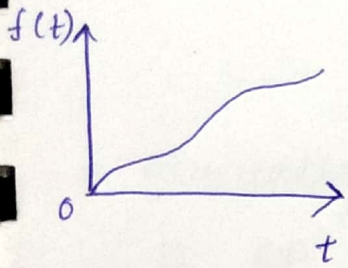
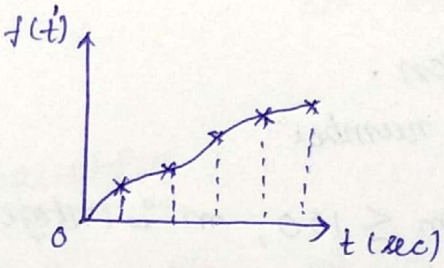
(finite)

DISCRETE MATHEMATICS (MATH231) 3.0.0.3

1. Rosen (TMH) 2. Shaum series (TMH)

continuous values $\xrightarrow{\text{Sampling}}$ Discrete values $\xrightarrow[\text{Binary Encoding.}]{\text{Quantization +}}$ Digital Data





$\Delta t = 0.5$

0 – 000
1 – 001
2 – 010
3 – 011
4 – 100

Quantization : converting into nearest integer (Round off).

$[3.6] = 4$, $[1.2] = 1$

Data : collection of raw facts

Data → Qualitative (Descriptive)

Data → Quantitative (Numerical) → Discrete, continuous

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SET THEORY

SET : well defined collection of finite objects which are distinct and can be unordered.

Representation of Set :

1) Roster or Tabular

Ex. $\{1, 2, 3, 4 \ldots\}$ (ii) $\{2, 3, 4, \ldots 100\}$.

2) Set builder Notation.

Ex. set of Natural number

$S = \{m \mid 2 \leq m \leq 100,\ m \text{ is integer}\}$

: and | → such that or where

* cardinality : Total number of elements in a set

(Finite set) (Infinite set)

Sub Set : a set A is said to be subset of B if and only if

$\forall x \in A \Rightarrow x \in B.$

$\Rightarrow A \subseteq B.$

$A \subseteq B \Leftrightarrow \forall x$, if $x \in A$ then $x \in B$.

$A \not\subseteq B \Leftrightarrow \exists x$ such that $x \in A$ & $x \notin B$.

Proper Subset : $A \subset B$ if

B has atleast one element more than A.

$\{A \neq B.\}$

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- Unit set or Singaltan set: only one element
- Equal set : $A \subseteq B$ and $B \subseteq A$.
- complement : $A' = U - A$.

- Mutually Exclusive :

if $A \cap B = \phi$ then A and B are mutually exlusive. or disjointset → No common element.

- Mutually exhaustive :

if $A \cup B = C$ the A and B are mutually exhaustive to C.

- Set Partitions :

A collection of non empty set $\{A_1, A_2 \ldots A_n\}$

if 1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$

2. $A_1, A_2 \ldots A_n$ are mutually disjoint.

Ex. $\{Z^+, Z^-, \{0\}\}$ is a partition of Z.

- Power set : collection of all possible subset. denoted by $P(A)$.

2^n = No. of subset in power set.

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Cartesian Products :

Given set A and B.

$a \in A$ and $b \in B$.

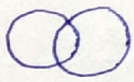
$A \times B = \{a, b\}$

collection of ordered pair of A and B.

VENN DIAGRAMS : Pictorial representation of set. or graphically representing a set.

symmetric difference, $A \oplus B = (A-B) \cup (B-A)$

$\{x : (x \in A \wedge x \notin B)$ and $y : (y \in B \wedge y \notin A)\}$.



1) Identity Law

2) Idempotent law $A \cup A = A$ & $A \cap A = A$.

3) Commutative Law. $A \cup B = B \cup A$, $A \cap B = B \cap A$.

3) Associative Law $A \cup (B \cup C) = (A \cup B) \cup C$.

4) Distributive Law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5) complement property $\phi' = U$, $U' = \phi$

$A \cup A' = U$, $A \cap A' = \phi$

$(A')' = A$.

6) De Morgan's Law : $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

7) Absorbtion Law.

$A \cap (A \cup B) = A$

$A \cup (A \cap B) = A$

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Proving Set Identity

1. Use of Venn diagrams.

2.

i) $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

ii) $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$

iii) $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

iv) $x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$.

v) $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

vi) $x \notin A - B \Leftrightarrow x \notin A$ or $x \in B$

vii) $x \in A \times B \Leftrightarrow x = (a, b)$ for some $a \in A$ & $b \in B$.

viii) $A \subseteq B \Leftrightarrow$ if $x \in A$, then $x \in B$.

ix) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

- Proof of associative law.

Let x be an arbitrary element of $(A \cup B) \cup C$

$\therefore x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$.

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$ {$\because$ Logical or is associative}

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$ ---- (I).

Now similarly let x be an arbitrary element of

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$. ---- (II)

from (I) & (II).

$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$ Hence proved.

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Proof of Distributive Law.

Let x be any arbitrary element of $A \cup (B \cap C)$

$\therefore x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

[$\because$ "or" is distributive over "and"]

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. —— (I)

Similarly.

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ —— (II)

- Demorgan's Law.

Let x be an arbitrary element of set $(A \cup B)'$

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B$

$\Rightarrow x \in A' \cap B'$ $\quad \therefore (A \cup B)' \subseteq A' \cap B'$ —— (I)

Let $x \in A' \cap B'$

$x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$ $\quad \therefore A' \cap B' \subseteq (A \cup B)'$

$\Rightarrow x \in (A \cup B)'$

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Ans. Prove II Demorgan's Law $(A \cap B)' = A' \cup B'$

Let x be an arbitary element of set $(A \cap B)'$

$x \in (A \cap B)' \Rightarrow x \notin A \cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\therefore (A \cap B)' \subseteq A' \cup B'$ — (I)

Let ~~A'~~ x be an arbitary element of $A' \cup B'$

$\Rightarrow x' \in A' \cup B'$

$\Rightarrow x' \in A'$ or $x \in B'$

$\Rightarrow x' \notin A$ or $x \notin B$

$\Rightarrow x' \notin$ ~~A∪B~~ $A \cap B$

$\Rightarrow x' \in (A \cap B)'$

$\Rightarrow \therefore A' \cup B' \subseteq (A \cap B)'$ — (II)

from (I) and (II) we can conclude that

$(A \cap B)' = A' \cup B'$

4. $A - (B \cap C) = (A - B) \cup (A - C)$

Let x be an arbitary element of set $A - (B \cap C)$

$\Rightarrow x \in A - (B \cap C)$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$ { and is distributive over or

$\Rightarrow x \in (A - B)$ or $x \in (A - C)$

$\Rightarrow x \in (A - B) \cup (A - C)$

$\Rightarrow A - (B \cap C) \subseteq (A - B) \cup (A - C)$ — (I)

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Let y be an arbitrary element of $(A-B) \cup (A-C)$

$\Rightarrow x \in (A-B) \cup (A-C)$

$\Rightarrow x \in (A-B)$ or $x \in (A-C)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin B \cap C)$

$\Rightarrow x \in A-(B \cap C)$

$\Rightarrow x \in A-(B \cap C)$ — (I)

$\Rightarrow (A-B) \cup (A-C) \subseteq A-(B \cap C)$ — (II)

from (I) and (II).

$A-(B \cap C) = (A-B) \cup (A-C)$.

Hence proved.

Ans. Prove the following properties of cartesian Product.

Let A, B and C be sets. Then,

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

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(1)

Let (x,y) be some arbitrary element of $A \times (B \cap C)$.

$\Rightarrow$ $x \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ and $y \in C)$ — $\{\because$ and is distributive over and

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow$ $x \in (A \times B)$ and $x \in (A \times C)$

$\Rightarrow$ $x \in (A \times B) \cap (A \times C)$.

$\Rightarrow$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ — (1)

Let (a,b) some arbitrary element of $(A \times B) \cap (A \times C)$

$\Rightarrow$ $a \in (A \times B)$ and $b \in (A \times C)$

$\Rightarrow$ $a \in A$

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Let (a,b) an arbitrary element of $(A \times B) \cap (A \times C)$

$\Rightarrow (a,b) \in (A \times B)$ and $(a,b) \in (A \times C)$

$\Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow a \in A$ and $b \in B$ and $a \in A$ and $b \in C$

$\Rightarrow \underbrace{a \in A \text{ and } a \in A}$ and $b \in B$ and $b \in C$ [associative law]

$\Rightarrow a \in A$ and $b \in B$ and $b \in C$ [using indempotant law]

$\Rightarrow a \in A$ and $b \in (B \cap C)$

$\Rightarrow (a,b) \in A \times (B \cap C)$

$\Rightarrow (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ —— (II)

from (I) and (II)

$(A \times B) \cap (A \times C) = A \times (B \cap C)$. H.P.

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1. $A \times (B-C) = (A \times B) - (A \times C)$

2. $(A-B) \cap (C-B) = (A \cap C) - B$

3. $(\bar{A} \times \bar{B}) = (\overline{A \times B})$

4. Use set properties to prove.

a) $A - (A \cap B) = A - B$

(b) $(\overline{A \cup B \cap C}) = (\bar{C} \cup \bar{B}) \cap \bar{A}$

Answers :

1. Let (x,y) be an arbitary element of $A \times (B-C)$ where, $x \in A$ and $y \in (B-C)$

$x \in A$ and $(y \in B$ and $y \notin C)$

$x \in A$ and $x \in A$ and $y \in B$ and $y \notin C$.

$\underbrace{x \in A \text{ and } y \in B}$ and $x \in A$ and $y \notin C$.

$(x, y) \in A \times B$ and $x, y \notin (A \times C)$

$x, y \in (A \times B) - (A \times C)$

Let (a,b) be an arbitary element of $(A \times B) - (A \times C)$

~~$a \in A \times$~~ $(a,b) \in (A \times B)$ and $(a,b) \notin (A \times C)$

$a \in A$ and $b \in B$ and ~~$a \in A$~~ $a \in A$ and $b \notin C$

$a \in A$ and $b \in B$ and $b \notin C$.

$a \in A$ and $b \in (B-C)$

$(a,b) \in A \times (B-C)$

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2. $(A-B) \cap (C-B) = (A \cap C) - B$

Let x, be an arbitary element of $(A-B) \cap (C-B)$

$\Rightarrow x \in (A-B)$ and $x \in (C-B)$

$\Rightarrow x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$.

$\Rightarrow x \in A$ and $x \in C$ and $x \notin B$ [Indempotance law]

$\Rightarrow x \in A \cap C$ and $x \notin B$

$\Rightarrow x \in (A \cap C) - B$. — (I)

Let y be an arbitary element of $(A \cap C) - B$

$\Rightarrow y \in A$ and $y \in C$ and $x \notin B$.

$\Rightarrow y \in A$ and $x \notin B$ and $x \notin B$ and $x \in C$.

$\Rightarrow y \in (A-B)$ and $y \in (C-B)$

$\Rightarrow y \in (A-B) \cap (C-B)$ — (II)

from (I) and (II) $(A-B) \cap (C-B) = (A \cap C) - B$.

3. $\bar{A} \times \bar{B} = \overline{A \times B}$

Let (x,y) be an arbitary element of $\bar{A} \times \bar{B}$

$\Rightarrow x \notin A$ and $y \notin B$.

$\Rightarrow (x,y) \notin A \times B$

$\Rightarrow (x,y) \in \overline{A \times B}$ — (I)

$\Rightarrow \bar{A} \times \bar{B} \subseteq \overline{A \times B}$

Let (x,y) be an arbitry element of $\overline{A \times B}$

$\Rightarrow x, y \in \overline{A \times B}$

$\Rightarrow (x,y) \notin A \times B$

$\Rightarrow x \notin A$ and $y \notin B$.

$\Rightarrow x \in \bar{A}$ and $y \in \bar{B}$

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$\Rightarrow (x,y) \in (\bar{A} \times \bar{B})$

$$\overline{A \times B} \subseteq \bar{A} \times \bar{B} \quad \text{— (ii)}$$

from (i) and (ii)

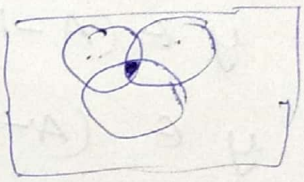
$$\bar{A} \times \bar{B} = \overline{A \times B}$$

4. (a) $A - (A \cap B) = A - B$

$\cancel{(A \cap A) - (A \cap B)}$

$$\begin{aligned} A - B &= A \cap B' \\ &= A \cap (U - B) \\ &= (A \cap U) - (A \cap B) \\ &= A - (A \cap B) \end{aligned}$$

(set difference is not distributive over $\cap$ or $\cup$)



(b) $(\overline{A \cup B \cap C}) = (\bar{C} \cup \bar{B}) \cap \bar{A}$

$$\begin{aligned} (\overline{A \cup Z}) &= \bar{A} \cap \bar{Z} \\ &= \bar{A} \cap (\overline{B \cap C}) \\ &= \bar{A} \cap (\bar{B} \cup \bar{C}) \end{aligned}$$

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4 (a) $A - (A \cap B)$ $\qquad A - B = (A \cap B')$

$= A \cap (A \cap B)'$ [Demorgan's Law]

$= \cancel{A \cap (B \cup B')}$

$= A \cap (A' \cup B')$

$= A \cap A' \cup A \cap B'$

$= \phi \cup A \cap B'$

$= (A - B)$

Relation

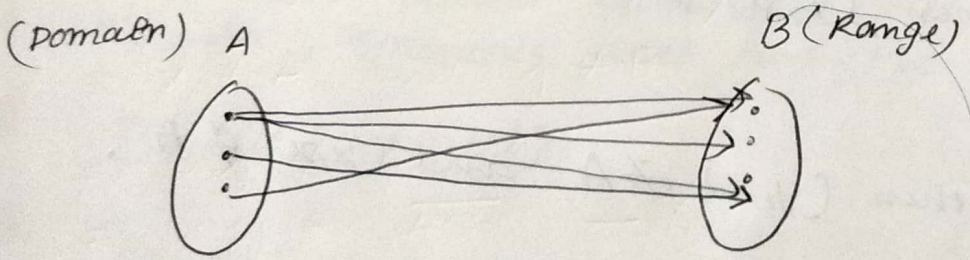
* subset of cartesian product of two sets which satisfy relation / condition / link b/w ordered pair.

