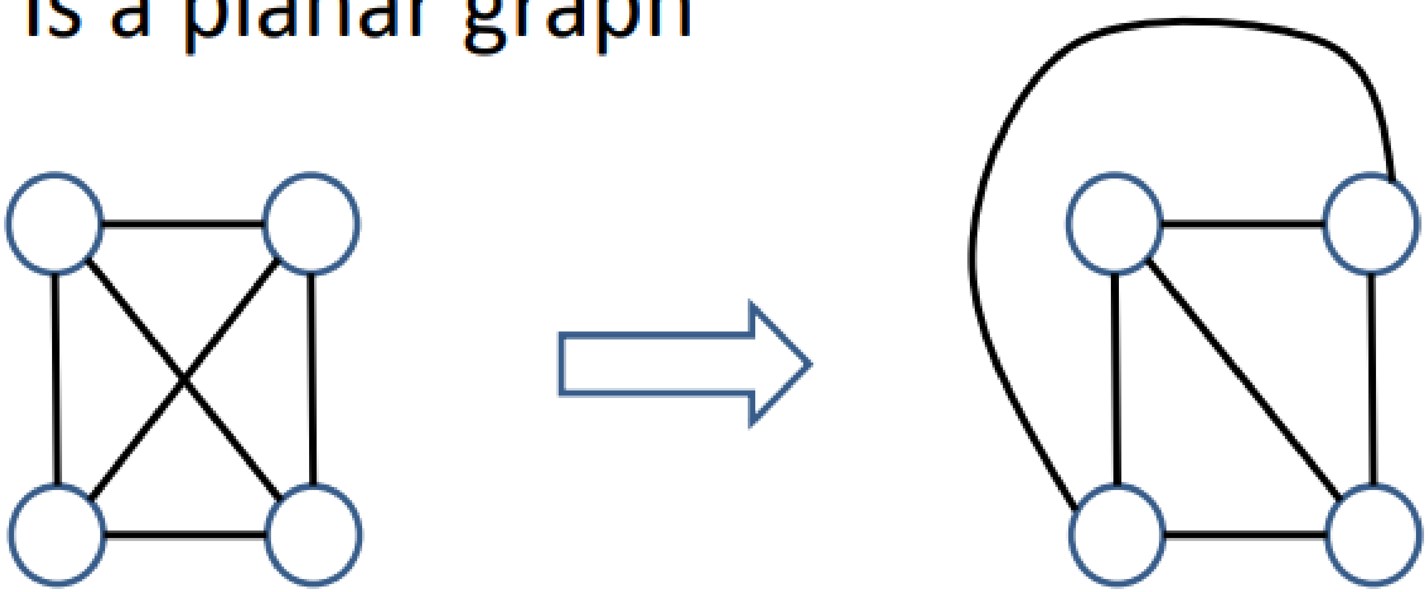
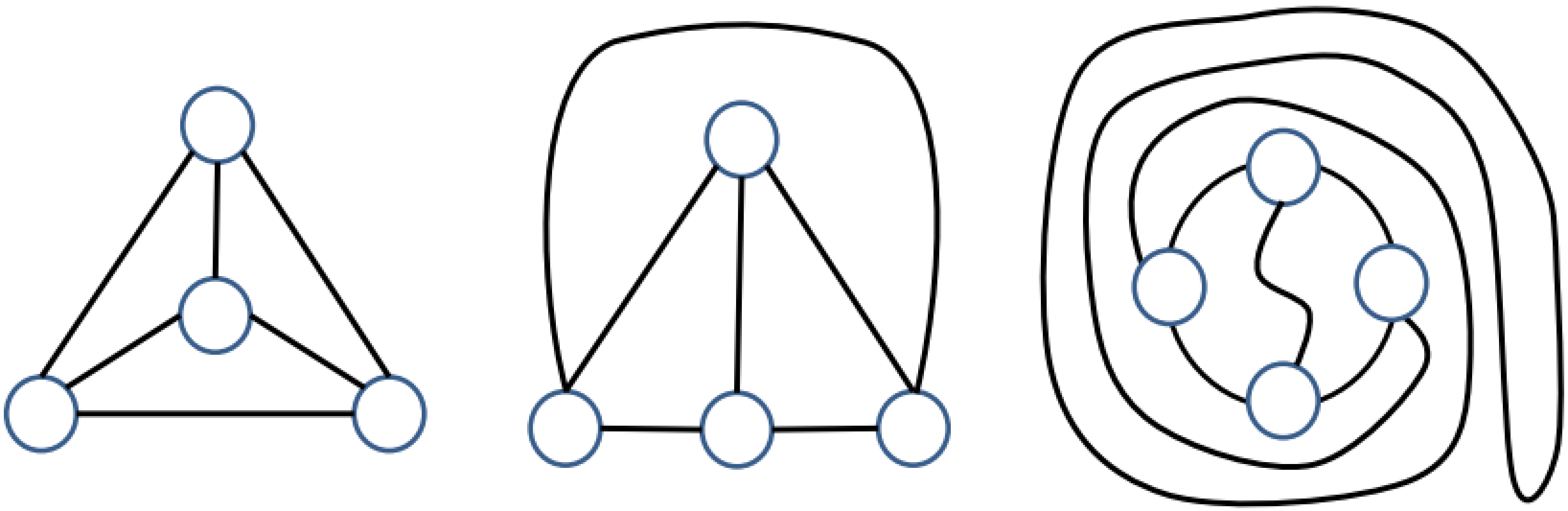
Planar Graphs

