Vector Space

Vector space

A ***vector space*** is a set V on which two operations + and · are defined, called ***vector addition*** and ***scalar multiplication***.

The operation + (vector addition) must satisfy the following conditions:

1. Closure: If **u** and **v** are any vectors in V, then the sum **u** + **v** belongs to V.

2. ***Commutative law:*** For all vectors **u** and **v** in V, **u** + **v** = **v** + **u**

3. ***Associative law:*** For all vectors **u**, **v**, **w** in V, **u** + (**v** + **w**) = (**u** + **v**) + **w**

4. ***Additive identity:*** The set V contains an *additive identity* element, denoted by **0**, such that for any vector **v** in V, **0** + **v** = **v** and **v** + **0** = **v**.

5. ***Additive inverses:*** For each vector **v** in V, the equations **v** + **x** = **0** and **x** + **v** = **0** have a solution **x** in V, called an *additive inverse* of **v**, and denoted by - **v**.

- The operation · (scalar multiplication) is defined between real numbers (or scalars) and vectors, and must satisfy the following conditions:
  6. Closure: If v in any vector in V, and c is any real number, then the product $c \cdot v$ belongs to V.
  7. Distributive law: For all real numbers c and all vectors u, v in V, $c \cdot (u + v) = c \cdot u + c \cdot v$
  8. Distributive law: For all real numbers c, d and all vectors v in V, $(c+d) \cdot v = c \cdot v + d \cdot v$
  9. Associative law: For all real numbers c,d and all vectors v in V, $c \cdot (d \cdot v) = (cd) \cdot v$
  10. Unitary law: For all vectors v in V, $1 \cdot v = v$

- Depending on the application, scalars may be real numbers or complex numbers.
- Vector spaces in which the scalars are complex numbers are called ***complex vector spaces***, and those in which the scalars must be real are called ***real vector spaces***.

Vectors in R^n

A vector in n−space is represented by an ordered n−tuple $(x_1,x_2,...,x_n)$.

The set of all ordered $n-$tuples is called the $n-$space and is denoted by $\mathbb{R}^n$. So,

1. $\mathbb{R}^1 = 1 - space =$ set of all real numbers,
2. $\mathbb{R}^2 = 2 - space =$ set of all ordered pairs (x_1, x_2) of real numbers
3. $\mathbb{R}^3 = 3 - space =$ set of all ordered triples (x_1, x_2, x_3) of real numbers
4. $\mathbb{R}^4 = 4 - space =$ set of all ordered quadruples (x_1, x_2, x_3, x_4) of real numbers. (*Think of space-time.*)
5. ......
6. $\mathbb{R}^n = n-space =$ set of all ordered ordered $n-$tuples $(x_1, x_2, \ldots, x_n)$ of real numbers.

Example 1: the set of real numbers (R) is a Vector Space.

Example 2: the set *V* of all 2 x 2 matrices with real entries is a vector space if addition is defined to be matrix addition and scalar multiplication is defined to be matrix scalar multiplication.

Example3: Let V be the set of all polynomials of degree ≤ n with real coefficients, let the field of scalars be R, and define vector addition and scalar multiplication as:

$$(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n)$$
$$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$$

and

$$c(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = ca_0 + ca_1 t + \cdots + ca_n t^n$$

Problem. Let V = {x, x /2: x real number} with standard operations. Is it a vector space. Justify your answer.

Solution: Yes, V is a vector space. We check all the properties, one by one:

1. Addition:

   (a) For real numbers x, y, We have

   $$\left(x, \frac{1}{2}x\right) + \left(y, \frac{1}{2}y\right) = \left(x + y, \frac{1}{2}(x + y)\right).$$

   So, V is closed under addition.

   (b) Clearly, addition is closed under addition.

   (c) Clearly, addition is associative.

   (d) The element $\mathbf{0} = (0, 0)$ satisfies the property of the zero element.

(e) We have $-\left(x, \frac{1}{2}x\right) = \left(-x, \frac{1}{2}(-x)\right)$ .So, every element in V has an additive inverse.

2. Scalar multiplication:

   (a) For a scalar c, we have

   $$c\left(x, \frac{1}{2}x\right) = \left(cx, \frac{1}{2}cx\right).$$

   So, V is closed under scalar multiplication.

   (b) The distributivity $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ works for $\mathbf{u}, \mathbf{v}$ in V.

   (c) The distributivity $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ works, for $\mathbf{u}$ in V and scalars c, d.

   (d) The associativity $c(d\mathbf{u}) = (cd)\mathbf{u}$ works.

   (e) Also $1\mathbf{u} = \mathbf{u}$.

Problem

The set V of vectors $[2x, x^2]$ with $x \in R^2$. Addition and scalar multiplication are defined in the same way as on vectors.

Ans:

No. Addition is not closed on V . For example, [2, 1] and [4, 4] are in V , but [6, 5] is not.

Problem

Problem 1. Let V be the set of vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in R$. Addition and scalar multiplication are defined in the same way as on vectors. Prove that V is a vector space.

Problem.

Determine which sets are vector spaces under the given operations. For those that are not vector spaces, list all axioms that fail to hold.

1. The set of all triples of real numbers (x, y, z) with the operations

   $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$ and $k(x, y, z) = (kx, y, z)$

The set of all pairs of real numbers (x, y) with the operations

$(x, y) + (x', y') = (x + x', y + y')$ and $k(x, y) = (2kx, 2ky)$

Vector Subspaces

Definition: Let V be a vector space, and let W be a subset of V. If W is a vector space with respect to the operations in V, then W is called a *subspace* of V.

Theorem: Let V be a vector space, with operations + and **·**, and let W be a subset of V. Then W is a subspace of V if and only if the following conditions hold.

Sub1 *W is nonempty*: The zero vector belongs to W.

Sub2 *Closure under* +: If **u** and **v** are any vectors in W, then **u + v** is in W.

Sub3 *Closure under* **·** : If **v** is any vector in W, and c is any real number, then c **· v** is in W.

Examples:

1. Let $W = \{(x, 0) : x \textit{ is real number}\}$. Then $W \subseteq \mathbb{R}^2$. (*The notation* $\subseteq$ *reads as 'subset of'*.) It is easy to check that W is a subspace of $\mathbb{R}^2$.

2. Let W be the set of all points on any given line $y = mx$ through the origin in the plane $\mathbb{R}^2$. Then, W is a subspace of $\mathbb{R}^2$.

3. Let P_2, P_3, P_n be vector space of polynomials, respectively, of degree less or equal to $2, 3, n$. Then P_2 is a subspace of P_3 and P_n is a subspace of P_{n+1}.