**CSC 2421H : Graphs, Matrices, and Optimization** Lecture 10 : 26 Nov. 2018

# Approxinate Cholesky Factorization

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## 1 Notation

We introduce the following notation:

- $S^{(k)}$: Schur complement at $k$ i.e., $S^{(k)} = S^{(k-1)} + c_k c_k^T$ where $c_k = \frac{1}{\sqrt{S^{(k-1)}_{\pi(k),\pi(k)}}} S^{(k-1)}_{:,\pi(k)}$.
- $Cl_k$: Clique at $k$.
- $(L)_v$: Laplacian of a star graph incident on $v$ with all edges belonging to the graph whose Laplacian is $L$. Therefore, we can write $S^{(k)} = S^{(k-1)} + \left(S^{(k-1)}\right)_{\pi(k)} + Cl_k$.
- For a matrix $M$, $\overline{M} = L^{-\frac{1}{2}} M L^{-\frac{1}{2}}$, where $L$ is the Laplacian of the original graph.
- We use $\mathbb{E}_{|k-1}[\cdot]$ to denote the expectation conditioned on everything that happened in the first $k-1$ iterations.

## 2 Clique Sampling

In the previous lecture, the following clique sampling algorithm was introduced.

**Algorithm 1** CliqueSample(G,v)

```
for every edge e = (v, u_1) ∈ E do
    Sample another edge (v, u_2) with probability w(v,u_2)/deg(v)
    if u_1 ≠ u_2 then
        Y_e ← [w(v,u_1)w(v,u_2) / (w(v,u_1)+w(v,u_2))] L_(u_1,u_2)
    end if
end for
Return Cl^ = Σ_e Y_e.
```

$$Y_e \leftarrow \frac{w(v,u_1)w(v,u_2)}{w(v,u_1)+w(v,u_2)} L_{(u_1,u_2)}, \qquad \widehat{Cl} = \sum_e Y_e$$

## 3 Approximate Cholesky Factorization Algorithm

Algorithm 2 is used for obtaining the approximate Cholesky factorization.

**Algorithm 2** ApproximateCholesky(G)

Replace each edge $e$ with $\rho$ parallel copies, each with weight $\frac{w(e)}{\rho}$.
Pick a random permutation $\pi$ on $V$.
**for** $k = 1$ **to** $n$ **do**
  "Record column": $c_k \leftarrow \frac{1}{\sqrt{S^{(k-1)}_{\pi(k),\pi(k)}}} S^{(k-1)}_{:,\pi(k)}$.
  "Eliminate and Sample": $\widehat{Cl}_k \leftarrow \text{CliqueSample}\left(S^{(k-1)}, \pi\left(k\right)\right)$,
  $\widehat{S}^{(k)} \leftarrow \widehat{S}^{(k-1)} + (\widehat{S}^{(k-1)})_{\pi(k)} + \widehat{Cl}_k$.
**end for**
**Return** $[c_1, c_2, \cdots, c_n]$.

# 4 Proof of Algorithm Approximation Result

Our goal is to show that the returned approximate normalized Laplacian satisfies the target approximation $\widehat{L}^n \approx_{\frac{1}{3}} L$, or equivalently

$$\begin{aligned}
&\frac{2}{3}L \preccurlyeq \widehat{L}^n \preccurlyeq \frac{4}{3}L \\
\Leftrightarrow &-\frac{1}{3}L \preccurlyeq \widehat{L}^n - L \preccurlyeq \frac{1}{3}L \\
\Leftrightarrow &-\frac{1}{3}\Pi \preccurlyeq \overline{\widehat{L}}^n - \overline{L} \preccurlyeq \frac{1}{3}\Pi \\
\Leftrightarrow &\|\overline{\widehat{L}}^n - \overline{L}\| \leq \frac{1}{3}.
\end{aligned}$$

In order to show this result, we first introduce a number of useful lemmas.

