsymbol{u}}_i \tag{11}$$

The rotational equation of motion is very similar to the balanced wheels EOM but has two additional torques due to the reaction wheel imbalance.

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\dot{\Omega}_i = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) + U_{s_i}\Omega_i^2[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{u}}_i + U_{d_i}\Omega_i^2\hat{\boldsymbol{v}}_i + \boldsymbol{L}_B \tag{12}$$

The motor torque equation can be seen below:

$$\dot{\Omega}_i = \frac{u_{\text{s}_i}}{J_{\text{s}_i}} - \hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{13}$$

Plugging Eq. (13) into Eq. (12)

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + ([I_{\text{sc},B}] - \sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - \sum_{i=1}^{N}(\hat{\boldsymbol{g}}_{\text{s}_i}u_{\text{s}_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times J_{\text{s}_i}\Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) - [I'_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + U_{s_i}\Omega_i^2[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{u}}_i + U_{d_i}\Omega_i^2\hat{\boldsymbol{v}} + \boldsymbol{L}_B \tag{14}$$

The following can be defined:

$$[A_{\text{contr}}] = [0_{3\times3}] \tag{15}$$

$$[B_{\text{contr}}] = [0_{3\times3}] \tag{16}$$

$$[C_{\text{contr}}] = [0_{3\times3}] \tag{17}$$

$$[D_{\text{contr}}] = -\sum_{i=1}^{N} J_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T \tag{18}$$

$$\boldsymbol{v}_{\text{trans,contr}} = U_{s_i}\Omega_i^2\hat{\boldsymbol{u}}_i \tag{19}$$

$$\boldsymbol{v}_{\text{rot,contr}} = U_{s_i}\Omega_i^2[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{u}}_i + U_{d_i}\Omega_i^2\hat{\boldsymbol{v}} \tag{20}$$

These are the contributions needed for the back-substitution method used in spacecraft plus.

1.2.3 Fully-Coupled Jitter

The translational equation of motion is

$$\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}\dot{\Omega}_i = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i}d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} \tag{21}$$

The rotational equation of motion is

$$m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N} + [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N}\Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\dot{\Omega}_i = \sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big)\Big] - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \tag{22}$$

The motor torque equation is (note that $J_{12_i} = J_{23_i} = 0$)

$$\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\ddot{\boldsymbol{r}}_{B/N} + \big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \big[J_{11_i} + m_{\text{rw}_i} d_i^2\big]\dot{\Omega}_i$$
$$= -J_{13_i}\omega_{w_{2_i}}\omega_{s_i} + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \quad (23)$$

The first step in the back-substitution method is to solve the motor torque equation for $\dot{\Omega}_i$ in terms of $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$

$$\dot{\Omega}_i = \frac{-m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T}{J_{11_i} + m_{\text{rw}_i} d_i^2}\ddot{\boldsymbol{r}}_{B/N} + \frac{-\big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\big]}{J_{11_i} + m_{\text{rw}_i} d_i^2}\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$+ \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\Big[\omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i}\omega_{w_{2_i}}\omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}\Big] \quad (24)$$

$$\boldsymbol{a}_{\Omega_i} = -\frac{m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}}{J_{11_i} + m_{\text{rw}_i} d_i^2} \quad (25)$$

$$\boldsymbol{b}_{\Omega_i} = -\frac{(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i} + J_{13_i}\hat{\boldsymbol{w}}_{3_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{w}}_{3_i}}{J_{11_i} + m_{\text{rw}_i} d_i^2} \quad (26)$$

$$c_{\Omega_i} = \frac{1}{J_{11_i} + m_{\text{rw}_i} d_i^2}\Big[\omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) - J_{13_i}\omega_{w_{2_i}}\omega_{s_i} - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}\Big] \quad (27)$$

$$\dot{\Omega}_i = \boldsymbol{a}_{\Omega_i}^T\ddot{\boldsymbol{r}}_{B/N} + \boldsymbol{b}_{\Omega_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + c_{\Omega_i} \quad (28)$$

Plugging the equation above into Eq. (21) and multiplying both sides by m_{sc}, (plug $\dot{\Omega}_i$ into translation)

$$\Big[m_{\text{sc}}[I_{3\times3}] + \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}\boldsymbol{a}_{\Omega_i}^T\Big]\ddot{\boldsymbol{r}}_{B/N} + \Big[-m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}\boldsymbol{b}_{\Omega_i}^T\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= m_{\text{sc}}\ddot{\boldsymbol{r}}_{C/N} - 2m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - m_{\text{sc}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \sum_{i=1}^{N} m_{\text{rw}_i} d_i\left(\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - c_{\Omega_i}\hat{\boldsymbol{w}}_{3_i}\right) \quad (29)$$

Moving on to rotation, (plug $\dot{\Omega}_i$ into rotation)

$$\Big[m_{\text{sc}}[\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{a}_{\Omega_i}^T\Big]\ddot{\boldsymbol{r}}_{B/N}$$
$$+ \Big[[I_{\text{sc},B}] + \sum_{i=1}^{N}\Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{b}_{\Omega_i}^T\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$$
$$= \sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big)$$
$$- \Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)c_{\Omega_i}\Big]$$
$$- [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \quad (30)$$

Now we have two equations containing $\ddot{\boldsymbol{r}}_{B/N}$ and $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$. Now the matrix contributions can be defined:

$$[A_{\text{contr}}] = \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{a}_{\Omega_i}^T \tag{31}$$

$$[B_{\text{contr}}] = \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \boldsymbol{b}_{\Omega_i}^T \tag{32}$$

$$[C_{\text{contr}}] = \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{a}_{\Omega_i}^T \tag{33}$$

$$[D_{\text{contr}}] = \sum_{i=1}^{N} \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) \boldsymbol{b}_{\Omega_i}^T \tag{34}$$

$$\boldsymbol{v}_{\text{trans,contr}} = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \left(\Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - c_{\Omega_i} \hat{\boldsymbol{w}}_{3_i} \right) \tag{35}$$

$$\boldsymbol{v}_{\text{rot,contr}} = \sum_{i=1}^{N} \Big[m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i, W_{c_i}}]' \Omega_i \hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}] \Big([I_{\text{rw}_i, W_{c_i}}] \Omega_i \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \boldsymbol{r}'_{W_{c_i}/B} \Big) - \Big([I_{\text{rw}_i, W_{c_i}}] \hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i [\tilde{\boldsymbol{r}}_{W_{c_i}/B}] \hat{\boldsymbol{w}}_{3_i} \Big) c_{\Omega_i} \Big] \tag{36}$$

This concludes the equations that are necessary to define the three different modes of the reaction wheel. Reference[1] explains in further detail the EOMs for the simple-jitter and fully-coupled modes. Reference[3] gives more details on the derivation for balanced reaction wheels.

1.3 Friction Model

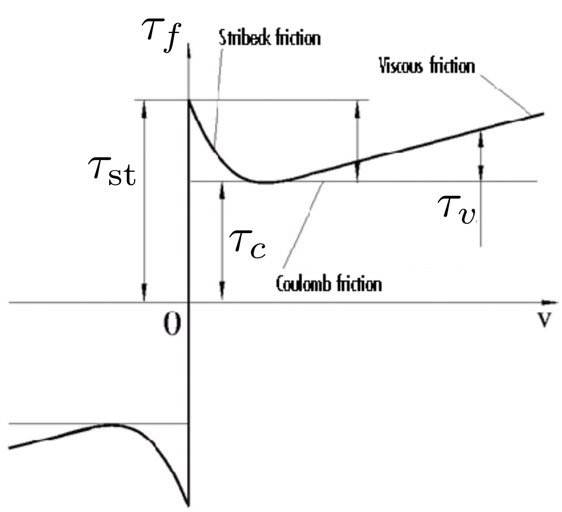


Fig. 2: Friction Torque Model - Reference[2]

The friction model used for reaction wheels uses a combination of static, Coulomb, and viscous friction and can be seen in Fig. 2. To incorporate all of these effects, the friction was adopted using the Stribeck friction model seen in Reference.[2] The following equation describes the calculation of the friction torque on the reaction wheels.

$$\tau_f = -\sqrt{2e}(\tau_{\text{st}} - \tau_c)e^{\left[-\left(\frac{\Omega}{\beta_{\text{st}}}\right)^2\right]} - \tau_c \tanh\left[\frac{10\Omega}{\beta_{\text{st}}}\right] - c_v\Omega \tag{37}$$

In Eq. (37), τ_f is the friction torque, τ_{st} is the static friction magnitude, τ_c is the coulomb friction magnitude, β_{st} is the Stribeck coefficient which modifies the peakedness of the Stribeck curve, and c_v is the viscous damping coefficient. These variables can also be seen in the variable descriptions in Fig. 2.

