**ECE 5397/6397: Introduction to Robotics, Spring 2016**

**HW #1 Solution**

**Discrete Grading Policy**. 5 points for each: 2 points for trying, 3 points if partial answer, 5 point if correct.

1. Consider the following sequence of rotations:
   1) Rotate by $\alpha_1$ about the current $x$-axis.
   2) Rotate by $\alpha_2$ about the world $y$-axis.
   3) Rotate by $\alpha_3$ about the current $z$-axis.
   4) Rotate by $\alpha_4$ about the world $x$-axis.

   Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication). Use the composition rule to derive each step.

   ***Solution:***

$$\begin{aligned}
R &= Rot_{x,\alpha_1}\left(R_1^0\right)^{-1} Rot_{y,\alpha_2} R_1^0 \left(R_2^0\right)^{-1} Rot_{z,\alpha_3} R_2^0 \\
&= \left[Rot_{x,\alpha_1}\left(Rot_{x,\alpha_1}\right)^{-1} Rot_{y,\alpha_2}\left(Rot_{x,\alpha_1}\right)\right]\left(R_2^0\right)^{-1} Rot_{z,\alpha_3} R_2^0 \\
&= \left(Rot_{y,\alpha_2} Rot_{x,\alpha_1}\right)\left(R_2^0\right)^{-1} Rot_{z,\alpha_3} R_2^0 \\
&= \left(Rot_{y,\alpha_2} Rot_{x,\alpha_1}\right)\left(Rot_{y,\alpha_2} Rot_{x,\alpha_1}\right)^{-1} Rot_{z,\alpha_3}\left(Rot_{y,\alpha_2} Rot_{x,\alpha_1}\right) \\
&= Rot_{y,\alpha_2} Rot_{x,\alpha_1}\left(Rot_{x,\alpha_1}\right)^{-1}\left(Rot_{y,\alpha_2}\right)^{-1} Rot_{z,\alpha_3} Rot_{y,\alpha_2} Rot_{x,\alpha_1} \\
&= Rot_{z,\alpha_3} Rot_{y,\alpha_2} Rot_{x,\alpha_1}
\end{aligned}$$

   Where

$$R_1^0 = Rot_{x,\alpha_1}, \quad R_2^0 = Rot_{y,\alpha_2} Rot_{x,\alpha_1}$$

2. Verify Equation (2-11) for similarity transformations, namely

$$B = \left(R_1^0\right)^{-1} A R_1^0 .$$

   (If $A$ is the matrix representation of a given linear transformation in $o_0x_0y_0z_0$ and $B$ is the representation of the same linear transformation in $o_1x_1y_1z_1$ then $A$ and $B$ are related as (2-11))

   ***Solution:***

   Given a point $p$, with $p^0$ and $p^1$ denoting its position in $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$, respectively. The frame $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ can be related by rotation matrix $R_1^0$. If $A$ is the matrix representation of a linear transformation for $p$ in $o_0x_0y_0z_0$, and $B$ is the matrix representation of the same linear transformation for $p$ in $o_1x_1y_1z_1$, we can represent the point after transformation as:

in $o_0x_0y_0z_0 : Ap^0$,

and in $o_1x_1y_1z_1$: $Bp^1$.

Now we represent the point after transformation in $o_0x_0y_0z_0$:

$$Ap^0 = R_1^0 Bp^1 = R_1^0 B\left(R_0^1 p^0\right)$$

We see that

$$A = R_1^0 B R_0^1 \qquad \Rightarrow \qquad B = \left(R_1^0\right)^{-1} A R_1^0$$

3. Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_3^2 = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix},\ R_3^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

Find the matrix $R_2^1$.

***Solution:***

$$R_2^1 = R_3^1 R_2^3 = R_3^1\left(R_3^2\right)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}\begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}\begin{bmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{bmatrix} = \begin{bmatrix} c_\theta & 0 & -s_\theta \\ -s_\phi s_\theta & c_\phi & -s_\phi c_\theta \\ c_\phi s_\theta & s_\phi & c_\phi c_\theta \end{bmatrix}$$

4. Given a rotation matrix $R \in SO(3)$

$$R = \begin{bmatrix} 0.9220 & -0.1546 & 0.3551 \\ 0.3807 & 0.5303 & -0.7575 \\ -0.0712 & 0.8336 & 0.5477 \end{bmatrix}$$

Let $V$ be a set of vectors, and $V \subset \mathbb{R}^3$, $V = \left\{ v \in V \mid \frac{Rv \cdot v}{\|v\|^2} = \frac{1}{2} \right\}$. You may use MATLAB, etc.

1) Use two-argument arctangent function to specify the Euler angles $(\phi, \theta, \varphi)$ of the rotation transformation $R$
2) Give a physical interpretation of the set $V$.