A **graph** is called ***planar*** if it can be drawn on a plane without edges crossing.
A ***plane graph*** is a drawing of planar graph in the plane.

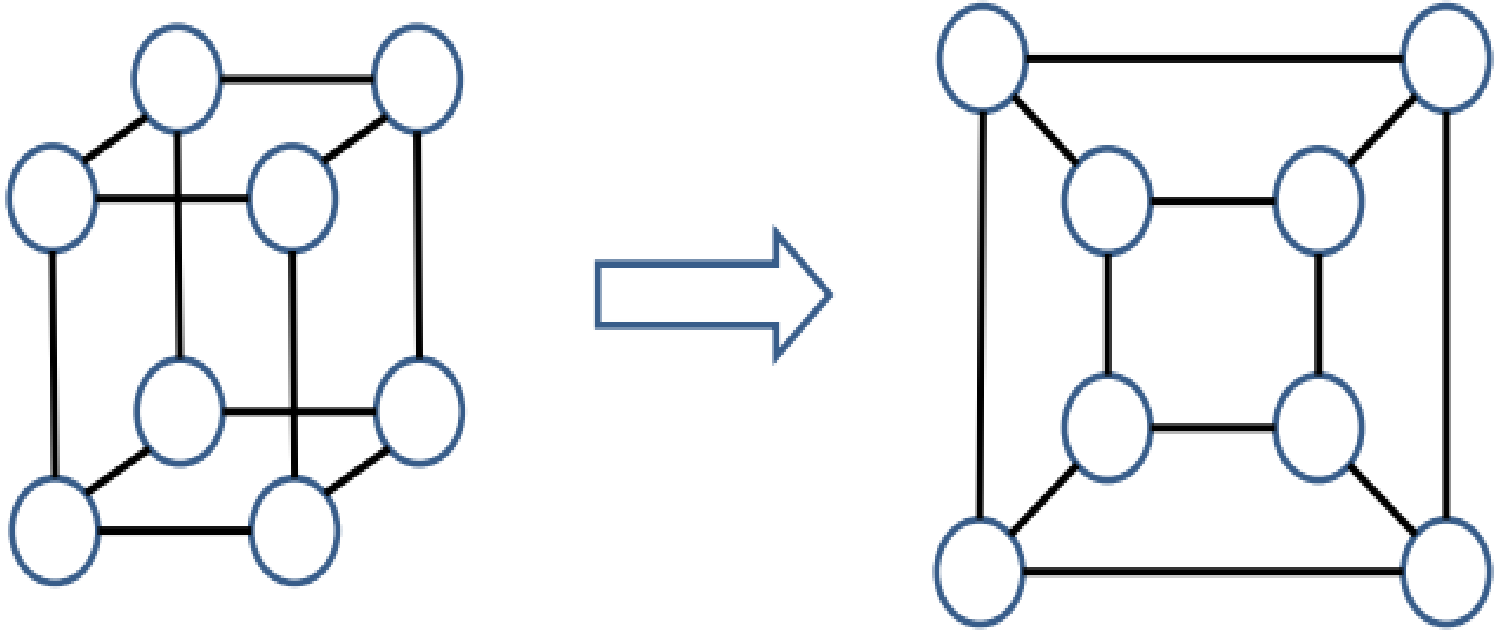
Ex : K_4 is a planar graph



- Ex : Other planar representations of K_4



- Ex : Q_3 is a planar graph



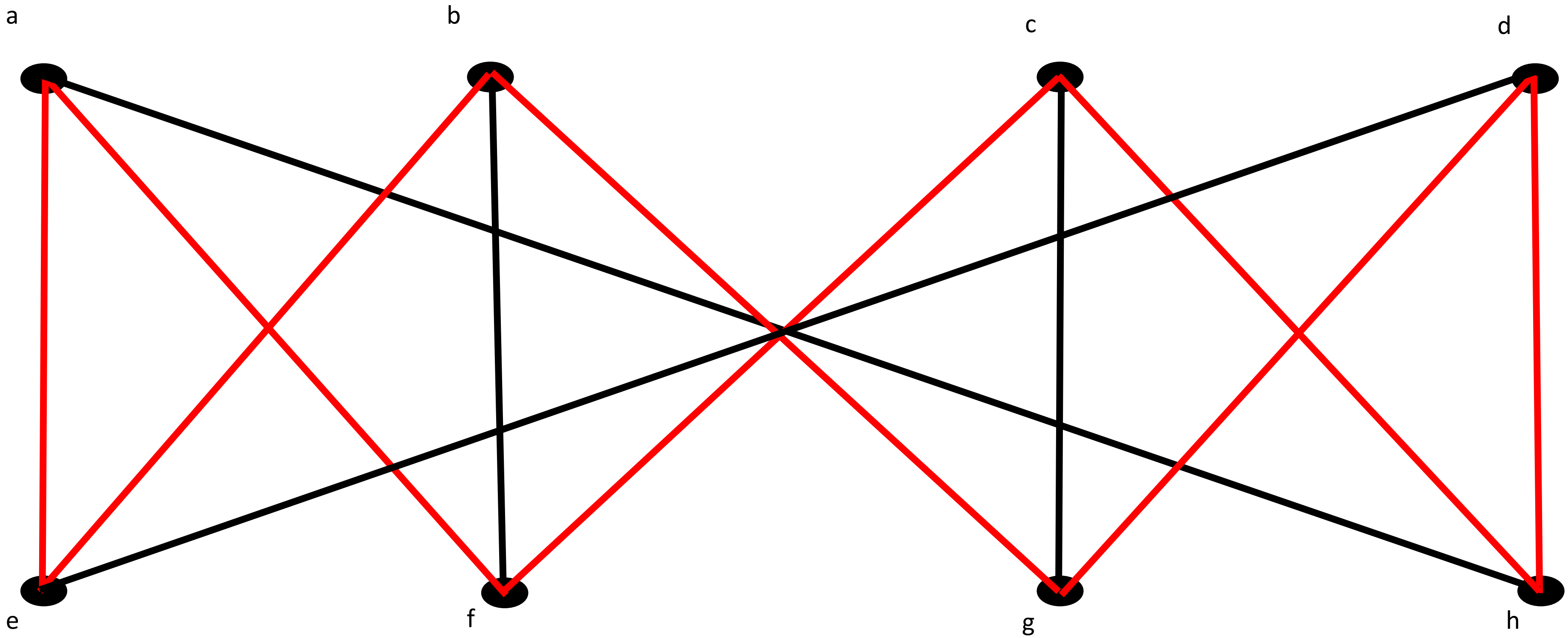
Here are two ways to determine if a graph is planar:

The first is a practical heuristic, which refers to as the **circle-chord method**. It consists of a step-by-step method of drawing the graph, edge-by-edge without crossing any edges.

The second consists of theoretical results, such as **Kuratowski's Theorem**, or **Euler's Formula**.