A binary relation from a set A to a set B is a subset $R \subseteq A \times B = \{(a,b) \mid a \in A, b \in B\}$

Relation v/s function.

* In a relation each $a \in A$ can map to multiple elements in B.
* Relation is more general than function.

when $(a,b) \in R$ we say aRb



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Type of Relations:

1) Reflexive Relation:

If all ordered pairs $(a,b)\ \forall\ a \in A$ appear in the relation, R is called reflexive.

A relation R on a set A is called reflexive if

$\forall a \in A\ (a,a) \in R$.

Ex. $<=$, $=$

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a,b \in N,\ a/b \in Z\}$

$R_2 = \{(a,b) \mid a,b \in N,\ a-b = 2\}$.

2) Symmetric Relation.

A relation is called symmetric if

$\forall\ a,b \in A\ \{(b,a) \in R \Leftrightarrow (a,b) \in R\}$

Ex: $\parallel$ (parallel) $(a \parallel b \Rightarrow b \parallel a)$

if $(x,y) \in A$ then (y,x) must also belong to the set

3) Asymmetric Relation:

$\forall\ (a,b) \in R$ then $(b,a) \notin R$.

Ex; $>$, $<$

if $(x,y) \in A$ then $(y,x) \notin A$ and $(x,x) \notin A$.

4) Antisymmetric

A relation R on a set A is called antisymmetric if

$\forall\ a,b \in A\ [(a,b) \in R \wedge (b,a) \in R \Rightarrow a = b]$

If $(x,y) \in A$ then $(y,x) \notin A$.

(x,x) may or may not be in the set

Ex. subset, / (Asymetric is a type of Antisymmetric)

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5) Transitive Relation

A Relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall\ a,b,c \in R$.

Ans. $R = \{(1,1),(1,2)\ (2,1),(3,3),(4,4)\}$

It is symmetric as $\forall\ a,b \in R$
$(b,a) \in R$.

Not antisymmetric as $(1,2)$ & $(2,1)$ but $1 \neq 2$.

Ans $R = \{(1,1)\}$ symmetric and anti symmetric.

Ans $R = \{(1,3)\ (3,2)\ (2,1)\}$ anti symmetric, asymmetric.

Ans. $\{(4,4), (3,3), (1,4)\}$ anti-symmetric.

Equivalence Relation:

A Relation is called equivalence relation if it is Reflexive, symmetric and transitive in nature.

Ex. // (parallel)

Partial order Relation:

If a Relation is Partial order Relation if it is Reflexive, Antisymmetric, transitive in nature.

Ex. / , <=

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* Congruence relation modulo n.

Given two integers a, b & a natural numbe $n > 1$ we say that a is congruent to b modulo n & write $a \equiv b \pmod{n}$ if $a-b$ is ~~divider~~ divisible by n.

Ex. $29 \equiv 8 \bmod 7$ and $60 \equiv \bmod 15$

Ex $3 \equiv 5 \bmod 2$

or $\boxed{a-b = nk}$

Where, $k \rightarrow$ some integer.

Is congruence relation is a Equivalence Relation

Ans. Reflexive. aRa.

$$a-a = nk$$
$$0 = nk.$$

hence it is is reflexive.

Symmetric: $aRb \Rightarrow bRa$.

$$a-b = nk.$$

$$bRa = \quad b-a = nk$$
$$-(a-b) = nk.$$

hence it is symmetric.

Transitive:

if aRb and $bRc \Rightarrow aRc$.

$a-b = nk_1$ and $b-c = nk_2$ — (ii)

— (i)

adding (i) & (ii)

$$a-b+b-c = n(k_1+k_2)$$
$$a-c = n(k_1+k_2)$$

$\Rightarrow aRc$. Hence Transitive.

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Hence it is a Equivalence relation.

Equivalence class

If $\sim$ is an equivalence relation on S and $x \in S$, the set

$$E_x = \{ y \in S : x \sim y \}$$

Every equivalence relation on a set generates a partition of the set, precisely the partition its equivalence class.

Ex: on $X = \{1, 2 \ldots 10\}$ the relation $x \equiv y$ defined by 3 divides $x - y$

It is an equivalence relation.

We can form $X/\equiv$ which is the set of all equivalence classes.

$[1] = \{1, 4, 7, 10\}$ or $[4]$ or $[7]$ or $[10]$

$[2] = [8] = \{2, 5, 8\}$ or $[5]$

$[3] = \{3, 6, 9\}$ or $[6]$ or $[9]$.

Ex. congruence relation modulo 3 : on It is a equivalence relation on set of integers.

$E_0 = \bar{0} = \{0, 3, 6, 9 \ldots, -3, -6, -9 \ldots\}$

$E_1 = \bar{1} = \{1, 4, 7, \ldots -1, -4, -7\}$

$E_2 = \bar{2} = \{2, 8, 5, \ldots, -2, -5, -8\}$.

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10 Let $A = \{0, 1, 2, 3\}$ and a Relation

$$R = \{(0,0)(0,1)(0,3)(1,0)(1,1)(2,2)(3,0),(3,3)\}$$

Yes, it is Reflexive, symmetrix ~~and transitive~~ but not transitive

<u>Example 2</u>: For $x, y \in R$ define $x \simeq y$ to mean that $x-y \in Z$. Prove that $\simeq$ is an equivalence relation.

$\forall x \in R$

$x - x = 0 \quad \in Z$

$\Rightarrow \cancel{xx} \; xRx$ (Reflexive)

$\forall x, y \in R$.

if $x - y \in nk\ (Z)$

$y - x \in -nk\ (Z)$

$\Rightarrow yRx$. (symmetrix)

$\forall x, y, z \in R$.

$x - y \in Z$ & $y - z \in Z$

$\Rightarrow x - y + y - z \in Z$

$\Rightarrow x - z \in Z$.

xRz (Transitive)

Hence it is an equivalence Relation

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Let $A = \{1, 2, 3 \ldots 9\}$ and R be the relation in $A \times A$ defined by $(a,b) R (c,d)$ if $(a+d) = b+c$ for $(a,b),(c,d)$ in $A \times A$.

Prove that R is an equivalence relation and also obtain the equivalent class $[(2,5)]$.

let $(a,b) \in R$.

$(a,b) R (a,b)$

$= (a+b) = (b+a)$

$\Rightarrow$ LHS = RHS $\therefore$ it is Reflexive.

let $(a,b) \in R$. and $(c,d) \in R$

$\Rightarrow a+d = c+b$. — (I)

$(c,d) R (a,b)$

$c+b = d+a$ — (II)

comparing (I) & (II)

$\because$ both are equal

$\therefore$ it is symmetric.

let $(a,b), (c,d), (e,f) \in R$

$(a,b) R (c,d)$ and $(c,d) R (e,f)$

$\Rightarrow a+d = b+c$ — (I)

& $c+f = d+e$ — (II)

adding (I) & (II)

$a+\cancel{d}+f+\cancel{c} = b+\cancel{f}+\cancel{d}+e$

$a+f = b+e \Rightarrow (a,b) R (e,f)$

$\Rightarrow$ It is a transitive Relation.

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$[(2,5)] = \{(2,5), \cancel{(5,2)}, (1,4), (4,7), (3,6),\}$

$2+d = 5+c$

$2+4$ $5+1$

$2+7$ $5+4$

$(2,5)$ $2+2$

$[(2,5)] = \{(2,5)\ (1,4)\ (4,7)\ (3,6)\}$

$a, b\ R\ (c,d)$

$a+d$

$2+6 = 5+$

$2,5\ R\ (3,6)$

$+6 = (5+3)$

$2+$

$2,5\ R\ 5,8$

Ans $A = \{1, 2, 3, 4, 5, 6\}$.

$R = \{(1,1), (2,2)\ (3,3), (4,4), (5,5)\ (6,6)$
$(2,3)\ (3,2), (4,5)\ (5,4)\ (4,6)\ (6,4)$
$(5,6)\ (6,5)\}$

$[1] = \{1\}$ $\qquad$ $[2] = \{2,3\} = [3]$

$[4] = \{4,5,6\} = [5] = [6]$.

Ans Define a relation on Z as xRy if & only if $3x-5y$ is even. Prove R is an equivalence Relation. Describe the equivalence class.

let $x \in R$.

$3x-5x = -2x$
$= 2(-x)$

$\Rightarrow xRx$ (Reflexive)

let $x, y \in R$.

$3x-5y$.

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* combining Relations:

$A = \{1, 2, 3, 4\}$ $B = \{1, 2, 3, 4\}$

$R_1 = \{(1,2)\ (1,3)\ (1,4)\ (2,2)\ (3,4)\ (4,1)\ (4,2)\}$

$R_2 = \{(1,1)\ (1,2)\ (1,3)\ (2,3)\}$

$R_1 \cap R_2 =$

$R_2 \cap R_1 =$

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Composition of Relation :

Let R_1 be a relation from the set A to B and R_2 be a relation from B to C, i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C.$$

the composition of R_1 and R_2 is the relation consist of ordered pair (a, c) where $a \in A$ & $c \in C$ for which $\exists$ an element $b \in B$

$$R_2 \circ R_1 \quad (R_1 \text{ composition on } R_2)$$

Power of Relations :

Using the composit way of combining relations allows us to recursively define power of a Relation R on

consider $R_1 = \{(1,1), (2,1), (3,2), (4,3)\}$

$$R^2 = \{(1,1)\ (2,1)\ (3,1)\ (4,2)\}$$

$R^2 = \{R\} \circ \{R\}$

$= \{(1,1)\ (2,1)\ (3,2)\ (4,3)\} \circ \{(1,1)\ (2,1)\ (3,2)\ (4,3)\}$

here $(a, \underline{b})\ (\underline{b}, c) = (a, c)$ should be in R

$R^2 = \{(1,1)\ (2,1)\ (3,1)\ (4,2)\}$

$R^3 = R^2 \circ R$

$= \{(1,1)\ (2,1)\ (3,1)\ (4,2)\} \circ \{(1,1)\ (2,1)\ (3,2)\ (4,3)\}$

$= \{(1,1)\ (2,1)\ (3,1)\ (4,1)\}$

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$R^4 = R^3 \circ R$

$= \{(1,1)(2,1)(3,1)(4,1)\} \circ \{(1,1)(2,1)(3,2)(4,3)\}$

$= \{(1,1), (2,1), (3,1)(4,1)\} = R^3.$

Note $R^n = R^3$ for $n = 4, 5, 6$.

Representation of Relations:

- 0-1 matrices
- Directed graphs.

0-1 matrix

The relation R can be represented.

$$m_{ij} = \begin{cases} 1 & \text{if } (a,b) \in R \\ 0 & \text{if } (a,b) \notin R \end{cases}$$

Ex.

$$\begin{array}{c|c} & \overbrace{\begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix}}^{B} \\ A \begin{cases} a_1 \\ a_2 \\ a_3 \\ a_4 \end{cases} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Matrix Representation: combining Relation.

Union: An entry in matrix of the union of two Relation $R_1 \cup R_2$ is

$$R_1 \cup R_2 = R_1 \vee R_2.$$

Intersection:

$$R_1 \cap R_2 = R_1 \wedge R_2$$

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Ans. $M_{R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $M_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $M_{R_1 \cap R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

composition Relations:

0-1 matrix are also useful.

$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $M_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$M_{R_2 \circ R_1} = M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 0+0+0 & 0+0+1 & 0\ 0\ 1 \\ 0+1+0 & 0+1+1 & 0+1+0 \\ 0+1+0 & 0+1+0 & 0+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Directed Graphs Representation.

set of vertices and edges.

A directed graph consist of

A set of vertices (or nodes) &

A set E of edges of ordered pair of elements of V.

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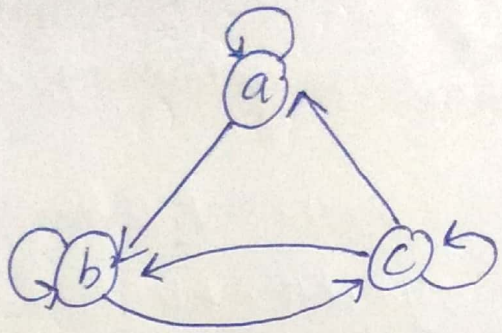
Ex. $R = \{(a,a)\ (b,b)\ (a,b)\ (b,c)\ (c,c)\ (c,b)\ (c,a)\}$.



Closures:

If a given Relation R

↳ is not reflexive (or symmetric, antisymmetric, transitive)

↳ How can we transform it into R' that is

Ex $R = \{(1,2)\ (2,1)\ (2,2), (3,1)\ (3,3)\}$.

Inserting a property on a relation is called closure

i.e $R \cup \{1,1\}$ (Reflexive closure).

Reflexive closures:

The so

Diagonal Relation.

Symmetric closure:

$R \cup R'$ is symmetric closure

Transitive closure:

$$R^+ = \bigcup_{j=1}^{\infty} R^j$$

$\infty \rightarrow$ till we get all are selected.

