Statistical Analysis of a Bidimensional Dark Energy Parametrization

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A new bidimensional parametrized equation of state for dark energy is studied and is tested against supernovae data from Union 2.1 and $H(z)$ data from Cosmic Clocks. It has a well defined ΛCDM limit and it correctly reproduces a typical feature of the standard solution: it has an exponential expansion of the Universe with a deceleration parameter such that $q(z = 0) \sim -0.55$. Also, we present the plots for the modulus distance and the Hubble parameter, which are very similar to the ones from the ΛCDM model. The regime (phantom or quintessence) of the corresponding exotic fluid is not yet settled down. More cosmological observations are needed in order to completely explore this new model.

I. INTRODUCTION

Current cosmological observations imply that the Universe is in an accelerated expansion stage. The standard cosmological model ΛCDM explain the accelerated expansion with the presence of a cosmological constant Λ that fills the entire Universe. Despite being the most successful model consistent with observations, the cosmological constant has some loose ends, for example its origin and nature which are still unknown. For these reasons, scientists have introduced a series of proposals dubbed *dark energy models* for exotic fluids to account for this kind of expansion. The principal idea behind these models is the existence of fluids with negative pressure characterized by a parametrized equation of state $w_X \equiv P_X/\rho_X$, where P_X is the pressure of the exotic fluid and ρ_X its energy density. [1]

These dark energy models rely on the standard cosmological models which are based on the cosmological principle, the Einstein's field equations and the Weyl's postulate. That is why another approach to solve the problem is to modify some of the previous assumptions. The most popular option is to use *modified gravity* models as dark energy instead of the original Einstein's equations to account for the accelerated expansion and other phenomena associated with dark energy. Nevertheless, in this work we will stick to the exotic fluid options.

Numerous dark energy models have an equation of state as a function of the redshift z and are parametrized by two parameters, which we call w_0 and w_1. These bidimensional parametrizations are dubbed w_0w_1CDM models (see e.g. [2] and references there in for a short review on the subject). Nevertheless, there are one-dimensional parametrizations [3] and even higher-dimensional parametrizations (see e.g. [4]) which have been considered as models for dark energy. One of the central tasks in this approach of a parametrized equation of state is to estimate the best-fit values for the parameters in order to see if the proposed model is good compared to the standard ΛCDM model.

In this work, we will explore a new dark energy model, based on an exotic fluid with a certain parametrized equation of state in a background FLRW flat metric. We will test the model with SNe IA data from Union 2.1 and $H(z)$ data from Cosmic Clocks, and we will present a comparison between this new model and ΛCDM model. This manuscript is organized as follows: in Section II we present the basic equations required for a model of dark energy with a parametrized equation of state and the equations needed for a statistical analysis; in Section III we present our results and their interpretation; finally, in Section IV we give our conclusions and final remarks. We will use natural units so that $c = 1$, with c the speed of light.

II. BASIC EQUATIONS

A. Dark Energy

Modern observations [5] indicate that the Universe is essentially filled with matter (cold dark matter + baryonic matter) and dark energy. The radiation (photons + neutrinos) component of the Universe is too small compared with the contents of matter and dark energy. Under these assumptions, the first Friedmann equation for the scale factor a

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in a flat Universe reads:

$$E(z)^2 \equiv \frac{H(z)^2}{H_0^2} = \Omega_m(1+z)^3 + \Omega_X f(z), \tag{1}$$

where $H(z) \equiv \dot{a}/a$ is the Hubble parameter and H_0 its present value, Ω_m is the density parameter of total matter today, Ω_X is the density parameter of dark energy today (with the constraint $\Omega_X = 1 - \Omega_m$), and

$$f(z) = \exp\left(3\int_0^z \frac{1+w(z')}{1+z'}\,dz'\right). \tag{2}$$

The last function comes from the continuity equation $\dot{\rho}_X = -3H(\rho_X + P_X)$, with ρ_X and P_X the energy density and pressure of the exotic fluid, such that:

$$\rho_X(z) = \rho_{X,0}\, f(z), \tag{3}$$

where $\rho_{X,0}$ is the energy density of dark matter today.