Circle-Chord Method--Planar

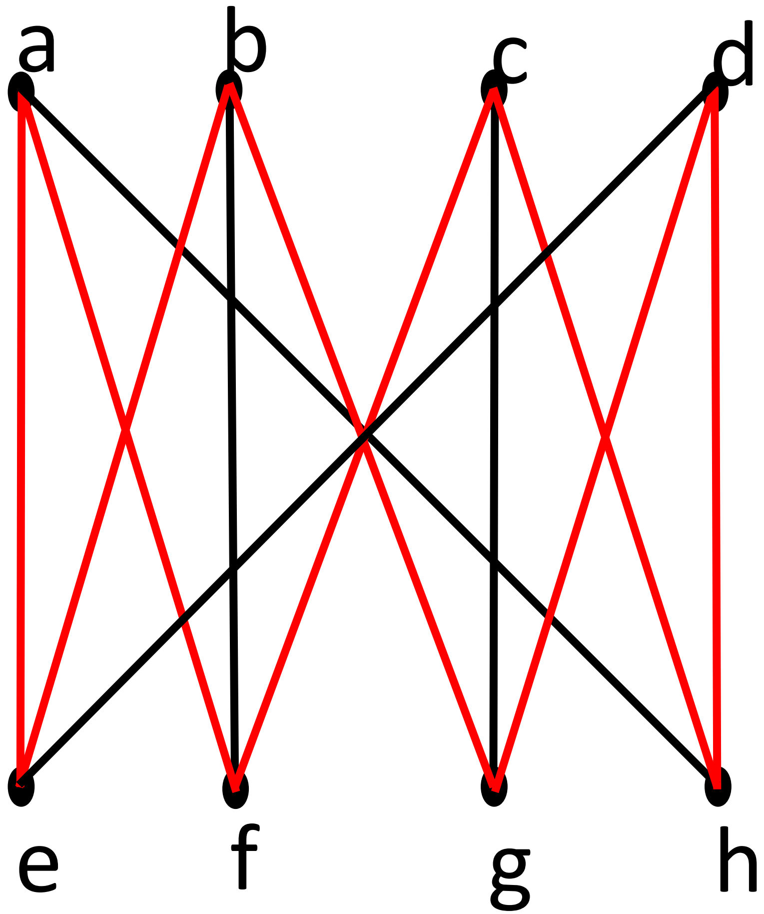
Step One: Find a **circuit** that contains all the vertices of the graph.