4 element R^4

3 elem R^3

The transitive closure of R is obtained by repeatedly adding (a,c) to R for each $(a,b) \in R$ & $(b,c) \in R$.

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Simple Algorithm for computing transitive closure.

$A = M_R$

$B = A$

for $i = 2$ to n do

$A = A \odot M_R$

$B = B \vee A$

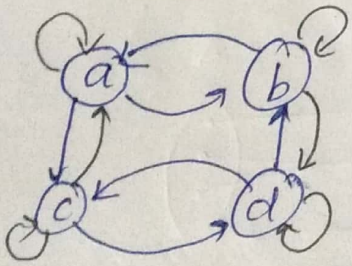
end for

return B.

$$M_R^* = M_R \vee M_R^2 \vee M_R^3 \vee \dots \vee M_R^n.$$

~~Use~~ Ans. For directed graph.

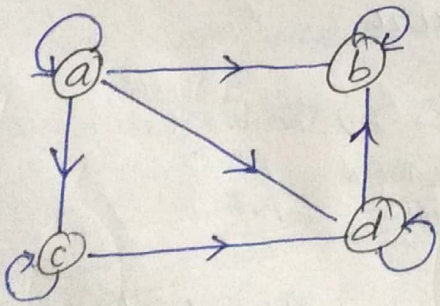
$R = \{(a,b)\ (b,a)\ (a,c)\ (c,d)\ (d,c)\ (d,b)\}$.



$R \cup \{(a,a)\ (b,b)\ (c,c)\ (d,d)\}$

$R \cup \{(c,a), (b,d)\}$.

Ans.



$R = \{(a,a)\ (b,b), (c,c), (d,d), (a,b), (a,c), (d,b)$
$(c,d)\ (a,d)\}$

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It is reflexive.
It is antisymmetric.
It is not transitive. $\because$ $(a,b)\ (b,c) \in R$
but $(a,c) \notin R$.

It is not POR.

Ans: $A = \{1,2,3,4\}$ $R = \{(1,2)\ (2,1)\ (2,3)\ (3,4)\ (4,1)\}$

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^4} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$R^+ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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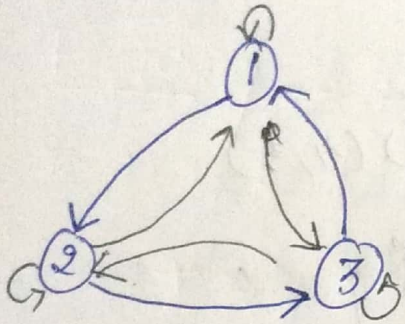
Ans. $R = \{(1,2)\ (2,3)\ (3,1)\}$

Find Reflexive symmetric and transitive closure of R.



Reflexive = $R \cup \{(1,1)\ (2,2)\ (3,3)$

Symmetric : $R \cup \{(2,1)\ (1,3)\ (3,2)\}$

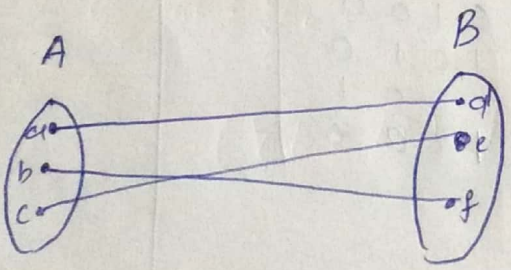
Transitive closure : $R \cup \{(1,3)\ (2,1)\ (3,2)\}$

Functions :

Functions are the subset of cartesian product.

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

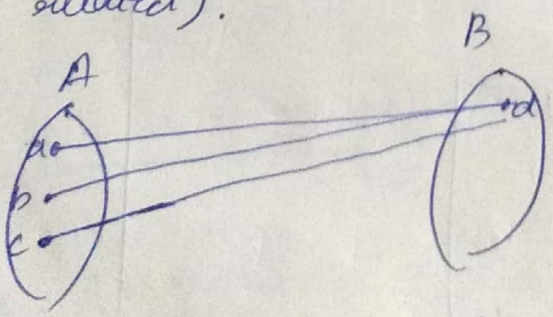
$f(a) = b$ if b is the unique element of B assigned by the function f to the element $a \in A$



(each pre image have unique ~~[illegible]~~ - image)

1 pre can have only 1 ~~[illegible]~~

each element of A is related to B. (Not all bs are necessarily related).



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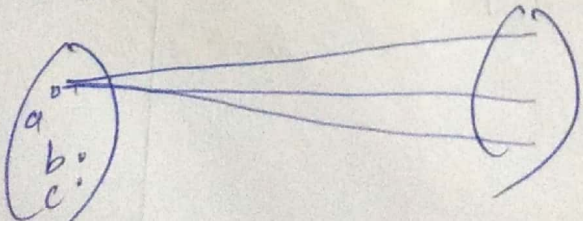

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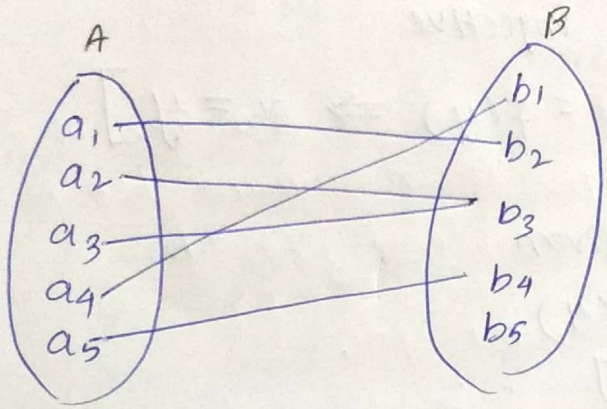
Let f_1 and f_2 be two functions from a set A to R
Then f_1+f_2 & f_1f_2 are also function from A to R

$$\left.\begin{aligned} -(f_1+f_2)(x) &= f_1(x)+f_2(x) \\ -\ f_1f_2(x) &= f_1(x)\cdot f_2(x). \end{aligned}\right\} \text{well defined}$$

Ex.



$f: A \rightarrow B$

Domain $(a_1\ a_2\ a_3\ a_4\ a_5)$
co-domain $(b_1\ b_2\ b_3\ b_4\ b_5)$
Range $(b_1, b_2\ b_3\ b_4)$.

$S = \{a_1\ a_3\} \longrightarrow \{b_2\ b_3\}$.
Range for S.

Types of Function :

i) Injective Function: $\forall x$ and y in the domain of f, $f(x) = f(y) \Rightarrow x = y$.

$\Rightarrow$ one-one function.

ii) Into function. $f: X \rightarrow Y$ is said to be an into function $\exists$ at least one element in co-domain which is not related to A.

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Surjection: (onto)

$\forall b \in B, \exists a \in A$ with $f(a) = b$.

Proving Injectivity:

To prove: $f: A \to B$ is injective

$(\forall x, y \in A)\ [f(x) = f(y) \Rightarrow x = y]$

Let $x, y \in A$ be given

assume $f(x) = f(y)$

[Logical Deduction] - - -

$\therefore\ x = y$ Hence f is injective.

Proving Surjectivity:

$f: A \to B$ is: $(\forall b \in B)\ \exists a \in A\ [f(a) = b]$

Let $b \in B$

Find an a that maps into the given element b

Show that $f(a) = b$.

Hence f is surjective.

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Q1. $f: Z \to Z$ be defined by $f(x) = 2x - 3$.

check wether the function is bijective or not.

let $x_1, x_2 \in Z$.

$f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\boxed{x_1 = x_2}$ hence the function is injective.

let $x \in Z$

$f(x) = 2x - 3$

$y = 2x - 3$

$x = \frac{y+3}{2}$

$\because$ for $y = 2$ $x \notin Z$.

$\therefore$ it is not surjective.

Q2. $f: N \to N$ $f(x)\ 2x - 3$

dom $N \to N$ codo

let $x_1, x_2 \in N$

$f(x_1) = f(x_2)$

$2x_1 - 3 = 2x_2 - 3$

$\boxed{x_1 = x_2}$ Hence, it is injective.

let $x \in N$.

$f(x) = 2x - 3$

$y = 2x - 3$

$x = \frac{y+3}{2}$

$\because$ for $y = 2$ $\nexists\ x \in N$

$\therefore$ it is not onto.

Range (

2

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$f(1) = -1 \notin N$

hence it is not a function

$R = \{-1, 1, 3, 5 - - - - \}$ here $-1 \notin N$

hence not a function.

Inverse Function:

$f: A \to B$ be a bijection. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

The inverse function is denoted f^{-1}.

When f is a bijection, its inverse exist

$f(a) = b$; $f^{-1}(b) = a$.

* When a function is inverted.

The domain and co-domain are interchanged.

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THEOREM PROVING TECHNIQUES :

1) Principle of Mathematical Induction.

It is a recursive approch in which we prove any expression by the use of hypothetical proof.

Theorem:

1. The statment is true for $n = 1$.
2. $P(K+1)$ is true when $P(K)$ is also true
3. Hence conclude $P(n)$ it is true for $P(n)$.

Ans. $3^n - 1$ is multiple of 2 for $n = 1, 2, 3 - - -$

for $n = 1$, $3^1 - 1 = 3 - 1 = 2$ which is multiple of 2.

Let assume $3^n - 1$ is true for $n = K$,

Hence, $3^K - 1$ is true i.e $3^K - 1 = 2\lambda$ —①.

So, for $n = K+1$.

$$3^{K+1} - 1 = 2\lambda.$$

$$= 3^K . 3 - 1 = 2\lambda$$

$$= 2.3^K + (3^K - 1) = 2\lambda.$$

Here, 2.3^K is alwas multiple of 2. & from ①
we have $(3^K - 1) = 2\lambda$.

$\Rightarrow$ $3^{K+1} - 1$ is a multiple of 2.

Hence, it is true for $3^n - 1$.

Hence proved.

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4. $1+2+2^2+2^3+\ldots 2^n = 2^{n+1}-1$

for $n=1$.

$1+2 = 2^{1+1}-1$

$3 = 3$ LHS = RHS

hence, it is true for $n=1$.

Let it be true for $n=k$

$\Rightarrow 1+2+2^2+\ldots 2^k = 2^{k+1}-1$ —— (1).

for $n=k+1$

$\underbrace{1+2+2^2+\ldots+2^k}+2^{k+1} = 2^{(k+1)+1}-1.$

$\Rightarrow 2^{k+1}-1+2^{k+1} = 2^{(k+1)+1}-1$

$\Rightarrow 2.2^{k+1}-1 = 2^{(k+1)}.2-1$

LHS = RHS.

$\therefore$ it is true for $n=k+1$

$\therefore$ we can conclude it is true for 'n' belongs to non-negative integers.

Hence proved.

4. $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots\frac{1}{(2n-1)(2n+1)} = n/2n+1$

for $n=1$; $\frac{1}{1\times3} = \frac{1}{2(1)+1}$

$\frac{1}{3} = \frac{1}{3}$ (LHS = RHS)

$\therefore$ it is true for $n=1$.

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Let it be true for $n = k$.

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots \frac{1}{(2k-1)(2k+1)} = \frac{k}{(2k+1)} \quad \text{—— (1)}$$

for $n = k+1$.

$$\underbrace{\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots \frac{1}{(2k-1)(2k+1)}} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{(k+1)}{(2(k+1)+1)}$$

$$\Rightarrow \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{(k+1)}{2(k+1)+1}$$

$$\Rightarrow \frac{k(2k+3)+1}{(2k+3)(2k+1)} = \text{''}$$

$$\Rightarrow \frac{2k^2+3k+1}{(2k+3)(2k+1)} = \text{''}$$

$$\Rightarrow \frac{2k^2+2k+k+1}{(2k+3)(2k+1)} = \text{''}$$

$$\Rightarrow \frac{2k(k+1)+1(k+1)}{(2k+3)(2k+1)} = \text{''}$$

$$= \frac{\cancel{(2k+1)}.(k+1)}{(2k+3)\cancel{(2k+1)}} = \text{''}$$

$$= \frac{(k+1)}{(2k+3)} = \frac{k+1}{(2k+3)} \qquad LHS = RHS$$

Hence it is true for $n = k+1$.

$\therefore$ it ~~will~~ it true for $n \in N$.

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Direct Proofs:

We derive the result by combining logically the given assumption.

Ex. Sum of two odd no.s are even

let $a = 2K_1 + 1$ & $b = 2K_2 + 1$

$$a + b = 2K_1 + 1 + 2K_2 + 1$$
$$= 2(K_1 + K_2 + 1)$$

Ex. If a and b are two consecutive integers, then the sum $a+b$ is odd

$b = a + 1$

$a + b = 2a + 1$

$a + b = 2K + 1$ (Hence always odd).

Proof By Contrapositive:

direct: $p \to q$

contrapositive: $\sim q \to \sim p$

Ex. if $n > 0$ and $4^n - 1$ is prime, then n is odd.

Assume: $n = 2K$ is even. Then

$$4^n - 1 = 4^{2K} - 1 = (4^K - 1)(4^K + 1).$$

Therefore, $4^n - 1$ factors (are both bigger than 1) and hence is not prime.

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Let a and b be even integer

i.e $a = 2K_1$ and $b = 2K_2$.

$a+b = 2(K_1+K_2)$

$\Rightarrow$ sum of two even no. is even or sum of two even no. is not odd.

a, b are odd $\rightarrow$ sum even

~~a, be are even ... sum~~

sum is odd $\rightarrow$ a, b are even.

let $a+b = (2K+1)$

P_1 no. is odd ; P_2 no. is odd

$P_1 \wedge P_2 \rightarrow Q$. (Direct)

$\sim Q \rightarrow \sim(P_1 \wedge P_2)$

$\sim Q \rightarrow \sim P_1 \vee \sim P_2$ (Contrapositive)

$\rightarrow$ P_1 and P_2 are odd and then sum is even

$\rightarrow$ sum is odd if either P_1 or P_2 is even.

