Functions

Outline

- Definitions & terminology
  - function, domain, co-domain, image, preimage (antecedent), range, image of a set, strictly increasing, strictly decreasing, monotonic
- Properties
  - One-to-one (injective), onto (surjective), one-to-one correspondence (bijective)
- Inverse functions (examples)
- Operators
  - Composition, Equality
- Important functions
  - identity, absolute value, floor, ceiling, factorial

Definition: Function

- **Definition**: A function f from a set A to a set B is an assignment of <u>exactly one</u> element of B to <u>each element</u> of A.
- We write f(a)=b if b is the unique element of B assigned by the function f to the element a$\in$A.
- If f is a function from A to B, we write

$$f: A \rightarrow B$$

  This can be read as 'f maps A to B'
- Note

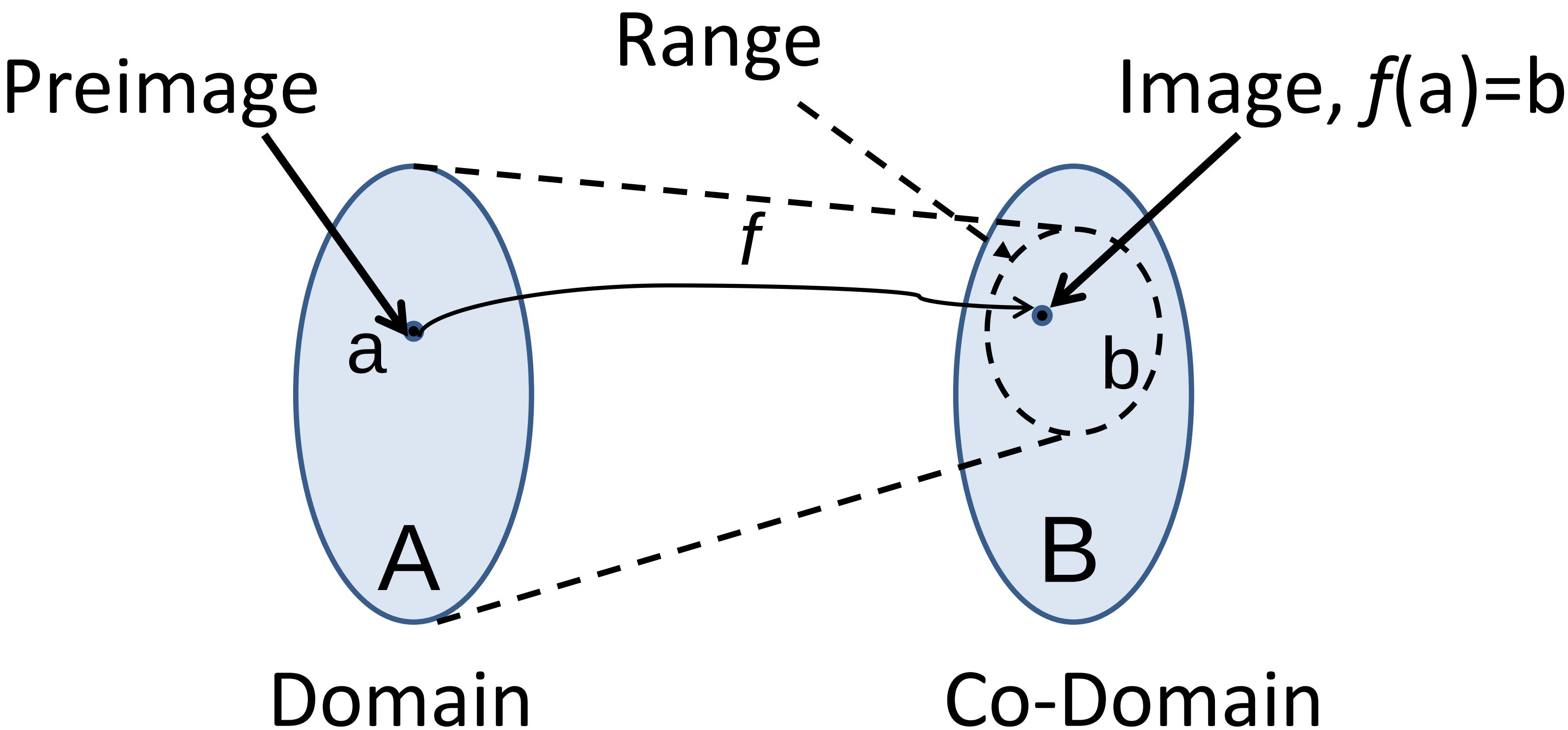
-Each and every element of A has a <u>single</u> mapping

