Program: **B.Tech. (CSE)**
Subject name: **Discrete Mathematical Structures**
Number of credits: **3**

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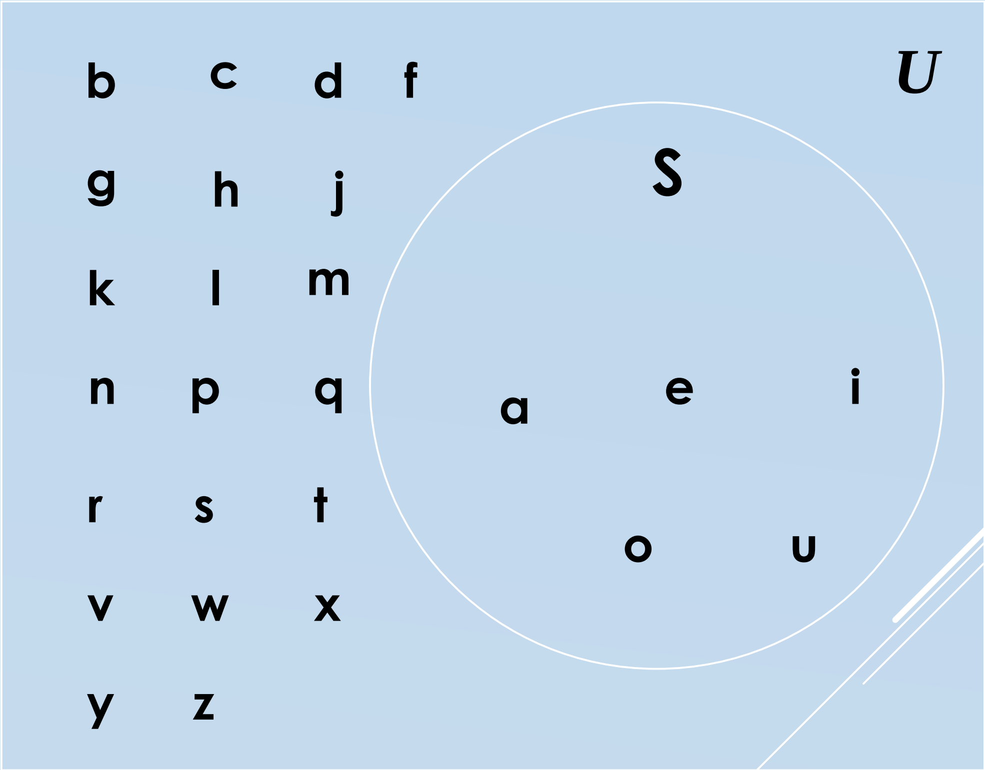
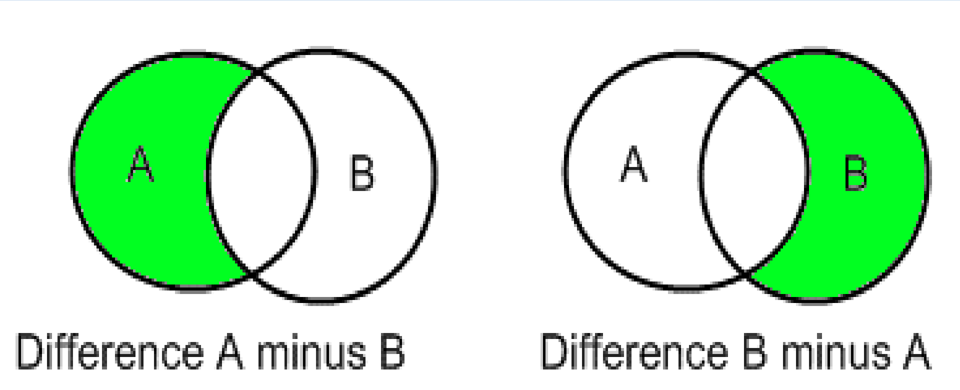
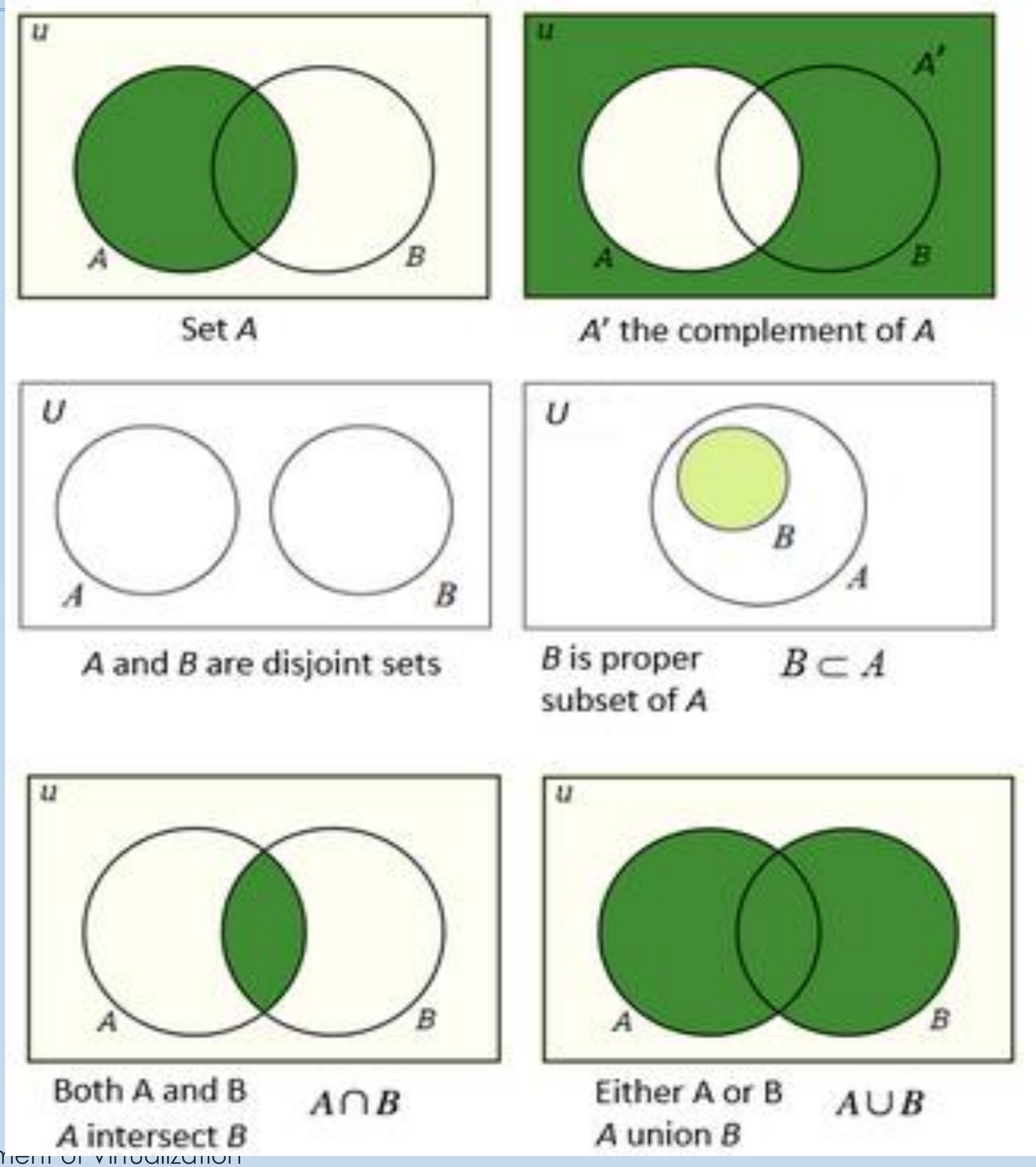
# VENN DIAGRAMS

