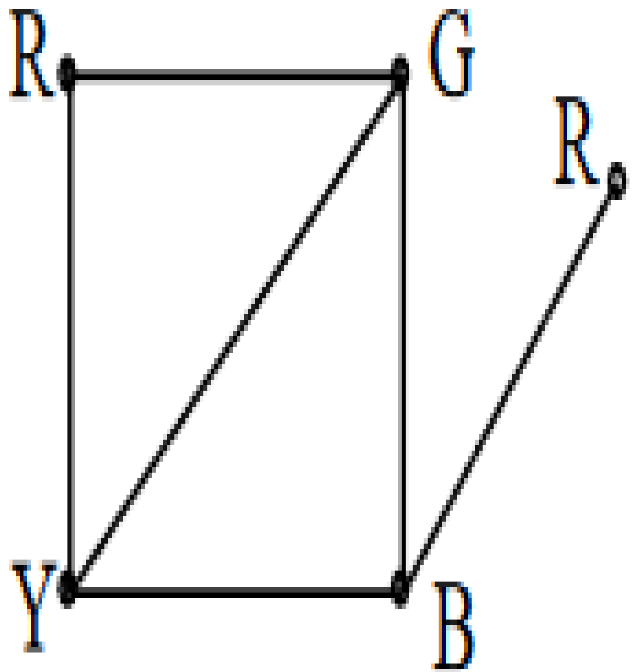
Graph coloring

Graph coloring

Graph coloring or more specifically vertex coloring means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

- A graph can be coloured by assigning a different colour to each of its vertices. However, for most graphs, a colouring can be found that uses fewer colours than the number of vertices in the graph. A colour with colours such that no two adjacent vertices have the same colour is called properly coloured graph.


R
G
R
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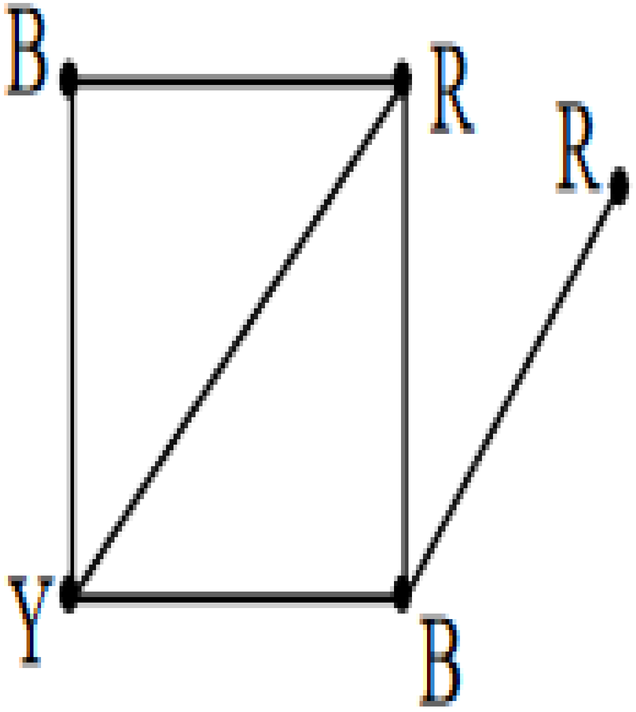
B
R
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Y
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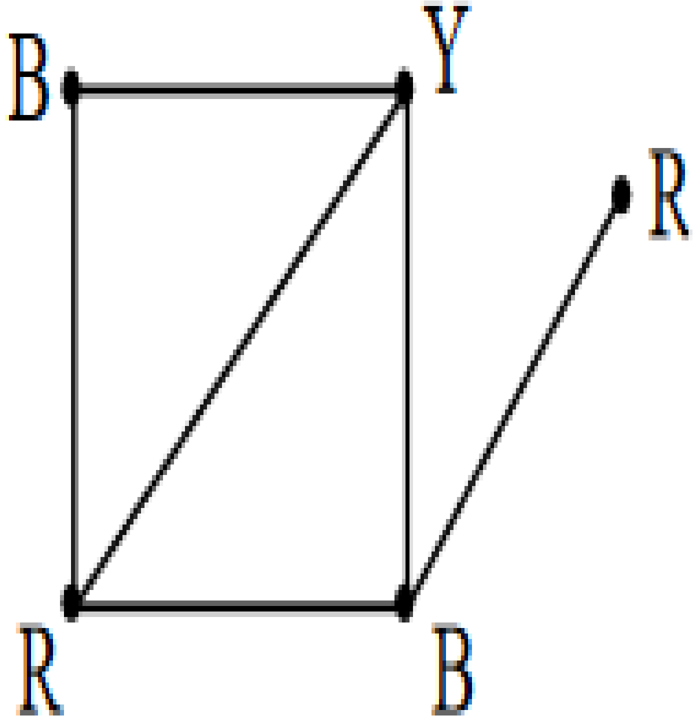