***Solution:***

1) In terms of *ZYZ* Euler angles,

$$\theta = \mathrm{Atan2}\left(r_{33}, \pm\sqrt{1 - r_{33}^2}\right) = \mathrm{Atan2}\left(0.5477, \pm\sqrt{1 - 0.5477^2}\right)$$

$$= \pm 0.9912 = \pm 56.7906^\circ$$

If we choose $s_\theta > 0$,

$\phi = \text{Atan2}(r_{13}, r_{23}) = \text{Atan2}(0.3551, -0.7575)$

$= -1.1324 = -64.8838^\circ$

$\varphi = \text{Atan2}(-r_{31}, r_{32}) = \text{Atan2}(0.0712, 0.8336)$

$= 1.4856 = 85.1181^\circ$

If we choose $s_\theta < 0$ ,

$\phi = \text{Atan2}(-r_{13}, -r_{23}) = \text{Atan2}(-0.3551, 0.7575)$

$= 2.0092 = 115.1162^\circ$

$\varphi = \text{Atan2}(r_{31}, -r_{32}) = \text{Atan2}(-0.0712, -0.8336)$

$= -1.6560 = -94.8819^\circ$

2) Axis/angle representation of R

$$R_{k,\theta} = R = \begin{bmatrix} 0.9220 & -0.1546 & 0.3551 \\ 0.3807 & 0.5303 & -0.7575 \\ -0.0712 & 0.8336 & 0.5477 \end{bmatrix},$$

$$\theta = \cos^{-1}\frac{trace(R_{k,\theta}) - 1}{2} = \cos^{-1}\frac{0.9220 + 0.5303 + 0.5477 - 1}{2} = \frac{\pi}{3} = 60^\circ$$

which indicates that $R_{k,\theta}$ is an operator rotating a given vector about the axis of rotation *k* for $60^\circ$. Any vector $v$ in the set $V = \left\{ v \in V \mid \frac{Rv \cdot v}{\|v\|^2} = \frac{1}{2} \right\}$ shows that after rotation ( $Rv$ ), the vector $v$ changes its direction by $\cos^{-1}\frac{Rv \cdot v}{\|v\|^2} = \cos^{-1}\frac{1}{2} = 60^\circ$, coinciding with the rotation angle $\theta = 60^\circ$. Therefore, the union of these vectors forms planes perpendicular to the rotation axis *k.*

5. Let $k = [0.5378,\ 0.2534,\ 0.8041]^T$ , $\theta = 30^\circ$. Find $R_{k,\theta}$.

$$R_{k,\theta} = \begin{bmatrix} 0.9048 & -0.3838 & 0.1846 \\ 0.4203 & 0.8746 & -0.2416 \\ -0.0688 & 0.2962 & 0.9527 \end{bmatrix}$$

6. Consider the diagram of a rotary inverted pendulum in Figure 1. The center of a rotary wheel is set up at the origin of a fixed frame $oxyz$. A frame $o_1x_1y_1z_1$ is fixed to the edge of the first arm, and another frame $o_2x_2y_2z_2$ is fixed to the edge of the second arm. Lengths and angles are specified in Fig.1. Find the homogeneous transformations relating each of these frames the world frame $oxyz$. Find the homogeneous transformations relating $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$.

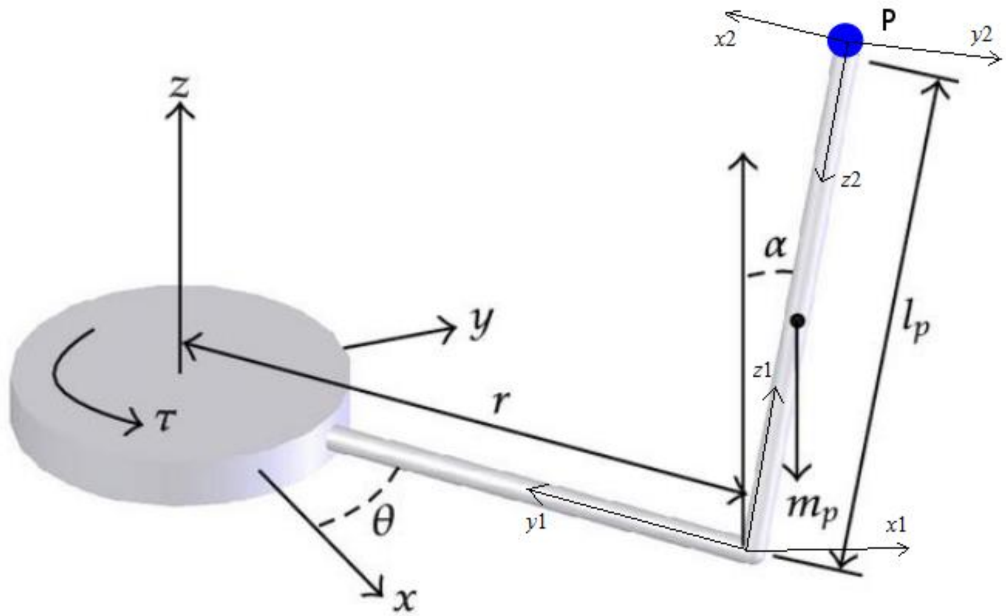


**Figure 1**. *A rotary inverted pendulum*

***Solution:***

$$H_1^0 = Rot_{z,\theta+\pi} Trans_{y,-r} Rot_{y,\alpha}$$

$$H_2^1 = Trans_{z,l_p} Rot_{z,\frac{\pi}{2}} Rot_{x,\pi}$$

$$H_2^0 = H_1^0 H_2^1 = Rot_{z,\theta+\pi} Trans_{y,-r} Rot_{y,\alpha} Trans_{z,l_p} Rot_{z,\frac{\pi}{2}} Rot_{x,\pi}$$