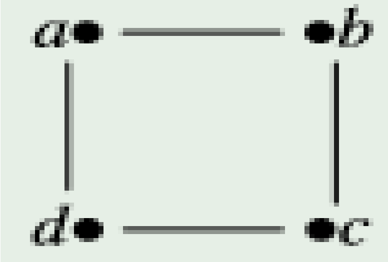
Graph Colouring

Example: Find chromatic Polynomial of Cyclic Graph with four nodes i.e. $P_{C4}(x)$.



There are two different cases to consider:

<u>Case 1</u>:

x choices for a

b and d get the same color since they are nonadjacnet

so $x - 1$ choices

1 choices for c,

same color

<u>Case 2</u>:

x choices for a

$x - 1$ choices for b

We force d to be a different color than b, so there ar

$x - 2$ choices

$x - 2$ choices for c

In math, 'or' means $+$, so we have

$P_{C_4}(x) = x(x-1)(x-2)^2 + x(x-1)^2 = x^4 - 4x^3 + 6x^2 - 3x$

Fundamental Reduction (decomposition) Theorem (Deletion- Contraction Theorem)

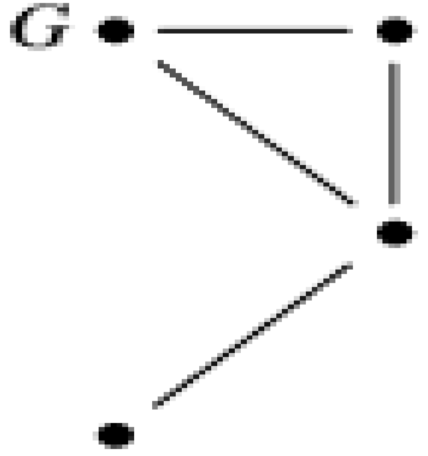
The Fundamental Reduction Theorem

$$P_G(x) = P_{G'(}x) - P_{G''}(x)$$

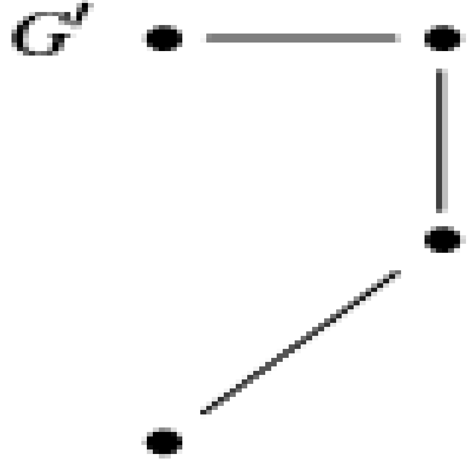
where G’ is a graph after deletion of an edge (e) from G and

G’’ is a graph after contraction (joining) of endpoints of e.

Example: Use the Fundamental Reduction Theorem to find the chromatic polynomial for the given graph.

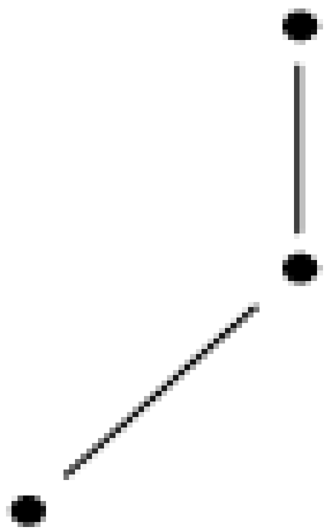


So now we have ...



$$P_{G'}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x$$

G''

