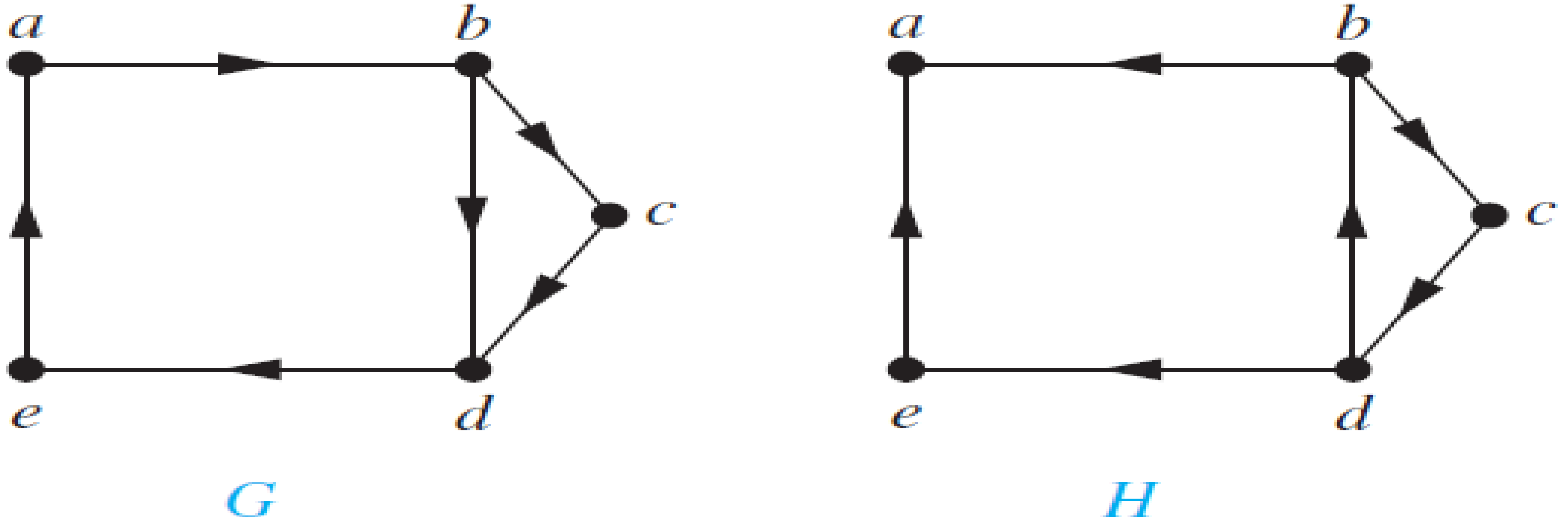
Graph Theory

Connectedness in Directed Graphs

- A directed graph is ***strongly connected*** if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph.
- A directed graph is **weakly connected** if and only if there is always a path between two vertices when the directions of the edges are disregarded.



G is strongly connected because there is a path between any two vertices in this directed graph (the reader should verify this). Hence, *G* is also weakly connected. The graph *H* is not strongly connected. However, *H* is weakly connected.

The Havel-Hakimi Algorithm

Take as input a degree sequence S and determine if that sequence is graphical.

That is, can we produce a graph with that degree sequence?

Assume the degree sequence is S

$$S = d_1, d_2, d_{3,} \cdots, d_n$$

$$d_i \geq d_{i+1}$$

1. If any $d_i \geq n$ then fail
2. If there is an odd number of odd degrees then fail
3. If there is a $d_i < 0$ then fail
4. If all $d_i = 0$ then report success
5. Reorder S into non-increasing order
6. Let $k = d_1$
7. Remove d_1 from S.
8. Subtract 1 from the first k terms remaining of the new sequence
9. Go to step 3 above

Example

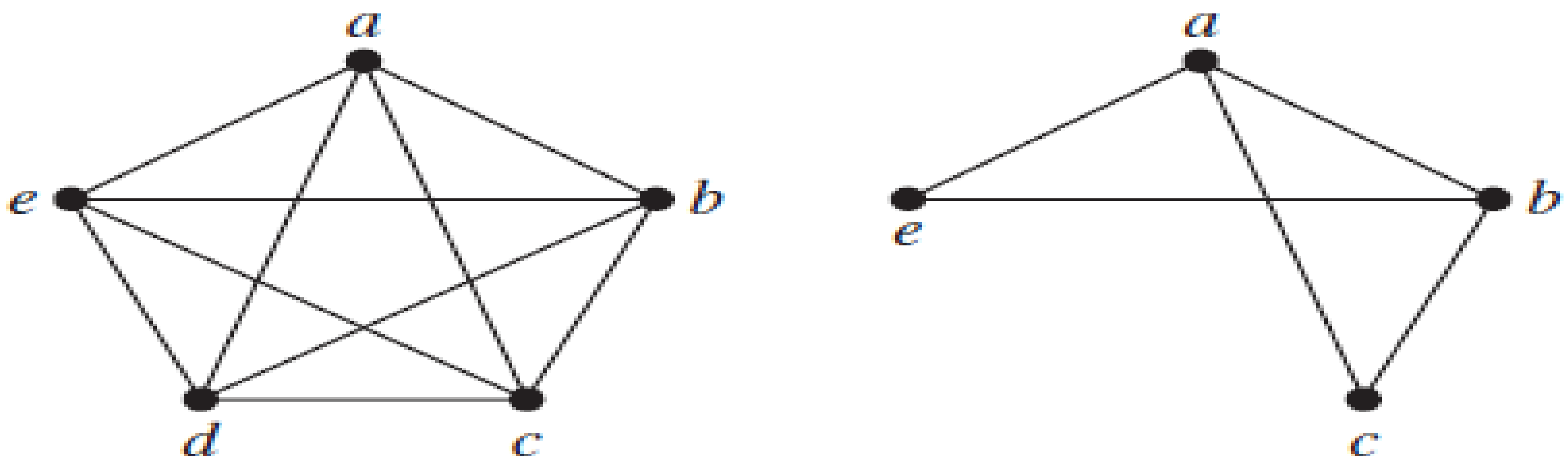
- Consider the sequence $5, 3, 3, 3, 2, 2, 1, 1$
- Is it graphical?
- Applying the Havel-Hakimi theorem
    - $5, 3, 3, 3, 2, 2, 1, 1$ is graphical iff
    - $2, 2, 2, 1, 1, 1, 1$ is graphical iff
    - $1, 1, 1, 1, 1, 1$ is graphical
- This last sequence is graphical!
- Hence, $5, 3, 3, 3, 2, 2, 1, 1$ is graphical

Example

- Is the sequence $6, 5, 5, 5, 4, 4, 2, 1$ graphical?
- Applying the Havel-Hakimi theorem
  - $6, 5, 5, 5, 4, 4, 2, 1$ is graphical iff
  - $4, 4, 4, 3, 3, 1, 1$ is graphical iff
  - $3, 3, 2, 2, 1, 1$ is graphical iff
  - $2, 1, 1, 1, 1$ is graphical iff
  - $0, 0, 1, 1$ is graphical
- This last sequence is graphical!

Sub Graph

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H = G$.