In this work, we will use two parametrizations:

$$w_{\Lambda\mathrm{CDM}} = -1 \qquad \rightarrow \qquad f(z) = 1, \tag{4}$$

$$w_X(z) = \frac{\ln w_1}{\ln\left(\frac{w_1}{1+z}\right)}(1+w_0) - 1 \qquad \rightarrow \qquad f(z) = \left(\frac{\ln w_1}{\ln\left(\frac{w_1}{1+z}\right)}\right)^{3\ln w_1(1+w_0)}. \tag{5}$$

Eq. (5) is the proposed new dark energy model to be explored. Note that for $z = 0$, then $w_X = w_0$, independently of the value of w_1 and z, and if $w_0 = -1$, we recover $w_X = w_{\Lambda\mathrm{CDM}}$. Also, from Eq. (5) we get the restriction $w_1 > 0$. Finally, note that for an early Universe[1] ($z \to \infty$) we get $w_X \to -1$. We chose this parametrization for two reasons: the first is the ΛCDM limit recovered when $w_0 = -1$ or when $z \to \infty$, and the second is due to a mathematical simplification that occurs when calculating $f(z)$ and the scale factor. In fact, we found this parametrization demanding that the scale factor had the form $a \propto \exp(Ht^\alpha)$, for some constant α, so that the expansion of the Universe is accelerating (we will address this point later in this section; see Eq. (9)) at a great or less rate, depending on the value of α.

Using the second Friedmann equation for a flat Universe,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \equiv \frac{H_0^2}{2}\sum_i^{\{m,X\}} \frac{8\pi G}{3H_0^2}\rho_i(1+3w_i) \equiv \frac{H_0^2}{2}\sum_i^{\{m,X\}} \Omega_i(1+3w_i) = H_0^2\,\frac{\Omega_m + \Omega_X[1+3w_X(z)]}{2}, \tag{6}$$

where G is Newton's gravitational constant, ρ and P are the total energy density and pressure, respectively, ρ_i and P_i are the energy density and pressure of the i-th component, respectively, and $w_m = 0$, we can calculate the deceleration parameter:

$$q(z) \equiv -\frac{\ddot{a}\,a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} = \frac{1}{2}\frac{\Omega_m + \Omega_X(1+3w(z))}{\Omega_m(1+z)^3 + \Omega_X f(z)}. \tag{7}$$

Furthermore, assuming an approximately dark energy-dominated Universe, we have from the first Friedmann equation and Eq. (3) that

$$\frac{\dot{a}}{a} \approx \sqrt{\frac{8\pi G}{3}\rho_{X,0}}\, f(a)^{1/2} = H_0\sqrt{\Omega_X}\left(\frac{\ln w_1}{\ln(w_1 a)}\right)^{3\ln w_1(1+w_0)/2}, \tag{8}$$

where we have used Eq. (5) and that $a = \frac{1}{1+z}$, with $a(t = t_0) = 1$ where t_0 is the age of the Universe. The integration in Eq. (8) can be done analytically due to our choice of the form of the parametrized equation of state. In this way, we find:

$$a(t) = \frac{1}{w_1}\exp\left\{\ln w_1\left[1 + \frac{1+\gamma}{\ln w_1}H_0\sqrt{\Omega_X}(t-t_0)\right]^{\frac{1}{\gamma+1}}\right\}, \tag{9}$$

with $\gamma \equiv \frac{3}{2}\ln w_1(1+w_0)$. Note that for $w_0 = -1$, we recover the well-known result $a(t) = \exp H(t-t_0)$. The best-fit values for w_0 and w_1 will give us an accelerated expansion, as we will see in Section III.

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[1] Actually, for an early Universe Eq. (1) has to be modified in order to include radiation which is an important component for the early Universe.

B. Statistics

We will work under the Bayes framework [6], which give us a relation for the posterior probability and the likelihood function for a model M, a set of data D and a set of parameters θ, through the Bayes theorem:

$$P(\theta|D,M) = \frac{P(D|\theta,M)P(\theta|M)}{P(D|M)}, \tag{10}$$

where $P(\theta|D,M)$ is the posterior probability, $P(D|\theta,M) \equiv \mathcal{L}(D|\theta,M)$ is the likelihood function, $P(\theta|M)$ is the prior and $P(D|M)$ is the evidence of the model.

