Program: **B.Tech. (CSE)**

Subject name: **Discrete Mathematical Structures**

Number of credits: **3**

Relations

Combining Relations

Relations are simply… sets (of ordered pairs); subsets of the Cartesian product of two sets

Therefore, in order to <u>combine</u> relations to create new relations, it makes sense to use the usual set operations

- Intersection ($R_1 \cap R_2$)
- Union ($R_1 \cup R_2$)
- Set difference ($R_1 \setminus R_2$)

Combining Relations: Example

Let

- $A=\{1,2,3,4\}$
- $B=\{1,2,3,4\}$
- $R_1=\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2)\}$
- $R_2=\{(1,1),(1,2),(1,3),(2,3)\}$

then

- $R_1 \cup R_2 =$ ……………………….
- $R_1 \cap R_2 =$……………………….
- $R_1 \setminus R_2 =$………………………..
- $R_2 \setminus R_1 =$………………………..

Composite of Relations

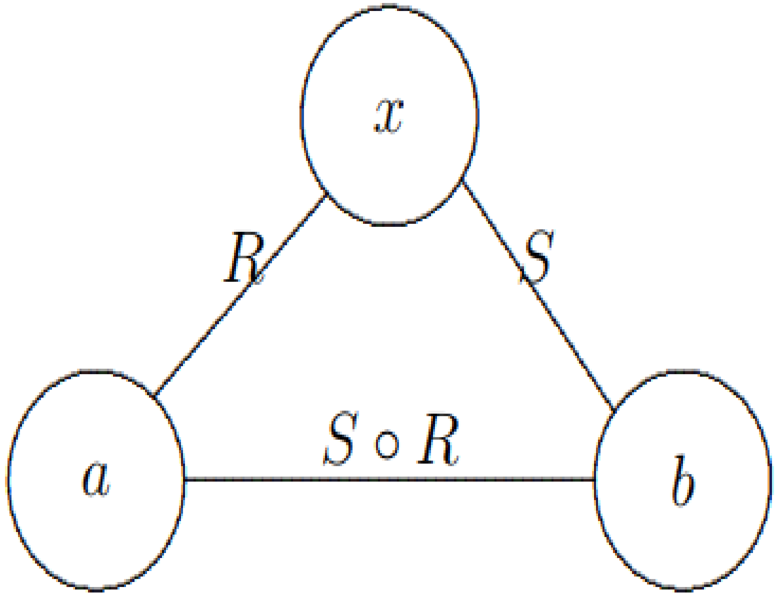
Definition: Let R_1 be a relation from the set A to B and R_2 be a relation from B to C, i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C$$

the composite of R_1 and R_2 is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_2 \circ R_1$$

In the language of the graphs composition means that we can reach b from a in two steps: an R-step from a to some element x and then an S-step from x to b.



Powers of Relations

Using the composite way of combining relations (similar to function composition) allows us to recursively define power of a relation R on a set A

Definition: Let R be a relation on A. The powers R^n, n=1,2,3,…, are defined recursively by

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

Powers of Relations: Example

Consider $R=\{(1,1),(2,1),(3,2),(4,3)\}$

$R^2=$ ……………………..

$R^3=$………………………..

$R^4=$…………………………

Note that $R^n=R^3$ for $n=4,5,6,\ldots$

Representing Relations

We have seen ways to graphically represent a function/relation between two (different) sets—Specifically a graph with arrows between nodes that are related

We will look at two alternative ways to represent relations

- 0-1 matrices (bit matrices)
- Directed graphs

0-1 Matrix

The relation R can be represented by a $(n \times m)$ sized 0-1 matrix $M_R=[m_{i,j}]$ as follows

$$m_{i,j} = \begin{cases} 1 \text{ if } (a_i,b_j) \in R \\ 0 \text{ if } (a_i,b_j) \notin R \end{cases}$$

Intuitively, the (i,j)-th entry if 1 if and only if $a_i \in A$ is related to $b_j \in B$

0-1 Matrix example

$$
\begin{array}{c|c}
 & \overbrace{\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array}}^{B} \\
A \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right. & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

Matrix Representation: Combining Relations

Combining relations is also simple: union and intersection of relations are nothing more than entry-wise Boolean opertions

