第三节 定积分的换元法与分部积分法-2

根据牛顿–莱布尼兹公式，求定积分可以转化为求原函数，而求原函数有换元积分法和分部积分法，所以求定积分也有换元积分法和分部积分法。

二、分部积分公式

设函数$u(x)$、$v(x)$在区间$[a,b]$上具有连续

导数，则有$\int_a^b udv = [uv]_a^b - \int_a^b vdu$.

定积分的分部积分公式

推导 $(uv)' = u'v + uv'$,

$$\int_a^b (uv)'dx = \int_a^b u'vdx + \int_a^b uv'dx,$$

$$[uv]_a^b = \int_a^b u'vdx + \int_a^b uv'dx = \int_a^b vdu + \int_a^b udv,$$

$$\therefore \int_a^b udv = [uv]_a^b - \int_a^b vdu.$$

例1　计算 $\int_0^{\frac{1}{2}} \arcsin x dx$.

解

$$\int_0^{\frac{1}{2}} \arcsin x dx = \left[x \arcsin x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x d \arcsin x$$

$$= \left[x \arcsin x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{6} - \left(-\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} d(1-x^2)$$

$$= \frac{\pi}{12} + \frac{1}{2} 2 \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

例2 计算 $\int_0^1 e^{\sqrt{x}}dx$

解 令$\sqrt{x}=t$，则$x=t^2, dx=2tdt$，

当$x=0$时，$t=0$； 当$x=1$时，$t=1$， 于是

$$\int_0^1 e^{\sqrt{x}}dx=\int_0^1 2te^t dt=2\int_0^1 tde^t$$

换元法

$$=2\left[(te^t)_0^1-\int_0^1 e^t dt\right]$$

分部积分法

$$=2[e-[e^t]_0^1]=2$$

-练习　计算$\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}$.

解　$\because 1+\cos 2x = 2\cos^2 x$,

$$\therefore \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x} = \int_0^{\frac{\pi}{4}} \frac{xdx}{2\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{x}{2} d(\tan x)$$

$$= \frac{1}{2}\left[x\tan x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{\pi}{8} + \frac{1}{2}\left[\ln\left|\cos x\right|\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

练习 计算 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx.$

解 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx = -\int_0^1 \ln(1+x)d\frac{1}{2+x}$

$$= -\left[\frac{\ln(1+x)}{2+x}\right]_0^1 + \int_0^1 \frac{1}{2+x}d\ln(1+x)$$

$$= -\frac{\ln 2}{3} + \int_0^1 \boxed{\frac{1}{2+x}\cdot\frac{1}{1+x}}dx \longrightarrow \frac{1}{1+x} - \frac{1}{2+x}$$

$$= -\frac{\ln 2}{3} + [\ln(1+x) - \ln(2+x)]_0^1 = \frac{5}{3}\ln 2 - \ln 3.$$

例3 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}dt$,求 $\int_0^1 xf(x)dx$.

解 因为$\frac{\sin t}{t}$没有初等形式的原函数，

无法直接求出$f(x)$，所以采用分部积分法

$$\int_0^1 xf(x)dx=\frac{1}{2}\int_0^1 f(x)d(x^2)$$

$$=\frac{1}{2}\left[x^2f(x)\right]_0^1-\frac{1}{2}\int_0^1 x^2df(x)$$

$$=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2f'(x)dx$$

$$\boxed{\text{原式} = \frac{1}{2} f(1) - \frac{1}{2}\int_0^1 x^2 f'(x)dx}$$

$$\boxed{f(x) = \int_1^{x^2} \frac{\sin t}{t} dt,}$$

$$f(1) = \int_1^1 \frac{\sin t}{t} dt = 0,$$

$$f'(x) = \frac{\sin x^2}{x^2} \cdot 2x = \frac{2\sin x^2}{x},$$

$$\therefore \int_0^1 xf(x)dx = \frac{1}{2} f(1) - \frac{1}{2}\int_0^1 x^2 f'(x)dx$$

$$= 0 - \frac{1}{2}\int_0^1 \frac{2x^2 \sin x^2}{x} dx = -\frac{1}{2}\int_0^1 2x \sin x^2 dx$$

$$= -\frac{1}{2}\int_0^1 \sin x^2 dx^2 = \frac{1}{2}\left[\cos x^2\right]_0^1 = \frac{1}{2}(\cos 1 - 1).$$

例4　设$f'(x)$在$[0,1]$上连续，求$\int_0^1[1+xf'(x)]e^{f(x)}dx$.