The parameter estimation is done with the maximization of the likelihood function $\mathcal{L}$ in order to obtain the most probable set of model parameters, given the data. In the Gaussian approximation, maximizing the likelihood is equivalent to minimizing the chi-square χ^2, related by

$$\mathcal{L} \propto e^{-\chi^2/2}. \tag{11}$$

We will use the Markov Chain Monte Carlo (MCMC) technique for the parameter inference. In particular, we use the ensemble sampler *emcee* [7] implemented in Python to build a chain in the parameter space of our model in order to evaluate the posterior of Eq. (10).

Two sets of data are used in this work, namely data of supernovae type Ia (SNe Ia) from Union 2.1 and $H(z)$ data from Cosmic Clocks (CC). For the former set of data, the chi-square function is given by:

$$\chi^2_{\rm SN} \equiv \sum_{i=1}^{N_{\rm Union\ 2.1}} \frac{[\mu(z_i;\Omega_m,w_0,w_1) - \mu_{\rm obs}(z_i)]^2}{\sigma^2_{\mu,i}}, \tag{12}$$

where $N_{\rm Union\ 2.1}$ is the total number of points in the set of data, $\mu_{\rm obs}(z_i)$ is the observed modulus distance for each supernova i and $\sigma_{\mu,i}$ its error, and the expression for the modulus distance is [8]:

$$\mu(z) = 5\log_{10}\left(\frac{(1+z)c}{H_0\,10\,\mathrm{pc}}\int_0^z \frac{1}{E(z')}\,dz'\right), \tag{13}$$

where we have included the c factor to have the correct units. We will use $H_0 = 69$ km/s Mpc and $c = 299,792.458$ km/s.

For the CC set of data, we will use

$$\chi^2_{\rm CC} \equiv \sum_{i=1}^{N_{\rm CC}} \frac{[H(z_i;\Omega_m,w_0,w_1) - H_{\rm obs}(z_i)]^2}{\sigma^2_{H,i}}, \tag{14}$$

where $N_{\rm CC}$ is the total number of points in the set of data, $H_{\rm obs}(z_i)$ is the observed Hubble parameter at each redshift z_i and $\sigma_{H,i}$ its error, and the expression for the Hubble parameter is calculated with Eq. (1). We will make three statistical analysis: one associated with $\chi^2_{\rm SN}$, another with $\chi^2_{\rm CC}$ and the third one with $\chi^2_{\rm SN} + \chi^2_{\rm CC}$.

III. RESULTS AND ANALYSIS

We calculated the probability distributions for the parameters Ω_m, w_0 and w_1 for the independent sets of data SNe Ia (32 walkers with 500 steps) and CC (32 walkers with 1000 steps), and for the joint data SNe Ia + CC (32 walkers with 500 steps)[2] for different uninformative (uniform) priors. The results of each independent set are shown in Appendix A. In Figure 1 we show the three sets of confidence contours, together with a comparison with the standard ΛCDM model with $w_0 = -1$ and $\Omega_m = 0.315 \pm 0.007$ [5]. The best-fit parameters are shown in Table 1, whose uncertainties are based on the 16th, 50th, and 84th percentiles of the samples in the marginalized distributions.

From Figure 1 we note that the confidence contours of the SNe Ia data are a little greater than the other sets, despite having more data than the CC set. Nevertheless this is a consequence of our chosen prior for the values of w_1 for the SNe Ia data, which was $0.3 < w_1 < 0.7$, compared to the chosen prior for the CC data, which was

[2] The choice of the number of walkers and steps was done according to our computational time resources.

$0.62 < w_1 < 0.7$, so that the MCMC method mapped a greater zone for the SNe Ia data. We see also see in Figure 1 a tension between the CC data and the other sets of data. Thus, more points (more data) are required to ensure the compatibility of our model with the observations.