The Stribeck function is only applicable when the reaction wheel is starting from rest. In contrast, when the reaction wheel starts from a non-zero speed, or has already broken free of static friction term, then the following equation is implemented:

$$\tau_f = -\tau_c\text{sgn}(\Omega) - c_v\Omega \tag{38}$$

This logic and math is implemented in the reaction wheel dynamics module.

2 Model Functions

This model is used to approximate the behavior of a reaction wheel. Below is a list of functions that this model performs:

- Compute it's contributions to the mass properties of the spacecraft
- Provides matrix contributions for the back substitution method
- Compute it's derivatives for θ and Ω
- Adds energy and momentum contributions to the spacecraft
- Convert commanded torque to applied torque. This takes into account friction, and minimum and maximum torque, and speed saturation
- Write output messages for states like Ω and applied torque

3 Model Assumptions and Limitations

Below is a summary of the assumptions/limitations:

- The reaction wheel is considered a rigid body
- The spin axis is body fixed, therefore does not take into account bearing flexing
- There is no error placed on the torque when converting from the commanded torque to the applied torque
- For balanced wheels and simple jitter mode the mass properties of the reaction wheels are assumed to be included in the mass and inertia of the rigid body hub, therefore there is zero contributions to the mass properties from the reaction wheels in the dynamics call.
- For fully-coupled imbalanced wheels mode the mass properties of the reaction wheels are assumed to not be included in the mass and inertia of the rigid body hub.

- For balanced wheels and simple jitter mode the inertia matrix is assumed to be diagonal with one of it's principle inertia axis equal to the spin axis, and the center of mass of the reaction wheel is coincident with the spin axis.
- For simple jitter, the parameters that define the static and dynamic imbalances are U_s and U_d.
- For fully-coupled imbalanced wheels the inertia off-diagonal terms, J_{12} and J_{23} are equal to zero and the remaining inertia off-diagonal term J_{13} is found through the setting the dynamic imbalance parameter U_d: $J_{13} = U_d$. The center of mass offset, d, is found using the static imbalance parameter U_s: $d = \frac{U_s}{m_{\text{rw}}}$
- The friction model is modeling static, Coulomb, and viscous friction. Other higher order effects of friction are not included.
- The speed saturation model only has one boundary, whereas in some reaction wheels once the speed boundary has been passed, the torque is turned off and won't turn back on until it spins down to another boundary. This model only can turn off and turn on the torque and the same boundary

4 Test Description and Success Criteria

The tests are located in `simulation/dynamics/reactionWheels/_UnitTest/test_reactionWheelStateEffector_integrated.py` **and** `simulation/dynamics/reactionWheels/_UnitTest/ test_reactionWheelStateEffector_ConfigureRWRequests.py`. Depending on the test, there are different success criteria. These are outlined in the following subsections:

4.1 Balanced Wheels Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 reaction wheels attached to the spacecraft, and the wheels are in "Balanced" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy (second half of the simulation)
- Achieving the expected final attitude
- Achieving the expected final position

4.2 Simple Jitter Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 reaction wheels attached to the spacecraft, and the wheels are in "Simple Jitter" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Achieving the expected final attitude
- Achieving the expected final position

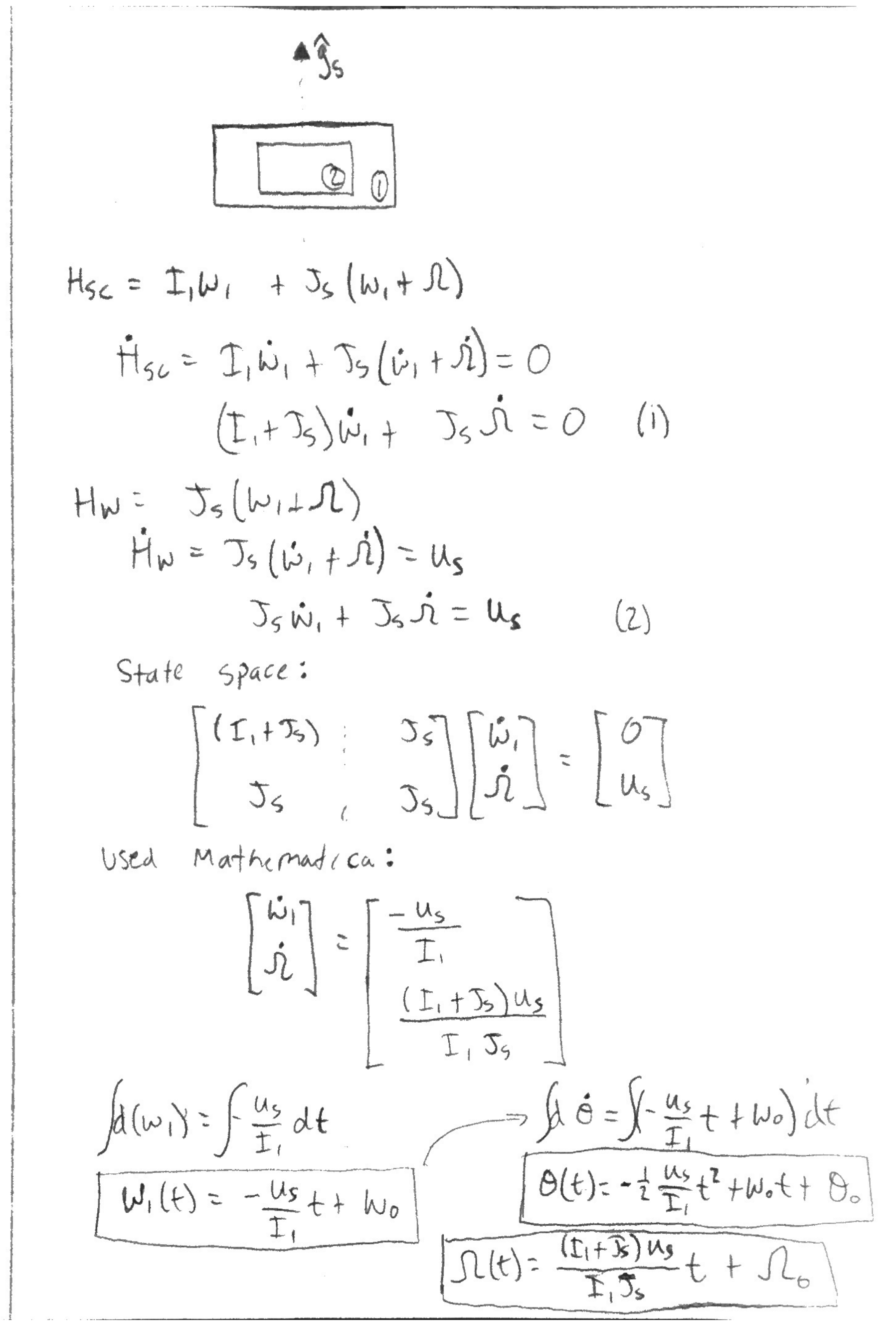
4.3 Fully Coupled Jitter Scenario - Integrated Test

In this test the simulation is placed into orbit around Earth with point gravity, has 3 reaction wheels attached to the spacecraft, and the wheels are in "Fully Coupled Jitter" mode. Each wheel is given a commanded torque for half the simulation and the rest of the simulation the torques are set to zero. The following parameters are being tested:

- Conservation of orbital angular momentum
- Conservation of orbital energy
- Conservation of rotational angular momentum
- Conservation of rotational energy (second half of the simulation)
- Achieving the expected final attitude
- Achieving the expected final position