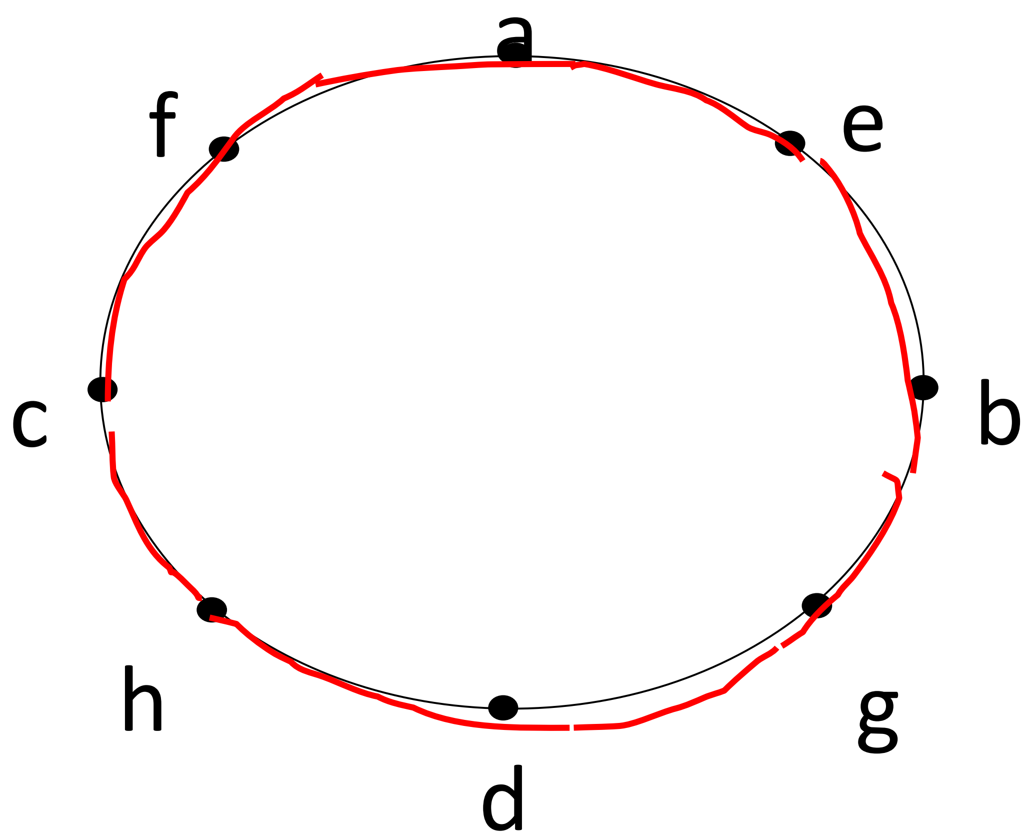


The **circuit** for this figure is highlighted in **red**

Circle-Chord Method--Planar

Step Two: Draw this circuit as a large circle.