-Each element of B may be mapped to by <u>several</u> elements in A or <u>not at all</u>

Terminology

- Let f: A $\rightarrow$ B and f(a)=b. Then we use the following terminology:
  - A is the <u>domain</u> of f, denoted dom(f)
  - B is the <u>co-domain</u> of f
  - b is the <u>image</u> of a
  - a is the <u>preimage</u> (<u>antecedent</u>) of b
  - The <u>range</u> of f is the set of all images of elements of A, denoted rng(f)

Function: Visualization



A function, f: A $\rightarrow$ B

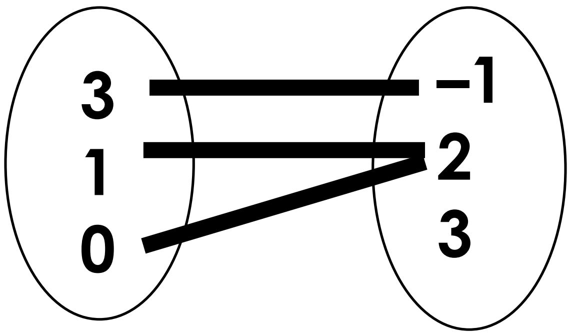
More Definitions

- **Definition**: Let f_1 and f_2 be two functions from a set A to $\mathbb{R}$. Then f_1+f_2 and $f_1 f_2$ are also function from A to R defined by:
  - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
  - $f_1 f_2(x)= f_1(x) f_2(x)$
- Example: Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
  - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
  - $f_1 f_2(x) = (x^4+2x^2+1)(2-x^2)= -x^6+3x^2+2$

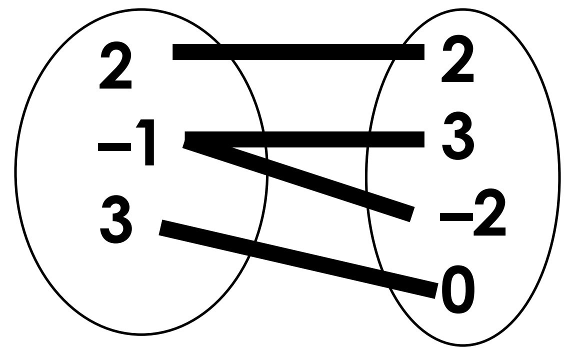
Example

Which mapping represents a function?

Choice One



Choice Two



?

Example

- Let:
  - $A = \{a_1,a_2,a_3,a_4,a_5\}$
  - $B = \{b_1,b_2,b_3,b_4,b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
- Draw a diagram for *f*
- What is the:
  - Domain, co-domain, range of *f*?
  - Image of S?