B
Y
R
R
B

- A graph G is said to be **k-colourable** if each vertex can be assigned one of k colours so that adjacent vertices get different colours.
- **Chromatic number** – least k for which G is k-colorable - $\chi(G)$

It is the minimum number of colors required for proper coloring of Graph.

A Graph is k-chromatic if $\chi(G) = k$

Applications of Graph Coloring

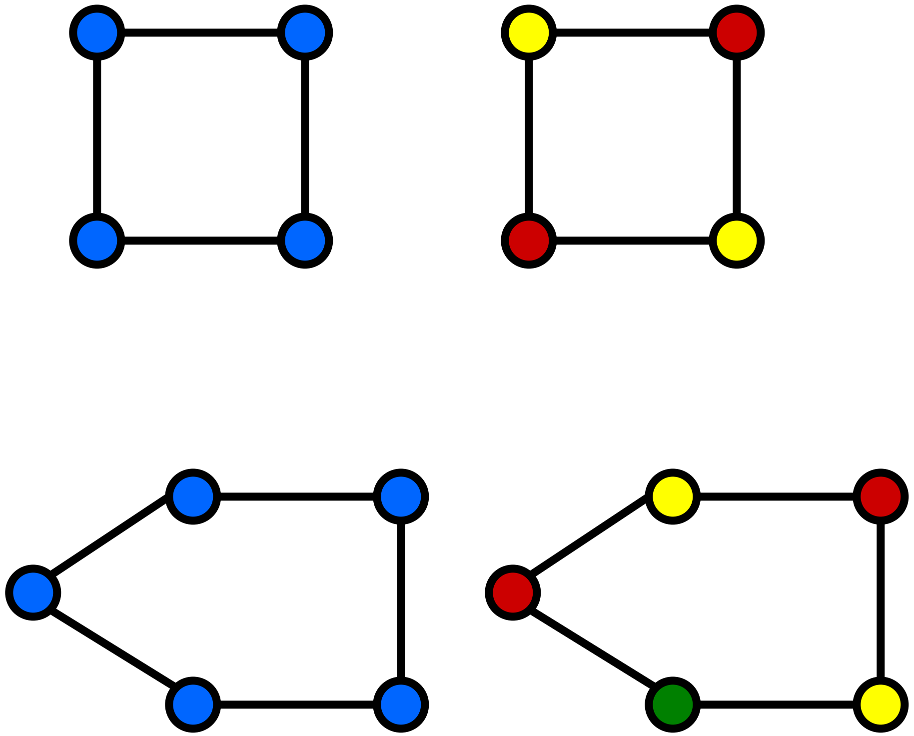
Graph coloring is one of the most important concepts in graph theory. It is used in many real-time applications of computer science such as –

- Clustering
- Data mining
- Image capturing
- Image segmentation
- Networking
- Resource allocation
- Processes scheduling

Chromatic Number

- Chromatic Number for a Null graph is 1.

