Question Bank

1. Let $U=\{1,2,3,4,5,6,7,8,9\}$ be the universal set. A, B, and C are three sets: $A = \{1,3,5,7,9\}$, $B=\{2,3,4,5\}$, $C=\{5,6,7\}$. Please explicitly show the members of the following sets.

(i) $A\cap B$,
(ii) $A\cup B$,
(iii) The complement of A,
(iv) A-C,
(v) C-A,
(vi) $C\cap (A\cup B)$,
(vii) $(C \cap A) \cup (C \cap B)$
(viii) $C\cup (A\cap B)$,
(ix) $(C \cup A) \cap (C \cup B)$

2. (i) Given any two sets A and B, is it true that the set A-B is always equal to the intersection of A and the complement of B. If true, explain why it is true. If not, give a counter example.
(ii) Please apply the distributive laws of set operations and the observation in (i) to prove that the set $(X\cup Y)-Z$ is always equal to the set $(X-Z) \cup (Y-Z)$ given any sets X, Y, and Z.

3. Given any 4 sets A, B, C, and D, is it true that $(A\cup C) \cap (A \cup D) \cap (B\cup C) \cap (B \cup D)$ is always equal to $(A \cap B) \cup (C \cap D)$? If true, use the distributive laws, the associative laws, and the idempotent laws to prove it. If not, give a counter example.

Use mathematical induction to prove the following statements:
4. For every integer $n\geq 1$, $(\sum_{1\leq i \leq n} i) = n(n+1)/2$.
In other words, the summation of the first n positive integers equals n(n+1)/2.

5. For every integer $n\geq 1$, $(\sum_{1\leq i \leq n} i^3) = (n(n+1)/2)^2$.
In other words, the summation of the cubes of the first n positive integers equals the square of the summation of the first n positive integers.

6. For every integer $n\geq 0$, $(\sum_{0 \leq i \leq n} r^i) = (r^{n+1} - 1) / (r - 1)$ where r is some real number and $r \neq 1$.

7. For every integer $n \geq 5$, we have $2^n > n^2$

8. Consider three specific sets $X=\{1,2\}$, $Y=\{a, b, c\}$, $Z=\{b, c, d\}$. Show what are the following sets. (Please explicitly list the members of these sets).
(i) **Power**(X), i.e. the power set of X, (ii) $X\times Y$, (iii) $X\times(Y\cap Z)$ and (iv) $(X\times Y) \cup (X\times Z)$

9. Consider ***any*** three sets X, Y, and Z.
(i) Is $X\times(Y\cap Z)$ always equal to $(X\times Y)\cap(X\times Z)$?
(ii) Is $X\times(Y\cup Z)$ always equal to $(X\times Y) \cup (X\times Z)$?
(iii) Is **Power**$(X\cap Y)$ always equal to **Power**(X) $\cap$ **Power**(Y)
(iv) Is **Power**$(X\cup Y)$ always equal to **Power**(X) $\cup$ **Power**(Y)

10. Consider the set $A=\{p, q, r\}$ and the two binary relations
$R=\{ (p, q), (q, r), (r, p) \}$ and $S= \{(q,p) (r,q), (p, r) \}$ on A.
do the followings:
(i) Put down the matrix representations of R and S respectively.

(ii) Draw the directed graphs representing R and S respectively.
(iii) What is R○S, i.e. the composition of R with S?
(iv) What is the relation $R^2 = R \circ R$, i.e. the composition of R with R?
(v) What is the relation $R^3 = R \circ (R \circ R)$, i.e. the composition of R with R^2?

11. Continue with problem #17 above and consider the five relations R, S, R○S, R^2, and R^3.
(i) Which of them are reflexive?
(ii) Which of them are symmetric?
(iii) Which of them are antisymmetric?
(iv)Which of them are transitive?