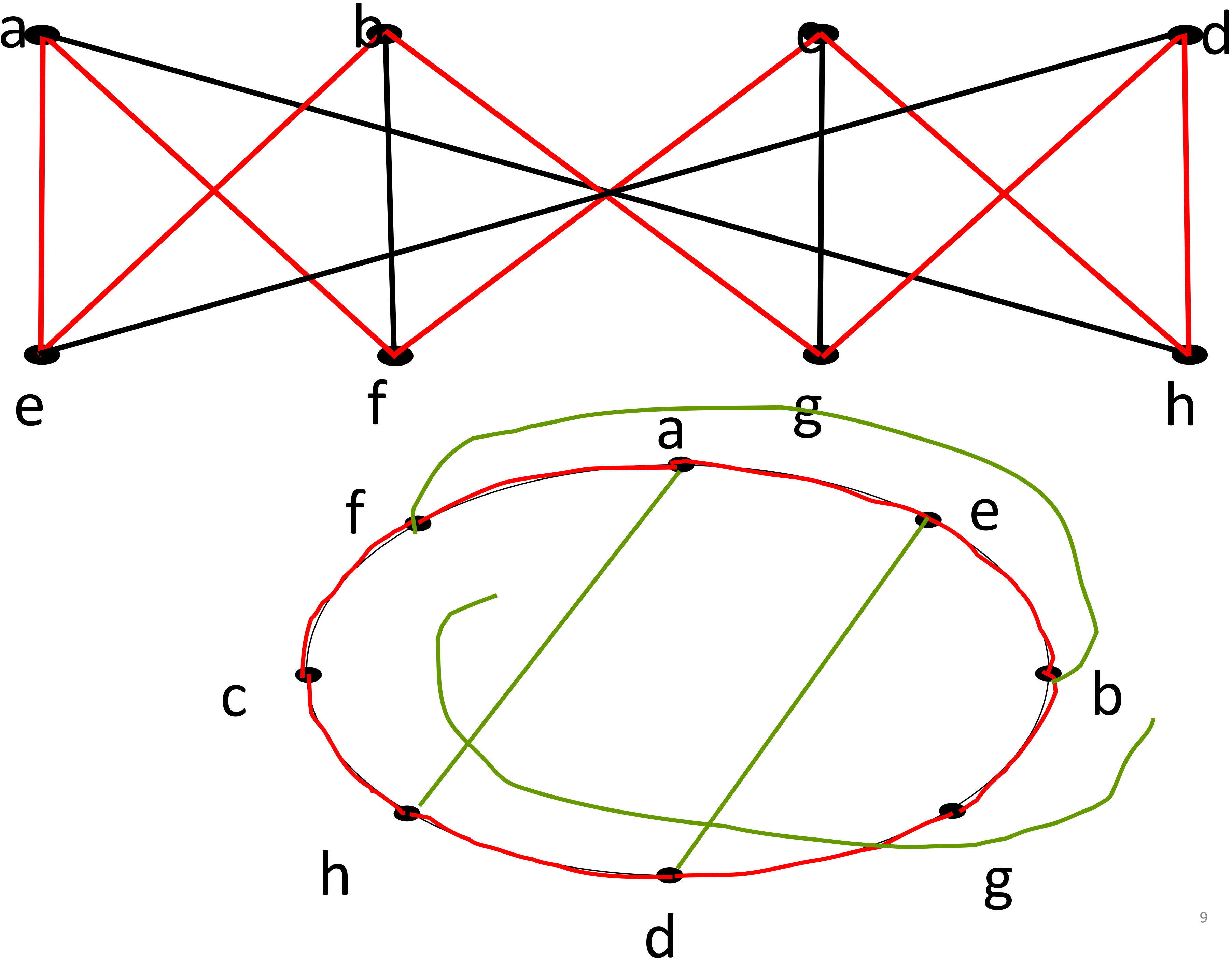




Circle-Chord Method--Planar

Step Three: Choose one **chord**, and decide to draw it either inside or outside the circle. If chosen correctly, it will force certain other chords to be drawn opposite to the circle. (Inside if the first chord was drawn outside, and vice versa.)

Since the chords could be drawn without crossing, this graph is planar.


a