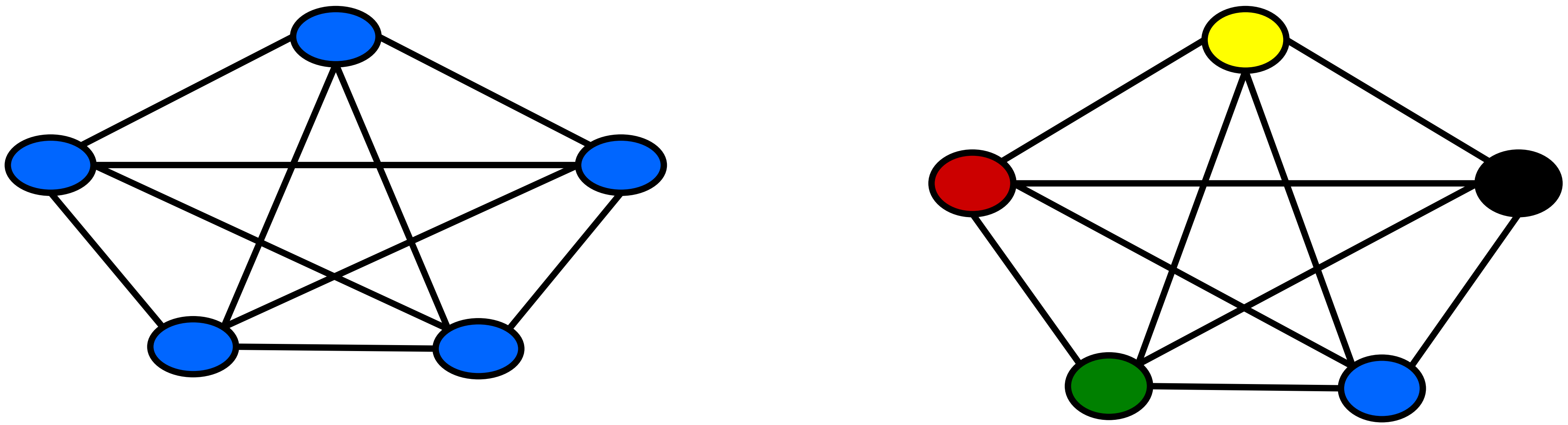
Chromatic number of cycles



$$\chi(C_{even}) = 2$$

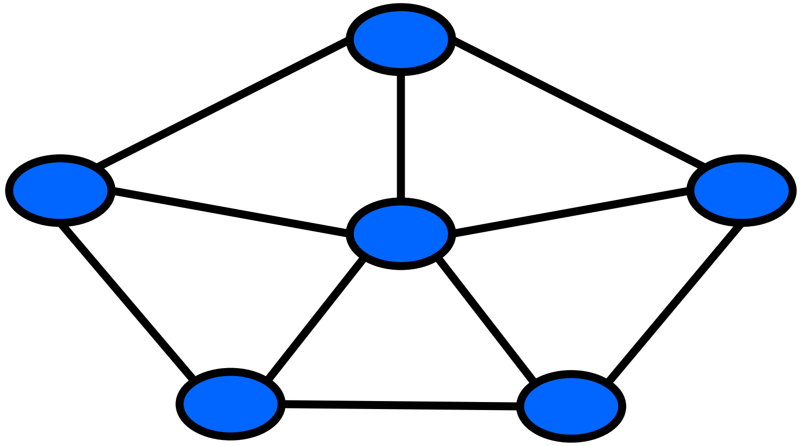
$$\chi(C_{odd}) = 3$$

Chromatic number of complete graph

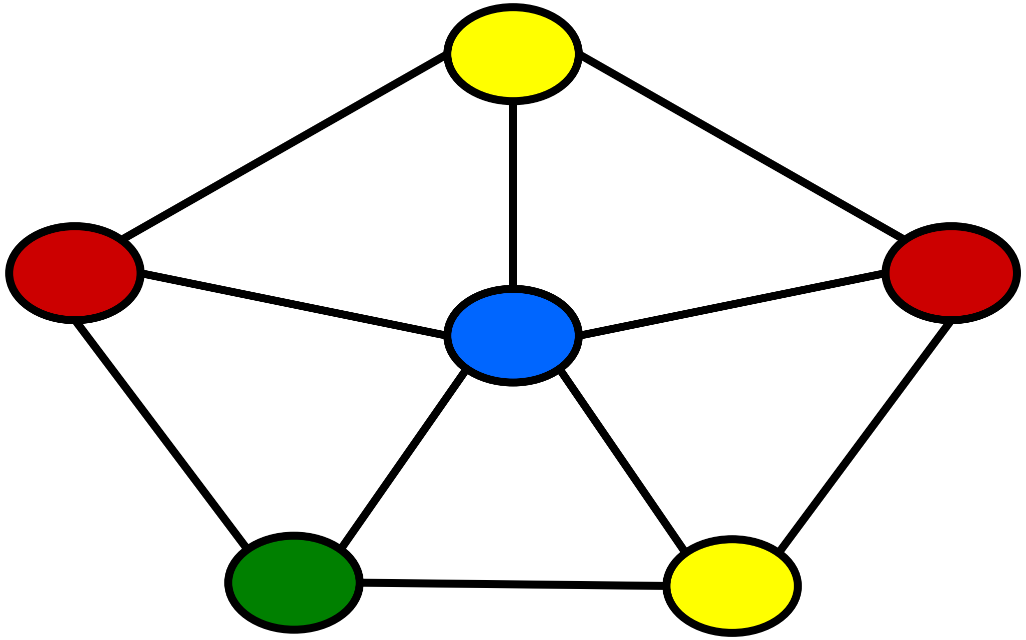


$$\chi(K_n) = n$$

Chromatic number of Wheels



W_5



$$\chi(W_{odd}) = 4 \qquad \chi(W_{even}) = 3$$

- Every tree with two or more vertices is 2 chromatic.
- A bipartite graph is 2 chromatic.

For any graph G, $\chi(G) \leq 1 + \Delta(G)$ where $\Delta(G)$ is the maximum degree of a vertex in G.

