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SCALING AND UNIVERSALITY IN THE DYNAMICS OF SEISMIC OCCURRENCE AND BEYOND

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ABSTRACT

We present a very brief review of recent findings about scale-invariant behavior in the structure of earthquake occurrence in time and size, leading to the proposal of a striking universal scaling law that has also been shown to describe fracture phenomena. The range of validity of this law in space and energy is enormous. Previously, we discuss the concepts of self similarity and scale invariance and relate them with power laws and scaling laws, explaining the difference between them. The connections between critical phenomena, crackling noise, self-organized criticality, and the possible existence of universality and universality classes out-of-equilibrium are mentioned as well.

1 SELF-SIMILARITY, SCALE INVARIANCE, POWER LAWS, AND SCALING LAWS

In the last decades, the study of the natural world has witnessed the emergence of the *fractal revolution* for the description of many existing physical and biological entities [1–4]. A fractal can be informally defined as a geometrical object that shows the same structure independently on the scale of observation; i.e., it does not matter the magnification of the microscope we apply to the system, we will always see something *similar*.

This *self-similarity* has the implication that any property of the system $f(x)$ depending on the scale of observarion x (for instance, the number of structures of size x or the auto-correlation at a distance x) must be *scale invariant*, i.e., it does not change when the scale of observation is changed. A change of scale is simply a linear transformation of the axes x and y (where $y = f(x)$), i.e.,

$$x \to x' \equiv bx,$$
$$y \to y' \equiv cy,$$

where b and c are the scale factors which expand each axis; for example, x can be the length measured in meters and x' the same quantity in mm, then $b = 1000$. It is clear that the change in the axes makes the function transform by means of the transformation $\top$ into

$$f(x) \to \top[f(x)] = cf(x'/b),$$

and then, the scale-invariance condition means that

$$f(x) = cf(x/b), \quad \forall b.$$

It is trivial to check that this functional equation has as a solution a *power law*,

$$f(x) = kx^{\alpha}, \tag{1}$$

with the exponent $\alpha \equiv \log c/\log b$ and k an arbitrary constant. Note that the solution holds for all b, but then c is not independent, $c = b^{\alpha}$. Moreover, the power law is not only a solution of the scale-invariance condition, but it is the only solution, see Appendix 1 or Refs. [3, 5]. Figure 1 illustrates the invariance of the power law under linear changes of scale, unlike to what happens for example for exponential and Gaussian functions. In some cases it is important to realize that the power-law solution is valid for complex (non-real) α, but then not all b factors are allowed, see Ref. [6].

The scale-invariance condition means that the system does not present a characteristic scale, in contrast to many phenomena in physics. Imagine that we are living on a fractal and, by means of some science-fiction procedure, our size is reduced by an unknown factor (this is equivalent to expand the x-axis by the

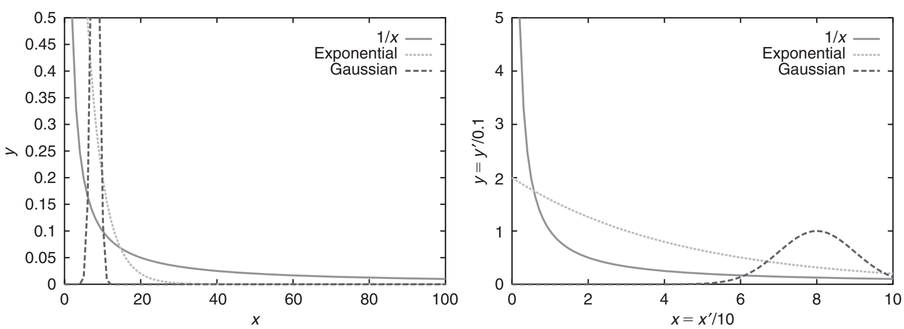

Figure 1: Scale invariance of the power law. In the left figure three functions are displayed: a power law ($1/x$), an exponential ($2 \cdot 10^{-x/10}$), and a Gaussian ($e^{-(x-8)^2/2}$). In the right figure, the same functions after expanding the x and y axes by factors $b = 10$ and $c = 0.1$ respectively (i.e., the y axis is contracted a factor $c^{-1} = 10$). The power law remains identical, in contrast to the other functions. Note that $\alpha = \log c/\log b = -1$.

same factor); we can try to investigate how much we have been reduced by measuring the property $f(x)$ on our (unknown) scale x and compare it with the same property on a close scale x/b, i.e., $f(x/b)$; then, the relative change on the property is given by $f(x/b)/f(x)=1/c$ and is therefore independent on the scale x. We can compare this with the case in which $f(x)$ is an exponential function, where the same procedure leads to a relation between the characteristic scale of the system and the scale of observation x which allows to calculate one of them if the other is known. For this reason, we can use radioactive decay to construct clocks (changing space x by time) but there is no way to define a standard of measurement in a scale-invariant system.

In fractal geometry, a power-law relation indicating self-similarity appears in the form $N(\ell)\propto 1/\ell^{d_f}$, where d_f is the box-counting fractal dimension and $N(\ell)$ is the number of boxes of linear size ℓ necessary to cover the object, which can be interpreted as the number of structures of size ℓ. So, in this case the exponent α is a fractal dimension, although the reciprocal is not true in general, not even in the case in which the function $f(x)$ is a cumulative distribution function.