b
c
d
e
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a
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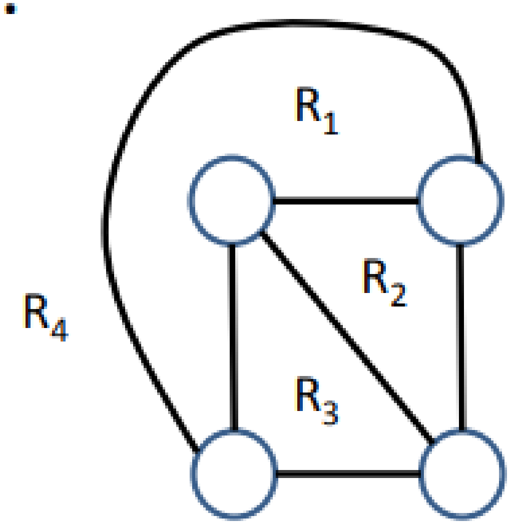
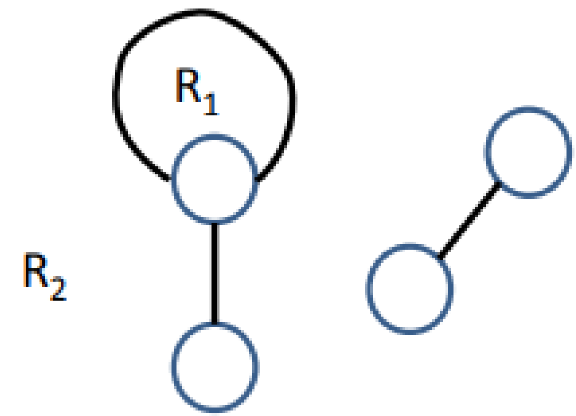
Euler's Planar Formula

Definition : A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

- Ex :



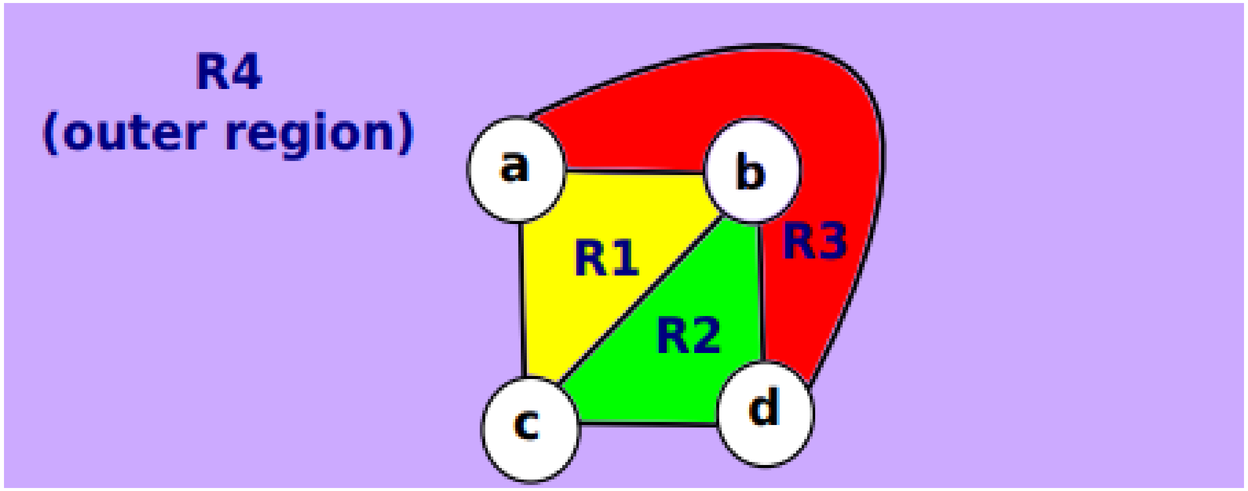
4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