CLASSIFICATION OF FUNCTIONS

Definition: Injection

- **Definition**: A function f is said to be <u>one-to-one</u> or <u>injective</u> (or an injection) if

$$\forall \text{ x and y in in the domain of } f,\ f(x)=f(y) \Rightarrow x=y$$

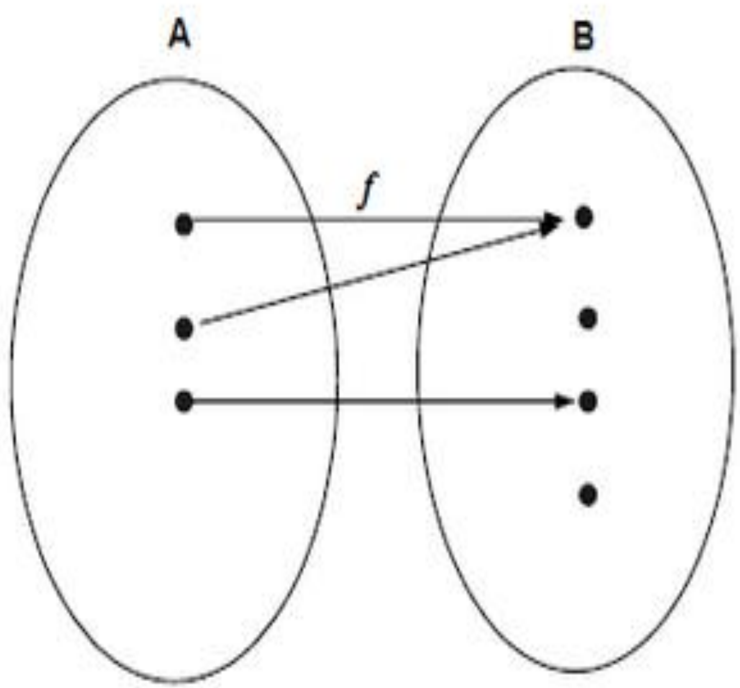
- Intuitively, an injection simply means that each element in the range has <u>at most</u> one preimage (antecedent)
- It may be useful to think of the contrapositive of this definition

$$x \neq y \Rightarrow f(x) \neq f(y)$$

Definition: Into

INTO FUNCTION: A function f: $X \rightarrow Y$ is said to be an into function if there exists at least one element in the co – domain Y which is not an image of any element in the domain X.

Into



Definition: Surjection

- **Definition**: A function *f*: A→B is called <u>**onto**</u> or <u>surjective</u> (or an surjection) if

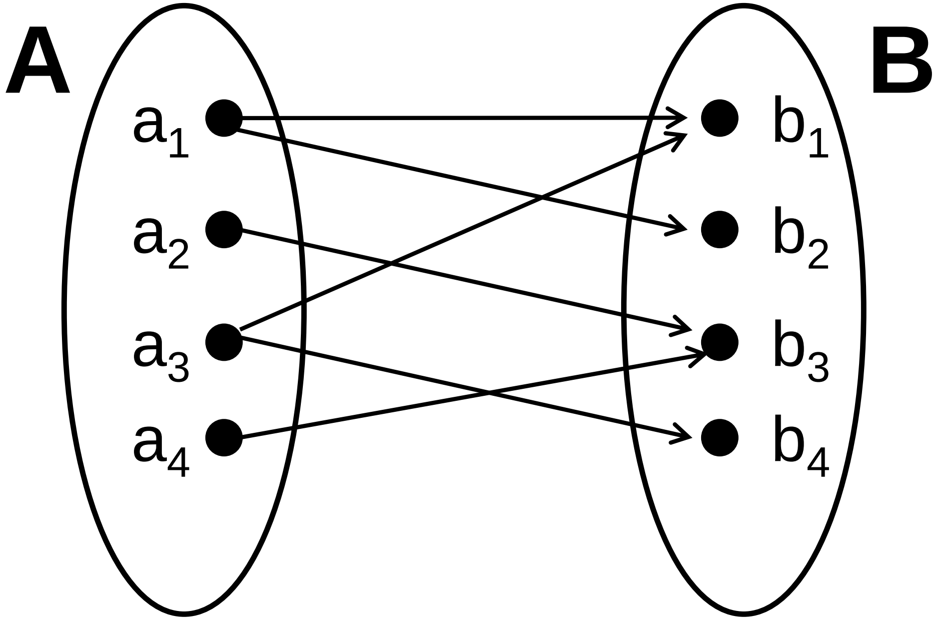
$$\forall\, b\in B,\ \exists\, a\in A \text{ with } f(a)=b$$

- Intuitively, a surjection means that every element in the codomain is mapped.
- Thus, the range is the same as the codomain

Definition: Bijection

- **Definition**: A function *f* is a one-to-one correspondence (or a bijection), if is both one-to-one (injective) and onto (surjective).