**Lemma 4.1.** $\mathbb{E}_{|k-1}\left[\widehat{Cl}_k\right] = Cl_k$

*Proof.*

$$\begin{aligned}
\mathbb{E}_{|k-1}[\widehat{Cl}_k] &= \mathbb{E}_{|k-1}\left[\sum_e Y_e\right] \\
&= \sum_e \mathbb{E}_{|k-1}[Y_e] \\
&= \sum_{u_1}\sum_{u_2} \frac{w\left(v,u_2\right)}{deg(v)} \frac{w\left(v,u_1\right)w\left(v,u_2\right)}{w\left(v,u_1\right)+w\left(v,u_2\right)} L_{(u_1,u_2)} \\
&= \sum_{u_1<u_2} \frac{w\left(v,u_1\right)w\left(v,u_2\right)}{deg(v)} \frac{w\left(v,u_1\right)+w\left(v,u_2\right)}{w\left(v,u_1\right)+w\left(v,u_2\right)} L_{(u_1,u_2)} \\
&= Cl_k
\end{aligned}$$

□

Consider the difference of the resulting Laplacians in two consecutive iterations:

$$
\begin{aligned}
\widehat{L}^{(k)} - \widehat{L}^{(k-1)} &= \widehat{S}^{(k)} - \widehat{S}^{(k-1)} + c_k c_k^T \\
&= -\left(\widehat{S}^{(k-1)}\right)_{\pi(k)} + \widehat{Cl}_k + c_k c_k^T \\
&= -c_k c_k^T - Cl_k + \widehat{Cl}_k + c_k c_k^T \\
&= \widehat{Cl}_k - Cl_k.
\end{aligned}
$$

Therefore, it follows from Lemma 4.1, that $\mathbb{E}_{|k-1}\left[\widehat{L}^{(k)}\right] = \widehat{L}^{(k-1)}$ (hence, $\{\widehat{L}^{(k)}\}_{k\geq 0}$ is a matrix martingale).

The following Lemma concerning a distance property of the effective resistance will prove useful.

**Lemma 4.2** (Triangle Inequality for Effective Resistance)**.** *$\forall a,b,c \in V$, $Reff(a,b) \leq Reff(a,c) + Reff(c,b)$, or equivalently, $\|\bar{L}_{a,b}\| \leq \|\bar{L}_{a,c}\| + \|\bar{L}_{c,b}\|$.*

*Proof.*

$$
\begin{aligned}
\mathrm{Reff}(a,b) &= \left(\mathbb{1}_a - \mathbb{1}_b\right)^T L^{\dagger}\left(\mathbb{1}_a - \mathbb{1}_b\right) \\
&= \left(\mathbb{1}_a - \mathbb{1}_c + \mathbb{1}_c - \mathbb{1}_b\right)^T L^{\dagger}\left(\mathbb{1}_a - \mathbb{1}_c + \mathbb{1}_c - \mathbb{1}_b\right) \\
&= \left(\mathbb{1}_a - \mathbb{1}_c\right)^T L^{\dagger}\left(\mathbb{1}_a - \mathbb{1}_c\right) + \left(\mathbb{1}_c - \mathbb{1}_b\right)^T L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right) + 2\left(\mathbb{1}_a - \mathbb{1}_c\right)^T L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right) \\
&= \mathrm{Reff}(a,c) + \mathrm{Reff}(c,b) + 2\left(\mathbb{1}_a - \mathbb{1}_c\right)^T L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right) \\
&\leq \mathrm{Reff}(a,c) + \mathrm{Reff}(c,b),
\end{aligned}
$$

where the inequality holds since $\left(\mathbb{1}_a - \mathbb{1}_c\right)^T L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right) \leq 0$. This is true since $L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right)$ corresponds to the voltage drops in a resistor network where an external current of one unit is sent from $c$ to $b$, hence $\left(\mathbb{1}_a - \mathbb{1}_c\right)^T L^{\dagger}\left(\mathbb{1}_c - \mathbb{1}_b\right)$ is the voltage drop from $a$ to $c$, that is $v(a) - v(c)$. Since $c$ is the source in this network, it has the highest potential in the network, therefore $v(c) \geq v(a)$. □

**Lemma 4.3.** *Any edge $e$ added in Algorithm 2 to $\widehat{S}^k$, satisfies $\|Y_e\| = \|w(e)\bar{L}_e\| = w(e)\|L^{-\frac{1}{2}} L_e L^{-\frac{1}{2}}\| \leq \frac{1}{\rho}$.*

