Theorem Proving Techniques

Theorem proving techniques

- **Principle of Mathematical Induction**
- **Direct Proofs**
- **Proof by Contrapositive**
- **Proof by Contradiction**

Principle of Mathematical Induction

Proofs by induction are often used when one tries to prove a statement made about natural numbers or integers. Here are examples of statements where induction would be used.

- For every natural number n, $1+2+3+...+n = \dfrac{n(n+1)}{2}$

Theorem (Induction) *Let $P(n)$ denote a statement about natural numbers with the following properties:*

1. *The statement is true when $n = 1$ i.e. $P(1)$ is true.*

2. *$P(k+1)$ is true whenever $P(k)$ is true for any positive integer k.*

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Remark *The case $n = 1$ is called the* ***base case****.*

Example

$3^n - 1$ is a multiple of 2 for n = 1, 2, ...

Solution

Step 1 – For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2

Step 2 – Let us assume $3^n - 1$ is true for $n = k$, Hence, $3^k - 1$ is true (It is an assumption)

We have to prove that $3^{k+1} - 1$ is also a multiple of 2

$$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3k - 1)$ is also true as our previous assumption.

Hence, $3^{k+1} - 1$ is a multiple of 2.

So, it is proved that $3^n - 1$ is a multiple of 2.

Direct Proofs

- We derive the result prove by combining logically the given assumptions (if any), definitions, axioms and known theorems.
- A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

Example

Prove that the sum of two odd integers is even.

Recall that an integer n is even if $n = 2k$ and it is odd if $n = 2k + 1$ for some integer k. We start with two odd integers we call a and b. This means that there exist integers k_1 and k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Now,

$$\begin{aligned} a + b &= 2k_1 + 1 + 2k_2 + 1 \\ &= 2k_1 + 2k_2 + 2 \\ &= 2\left(k_1 + k_2 + 1\right) \end{aligned}$$

If k_1 and k_2 are integers, $k_1 + k_2 + 1$ is also an integer. Hence, $a + b$ is even.

Example

If a and b are consecutive integers, then the sum a + b is odd.

- Assume that a and b are consecutive integers. Because a and b are consecutive we know that b = a + 1.
- Thus, the sum a + b may be re-written as 2a + 1.
- Thus, there exists a number k such that a + b = 2k + 1 so the sum a + b is odd.

Proof by Contrapositive (Indirect Proof)

Proof by contrapositive takes advantage of the logical equivalence between **"P implies Q"** and **"Not Q implies Not P".** For example, the assertion **"If it is my car, then it is red"** is equivalent to **"If that car is not red, then it is not mine".** So, to prove "If P, Then Q" by the method of contrapositive means to prove "If Not Q, Then Not P".

So the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.

Example

If x, y are integers and x and y are both odd, then x + y is even.

Restate the original statement to be as follows:

If x + y is odd, then x and y are not both odd

Consider $x + y = 2n + 1$ by definition of odd for some integer n. Then we have that $x = (2n + 1) – y$. We will now consider the possible cases for y.

Case 1: If y is odd then $y = 2a + 1$ for some integer a and the difference of two odd numbers is $(2n +1) – (2m +1) = 2(n+m)$ which makes x even.

Case 2: If y is even then $y = 2a$ for some integer a and the difference $x = (2n + 1) – 2a = 2(n +a) +1$ is odd.

Example

- Prove the following statement by contraposition:

If x and y are two integers for which x+y is even, then x and y have the same parity.

The contrapositive version of this theorem is

"If x and y are two integers with opposite parity, then their sum must be odd.“

Example

If $n > 0$ and $4^n - 1$ is prime, then n is odd: Assume $n = 2k$ is even. Then

$$4^n - 1 = 4^{2k} - 1 = (4^k - 1)(4^k + 1).$$

Therefore, $4^n - 1$ factors (are both factors bigger than 1?) and hence is not prime.

Example

Prove that if n^2 *is even, so is* n.

Since a number is odd, the contrapositive of this statement is "if n *is odd so is* n^2. *We prove that instead.*

$$
\begin{aligned}
n \text{ odd} \quad &\Longrightarrow \quad n = 2k+1 \text{ for some integer } k \\
&\Longrightarrow \quad n^2 = (2k+1)^2 \\
&\Longrightarrow \quad n^2 = 4k^2 + 4k + 1 \\
&\Longrightarrow \quad n^2 = 2\left(2k^2 + 2k\right) + 1 \\
&\Longrightarrow \quad n^2 \text{ is odd since } 2k^2 + 2k \text{ is an integer}
\end{aligned}
$$

Proof by Contradiction

it is another type of indirect proof. We prove that under the given assumptions, assuming a statement is true leads to some contradiction. Hence, the statement cannot be true.

- Suppose we want to prove that $P \to Q$ is true by contradiction.
- The proof will look something like this:
  - Assume that ***P* is true and *Q* is false**.
  - Using this assumption, derive a contradiction.
  - Conclude that $P \to Q$ must be true.

Example

EXAMPLE 4: IF THE SQUARE OF N IS EVEN, THEN N ITSELF MUST BE EVEN.

To prove this by contradiction, we assume that N^2 is even, but N is odd.

Then N must be of the form $2k+1$ where k is an integer.

Then N^2 becomes:

$$
\begin{aligned}
N^2 &= (2k+1)^2 \\
&= 4k^2+4k+1 \\
&= 2\left(2k^2+2k\right)+1 \\
&= 2t+1
\end{aligned}
$$

But then this means that N^2 is also odd! But our assumption was that N^2 was even. This is a contradiction, and so our our assumption that N^2 is even and N is odd is incorrect.

Example

If x, y are integers and x and y are both odd, then x + y is even.

Restate the original to be that

x and y are odd integers and the sum x + y is also odd.

Class Exercise 1

Use mathematical induction to show that

$1 + 2 + 2^2 + \cdot\cdot\cdot + 2^n = 2^{n+1} - 1$

for all nonnegative integers *n*.

Class Exercise 2

Use mathematical induction to prove that

$1/1.3 + 1/3.5 + 1/5.7 + \ldots\ldots + 1/(2n-1)(2n+1) = n/(2n + 1)$

For all $n \in N$

Class Exercise 3

Give a direct proof that if *m* and *n* are both perfect squares, then *nm* is also a perfect square. (An integer *a* is a perfect square if there is an integer *b* such that $a = b^{2}$).

Class Exercise 4

- **Prove the sum of two even numbers is always even using direct method.**

Class Exercise 5

Use mathematical induction to prove De Moivre's theorem

$$[\, R\,(\cos t + i \sin t)\,]^n = R^n(\cos nt + i \sin nt)$$

for n a positive integer.

Class Exercise 6

Prove by contrapositive that if n is an integer and $3n + 2$ is odd, then n is odd.

Class Exercise 7

Give a proof by contradiction of the theorem "If 3*n* + 2 is odd, then *n* is odd."

Class Exercise 8

Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a a proof by contraposition

b a proof by contradiction

Class Exercise 9

- The product of any even integer and any other integer is even. Prove by direct method.

(two cases)