4.4 BOE Calculation Scenario - Integrated Test

The BOE for this scenario can be seen in Figure 3. This involves a rigid body hub connected to a reaction wheel with the spin axis being aligned with both the hub's center of mass and the reaction wheel's center of mass. This problem assumes the hub and reaction wheel are fixed to rotate about the the spin axis and so it is a two degree of freedom problem. The analytical expressions for the angular velocity of the hub, ω_1, the angle of the hub, θ and the reaction wheel speed, Ω are shown in Figure 3. The test sets up Basilisk so that the initial conditions constrain the spacecraft to rotate about the spin axis. The results confirm that the analytical expressions agree with the Basilisk simulation.



$$H_{sc} = I_1 \omega_1 + J_s(\omega_1 + \Omega)$$

$$\dot{H}_{sc} = I_1 \dot{\omega}_1 + J_s(\dot{\omega}_1 + \dot{\Omega}) = 0$$

$$(I_1 + J_s)\dot{\omega}_1 + J_s \dot{\Omega} = 0 \quad (1)$$

$$H_W = J_s(\omega_1 + \Omega)$$

$$\dot{H}_W = J_s(\dot{\omega}_1 + \dot{\Omega}) = u_s$$

$$J_s \dot{\omega}_1 + J_s \dot{\Omega} = u_s \quad (2)$$

State space:

$$\begin{bmatrix} (I_1 + J_s) & J_s \\ J_s & J_s \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\Omega} \end{bmatrix} = \begin{bmatrix} 0 \\ u_s \end{bmatrix}$$

Used Mathematica:

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\Omega} \end{bmatrix} = \begin{bmatrix} -\frac{u_s}{I_1} \\ \frac{(I_1 + J_s)u_s}{I_1 J_s} \end{bmatrix}$$

$$\int d(\omega_1) = \int -\frac{u_s}{I_1} dt$$

$$\boxed{\omega_1(t) = -\frac{u_s}{I_1} t + \omega_0}$$

$$\int d\dot{\theta} = \int \left(-\frac{u_s}{I_1} t + \omega_0\right) dt$$

$$\boxed{\theta(t) = -\frac{1}{2}\frac{u_s}{I_1} t^2 + \omega_0 t + \theta_0}$$

$$\boxed{\Omega(t) = \frac{(I_1 + J_s)u_s}{I_1 J_s} t + \Omega_0}$$

Fig. 3: Back of the envelope calculation for RWs

4.5 Friction - Integrated Tests

In this test the goal is to validate that the friction model is matching the desired static, Coulomb and linear friction model. This is done by setting a spacecraft with two identical reaction wheels with identical spin axes. There are two scenarios being tested: one when the reaction wheels start with a non-zero angular velocity and are spinning down to rest, and the other scenario is when both reaction wheels are starting from rest and applying torque to them to break away from the static friction. These two different friction models for spinning up and spinning down reaction wheels can be seen in Section 1.3.

4.5.1 Spin Down Friction Test

In this test, the spacecraft is set to have no initial rotation. The wheel speeds are equal in magnitude but opposite in direction. The expected results are that the spacecraft will not rotate and the wheel speeds will spin down to zero while matching the function seen in Eq. (38). The test verifies that the math in Eq. (38) is being computed properly by calculating this in python and ensuring that the results match the Basilisk output.

4.5.2 Spin Up Friction Test

In this test, the spacecraft and wheel speeds are all have zero angular velocities. The reaction wheels are given equal but opposite applied torques that are greater in magnitude than the static friction torque. The expected results are that the spacecraft will not rotate and the wheel speeds will spin up while matching the function seen in Eq. (37) and Figure 2. The test verifies that the math in Eq. (37) is being computed properly by calculating this in python and ensuring that the results match the Basilisk output.

4.6 Saturation - Unit Test

This test is ensuring that when a commanded torque requests a torque above the max torque of the wheels, the applied torque is set to the max torque. The logic can be seen in the following equation:

$$\begin{aligned}
&\textbf{if} \quad u_{\text{cmd}} > u_{\text{max}} \quad \textbf{then}\\
&\quad u_s = u_{\text{max}}\\
&\textbf{else if} \quad u_{\text{cmd}} < -u_{\text{max}} \quad \textbf{then}\\
&\quad u_s = -u_{\text{max}}\\
&\textbf{else}\\
&\quad u_s = u_{\text{cmd}}\\
&\textbf{end if}
\end{aligned} \tag{39}$$

The test gives two commanded torques, one above u_{max} and one below, and ensures that the correct values are being set for u_s.

4.7 Minimum Torque - Unit Test

This test is ensuring that when a commanded torque requests a torque below the minimum torque of the wheels, the applied torque is set to zero. The logic can be seen in the following equation:

$$\begin{aligned}
&\textbf{if} \quad |u_{\text{cmd}}| < u_{\text{min}} \quad \textbf{then}\\
&\quad u_s = 0.0\\
&\textbf{end if}
\end{aligned} \tag{40}$$

The test gives two commanded torques, one above u_{min} and one below, and ensures that the correct values are being set for u_s.

4.8 Speed Saturation - Unit Test

This test is ensuring that when the current reaction wheel speed is greater than or equal to the maximum allowable wheel speed, and the applied torque is trying to increase the wheel speed further, the applied torque is set to zero. The logic can be seen in the following equation:

$$\begin{aligned} &\textbf{if} \quad |\Omega| >= \Omega_{\text{max}} \text{ and } \Omega u_s >= 0 \quad \textbf{then} \\ &\quad u_s = 0.0 \\ &\textbf{end if} \end{aligned} \tag{41}$$

The test requests a non-zero commanded torque u_{cmd}, gives two possibilities for Ω, one above Ω_{max} and one below, and ensures that the correct values are being set for u_s.

5 Test Parameters

Since this is an integrated test, the inputs to the test are the physical parameters of the spacecraft along with the initial conditions of the states. These parameters are outlined in Tables 2- 8. Additionally, the error tolerances can be seen in Table 9. The error tolerances are different depending on the test. The energy-momentum conservation values will normally have an agreement down to 1e-14, but to ensure cross-platform agreement the tolerance was chose to be 1e-10. The position and attitude checks have a tolerance set to 1e-7 and is because 8 significant digits were chosen as the values being compared to. The BOE tests depend on the integration time step but as the time step gets smaller the accuracy gets better. So 1e-8 tolerance was chosen so that a larger time step could be used but still show agreement. The Friction tests give the same numerical outputs down to 1e-15 between python and Basilisk, but 1e-10 was chosen to ensure cross platform agreement. Finally, the saturation and minimum torque tests have 1e-10 to ensure cross-platform success, but these values will typically agree to machine precision.

Table 2: Spacecraft Hub Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
mHub	mass	750.0	kg
IHubPntBc_B	Inertia in $\mathcal{B}$ frame	$\begin{bmatrix} 900.0 & 0.0 & 0.0 \\ 0.0 & 800.0 & 0.0 \\ 0.0 & 0.0 & 600.0 \end{bmatrix}$	kg-m^2
r_BcB_B	CoM Location in $\mathcal{B}$ frame	$\begin{bmatrix} -0.0002 & 0.0001 & 0.1 \end{bmatrix}^T$	m

Table 3: Reaction Wheel 1 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\begin{bmatrix} 1.0 & 0.0 & 0.0 \end{bmatrix}^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$\begin{bmatrix} 0.1 & 0.0 & 0.0 \end{bmatrix}^T$	m

Table 4: Reaction Wheel 2 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$[0.0 \quad 1.0 \quad 0.0]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.0 \quad 0.1 \quad 0.0]^T$	m

Table 5: Reaction Wheel 3 Parameters for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
U_s	Static Imbalance	4.8E-6	kg-m
U_d	Dynamic Imbalance	15.4E-7	kg-m^2
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 1.0]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 0.1]^T$	m

Table 6: Reaction wheel 1 parameters for friction tests

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\left[\frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}\right]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[0.5 \quad -0.5 \quad 0.5]^T$	m