$a+b = (2K_1)+(2K_2+1)$

$= 2(K_1+K_2)+1$

$=$ odd.

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Solving of Recurrence Relation :

A recurrence relation for a sequence $\{a_n\}$ is an equation that express a_n in terms of more or previous terms of sequence.

A recursive definition of a sequence specifies :

↳ Initial condition

↳ Recurrence relation

$a_0 = 0$ and $a_1 = 3$ ⟶ initial condition

$a_n = 2a_{n-1} - a_{n-2}$

$a_n = 3n$

$3n$ is satisfying the solution as it is satisfying sequence giving each term of a sequence of

$2a_{n-1} - a_{n-2}$

- Iterative method
- Recursive method
- characteristics equations.
- Generating function.

- Iterative Method :

$a_n - 2a_{n-1} = 0$ for $n >= 1$ and $a_0 = 1$

solu^n ⇒ $a_n - 2a_{n-1} = 0$

$a_n = 2a_{n-1}$ $n = 1$

$a_1 = 2$

$a_2 = 2a_1$
$= 4$

$a_r = 2^r$

for $n >= 1$

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• Recursive method:

$a_n - 2a_n - 1 = 0$ for $n >= 1$ & $a_0 = 1$

$$a_n = 2a_n - 1$$
$$= 2^2 a_{n-2}$$
$$= 2^3 a_{n-3}$$
$$\vdots$$
$$= 2$$

Ans 1. $a_n = a_{n-1} + 3$ for $n >= 1$ & $a_0 = 1$

(iterative)

$$a_1 = a_0 + 3$$
$$= 1 + 3$$
$$= 4$$

$$a_2 = a_1 + 3$$
$$= 4 + 3 = 7.$$

$$a_3 = a_2 + 3$$
$$= 7 + 3 = 10$$

$$a_n = a_{n-1} + 3$$

$$a_1 = a_0 + 3 = 1 + 3$$
$$a_2 = a_1 + 3 = 1 + 3 + 3. = 1 + 2.3.$$
$$a_3 = a_2 + 3 = 1 + 2.3 + 3 = 1 + 3.3$$

$$a_r = a_{n-1} + 3 = 1 + (n-1).3 + 3 = 1 + 3n.$$

(Recursive).

$$a_n = a_{n-1} + 3$$
$$= a_{n-2} + 3 + 3$$
$$= a_{n-2} + 2.3$$
$$= a_{n-3} + 3 + 2.3$$
$$= a_{n-3} + 3.3.$$

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$a_r = a_{r-r} + 3 + (n-1).3$

$a_r = 1 + 3n.$

Linear Homogeneous Recurrences :

(can be solved with characteristics eqn).

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots c_k a_{n-k}$$

with $c_1, c_2, c_3 \ldots c_k \in R$.

* it is homogenous as each term is depending on some previous term.

Order / Degree : 'k' is the order i,e a_n depends on 'k' previous terms.

Ex Determine ~~the~~ wether the following are linear homogeneous recurrance relation or not.

(i) $P_n = (1.11) P_{n-1}$ ✓

(ii) $a_n = a_{n-1} + a^2_{n-2}$. ✗ (not linear)

(iii) $f_n = f_{n-1} + f_{n-2}$. ✓

(iv) $H_n = 2H_{n-1} + 1$ ✗ not homogenous $\because$ 1 is not depending on any a_k.

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(V) $a_n = a_{n-6}$ ✓ $K = 6$.

, LHRR

(VI) $B_n = n B_{n-1}$. ✗ coefficients are not constant.

Solution of L-H-R-R.

Let $a_n = r^n$ be the solun of the recurrance relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots C_k a_{n-k}.$$

put $a_n = r^n$;

$$r^n = C_1 r^{n-1} + C_2 r^{n-2} + \cdots C_k r^{n-k}.$$

divide both side by r^{n-k}.

if α and β are distinct & real roots.

$$x_n = C_1 \alpha^n + C_2 \beta^n.$$

~~$x_n = C_1 \alpha^n + C_2$~~

if α, β & γ are distinct & real roots

$$x_n = C_1 \alpha^n + C_2 \beta^n + C_3 \gamma^n.$$

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* If roots are real and equal.

$\alpha = \beta = r$

$$x_n = c_1 r^n + c_2 n r^n.$$

* If roots are imaginary.

$$x_n = K_1 r^n \cos n\alpha + K_2 r^n \sin n\alpha.$$

Q. Find a solution to

$a_n = 5a_{n-1} - 6a_{n-2}$

with initial condition $a_0 = 1, a_1 = 4$.

Let $a_n = r^n$ is solution of the eqn

substitute the value of a_n in the given eqn.

$$r^n = 5r^{n-1} - 6r^{n-2} \quad \text{—(1)}$$

devide (1) by r^{n-2} on both side

$$r^2 = 5r - 6.$$

$r^2 - 5r + 6 = 0$

$r^2 - 2r - 3r + 6 = 0$

$r(r-2) - 3(r-2) = 0$

$r = 3 ; r = 2$

$x_n = c_1 \alpha^n + c_2 \beta^n$

$x_n = c_1 (3)^n + c_2 (2)^n$

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$a_n = c_1(3^n) + c_2(2)^n$

put $n=0$.

$a_0 = c_1(3)^0 + c_2(2)^0$

$1 = c_1 + c_2$ — (a)

put $n=1$

$a_1 = c_1(3)^1 + c_2(2)^1$

$4 = c_1(3) + c_2(2)$

$2c_1 + 2c_2 = 4$

$(-)\ 2c_1 + 3c_2 = 2$

$(-) \quad (-)$

$\boxed{c_1 = 2}$ $\boxed{c_2 = -1}$

$a_n = 2(3)^n + (-1)2^n$

Q. $a_n = 8a_{n-1} - 16a_{n-2}$

$a_0 = 1, \quad a_1 = 7.$

Let $a_n = r^n$ be a solution.

$r^n = 8r^{n-1} - 16r^{n-2}$

$r^2 = 8r - 16$

$r^2 - 8r + 16 = 0$

$r^2 - 4r - 4r + 16 = 0$

$r(r-4) - 4(r-4) = 0$

$r = 4, 4.$

$a_n = c_1(4)^n + c_2(4)^n.$

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$a_0 = c_1(4)^0 + c_2(4)^0$

$\cancel{c_1 + c_2 = 1 \quad (I)}$

$\cancel{a_1 = c_1(4)^1 + c_2(4)^1}$

$\cancel{4c_1 + 4c_2 = 7 \quad (II)}$

$\cancel{4(1-c_2) + 4c_2 = 7.}$

$4c_1 + 4c_2 = 7$

$a_0 = c_1(4)^0 + nc_2(4)^0$

$\boxed{c_1 = 1}$

$a_1 = c_1(4)^1 + c_2(4)^1$

$4c_1 + 4c_2 = 7.$

$\boxed{c_2 = 3/4}$

$$\boxed{a_n = 1(4)^n + \frac{3}{4}n(4)^n}$$

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4. Suppose that a person deposit Rs 10000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 3 years?

- Find the recurrence relation
- solve using method of iteration and recursive approach
- Apply method of characteristic eqn if applicable.

$$A_n = 10000\left(1+\frac{11}{100}\right)^T$$

$$= 10000\left(1+\frac{11}{100}\right)^1$$

$$A_n = A_{n-1} + A_{n-1}\left(\frac{11}{100}\right)$$

~~$A_n = A_{n-1} + 11\% \text{ of } A_{n-1}$~~

$$A_n = 1.11\, A_{n-1}$$

$$A_1 = 1.11\,(10000)$$

$$A_2 = (11100)\,1.11$$

$$= 111 \times 111$$

$$= 12321$$

$$A_3 = 1.11 \times (12321).$$

$$= 13676.31.$$

$$A_n = 1.11 A_{n-1}.$$

$10000 + \left(\frac{11}{100} \times 10000\right.$

$+$

$T(n) = T(n) + \frac{11}{100} T(n)$

$P_1 \quad (10K + \frac{11}{100} \times 10K.)$

$P_1 = 10K + 1100$

$P_2 = P_1 + 11\% P_1$

$P_3 = P_2 + 11\% P_2.$

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$x^n = 1.11\, x^{n-1}$

$x = 1.11$

$A_n = 1.11 \cdot A_{n-1}$

$r - 1.11 = 0$

$r = 1.11$

$\boxed{a_n = c_1 (1.11)^n}$

$n = 0$

$a_0 = c_1 = 10,000$

Iterative:

Example:

An employee joined a company in 2009 with a starting salary of 50,000. every year this employee received of 1000 plus 5% of the salary of previous year.

$A_0 = 50,000$

2009
2019

$A_1 = A_0 + 1000 + \frac{5}{100} A_0$.

$A_n = A_{n-1} + 1000 + \frac{5}{100} A_{n-1}$

$A_n = 1.05 A_{n-1} + 1000$

can't apply C.E method $\because$ it is not homogeneous.

$A_{2009} = 50,000$

$A_n = 1.05 (1.05 A_{n-2} + 1000) + 1000$

$= (1.05)^2 A_{n-2} + 1000 + 1.05 (1000)$

$= (1.05)^3 [A_{n-3}] + 1000] + (1.05)(1000) + 1000 (1.05)^0$

$n = 1$:

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$$= (1.05)^{k}[A_{n-k}] + (1.05)^{k-1}(1000) + (1.05)^{k-2}(1000) + \ldots\ldots + 1000$$

~~for k=9~~ for $k=n$

$\cancel{(1.05)^{8}[A}$

$$A_n = (1.05)^{n}[A_0] + (1.05)^{n-1}(1000) + (1.05)^{n-2}(1000) + \ldots\ldots 1000$$

$$= (1.05)^{n}[A_0] + 1000\,[(1.05)^{n-1} + (1.05)^{n-2} + \ldots (1.05)^{0}]$$

$$= (1.05)^{n}[A_0] + \frac{1000}{1.05}\,[(1.05)^{n} + (1.05)^{n-1} + \ldots (1.05)]$$

$$= (1.05)^{n}[A_0] + 1000\,[(1.05)^{n-1} + (1.05)^{n-2} + \ldots 1].$$

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Ans $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$.

$a_0 = 8, a_1 = 6, a_2 =$

Ans. $a_n = 8a_{n-2} - 16a_{n-4}$

let r^n be a solun

$r^n = 8 r^{n-2} - 16 r^{n-4}$

$r^4 = 8r^2 - 16$

$\therefore r^4 - 8r^2 + 16 = 0$

$r^2(r^2 - 8) + 16 = 0$

~~$r^2(r^2-8)=0$~~

for $r = 2, 2, -2, -2$

$a_n = c_1(2)^n + nc_2(2)^n + c_3(-2)^n + nc_4(-2)^n$

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Solution of Non-Homogeneous Linear Recurrence Relation

$f(n)$	Trial functions
1. constant	1. A
2. b^n (b is not root of C.E)	2. Ab^n
3. $P(n)$ (polynomial of degree m)	3. $A_0 + A_1 n + A_2 n^2 + \dots A_m n^m$
4. b^n (b is not of C.E with multiplicity m)	4. $A_0 n^m b^n$
5. $b^n P(n)$ (b is not root of C.E)	5. $b^n (A_0 + A_1 n + \dots A_m n^m)$
6. $b^n (P(n))$ (b is root of C.E with multiplicity m)	6. $n^m b^n (A_0 + A_1 n + \dots A_m n^m)$

* $a_n = 5a_{n-2} - 3a_{n-1} + f(n)$

$a_n = a_n^h + a_n^p$

a_n^h = Homogeneous solution

a_n^p = particular solu^n.

Ex. $a_{n+2} - 5a_{n+1} + 6a_n = 2$. — (1) $a_0 = 1, a_1 = -1$.

For homogeneous part :

$a_{n+2} - 5a_{n+1} + 6a_n = 0$

let $a_n = r^n$ be a solu^n.

$r^{n+2} - 5r^{n+1} + 6 = 0$

divide by r^n both side

$r^2 - 5r + 6 = 0$

$r^2 - 2r - 3r + 6 = 0$

$(r-2)(r-3)$

$\boxed{r = 2, 3}$

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so, $a_n = c_1 2^n + c_2 3^n$ — (B)

here, $f(n) = 2$ (constant)

$\therefore a_n = A$ (substituted).

substitute all $a_{n+m} = A$. $(m = 0, 1, 2 \ldots)$ in (1).

$A - 5A + 6A = 2$

$\Rightarrow A = 1$

$\boxed{a_n^p = 1.}$

$a_n = a_n^h + a_n^p$

$a_n = c_1 2^n + c_2 3^n + 1$

$a_0 = 1$; $c_1 + c_2 + 1 = 1$ — (a)

$\cancel{c_1 2^{-1} + c_2 3^{-1} + 1 = -1}$

$\cancel{c_1/2 + c_2/3 + 2 = 0.}$

$c_1 = -c_2$ from (a).