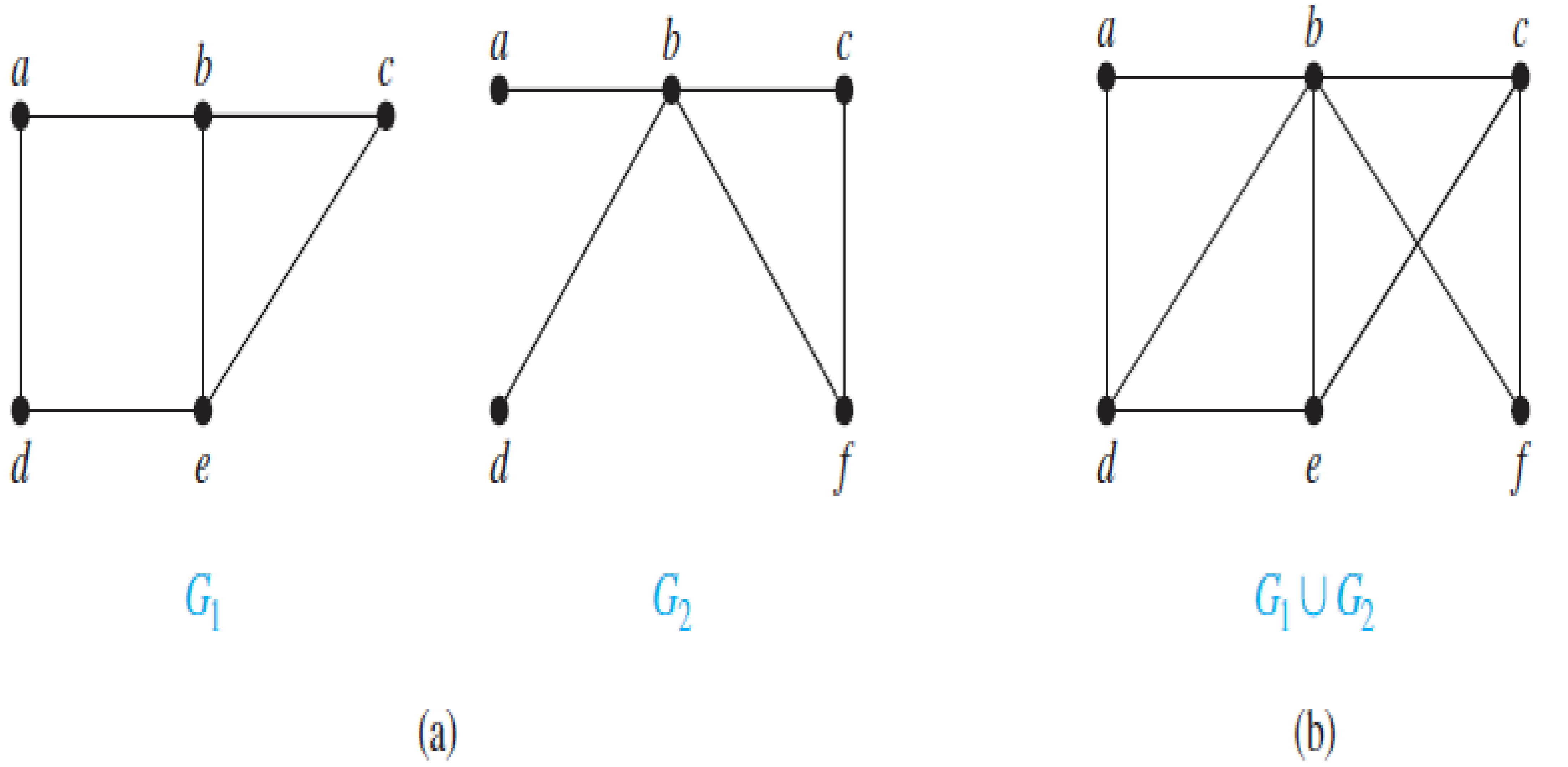
A Subgraph of K_5.

1. Every graph is its own subgraph.
2. If Z is a subgraph of Y and Y is a subgraph of X then Z is also a subgraph of X.
3. A single vertex of a graph is also its subgraph.
4. A single edge of a graph together with its incident vertices is also its subgraph.

Graph Operations

GRAPH UNIONS Two or more graphs can be combined in various ways. The new graph that contains all the vertices and edges of these graphs is called the **union** of the graphs.

The *union* of two simple graphs $G1 = (V1, E1)$ and $G2 = (V2, E2)$ is the simple graph with vertex set $V1 \cup V2$ and edge set $E1 \cup E2$. The union of $G1$ and $G2$ is denoted by $G1 \cup G2$.

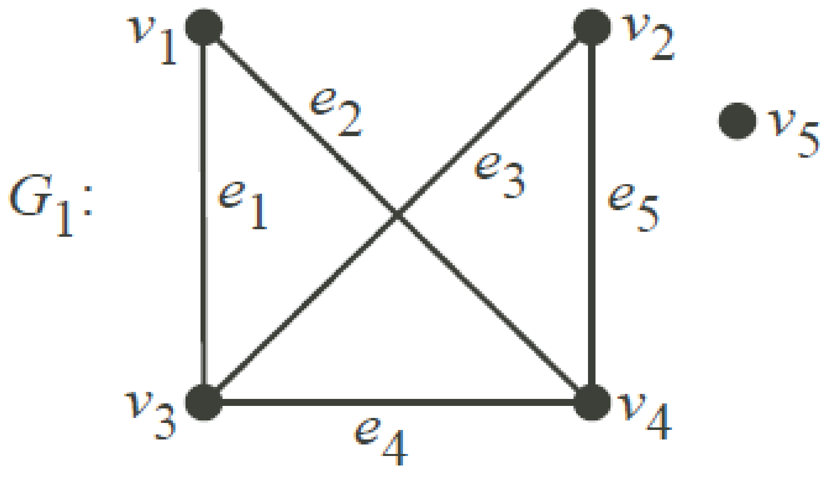


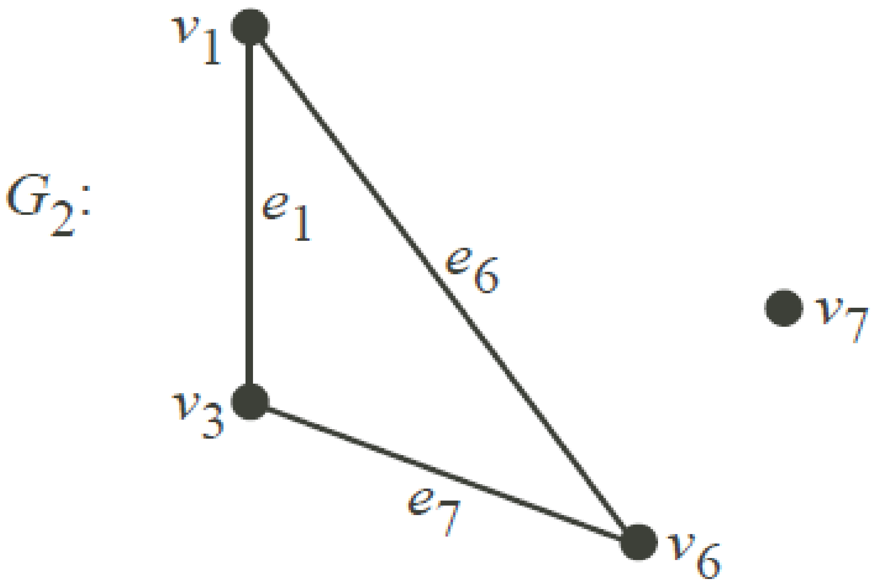
(a) The Simple Graphs G_1 and G_2; (b) Their Union $G_1 \cup G_2$.