Welch and Powell Graph Coloring Algorithm

- Welch-Powell algorithm is an efficient algorithm for coloring graph G. This algorithm only give us upper bound of chromatic number of G. Thus, this algorithm does not always give the number of minimum color needed in graph coloring

Let G is a simple connected graph.

Step 1. Order the vertices of G according to decreasing degrees.

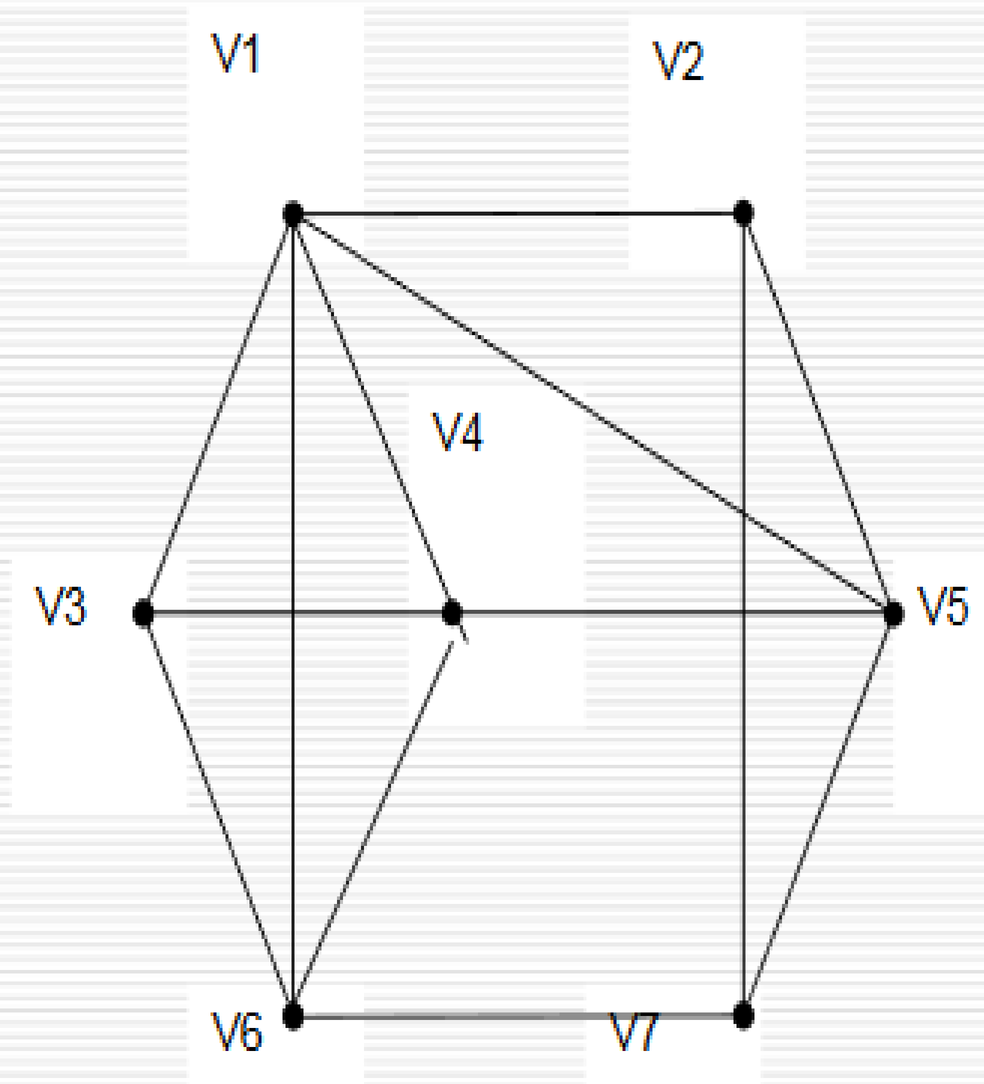
Step2. Assign the first color c1 to the first vertex and then, in sequential order, assign c1 to each vertex, which is not adjacent to a previous vertex.