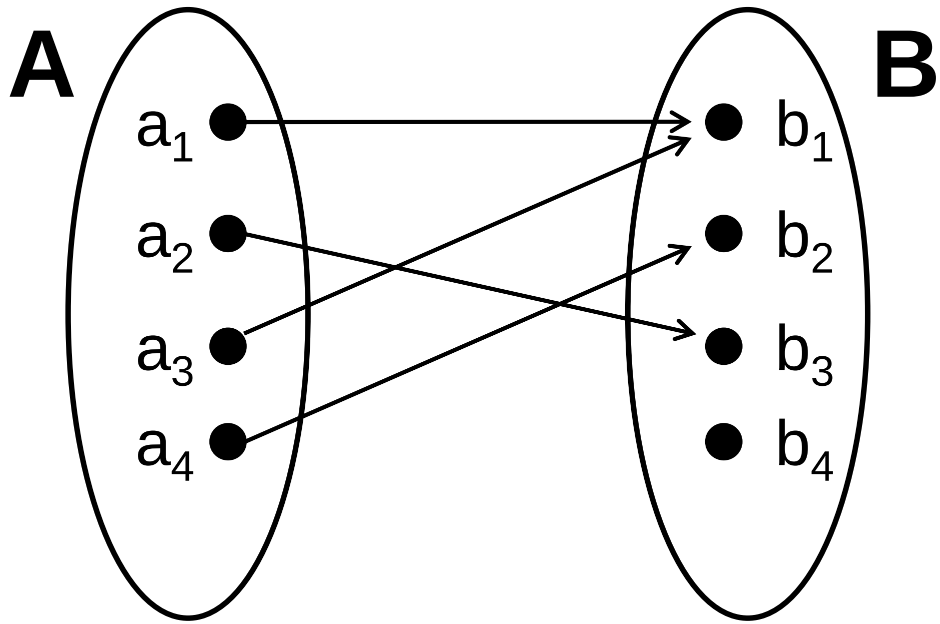
Functions: Example 1



- Is this a function? Why?