In this work it will be of special importance the case in which the function that displays scale invariance depends on more than one variable, let us say, $z=f(x,y)$. We will consider a change of scale of the axes of the form

$$\begin{aligned} x &\to x' \equiv ax, \\ y &\to y' \equiv by, \\ z &\to z' \equiv cz, \end{aligned}$$

with scale factors of expansion a, b, and c. The scale transformation $\top$ acts now as

$$f(x,y) \to \top[f(x,y)] = cf(x'/a, y'/b),$$

and the scale-invariant condition is $f(x,y)=cf(x/a,y/b)$, $\forall a$. Although the situation is very similar to the previous case, the solution of the problem is certainly different. It can be shown that all these solutions have to be *scaling laws* [5], i.e.,

$$f(x,y) = x^{\alpha} F(y/x^{\beta}), \tag{2}$$

where F, the *scaling function*, is an arbitrary monovariate function, and the exponents are related to the scale factors by $\alpha \equiv \ln c/\ln a$ and $\beta \equiv \ln c/\ln b$, i.e., only one scale factor is independent. It follows that the condition of scale invariance is fully equivalent to the fact that $f(x,y)$ is a generalized homogeneous function. Note that the solution can be written in different forms, for instance, $f(x,y)=y^{\alpha/\beta}G(y/x^{\beta})$, which is identical to the previous one if $F(\theta)=\theta^{\alpha/\beta}G(\theta)$. The argument of the scaling function can also be replaced by $y^{1/\beta}/x$, for instance.

It is worth to discuss the distinction between power laws, $f(x)=x^{\alpha}$ [Eq. (1)], and scaling laws, $f(x,y)=x^{\alpha}F(y/x^{\beta})$ [Eq. (2)]. We can see that power laws are special cases of scaling laws, where the scaling function is simply a constant and then there is no dependence on the second variable. Although in the literature there is some confusion between power laws and scaling laws, we will keep the distinction clear in this work.

2 CRACKLING NOISE, CRITICALITY, AND SELF-ORGANIZED CRITICALITY

It turns out that many physical phenomena, in particular natural hazards (like earthquakes, forest fires, solar flares, etc. [7, 8]) and also similar phenomena in condensed-matter physics (as the motion of domain walls in soft magnetic materials or the sound emitted during martensitic phase transitions [9]) are constituted by bursty, episodic events whose size spans a broad-range of scales, following a power-law distribution. This behavior has been labeled as *crackling noise* [9], but notice that, in contrast to fractals, in this case it is not clear how the power law can be related to a geometrical structure, i.e., we have scale invariance (and therefore a lack of characteristic scales) but without an interpretation in terms of geometrical self similarity.

The notion of *self-organized criticality* goes one step beyond, as it proposes a mechanism for the emergence of the scale-invariant response of these systems (i.e., crackling noise) as a spontaneous organization of the dynamics of the system towards a very particular state, analogous to the critical point found in equilibrium phase transitions [7]. The idea is explained by means of the sandpile metaphor: if we add grains very slowly over a finite support, a pile starts to grow, as there is a positive balance between the added grains and the grains that fall off the pile (the former win). The growth process continues until the slope reaches a *critical* value, where the addition and the falling of grains are perfectly balanced, on average. On the other hand, if, by means of some artificial mechanism, the slope of the pile is larger than the critical value, then, the dissipation of grains at the borders predominates and the slope of the pile is reduced until it reaches the critical value. At the end, the slope of the pile fluctuates around this value, which means that the critical state can be considered an *attractor* of the dynamics.

Note that we have used the term critical as referring to a separation between two different behaviors; however, this nonequilibrium critical state of the pile should have analogous properties to equilibrium systems at the critical point of a phase transition. Let us consider a magnetic phase transition in an idealized case [5]. We assume that in a magnetic material, even if no magnetic field is applied, there exist a spontaneous (non-zero) magnetization at low temperatures (this constitutes the ferromagnetic phase). However, for high enough temperatures it is well known that there is no neat magnetization (this behavior defines the paramagnetic phase). In fact, there exists a critical value of the temperature for which an abrupt transition takes place between the spontaneous magnetic behavior and the zero-magnetization case (the transition is so sharp that magnetization goes to zero with infinite slope, hence analytic functions are unable to describe the magnetic behavior in this case; this is a key point in critical phenomena). It turns out that above the critical temperature, small clusters of aligned particles with a neat magnetization exist, but their magnetization cancels with each other; in contrast, below the critical temperature there exists an *infinite cluster* of aligned units, providing the neat magnetization of the material. The transition takes place in such a way that, precisely at the critical temperature, there are clusters of all sizes, but without a characteristic scale, and therefore the critical points of equilibrium phase transitions show fractal properties. Because the value of the magnetization allows to distinguish between an ordered phase (the ferromagnetic one) and a disordered one (paramagnetic), m is considered an *order parameter*. This should be what happens as well in the critical state of a sandpile, providing a metaphor for the behavior of many nonequilibrium systems, which has been verified for diverse computer models. Nevertheless, for real sandpiles the picture is not always correct, but this is a different story [10].