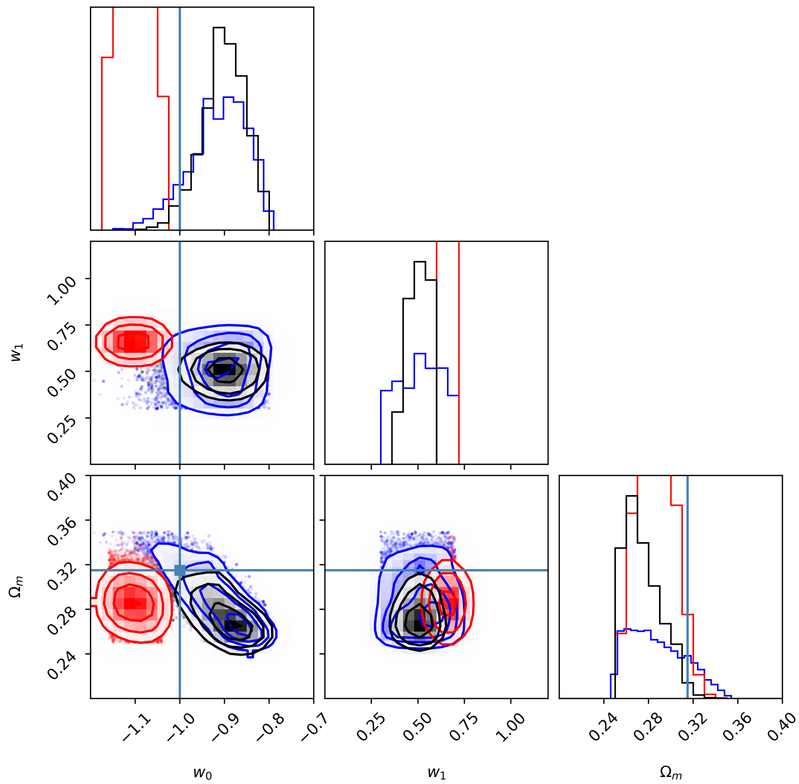


Figure 1: Confidence contours for the parameter estimation of each set of data: SNe Ia (blue), CC (red) and SNe Ia + CC (black). The horizontal and vertical blue lines indicate the ΛCDM limit which corresponds to $w_0 = -1$ and $\Omega_m = 0.315 \pm 0.007$.

Model	Ω_m	w_0	w_1
SNe Ia	$0.284^{0.032}_{-0.023}$	$-0.908^{0.058}_{-0.070}$	$0.514^{0.128}_{-0.136}$
CC	$0.286^{0.017}_{-0.016}$	$-1.102^{0.040}_{-0.039}$	$0.659^{0.027}_{-0.026}$
SNe Ia + CC	$0.272^{0.021}_{-0.013}$	$-0.900^{0.046}_{-0.050}$	$0.504^{0.065}_{-0.068}$

Table 1: Best-fit values for different sets of data calculated with a MCMC method.

With the best-fit values for the parameters we plot the parametrized equation of state, Eq. (5). For the CC fit we get a phantom behaviour for our theoretical exotic fluid, while for the SNe Ia and SNe Ia + CC fits we get a quintessence behaviour. This contradictory result could be alleviated with a greater number of data from the same type or different observations, as in the case of the tension between CC and SNe Ia data. In this way, we could determine with more certainty the nature of the exotic fluid. For each set of values for the parameters we recover the ΛCDM limit at large redshifts, as we expect from the form of Eq. (5). Nevertheless, this is an extrapolation because the redshifts used are low. Observations with large redshifts could help us to determine better distributions for the parameters.

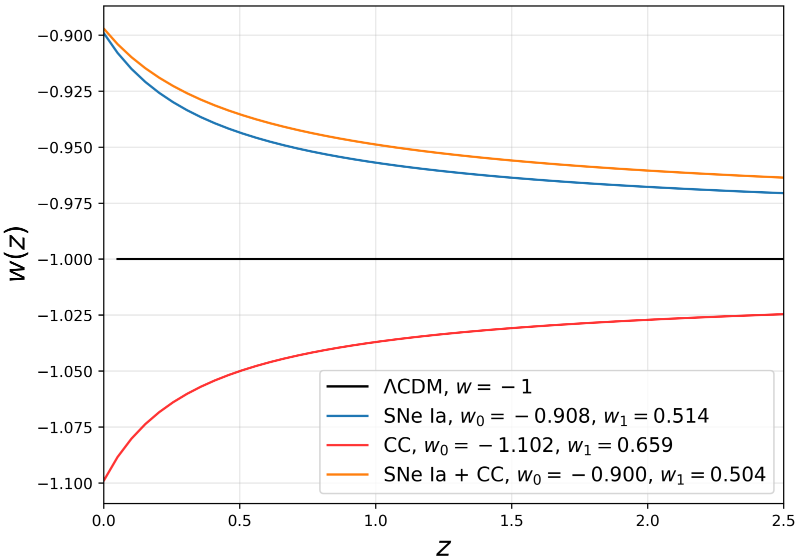


Figure 2: Parametrized equation of state (Eq. (5)) for each set of best-fit values.