12. For each of the following relations *R* on Z, determine whether the relation is reflexive, symmetric or transitive, and specify the equivalence classes if *R* is an equivalence relation on Z:
a) (*a*, *b*) € *R* if *a* divides *b*   b) (*a*, *b*) €*R* if *a* + *b* is even
c) (*a*, *b*) € *R* if *a* + *b* is odd   d) (*a*, *b*) € *R* if *a* = *b*
e) (*a*, *b*) €. *R* if *a*2 = *b*2 f) (*a*, *b*) . *R* if *a* < *b*

13. Let *A* = {1, 2, 4, 5, 7, 11, 13}. Define a relation *R* on *A* by writing (*x*, *y*) € *R* if and only if *x* - *y* is a multiple of 3.
a) Show that *R* is an equivalence relation on *A*.
b) How many equivalence classes of *R* are there?

14. Let *f*, *g* and *h* be functions from N to N defined by
$f(x) = 1$ if $x > 100$,
  2 if $x = 100$,
$g(x) = x^2 + 1$ and $h(x) = 2x + 1$ for every *x* € N.
a) Determine whether each function is one-to-one or onto.
b) Find *h o* (*g o f*) and (*h o g*) o *f*, and verify the Associative law for composition of functions.

15. Consider the function *f* : N →N, given by $f(x) = x + 1$ for every *x* € N.
a) What is the domain of this function?
b) What is the range of this function?
c) Is the function one-to-one?
d) Is the function onto?

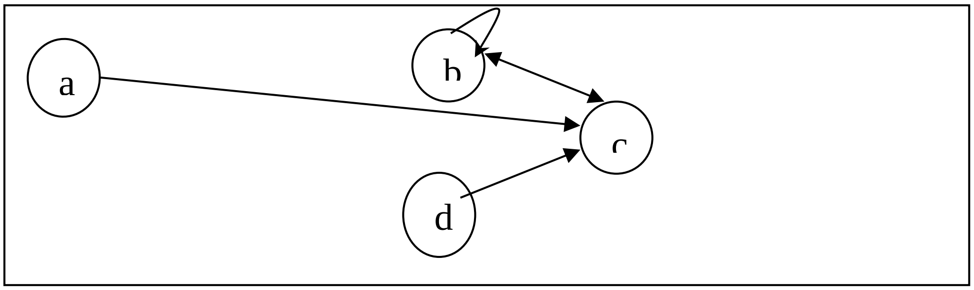
16. Let $f : A \to B$ and $g : B \to C$ be functions. Prove each of the following:
a) If *f* and *g* are one-to-one, then *g o f* is one-to-one.
b) If *g o f* is one-to-one, then *f* is one-to-one.
c) If *f* is onto and *g o f* is one-to-one, then *g* is one-to-one.
d) If *f* and *g* are onto, then *g o f* is onto.
e) If *g o. f* is onto, then *g* is onto.
f) If *g o f* is onto and *g* is one-to-one, then *f* is onto.

17. Let *A* = {1, 2} and *B* = {2, 3, 4, 5}. Write down the number of elements in each of the following sets:
a) $A \times A$
b) the set of functions from *A* to *B*
c) the set of one-to-one functions from *A* to *B*
d) the set of onto functions from *A* to *B*
e) the set of relations on *B*
f) the set of equivalence relations on *B* for which there are exactly two equivalence classes
g) the set of all equivalence relations on *B*
h) the set of one-to-one functions from *B* to *A*

i) the set of onto functions from B to A
j) the set of one-to-one and onto functions from B to B

18. *Suppose that A, B, C and D are sets, and that* $f : A \rightarrow B$, $g : B \rightarrow C$ *and* $h : C \rightarrow D$ *are functions. Then prove that* $h \circ (g \circ f) = (h \circ g) \circ f$.

19.



(i) Find the relation R determined by above directed graph.
(ii) Find the Reflexive, Symmetric and Transitive closure of R.
(iii) How many more edges are needed in the figure to extend (A,R) to a partial order.