Table 7: Reaction wheel 2 parameters for friction tests

Name	Description	Value	Units
Js	Spin Axis Inertia	0.159	kg-m^2
mass	mass	12.0	kg
gsHat_B	Spin Axis in $\mathcal{B}$ frame	$\left[\frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{3}}{3}\right]^T$	-
rWB_B	Location of Wheel in $\mathcal{B}$ frame	$[-0.5 \quad 0.5 \quad -0.5]^T$	m

Table 8: Initial Conditions for Energy Momentum Conservation Scenarios

Name	Description	Value	Units
(RW 1) OmegaInit	(RW 1) Initial Ω	500	RPM
(RW 2) OmegaInit	(RW 2) Initial Ω	200	RPM
(RW 3) OmegaInit	(RW 3) Initial Ω	-150	RPM
r_CN_NInit	Initial Position of S/C	$[-4020339 \quad 7490567 \quad 5248299]^T$	m
v_CN_NInit	Initial Velocity of S/C	$[-5199.78 \quad -3436.68 \quad 1041.58]^T$	m/s
sigma_BNInit	Initial MRP of $\mathcal{B}$ frame	$[0.0 \quad 0.0 \quad 0.0]^T$	-
omega_BN_BInit	Initial Angular Velocity of $\mathcal{B}$ frame	$[0.08 \quad 0.01 \quad 0.0]^T$	rad/s

Table 9: Error Tolerance - Note: Relative Tolerance is $\text{abs}(\frac{\text{truth}-\text{value}}{\text{truth}})$

Test	Relative Tolerance
Energy and Momentum Conservation	1e-10
Position, Attitude Check	1e-7
BOE	1e-8
Friction Tests	1e-10
Saturation Tests	1e-10
Minimum Torque	1e-10