- No, because each of a_1, a_2 has two images

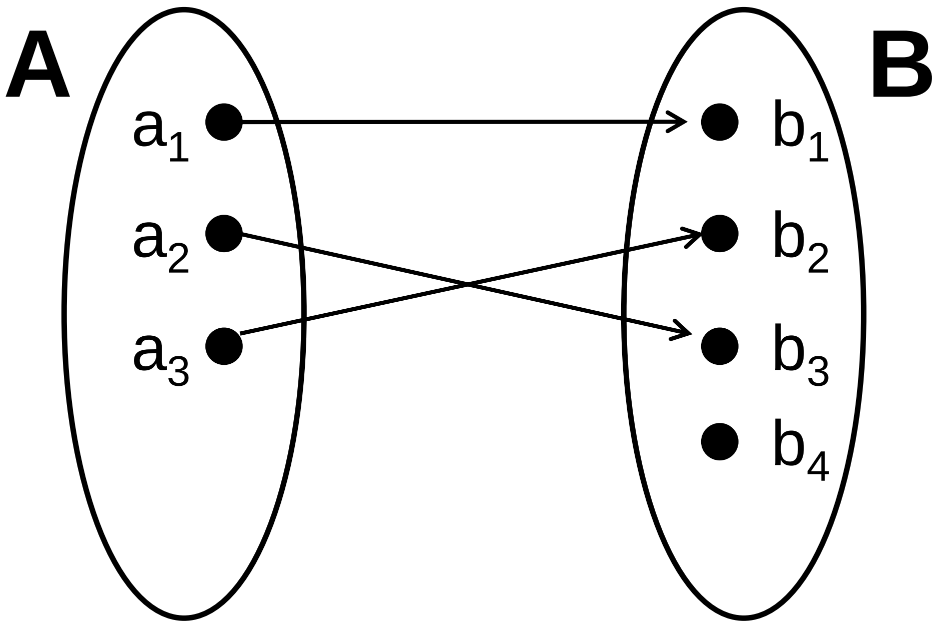
Functions: Example 2



- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

- No, b_1 has 2 preimages
- No, b_4 has no preimage

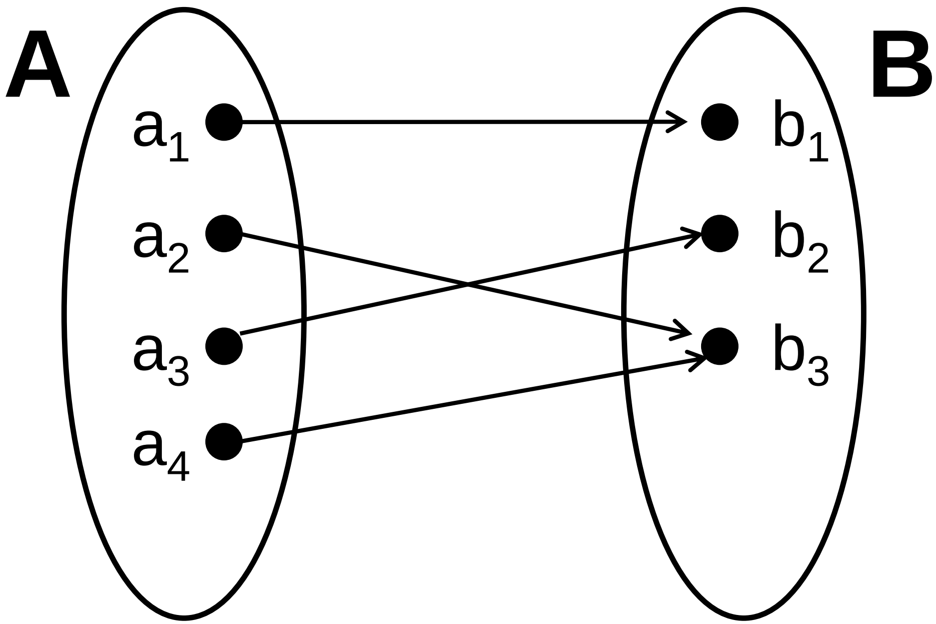
Functions: Example 3



- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

- Yes, no b_i has 2 preimages
- No, b_4 has no preimage

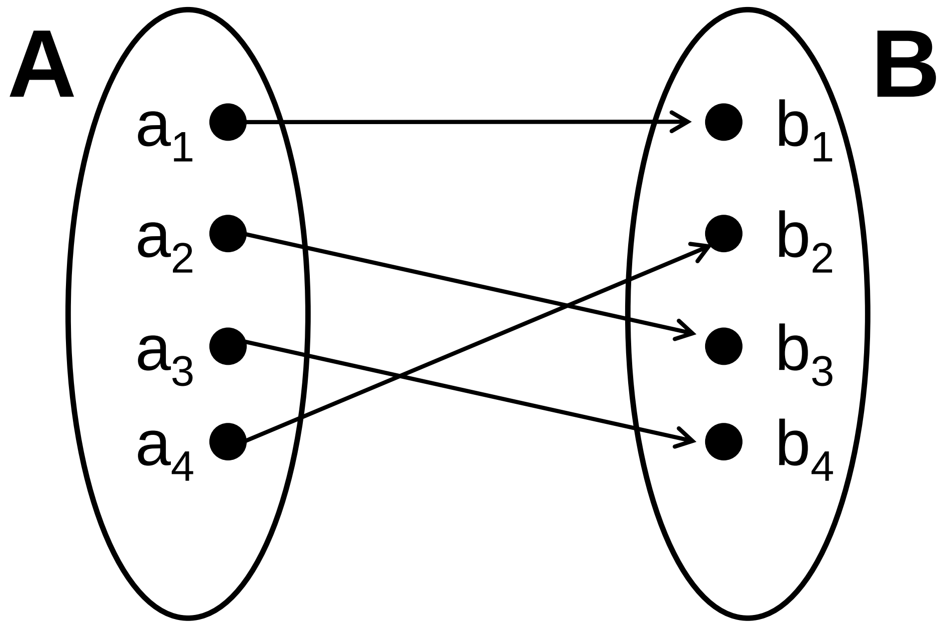
Functions: Example 4



- Is this a function
  - One-to-one (injective)? Why?
  - Onto (surjective)? Why?

- No, b_3 has 2 preimages
- Yes, every b_i has a preimage

Functions: Example 5



- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

Thus, it is a bijection or a

one-to-one correspondence

Proving Injectivity

To prove that a function $f : A \to B$ is injective

$$(\forall x, y \in A)[f(x) = f(y) \implies x = y]$$

This translates into a proof of the following form:

Let $x, y \in A$ be given.
Assume $f(x) = f(y)$.
... [Logical deductions] ..
Therefore $x = y$.
Hence f is injective.

Proving surjectivity

The definition of surjectivity of a function $f : A \to B$ is:

$$(\forall b \in B)(\exists a \in A)[f(a) = b]$$

This translates into a proof of the following form:

Let $b \in B$ be given.
... [Find an a that maps into the given element b.] ...
... [Show that $f(a) = b$.] ...