*Proof.* We prove the lemma by induction. At $k=0$, before "splitting" $w(e)\bar{L}_e \leq \Pi$, therefore after "splitting" $\|\frac{w(e)}{\rho}\bar{L}_e\| \leq \frac{1}{\rho}$. Now, suppose that the lemma holds for step $k-1$. Suppose edge $(u_1,u_2)$ is added at step $k$, then

$$
\begin{aligned}
\frac{w(v,u_1)\,w(v,u_2)}{w(v,u_1)+w(v,u_2)}\|\bar{L}_{u_1,u_2}\| &\leq \frac{w(v,u_1)\,w(v,u_2)}{w(v,u_1)+w(v,u_2)}\left(\|\bar{L}_{v,u_1}\| + \|\bar{L}_{v,u_2}\|\right) \\
&= \frac{w(v,u_2)}{w(v,u_1)+w(v,u_2)}\left(w(v,u_1)\,\|\bar{L}_{v,u_1}\|\right) \\
&\quad + \frac{w(v,u_1)}{w(v,u_1)+w(v,u_2)}\left(w(v,u_2)\,\|\bar{L}_{v,u_2}\|\right) \\
&\leq \frac{w(v,u_2)}{w(v,u_1)+w(v,u_2)}\frac{1}{\rho} + \frac{w(v,u_1)}{w(v,u_1)+w(v,u_2)}\frac{1}{\rho}
\end{aligned}
$$

$$= \frac{1}{\rho},$$

where the first inequality follows from Lemma 4.2 and the second from the inductive hypothesis. $\square$

Finally, in order to prove the approximation result, we will make use of the following matrix concentration result.

**Theorem 4.4** (Matrix Freedman Inequality)**.** *Consider a matrix martingale* $\{Y_k \in \mathbb{R}^{n\times n}\}_{k\geq 0}$ *with* $Y_0 = 0$ *(i.e., a random process satisfying* $\mathbb{E}_{|k-1}[Y_k] = Y_{k-1}$*). Define* $X_k = Y_k - Y_{k-1}$ *such that* $\lambda_{max}(X_k) \leq R$ *for all* $k$*. Further, define the "predictable quadratic variation"* $W_k = \sum_{i=1}^{k} \mathbb{E}_{|i-1}[X_i^2]$*. Then,* $\forall t, \sigma^2 \geq 0$*,*

$$\Pr\left(\exists j : \lambda_{max}(Y_j) \geq t \text{ and } \lambda_{max}(W_k) \leq \sigma^2\right) \leq n \exp\left\{-\frac{\frac{t^2}{2}}{\sigma^2 + \frac{tR}{3}}\right\}.$$

*Proof of Algorithm Approximation Result.* In the following we will consider each edge sampling individually. We introduce a notation to indicate the call index to CliqueSample and sampled edge index. For example, $Y_e^{(k)}$ is the Laplacian of the edge $e$ sampled at the $k^{th}$ call to CliqueSample. We will use $e-1$ to indicate the edge sampled before $e$ (i.e., $Y_{e-1}^{(k)}$ is the Laplacian of the edge $e-1$ sampled just before edge $e$, whose Laplacian is $Y_e^{(k)}$).

We had previously shown that

$$\begin{aligned}
\overline{\widehat{L}}^{(k)} - \overline{\widehat{L}}^{(k-1)} &= \overline{\widehat{Cl}}_k - \overline{Cl}_k \\
&= \sum_e \overline{Y}_e + \sum_e \mathbb{E}_{|k-1}\left[\overline{Y}_e\right] \\
&= \sum_e \overline{Y}_e + \mathbb{E}_{|k-1}\left[\overline{Y}_e\right],
\end{aligned}$$

from which it follows that

$$\overline{\widehat{L}}^{(k)} - \overline{\widehat{L}}^{(0)} = \sum_{i=1}^{k} \sum_e \overline{Y}_e^{(i)} + \mathbb{E}_{|i-1}\left[\overline{Y}_e^{(i)}\right], \tag{1}$$

where the summation of the edges is over **all** the edges in the each $i^{th}$ call of CliqueSample (i.e., in the $j^{th}$ call to CliqueSample we consider all the edges incident on vertex $v$ in Algorithm 1).

