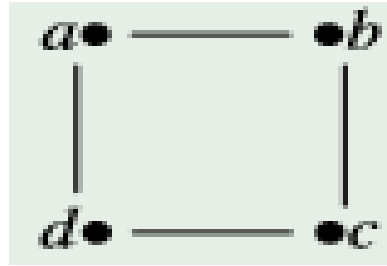


Graph Colouring

Example: Find chromatic Polynomial of Cyclic Graph with four nodes i.e. $P_{C_4}(x)$.



There are two different cases to consider:

Case 1:

x choices for a

b and d get the same color since they are nonadjacent

so $x - 1$ choices

1 choices for c ,

same color

Case 2:

x choices for a

$x - 1$ choices for b

We force d to be a different color than b , so there are

$x - 2$ choices

$x - 2$ choices for c

In math, 'or' means $+$, so we have

$$P_{C_4}(x) = x(x - 1)(x - 2)^2 + x(x - 1)^2 = x^4 - 4x^3 + 6x^2 - 3x$$

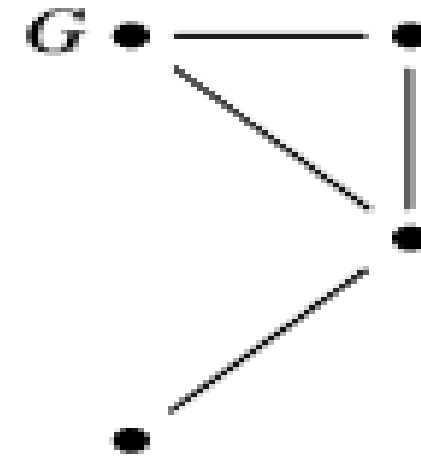
Fundamental Reduction (decomposition) Theorem (Deletion- Contraction Theorem)

The Fundamental Reduction Theorem

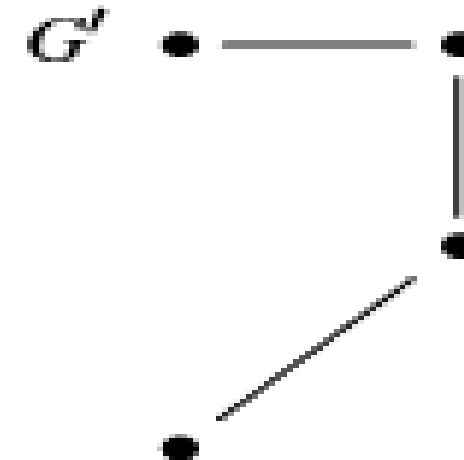
$$P_G(x) = P_{G'}(x) - P_{G''}(x)$$

where G' is a graph after deletion of an edge (e) from G and G'' is a graph after contraction (joining) of endpoints of e .

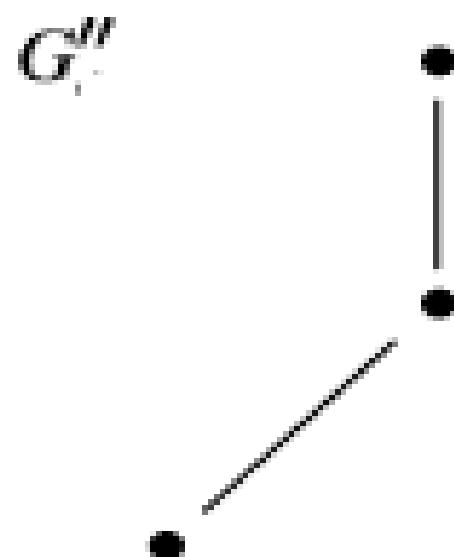
Example: Use the Fundamental Reduction Theorem to find the chromatic polynomial for the given graph.



So now we have ...



