Solution of Recurrence Relation

Recurrence Relations

- **Definition**: A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms in the sequence:

$$a_0, a_1, a_2, \ldots, a_{n-1}$$

  for all integers $n \geq n_0$ where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence if its terms satisfy the recurrence relation

A recursive definition of a sequence specifies

- Initial conditions
- Recurrence relation

Example:

$a_0=0$ and $a_1=3$ $\Rightarrow$ Initial conditions

$a_n = 2a_{n-1} - a_{n-2}$ $\Rightarrow$ Recurrence relation

$a_n = 3n$ $\Rightarrow$ Solution

Solving Recurrences

There are several methods for solving recurrences

- Iterative method
- Recursive Method
- Characteristic Equations
- Generating Function
- ……………………

Example: Iterative Method

a_r-2a_{r-1}=0 for r>=1 and a_0=1

Solution:

$a_r - 2a_{r-1} = 0$

$\Rightarrow a_r = 2a_{r-1}$

$\Rightarrow a_1 = 2$

$\Rightarrow a_2 = 2.2 = 2^2$

$\Rightarrow a_3 = 2.2.2 = 2^3$

$\Rightarrow$ ....

$\Rightarrow a_r = 2^r$ for r>=1

Class Exercise 1

- Find the solution of the following recurrence relation using iterative method

$\mathbf{a_r=a_{r-1}+3}$ **for r>=1 and** $\mathbf{a_0=1}$

Example: Recursive Method

$a_r - 2a_{r-1} = 0$ for $r >= 1$ and $a_0 = 1$

Solution:

$a_r = 2a_{r-1}$

$a_r = 2^2 a_{r-2}$

$a_r = 2^3 a_{r-3}$

……………………………………

……………………………………

$a_r = 2^r a_{r-r} = 2^r a_0 = 2^r$

Class Exercise 2

- Find the solution of the following recurrence relation using recursive method

$\mathbf{a_r = a_{r-1} + 3}$ **for r>=1 and** $\mathbf{a_0 = 1}$

Linear Homogeneous Recurrences

- **Definition**: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

with $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Linear Homogeneous Recurrences: Examples

Determine if the following recurrence relations are linear homogene recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$

  a linear homogeneous recurrence relation of degree one
- $a_n = a_{n-1} + a^2_{n-2}$

  not linear
- $f_n = f_{n-1} + f_{n-2}$

  a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1}+1$

  not homogeneous
- $a_n = a_{n-6}$

  a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$

  does not have constant coefficient

Solving Linear Homogeneous Recurrences

- We want a solution of the form $a_n = r^n$ where r is some real constant
- We observe that $a_n = r^n$ is a solution to a linear homogeneous recurrence *if and only if*

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

- We can now divide both sides by r^{n-k}, collect terms and we get a k-degree polynomial

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

- This equation is called the <u>characteristic equation</u> of the recurrence relation
- The roots of this polynomial are called the <u>characteristics roots</u> of the recurrence relation. They can be used to find the solutions (if they exist) to the recurrence relation.

1. *Distinct Real Roots.* In this case the general solution of the recurrence relation is

$$x_n = c_1\, r_1^n + c_2\, r_2^n\,,$$

where c_1, c_2 are arbitrary constants.

2. *Double Real Root.* If $r_1 = r_2 = r$, the general solution of the recurrence relation is

$$x_n = c_1\, r^n + c_2\, n\, r^n\,,$$

where c_1, c_2 are arbitrary constants.

3. *Complex Roots.* In this case the solution is

$$x_n = k_1\, r^n \cos n\alpha + k_2\, r^n \sin n\alpha$$

Example (1)

- Find a solution to

$$a_n = 5a_{n-1} - 6a_{n-2}$$

  with initial conditions $a_0=1$, $a_1=4$
- The characteristic equation is

$$r^2 - 5r + 6 = 0$$

- The roots are $r_1=2$, $r_2=3$

$$r^2 - 5r + 6 = (r-2)(r-3)$$

- So we have a solution

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

- Given the solution

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

- We plug in the two initial conditions to get a system of linear equations

$$a_0 = \alpha_1 2^0 + \alpha_2 3^0$$

$$a_1 = \alpha_1 2^1 + \alpha_2 3^1$$

- Thus:

$$1 = \alpha_1 + \alpha_2$$

$$4 = 2\alpha_1 + 3\alpha_2$$

$$1 = \alpha_1 + \alpha_2$$
$$4 = 2\alpha_1 + 3\alpha_2$$

- Solving for $\alpha_1 = (1 - \alpha_2)$, we get

$$4 = 2\alpha_1 + 3\alpha_2$$
$$4 = 2(1-\alpha_2) + 3\alpha_2$$
$$4 = 2 - 2\alpha_2 + 3\alpha_2$$
$$2 = \alpha_2$$

- Substituting for α_1: $\alpha_1 = -1$
- Putting it back together, we have

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$
$$a_n = -1 \cdot 2^n + 2 \cdot 3^n$$

Example (2)

- Solve the recurrence

$$a_n = -2a_{n-1} + 15a_{n-2}$$

  with initial conditions $a_0 = 0$, $a_1 = 1$

- If we did it right, we have

$$a_n = 1/8\ (3)^n - 1/8\ (-5)^n$$

Single Root Case: Example

- What is the solution to the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 7$?

- The characteristic equation is:

$$r^2 - 8r + 16 = 0$$

- Factoring gives us:

$$r^2 - 8r + 16 = (r-4)(r-4), \text{ so } r_0 = 4$$

- Applying the theorem we have the solution:

$$a_n = \alpha_1(4)^n + \alpha_2 n(4)^n$$

- Given: $a_n = \alpha_1(4)^n + \alpha_2 n(4)^n$
- Using the initial conditions, we get:

$$a_0 = 1 = \alpha_1(4)^0 + \alpha_2 0(4)^0 = \alpha_1$$

$$a_1 = 7 = \alpha_1(4) + \alpha_2 1(4)^1 = 4\alpha_1 + 4\alpha_2$$

- Thus: $= \alpha_1 = 1, \alpha_2 = 3/4$

- The solution is

$$a_n = (4)^n + ¾\, n\, (4)^n$$

Class Exercise 3

- Suppose that a person deposits Rs.10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 3 years?
- Find the recurrence relation.
- Solve using method of iteration and recursive approach.
- Apply method of CE if applicable.

$$P_n = P_{n-1} + 0.11 P_{n-1} = (1.11) P_{n-1}.$$

$$\begin{aligned}
P_1 &= (1.11) P_0 \\
P_2 &= (1.11) P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11) P_2 = (1.11)^3 P_0 \\
&\vdots \\
P_n &= (1.11) P_{n-1} = (1.11)^n P_0.
\end{aligned}$$

Class Exercise 4

An employee joined a company in 2009 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

- **a)** Set up a recurrence relation for the salary of this employee *n* years after 2009.
- **b)** What will the salary of this employee be in 2019?
- **c)** Find an explicit formula for the salary of this employee *n* years after 2009.

Class Exercise 5

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0=0$ and $f_1=1$?

$a_n = 1/\sqrt{5} \cdot ((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$ is a solution.

Class Exercise 6

What is the solution of the recurrence relation

$$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$

with $a_0=8$, $a_1=6$ and $a_2=26$?

$a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.

Class Exercise 7

- Solve $$a_n = 8a_{n-2} - 16a_{n-4}$$