Consider the random matrix $T_{k,e'}$ sequence defined as

$$T_{k,e'} = \left(\sum_{i=1}^{k-1} \sum_e \overline{Y}_e^{(i)} + \mathbb{E}_{|i-1}\left[\overline{Y}_e^{(i)}\right]\right) + \left(\sum_{e\leq e'} Y_e^{(k)} - \mathbb{E}_{|k-1}\left[\overline{Y}_e^{(k)}\right]\right), \tag{2}$$

where the summation over edges in the $k^{th}$ call to CliqueSample is over all the edges up to and including edge $e'$. To reiterate, the difference between equations 1 and 2 is that in equation 1 we consider the summation over all the edges in each of the calls of CliqueSample, whereas in equation 2 we consider all the sampled edges in the first $k-1$ calls and the sum of all the edges sampled "**so far**" in the $k^{th}$ call to CliqueSample.

To simplify notation, we let $X_e^{(k)} = \overline{Y}_e^{(k)} - \mathbb{E}_{|i-1}\left[\overline{Y}_e^{(k)}\right]$ and write $T_{k,e'} = \sum_{(i,e)\leq(k,e')} X_e^{(i)}$. Furthermore, we use $\mathbb{E}_{|<(k,e')}[\cdot]$ to denote the expectation conditioned on all edge samplings until (and including) the sampling of edge $e'$ in the $k^{th}$ call to CliqueSample.

Note that $\|T_{k,e'}\| \leq \frac{1}{3}$ for all $k$ and $e'$ implies $\|\overline{\widehat{L}}^k - \overline{L}\| \leq \frac{1}{3}$. Therefore, we will use $T_{k,e'}$ to prove the algorithm approximation result. First note that $T_{k,e'}$ is a martingale of the form described in Theorem 4.4, since trivially $T_{0,0} = 0$ and

$$
\begin{aligned}
\mathbb{E}_{|<(k,e')}[T_{k,e'}] &= \mathbb{E}_{|<(k,e')}\left[\sum_{(i,e)\leq(k,e')} X_e^{(i)}\right] \\
&= \mathbb{E}_{|<(k,e')}\left[\sum_{(i,e)<(k,e')} X_e^{(i)}\right] - \mathbb{E}_{|<(k,e')}\left[X_{e'}^{(k)}\right] \\
&= \sum_{(i,e)<(k,e')} X_e^{(i)} + \mathbb{E}_{|<(k,e')}\left[\overline{Y}_{e'}^{(k)} - \mathbb{E}_{|k-1}\left[\overline{Y}_{e'}^{(k)}\right]\right] \\
&\overset{(*)}{=} \sum_{(i,e)<(k,e')} X_e^{(i)} + \mathbb{E}_{|k-1}\left[\overline{Y}_{e'}^{(k)}\right] - \mathbb{E}_{|k-1}\left[\overline{Y}_{e'}^{(k)}\right] \\
&= \sum_{(i,e)<(k,e')} X_e^{(i)} \\
&= T_{k,e-1},
\end{aligned}
$$

where $\mathbb{E}_{|<(k,e')}\left[\overline{Y}_{e'}^{(k)}\right] = \mathbb{E}_{|k-1}\left[\overline{Y}_{e'}^{(k)}\right]$ in $(*)$ follows from the fact that sampling of each edge within a call to CliqueSample is independent of any other edge sampling in the same call.

Now, note that $T_{k,e'} - T_{k,e'-1} = X_{e'}^{(k)}$, then

$$
\begin{aligned}
\|X_{e'}^{(k)}\| &= \|Y_{e'}^{(k)} - \mathbb{E}_{|k-1}\left[Y_{e'}^{(k)}\right]\| \\
&\leq \|Y_{e'}^{(k)}\| \\
&\leq \frac{1}{\rho},
\end{aligned}
$$

where the last inequality follows from Lemma 4.3. Therefore, when can bound the maximum eigenvalue of $X_{k,e'}$ as per Theorem 4.4, $\lambda_{\max}(X_{e'}^{(k)}) \leq R = \frac{1}{\rho}$.