Union: An entry in the matrix of the union of two relations $R_1 \cup R_2$ is 1 *iff* at least one of the corresponding entries in R_1 or R_2 is 1. Thus

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$$

Intersection: An entry in the matrix of the intersection of two relations $R_1 \cap R_2$ is 1 *iff* both of the corresponding entries in R_1 and R_2 are 1. Thus

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$$

Combining Relations: Example

What is $M_{R1 \cup R2}$ and $M_{R1 \cap R2}$?

$$M_{R1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad M_{R2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$M_{R1 \cup R2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad M_{R1 \cap R2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Composing Relations: Example

0-1 matrices are also useful for composing matrices.

$$\mathbf{M}_{R1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{M}_{R2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{M}_{R2 \circ R1} = \mathbf{M}_{R1} \cdot \mathbf{M}_{R2} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Directed Graphs Representation

Definition: A *G* graph consists of

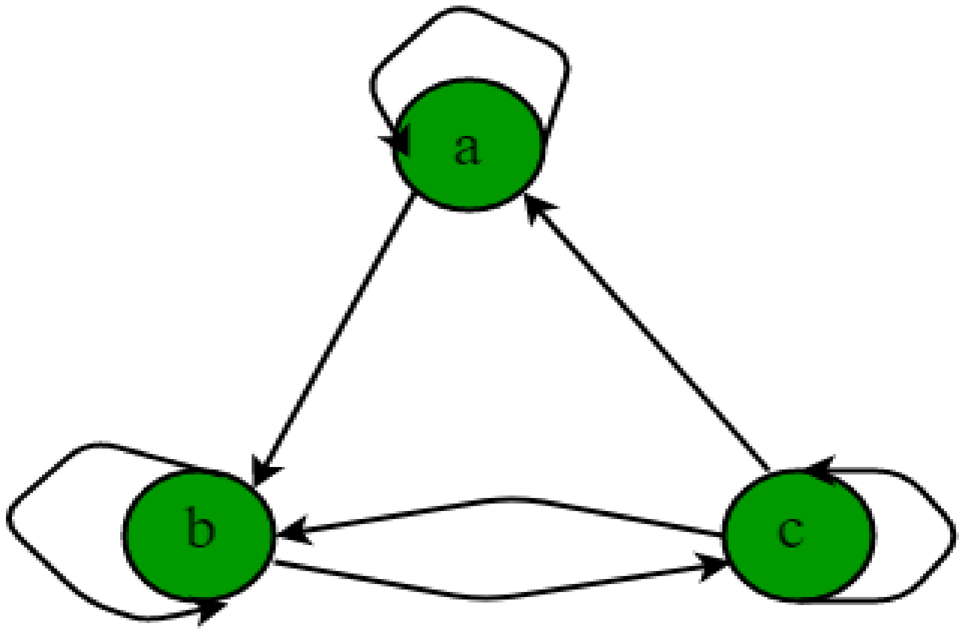
- A set *V* of vertices (or nodes), and
- A set *E* of edges (or arcs)
- We note: $G=(V,E)$

Definition: A directed *G* graph (digraph) consists of

- A set *V* of vertices (or nodes), and
- A set *E* of edges of ordered pairs of elements of V (of vertices)

Directed Graphs Representation

The directed graph of relation R = {(a,a),(a,b),(b,b),(b,c),(c,c),(c,b),(c,a)} is represented as :



Closures: Definitions

If a given relation R

- is not reflexive (or symmetric, antisymmetric, transitive)
- How can we transform it into a relation R' that is?

Example: Let $R=\{(1,2),(2,1),(2,2),(3,1),(3,3)\}$

- How can we make it reflexive?
- In general we would like to change the relation as little as possible
- To make R reflexive, we simply add (1,1) to the set

Inducing a property on a relation is called its closure.

Above, $R'=R \cup \{(1,1)\}$ is called the reflexive closure

Reflexive Closure

In general, the reflexive closure of a relation R on A is $R \cup \Delta$ where $\Delta = \{ (a,a) \mid a \in A\}$

Δ is the diagonal relation (Identity relation) on A.