The models for $\mu(z)$ (Eq. (13)) and $H(z)$ (which is related to Eq. (1)), together with the data from Union 2.1 and Cosmic Clocks are shown in Figures 3 and 4. In Figure 3, we can appreciate a very good fit compared with the ΛCDM limit for each set of data at lower redshifts. A small difference begin to grows at greater redshifts, as can be seen in 3 (right). Nevertheless, it is required a greater set of data to extrapolate our results for even greater redshifts (in fact, for larger redshifts we need to change the model to include the presence of radiation; see Footnote 1). In Figure 4 we see the same kind of behaviour except for the CC data which differs from the other curves even at low redshifts.

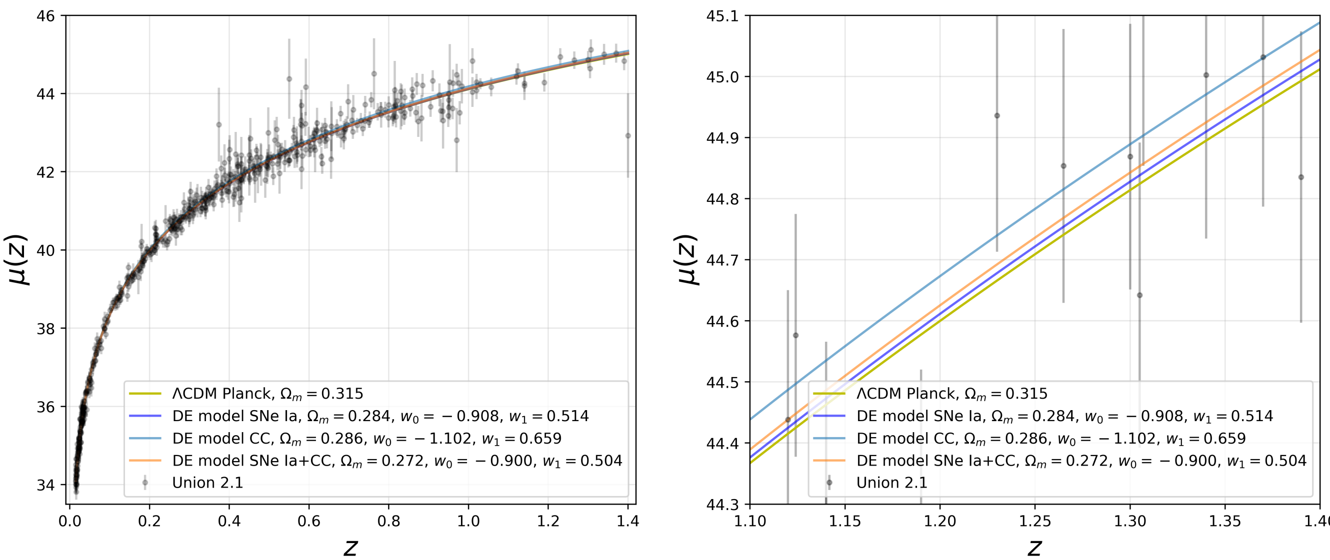


Figure 3: The set of data from Union 2.1 of supernovae and the plot of the modulus distance (Eq. (13)) for the best-fit parameters for each set of data. For comparison, we plot the ΛCDM limit with Ω_m from Planck results [5]. The figure on the right is an amplification of the left plot at the greatest redshifts of the set of data.