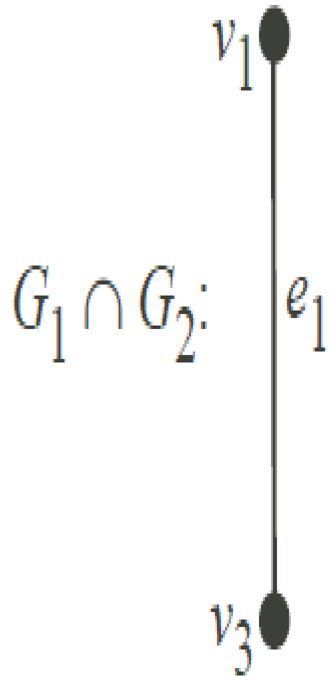
Graph Intersection

The intersection is $G1 \cap G2 = (V1 \cap V2, E1 \cap E2)$

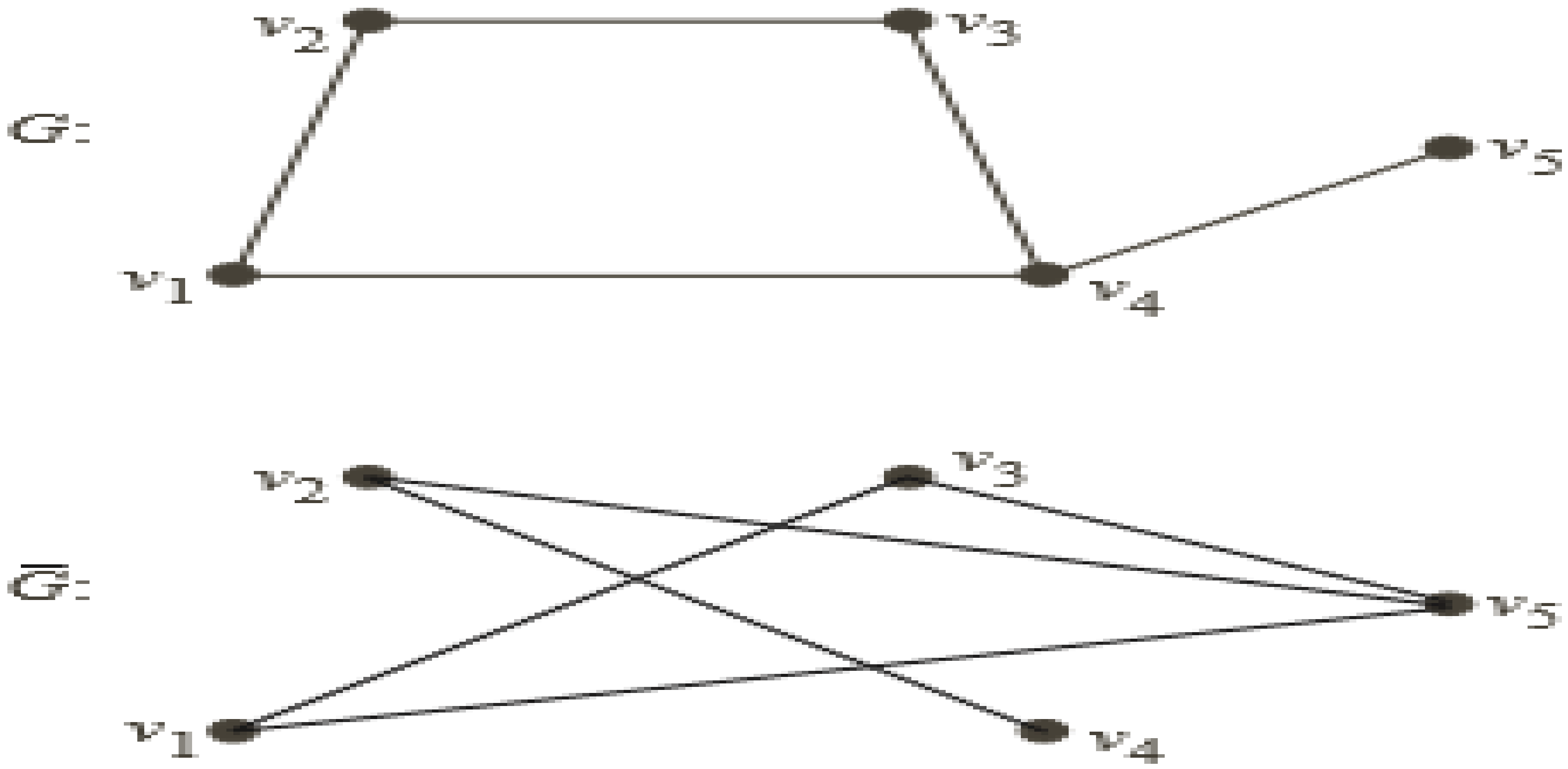
ple. *For the graphs*







The **complementary graph** *G* of a simple graph *G* has the same vertices as *G*. Two vertices are adjacent in *G* if and only if they are not adjacent in *G*.



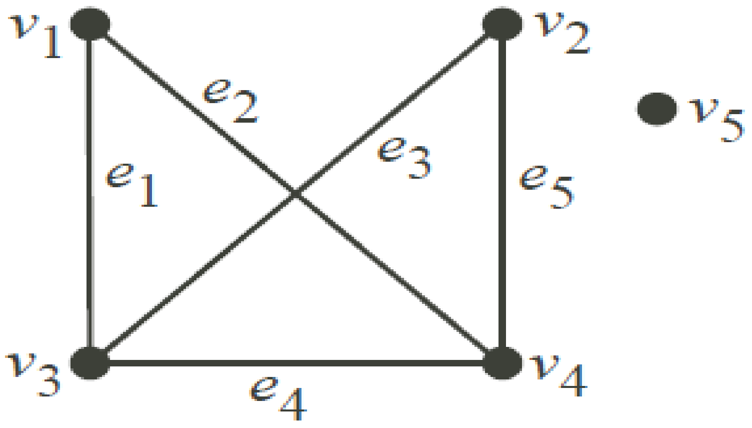
Graph Ring Sum

The *ring sum* $G1 \oplus G2$ is the subgraph of $G1 \cup G2$ induced by the edge set $E1 \oplus E2$.
Note! The set operation $\oplus$ is the *symmetric difference*, i.e.
$E1 \oplus E2 = (E1 - E2) \cup (E2 - E1)$

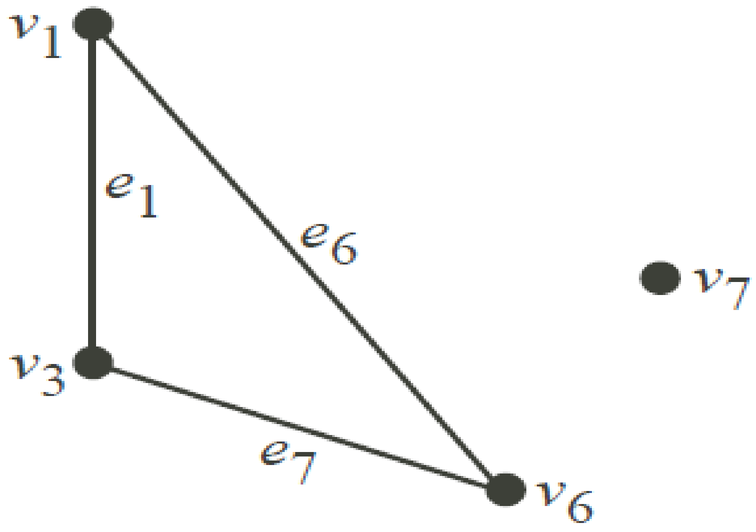
So, the ring sum is a subgraph induced by an edge set.