Two remarkable properties of equilibrium systems close to their critical point are *scaling* and *universality* [11, 12]. Following with the example of a magnetic material, an equation of state of the form $m = f(T, H)$ relates the magnetization m with the temperature T and the magnetic field H. However, in the vicinity of the critical point, which is given by $T = T_c$, $H = 0$, and $m = 0$, the equation of state fulfills a scaling law, identical to Eq. (2) [5],

$$\frac{m}{h^{1/\delta}} = F\left(\frac{\tau}{h^{1/\beta\delta}}\right),$$

where $\tau \equiv (T - T_c/T_c)$ is the reduced temperature (which measures the deviation of the temperature with respect the critical temperature T_c in units of T_c), h is a dimensionless measure of H, as a balance between the magnetic energy and the thermal energy, and the exponents δ and β are called *critical exponents* (for which the so-called mean-field theory yields $\delta = 3$ and $\beta = 1/2$). This means that the plots of magnetization versus reduced temperature for different applied fields yield a unique curve if m is rescaled by $h^{1/\delta}$ and τ by $h^{1/\beta\delta}$. In other words, the dependence of m on τ is *similar* for different h, except for a change of scale, at least close to the critical point, and we can say that m and τ *scale* with h as $h^{1/\delta}$ and $h^{1/\beta\delta}$, respectively. The unique, monovariate function describing this behavior is the scaling function F, for which in mean-field theory its inverse is $F^{-1}(\sigma) = 1/\sigma - \sigma^2/3$ (assuming that m is dimensionless). The scaling of the equation of state close to the critical point has as a consequence that when it is restricted to particular conditions (isothermal, or zero field), the remaining variables, as well as the response functions deriving from them (the susceptibility for instance) are related by means of simple power laws.

When we compare different magnetic systems it turns out that, although the critical temperatures are different for each one, the critical exponents and the scaling function are the same. This property is called *universality*, and is valid even for different types of phase transition; for instance, magnetic phase transitions and liquid-gas transitions in diverse materials share as well exponents and scaling functions. If we think about it, it is astonishing how universal properties may emerge independently of the variability of microscopic, system-specific details. (In practice, the scaling of m and τ with h involves two extra constants that may change for different materials although the exponents and the scaling function do not change.) Nevertheless, there exist other cases for which the exponents and the functions are different, defining each group with the same properties what is (strangely) labeled as a *universality class*. In any case, microscopic details are not important to distinguish different universality classes, rather, they depend on general properties like the dimensionality of the system (one of the universality classes is the mean-field one, for which the value of two exponents and the scaling function are given above). Still it is an open question up to what point universality and universality classes are a fundamental issue for out-of-equilibrium systems.

3 SCALE INVARIANCE OF EARTHQUAKE SIZES

We have mentioned that earthquakes display crackling noise, and they have been proposed also as a prototypical example of a self-organized critical system. However, it is necessary to clarify that, whereas the property of crackling noise is easy to test from the response of the system (one only needs to perform the statistics of event sizes and check if it yields a power law), for the verification of the existence of self-organized criticality one has to have access to the internal variables of the system in order to measure the fluctuations of these variables around a critical state, introducing perhaps an order parameter [13].

In any case, let us start with the statistics of earthquake sizes. If we consider magnitude as a measure of size (which indeed it is), and count the number of earthquakes in a fixed region and during a sufficiently long period of time, we obtain that for each earthquake of magnitude 6 there are about 10 earthquakes of magnitude 5, 100 earthquakes of magnitude 4, and so on. This is a fundamental property of seismicity, known as the *Gutenberg-Richter law* [4, 14–16] and can be formulated more formally as,

$$n_\Delta(M) \propto 10^{-bM} \propto e^{-b \ln 10\, M},$$

where M is magnitude, $n_\Delta(M)$ is the number of earthquakes between M and $M + \Delta$, the b-value is close to 1 (do not confound it with the scale factor b of the first section), and the symbol $\propto$ stands for proportionality, the coefficient of proportionality would depend on the activity of the region. We can conclude that the Gutenberg-Richter law brings good news, as most of the earthquakes are very small, and only very few of them are large or catastrophic (we will see below that, unfortunately, this interpretation is not appropriate). Figure 2

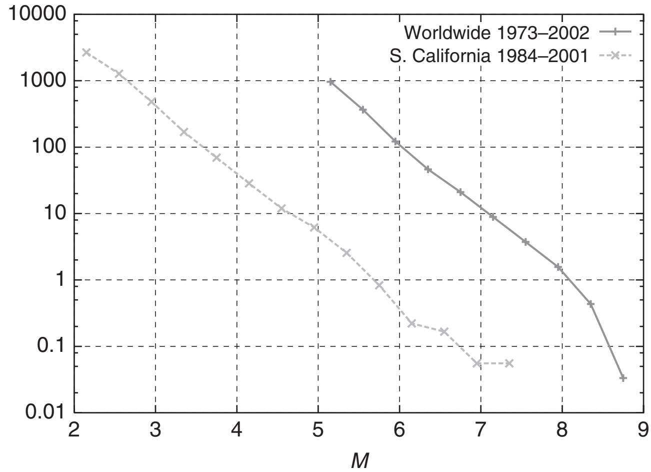

Figure 2: Average number of earthquakes per year in Southern California, for the years 1984–2001, and worldwide, for 1973–2002, for magnitude ranges of extend $\Delta = 0.4$. The straight-line behavior in the logarithmic-linear axes is an indication of exponential behavior, the slope b (when the logarithm is decimal) is very close to 1. The lack of data at the left of each curve is a consequence of the incompleteness of the catalogs for small events; for regions with higher density of instrumentation the law can be shown valid well below magnitude 2. Note that $M \simeq 2/3 \log E/60$, if E is measured in kJ.

illustrates the Gutenberg-Richter law for Southern California and for worldwide seismicity.