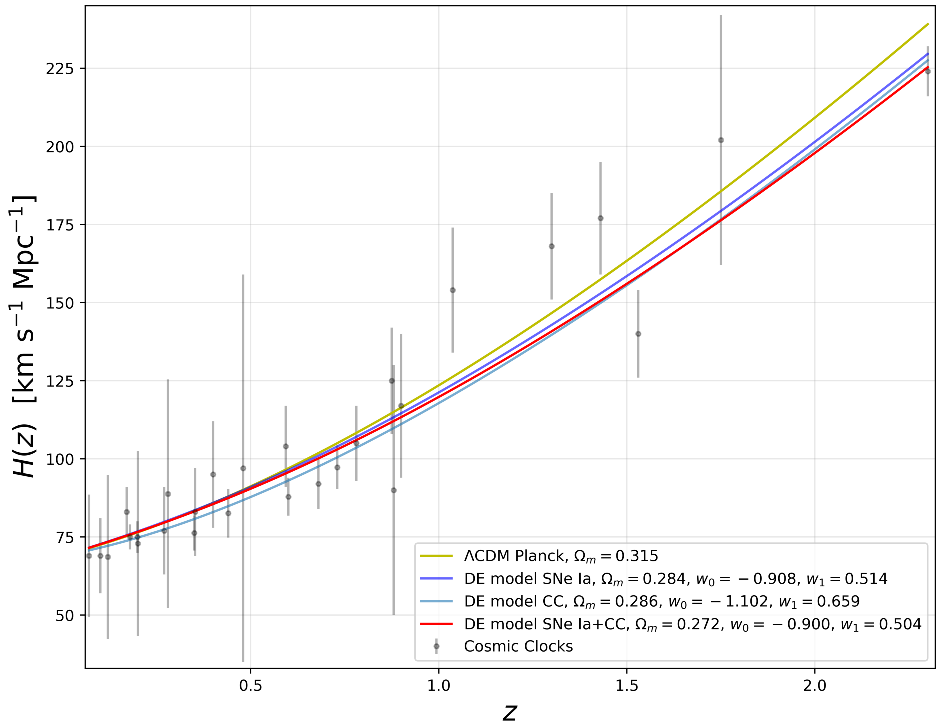
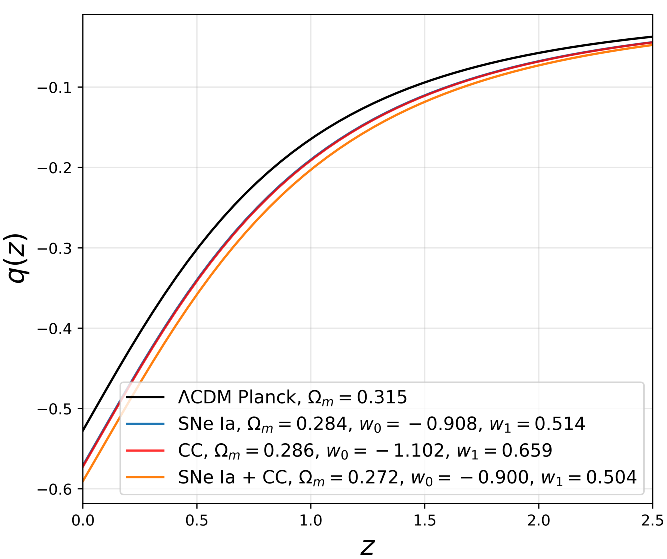


Figure 4: The set of points from $H(z)$ data of Cosmic Clocks and the plot of the Hubble parameter (related to Eq. (1)) for the best-fit parameters for each set of data and the ΛCDM limit with Ω_m from Planck results [5].

Finally, we plot the deceleration parameter and the scale factor, Eqs. (7) and (9), in Figure 5. Each set of results produce an accelerated expansion greater than the one predicted by the ΛCDM model, as we expected from our proposal of the parametrized equation of state, and in general they reproduce a value of $q_{\rm today} \sim -0.55$.



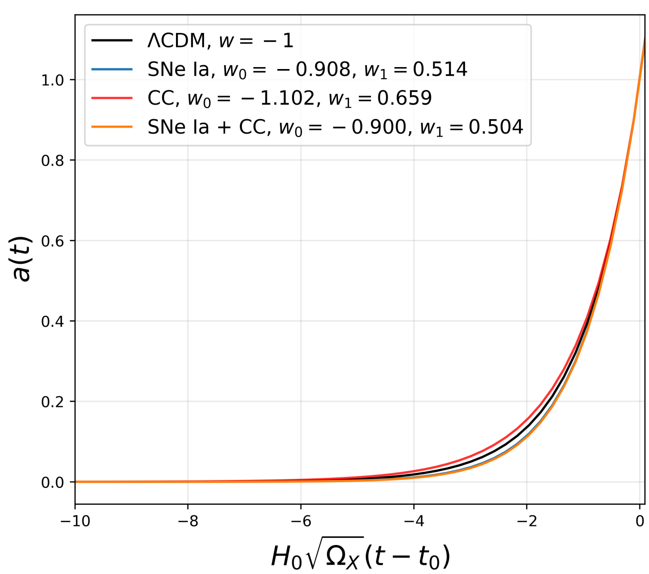


Figure 5: Deceleration parameter, Eq. (7), and scale factor, Eq. (9), for the best-fit parameters for each set of data, together with the ΛCDM limit.