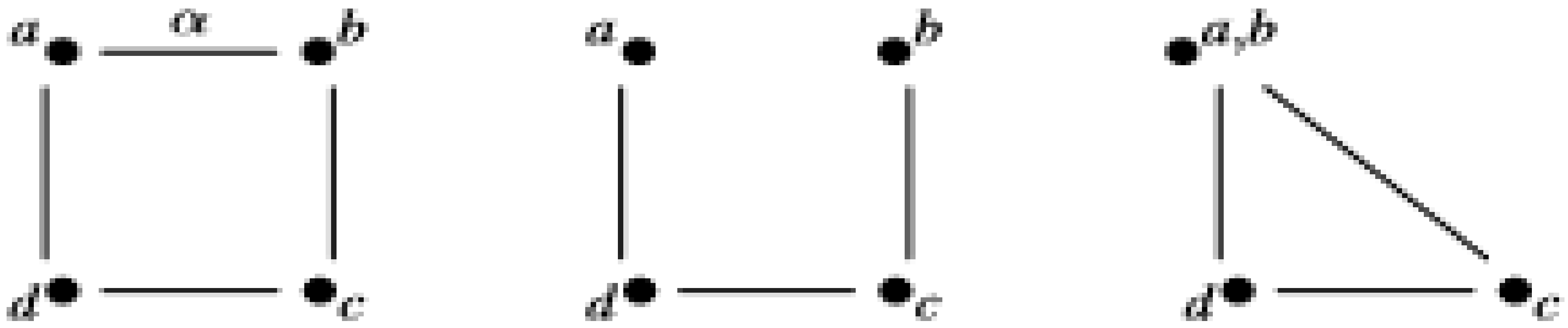


$P_{G''}(x) = x(x-1)^2 = x^3 - 2x^2 + x$

Putting this together, we have

$$\begin{aligned} P_G(x) &= x^4 - 3x^3 + 3x^2 - x - (x^3 - 2x^2 + x) \\ &= x^4 - 4x^3 + 5x^2 - 2x \end{aligned}$$

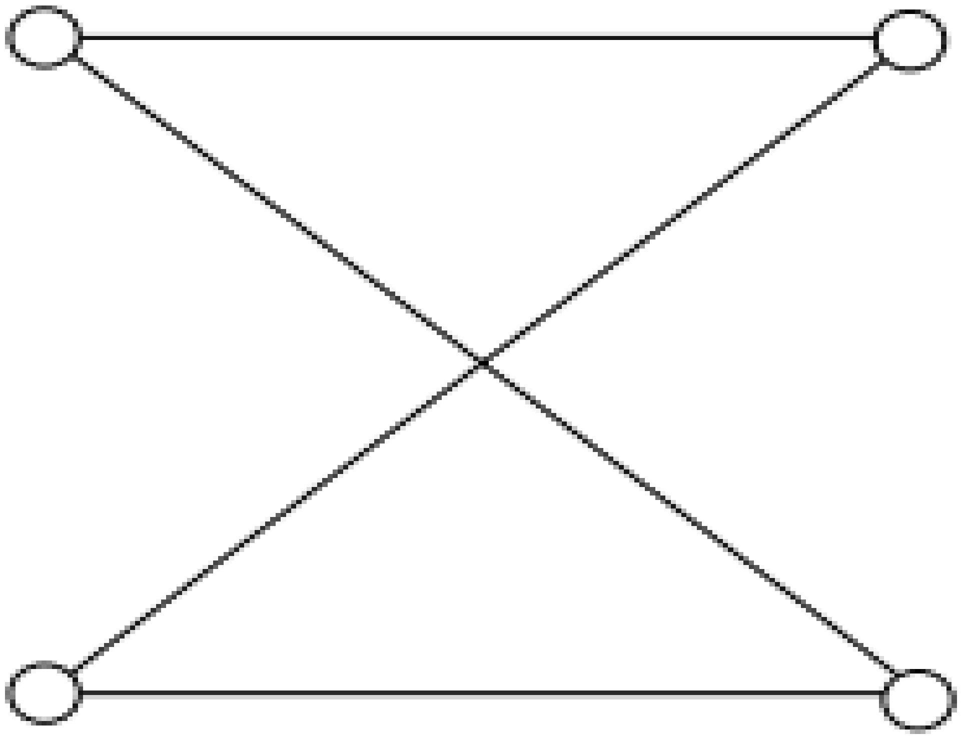
Example: Find $P_{C4}(x)$ using the Fundamental Reduction Theorem.



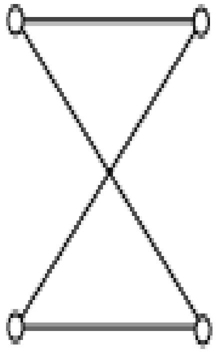
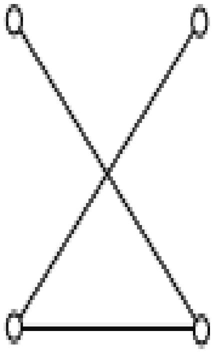
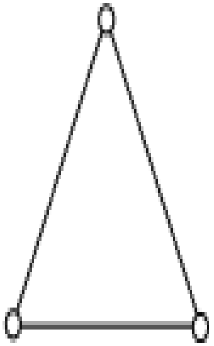
$$P_{G'_\alpha}(x) = x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x$$
$$P_{G''_\alpha}(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$
$$P_G(x) = x^4 - 3x^3 + 3x^2 - x - (x^3 - 3x^2 + 2x)$$
$$= x^4 - 4x^3 + 6x^2 - 3x$$

Exercise1

- Find the chromatic polynomials of the three graphs below using reduction theorem.