6 Test Results

6.1 Balanced Wheels Scenario - Integrated Test Results

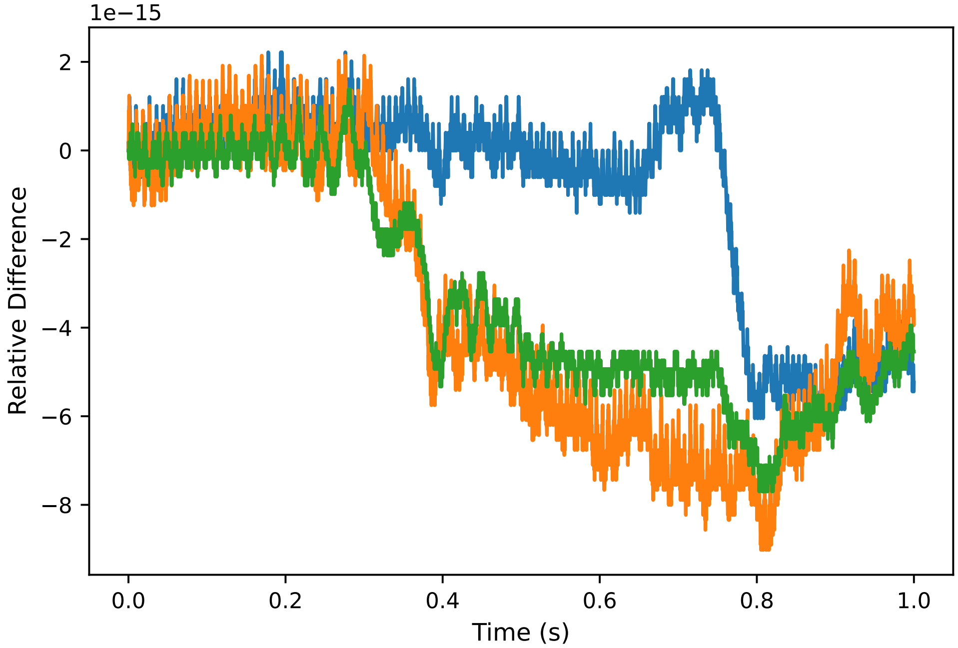


Fig. 4: Change in Orbital Angular Momentum BalancedWheels

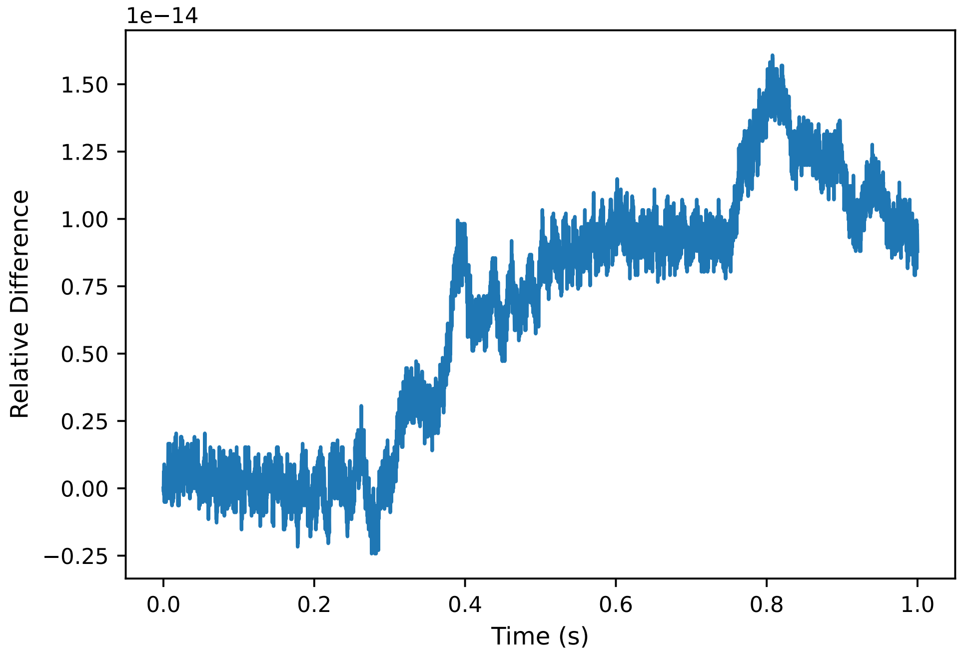


Fig. 5: Change in Orbital Energy BalancedWheels

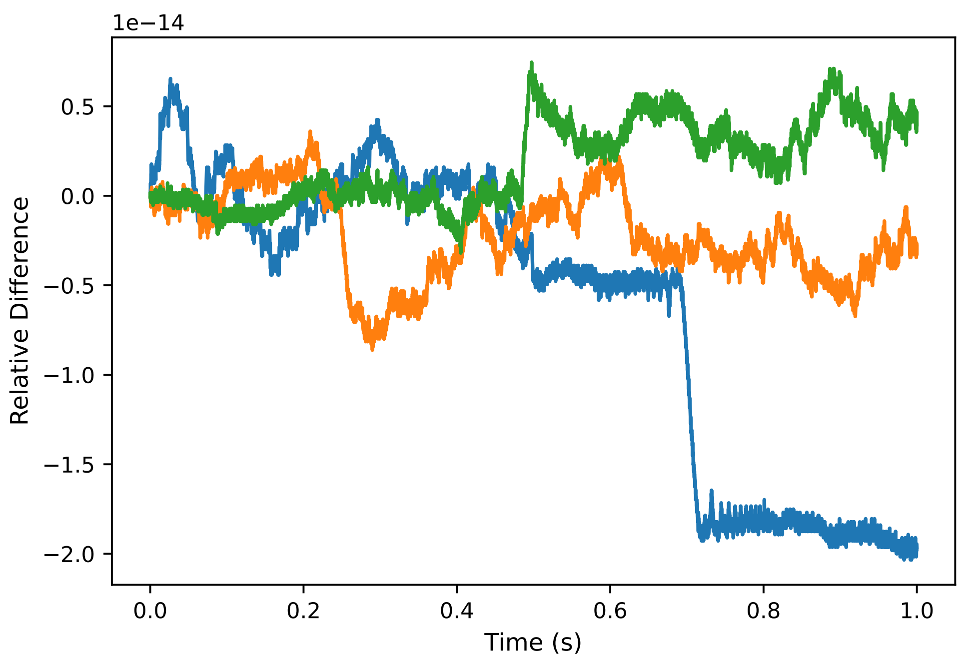


Fig. 6: Change in Rotational Angular Momentum BalancedWheels

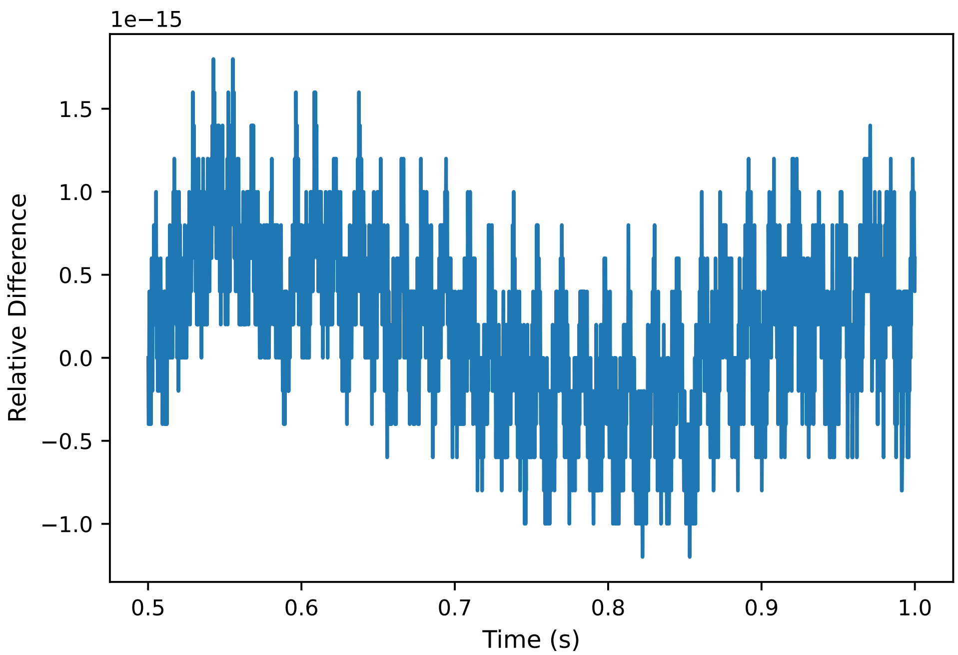


Fig. 7: Change in Rotational Energy BalancedWheels

6.2 Fully Coupled Jitter Scenario - Integrated Test Results

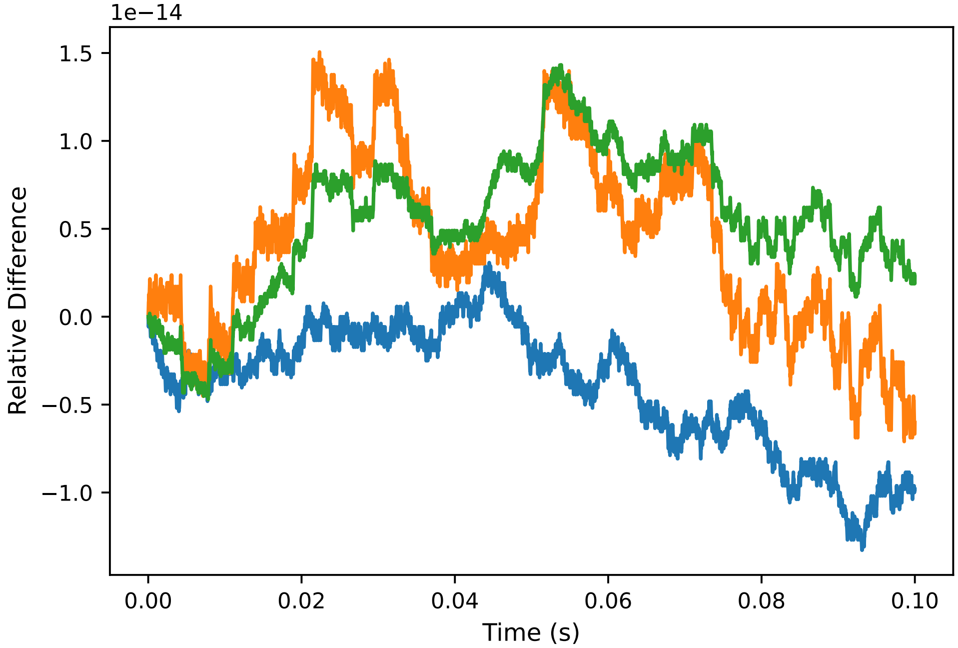


Fig. 8: Change in Orbital Angular Momentum JitterFullyCoupled

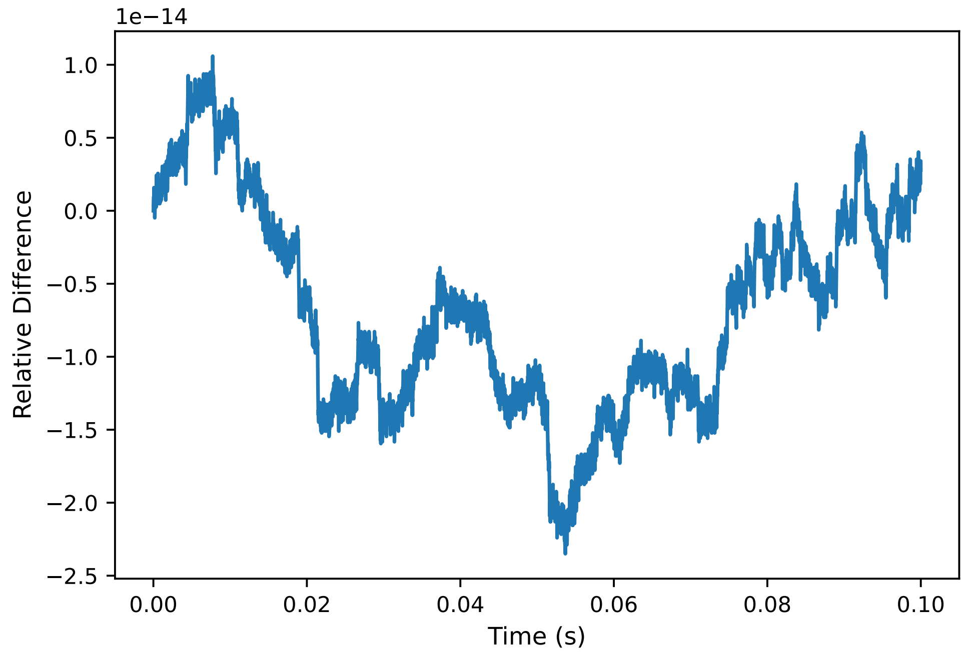


Fig. 9: Change in Orbital Energy JitterFullyCoupled

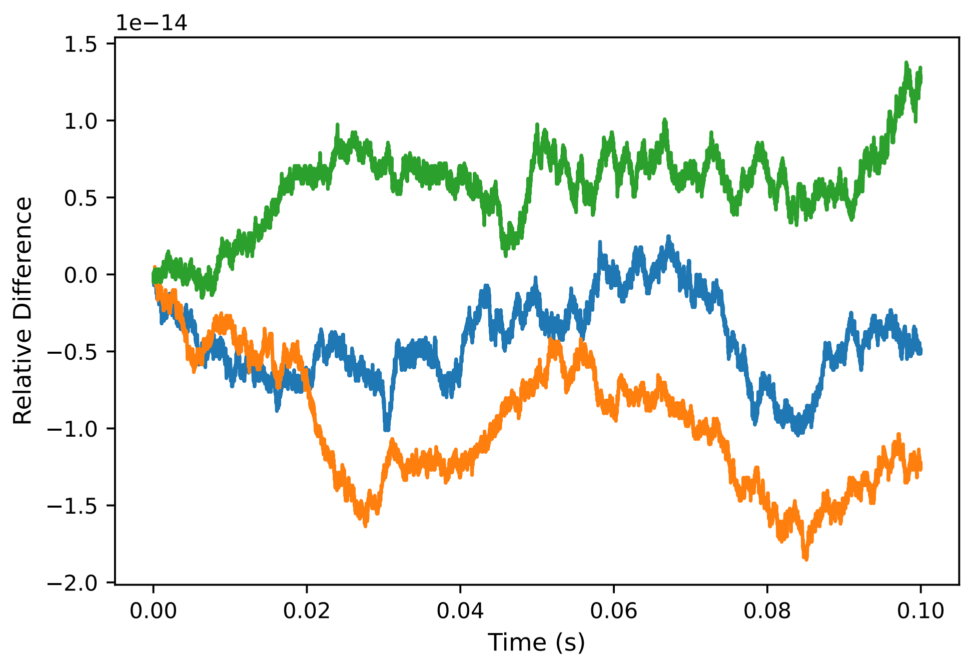


Fig. 10: Change in Rotational Angular Momentum JitterFullyCoupled

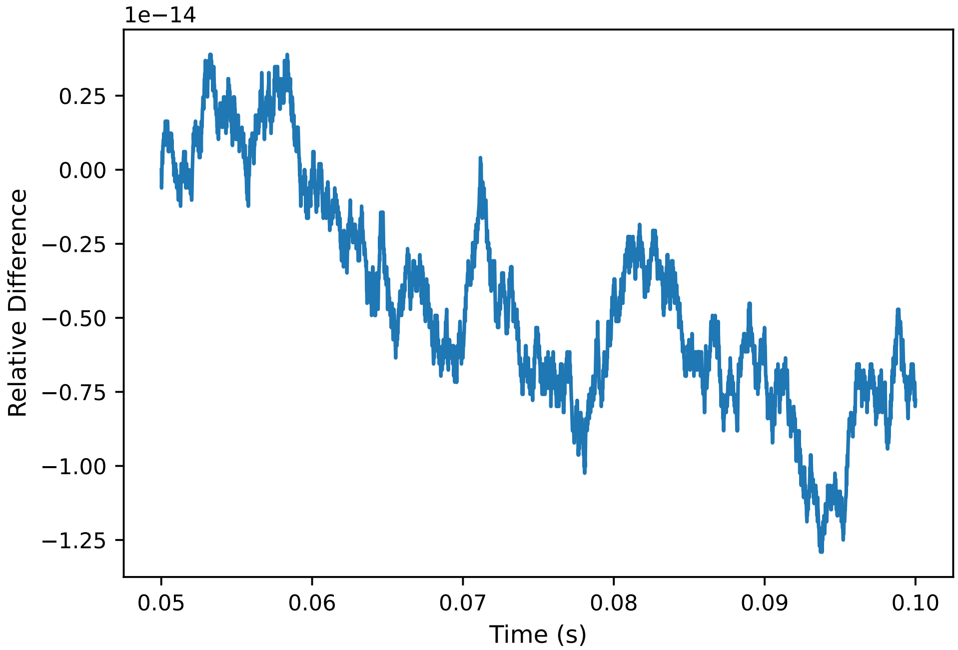


Fig. 11: Change in Rotational Energy JitterFullyCoupled