Example. *For the graphs*

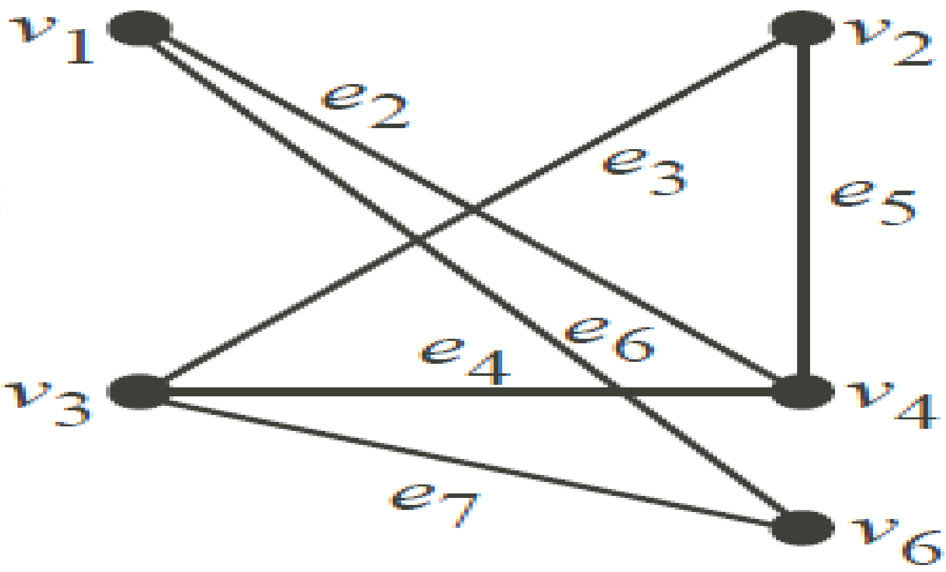
G_1:



G_2:



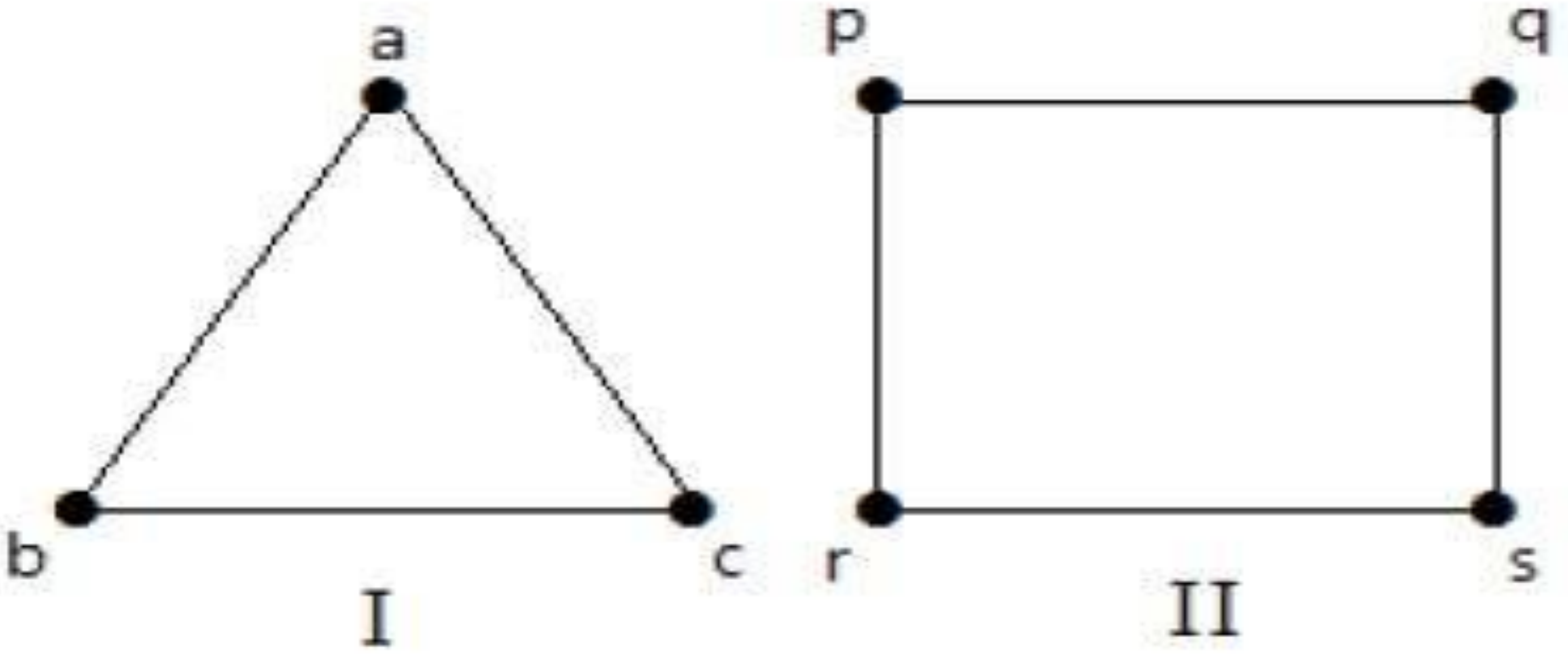
$G_1 \oplus G_2$:



Regular Graph

A graph G is said to be regular, **if all its vertices have the same degree**. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

Example



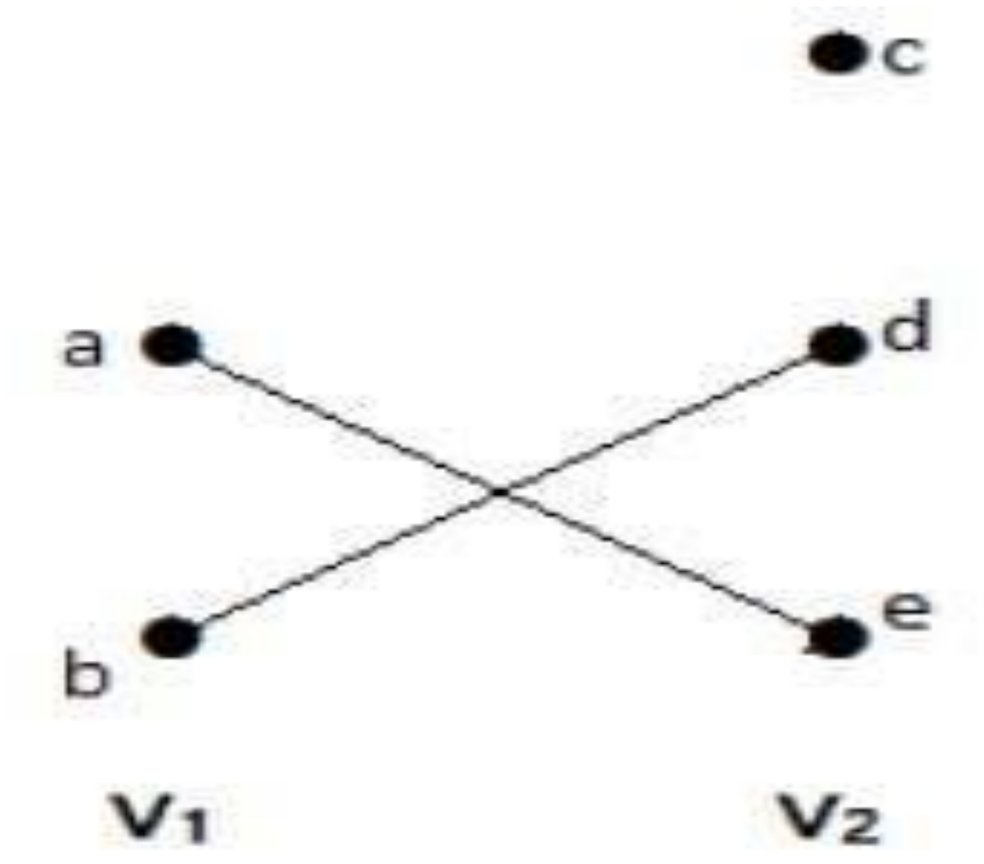
In both the graphs, all the vertices have degree 2. They are called 2-Regular Graphs.

