Program: **B.Tech. (CSE)**

Subject name: **Discrete Mathematical Structures**

Number of credits: **3**

Relations

Outline

Relation:

- Definition, representation, relation on a set

Properties

- Reflexivity, symmetry, antisymmetric, irreflexive, asymmetric

Combining relations

- $\cap$, $\cup$, $\setminus$, composite of relations

Representing relations

- 0-1 matrices, directed graphs

Closure of relations

- Reflexive closure, diagonal relation, Transitive closure,

Introduction

- A relation between elements of two sets is a subset of their Cartesian products (set of all ordered pairs)
- **Definition**: A binary <u>relation</u> from a set A to a set B is a subset $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B \}$
- Relation versus function
  - In a relation, each $a \in A$ can map to <u>multiple</u> elements in B
  - Relations are more general than functions
- When $(a,b) \in R$, we say that a is <u>related</u> to b.
- Notation: $a R b$, $a \not{R} b$

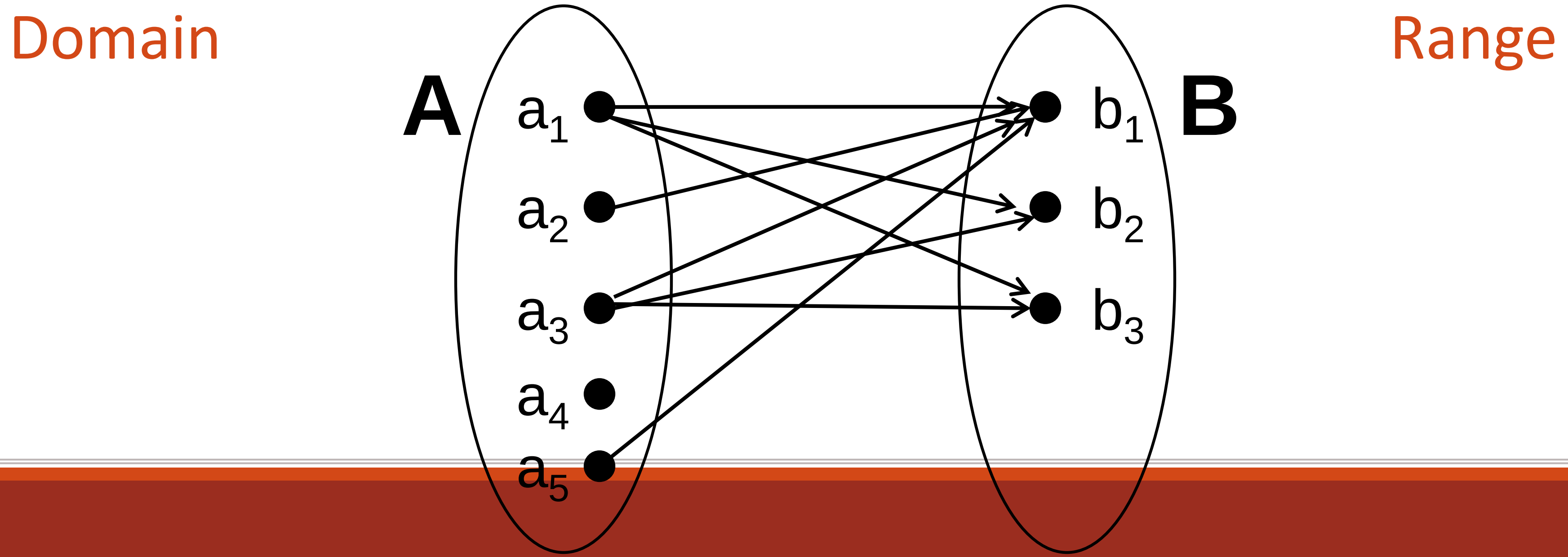
Relations: Representation

Example

- Let $A=\{a_1,a_2,a_3,a_4,a_5\}$ and $B=\{b_1,b_2,b_3\}$
- Let R be a relation from A to B defined as follows

$$R=\{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_5,b_1)\}$$

We can represent this relation graphically



Relations on a Set

Definition: A relation on the set A is a relation from A to A and is a subset of $A \times A$

Example: The following are binary relations on $\mathbb{N}$

$$R_1=\{ (a,b) \mid a \leq b \}$$

$$R_2=\{ (a,b) \mid a,b \in \mathbb{N},\ a/b \in \mathbb{Z} \}$$

$$R_3=\{ (a,b) \mid a,b \in \mathbb{N}, a-b=2 \}$$

Types of relation

- Reflexive
- Symmetric
- Transitive
- Antisymmetric
- Asymmetric
- Equivalence Relation
- Partial Order Relation

Reflexive Relation

In a relation on a set, if all ordered pairs (a,a) for every $a \in A$ appears in the relation, R is called reflexive

Definition: A relation *R* on a set A is called reflexive iff

$$\forall a \in A \ (a,a) \in R$$

Example: <=

Reflexivity: Examples

Recall the relations below, which is reflexive?

$$R_1=\{ (a,b) \mid a \leq b \}$$

$$R_2=\{ (a,b) \mid a,b\in \mathbb{N},\ a/b\in \mathbb{Z} \}$$

$$R_3=\{ (a,b) \mid a,b\in \mathbb{N}, a-b=2 \}$$

- R_1 is reflexive since for every $a\in \mathbb{N}$, $a \leq a$
- R_2 is reflexive since $a/a=1$ is an integer
- R_3 is not reflexive since $a-a=0$ for every $a\in \mathbb{N}$

Symmetric Relation

Definitions:

- A relation R on a set A is called symmetric if

$$\forall a,b \in A\ (\ (b,a) \in R \Leftrightarrow (a,b) \in R\ )$$

- Example: ||

So, Symmetric relation -

If (x,y) is in set, then (y,x) must be in the set.