20. Prove De Morgan's rule

$(A \cup B)^c = A^c \cap B^c$

by considering an element x of both sides of the equation. Do not use a Venn diagram.

21. Let X = {a,b,c}.Define f: X → X such that

f = {(a,b),(b,a),(c,c)}

Find :
(i) f^{-1}
(ii) f^2
(iii) f^3
(iv) f^4

22. If R and S are equivalence relations on the set A, show that the following are equivalence relation:-
(i) $R \cap S$
(ii) $R \cup S$

23. Determine which of the following statements are true in the case of three arbitrary sets P, Q, and R.

(a) If P is an element of Q and if Q is a subset of R, then P is an element of R.
(b) If P is an element of Q and if Q is a subset of R, then P is also a subset of R.
(c) If P is a subset of Q and Q is an element of R, then P is an element of R.
(d) If P is a subset of Q and Q is an element of R, then P is a subset of R.

24. Let S be a set and suppose that x *not element of* S. Define the function

$f : P(S) \rightarrow (P(S) \cup \{x\})$ by

$f(A) = A \cup \{x\}$ for all A € P(S). Is this function f one-to-one? Is this function f onto? Explain your answers.

25. let a= R X R (R is a set of real no.) and define the following relation on A: (a,b)R(c,d) iff $a^2 + b^2 = c^2 + d^2$

(i) Verify that (A,R) is an equivalence relation.
(ii) Describe what the equivalence classes are for this relation

26. Let $A = \{-2,-1,0,1,2\}$, $B = \{-2,0,1,3\}$, $C = \{0,1,4,9\}$ and define f by the formula $f(x) = x^2$. Which of the following statements are true and which false? Justify your answers.
(i) The function $f : A \to C$ is injective.
(ii) The function $f : A \to C$ is surjective.
(iii) The function $f : B \to C$ is injective.
(iv) The function $f : B \to C$ is surjective

27. Solve the recurrences:
(i) $T(n) = T(n-1) + T(n-2)$
$T(1) = T(0) = 1$
(ii) $g(n) = g(n-1) + 2n - 1$
$g(0) = 0$

28. Solve the following recurrence relations.

i. $a_n = a_{n-1} + 6a_{n-2}$; $n >= 2$ $a_0 = 1$, $a_1 = 1$
ii. $a_n = 7a_{n-1} - 10a_{n-2}$; $n >= 2$ $a_0 = 4$, $a_1 = 17$

29. Solve the following recurrence relations.

i. $a_n = 5a_{n-1} - 6a_{n-2} + 1$
ii. $a_n - 6a_{n-1} + 8a_{n-2} = 3$

30. Solve the following recurrence relations.

i. $a_{n+2} - 2a_{n+1} + a_n = 3n + 5$
ii. $a_n + 4a_{n-1} + 4a_{n-2} = n^2 - 3n + 5$
iii. $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$

31. Show that $a_n = -2^{n+1}$ is the solution of non-homogeneous recurrence relation $a_n = 3a_{n-1} + 2^n$.

32. Show that $a_n = C_1 + C_2 2^n - n$ is the solution of recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 1$.

33. Find the generating function for the following sequences:
i. 1, 1, 1, 1, 1, 1,…………………….
ii. 1, 2, 3, 4, 5, ……………………..
iii. 0, 0, 1, 1, 1, 1, 1, ……………….
iv. 1, 3, 3^2,3^3,3^4,…………………..

34. Use generating function to solve the following recurrences.
i. $a_n = 3a_{n-1}$; $n >= 1$ $a_0 = 1$
ii. $a_n = 3a_{n-1} + 2$; $n >= 1$ $a_0 = 1$
iii. $a_n - 9a_{n-1} + 20a_{n-2} = 0$; $n >= 2$ $a_0 = -3$, $a_1 = -10$
iv. $a_{n+2} - 2a_{n+1} + a_n = 2^n$ $a_0 = 2$, $a1 = 1$

35. Solve the recurrence relation $a_n = 3a_{n-1}$; $n >= 1$ with initial condition $a_0 = 1$ by **the iterative** and **recursive** approaches.