Bipartite Graphs

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets $V1$ and $V2$ such that every edge in the graph connects a vertex in $V1$ and a vertex in $V2$ (so that no edge in G connects either two vertices in $V1$ or two vertices in $V2$).

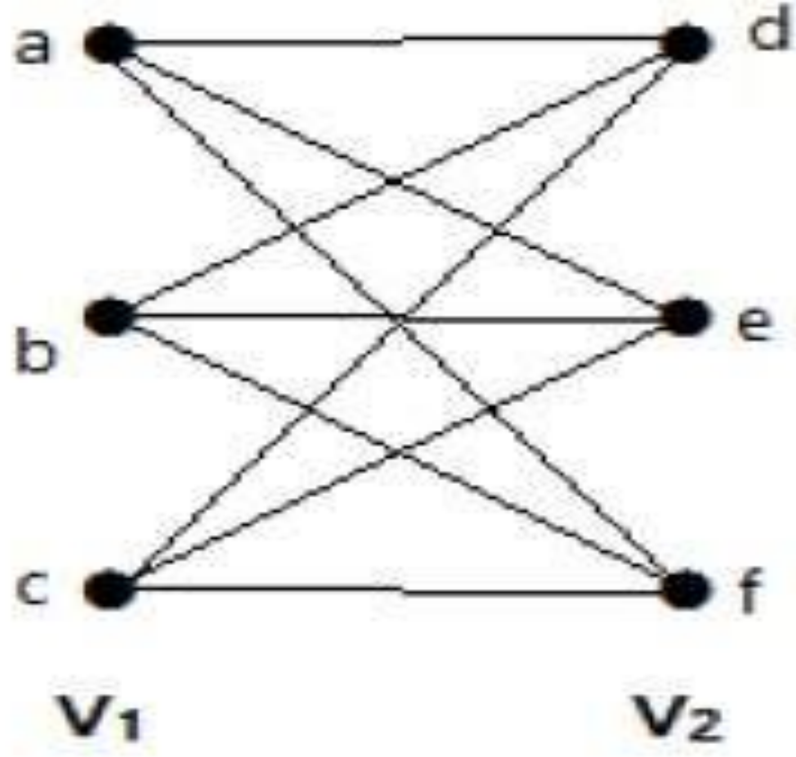
When this condition holds, we call the pair $(V1, V2)$ a *bipartition* of the vertex set V of G.

Example



In this graph, you can observe two sets of vertices – V_1 and V_2. Here, two edges named 'ae' and 'bd' are connecting the vertices of two sets V_1 and V_2.

Complete Bipartite Graph



A bipartite graph 'G', G = (V, E) with partition V = {V_1, V_2} is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2.

In general, a complete bipartite graph connects each vertex from set V_1 to each vertex from set V_2.

If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m, n}$.

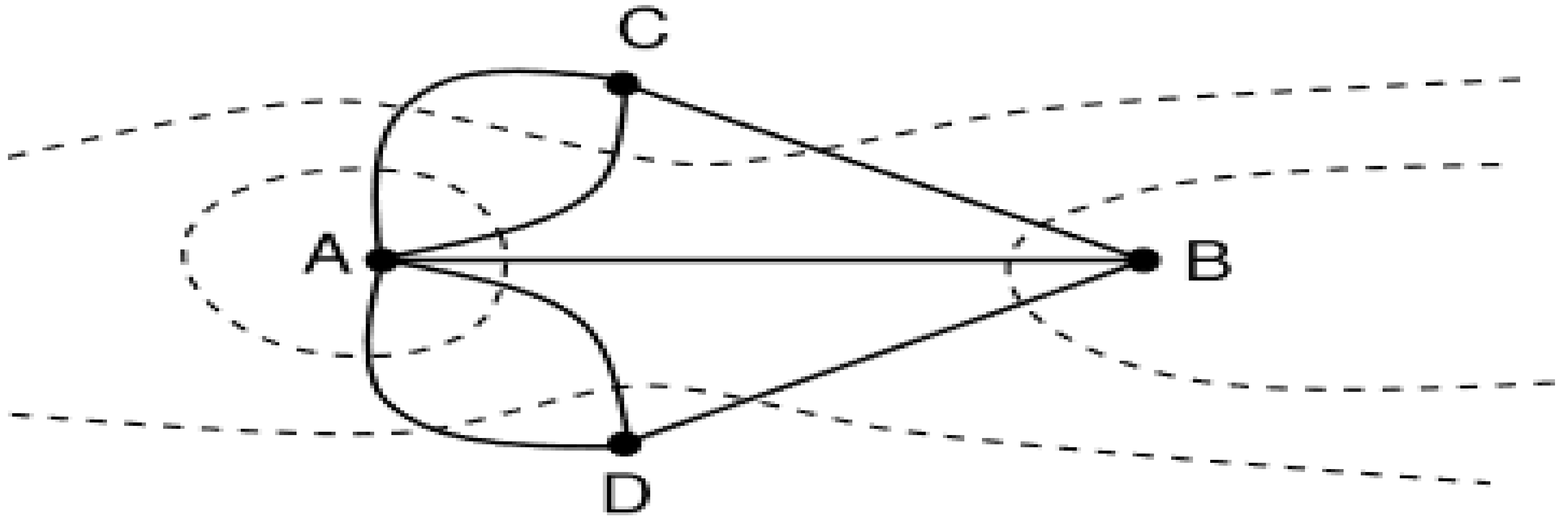
- $K_{m,n}$ has $(m+n)$ vertices and (mn) edges.

- $K_{m,n}$ is a regular graph if $m=n$.

Eulerian Graphs

The following problem, often referred to as the bridges of Konigsberg problem, was first solved by Euler in the eighteenth century. The problem was rather simple — the town of Konigsberg consists of two islands and seven bridges. **Is it possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not crossing any bridge twice?**

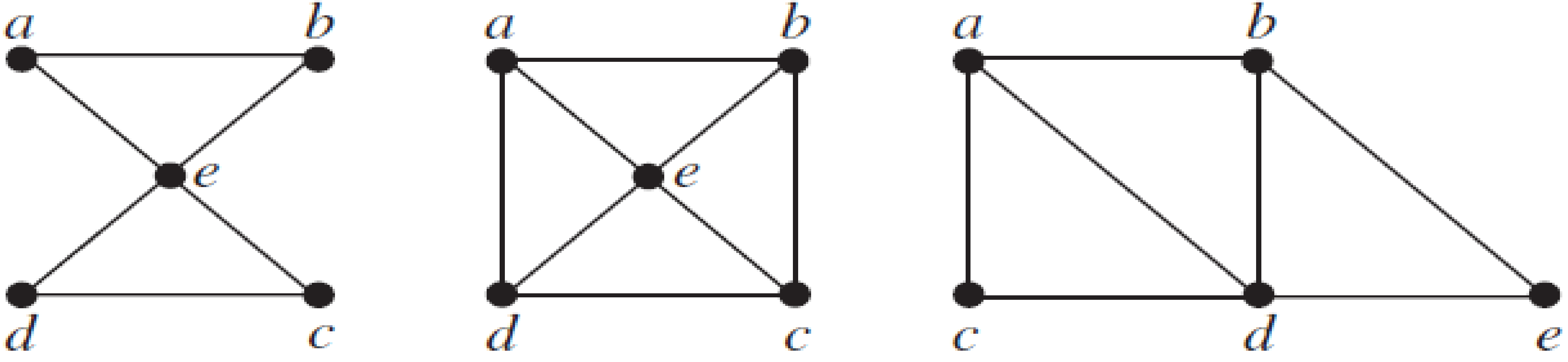
The problem is same as following question. “Is it possible to draw a given graph without lifting pencil from the paper and without tracing any of the edges more than once”.