Hence f is surjective.

Exercice 1

- Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by

$$f(x)=2x-3$$

- What is the domain, codomain, range of f?
- Is f one-to-one (injective)?
- Is f onto (surjective)?
- Clearly, $\mathrm{dom}(f)=\mathbb{Z}$. To see what the range is, note that:

$$\begin{aligned} b \in \mathrm{rng}(f) &\Leftrightarrow b=2a-3, \text{ with } a \in \mathbb{Z} \\ &\Leftrightarrow b=2(a-2)+1 \\ &\Leftrightarrow b \text{ is odd} \end{aligned}$$

Exercise 1 (cont'd)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., rng(f) $\neq \mathbb{Z}$), we can conclude that f is not onto (surjective)
- However, f is one-to-one injective. Using simple algebra, we have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1 - 3 = 2x_2 - 3 \Rightarrow x_1 = x_2$$

Exercise 2

- Let f be as before

$$f(x)=2x-3$$

  but now we define $f:\mathbb{N}\rightarrow\mathbb{N}$
- What is the domain and range of f?
- Is f onto (surjective)?
- Is f one-to-one (injective)?

- f is not even a function anymore. Indeed, $f(1)=2\cdot 1-3=-1\notin\mathbb{N}$

Inverse Functions (1)

- **Definition**: Let f: $A \rightarrow B$ be a bijection. The <u>inverse</u> function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a)=b$
- The inverse function is denote f^{-1}.
- When f is a bijection, its inverse exists and