Asymmetric Relation

A relation is **asymmetric** if, for every $(a,b) \in R$, then $(b,a) \notin R$

- Asymmetry is the opposite of symmetry

Examples of asymmetric relations:

- <, >

so, Asymmetric relation -

If (x,y) is in set then, (y,x) must not be in the set.

(x,x) must not be in the set.

Antisymmetric relation

- A relation R on a set A is called **antisymmetric** if

$$\forall a,b \in A \ [(a,b) \in R \land (b,a) \in R \Rightarrow a=b]$$

Example: Subset, /

So, Antisymmetric relation -

If (x,y) is in set, then (y,x) must not be in the set.

(x,x) may or may not be in the set.

Every asymmetric relation is also antisymmetric. But if antisymmetric relation contains pair of the form (a,a) then it cannot be asymmetric.

an asymmetric relation is just one that is both antisymmetric and irreflexive.

Transitive Relation

Definition: A relation R on a set A is called <u>transitive</u> if whenever $(a,b)\in R$ and $(b,c)\in R$ then $(a,c)\in R$ for all $a,b,c \in A$

$$\forall a,b,c\in A\ ((aRb) \wedge (bRc)) \Rightarrow aRc$$

EXAMPLE: /, ||

Properties of Relations

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)}	symmetric
R = {(1, 1)}	sym. and antisym.
R = {(1, 3), (3, 2), (2, 1)}	antisym. and asym.
R = {(4, 4), (3, 3), (1, 4)}	antisym.

Equivalence Relation

A relation is Equivalence Relation if it is

i. Reflexive

ii. Symmetric

iii. Transitive

EXAMPLE: ||

Partial Order Relation

A relation is Partial Order Relation if it is

i. Reflexive

ii. Antisymmetric

iii. Transitive

example: <=, /

Congruence relation modulo n

Definition. Given two integers a, b and a natural number $n > 1$. We say that a *is congruent to* b *modulo* n and write $a \equiv b \pmod{n}$ if $a - b$ is divisible by n.

For example, $29 \equiv 8 \bmod 7$, and $60 \equiv 0 \bmod 15$.

The Relation Modulo n is an Equivalence Relation on $\mathbb{Z}$.

Let a, b, and c be integers. Then

1. $a \equiv a \pmod{n}$ (modulo n is reflexive);
2. if $a \equiv b \pmod{n}$, then also $b \equiv a \pmod{n}$ (modulo n is symmetric);
3. if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$ (modulo n is transitive).

Equivalence Class

If $\sim$ is an equivalence relation on S and $x \in S$, the set

$$E_x = \{y \in S : x \sim y\}$$

Every equivalence relation on a set generates a partition of the set, precisely the partition into its equivalence classes.

Example: on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ the relation $x \equiv y$ defined by

$$3 \;\; divides \;\; x - y$$

is an equivalence relation

We can now form $X/\equiv$ which is the set of all *equivalence classes*

$$[1] = [4] = \{1, 4, 7, 10\} \qquad [2] = [8] = \{2, 5, 8\} \qquad [3] = \{3, 6, 9\}$$

notations, $[x]$ and, $\bar{x}$.

Example: Congruence relation modulo 3. The congruence relation modulo 3 is an equivalence relation with the following equivalence classes:

$$E_0 = \bar{0} = \{0, 3, 6, 9, \ldots, -3, -6, -9, \ldots\},$$

$$E_1 = \bar{1} = \{1, 4, 7, 10, \ldots, -2, -5, -8, \ldots\},$$

$$E_2 = \bar{2} = \{2, 5, 8, 11, \ldots, -1, -4, -7, \ldots\}.$$

Every integer belongs to exactly one of these three classes, and any other equivalence class $\bar{x}$ is equal to one of the above classes. For example, the equivalence class of 3, i.e., $\bar{3}$, is the same as $\bar{0}$.

Class Exercise 1

Let $A=\{0,1,2,3\}$ and a relation R on A be given by

$$R=\{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}.$$

Is R reflexive? symmetric? transitive?

Class Exercise 2

For $x, y \in \mathbf{R}$ define $x \simeq y$ to mean that $x - y \in \mathbf{Z}$. Prove that $\simeq$ is an equivalence relation on $\mathbf{R}$.

Class Exercise 3

Let A={1,2,3....9} and R be the relation in AxA defined by (a,b)R(c,d) if a+d=b+c for (a,b),(c,d) in AxA.

Prove that R is an equivalence relation and also obtain the equivalent class[(2,5)].

Class Exercise 4

Let A = {1,2,3,4,5,6}, and consider the following equivalence relation on A:

R = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)}.

List the equivalence classes of R.

Class Exercise 5

Define a relation R on Z as xR y if and only if 3x – 5y is even.

Prove R is an equivalence relation. Describe its equivalence classes.