The *reflexive closure* of a relation R on A is obtained by adding (a, a) to R for each $a \in A$.

Symmetric Closure

Similarly, we can create the symmetric closure using the inverse of the relation R.

The symmetric closer is, R∪R' where R' is the inverse relation i.e.

$$R' = \{ (b,a) \mid (a,b) \in R \}$$

The *symmetric closure* of R is obtained by adding (b, a) to R for each $(a, b) \in R$.

Transitive Closure

The *transitive closure* of R is obtained by repeatedly adding (a, c) to R for each $(a, b) \in R$ and $(b, c) \in R$.

The transitive closure of a relation R on a set S is the relation

$$R^* = \bigcup_{j=1}^{\infty} R^j$$

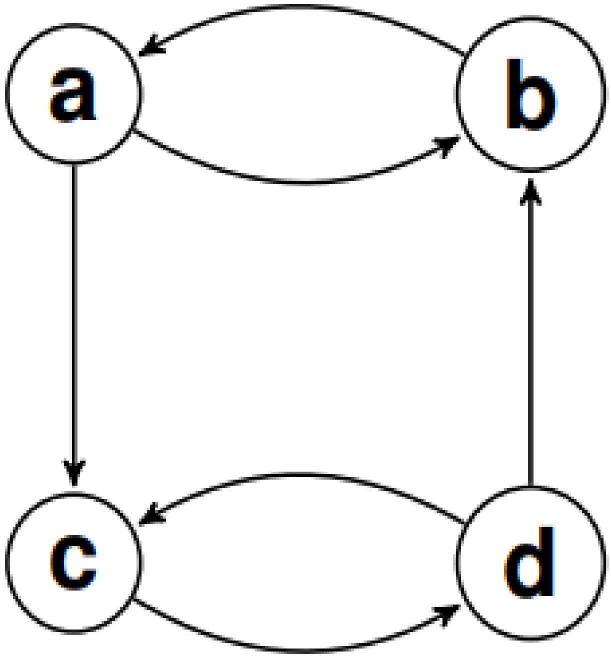
Simple Algorithm for Computing Transitive Closure

Algorithm $transitive\_closure(M_R$: zero-one $n \times n$ matrix)

$A = M_R$
$B = A$
for $i = 2$ **to** n **do**
  $A = A \odot M_R$
  $B = B \vee A$
end for
return B {B is the zero-one matrix for R^*}

Class Exercise 1

For the directed graph shown

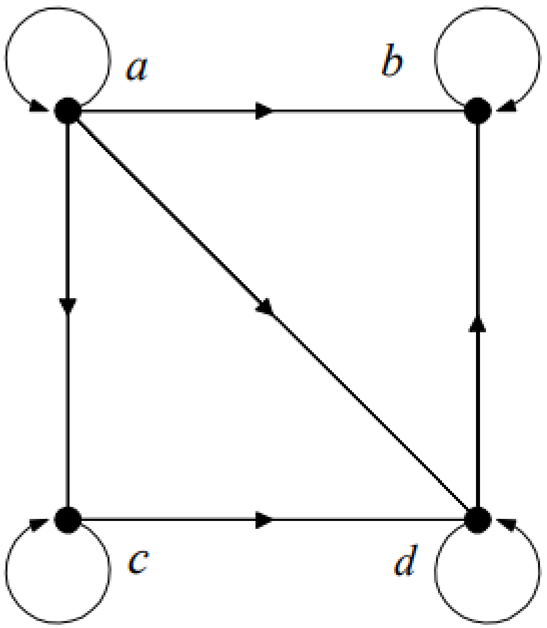


a) Find the reflexive closure

b) Find the symmetric closure

Class Exercise 2

Determine whether the relation with the directed graph shown is a partial order.



Class Exercise 3

Use Algorithm to find the transitive closure of relations on $\{1, 2, 3, 4\}$.

$\{(1,2),(2,1),(2,3),(3,4),(4,1)\}$

Class Exercise 4

Let R={ (1,2),(2,3),(3,1)} is defined on a set A={1,2,3}. Find the reflexive, symmetric and transitive Closure of R using:

1. R
2. matrix M_R
3. Graphical representation of R