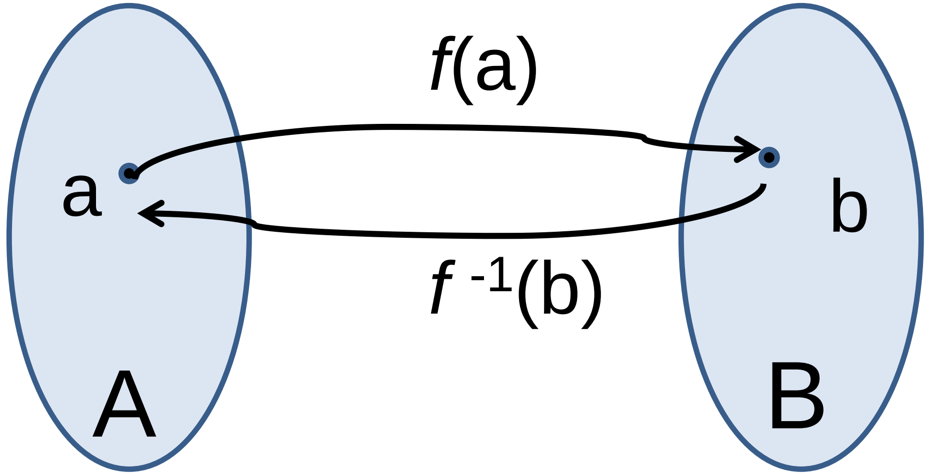
$$f(a)=b \Leftrightarrow f^{-1}(b)=a$$

Inverse Functions (2)

- Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is <u>invertible</u>

- Why must a function be bijective to have an inverse?
  - Consider the case where f is not one-to-one (not injective). This means that some element $b \in B$ has more than one antecedent in A, say a_1 and a_2. How can we define an inverse? Does $f^{-1}(b)=a_1$ or a_2?
  - Consider the case where f is not onto (not surjective). This means that there is some element $b \in B$ that does not have any preimage $a \in A$. What is then $f^{-1}(b)$?

Inverse Functions: Representation



Domain

Co-Domain

A function and its inverse

Inverse Functions: Example

- Let f:$\mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = 2x - 3$$

- What is f^{-1}?

1. We must verify that *f* is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.
2. To find the inverse, we use the substitution