We find it more practical to describe the distribution of the size of earthquakes by means of the *probability density*, $D(M)$, defined as

$$D(M) \equiv \frac{\text{Prob}\,[M \leq \text{magnitude} < M + \Delta]}{\Delta},$$

where Prob denotes probability, and Δ should tend to zero, in the ideal case. It is straightforward to estimate $D(M)$ from $n_\Delta(M)$ as $D(M) = \Delta^{-1}N^{-1}n_\Delta(M)$, where N is the total number of earthquakes (of any size); in practice, Δ should be small, in order not to modify the continuous nature of $D(M)$, but also large enough to provide statistical significance in any interval it defines. In any case, the probability density removes the dependence of the size distribution on Δ, as well as on the seismic activity and extension of the spatial region under analysis, allowing the study of possible universal properties (properties that would be the same for very different systems or tectonic environment, see previous section). In this way, in terms of the probability density, the Gutenberg-Richter law still gives an exponential behavior. Even more, due to the special property of exponential functions, the cumulative distribution of sizes, defined by $S(M) \equiv \text{Prob}\,[\text{magnitude} > M] = \int_M^\infty D(M)\,dM$ is also an exponential function, with the same characteristic decay (given by $b \ln 10$). For this reason, many times it is not clearly stated in the literature if cumulative or non-cumulative distributions are used, but this ambiguity is only possible (and tolerable) for exponential distributions.

At the beginning of this text we have argued with great excitation about the special properties of power laws, but the distribution of earthquake magnitudes is not power law but exponential. Then, how can earthquakes be an example of scale invariance, crackling noise, and self-organized criticality? The reason is because earthquake magnitude is not a good representation of earthquake size. After all, magnitude has no dimensions, i.e., physical units. A much more natural measure of earthquake size would be the energy it releases; however, this is a quantity very difficult to obtain in practice. Nowadays, the most reliable measurement of earthquake size is *seismic moment* [16], represented by $\mathscr{M}$ here, which turns out to be equal to the product of the rigidity μ, the spatially averaged slip d, and the fault area A ruptured by the earthquake, i.e., $\mathscr{M} = \mu A d$.

The relation between seismic moment and magnitude is given, when the formed is measured in Newtons $\times$ meters by $\mathscr{M} = 10^{1.5(M+6.07)}$, which is in fact a definition of the so called moment magnitude from the seismic moment; other magnitudes, defined in many different ways, are close to the moment magnitude, except for saturation effects. Notice that $\mathscr{M}$ has units of energy ($1\ \text{J} = 1\,\text{N}\,\text{m}$), but it is not an energy. The energy E radiated by seismic waves is believed by some authors to be roughly proportional to the seismic moment, i.e., $E \simeq 5 \cdot 10^{-5}\, \mathscr{M}$, although this question is not solved at present [16]. If we accept this proportionality as a first approximation, we obtain that the energy (in kilo-Joules) relates with the (moment) magnitude as $E \simeq 60 \cdot 10^{1.5M}$, i.e., the energy is an exponential function of the magnitude. (For each increase in one unit of the magnitude, the energy is multiplied by a factor $10^{1.5} \simeq 32$.) As the magnitude is exponentially distributed, it is straightforward to show that E must be power-law distributed, with an exponent of the probability density equal to $1 + 2b/3$. Indeed,

$$D(M)dM = D(E)dE \quad \Rightarrow \quad D(E) = D(M)\frac{dM}{dE} \propto 10^{-bM}\frac{1}{E}$$

and then

$$D(E) \propto \frac{1}{E^{1+2b/3}},$$

where we have used the same symbol D for the densities of magnitudes and energies, although of course they are different functions (exponential and power law, respectively). For the cumulative energy distribution the exponent is reduced in one unit, i.e., $S(E) = \text{Prob [energy} > E] \propto 1/E^{2b/3}$. If the proportionality between seismic moment and radiated energy were not true, the power-law behavior, with the same exponent, would at least be valid for the seismic moment. It is important to realize that there is a fundamental conflict between scale invariance and normalization of probability, and so the power law cannot hold for very small values of the variable (E) if the exponent of the density is larger than 1, and then a minimum cutoff is necessary in the distribution.

Other possible physical measurements of size, in addition to seismic moment and radiated energy are the rupture duration T (size in time), the rupture length ℓ, the rupture area A (mentioned above), and the slip d (also mentioned above). Let us assume, as a working hypothesis, that earthquakes are dynamically similar, i.e., from the evolution of the rupture it is not possible to distinguish between small and large earthquakes; this means that in particular the rupture velocity $v = \ell/T$ and the power radiated per unit of fault area $E/(TA)$ are independent of size, which implies that ℓ and T are proportional to each other; supposing the area A goes as ℓ^2 (i.e., the earthquake propagates in the same way in all directions in the fault), this leads to the fact that the energy fulfills $E \propto T^3$ and $E \propto \ell^3$. This is in agreement with observations, at least replacing E by $\mathscr{M}$ [16]. From these relations one can obtain the distributions of T, ℓ, and A, and also that of d using that of the seismic moment. It turns out that all of them are power-law distributed,

$$\begin{aligned} D(T) &= D(\mathscr{M})d\mathscr{M}/dT \propto 1/T^{1+2b}, \\ D(\ell) &= D(\mathscr{M})d\mathscr{M}/d\ell \propto 1/\ell^{1+2b}, \\ D(A) &= D(\mathscr{M})d\mathscr{M}/dA \propto 1/A^{1+b}, \\ D(d) &= D(\mathscr{M})d\mathscr{M}/dd \propto 1/d^{1+2b}, \end{aligned}$$

as power-law changes of variables preserve power-law distributions, but changing the exponent. From these power-law distributions of sizes the scale-invariance and the crackling-noise behavior of earthquakes follow directly. We need to mention that the dynamical similarity of earthquakes is however still a debatable topic.