- Represents sets graphically
  - The box represents the universal set
  - Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet



## Set operations and Venn diagrams



The ***symmetric difference***, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$

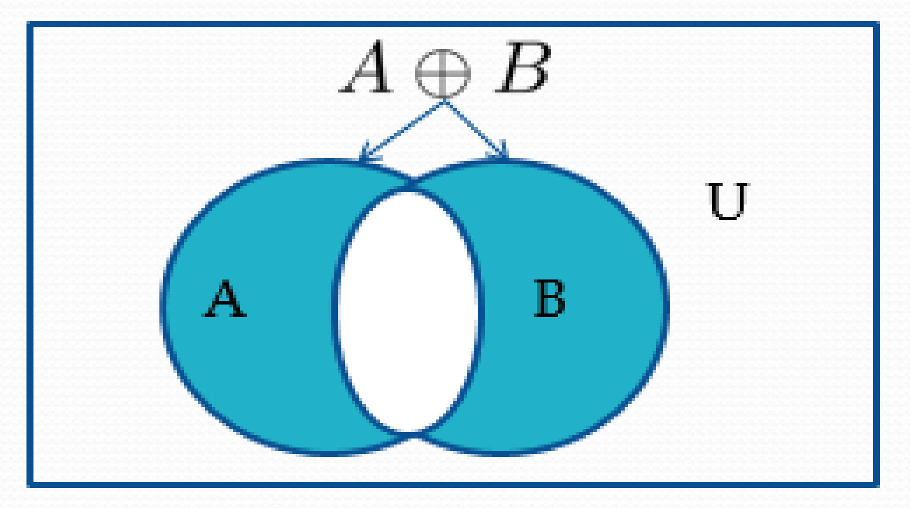
**Example**:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is: $A \oplus B$

**Solution**: $\{1,2,3,6,7,8\}$



# SET PROPERTIES (LAW OF ALGEBRA OF SETS)

- **Property 1 (*Identity Laws*)**

  $A \cup \emptyset = A$ , $A \cap U = A$

  $A \cup U = U$ , $A \cap \emptyset = \emptyset$

- **Property 2 ( *The idempotent properties*)**

  $A \cup A = A$ , $A \cap A = A$

- **Property 3 (*The commutative properties*)**

  $A \cup B = B \cup A$ , $A \cap B = B \cap A$

- **Property 4 (*The associative properties*)**

  $A \cup (B \cup C) = (A \cup B) \cup C$

  $A \cap (B \cap C) = (A \cap B) \cap C$

- **Property 5 (*The distributive properties)***

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **Property 6 (*Properties of the complement*)**

  $\emptyset^C = U \quad , \quad U^C = \emptyset$

  $A \cup A^C = U \quad , \quad A \cap A^C = \emptyset$

  $(A^C)^C = A$

- **Property 7 (*De Morgan's laws*)**

  $(A \cup B)^C = A^C \cap B^C$

  $(A \cap B)^C = A^C \cup B^C$

- **Property 8 *(Absortion laws)***

  $A \cap (A \cup B) = A$

  $A \cup (A \cap B) = A$

# PROVING SET IDENTITIES

- Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other using propositional logic.
2. Use of Venn diagrams.

You should be familiar with the basic set-theoretic operations and relations and know their precise definitions. The following table summarizes the key definitions and shows how to correctly define expressions like “x ∊ A∪B”, “x ∊ A−B”, etc.

| | |
|---|---|
| (1) | $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$ |
| (2) | $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$ |
| (3) | $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$ |
| (4) | $x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$ |
| (5) | $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$ |
| (6) | $x \notin A - B \Leftrightarrow x \notin A$ or $x \in B$ |
| (7) | $x \in A \times B \Leftrightarrow x = (a, b)$ for some $a \in A$ and $b \in B$ |
| (8) | $A \subseteq B \Leftrightarrow$ If $x \in A$, then $x \in B$. |
| (9) | $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. |

# PROOF OF ASSOCIATIVE LAW

Let x be any arbitrary element of $(A \cup B) \cup C$.