$$P_{G'}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x$$

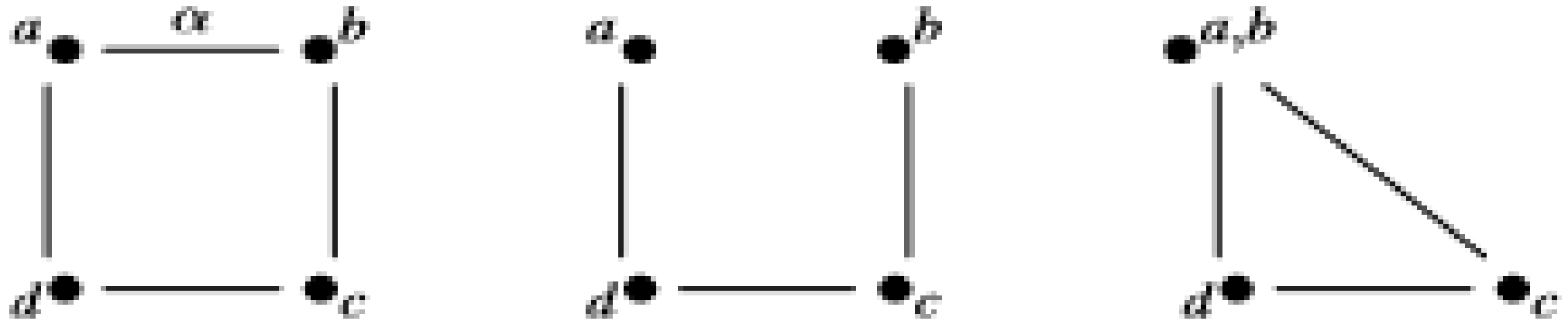


$$P_{G''}(x) = x(x-1)^2 = x^3 - 2x^2 + x$$

Putting this together, we have

$$\begin{aligned} P_G(x) &= x^4 - 3x^3 + 3x^2 - x - (x^3 - 2x^2 + x) \\ &= x^4 - 4x^3 + 5x^2 - 2x \end{aligned}$$

Example: Find $P_{C_4}(x)$ using the Fundamental Reduction Theorem.



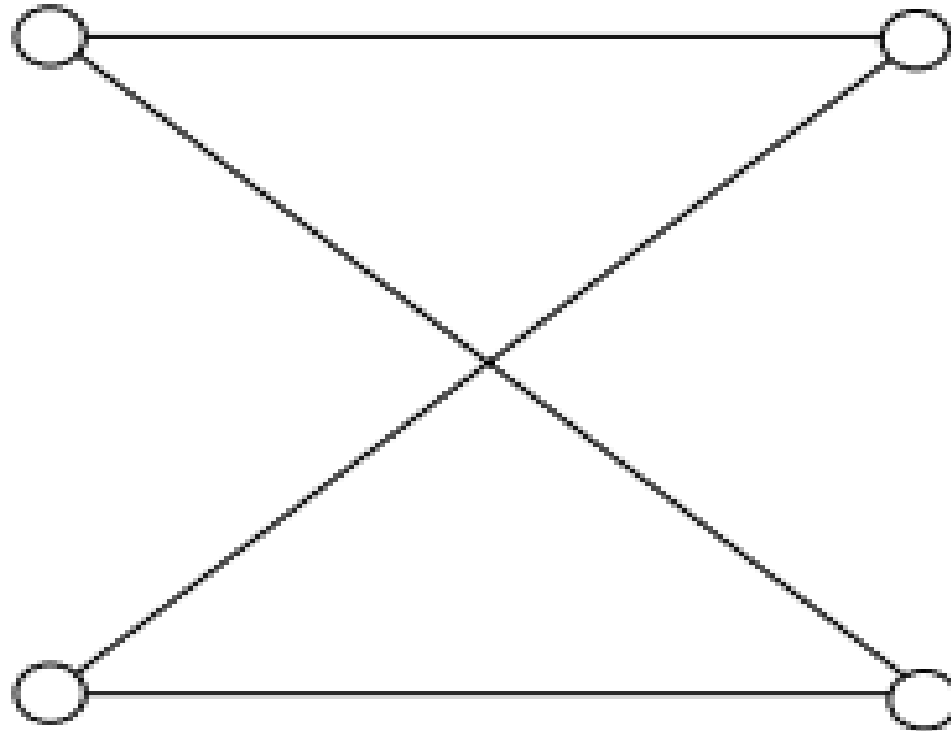
$$P_{G'_\alpha}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x$$

$$P_{G''_\alpha}(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$

$$\begin{aligned} P_G(x) &= x^4 - 3x^3 + 3x^2 - x - (x^3 - 3x^2 + 2x) \\ &= x^4 - 4x^3 + 6x^2 - 3x \end{aligned}$$

Exercise1

- Find the chromatic polynomials of the three graphs below using reduction theorem.