6.3 BOE Calculation Scenario - Integrated Test Results

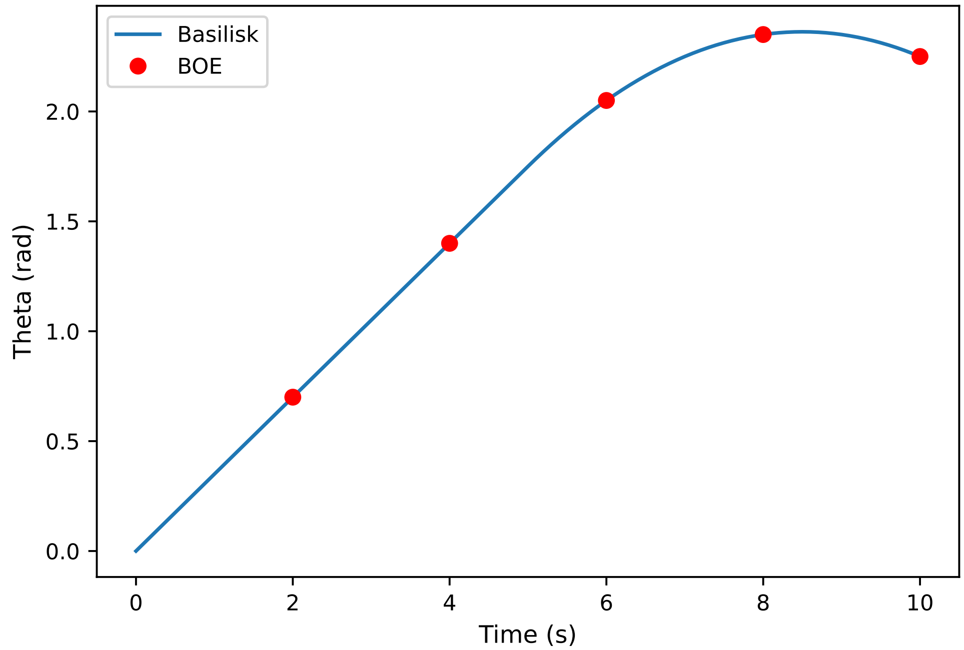


Fig. 12: Reaction Wheel BOE Theta

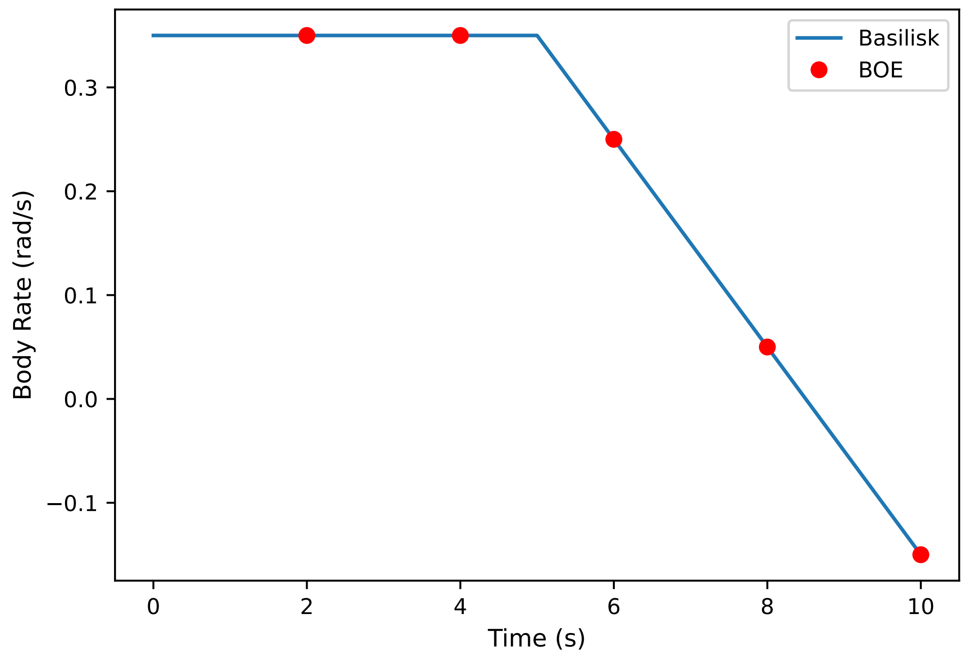


Fig. 13: Reaction Wheel BOE Body Rate

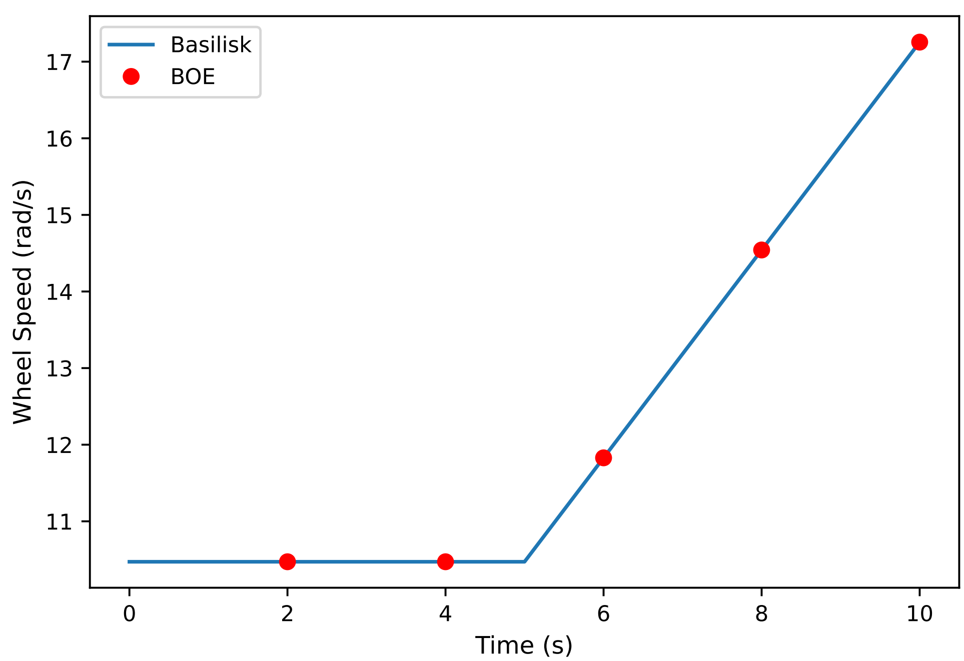


Fig. 14: Reaction Wheel BOE RW Rate

6.4 Friction Spin Down Integrated Test Results

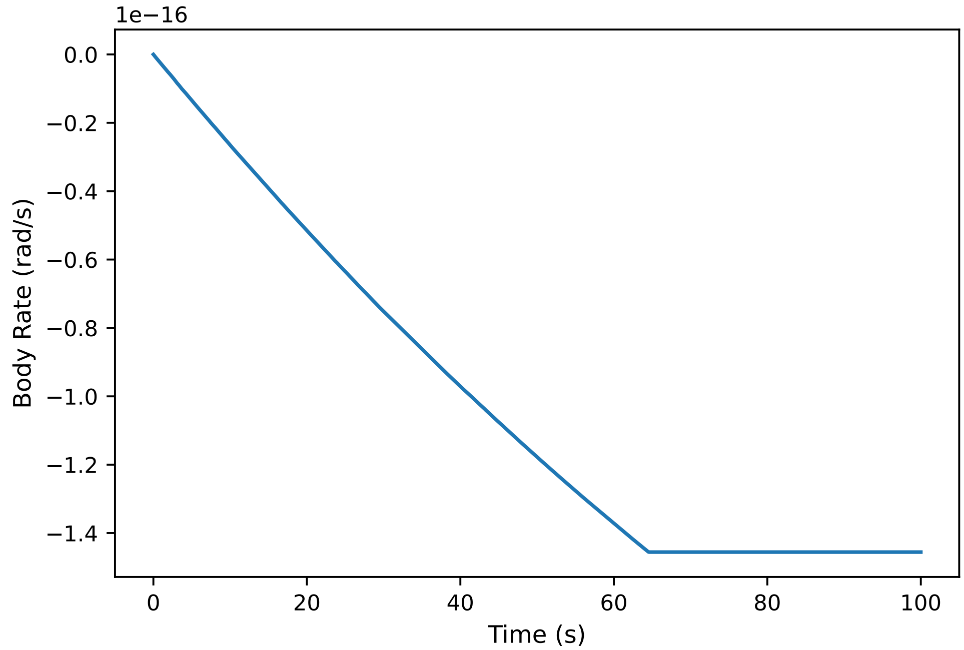


Fig. 15: Reaction Wheel FrictionSpinDown Test Body Rates

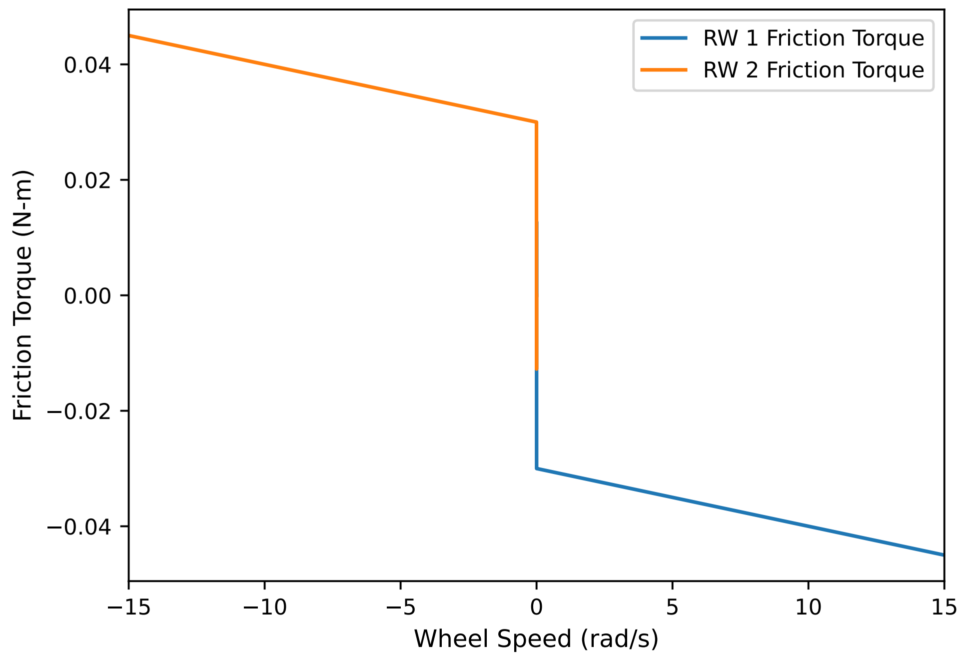


Fig. 16: Reaction Wheel FrictionSpinDown Test Friction Torque

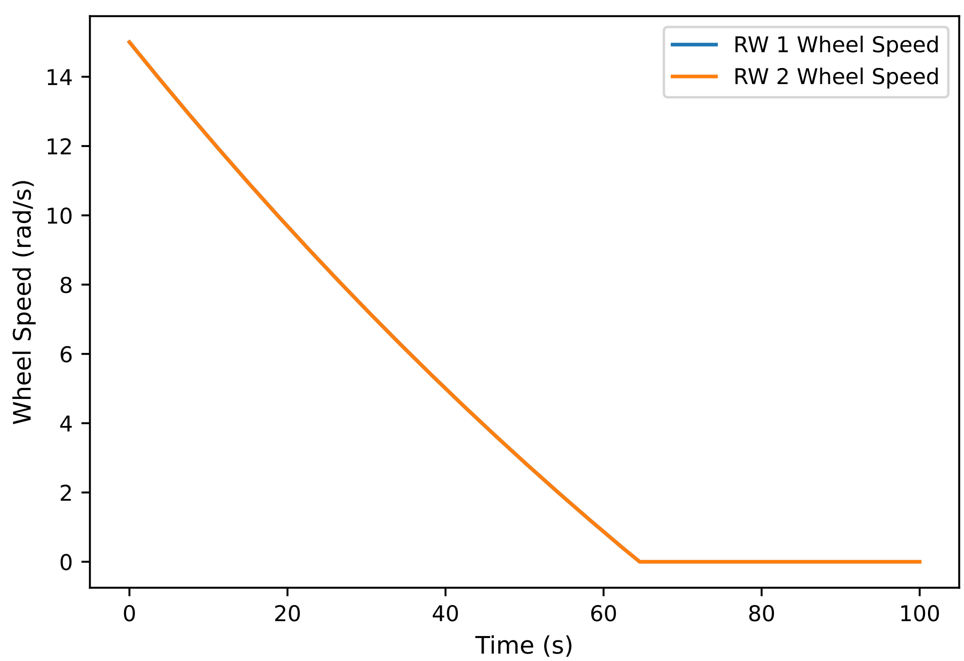


Fig. 17: Reaction Wheel FrictionSpinDown Test Wheel Speed

6.5 Friction Spin Up Integrated Test Results

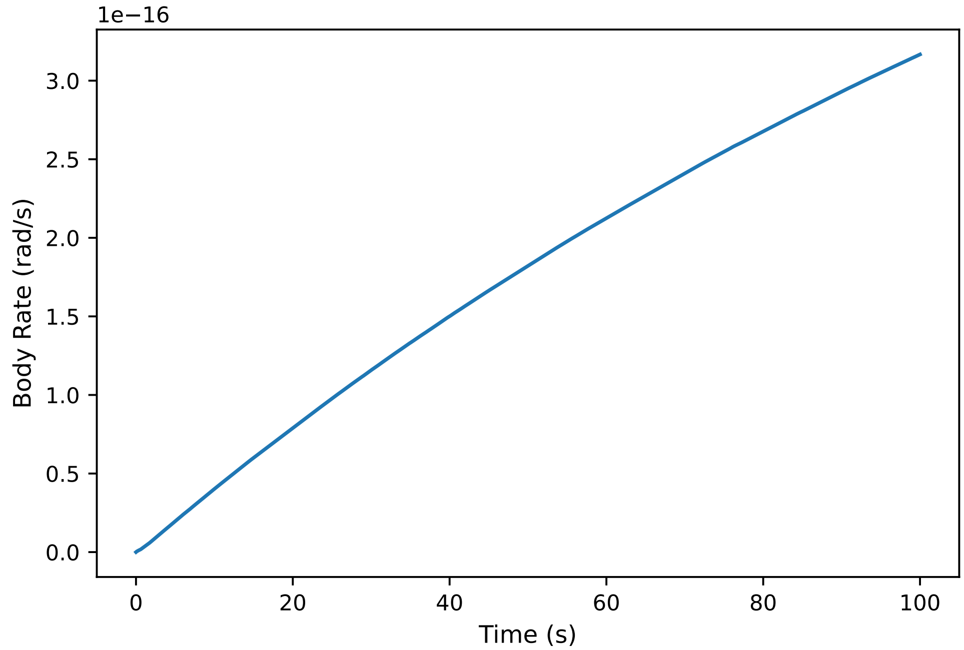


Fig. 18: Reaction Wheel FrictionSpinUp Test Body Rates

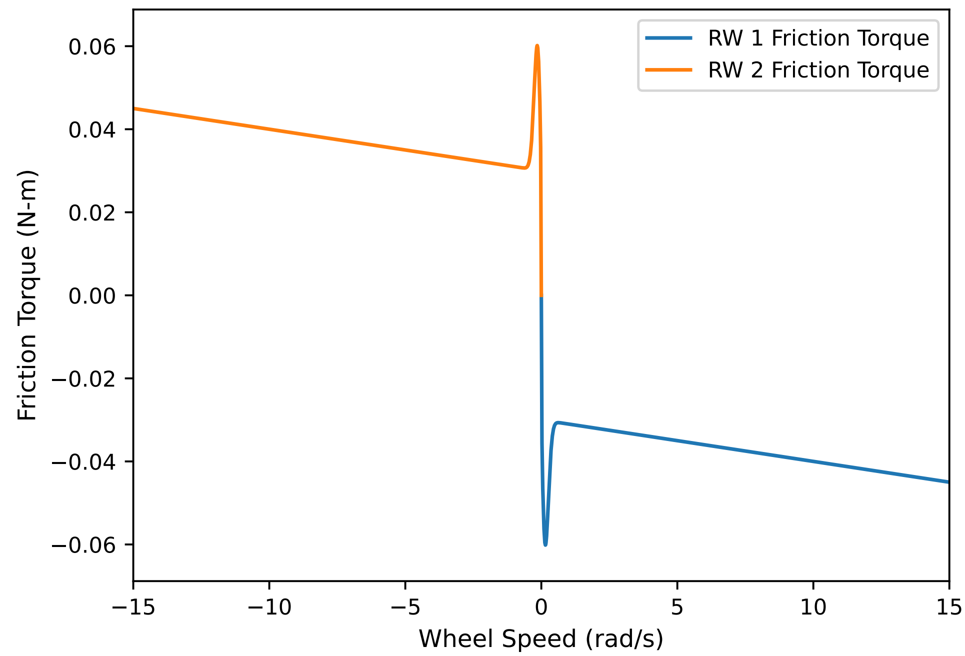


Fig. 19: Reaction Wheel FrictionSpinUp Test Friction Torque