Let $f^{-1}(y)=x$

And $y=2x-3$, which we solve for x. Clearly, $x=(y+3)/2$

So, $f^{-1}(y)=(y+3)/2$

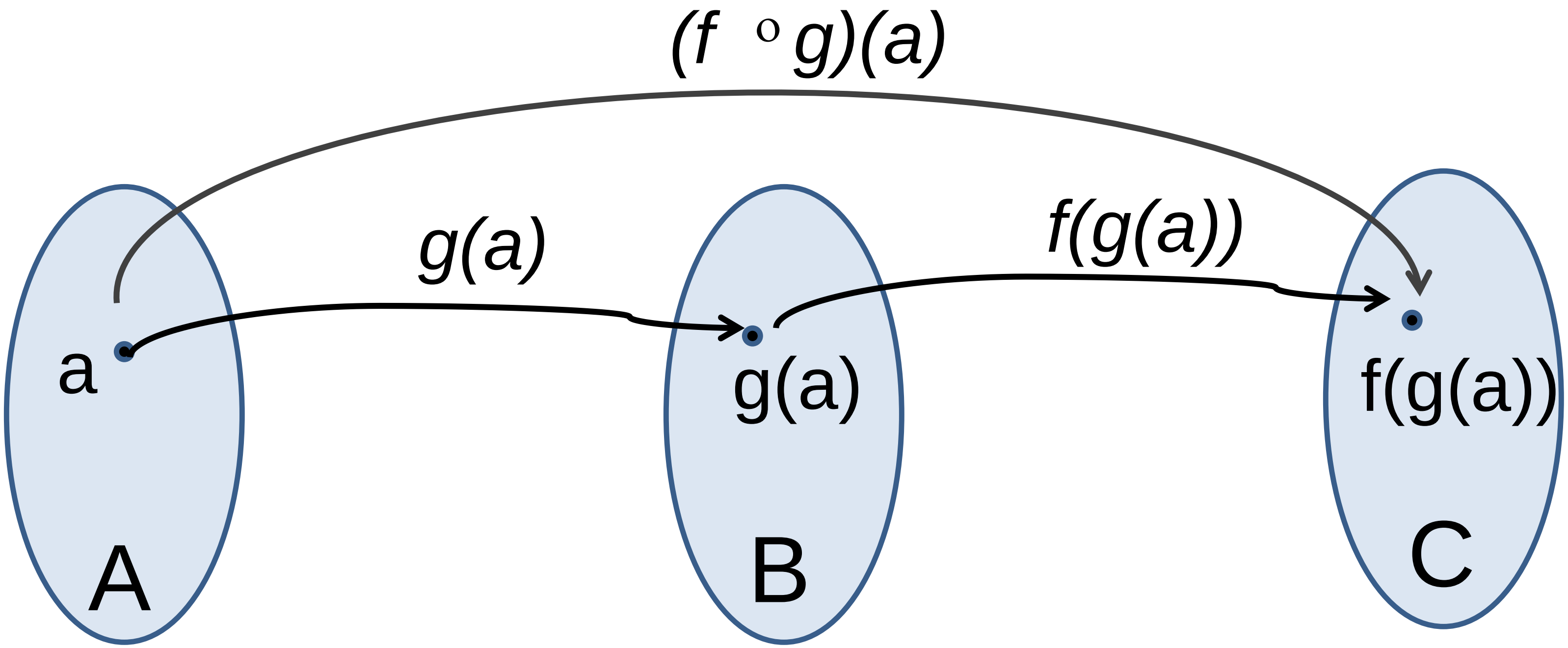
Function Composition

- The value of functions can be used as the input to other functions
- **Definition**: Let g:A$\rightarrow$B and f:B $\rightarrow$C. The <u>composition</u> of the functions f and g is

$$(f \circ g)\,(x)=f(g(x))$$

- $f \circ g$ is read as 'f circle g', or 'f composed with g', 'f following g', or just 'f of g'

Composition: Graphical Representation



The composition of two functions

Composition: Example 1

- Let f, g be two functions on $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 2x - 3$$

$$g(x) = x^2 + 1$$

- What are $f \circ g$ and $g \circ f$?

Composition: Example 1 (cont')

- Given $f(x) = 2x – 3$ and $g(x) = x^2 + 1$
- $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$
  $= 2x^2 - 1$
- $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 +1$
  $= 4x^2 - 12x + 10$

Function Equality

Two functions *f* and *g* are <u>equal</u> if and only

- dom(*f*) = dom(*g*)
- $\forall$ a $\in$dom(*f*) (*f*(a) = *g*(a))

Example: 2x and x+x defined for set of integers

Associativity

- The composition of function is not commutative ($f \circ g \neq g \circ f$), it is associative
- **Lemma**: The composition of functions is an associative operation, that is

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Important Functions: Identity

- **Definition**: The <u>identity</u> function on a set A is the function

$$\iota: A \rightarrow A$$

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defined by $\iota(a)=a$ for all $a \in A$.

- One can view the identity function as a composition of a function and its inverse:

$$\iota(a) = (f \circ f^{-1})(a) = (f^{-1} \circ f)(a)$$

- Moreover, the composition of any function f with the identity function is itself f:

$$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$

Inverses and Identity

- The identity function, along with the composition operation, gives us another characterization of <u>inverses</u> when a function has an inverse
- **Theorem**: The functions f: A$\rightarrow$B and g: B$\rightarrow$A are <u>inverses</u> if and only if

$$(g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B$$

where the ι_A and ι_B are the identity functions on sets A and B.

Class Exercise 1

Let f: R→R, g: R→R, h: R→R. If f(x) = x2+2, g(x) = 3x-1, h(x) = 1-x2.

Find

a) ho(gof)(-5)

b) (hog)of (-5)

c) rules for ho(gof).

Class Exercise 2

Prove the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n + 4$ is injective.

Class Exercise 3

Prove the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) = \ln(x)$ is surjective.

Class Exercise 4

Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if f and g are injective, then $g \circ f$ is injective.

(b) Prove that if f and g are surjective, then $g \circ f$ is surjective.