$\therefore\ x \in (A \cup B) \cup C \quad \Rightarrow\ x \in (A \cup B)\ \text{or}\ x \in C$

$\Rightarrow (x \in A\ \text{or}\ x \in B)\ \text{or}\ x \in C \quad \Rightarrow x \in A\ \text{or}\ (x \in B\ \text{or}\ x \in C)$

$\Rightarrow\ x \in A\ \text{or}\ (x \in B \cup C) \quad \Rightarrow x \in A \cup (B \cup C)$

$\therefore\ (A \cup B) \cup C \subseteq A \cup (B \cup C) \quad \ldots(i)$

Similarly, let y be any arbitrary element of $A \cup (B \cup C)$.

$\therefore\ y \in A \cup (B \cup C) \quad \Rightarrow\ y \in A\ \text{or}\ y \in (B \cup C)$

$\Rightarrow\ y \in A\ \text{or}\ (y \in B\ \text{or}\ y \in C) \quad \Rightarrow\ (y \in A\ \text{or}\ y \in B)\ \text{or}\ y \in C$

$\Rightarrow (y \in A \cup B)\ \text{or}\ y \in C \quad \Rightarrow\ y \in (A \cup B) \cup C$

$\therefore\ A \cup (B \cup C) \subseteq (A \cup B) \cup C \quad \ldots(ii)$

From (i) and (ii)

$(A \cup B) \cup C = A \cup (B \cup C)$ Proved.

# PROOF OF DISTRIBUTIVE LAW

Let x be any arbitrary element of $A \cup (B \cap C)$.

$\therefore\ x \in A \cup (B \cap C) \quad \Rightarrow x \in A \ \text{ or } \ x \in (B \cap C)$

$\Rightarrow x \in A \ \text{ or } \ (x \in B \ \text{ and } \ x \in C)$

$\Rightarrow (x \in A \ \text{ or } \ x \in B) \ \text{ and } \ (x \in A \ \text{ or } \ x \in C)$

[$\because$ 'or' is distributive over 'and']

$\Rightarrow x \in (A \cup B) \ \text{ and } \ x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore\ A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \qquad \text{...(i)}$

Similarly, let y be any arbitrary element of $(A \cup B) \cap (A \cup C)$.

$\therefore\ y \in (A \cup B) \cap (A \cup C) \quad \Rightarrow y \in (A \cup B) \ \text{ and } \ y \in (A \cup C)$

$\Rightarrow (y \in A \ \text{ or } \ y \in B) \text{ and } (y \in A \ \text{ or } \ y \in C)$

$\Rightarrow y \in A \ \text{ or } \ (y \in B \ \text{ and } \ y \in C)$

$$\Rightarrow \; y \in A \;\text{ or }\; (y \in B \cap C)$$

$$\Rightarrow \; y \in A \cup (B \cap C)$$

$$\therefore \; (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \qquad \ldots \text{(ii)}$$

From (i) and (ii),

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad \text{Proved.}$$

# PROOF OF FIRST DEMORGAN LAW

Let x be an arbitrary element of the set $(A \cup B)'$.

$\therefore\ x \in (A \cup B)'$ $\qquad \Rightarrow\ x \notin (A \cup B)$

$\Rightarrow\ x \notin A$ and $x \notin B$ $\qquad \Rightarrow\ x \in A'$ and $x \in B'$

$\Rightarrow\ x \in A' \cap B'$ $\qquad \therefore\ (A \cup B)' \subseteq A' \cap B'$ ...(i)

Again let y be an arbitrary element of $A' \cap B'$.

$\therefore\ y \in A' \cap B'$ $\qquad \Rightarrow\ y \in A'$ and $y \in B'$

$\Rightarrow\ y \notin A$ and $y \notin B$ $\qquad \Rightarrow\ y \notin A \cup B$

$\Rightarrow\ y \in (A \cup B)'$

$\therefore\ A' \cap B' \subseteq (A \cup B)'$ ...(ii)