In any case, the power-law distributions indicate that all these measures of earthquake size (radiated energy, seismic moment, duration, rupture length, etc.) do not have a characteristic scale, which has to be considered right up to the scale imposed by the size of the system, in this case the Earth crust. On the other hand, we can evaluate from the distributions the mean earthquake size, however, there is a problem for the energy and the seismic moment (and for the area if $b \le 1$), which is that their mean values become infinite. So, despite the fact that there are no characteristic scales, mean values for durations and lengths exist, but mean values for the other variables do not. Taking the energy as an example, this means that the dissipation of energy is leaded by large earthquakes; if we try to calculate its mean value from the catalogs it turns out that this quantity does not converge: when a large earthquake takes place the energy it radiates is much larger than that of the rest of smaller earthquakes. In this way, the Gutenberg-Richter law expressed in terms of the energy brings very bad news: although large earthquakes are a tiny fraction of all earthquakes, they dissipate most of the energy, providing the tragic reputation of the phenomenon.

We have mentioned that a power-law distribution does not necessarily imply the existence of a spatial fractal structure. Nevertheless, in the case of earthquakes there are clear evidences for the existence of such structures. First, the distribution of earthquake hypocenters shows fractal (and even multifractal) properties [14]; and second, faults, which are the structures where earthquakes take place, form complex fractal networks [4, 17, 18]; in addition, their lengths are power-law distributed [4, 19], the same as the extension of tectonic plates (which are both responsible for and modeled by earthquakes) [20]. In conclusion, earthquakes provide a fantastic natural laboratory for the study of fractal properties.

4 SCALE INVARIANCE IN THE TIMING OF EARTHQUAKES

Neither in the research on earthquakes nor in the studies of crackling noise and self-organized criticality there has been a clear picture for the temporal properties of these systems. Notice for instance that crackling noise is defined only by means of the size of the response of the system, without any reference to its time evolution. In the case of self-organized critical systems it was believed that the dynamics was trivial, with the response of the system (the avalanches) following a Poisson process (i.e., at each time step a dice would decide if there is an avalanche or not). In fact, this seems to be just a pathological characteristic of very small avalanches in some popular sandpile models; larger events do not follow this trend [21].

The time properties of earthquakes are more puzzling. It is true that the Omori law, which states that the seismic rate decays as a power law after a big earthquake in a certain area around it, is well established, but when we go deeper than the measurement of rates and care about the timing of individual earthquake occurrence the issue becomes fuzzy, as there is a clear lack of a coherent phenomenology [22]. It is widely believed that fault segments produce *characteristic* earthquakes; these are events of not only the same size but also similar seismographic fingerprints that take place at more or less regular intervals, in which it is known as the *seismic cycle* [23, 24]. However, the characteristic-earthquake view has been challenged recently, mainly with the failure of the Parkfield prediction experiment [25–27], but also because paleoseismological studies do not allow to distinguish regular occurrence from a Poisson process [28]. In any case, this perspective is rooted on the assumption that faults do not interact with each other, or at least in the fact that the interaction is weak enough in order that the seismic cycle is not significantly altered.

Recently, an alternative, non-reductionistic approach inspired by the physics of complex systems has emerged for the study of earthquakes. Instead on concentrating on single fault segments, a more global perspective, considering large extended regions as well defined systems is necessary. It was believed that for such large regions the occurrence of mainshocks was totally random, described by a Poisson process [29]. Although other approaches have been later introduced for the statistics of mainshocks, again the complex-system perspective questions the separation of earthquakes into different processes (mainshocks and foreshocks): all earthquakes are generated by the same process, and even those events considered as aftershocks can trigger successive events and become mainshocks [30].

A major conceptual advance from this point of view has been the recent work of Bak *et al.* [31], which focused on common universal properties rather than on different peculiarities associated to particular tectonic settings, and combined the size distribution of earthquakes, the fractal structure of its spatial occurrence and its temporal occurrence into a unique unified scaling law, similar in spirit to those used to describe critical systems in statistical physics.

In the same line, we have developed a simpler approach to analyze the complexity of seismicity in time, space, and size. A fundamental advance introduced by Bak *et al.* that we take advantage of it was to consider, for a given spatial region, only events above a minimum magnitude M_c, and study how the dynamics of occurrence changes as a function of M_c. For each M_c-value, those events with $M \ge M_c$ define a point process, $t_0, t_1, t_2 \ldots$ where t_i is the time of occurrence of the i-th event above the minimum magnitude in the region. Then we compute the *recurrence time* τ (also called interevent time, interoccurrence time, waiting time, etc.), which is nothing else than the time between consecutive events, i.e., $\tau_i \equiv t_i - t_{i-1}$; notice that both occurrence and recurrence times depend on M_c, although, for simplicity, this dependence is not reflected in the notation. In the same way, the probability density of the recurrence time will depend on M_c, or on the minimum radiated energy, $E_c \propto 10^{1.5M_c}$, as well as on the spatial region $\mathscr{R}$ under study; we will denote this density as $D(\tau; M_c, \mathscr{R})$, as a function of the minimum magnitude or $D(\tau; E_c, \mathscr{R})$, in terms of the minimum energy.