IV. CONCLUSIONS

We presented a new parametrized equation of state in the framework of the w_0w_1-CDM models for dark energy based on an exotic fluid and we tested it with supernovae data from Union 2.1 and $H(z)$ data from Cosmic Clocks. This model has a corresponding ΛCDM limit, and it reproduces in general the behaviour of the standard solution, namely it has an accelerated expansion with a deceleration parameter such that $q(z=0) \sim -0.55$, an exponential growth of the scale factor, and reproduces very similar curves for the modulus distance and the Hubble parameter. The equation of state belongs to the phantom regime for the CC data, but for the SNe Ia data and joint SNe Ia+CC data it belongs to the quintessence regime. Furthermore, our model presented tension between the two independent sets of data, which suggests that more observations are necessary in order to settle down the nature of the exotic fluid. In particular, observations at larger redshifts could help us to determine the validity of the ΛCDM limit.

We suggest a future test of our model with more cosmological observations to study its validity. Another suggestion is to fix the matter density parameter Ω_m to is estimated Planck value [5] and find the best-fit values for the parameters w_0 and w_1. This could in principle alleviate the tension between the two sets of data used in this work.

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Appendix A: Confidence contours for each set of data

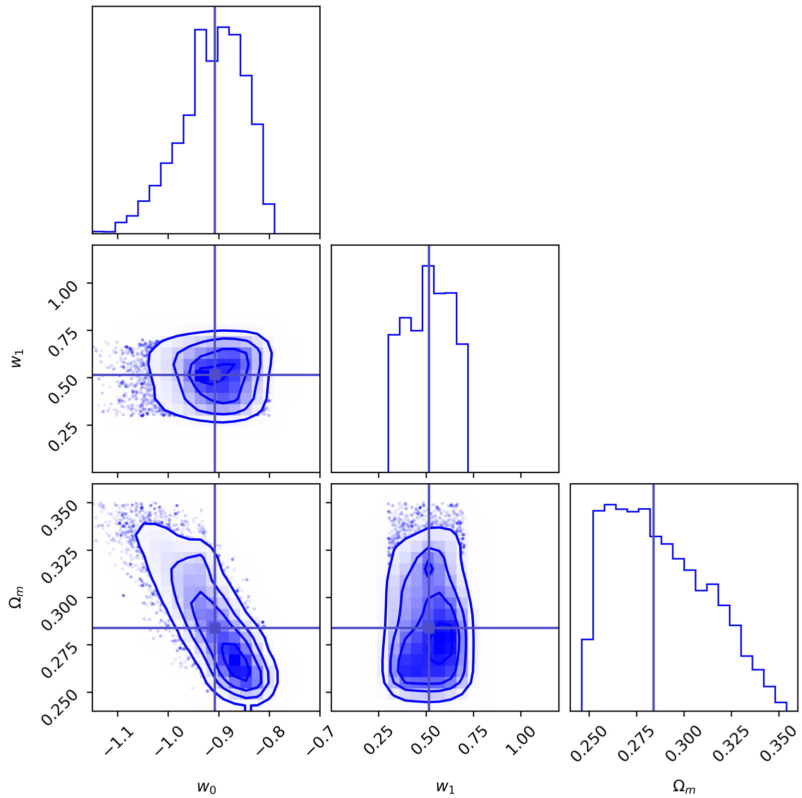

Figure 6: Confidence contours for the parameter estimation of SNe Ia data. The horizontal and vertical blue lines indicate the maximum likelihood first estimate which served as the initial point for the MCMC method.