The bridges of Königsberg problem

Euler Paths and Circuits

- An *Euler circuit* in a graph *G* is a simple circuit containing every edge of *G*.
- An *Euler path* in *G* is a simple path containing every edge of *G*.



G_1 G_2 G_3

The graph *G*1 has an Euler circuit, for example, *a, e, c, d, e, b, a*. Neither of the graphs *G*2 or *G*3 has an Euler circuit (the reader should verify this). However, *G*3 has an Euler path, namely, *a, c, d, e, b, d, a, b*. *G*2 does not have an Euler path

Necessary and sufficient conditions for Euler circuits

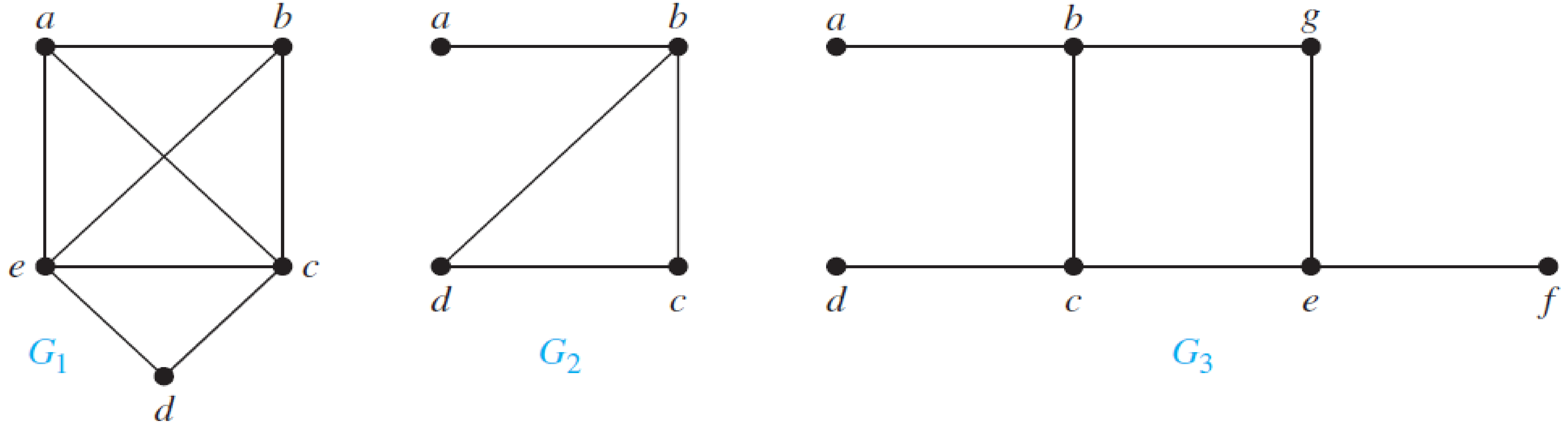
THEOREM: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Necessary and sufficient conditions for Euler paths

THEOREM: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Hamilton Paths and Circuits

A simple path in a graph G that passes through every vertex exactly once is called a ***Hamilton path***, and a simple circuit in a graph G that passes through every vertex exactly once is called a ***Hamilton circuit***.



$G1$ has a Hamilton circuit: *a, b, c, d, e, a*. There is no Hamilton circuit in $G2$ but $G2$ does have a Hamilton path, namely, *a, b, c, d*. $G3$ has neither a Hamilton circuit nor a Hamilton path

Sufficient Condition for a Hamilton Circuit

DIRAC'S THEOREM If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

ORE'S THEOREM If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Eulerian and Hamiltonian Graphs

A graph is called **Eulerian Graph** when it contains an Eulerian circuit.

If a graph has a Hamiltonian circuit, then the graph is called a **Hamiltonian graph**.

Class Exercise 1

A sequence $d1, d2, \ldots, dn$ is called **graphic** if it is the degree sequence of a simple graph.

Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0

b) 6, 5, 4, 3, 2, 1

c) 2, 2, 2, 2, 2, 2

d) 3, 3, 3, 2, 2, 2

e) 3, 3, 2, 2, 2, 2

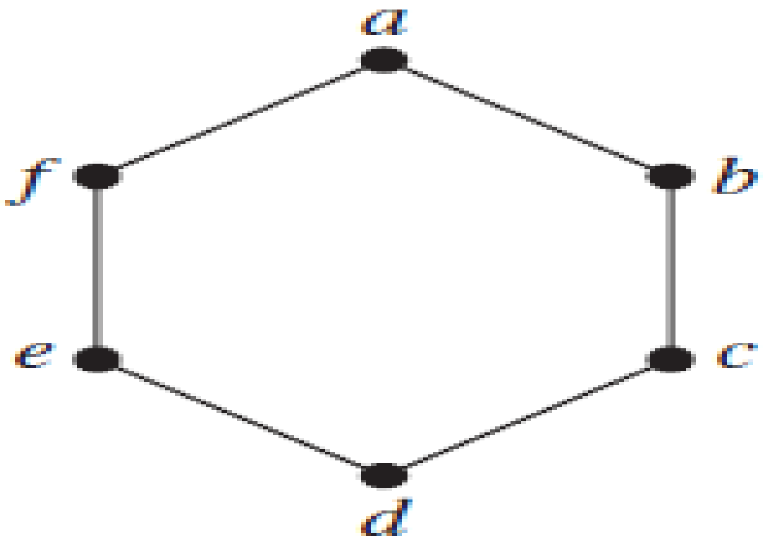
f) 1, 1, 1, 1, 1, 1

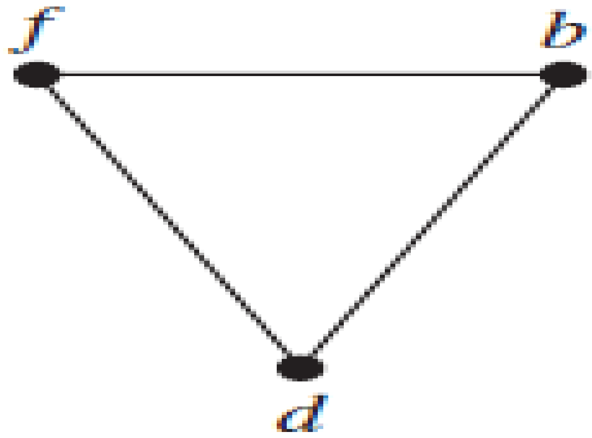
g) 5, 3, 3, 3, 3, 3

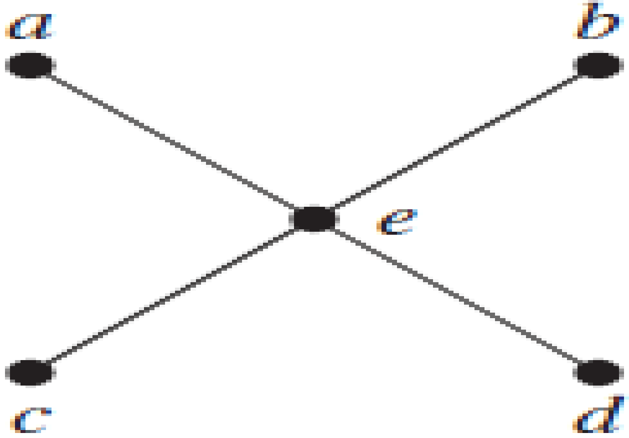
h) 5, 5, 4, 3, 2, 1

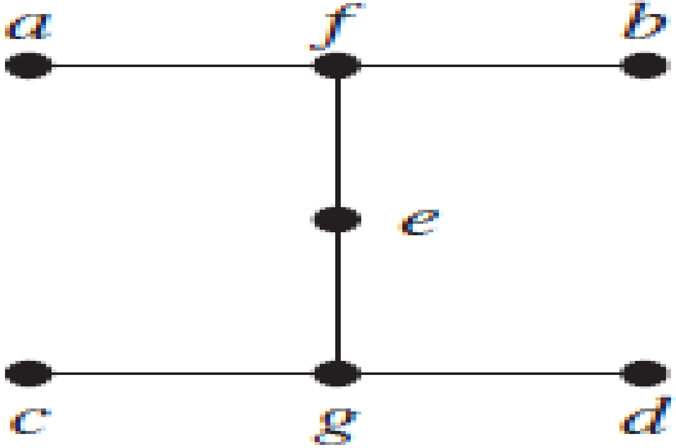
Class Exercise 2

Find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)