解：
$$原式=\int_0^1 e^{f(x)}dx+\int_0^1 xf'(x)e^{f(x)}dx$$

$$=\int_0^1 e^{f(x)}dx+\int_0^1 xe^{f(x)}df(x)$$

$$=\int_0^1 e^{f(x)}dx+\int_0^1 xde^{f(x)}$$

$$=\int_0^1 e^{f(x)}dx+[xe^{f(x)}]_0^1-\int_0^1 e^{f(x)}dx$$

$$=[xe^{f(x)}]_0^1 \quad =e^{f(1)}$$

一例. **设$f(x)$在$[a,b]$上有连续的二阶导数，且$f(a)=$**
$f(b)=0$，试证 $\int_a^b f(x)\,\mathrm{d}x=\frac{1}{2}\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x$

解：右端 $=\frac{1}{2}\int_a^b (x-a)(x-b)\,\mathrm{d}f'(x)$ 分部积分

$$=\frac{1}{2}\left[(x-a)(x-b)f'(x)\right]\Big|_a^b-\frac{1}{2}\int_a^b f'(x)d(x^2-(a+b)x+ab)$$

$$=-\frac{1}{2}\int_a^b f'(x)(2x-a-b)dx$$

$$=-\frac{1}{2}\int_a^b (2x-a-b)\,\mathrm{d}f(x)$$ 再次分部积分

$$=-\frac{1}{2}\left\{\left[(2x-a-b)f(x)\right]\Big|_a^b-2\int_a^b f(x)\,\mathrm{d}x\right\}\quad=\text{左端}$$

例5. 证明 $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{ 为偶数}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}, & n\text{ 为奇数}\end{cases}$$

证: 令 $t=\frac{\pi}{2}-x$, 则

$$\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=-\int_{\frac{\pi}{2}}^{0}\sin^n\left(\tfrac{\pi}{2}-t\right)\mathrm{d}t=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$$

令 $u=\sin^{n-1}x,\ v'=\sin x$, 则 $u'=(n-1)\sin^{n-2}x\cos x$,

$v=-\cos x$

$$\therefore\quad I_n=\underbrace{\left[-\cos x\cdot\sin^{n-1}x\right]\Big|_0^{\frac{\pi}{2}}}_{=\,0}+(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x\,\mathrm{d}x$$

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \,\mathrm{d}x$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x)\,\mathrm{d}x$$

$$= (n-1) I_{n-2} - (n-1) I_n$$

$$\boxed{I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x}$$

由此得递推公式 $I_n = \frac{n-1}{n} I_{n-2}$

于是 $$I_{2m} = \frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot I_0$$

$I_{2m+1} = \frac{2m}{2m+1}\cdot$ [redacted] $\cdot\frac{2}{3}\cdot I_1$

而 $$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}, \qquad I_1 = \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x = 1$$

故所证结论成立 .

小结

定积分的分部积分公式

$$\int_a^b u dv = \left[uv\right]_a^b - \int_a^b v du.$$

作业

P254　7(4);(9);(10)

备用题

1. 证明 $f(x)=\int_{x}^{x+\frac{\pi}{2}}|\sin x|\,\mathrm{d}x$ 是以π为周期的函数.

证：$f(x+\pi)=\int_{x+\pi}^{x+\pi+\frac{\pi}{2}}|\sin u|\,\mathrm{d}u$

令$u=t+\pi$

$$=\int_{x}^{x+\frac{\pi}{2}}|\sin(t-\pi)|\,dt$$

$$=\int_{x}^{x+\frac{\pi}{2}}|\sin t|\,\mathrm{d}t=\int_{x}^{x+\frac{\pi}{2}}|\sin x|\,\mathrm{d}x$$

$$=f(x)$$

$f(x)$是以π为周期的周期函数.