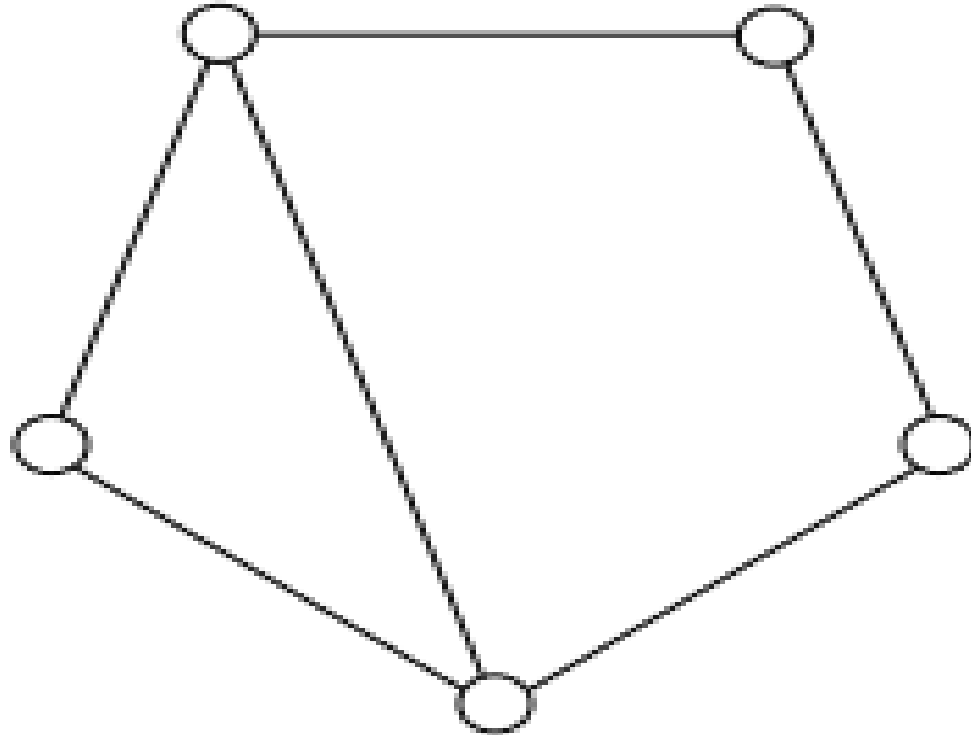
Solution:

$$\begin{aligned}
 p\left(\begin{array}{c} \circ & & \circ \\ & \diagdown & \diagup \\ & \times & \\ & \diagup & \diagdown \\ \circ & & \circ \end{array}\right) &= p\left(\begin{array}{c} \circ & & \circ \\ & \diagdown & \diagup \\ & \times & \\ & \diagup & \diagdown \\ \circ & & \circ \end{array}\right) - p\left(\begin{array}{c} & \circ & \\ & \diagdown & \diagup \\ & \times & \\ & \diagup & \diagdown \\ \circ & & \circ \end{array}\right) \\
 &= k(k-1)^3 - k(k-1)(k-2)
 \end{aligned}$$

After simplifying, we see that $p_{G_1}(k) = k(k-1)((k-1)^2 - (k-2))$.

Exercise2

- Find the chromatic polynomials of the three graphs below using reduction theorem.



Solution

$$\begin{aligned}
 p(G_2) &= p\left(\text{Diagram 1}\right) - p\left(\text{Diagram 2}\right) \\
 &= p\left(\text{Diagram 1}\right) - p\left(\text{Diagram 3}\right) - p\left(\text{Diagram 4}\right) + p\left(\text{Diagram 5}\right) \\
 &= k(k-1)^4 - k(k-1)^3 - k(k-1)(k-2)(k-1) + k(k-1)(k-2) \\
 &= k(k-1)^4 - k(k-1)^3 - k(k-1)^2(k-2) + k(k-1)(k-2)
 \end{aligned}$$

Diagram 1: A graph with 5 vertices and 6 edges. It consists of a central vertex connected to four other vertices, which are also connected in a cycle.

Diagram 2: A graph with 5 vertices and 7 edges. It consists of a central vertex connected to four other vertices, which are also connected in a cycle, and an additional edge between two of the cycle vertices.

Diagram 3: A graph with 5 vertices and 4 edges. It consists of a central vertex connected to four other vertices, which are also connected in a cycle.

Diagram 4: A graph with 5 vertices and 5 edges. It consists of a central vertex connected to four other vertices, which are also connected in a cycle, and an additional edge between two of the cycle vertices.

Diagram 5: A graph with 5 vertices and 3 edges. It consists of a central vertex connected to four other vertices, which are also connected in a cycle.

Exercise3

- Find the chromatic polynomials of the three graphs below using reduction theorem.