Regions of a Planar Graph

- The planar representation of a graph splits the plane into regions (sometimes also called faces):



- Every planar graph has an outer region, which is unbounded.
- Degree of a region R, written $\deg(R)$, is the number of edges adjacent to R
- What is degree of $R1, R2, R3, R4$?

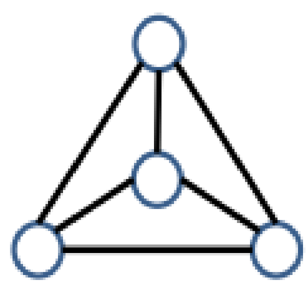
Euler's Planar Formula

- Let G be a connected planar graph, and consider a planar representation of G. Let

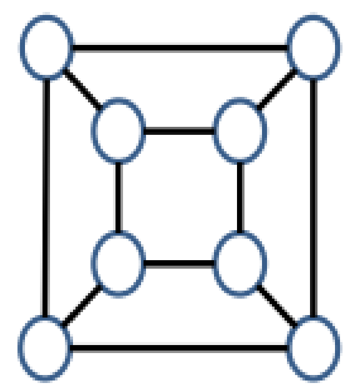
  V = # vertices, E = # edges, F = # regions.

Theorem : $V + F = E + 2.$

- Ex :



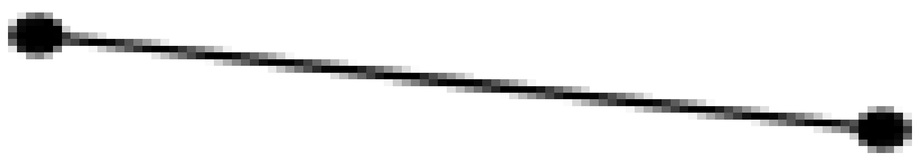
$V = 4,\ F = 4,\ E = 6$



$V = 8,\ F = 6,\ E = 12$

Proof by Induction on edges

Base Step: We proceed by induction on the number of edges. For e=1, we can only have this graph, which has v=2 and r=1, so the theorem holds.