From (i) and (ii),

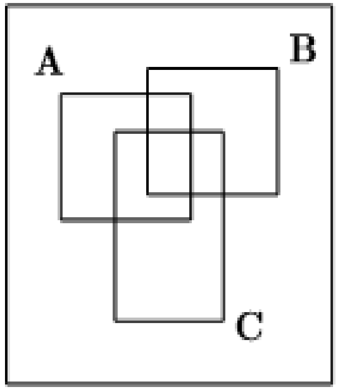
$(A \cup B)' = A' \cap B'$

# Proof of Distributive Laws of Boolean Algebra by Venn diagrams

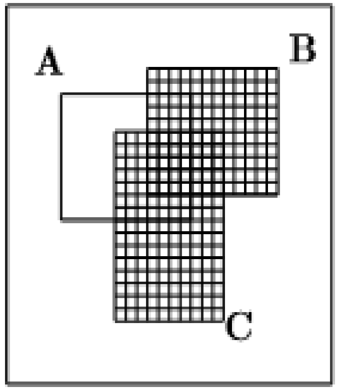
to verify that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

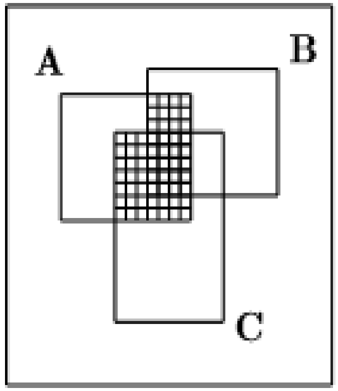
we draw the following diagrams.



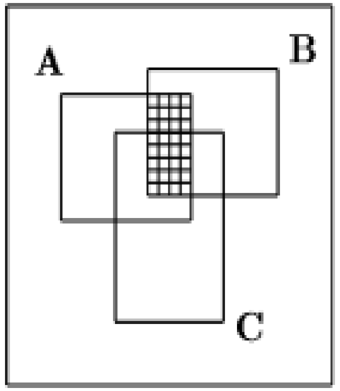

(a) $\mathbf{A, B, C}$



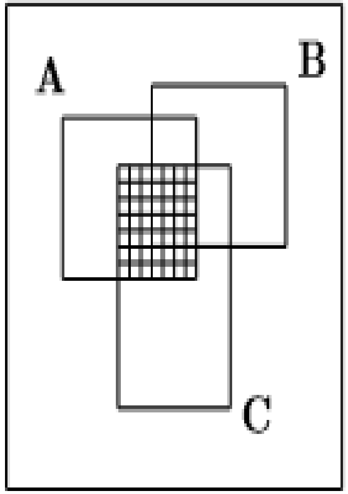

(b) $\mathbf{B \cup C}$



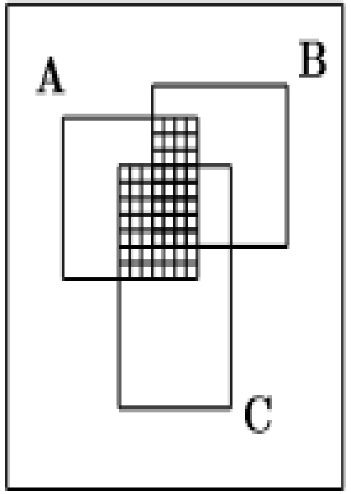

(c) $\mathbf{A \cap (B \cup C)}$




(d) $\mathbf{A \cap B}$




(e) $\mathbf{A \cap C}$




(f) $\mathbf{(A \cap B) \cup (A \cap C)}$

Equality holds because diagrams (c) and (f) are the same.

# CLASS EXERCISE - 1

Prove Second Demorgan's Law i.e.

$$(A \cap B)^C = A^C \cup B^C$$

# CLASS EXERCISE - 2

**If A, B and C are any three sets, then prove that**

$$A-(B\cap C)=(A-B)\cup(A-C).$$

# Discrete Mathematical Structures



## Solution

Let x be any element of $A-(B\cap C)$.

$\therefore\ x\in A-(B\cap C)\Rightarrow x\in A$ and $x\notin(B\cap C)$

$\Rightarrow\ x\in A$ and $(x\notin B$ or $x\notin C)$

$\Rightarrow\ (x\in A$ and $x\notin B)$ or $(x\in A$ and $x\notin C)$

$\Rightarrow\ x\in(A-B)$ or $x\in(A-C)$

$\Rightarrow\ x\in(A-B)\cup(A-C)$

$\therefore\ A-(B\cap C)\subseteq(A-B)\cup(A-C)$ ...(i)

Again y be any element of $(A-B)\cup(A-C)$.