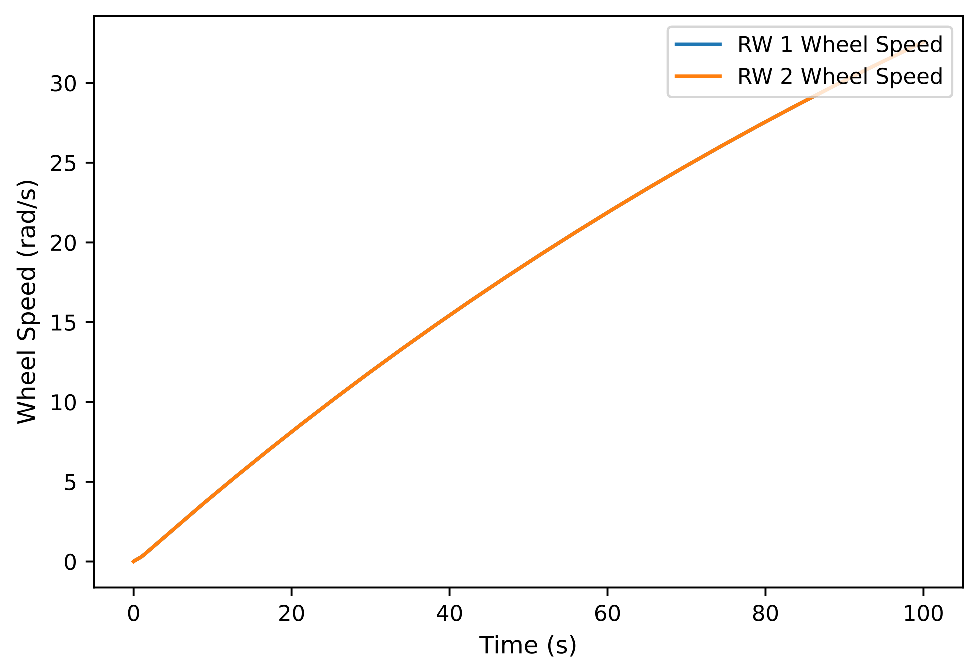


Fig. 20: Reaction Wheel FrictionSpinUp Test Wheel Speed

6.6 Simple Jitter, Saturation and Minimum Torque Tests Results

Table 10: Test results.

Test	Pass/Fail
Simple Jitter	PASSED
Saturation	PASSED
Minimum Torque	PASSED

7 User Guide

This section is to outline the steps needed to setup a reaction wheel state effector in python using Basilisk.

1. Import the reactionWheelStateEffector class, the spacecraft class, and the simIncludeRW python module:
   import reactionWheelStateEffector, *import spacecraft* and *import simIncludeRW*

2. Define an instantiation of a rwFactory:
   rwFactory = simIncludeRW.rwFactory()

3. Create a reaction wheel (Honeywell HR16 as an example):
   rwFactory.create()
   'Honeywell_HR16'
   ,[1,0,0]
   ,Omega = 500.
   ,rWB_B = [0.1,0.,0.]
   ,maxMomentum = varMaxMomentum
   ,RWModel= varRWModel)

4. To include stribeck friction effects, include *betaStatic* within the rwFactory.create() and assign it a non-zero value. By default, betaStatic is set to -1 and stribeck friction is ignored.

5. Create an instantiation of a reaction wheel state effector:
   rws = reactionWheelStateEffector.ReactionWheelStateEffector()

6. Create an instantiation of a spacecraft:
   scObject = spacecraft.Spacecraft()

7. Finally, add the reaction wheel object to your spacecraft:
   rwFactory.addToSpacecraft("ReactionWheels", rwStateEffector, scObject). See spacecraft documentation on how to set up a spacecraft object.

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[2] H. Olsson, K.J. Å ström, C. Canudas de Wit, M. Gäfvert, and P. Lischinsky. Friction models and friction compensation. *European Journal of Control*, 4(3):176 – 195, 1998.

[3] Hanspeter Schaub and John L. Junkins. *Analytical Mechanics of Space Systems*. AIAA Education Series, Reston, VA, 3rd edition, 2014.