$\cancel{c_1/2 = -2 - c_2/3}$

$\cancel{-c_2/2 = -2 + c_2/3}$

$\cancel{-c_2/3 + c_2/2 = 2}$

$\cancel{\frac{-2c_2 + 3c_2}{6} = 2}$

$c_2 =$

$a_1 = -1$

$a_1 = c_1 2^1 + c_2 3^1 + 1$

$2c_1 + 3c_2 + 1 = -1$

$2(-c_2) + 3c_2 + 1 = -1$

$\boxed{c_2 = -2}$ $\boxed{c_1 = 2}$

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$a_n = 2.(2)^n + (-2)(3^n) + 1$

$a_n = 2^{n+1} - 2(3)^n + 1.$

4. $y_{n+2} - y_{n+1} - 2y_n = n^2.$

for homogeneous part: $y_{n+2} - y_{n+1} - 2y_n = 0.$

let $y_n = a_n$ be a solun

$a^2 - a - 2 = 0$

$a^2 - 2a + a - 2 = 0$

$a(a-2) + 1(a-2) = 0$

$\boxed{a = -1, 2}$

$a_n^h = C_1(-1)^n + C_2(2)^n.$

here $f(n) = n^2.$

let $a_n = A_0 + nA_1 + n^2A_2.$

or $A + Bn + Cn^2. \quad — (*)$ in (I)

$a_n^{n^2} = A + B(n+2) + C(n+2)^2 - [A + B(n+1) + C(n+1)^2] - 2[A + Bn + Cn^2]$

$= A + B(n+2) + C(n+2)^2 - A - B(n+1) - C(n+1)^2 - 2A - 2Bn - 2Cn^2$

$= \cancel{A} + \cancel{Bn} + \cancel{2}B + \cancel{Cn^2} + \overset{1}{\cancel{4}}C + \overset{2}{\cancel{4}}Cn - \cancel{A} - \cancel{Bn} - \cancel{B} - \cancel{Cn^2} - \cancel{2}Cn - \cancel{C} - 2A$
$- 2Bn - 2Cn^2$

$= B + 3C + 2Cn - 2A - 2Bn - 2Cn^2.$

$= (-2C)n^2 + 2(C-B)n - 2A + B + 3C. = n^2$

$= (-2C)n^2 + 2(C-B)n - (2A - B - 3C) = n^2$

$-2C = 1$	$2C - 2B = 0$	$-(2A - B - 3C) = 0$
$\boxed{C = -1/2}.$	$-1 - 2B = 0$	$-(2A + 1/2 + 3/2) = 0$
	$\boxed{B = -1/2}$	$-2A + 2 = 0$
		$\boxed{A = -1}$

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$$a_n^p = A + Bn + Cn^2$$
$$= -1 - \tfrac{1}{2}n - \tfrac{1}{2}n^2$$

$$a_n = a_n^h + a_n^p$$
$$= c_1(2)^n + c_2(-1)^n - \left(\tfrac{n^2}{2} + \tfrac{n}{2} + 1\right)$$

Qs. $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$

$$m^2 - 4m + 4 = 0$$
$$(m-2)(m-2) = 0$$
$$m = 2, 2.$$

$$a_n^h = c_1(2)^n + n c_2 (2)^n.$$

Let $a_n = An^2 2^n$.

non homogenous part $= 2^n$.

Let $a_n = An^2 2^n$.

$$A(n+2)^2 2^{n+2} - 4A(n+1)^2 2^{n+1} + 4A \cdot n^2 2^n = 2^n.$$
$$4A(n+2)^2 - 8A(n+1)^2 + 4An^2 = 1$$
$$n^2(4A - 8A + 4A) + n(16A + (-16A)) + (16A - 8A) = 1$$

$$8A = 1$$
$$\boxed{A = 1/8}$$

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$$a_n^p = \frac{n^2 2^n}{8}$$

$$\boxed{a_n = c_1 (2)^n + n c_2 (2)^n + \frac{n^2 2^n}{8}}$$

Ex. $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 \quad (f(n))$.

$r^2 + 5r + 6 = 0$ (aux. eqn)

$r^2 + 2r + 3r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2 ; \; r = -3$.

$a_n^h = c_1 \alpha^n + c_2 (\beta)^n$

$= c(-2)^n + c_2(-3)^n$

$f(n) = 3r^2$.

$A + Bx + Cx^2 = n^2$

$= A + Bx + Cx^2 + 5[A + B(x-1) + C(x-1)^2] + 6[A + B(x-2) + C(x-2)^2]$

$= (A + 5A + 6A + 5(-B) + 5C + (-12B) + 24C) + (B + 5B - 2C + (-12B) - 24C)\, x$.

$+ (C + 5C + 6C)x^2$.

$12C = 1$

$\boxed{C = 1/12}$

$6B - 12B - 26C = 0$

$-6B = 26C$

$-6B = \frac{26}{12} \cdot 13$

$B = \frac{13}{36} = \frac{1}{3}$

$12A - 17B + 29C = 0$.

$12A - \frac{17}{3} + \frac{29}{12} = 0$

$A = \frac{1}{12}\left(\frac{17}{3} - \frac{29}{12}\right)$

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Q6. $a_n + 5a_{n-1} + 6a_{n-1} = 2^n \cdot n$

Q7. $a_n + 5a_{n-1} + 6a_{n-2} = 2^n + n^2$

Solution of Recurrence Relation using Generating Functions :

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k \, a^{n-k} b^k$$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k = \sum_{k=0}^{n} {}^nC_k (1)^{n-k} (x)^k = \sum_{k=0}^{n} {}^nC_k x^k$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$(1-x)^n = \sum_{k=0}^{n} {}^nC_k (+1)^{n-k} (-x)^k$$

$$= 1 - nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$\frac{1}{(1-x)} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

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$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$$

$$\frac{1}{1+ax} = 1 - ax + a^2x^2 - a^3x^3 + \cdots$$

Generating function: G.F. for a sequence $a_0, a_1, a_2, a_3, a_4, \cdots$ (sequence) is given by.

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$\boxed{G(x) = \sum_{n=0}^{\infty} a_n x^n}$$

Here, for the series $1, 1, 1, 1, 1, \cdots$ $(1 + x + x^2 + x^3 + \cdots)$

$$G.F(x) = \frac{1}{1-x}$$

For the series $1, -1, 1, -1, 1, -1 \cdots$ $(1 - x + x^2 - x^3 \cdots)$

$$G(x) = \frac{1}{(1+x)}$$

For series: $1, 2, 3, 4, 5, \cdots$ $(1 + 2x + 3x^2 + 4x^3 + \cdots)$

$$G(x) = \frac{1}{(1-x)^2}.$$

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Ans. Find G.F for

1) 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0 - - -

2) 1, 1, 1, 1, 1, 1, - - - - - -

3) 1, 3, 3, 1, 0, 0, - - - - - -

1) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$= 1x^0 + x + x^2 + x^3 + x^4 + x^5 + \underbrace{0 + 0 + 0 + \cdots}_{\text{ignore}}$

$= 1 + x + x^2 + x^3 + x^4 + x^5$

$S_n = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{x^n - 1}{x - 1} = \dfrac{x^6 - 1}{x - 1}$

2) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$= a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= 1 + x + x^2 + x^3 + \cdots$

$= \dfrac{1}{1 - x}.$

3) $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + 0 + 0 + 0 \cdots$

$= 1 + 3x + 3x^2 + x^3 \quad [1^3 + 3(1)^2 x + 3(1)(x)^2 + x^3]$

$= (1 + x)^3$

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Ex. $a_n - 5a_{n-1} + 6a_{n-2} = 0$ $\quad a_0 = 1, a_1 = -1$ $(n \geq 2)$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n$$

$$= 1 + -1(x) + \sum_{n=2}^{\infty} a_n x^n \quad \text{— (a)}$$

$$a_n = 5a_{n-1} - 6a_{n-2} \} \quad \text{— (1)}$$

$$= 1 - x + 5\sum_{n=2}^{\infty} a_{n-1} x^n - 6\sum_{n=2}^{\infty} a_{n-2} x^n$$

$$G(x) = 1 - x + 5[a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots] - 6[a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots]$$

$$G(x) = 1 - x + 5x[a_1 x + a_2 x^2 + a_3 x^3 + \dots] - 6x^2[a_0 + a_1 x + a_2 x^2 + \dots]$$

$$G(x) = 1 - x + 5x[G(x) - a_0] - 6x^2[G(x)]$$

$$G(x) = 1 - x + 5x[G(x) - a_0] - 6x^2[G(x)]$$

$$G(x) = 1 - x + 5x[G(x) - 1] - 6x^2[G(x)] \quad [\because a_0 = 1]$$

$$G(x) = 1 - x + 5x\,G(x) - 5x - 6x^2 G(x)$$

$$G(x) - 5x\,G(x) + 6x^2 G(x) = 1 - x - 5x$$

$$(1 - 5x + 6x^2)\,G(x) = 1 - 6x$$

$$G(x) = \frac{1 - 6x}{1 - 5x + 6x^2} = \frac{1 - 6x}{(3x - 1)(2x - 1)}$$

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$$\Rightarrow \frac{A}{(3x-1)} + \frac{B}{2x-1} = \frac{1-6x}{(3x-1)(2x-1)}$$

$$\Rightarrow A(2x-1) + B(3x-1) = 1-6x$$

$$2Ax - A + 3Bx - B = 1-6x$$

$$(2A+3B)x - (A+B) = 1-6x$$

$$2A+3B = -6 \qquad -A-B = 1$$

$$B = -A-1$$

$$\Rightarrow 2A+3(-A-1) = -6$$

$$\Rightarrow 2A-3A-3 = -6$$

$$\Rightarrow -A = -3$$

$$\boxed{A = 3}$$

$$B = -3-1$$

$$= -4$$

$$\boxed{B = -4}$$

$$= \frac{3}{(3x-1)} + \frac{(-4)}{(2x-1)}$$

$$\boxed{G(x) = \frac{3}{(3x-1)} - \frac{4}{(2x-1)}}$$

or $\quad G(x) = \left.\frac{4}{1-2x} - \frac{3}{1-3x}\right\}.$

now $\quad \frac{4}{1-2x} = 4\left(\frac{1}{1-2x}\right) = 4\left[\sum_{n=0}^{\infty} 2^n x^n\right]$

$$\left\{\because \frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n\right\}$$

Similarly. $\quad 3\left(\sum_{n=0}^{\infty} 3^n x^n\right).$

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$$G(x) = 4 \cdot \sum_{n=0}^{\infty} 2^n x^n - 3 \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (4 \cdot 2^n - 3 \cdot 3^n) x^n$$

$$= \sum_{n=0}^{\infty} (4 \cdot 2^n - 3^{n+1}) x^n$$

$$= \sum_{n=0}^{\infty} (2^{n+2} - 3^{n+1}) x^n$$

So, $\boxed{a_n = 2^{n+2} - 3^{n+1}}$ $\left[\because G(x) = \sum_{n=0}^{\infty} a_n x^n\right.$

4s. $a_n - 3a_{n-1} + 2a_{n-2} = 0$ for $n \geq 2$

$a_0 = 2,\ a_1 = 3.$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

$$= 2 + 3 + \sum_{n=2}^{\infty} a_n x^n$$

$$= 2 + 3 + \cancel{a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots} \sum_{n=2}^{\infty} a_n x^n$$

$\{ a_n = 3a_{n-1} - 2a_{n-2} \}$

$$= 2 + 3x + 3 \sum_{n=2}^{\infty} a_{n-1} x^n - 2 \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$= 2 + 3x + 3 [a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots] - 2 [a_0 x^2 + a_1 x^3 + \dots]$$

$$= 2 + 3x + 3x [a_1 x + a_2 x^2 + a_3 x^3 + \dots] - 2x^2 [a_0 + a_1 x + \dots]$$

$$= 2 + 3x + 3x [G(x) - a_0] - 2x^2 [G(x)]$$

$$= 2 + 3x + 3x G(x) - 6x - 2x^2 G(x) \quad [\because a_0 = 2]$$

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$$(2x^2-3x+1)\,G(x) = 2-3x.$$

$$G(x) = \frac{2-3x}{2x^2-3x+1} = \frac{2-3x}{(2x^2-2x-x+1)}$$

$$= \frac{2-3x}{2x(x-1)-1(x-1)}$$

$$= \frac{2-3x}{(2x-1)(x-1)} = \frac{A}{(2x-1)} + \frac{B}{(x-1)}$$

$$2-3x = A(x-1) + B(2x-1)$$

for $2-3x = Ax - A + 2Bx - B$

$$2-3x = (A+2B)x + (-A-B)$$

$-A-B = 2$; $A+2B = -3$.