As we are interested in a global perspective, let us start at the largest possible scale, which is that of worldwide earthquake occurrence. The recurrence-time distributions for M_c ranging from 5 to 6.5 are shown in Fig. 3(a). Obviously, the larger M_c, the less earthquakes there will be and the longest the recurrence times between them. However, for each M_c, we can measure the time in units of the mean recurrence time, this also changes the units of the density; in other words, we perform the transformation $\tau \to R(M_c, \mathscr{R})\tau$ and $D \to D/R(M_c, \mathscr{R})$, where $R(M_c, \mathscr{R})$ is the mean seismic rate, defined as the total number of earthquakes with $M \ge M_c$ in the region $\mathscr{R}$ divided by the time interval under study;

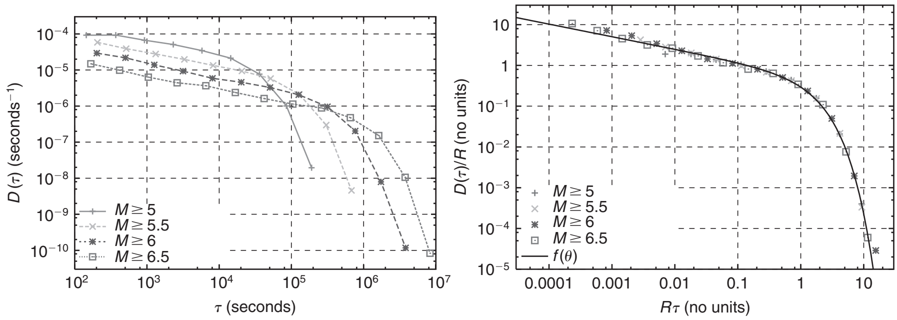

Figure 3: Left: Probability densities of recurrence times of earthquakes worldwide with $M \geq M_c$ during the period 1973–2002, for M_c ranging from 5 to 6.5. Right: The same distributions rescaled by its rate. The data collapse is the signature of the scaling law. The continuous function is the gamma distribution explained in the text. $D(\tau)$ is $D(\tau; M_c, \mathscr{R})$ and R is $R(M_c, \mathscr{R})$. Taken from Ref. [34].

in other words, R is the inverse of the mean recurrence time. The collapse of the distributions into a unique curve shown in Fig. 3(b) is an indication of the fulfillment of a scaling law [32],

$$D(\tau; M_c, \mathscr{R}) = R(M_c, \mathscr{R}) F_{\mathscr{R}}(R(M_c, \mathscr{R})\tau),$$

where the scaling function in region $\mathscr{R}$, $F_{\mathscr{R}}$, is independent on M_c. Using the Gutenberg-Richter law, it turns out that $R(M_c, \mathscr{R}) \propto 10^{-bM_c}$ or $R(E_c, \mathscr{R}) \propto 1/E_c^{2b/3}$, to be concrete, $R(E_c, \mathscr{R}) = R_0(\mathscr{R})/E_c^{2b/3}$, where $R_0(\mathscr{R})$ is the rate of occurrence of events with energy greater than $E_c = 1$ in the region (in whatever units we chose for the energy, in the case in which the unit is 60 kJ this corresponds to $M_c = 0$); then the scaling law can be written as

$$D(\tau; E_c, \mathscr{R}) = \frac{R_0(\mathscr{R})}{E_c^{2b/3}} F_{\mathscr{R}} \left(\frac{R_0(\mathscr{R})\tau}{E_c^{2b/3}} \right).$$

Although it is not obvious at a first sight, this scaling law is a remarkable result. In general, when events are removed from a point process (as we do raising M_c), the properties of the process change; in other words, the process will *renormalize* to a new process. In many cases this renormalization leads to a Poisson process: if there are short range correlations, or if the (nonstationary) rate of the process shows a characteristic time scale, when a sufficiently large number of events have been eliminated the characteristic scales are lost and the process becomes structureless and therefore Poisson. The existence of the scaling law means that the process is invariant, or a fixed point of a renormalization-group transformation, which constitutes a very particular solution in the space of all possible processes [33]. The same happens at the critical points of equilibrium phase transitions [5].

As the plot indicates, the scaling function turns out to be a decreasing power law up to the largest times, where the behavior changes to an exponential decay. A gamma function can account for this behavior,

$$F_{\mathscr{R}}(\theta) = \frac{1}{\Gamma(\gamma)a}\left(\frac{a}{\theta}\right)^{1-\gamma} e^{-\theta/a},$$

where θ is the recurrence time in units of its mean, i.e., $\theta \equiv R(M_c, \mathscr{R})\tau$, $\Gamma()$ is the gamma function, a is a scale parameter, and γ is the shape parameter of the distribution. In fact, there is only an independent parameter, as the mean of θ is reinforced by the rescaling of the axes to be one, and then $\overline{\theta} = \gamma a = 1$. A fit yields $\gamma \simeq 0.7$, which implies that the exponent describing the power-law decay is about 0.3. This behavior is an indication of *clustering*, and means that there are more earthquakes separated by short recurrence intervals than what corresponds to a random, memoryless occurrence (i.e., a Poisson process with the same rate); in other words, the risk of occurrence is higher right after an earthquake, and then decreases for long times, which leads to the formation of clusters of events. This behavior was previously found by Kagan in a different approach, and is the opposite of the regularity or quasi-periodicity derived from the idea of the seismic cycle [14, 35]. One surprising consequence of clustering is the paradoxical result that *the longer it has been since the last earthquake, the longer the expected time till the next* [36].