Step 3. Repeat step 2 with a second color c2 and the subsequence of non-color vertices.

Step4. Repeat step 3 with a third color c3, then a forth color c4 and so on until all vertices are colored.

Step5. Stop.

Example:



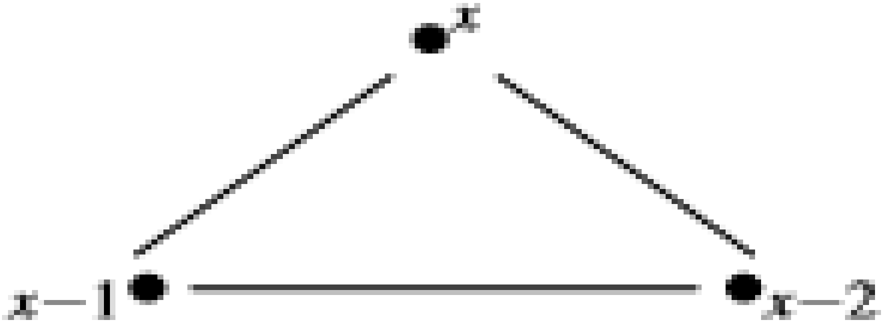
Vertices	V1	V4	V5	V6	V2	V3	V7
Degree	5	4	4	4	3	3	3
Color	a	b	c	d	b	c	a

Thus $\chi(H) = 4$

Chromatic Polynomial

$P_G(x)$ is the chromatic polynomial for the graph G. It counts the number of ways to color G in at most x colors.

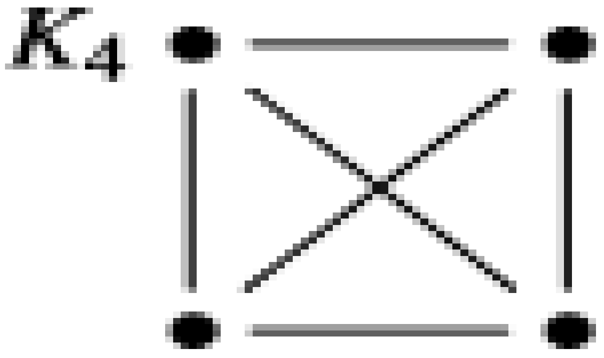
Example: Find $P_{K3}(x)$ and find out how many colorings there are on 1, 3 and 5 colors

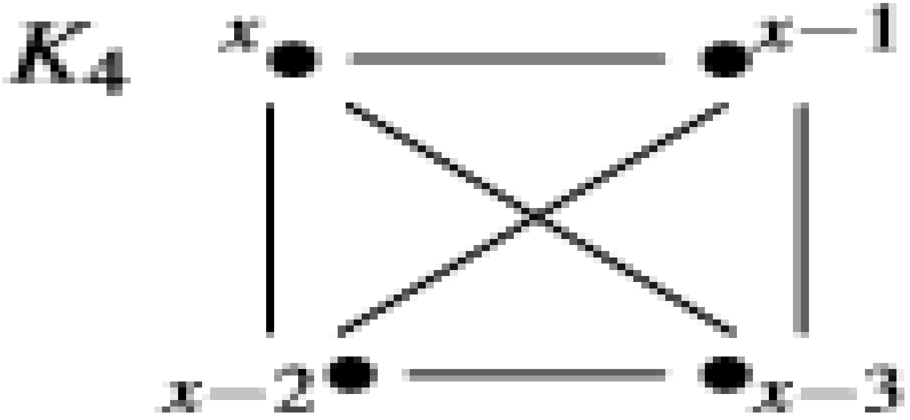


$$P_{K_3}(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

- For 1 color, there are no proper colorings
- For 3 colors, there are 6 proper colorings
- For 5 colors, there are