$A = -B-2$

$-B-2+2B = -3$

$\boxed{B=-1}$

$\boxed{A=-1}$

$$G(x) = \frac{1}{(1-2x)} + \frac{1}{(1-x)} \quad \left[\because \frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n\right].$$

$$= \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} 1^n x^n$$

$$= \sum_{n=0}^{\infty} (1+2^n)x^n.$$

where,

$$\boxed{a_n = 1+2^n}$$

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Solving Non-Homogeneous Linear Recurrence Relation using Generating Function:

Qs. $a_{n+2} - 5a_{n+1} + 6a_n = 2 \quad n \geq 0, \; a_0 = 1, \; a_1 = 3$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= \sum_{k=0}^{\infty} a_k x^k$$

$$\boxed{a_n = \frac{2 - a_{n+2} + 5a_{n+1}}{6}}$$

$$G(x) = \frac{1}{6} \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) = \frac{1}{6} \sum_{n=0}^{\infty} (2 - a_{n+2} + 5a_{n+1}) x^n$$

$$6G(x) = -\sum_{n=0}^{\infty} a_{n+2} x^n + 5 \sum_{n=0}^{\infty} 5a_{n+1} x^n + 2 \sum_{n=0}^{\infty} x^n.$$

$$6G(x) = -[a_2 + a_3 x + a_4 x^2 + \cdots] + 5[a_1 + a_2 x + a_3 x^2 \cdots] + 2[1 + x + x^2 + \cdots]$$

$$6G(x) = -[a_2 + a_3 x + a_4 x^2 + \cdots] + 5[a_1 + a_2 x + a_3 x^2 \cdots] + \frac{2}{1-x}$$

& multiply x^2 and x and divide the same

$$6G(x) = \frac{-1}{x^2}[a_2 x^2 + a_3 x^3 + \cdots] + \frac{5}{x}[a_1 x + a_2 x^2 + \cdots] + \frac{2}{1-x}.$$

$$6G(x) = -\frac{1}{x^2}[G(x) - a_0 - a_1 x] + \frac{5}{x}[G(x) - a_0] + \frac{2}{1-x}.$$

$$6G(x) = -\frac{1}{x^2}[G(x) - 1 - 3x] + \frac{5}{x}[G(x) - 1] + \frac{2}{1-x}.$$

$$6x^2 = -G(x) + 1 + 3x + 5x\,G(x) - 5x + \frac{2x^2}{1-x}$$

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$$(6x^2-5x+1)\,G(x) = 1-2x+\frac{2x^2}{1-x}.$$

$$G(x) = \frac{1-2x}{6x^2-5x+1} + \frac{2x^2}{(1-x)(6x^2-5x+1)}$$

$$= \frac{1-x-2x+2x^2+2x^2}{(1-x)(6x^2-5x+1)}$$

$$= \frac{1-3x+4x^2}{(1-x)(1-2x)(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-2x)} + \frac{C}{(1-3x)}$$

$$A(1-2x)(1-3x) + B(1-x)(1-3x) + C(1-x)(1-2x) = 1-3x+4x^2$$

$$A(1-5x+6x^2) + B(1-4x+3x^2) + C(1-3x+2x^2) = 1-3x+4x^2$$

$$(6A+3B+2C)x^2 + (-5A-4B-3C)x + (A+B+C) = 1-3x+4x^2$$

$$6A+3B+2C = 4 \;;\quad -5A-4B-3C = -3 \;;\quad A+B+C = 1$$

$$6A+3B+2C = 4 \qquad A = 1-B-C.$$

$$6A+6B+6C = 1$$

$$(-)\quad (-)\quad (-)$$

$$\boxed{-3B-4C = 3}$$

$$-5+5B+5C-4B-3C = -3$$

$$2C-B = 2$$

$$B = 2C-2.$$

$$\frac{A}{} = \frac{B}{} = \frac{C}{}$$

$$6A+3B-2C = 4$$

$$6(1-B-C)+3(2C-2)-2C = 4$$

$$6(1-2C+2-C)+3(2C-2)-2C = 4$$

$$\cancel{6}-12C+12-\cancel{6}C+\cancel{6}C-\cancel{6}-2C = 4$$

$$-14C = 16$$

$$C = -2;\; B = 2;\; A = 1.$$

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$$G(x) = \frac{1}{1-x} + \frac{2}{1-3x} - \frac{2}{1-2x}$$

$$G(x) = \sum_{n=0}^{\infty} 1.x^n + 2\sum_{n=0}^{\infty} 3^n x^n + (-2)\sum_{n=0}^{\infty} 2^n . x^n$$

$$a_n = 1 + 2.3^n + (-2)2^n$$
$$= 1 + 2.3^n - 2^{n+1}.$$

<u>4s</u>. $a_{n+2} - 2a_{n+1} + a_n = 2^n$. $\quad a_0 = 2,\ a_1 = 1 \quad n \geq 0$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$
$$= \sum_{n=0}^{\infty} a_n x^n .$$

$$a_n = 2^n - a_{n+2} + 2a_{n+1}$$

$$G(x) = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} a_{n+2} x^n + 2\sum_{n=0}^{\infty} a_{n+1} x^n$$

$$= [2^0 x^0 + 2x + 2^2 x^2 + \dots] - [a_2 x^0 + a_3 x + a_4 x^2 + \dots] + 2[a_1 x^0 + a_2 x^1 + \dots$$

$$= \left[\frac{1}{1-2x}\right] - \frac{1}{x^2}[a_2 x^2 + a_3 x^3 + \dots] + \frac{2}{x}[a_1 x + a_2 x^2 + \dots]$$

$$= \frac{1}{(1-2x)} - \frac{1}{x^2}[G(x) - a_0 - a_1 x] + \frac{2}{x}[G(x) - a_0]$$

$$G(x) = \frac{1}{(1-2x)} - \frac{1}{x^2}[G(x) - 2 - x] + \frac{2}{x}[G(x) - 2].$$

$$x^2 G(x) = \frac{x^2}{(1-2x)} - G(x) + 2 + x + 2x\,G(x) - 4x.$$

$$(x^2 - 2x + 1)\,G(x) = \frac{x^2}{1-2x} + 2 - 3x$$

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$$G(x) = \frac{x^2 + 2(1-2x) - 3x(1-2x)}{(x^2-2x+1)(1-2x)}$$

$$= \frac{x^2+2-4x-3x+6x^2}{(x^2-2x+1)(1-2x)}$$

$$= \frac{7x^2-7x+2}{(x^2-2x+1)(1-2x)} = \frac{7x^2-7x+2}{(1-2x)(x-1)^2}.$$

$$\frac{A}{(1-2x)} + \frac{Bx+C}{(x-1)^2} + \frac{D}{(x-1)} = \frac{7x^2-7x+2}{(1-2x)(x-1)^2}.$$

$$A(x-1)^2 + (Bx+C)(1-2x) + D(1-2x)(x-1) = 7x^2-7x+2$$

$$A[x^2+1-2x] + [Bx-2Bx^2+C-2Cx] + D[x-1-2x^2+2x]$$

$$A[x^2+1-2x] + B(x-2x^2) + C(1-2x) + D(-2x^2+3x-1)$$

$$(A-2B-2D)x^2 + (-2A+B-2C+3D)x + (A+C-D) = 7x^2-7x+2$$

$$A-2B-2D = 7 \; ; \; -2A+B-2C+3D = -7 \; ; \; A+C-D = 2$$

$$A-2B-2A-2C+4 = 7 \qquad -2A+B-2C+3A+3C-6 = -7 \qquad \boxed{D = A+C-2}$$

$$-A-2B-2C =$$

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$$\frac{7x^2-7x+2}{(x-1)^2(1-2x)} = \frac{Ax+B}{(x-1)^2} + \frac{C}{1-2x}$$

$$(Ax+B)(1-2x) + C(x-1)^2 = 7x^2-7x+2.$$

$$Ax - 2Ax^2 + B - 2Bx + Cx^2 - 2Cx + C = 7x^2-7x+2$$

$$(C-2A)x^2 + (A-2B-2C)x + B+C = 7x^2-7x+2$$

$C-2A = 7$; $A-2B-2C = -7$; $B+C=2$

$C = 7+2A.$ $\quad A-2(2-C)-2C = -7$ $\quad (B = 2-C)$

$A-4+2C-2C = -7$

$\boxed{A = -3}$ $\quad \boxed{B = 1}$

$C = 7-6 = 1$

$\boxed{C = 1}$

$$G(x) = \frac{-3x+1}{(x-1)^2} + \frac{1}{(1-2x)}$$

or $$G(x) = \frac{-3x}{(x-1)^2} + \frac{1}{(x-1)^2} + \frac{1}{(1-2x)}$$

$$G(x) = -3\sum_{n=0}^{\infty} n x^n + \sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=0}^{\infty} 2^n x^n$$

$$a_n = -3n + n+1 + 2^n.$$
$$= -2n+1+2^n.$$

$$-3\left[x+2x^2+3x^2+4x^3+\cdots\right]$$

$$\sum_{n=0}^{\infty} n.x^n$$

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Tutorial

1. Solve $a_n = 3a_{n-1} + 2^n$

2. Prove $(A-B) \times C = (A \times C)$

3. In a class, 7 students like to play game-A and 8 like to play game-B. 3 students like to play A & B. How many student like to play A, B or both? Use Venn diagram.

4. Prove $(a,b)(c,d) \in R$ if & only if $ad = bc$ is equivalence

5. Suppose that no. of bacteria in a colony triples every hour.

   1) Find RR for the no. of bacteria after n hours.

   2) Solve RR using C.E and GF. Assume 100 bacteria initially

   3) No. of bacteria after 10 hours?

6. Find G.F. for a) 0, 0, 1, 1, 1, 1, ...

   b) 1, 3, 9, 27, ...

7. Apply mathematical induction to prove

$$\sum_{1 \le i \le n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ for } i \in N.$$

8. Let $X = \{a, b, c\}$ and $f : X \to X$ such that $f = \{(a,b), (b,a), (c,c)\}$

   Find (i) f^{-1}

   (ii) f^2

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Answers :

1. $a_n - 3a_{n-1} = 2^n$.

$a_n^h = ?$ $\qquad$ $a_n^p = ?$

$a_n - 3a_{n-1} = 0$

let $a_{n-1} = r^{n-1}$

$r^n - 3r^{n-1} = 0$

$r - 3 = 0$

$\boxed{r = 3}$

$a_n^h = c_1 (3)^n$ — (1).

here $f(n) = 2^n$

$a_n^p = Ab^n$

$Ab^n - 3Ab^{n-1} = 2^n$ $\qquad$ $b = 2$;

$\cancel{2A - 3A = 2}$

$A2^n - 3A2^{n-1} = 2^n$ $\qquad$ divide by 2^{n-1}

$2A - 3A = 2$

$\boxed{A = -2}$

$a_n^p = Ab^n = -2 \cdot 2^n$
$= -2^{n+1}$

$a_n = a_n^h + a_n^p = c_1 (3)^n - 2^{n+1}$

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2. Let (x,y) an arbitary element $\in (A-B)\times C$

$x \in (A-B)$ and $y \in C$.

$x \in A$ and $x \notin B$ and $y \in C$.

$x \in A$ and $x \notin B$ any $y \in C$ and $y \in C$

$x \in A$ and $y \in C$ and $x \notin B$ and $y \in C$.

$(x,y) \in (A\times C)$ and $(x,y) \notin (B\times C)$

$(x,y) \in (A\times C)-(B\times C)$

$\Rightarrow \quad (A\times C)-(B\times C) \subseteq (A-B)\times C$

Let (x,y) be arbitary element $\in (A\times C)-(B\times C)$

$(x,y) \in (A\times C)$ and $(x,y) \notin (B\times C)$

$x \in A$ and $y \in C$ and $x \notin B$ and $y \in C$.

$x \in A$ and $x \notin B$ and $y \in C$

$x \in (A-B)$ and $y \in C$

$x \in (A-B)\times C$.

$\Rightarrow \quad (A-B)\times C \subseteq (A\times B)-(B\times C)$

$\Rightarrow \quad LHS = RHS$.

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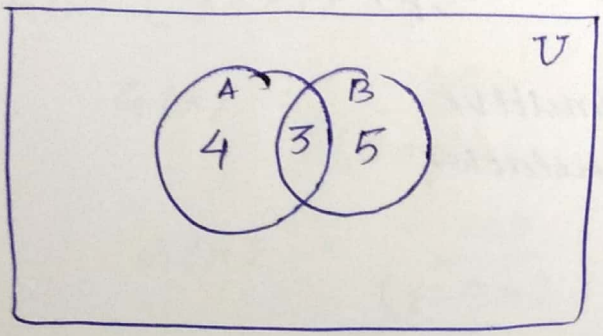
3. $A = 7$; $B = 8$; $A \cap B = 3$.

$$A \cup B = nA + nB - n(A \cap B)$$
$$= 7 + 8 - 3$$
$$= 12$$



$$A \cup B = 4 + 3 + 5$$
$$= 12$$

4. Reflexive : Let $(a, b) \in R$.

$\Rightarrow (a, b) R (a, b)$

$\Rightarrow ab = ba$ (LHS = RHS)

hence it is reflexive.

Symetric : Let (a, b) & $(c, d) \in R$.

$(a, b) R (c, d)$

$\Rightarrow ad = bc$. — (I)

$\cancel{\Rightarrow da = cb}$

$(c, d) R (a, b)$

$\Rightarrow cb = ad$ — (II)

comparing (I) & (II).

it is symmetric.

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Transitive: let (a,b) (c,d) & $(e,f) \in R$.

$(a,b) R (c,d)$ and $(c,d) R (e,f)$

$ad = bc$ — (I) $\qquad cf = de$. — (II)

$adcf = bcde$

$af = be$. $\quad = \quad (a,b) R (e,f)$.

Hence it is transitive.

Hence, it is equivalence relation

5. i) $a_n = 3a_{n-1}$

(ii) C.E. $\quad a_n = 3a_{n-1}$

$r = 3$

$a_n = c_1 3^n$

$= (100)3^n$ at (100 hours).

GF $\quad a_n - 3a_{n-1} = 0$

or $\quad a_{n+1} - 3a_n = 0$.

$a_n = a_{n+1}/3$.

$G(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

or $\sum_{n=0}^{\infty} a_n x^n$

$G(x) = \frac{1}{3}\sum_{n=0}^{\infty} a_{n+1}x^n$

$G(x) = \frac{1}{3}[a_1 + a_2x + a_3x^2 + \cdots]$

$= \frac{1}{3x}[a_1x + a_2x^2 + a_3x^3 + \cdots]$

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$$G(x) = \frac{1}{3x}\left[G(x) - a_0\right]$$

$$G(x) = \frac{1}{3x}\left[G(x) - 100\right]$$

$$G(x)(3x-1) = -100$$

$$G(x) = \frac{-100}{(3x-1)}$$

$$G(x) = \frac{100}{(1-3x)}$$

$$G(x) = 100\left[\frac{1}{1-3x}\right]$$

$$G(x) = 100(1-3x)^{-1}$$

$$= 100\sum_{n=0}^{\infty} 3^n x^n$$

$$\boxed{a_n = 3^n(100)}$$

(iii) $a_{10} = (100)\,3^{10}$

6. $0, 0, 1, 1, 1, \ldots$

$$G(x) = 0 + 0x^1 + 1.x^2 + 1.x^3 + 1.x^4 + \cdots$$

$$= x^2 + x^3 + x^4 + \cdots$$

$$= x^2\left[1 + x + x^2 + \cdots\right]$$

$$= \frac{x^2}{(1-x)}$$

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11) $1, 3, 9, 27 \ldots$

$$G(x) = 3^0 + 3^1 + 3^2 + 3^3 + \ldots$$

$$= 3^0 + 3^1x^1 + 3^2x^2 + 3^3x^3 + \ldots$$

$$= \frac{1}{(1-3x)}$$

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UNIT – III

GRAPH THEORY :

Graph is a set of tuples consist of vertices and edges.