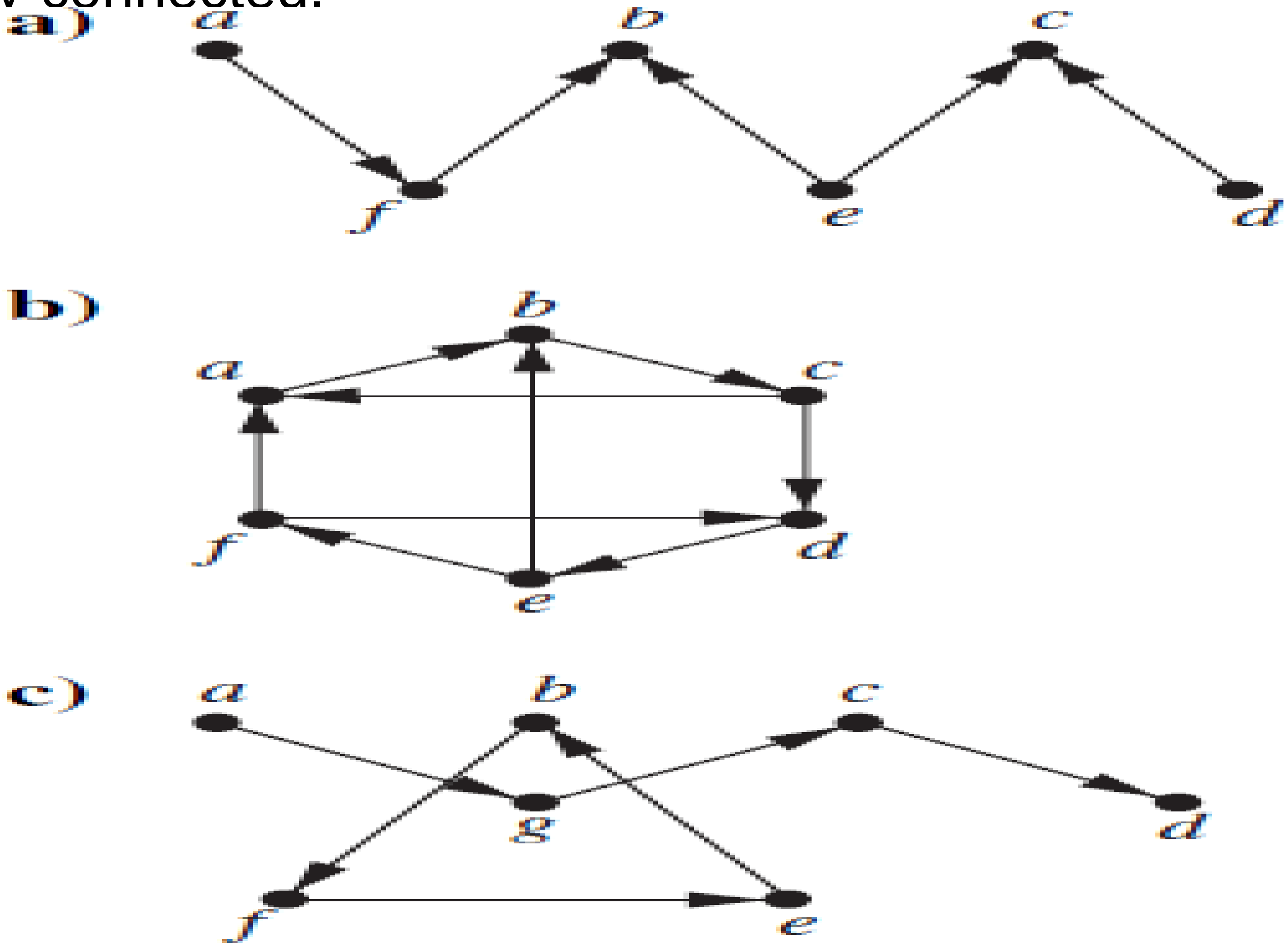
Class Exercise 3

If the degree sequence of the simple graph *G* is 4, 3, 3, 2, 2, what is the degree sequence for complement of *G*?

There are five vertices - so there are four possible edges for each vertex. If a vertex is joined to two vertices in the graph, it is joined to the other $4-2=2$ *vertices in the complement.*

Class Exercise 4

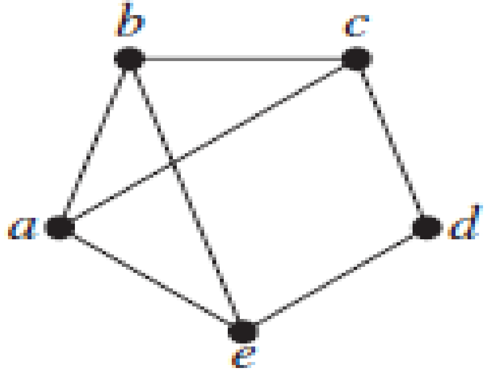
Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.



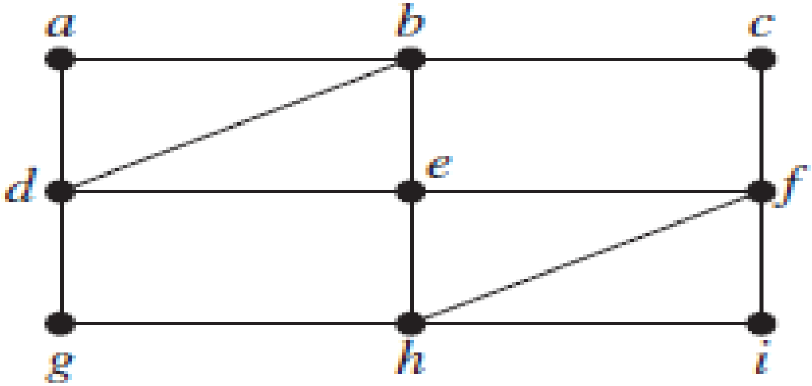
Class Exercise 5

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

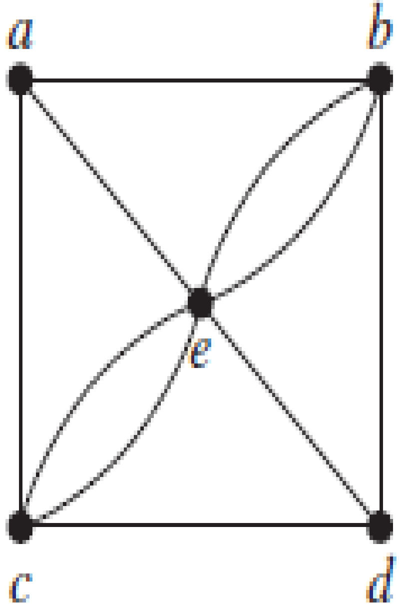
1.