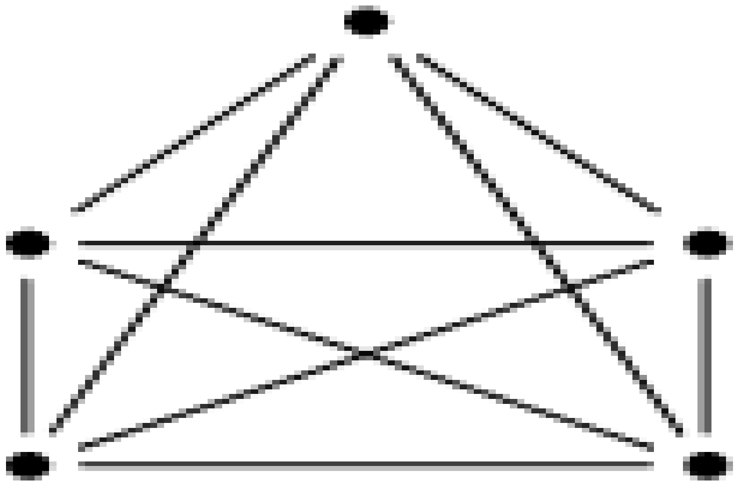
Example: Chromatic Polynomial of complete graph K_4.

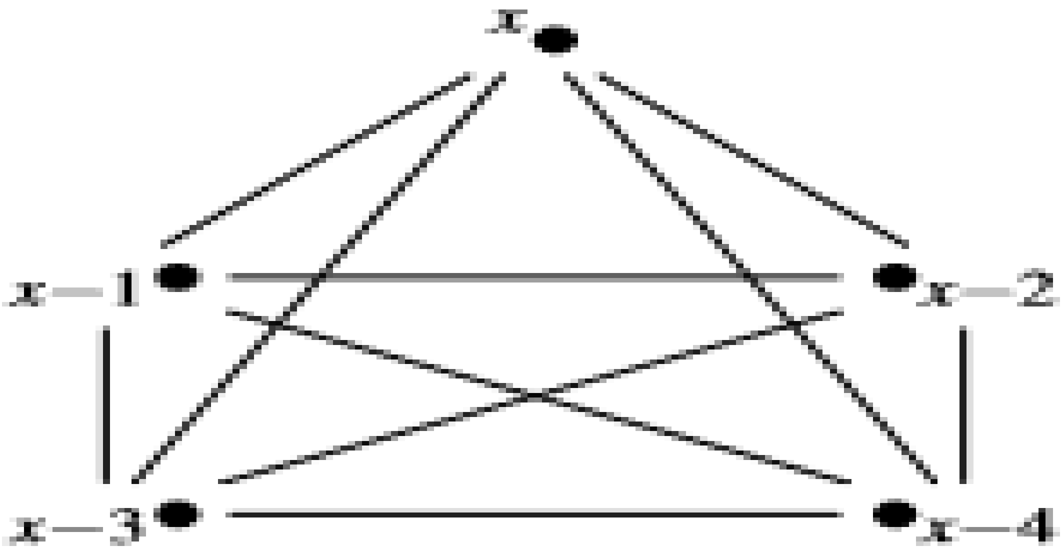




$$P_{K_4}(x) = x(x-1)(x-2)(x-3)$$

Example: Chromatic Polynomial of complete graph K_5.