35. Solve the recurrence relation $a_n = a_{n-1} + 3$; $n >= 1$ with initial condition $a_0 = 1$ by **the iterative** and **recursive** approaches.

36. A person deposits Rs. 500 in a saving account at a bank. The interest rate is 9% per year with interest compounded annually. Let a_r be the total amount after r years. Find the recurrence relation for a_r and solve it. How much amount the person will receive after 10 years.

37. A company initially invests Rs. 10,000 in the share market. After the first year, the company decides to invest 10% of the previous year amount and additional Rs. 500. Let a_r be the amount to invest after r years. Find the recurrence relation for a_r . Also find explicit formula for the a_r.

38. Suppose that the number of bacteria in a colony triples every hour.
a) Set up a recurrence relation for the number of bacteria after *n* hours have elapsed.
b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

39. Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.
a) Set up a recurrence relation for the population of the world *n* years after 2010.
b) Find an explicit formula for the population of the world *n* years after 2010.
c) What will the population of the world be in 2030?

40. Find the smallest relation containing the relation
$\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is
a) reflexive and transitive.
b) symmetric and transitive.
c) reflexive, symmetric, and transitive.

41. Find the reflexive, symmetric and transitive closures of these relations on $\{1, 2, 3, 4\}$ using Matrix representation of relations.
a) $\{(1, 2), (2,1), (2,3), (3,4), (4,1)\}$
b) $\{(2, 1), (2,3), (3,1), (3,4), (4,1), (4, 3)\}$

42. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

43. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

44. Let m be an integer with $m > 1$. Show that the relation
$R = \{(a, b) \mid a \equiv b \pmod{m}\}$
is an equivalence relation on the set of integers.

45. What are the sets in the partition of the integers (equivalence classes) arising from congruence modulo 4?

46. Determine whether each of these functions is a bijection from **R** to **R**.
a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$

47. Let f be a function from A to B. Let S and T be subsets of B. Show that
a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

48. Prove the second distributive law from, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

49. The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

50. Draw the Venn diagrams for each of these combinations of the sets *A*, *B*, and *C*.
a) $A \cap (B - C)$
b) $(A \cap B) \cup (A \cap C)$
c) $(A \cap B') \cup (A \cap C')$

51. Show that if *A* and *B* are sets, then
a) $A - B = A \cap B$.
b) $(A \cap B) \cup (A \cap B) = A$.

52. Let A, B, C and D are four sets. Prove the following:
i. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
ii. $(A - B) \times C = (A \times C) - (B \times C)$

53. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

54. In a class,7 students like to play Basketball and 8 like to play Cricket. 3 students like to play on both Basketball and Cricket. How many students like to play Basketball, Cricket, or both? Use the Venn diagram.

55. A college has 63 students studying Political Science, Chemistry and Botany. 33 students study Political Science, 25 Chemistry and 26 Botany. 10 study Political Science and Chemistry, 9 study Botany and Chemistry while 8 study both Political Science and Botany. Same number of students study all three subjects as those who learn none of the three.
a) How many students study all the three subjects?
b) How many students study only one of the three subjects?

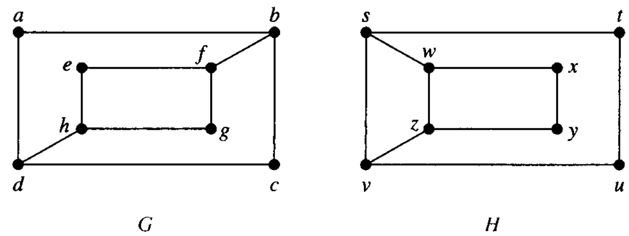
56. There are 79 Grade 10 learners at school. All of these take some combination of Maths, Geography and History. The number who take Geography is 41; those who take History is 36; and 30 take Maths. The number who take Maths and History is 16; the number who take Geography and History is 6, and there are 8 who take Maths only and 16 who take History only.