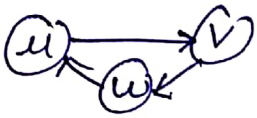
Graph $G = (V, E)$ $V \rightarrow$ vertices
$E \rightarrow$ edges.

Directed edges : ordered pair of vertices. Represented as (u, v) directed from u to v

Undirected edges : unordered pairs $\{u, v\}$.

Loop : A loop is an edge whose starting & endpoint is same.

Multiple Edges :

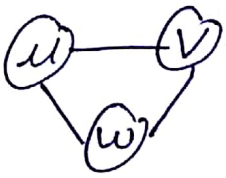


Types of Graphs :

Simple graph.

A graph in which different vertices are connected via edges.

there is no self loop
no multiple edges.



Pendent : 1 degree

Isolated : 0 degree

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maximum no. of edges possible in a simple graph with 'n' vertices is ${}^nC_2$ where ${}^nC_2 = n(n-1)/2$.

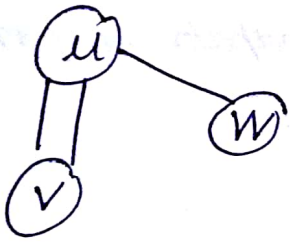
Maximum no. of simple graphs than can be constructed using 'n' vertices is

$$2^{{}^nC_2}$$

Ex. Let $n = 3$ $\quad N = 2^{{}^nC_2} = 2^{{}^3C_2} = 2^{3(3-1)/2}$

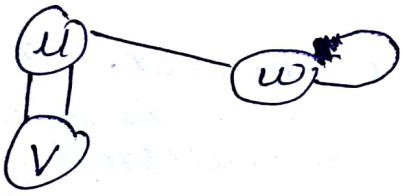
$$= 2^3 = 8.$$

Multigraph : multiple edges connecting same vertices.



Pseudograph :

Both multi edges and loops are their



Directed Graph :

Directed multigraph :

both (multi edges & loops) directed are allowed.

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Mixed Graph:

consist both directed and undirected

Terminology

Degree of vertex : No. of edges connected to the vertex.
→ loop counted twice.

Pendent : 1 to 1

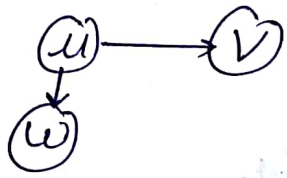
Isolated : No connection

For directed:

Degree of vertex.

In-degree :

out-degree :



u : OD = 2 , ID = 0.

v : OD = 0 , ID = 1

w : OD = 0 , ID = 1.

Theorem.1:

The handshaking theorem :

$$2e = \sum_{v \in V} \deg(v)$$

only applicable for simple undirected graph.

⇒ every edge connects 2 vertices.

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Theorem 2 :

An undirected graph has even no. of vertices with odd no. degree.

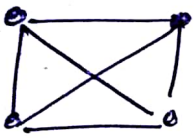
2e = Proof: V_1 is the set of even degree vertices and and V_2 is the set of odd degree vertices.

$$2e = \sum deg(V) = \sum_{u \in V_1} deg(u) + \sum_{V = V_2} deg(V)$$

deg. of (V) is even for $V \in V_1$

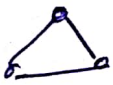
Simple Graph :

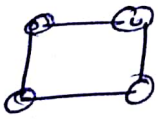
① complete Graph :
when each node is connected with there is direct edge b/w two distinct nodes



K_4.

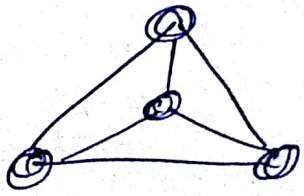
② Cyclic Graph : C_n, $n \geq 3$ consist on n vertices $v_1, v_2, v_3 \ldots v_n$ and edges $\{v_1, v_2\}$, $\{v_2 v_3\}$ -----

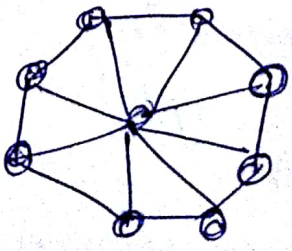




③ Wheel : C_3 or C_n have 1 node at centre.

⇒ W_3 =





W_8.

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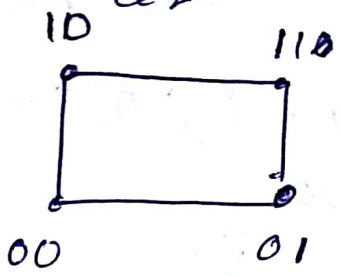
cube graphs. Q_n, vertices represented by 2^n bits.
two nodes are connected when the nodes differ by 1 bit position. i.e 00 01 10 00 11 10

Ex. Q_1

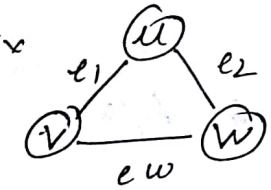
Q_2



Representation:

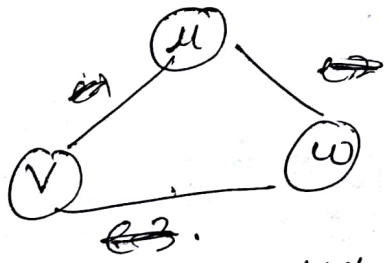
① Incidence Graph: column represent edges, row represent nodes.

Ex



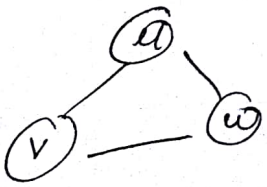
	e_1	e_2	e_3
v	1	0	1
u	1	1	0
w	0	1	1

② Adjacency matrix:



	v	u	w
v	0	1	1
u	1	0	1
w	1	1	0

③ Adjacency List



node	
v	u, w
u	v, w
w	u, v

simple ~~path~~ path: one edge only once.

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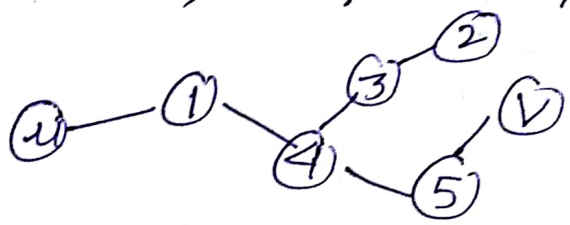
Path : A path is a sequence of edges that begins at a vertex of any graph and travels along edges of the graph, always connecting pair of adjacent vertices.

Ex. { {u,1}, {1,4}, {4,5}, {5,v}, {4,3}, {3,2} }



path (u, 4) = length = 2 .

circuit / cycle : $u \equiv v$ starting and ending point is same & $l > 0$.

Connected: When we have a simple path b/w every pair is called connected.

Ex. How many edges are there in a graph with 10 vertices and six degree.

$$2e = \sum v(d).$$

total no. of degree = 10×6
$= 60$.

$2e = 60$

$\boxed{e = 30}$

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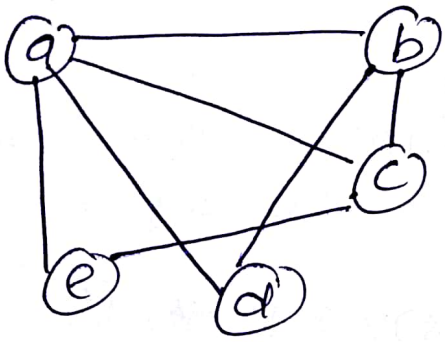
4. How many edges does a graph have if its degree sequence 4, 3, 3, 2, 2? Draw such graph,

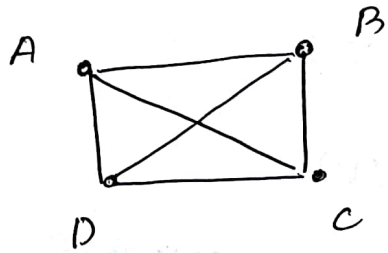
$2e = \Sigma$ (degrees)

$2e = 4+3+3+2+2$

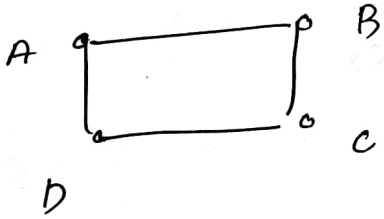
$\boxed{e = 7}$



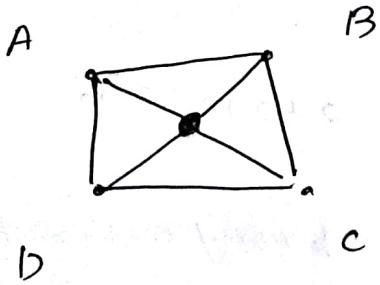
4. (a) K_4



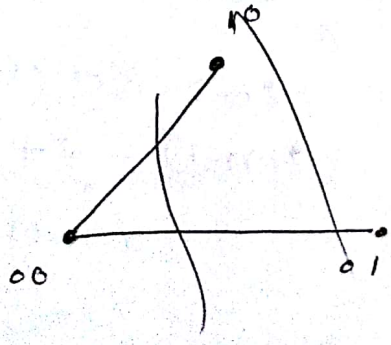
(b) C_4



(c) W_4



(d) Q_3.



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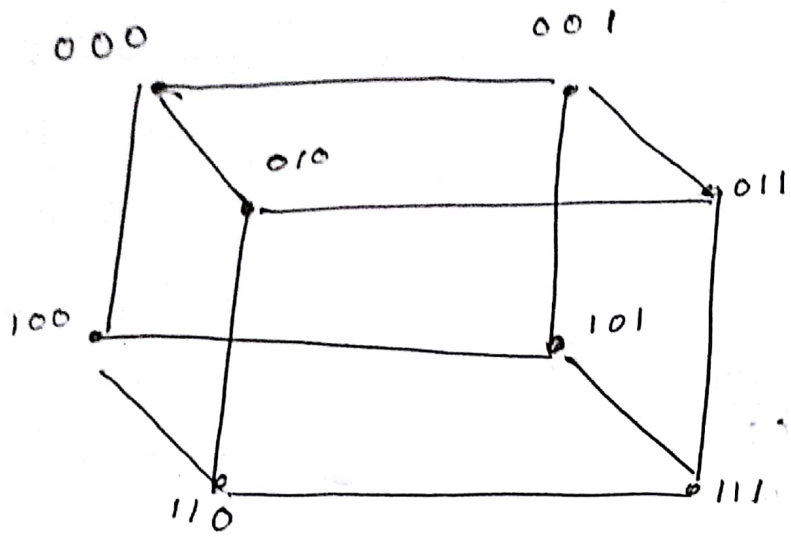
Q_3 (Q_n) $n = 2^3 = 8$



As a Graph has 12 e, 2 V of d(3), 2 V of d(4) and n V of d(5).

$2e = \Sigma d(V)$

$2(12) = 2\times3 + 2\times4\times n5$

$24 = 6 + 8 + 5n$

$5n = 10$

$\boxed{n = 2}$

no. of ~~edge~~ vertices : $2+2+2 = 6$.

Prove that in a full binary tree with n vertices, the number of pendant vertices is $(n+1)/2$.

n vertices.

$(n-1)$ edges.

$2e = \Sigma d(V)$

$2(n-1) = \underset{d(\text{root})}{2} + \underset{\text{leaf nodes (pendent)}}{K(1)} + \underset{\text{Remaining internal nodes}}{3(n-K-1)}$

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$2n - 2 - 2 + 1 + 1 = 3n - 3K - 3$

$2K = n + 4 - 3$

$$\boxed{K = (n+1)/2}$$

No. of pendent nodes in full binary tree is $(n+1)/2$.

Strongly connected: if there is a path from a to b and a path from b to a then it is strongly connected.
And this condition is available for all the nodes.

Weakly connected: if there is a path from a to b but no path from b to a.

The Havel - Hakimi Algorith:

Used to find if a graph is possible or not based on degree.
i.e. wether a degree sequence is graphical or not.

Assume the degree sequence is S.

$S = d_1, d_2, d_3 \ldots d_n$

$d_i \geq d_{i+1}$.

1. If any $d_i \geq n$ then fail.
2. If there is an odd no. of odd degree then fail.
3. If there is a $d_i < 0$ then fail.
4. If all $d_i = 0$ then report success.
5. Reorder S into non increasing order.
6. Let $K = d_1$
7. Remove d_1 from S.
8. subtract 1 from 1st K terms in the sequence & remaining degree

9. repeat from step 3.

Ex 5, 3, 3, 3, 2, 2, 1 1

2 2 2 1 1 ✓

1 1 1 1 ✓

4. 6, 5, 5, 5, 4, 4, 2, 1.