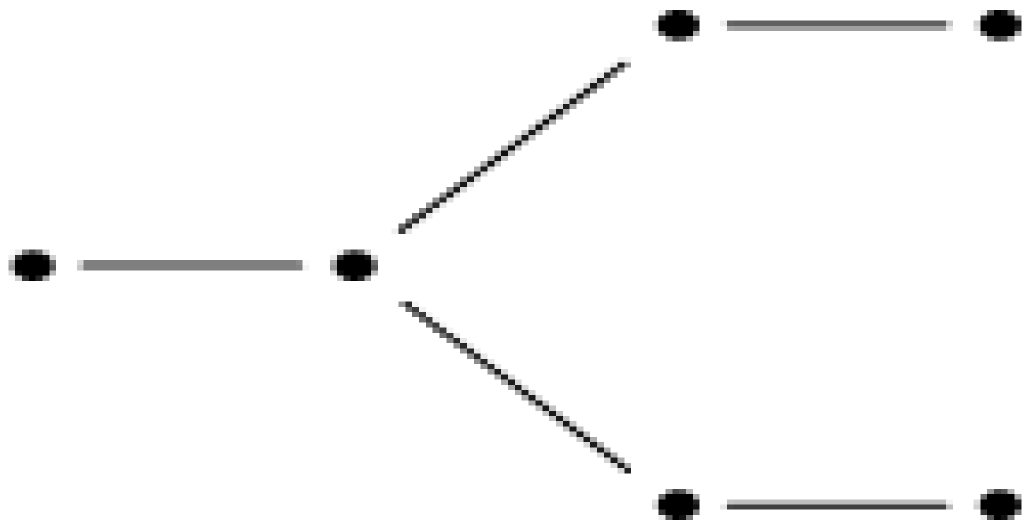


$$P_{K_5}(x) = x(x-1)(x-2)(x-3)(x-4)$$

- Chromatic Polynomial of complete graph with n nodes K_n

$$P_{K_n}(x) = x(x-1)(x-2)\cdots(x-(n-1))$$

Example: Find $P_G(x)$ for the following graph.