1. Draw a Venn diagram to illustrate all this information.
2. How many learners take Maths and Geography but not History?
3. How many learners take Geography only?
4. How many learners take all three subjects?

57. Prove that$\sqrt{2}$ is irrational by giving a proof by contradiction.
58. Prove the theorem “If n is an integer, then n is odd if and only if n2 is odd.”
59. Use a direct proof to show that the product of two odd numbers is odd.
60. Prove that if n is an integer and 3n+2 is even, then n is even using a) a proof by contraposition. b) a proof by contradiction.
61. Show that the square of an even number is an even number using a direct proof.
62. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
a) a proof by contraposition. b) a proof by contradiction.

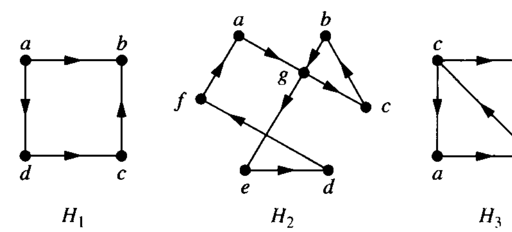
Question Bank (Graph Theory)

1. Consider the graph shown in figure.



a) Determine whether the graphs (G and H) shown in figure are isomorphic or not.
b) Represent the graph shown in figure with an incidence matrix. You can take your own incidence names (if required) for computing incidence matrix.
c) Represent adjacency matrix for the graph shown in figure.

2. Consider the graph shown in figure.



a) Which of the directed graph shown in figure have an Euler circuit? Of those that do not, which have an Euler path?
b) Represent adjacency matrix for H_2 graph.
c) Find the in-degree and out-degree of each vertex in the graph H_2 with directed edges shown in figure.
d) Out of H_1, H_2 and H_3 graphs, identify strongly and weakly connected graphs.

3. Write short note on:
   1. Complete graph
   2. Complement graph
   3. Subgraph
   4. Cubic graph
   5. Cyclic graph
   6. Bipartite Graph
   7. Wheel graph
   8. Euler Graph
   9. Hamiltonian Graph
   10. Matrix Representation of Graph
4. Discuss the necessary and sufficient condition for Euler circuit with suitable example.
5. Discuss the sufficient condition **(Dirac and Ore theorems)** for Hamilton circuit with suitable example.
6. How many vertices and how many edges do these graphs have?
   a) K_3
   b) C_5
   c) C_3
   d) W_3
   e) W_4

f) $K_{3,3}$
g) Q_2
h) Q_3

7. Apply Havel-Hakimi algorithm and determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3,3,3,3,2
b) 1,1,1,1,1

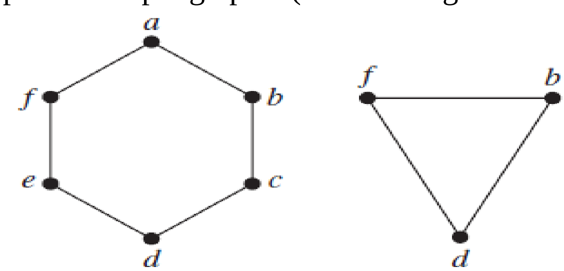
8. Prove the following:

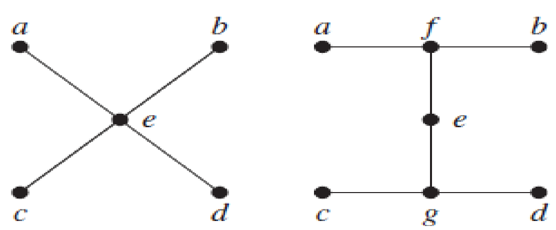
a) Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.
b) A graph is called 5-regular if every vertex has degree 5. Prove that a 5-regular graph has an even number of vertices.