We can explore what happens beyond worldwide seismicity, at the regional and local levels. In doing this one needs to have in mind that at the world scale seismicity is a stationary process (at least from 1973 to 2002), in the sense that the rate of occurrence is independent of the time window selected. In contrast, when we go to smaller scales, local inhomogeneities appear, due to the presence of aftershock sequences that at the global level are mixed between them and not apparent. So, at the regional and local scales seismicity is not stationary in general, and periods of variable rate exist, but followed sometimes by stationary periods. We will concentrate only on those stationary periods (as for instance 1988–1991 for Southern California). Figure 4 shows rescaled recurrence time distributions for many regions with stationary seismicity at different scales, from worldwide to small areas in Southern California and Japan, mainly. All cases in the figure, and others not shown [36], seem to be in agreement with the scaling law presented above for the worldwide case, covering a range of minimum magnitudes from 1.5 to 7.5 and from worldwide scales up to about 20 km. Therefore the scaling function can be considered the same for different regions, which means that it can be written as

$$D(\tau; M_c, \mathscr{R}) = R(M_c, \mathscr{R}) F(R(M_c, \mathscr{R})\tau),$$

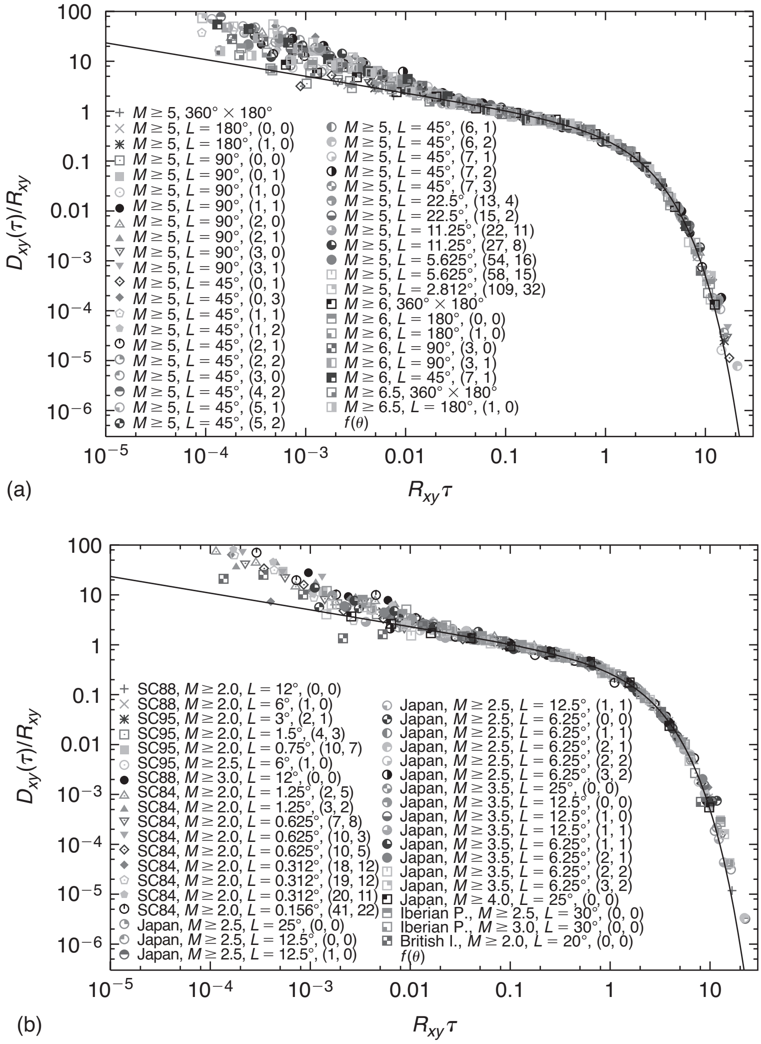

Figure 4: Rescaled probability densities of recurrence times in large regions worldwide (top) and smaller areas in Southern California (SC), Japan, the Iberian Peninsula, and the British Islands (bottom), using different minimum magnitudes and selecting time periods for which seismicity is stationary. The linear size of the region is given by L, R_{xy} refers to $R(M_c, \mathscr{R})$ and $D_{xy}(\tau)$ to $D(\tau; M_c, \mathscr{R})$. Taken from Ref. [32].

where F is independent on $\mathscr{R}$, or, simplifying the notation, $D_w(\tau) = R_w F(R_w \tau)$ with $w \equiv \{M \geq M_c, \mathscr{R}\}$. As the regions analyzed present very different tectonic properties, we may consider the results as universal, in the same sense as in equilibrium critical phenomena.

Several authors have argued that the only possible form for the scaling function is provided by the trivial case of a Poisson process if the scaling law is exact [37], although an approximate scaling may be valid, weakly dependent on the rate [38]. These results are based on particular simple models of seismicity or on the hypothesis that contiguous regions may be considered as independent between them. In any case, this is an interesting open problem.

5 RELATION WITH FRACTURES: UNIVERSALITY

It is a remarkable fact that the results exposed here for earthquakes are representative of somehow related processes at a much smaller scale. Davidsen *et al.* [39] have measured the acoustic emission from laboratory rock fractures, using different materials and diverse experimental procedures. Their findings are displayed in Fig. 5 (left), showing how a scaling law for the distributions of time intervals between detected emissions holds, with a scaling function surprisingly compatible with that of earthquakes. The materials considered in the plot are a couple of sandstones (at wet conditions), three samples of granite, and Etna basalt (at dry conditions). The confined pressures were in the range from 5 to 100 MPa and the loading conditions were constant displacement rate, acoustic emission activity feedback control of loading, and punch-through loading. The authors concentrated only in the periods of stationary activity.