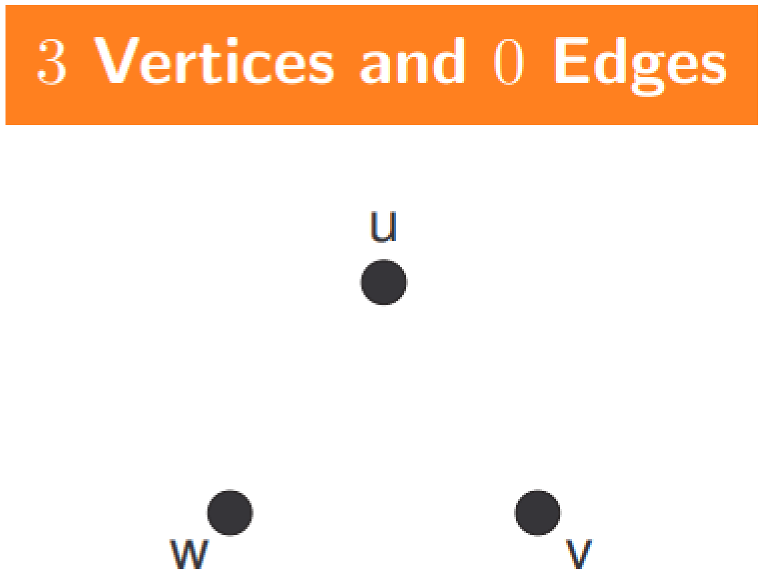


$$P_G(x) = x(x-1)^5$$

Chromatic Polynomial for a Tree of n nodes $P_{T_n}(x) = x(x-1)^{n-1}$

Exercise1

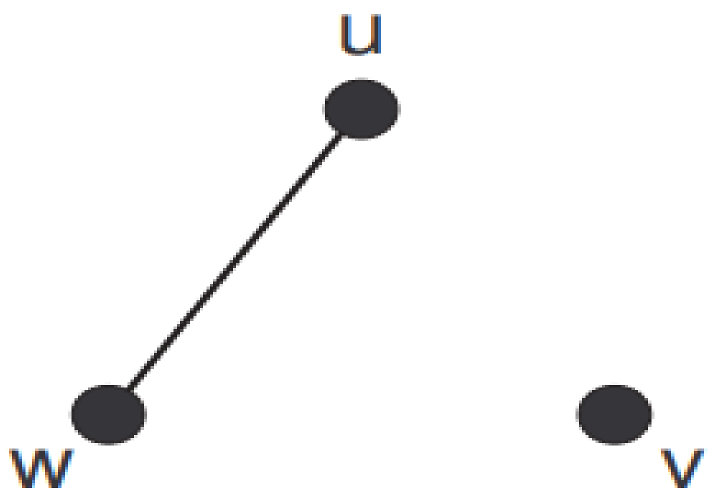
- Find Chromatic polynomial for



Exercise2

- Find Chromatic polynomial for

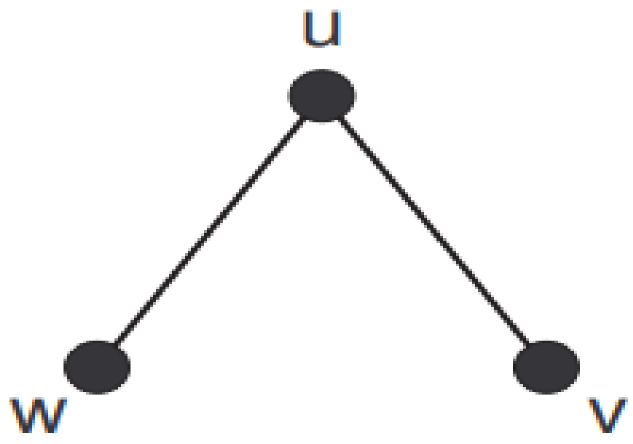
3 Vertices and 1 Edge



Exercise3

- Find Chromatic polynomial for

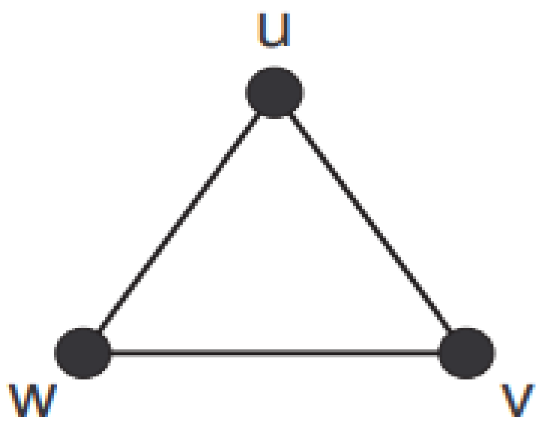
3 Vertices and 2 Edges



Exercise4

- Find Chromatic polynomial for

3 Vertices and 3 Edges



Exercise5

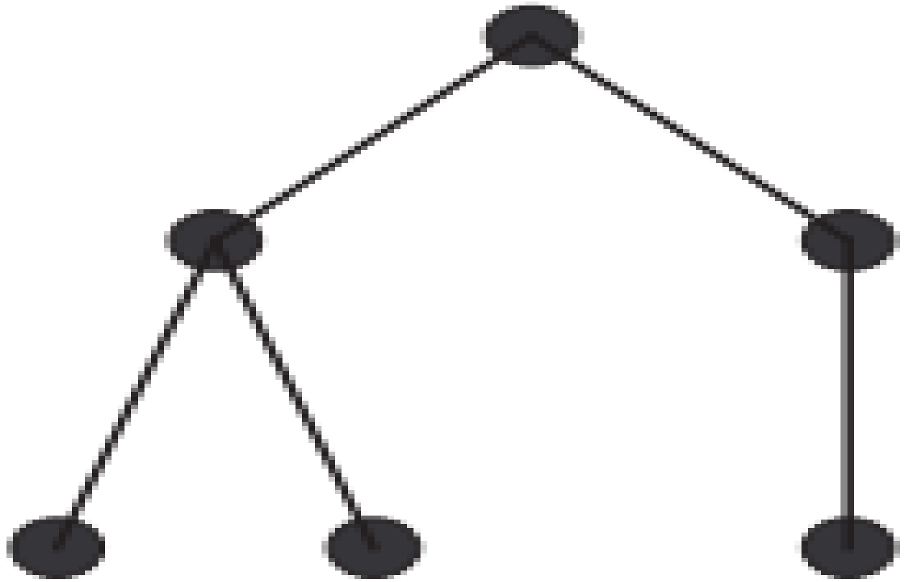
- Find Chromatic polynomial for star graph

Exercise6

- Find Chromatic polynomial for path graph

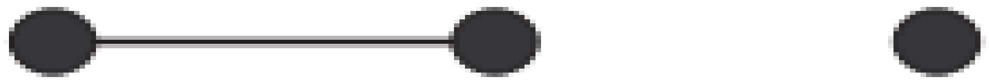
Exercise6

- Find Chromatic polynomial for tree



Exercise7

- Find Chromatic polynomial for disconnected graph



Exercise 8

Apply welsh Powell algorithm to colour the following graph. Find the number of colours required?