9. A sequence $d1, d2, \ldots, dn$ is called **graphic** if it is the degree sequence of a simple graph. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0
b) 6, 5, 4, 3, 2, 1
c) 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2

10. Find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

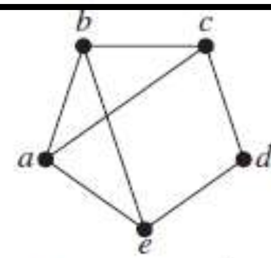




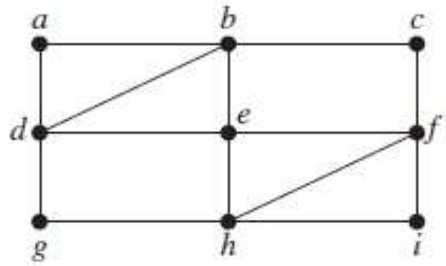
11. If the degree sequence of the simple graph G is 2, 2, 2, 1, 1, what is the degree sequence for complement of G?

12. Determine whether the given graph has an Euler circuit/Euler path.

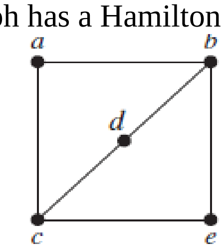
1.

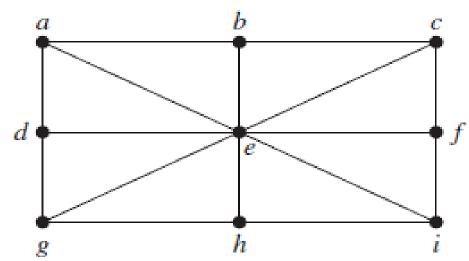


2.

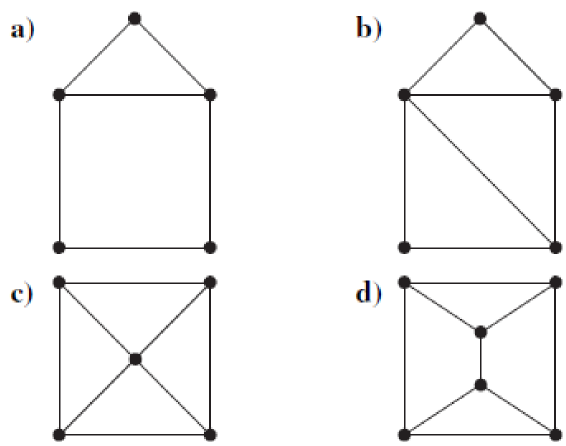


13. Determine whether the given graph has a Hamilton circuit.

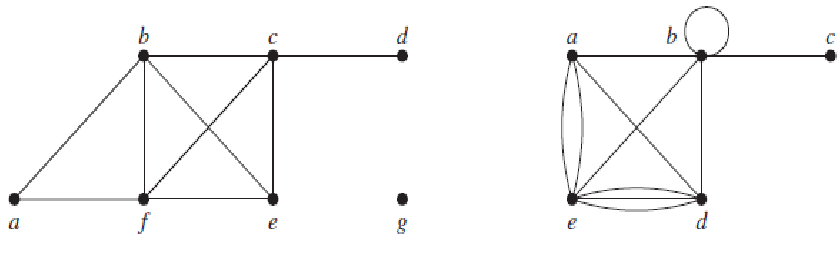




14. For each of these graphs, determine whether Dirac’s/ Ore’s theorem can be used to show that the graph has a Hamilton circuit.