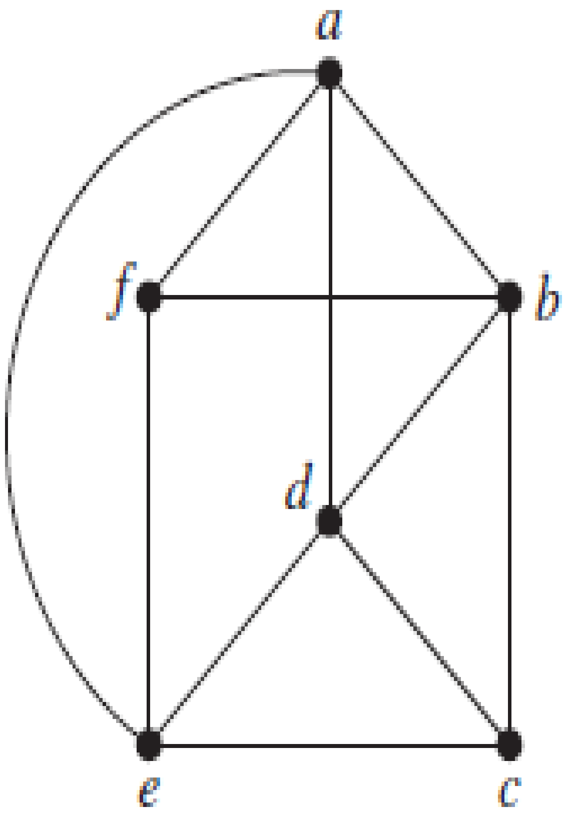
2.



3.

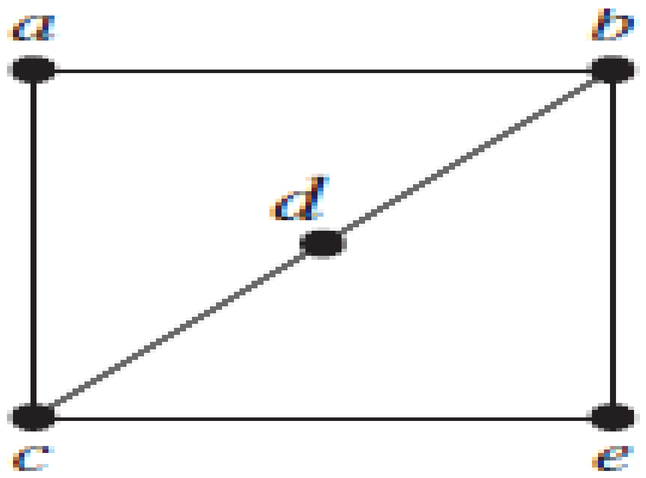


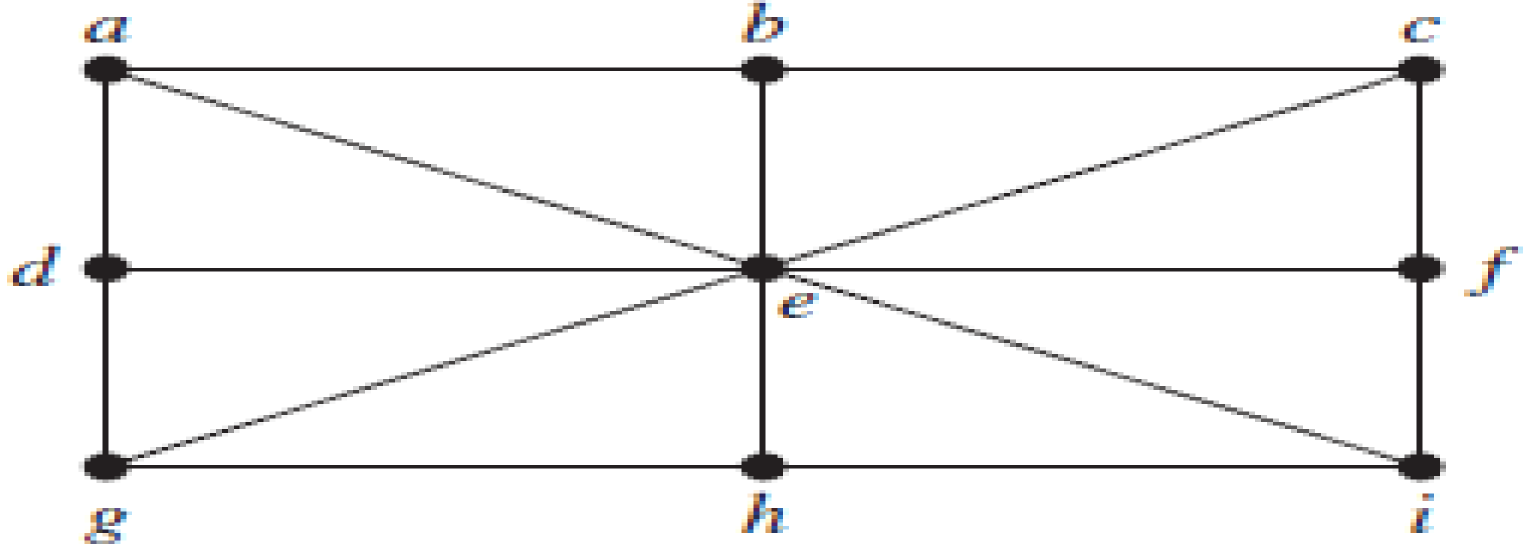
4.



Class Exercise 6

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.

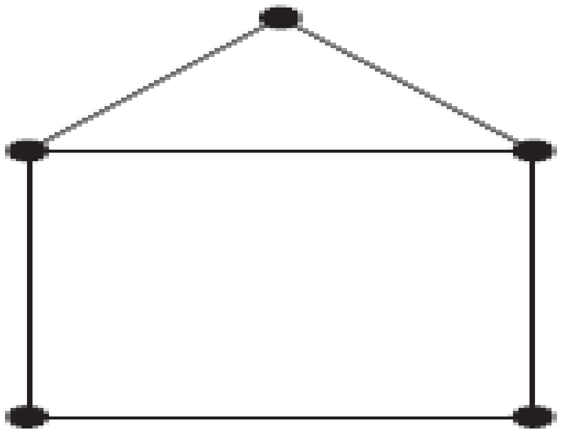




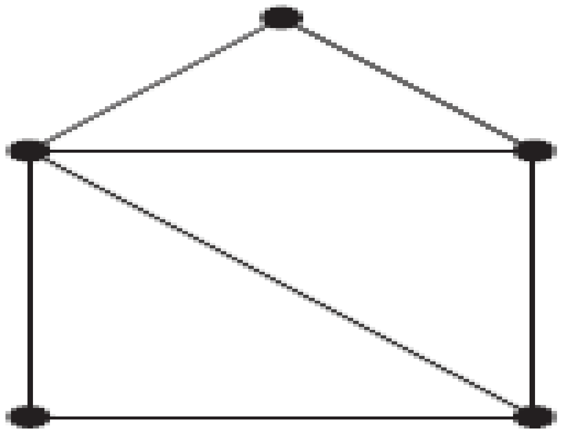
Class Exercise 7

For each of these graphs, determine (*i*) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (*ii*) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (*iii*) whether the graph has a Hamilton circuit.

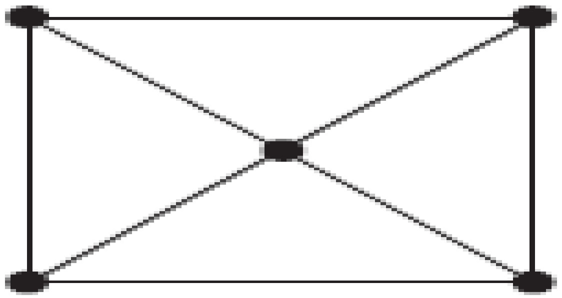
a)



b)



c)



d)