Solution:

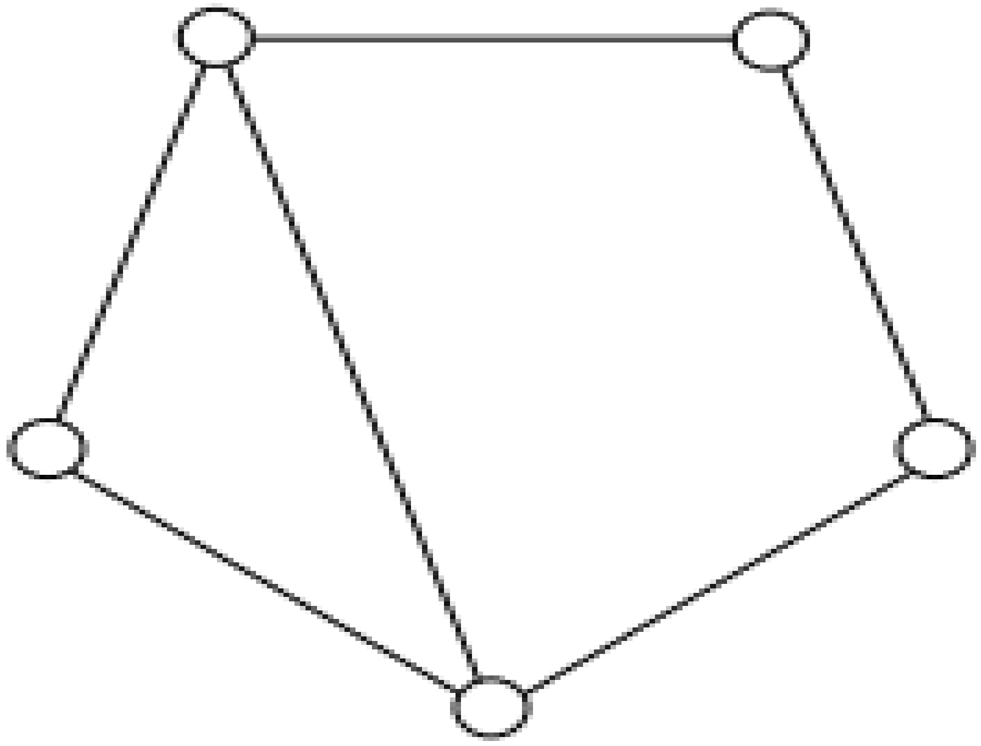
$p\Big($  $\Big) = p\Big($  $\Big) - p\Big($  $\Big)$

$$= k(k-1)^3 - k(k-1)(k-2)$$

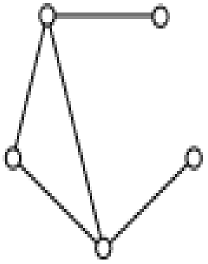
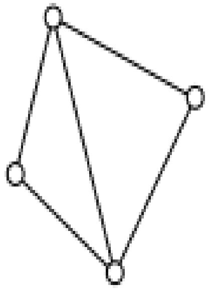
After simplifying, we see that $p_{G_1}(k) = k(k-1)\big((k-1)^2 - (k-2)\big)$.

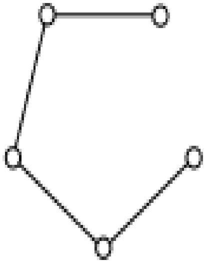
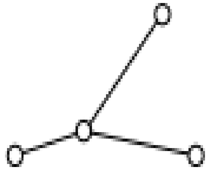
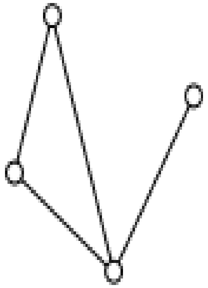
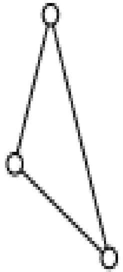
Exercise2

- Find the chromatic polynomials of the three graphs below using reduction theorem.



Solution

$p(G_2) = p\Big($  $\Big) - p\Big($  $\Big)$

$= p\Big($  $\Big) - p\Big($  $\Big) - p\Big($  $\Big) + p\Big($  $\Big)$

$$= k(k-1)^4 - k(k-1)^3 - k(k-1)(k-2)(k-1) + k(k-1)(k-2)$$

$$= k(k-1)^4 - k(k-1)^3 - k(k-1)^2(k-2) + k(k-1)(k-2)$$

Exercise3

- Find the chromatic polynomials of the three graphs below using reduction theorem.