$\therefore\ y\in(A-B)\cup(A-C)\Rightarrow y\in(A-B)$ or $y\in(A-C)$

$\Rightarrow\ (y\in A$ and $y\notin B)$ or $(y\in A$ and $y\notin C)$

$\Rightarrow\ y\in A$ and $(y\notin B$ or $y\notin C)$

$\Rightarrow\ y\in A$ and $(y\notin(B\cap C))$ $\Rightarrow\ y\in A-(B\cap C)$

$\therefore\ (A-B)\cup(A-C)\subseteq A-(B\cap C)$ ...(ii)

From (i) and (ii),

$A-(B\cap C)=(A-B)\cup(A-C)$ Proved.

# CLASS EXERCISE - 3

Illustrate the first Demorgan's Law using Venn diagrams.

# CLASS EXERCISE - 4

**Prove following properties of Cartesian Product.**

Let $A, B$ and $C$ be sets. Then,

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;

(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$;

(d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

# PRINCIPLE OF INCLUSION & EXCLUSION

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- The above equation represents the principle of inclusion and exclusion for two sets A and B.
- The name comes from the fact that to calculate the elements in a union, we include the individual elements of A and B but subtract the elements common to A and B so that we don't count them twice.
- This principle can be generalized to *n* sets.

# EXAMPLE: INCLUSION AND EXCLUSION PRINCIPLE

- Example 1: How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?
  - Let us assume that A = set of all integers from 1 to 1000 that are multiples of 3.
  - Let us assume that B = set of all integers from 1 to 1000 that are multiples of 5.
  - $A \cup B$ = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
  - $A \cap B$ = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.
- To use the inclusion/exclusion principle to obtain $|A \cup B|$, we need $|A|$, $|B|$ and $|A \cap B|$.
  - From 1 to 1000, every third integer is a multiple of 3, each of this multiple can be represented as $3p$, for any integer $p$ from 1 through 333, Hence $|A| = 333$.
  - Similarly for multiples of 5, each multiple of 5 is of the form $5q$ for some integer $q$ from 1 through 200. Hence, we have $|B| = 200$.

# EXAMPLE: INCLUSION/EXCLUSION PRINCIPLE FOR 3 SETS

- To determine the number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form 15*r* for some integer *r* from 1 through 66.
- Hence, $|A \cap B| = 66$.

- From the principle, we have the number of integers either multiples of 3 or multiples of 5 from 1 to 1000 given by

  $|A \cup B| = 333 + 200 - 66 = 467.$

- <u>Example 2:</u> In a class of students undergoing a computer course the following were observed.
  - Out of a total of 50 students: 30 know Pascal, 18 know Fortran, 26 know COBOL, 9 know both Pascal and Fortran, 16 know both Pascal and COBOL, 8 know both Fortran and COBOL, 47 know at least one of the three languages.
- From this we have to determine
  - a. How many students know none of these languages?
  - b. How many students know all three languages?

# EXAMPLE: INCLUSION/EXCLUSION PRINCIPLE FOR 3 SETS

a. We know that 47 students know at least one of the three languages in the class of 50. The number of students who do not know any of three languages is given by the difference between the number of students in class and the number of students who know at least one language.
   - Hence, the students who know none of these languages = 50 – 47 = 3.

b. Students know all three languages, so we need to find $|A \cap B \cap C|$.
   - A = All the students who know Pascal in class.
   - B = All the students who know COBOL in the class.
   - C = All the students who know FORTRAN in class.

- We have to derive the inclusion/exclusion formula for three sets

$$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$$

- $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$
  $= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$
  $= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|)$
  $= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
- Given in the problem are the following:
  $|B \cap C| = 8$
  $|A \cap B| = 9$
  $|A \cap C| = 16$
  $|A \cup B \cup C| = 50$
- Hence, using the above formula, we have
  $47 = 30 + 26 + 18 - 9 - 16 - 8 + |A \cap B \cap C|$
  Hence, $|A \cap B \cap C| = 6$