Next, we wish to bound $W_{k,e'} = \sum_{(i,e)\leq(k,e')} \mathbb{E}_{|<(i,e)}\left[\left(X_{e'}^{(k)}\right)^2\right]$,

$$
\begin{aligned}
W_{k,e'} &\preccurlyeq \sum_{i=1}^{n}\sum_{e} \mathbb{E}_{|i-1}\left[\left(X_e^{(k)}\right)^2\right] \\
&= \sum_{i=1}^{n}\sum_{e} \mathbb{E}_{|i-1}\left[\overline{(Y_e^{(i)})}^2 - \left(\mathbb{E}_{|i-1}\left[\overline{Y_e^{(i)}}\right]\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&\preccurlyeq \sum_{i=1}^{n}\sum_{e}\mathbb{E}_{|i-1}\left[\overline{(Y_e^{(i)}})^2\right] \\
&\preccurlyeq \frac{1}{\rho}\sum_{i=1}^{n}\sum_{e}\mathbb{E}_{|i-1}\left[\overline{Y_e^{(i)}}\right] \\
&= \frac{1}{\rho}\sum_{i=1}^{n}\overline{Cl}_i.
\end{aligned}
$$

We know construct a new martingale to bound $W_{k,e'}$ by applying Theorem 4.4 a second time. Let $Z_j = \frac{1}{\rho}\sum_{i=1}^{j}\overline{Cl}_i$ and

$$
\begin{aligned}
A_j &= \left(Z_j - \mathbb{E}_{|j-1}\left[Z_j\right]\right) + A_{j-1} \\
&= \frac{1}{\rho}\sum_{i=1}^{j}\overline{Cl}_i - \frac{1}{\rho}\sum_{i=1}^{j-1}\overline{Cl}_i - \frac{1}{\rho}\mathbb{E}_{|j-1}\left[\overline{Cl}_j\right] + A_{j-1} \\
&= \frac{1}{\rho}\left(\overline{Cl}_j - \mathbb{E}_{|j-1}\left[\overline{Cl}_j\right]\right) + A_{j-1}.
\end{aligned}
$$

$A_j$ is a martingale of the type used in Theorem 4.4, since $A_0 = 0$ and

$$
\mathbb{E}_{|j-1}\left[A_j\right] = \frac{1}{\rho}\left(\mathbb{E}_{|j-1}\left[\overline{Cl}_j\right] - \mathbb{E}_{|j-1}\left[\overline{Cl}_j\right]\right) + \mathbb{E}_{|j-1}\left[A_{j-1}\right] = A_{j-1}.
$$

Since $\overline{Cl}_j \preccurlyeq \left(\overline{L}\right)_j \preccurlyeq \overline{\widehat{L^{j-1}}} \preccurlyeq \frac{4}{3}\Pi$, we have $0 \preccurlyeq \frac{1}{\rho}\overline{Cl}_j \preccurlyeq \frac{4}{3\rho}\Pi$ and similarly $0 \preccurlyeq \mathbb{E}_{|j-1}\left[\frac{1}{\rho}\overline{Cl}_j\right] \preccurlyeq \frac{4}{3\rho}\Pi$. Therefore, $-\frac{4}{3\rho}\Pi \preccurlyeq \frac{1}{\rho}\overline{Cl}_j - \mathbb{E}_{|j-1}\left[\frac{1}{\rho}\overline{Cl}_j\right] \preccurlyeq \frac{4}{3\rho}\Pi$. Hence,

$$
\|A_j - A_{j-1}\| = \frac{1}{\rho}\|\overline{Cl}_j - \mathbb{E}_{|j-1}\left[\overline{Cl}_j\right]\| \leq \frac{4}{3\rho},
$$

so we choose $\lambda_{\max}(A_j - A_{j-1}) \leq \frac{4}{3\rho} = R$.

Now we find a value for the bound $\sigma^2$,

$$
\begin{aligned}
\sum_{j=1}^{n}\mathbb{E}_{|j-1}\left[\left(A_j - A_{j-1}\right)^2\right] &= \sum_{j=1}^{n}\mathbb{E}_{|j-1}\left[\left(\frac{1}{\rho}\left(\overline{Cl}_j - \mathbb{E}_{|j-1}\left[\overline{Cl}_j\right]\right)\right)^2\right] \\
&\preccurlyeq \sum_{j=1}^{n}\mathbb{E}_{|j-1}\left[\frac{4}{3\rho^2}\overline{Cl}_j\right] \\
&\preccurlyeq \frac{4}{3\rho^2}\sum_{j=1}^{n}\mathbb{E}_{|j-1}\left(\overline{L}\right)_j \\
&= \frac{4}{3\rho^2}\sum_{j=1}^{n}\frac{1}{n-j+1}2\overline{\widehat{L^{(j-1)}}} \\
&\preccurlyeq \frac{4}{3\rho^2}\frac{4}{3}\sum_{j=1}^{n}\frac{2}{n-j+1}\Pi
\end{aligned}
$$

