Vector Space

Question Bank

1. Let V be the set of vectors $[2x − 3y, x + 2y, −y, 4x]$ with $x, y \in R^2$. Addition and scalar multiplication are defined in the same way as on vectors. Prove that V is a vector space.

2. For each of the following sets, indicate whether it is a vector space. If so, prove it; otherwise, point out which vector-space property is violated.

a. The set V of vectors $[2x, x^2]$ with $x \in R^2$. Addition and scalar multiplication are defined in the same way as on vectors.

b. The set V of vectors $[x, y, z] \in R^3$ satisfying $x + y + z = 3$ and $x − y + 2z = 6$. Addition and scalar multiplication are defined in the same way as on vectors.

c. The set V of symmetric 2×2 matrices. Addition and scalar multiplication are defined in the same way as on matrices.

3. What is vector space? Discuss the properties of vector space.

4. Prove that the set of real numbers (R) is a Vector Space.

5. Prove that the set *V* of all 2 x 2 matrices with real entries is a vector space if addition is defined to be matrix addition and scalar multiplication is defined to be matrix scalar multiplication.

6. Let V be the set of all polynomials of degree ≤ n with real coefficients, let the field of scalars be R, and define vector addition and scalar multiplication as:

$$(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) + (b_0 + b_1t + b_2t^2 + \cdots + b_nt^n)$$
$$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$$

and

$$c(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) = ca_0 + ca_1t + \cdots + ca_nt^n$$

Prove that it is a vector space.

7. Let V = {x, x /2: x real number} with standard operations. Is it a vector space. Justify your answer.

8. The set V of vectors $[2x, x^2]$ with $x \in R^2$. Addition and scalar multiplication are defined in the same way as on vectors. Does it form a vector space.

9. Let V be the set of vectors $[2x − 3y, x + 2y, −y, 4x]$ with $x, y \in R$. Addition and scalar multiplication are defined in the same way as on vectors. Prove that V is a vector space.

10. Determine which sets are vector spaces under the given operations. For those that are not vector spaces, list all axioms that fail to hold.

I. The set of all triples of real numbers (x, y, z) with the operations
$$(x,y,z) + (x',y',z') = (x+x', y+y', z+z') \quad \text{and} \quad k(x,y,z) = (kx, y, z)$$

The set of all pairs of real numbers (x, y) with the operations
$$(x,y) + (x',y') = (x+x', y+y') \quad \text{and} \quad k(x,y) = (2kx, 2ky)$$

11. Discuss the necessary and sufficient conditions for a vector subspace.

12. Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

13. Following sets are not a subspace of the specified vector space. Give a reason why it is not a subspace.

$$S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 \geq 0 \right\}$$

in the vector space $\mathbb{R}^3$.

14. Following sets are not a subspace of the specified vector space., Give a reason why it is not a subspace.

$$S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 - 4x_2 + 5x_3 = 2 \right\}$$

in the vector space $\mathbb{R}^3$.

15. Is V = {(a + 2b, 2a − 3b) : a and b are real} a subspace of R^2? Why or why not?

16. Is $H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$? Why or why not?

17. Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a + b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Where a and b are real numbers.

18. Determine whether the function T: R^2->R^2 is linear transformation. If yes, prove it; if not, provide a counterexample to one of the properties:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

19. Determine whether the function T: R^2->R^2 is linear transformation. If yes, prove it; if not, provide a counterexample to one of the properties:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$$

20. Determine whether the function T: R^2->R^2 is linear transformation. If yes, prove it; if not, provide a counterexample to one of the properties:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

21. Let $V=\mathbf{R}^2$, and let $u,v \in V$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define addition component-wise, that is $u+v = (u_1+v_1, u_2+v_2)$ and define scalar multiplication by scalar a to be $au = (au_1, 0)$. Determine whether or not this set under these operations is a vector space.

22. Sketch the image of the rectangle with vertices (0, 0), (1, 0), (1, 2), and (0, 2) under

(a) A reflection about the x-axis
(b) A reflection about the y-axis
(c) A compression of factor k=1/4 in the y-direction
(d) An expansion of factor k=2 in the x-direction
(e) A shear of factor k=3 in the x-direction
(f) A shear of factor k=2 in the y-direction

23. Sketch the image of the rectangle with vertices (0, 0), (2, 0), (2, 2), and (0, 2) under
(a) A reflection about the x-axis
(b) A reflection about the y-axis
(c) A compression of factor k=1/2 in the x-direction
(d) An expansion of factor k=3 in the x-direction
(e) A shear of factor k=1 in the x-direction
(f) A shear of factor k=.5 in the y-direction

24. Determine whether $T_1 o T_2 = T_2 o T_1$
(a) T_1: $R^2 \rightarrow R^2$ is the orthogonal projection on the y-axis, and T_2:$R^2 \rightarrow R^2$ is the orthogonal projection on the x-axis.
(b) T_1: $R^2 \rightarrow R^2$ is the clockwise rotation through an angle α_1, and T_2:$R^2 \rightarrow R^2$ is the anticlockwise rotation through an angle α_2.
(c) T_1: $R^2 \rightarrow R^2$ is the orthogonal projection on the y-axis, and T_2:$R^2 \rightarrow R^2$ is the rotation through an angle α.

25. Find the standard matrix for the stated composition of linear operators on R^2
(a) A rotation of 90^0, followed by a reflection about line y=x.
(b) An orthogonal projection on the y-axis, followed by a contraction with factor k= 0.5.
(c) A reflection about the x-axis, followed by a dilation with factor k=3.

26. Find the required transforms on an object, such that each point (x, y), becomes (2x + 4, 3y + 3).
27. Triangle ABC has coordinates A(−1,2), B(6,2), and C(3,4). Plot and label ABC. Draw Graph and state the coordinates of A′B′C′, the image of ABC after the composition R_{90° o $r_{x-\text{axis}}$. Write a transformation matrix.
28. Construct a transformation matrix to rotate -60˚ about the x-axis.
29. Construct a transformation matrix to rotate 30˚ about the y-axis.
30. Construct a transformation matrix that doubles the height, width, and length of an object.
31. Construct a composite transformation matrix with rotation -60˚ about the x-axis followed by reflection about y-axis and followed by scaling with factor 2 in R^2 space.
32. Let

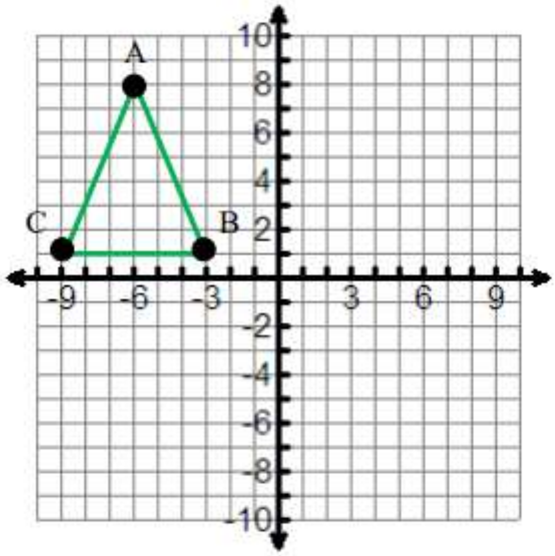
L: R2 ---> R3

be defined by

$$L(x,y) = (y, x, x + y)$$

Show that L is a linear transformation.

33. Given the following triangle, perform each composite transformation in the order specified.



a) Rotate 90° Clockwise
b) Reflect on x axis

35. What is a composite transformation? When doing a composite transformation, does the order in which you perform the transformations matter? Show with suitable example.