Another important result was found by chance during preliminary runs of the CRESST project for dark-matter search at the Gran Sasso Laboratory [40, 41]. A cryogenic detector, made by a single crystal of sapphire, recorded an unexpected series of pulses of a high rate. Initially, the researches believed that the signal was due to an unknown radioactive contamination, but the time distribution of the pulses was not Poisson but bursty, which clearly discarded radioactivity. Finally, the origin of the signal was traced to nanofractures in the crystal due to the tight clamping of the detector. The size of the fractures was rather small, involving in some cases the breaking of only several hundreds of covalent bonds. Figure 5 (right) shows the astonishing similarity between the recurrence times in nanofractures and earthquakes, where the same scaling function can describe both phenomena. Note therefore the enormous range of values for which the same law is valid, from earthquakes with estimated radiated energy larger than $2 \cdot 10^{15}$ J (magnitude larger than 7) to nanofractures of minimum absolute dissipated energy of about 5 keV, or $8 \cdot 10^{-16}$ J, which yields a range of more than 30 orders of magnitude of validity. It is noteworthy as well the profund differences between the homogeneity and regularity of a monocrystal at milli-Kelvin temperatures and the heterogeneity of fault gouge producing (and produced by) earthquakes.

Other systems outside this field verify this complex dynamics, in particular extreme climatic records [42], with a scaling function similar to the one of

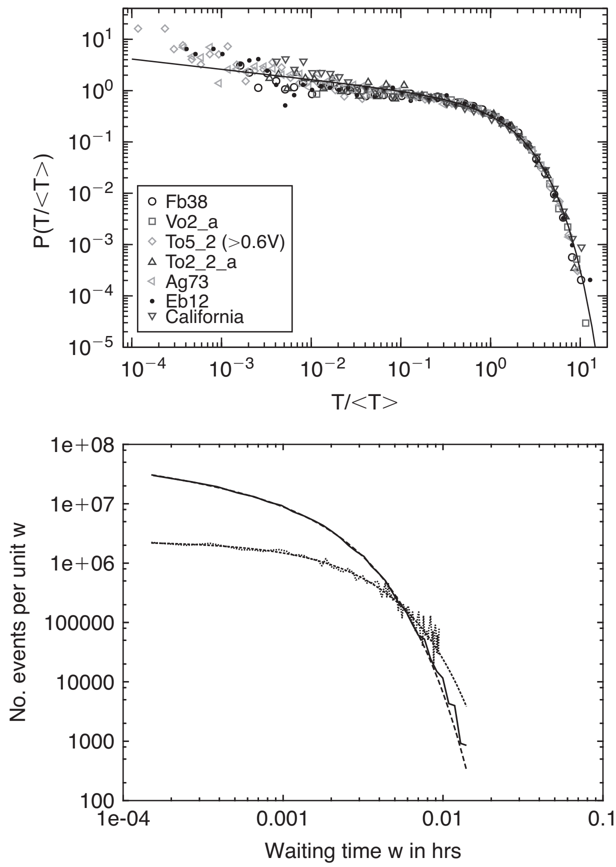

Figure 5: Top: rescaled probability densities of recurrence times of acoustic emission signals measured in laboratory experiments of rock fractures, using different samples. The continuous line is a gamma function with parameter $\gamma = 0.8$, which is compatible with the value obtained for earthquakes. Taken from Ref. [39]. Bottom, continuous curve: probability density (not normalized) of recurrence times of nanofractures in a monocrystal of sapphire at a temperature of about 10 milli-Kelvin. The fit is a gamma distribution with parameter $\gamma = 0.67$, practically the same as the value for earthquakes. Right, dashed curve: The same for radioactive induced events. The fit is an exponential (Poisson) function. From Ref. [40].

earthquakes, as well as solar flares [43] and financial time series [44], where a scaling law for recurrence times is valid but the scaling functions are different. It would be interesting to explore if they correspond to different universality classes.

In conclusion, the dynamics of point-like process representing the sudden appearance of several natural disasters and the explosive behavior of many nonequilibrium solid-state responses is an open field of research, that will allow a rich interchange of ideas and will certainly grow in the next years, extending perhaps to biological systems and even human behavior [45].

I am grateful to J. Davidsen and L. Stodolsky for their comments and the permission to use their figures.

APPENDIX

We are going to demonstrate the fact that scale-invariant systems are characterized by power laws and vice versa. Let us consider a one-dimensional function $f(x)$. As we have explained in the main text, a *scale transformation* is given by the application $\top$:

$$f(x) \longrightarrow \top[f(x)] = cf(x'/b),$$

where the space expands a factor b in the x axis and a factor c in the y axis. The condition of *scale invariance* is

$$f(x) = cf(x/b) \quad \forall b, \tag{3}$$

which turns out to be a functional equation for $f(x)$ whose solutions will give the analytical form of a scale-invariant function. It means that graphs of $f(x)$ "centered" at different scales can be superimposed by a simple change of scale. To find its solutions, let us derive both sides respect x and isolate c to obtain [3]

$$c = b\frac{f'(x)}{f'(x/b)} \quad \forall b,$$

where the prime denotes the derivative. Substituting c in the original equation (3) and multiplying both sides by x one finds

$$\frac{x}{b}\frac{f'(x/b)}{f(x/b)} = x\frac{f'(x)}{f(x)} \quad \forall b.$$

As the equation holds for all b, it is clear that $xf'(x)/f(x)$ must be a constant, i.e.

$$x\frac{f'(x)}{f(x)} = \alpha.$$

In this way, the functional equation has been transformed into a differential one of straightforward solution, which is,

$$f(x) = kx^{\alpha},$$

with k an arbitrary constant. Thus, the only scale-invariant function in one dimension is the *power law*. Substituting into the original equation (3), we find that it is required that

$$c(b) = b^{\alpha},$$

and hence the scale-invariant condition (3) can be rewritten as

$$f(x) = b^{\alpha} f(x/b) \quad \forall b, \tag{4}$$

which means that $f(x)$ is an *homogeneous* function. Note that only when $f(x)$ is linear, i.e. in the case $\alpha = 1$, both axis are transformed by the same factor.

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