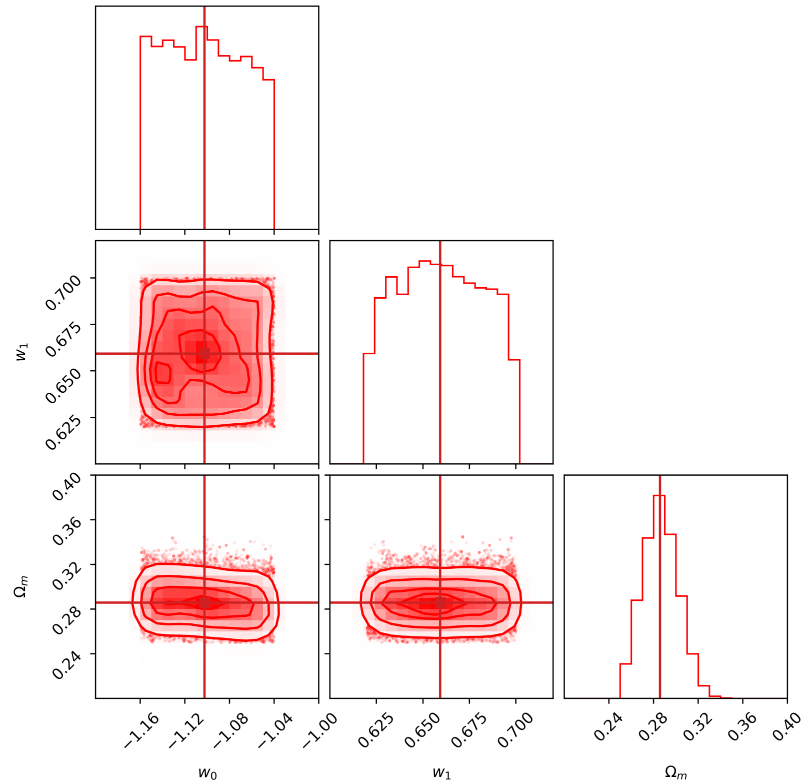

Figure 7: Confidence contours for the parameter estimation of CC data. The horizontal and vertical blue lines indicate the maximum likelihood first estimate which served as the initial point for the MCMC method.

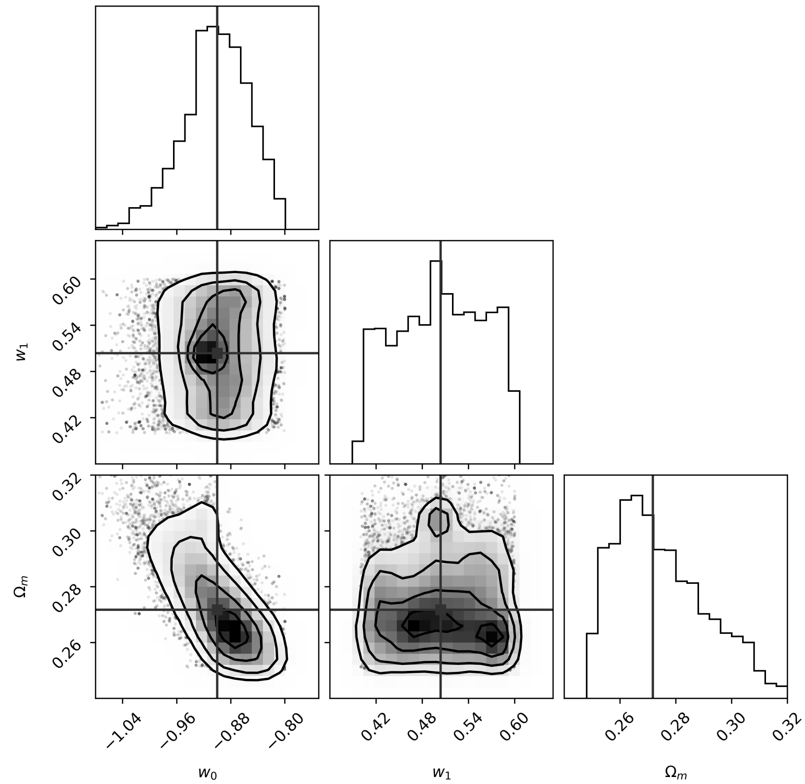

Figure 8: Confidence contours for the parameter estimation of SNe Ia + CC data. The horizontal and vertical blue lines indicate the maximum likelihood first estimate which served as the initial point for the MCMC method.