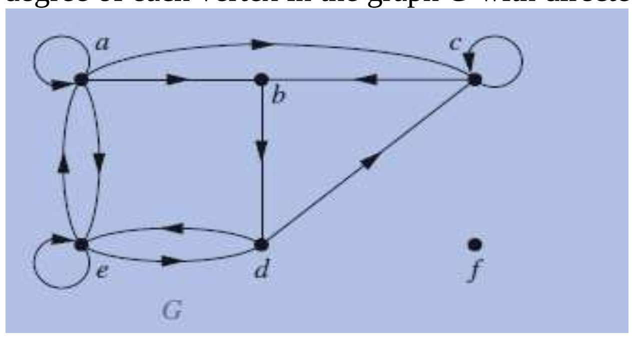


15. Prove undirected graph has even number of vertices with odd degree.
16. What are the degrees and what are the neighborhoods of the vertices in the graphs *G* and *H* displayed in Figure?



G H

17. How many edges are there in a graph with 10 vertices each of degree six?
18. Find the in-degree and out-degree of each vertex in the graph *G* with directed edges shown in Figure.

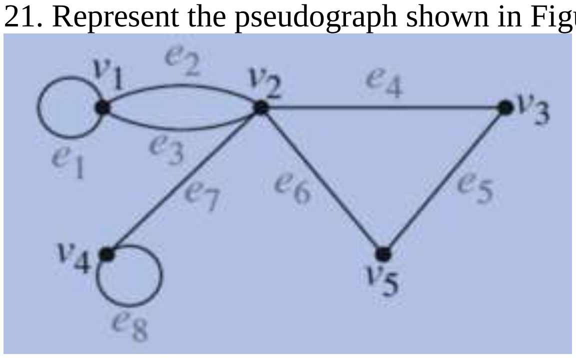


19. How many edges does a graph have if its degree sequence is 2, 2, 2, 1, 1? Draw such a graph.
20. Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

21. Represent the pseudograph shown in Figure using an incidence matrix.



22. Find the degree sequence of each of the following graphs.

a) *K*4

b) *C*4

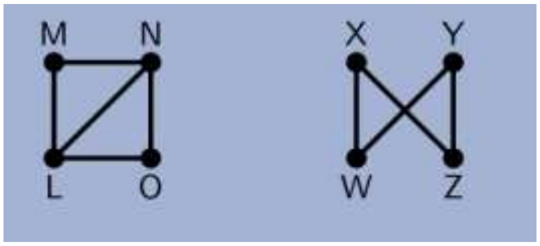
c) *W*4

d) *Q*3

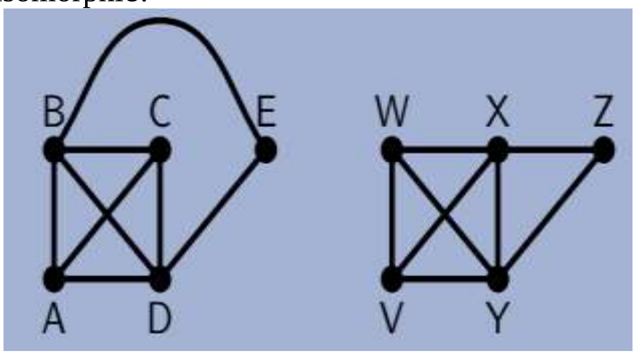
23. Prove that in a full binary tree with n vertices, the number of pendant vertices is (n+1)/2.

24. Write short note on
a. Tree
b. pendant vertices
c. isolated vertices
b. Binary Tree
c. Cut set
d. cut vertices

24. Prove a graph G with n vertices, n−1 edges and no cycles is connected.
25. Draw the graph with vertices A, B, C, D and edge set {AB, AC, AD, BC, BD} Is your graph isomorphic to one of the graphs below?



26. Are the two graphs below isomorphic?



27. Show a connected graph on n vertices has at least (n - 1) edges.
28. An acyclic graph on n vertices has at most (n - 1) edges.
29. Show that in any digraph the sum of all the outdegrees is equal to the sum of all the in-degrees.
30. If a graph has 5 vertices, can each vertex have degree 3?
31. If T is a full binary tree with i internal vertices, then T has i + 1 terminal vertices and 2i + 1 total vertices. Prove by hand shaking lemma.