$$\preccurlyeq \frac{64}{9\rho^2}\log n\Pi.$$

Hence, we can choose $\sigma^2 = \frac{64}{9\rho^2}\log n$ (and, as stated above, $R = \frac{4}{3\rho}$) giving us according to Theorem 4.4 that $\lambda_{\max}(A_j) \geq t$ with probability $\leq n\exp\left\{-\frac{\frac{t^2}{2}}{\sigma^2+\frac{tR}{3}}\right\}$. We let $n\exp\left\{-\frac{\frac{t^2}{2}}{\sigma^2+\frac{tR}{3}}\right\} \leq \frac{1}{n^3}$, which implies that we need $\frac{t^2}{\sigma^2+\frac{tR}{3}} \geq 8\log n$. This in turn implies that we need to choose $t \geq \max\{\sqrt{\sigma^2 8\log n}, \frac{R}{3}8\log n\} = \max\{\frac{16}{3\rho}\sqrt{2\log n}, \frac{32}{9\rho}\log n\}$. By choosing $t = \frac{32}{3\rho}\log n$, we get that $\lambda_{\max}(A_j) \leq t$ with high probability. Therefore, by induction $\lambda_{\max}(A_n) \leq \frac{32}{3\rho}\log n$.

Furthermore,

$$\begin{aligned}
A_n &= \frac{1}{\rho}\left(\sum_{j=1}^{n}\overline{Cl}_j - \sum_{j=1}^{n}\mathbb{E}_{|j-1}\left[\overline{Cl}_j\right]\right) \\
&\succcurlyeq \frac{1}{\rho}\left(\sum_{j=1}^{n}\overline{Cl}_j - \sum_{j=1}^{n}\mathbb{E}_{|j-1}\left[\left(\overline{L}\right)_j\right]\right) \\
&= \frac{1}{\rho}\left(\sum_{j=1}^{n}\overline{Cl}_j - \sum_{j=1}^{n}\frac{1}{n-j+1}2\overline{\widehat{L^{(j-1)}}}\right) \\
&\succcurlyeq \frac{1}{\rho}\sum_{j=1}^{n}\overline{Cl}_j - \frac{1}{\rho}\frac{4}{3}4\log n\Pi \\
&= Z_n - \frac{16}{3\rho}\log n\Pi,
\end{aligned}$$

Going back to the first application of Theorem 4.4 we find

$$W_{k,e'} \preccurlyeq Z_n \preccurlyeq A_n + \frac{16}{3\rho}\log n\Pi \preccurlyeq \frac{16}{\rho}\log n\Pi.$$

Therefore, to apply Theorem 4.4 on the original martingale we have (from before) $R = \frac{1}{\rho}$ and $\sigma^2 = \frac{16}{\rho}\log n$. As before, we need to choose $t \geq \max\{\sqrt{\sigma^2 8\log n}, \frac{R}{3}8\log n\} = \max\{\sqrt{\frac{2}{\rho}}8\log n, \frac{1}{3\rho}8\log n\}$.

By choosing $t = \frac{16}{3\rho}\log n$, $\lambda_{\max}(T_{k,e'}) \leq t$. To get our result, we want to choose $t \leq \frac{1}{3}$. And so by choosing $\rho = \frac{16}{9}\log n$, we get that $\|T_{k,e'}\| \leq \frac{1}{3}$ implying $\|\overline{\widehat{L}}^k - \overline{L}\| \leq \frac{1}{3}$. Using induction, we find $\|\overline{\widehat{L}}^n - \overline{L}\| \leq \frac{1}{3}$, that is $\frac{2}{3}L \preccurlyeq \widehat{L}^n \preccurlyeq \frac{4}{3}L$. $\square$