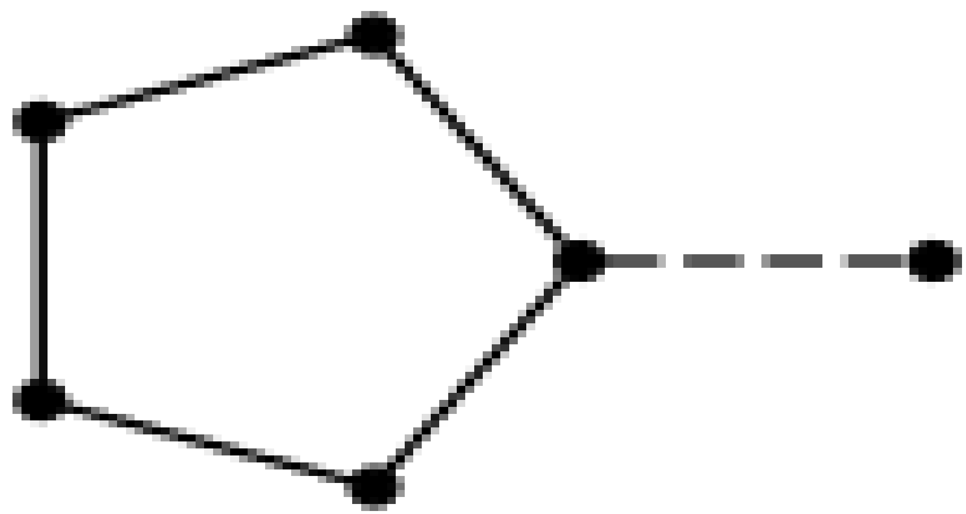


i.e. 2+1=1+2

Hypothesis: Assume the conclusion for $e-1$ edges,

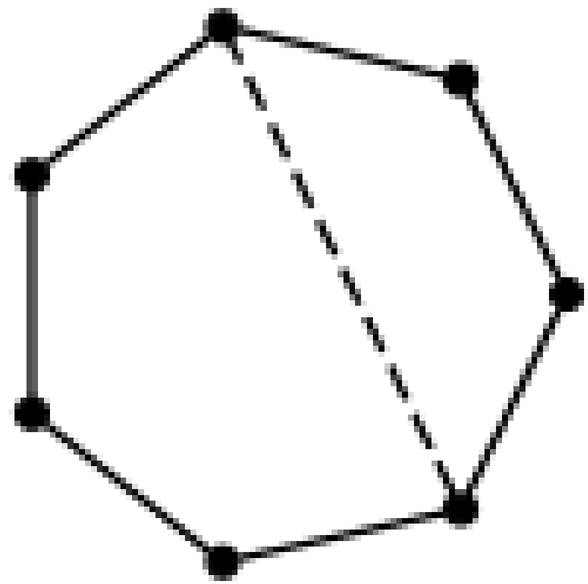
Inductive step: add an edge to get our planar graph, there are only two possibilities to consider.

Case 1: we add a pendant vertex and edge. In this case *e* and *v* both increase by one and the equality still holds.



i.e. $(V+1)+F=(E+1)+2$

Case 2: we add an edge across two existing vertices (that doesn't cross any other edges). Here, *r* and *e* both increase by one, preserving the equality.



i.e. $V+ (F+1)=(E+1)+2$

Exercise

- A connected planar graph has 8 vertices and 12 edges. How many faces are there?

- A connected planar graph has 6 vertices and 4 faces. How many edges are there?

- A connected planar graph has 8 vertices with degrees: 1, 1, 2, 2, 3, 3, 4, 4. How many edges are there?

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- A connected planar graph has 6 faces and 12 edges. How many vertices are there?

- A connected planar graph has 4 faces with degrees: 3, 3, 4, 4. How many edges are there?

- A connected planar graph has 4 faces with degrees: 3, 3, 4, 4. How many vertices are there?

Theorem: For a simple connected planar graph with $v \geq 3$ vertices and e edges, $e \leq 3v-6$.

Proof: Let r be the number of regions in a planar representation of the graph, and for a region R, let $\deg(R)$ be the number of edges that are adjacent to the region, so each edge is adjacent to two regions.

We know that $\deg(R) \geq 3$ for each region since the graph doesn't have multiple edges and has at least three vertices (for the exterior region).

Since each edge is adjacent to two regions,

$$2e = \sum \deg(R) \geq 3r\,.$$

We can use this with Euler's formula ($r = e - v + 2$) to get

$$\begin{aligned} 3r &\leq 2e \\ 3(e - v + 2) &\leq 2e \\ e &\leq 3v - 6\,. \end{aligned}$$

This theorem gives necessary condition for planar graph.

Show that K_5 is non-planar.

Proof Suppose that K_5 is a planar graph. Since K_5 has 5 vertices and 10 edges it follows from theorem **e<=3v-6** that

$10 > (3 \times 5) - 6$

This contradiction shows that K_5 is non planar.

Theorem: For a simple connected planar graph with $v \geq 3$ vertices and e edges, and no circuits of length three, $e \leq 2v - 4$.

Proof: The sum of the degrees of the regions is equal to twice the number of edges.
But each region must have degree ≥ 4 because we have no circuits of length 3.
So we have 2e ≥ 4f. Then 1/2 e ≥ f.

Euler’s formula says that

$v - e + f = 2.$

or $e - v + 2 = f.$

Combining this with $1/2\, e \geq f$, we get

$e - v + 2 \leq 1/2\, e$

So $e/2 - v + 2 \leq 0.$

So $e/2 \leq v - 2$. Therefore $e \leq 2v - 4.$

This theorem gives necessary condition for planar graph.

Show that $K_{3,3}$ is non-planar.

Proof Suppose that $K_{3,3}$ is a planar graph. Since $K_{3,3}$ has 6 vertices and 9 edges and no triangles, it follows from theorem $\mathbf{e \leq 2v - 4}$ that

$9 \leq (2\times 6) - 4 = 8.$

This contradiction shows that $K_{3,3}$ is non-planar.