4, 4, 4, 3, 3, 1, 1

3, 3, 2, 2, 1, 1

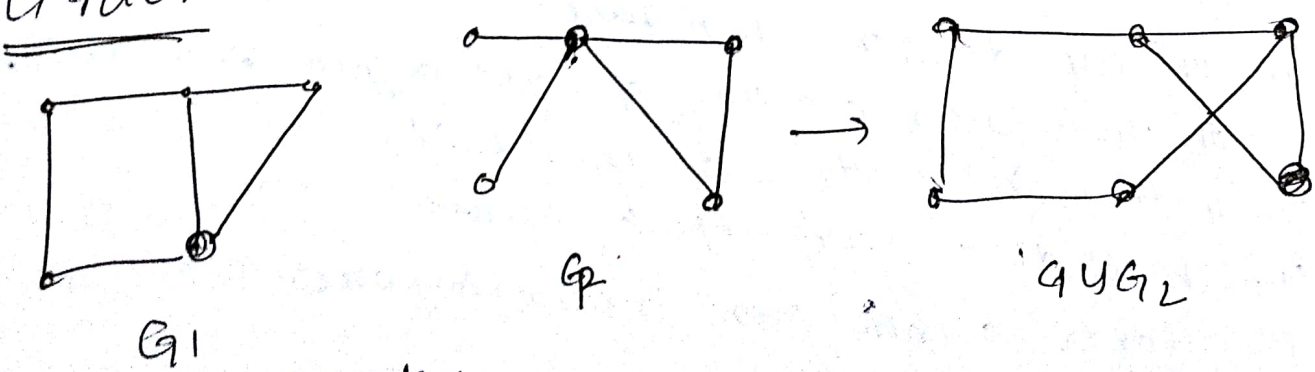
2, 1, 1, 1, 1

0, 0, 1, 1. ✓ (This sequence is possible

Subgraph of n vertices:

$$n(\underline{\text{Sub } G}) = 2^{{}^nC_2}$$

Union:



all edges.

Intersection of graph: (Take common vertices & edges of two graph)

Complement of graph: G is simple graph we will ~~sub~~ remove the existing node & make graphs edges formation.

Graph Ring Sum:
ring sum is
$G_1 \oplus G_2$ is the subgraph of $G_1 \cup G_2$ edge set of $E_1 \oplus E_2$.

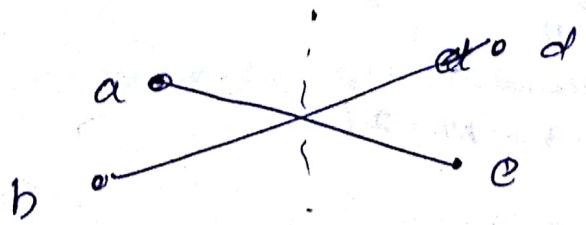
$$E_1 \oplus E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

(Take union of graph and union of edges & remove common edges.)

Regular graph: if all its vertices ~~have~~ have same degree.

Bipartite graph: if its vertex set V can be partitioned into two disjoint

Ex



only connection b/w one partition to other partition.

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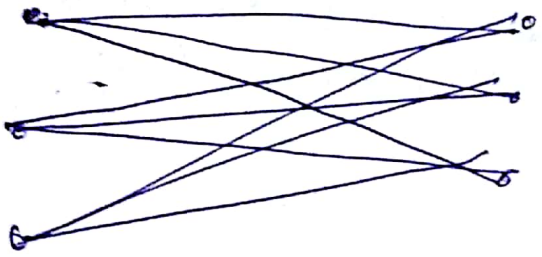
Complete Bipartite Graph :

Each vertices of 1st partition is connected to every vertices of 2nd partion.



$(m+n)$ = total no. of vertices

$(m \cdot n)$ = total no. of edge

4s. If T is a full binary tree with i internal vertices then i+1 terminal vertices and 2i+1 total vertices. Prove by hand shaking lemma.

$l \rightarrow$ total no. of nodes including root.

$$2e = \sum d(V)$$

$$2e = \cancel{1\times2 + 3\times3 + 2(l-1)}$$

$$2e = 1\times2 + 3(l-1) + 2(n-l)$$

$$2(n-1) = \underbrace{2}_{\text{root}} + \underbrace{3(l-1)}_{\underbrace{(i-1)}_{\text{internal nodes}}} + \underbrace{2(n-l)}_{\text{leaf nodes.}}$$

$$2n-2 = 2 + 3l - 3 + 2n - 2l$$

$$n = 3l - l + 2 - 3 + 2$$

$$\boxed{n = 2l + 1}$$

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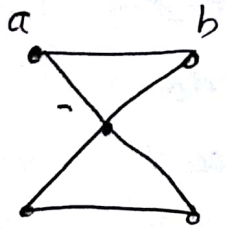
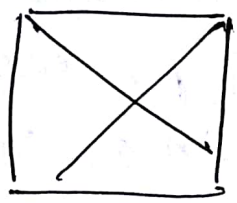
Eulerian Graphs :

Euler circuit : It is a simple circuit containing every edge of G (no repetion of edge) (come back to same start)

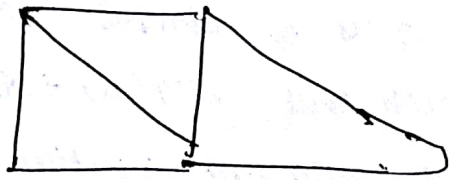
Euler path : In G is a simple path containing every edge of G.



✓ EC
✓ EP



✗ EC
✗ EP



✗ EC
✓ EP

Theorem : Euler ~~path~~ circuit will exist (Not path) if and only if each vertice has even degree.

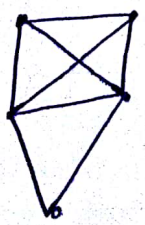
Theorem : A connected multi graph has an euler path but not euler circuit if and only if we have ~~two~~ exactly two vertices with odd degree.

Hamilton Path

A simple path in a Graph G that passes through each vertices only once then it is hamilton path.

Hamilton circuit :

A simple circuit in a Graph G that passes through each vertices only once then it is hamilton circuit



H

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Dirac's Theorem :

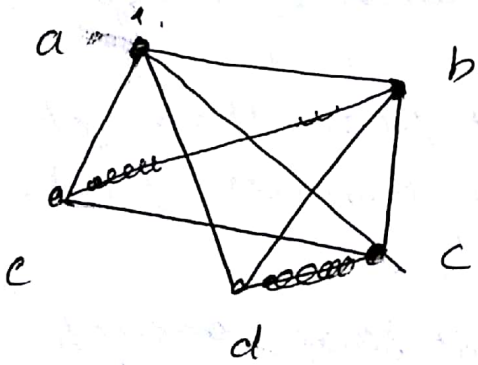
If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is atleast $n/2$, then G has a Hamilton circuit.

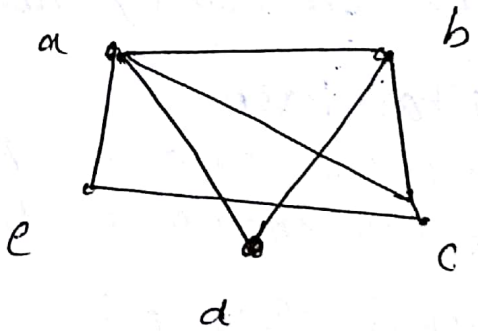
Ore's Theorem :

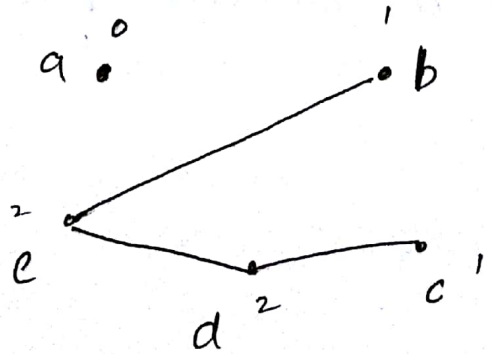
If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non adjacent vertices u and v in G, then G has Hamilton circuit.

Sufficient conditions for Hamilton theorem.

4. If degree sequence of $G = \overset{a}{4}, \overset{b}{3}, \overset{c}{3}, \overset{d}{2}, \overset{e}{2}$.
Find degree sequence for $G' = ?$







$(0, 1, 1, 2, 2)$

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Cut edge (Bridge). A bridge is a single edge whose removal disconnects a graph.

Cut set : It is a set of edges which makes a graph disconnected.

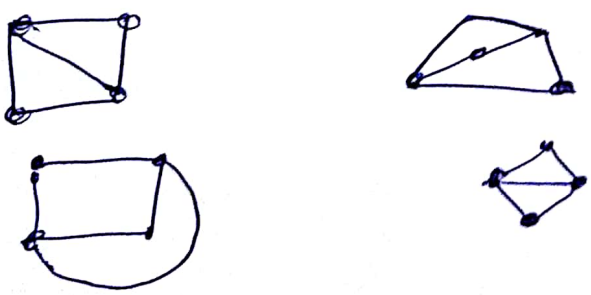
Edge connectivity :
Let G be a connecting graph. The min. no. of edges which can make a graph disconnected. denoted by λ.

Cut Vertex : remove one vertex graph become disconnected

Cut vertex set : set of vertices whose removal makes graph disconnected.

(when vertex is removed edge is also removed)

Graph Isomorphism :
Graph isomorphism is a phenomenon of existing the same graph in more than one forms.



$G1 = (V1, E1)$ and $G2 = (V2, E2)$ are isomorphic if $\exists$ one-one & onto function f from $V1$ to $V2$ with the property that a and b are adjacent in $G1$ if and only if $f(a)$ & $f(b)$ are adjacent in $G(2)$.

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Necessary condition for Isomorphic graph.

→ same no. of V
→ same no. of E
→ same no. of d (V) for corresponding vertices
→ the same number of connected components
→ same no. of loops
→ same no. of multi edge
→ the same circuit of given length.

Sufficient condition for 2 Graphs to be Isomorphic.

1 : If their complement graph are isomorphic

2 : If their adjacency matrices are same.

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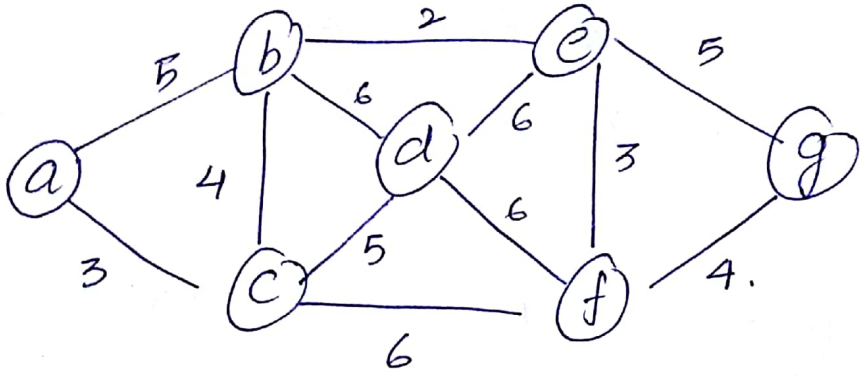
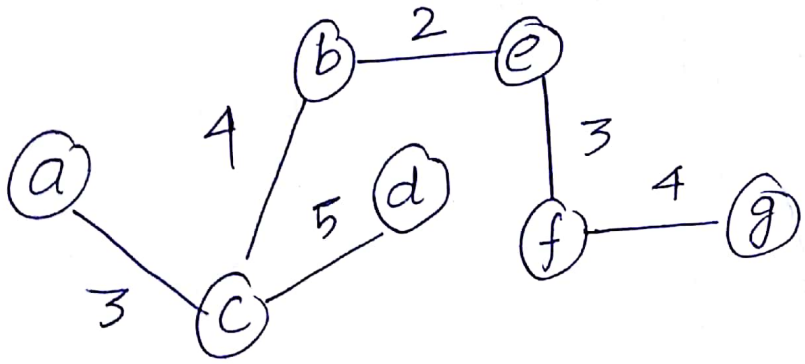
Minimum Spanning Tree :

Ans. Consider the following graph. Find MST using Kruskal's and Prim's.

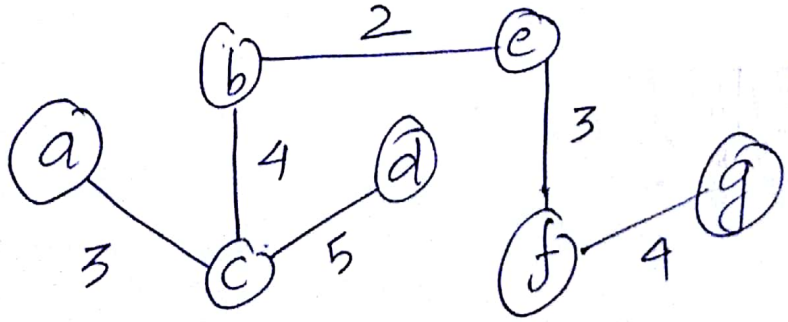


1) Kruskal's



minimum cost = 21

2) Prim's



minimum cost = 21

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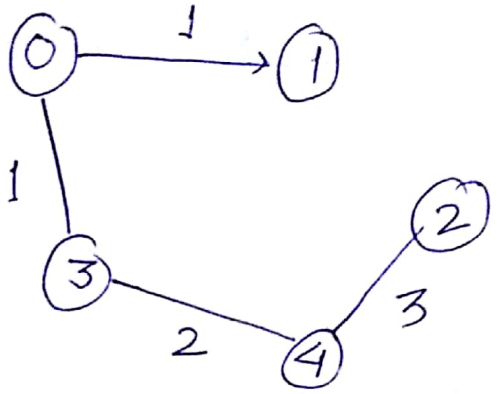
4e.

$$w = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 8 & 1 & 4 \\ 1 & 1 & 0 & 12 & 4 & 9 \\ 2 & 8 & 12 & 0 & 7 & 3 \\ 3 & 1 & 4 & 7 & 0 & 2 \\ 4 & 4 & 9 & 3 & 2 & 0 \end{array}$$

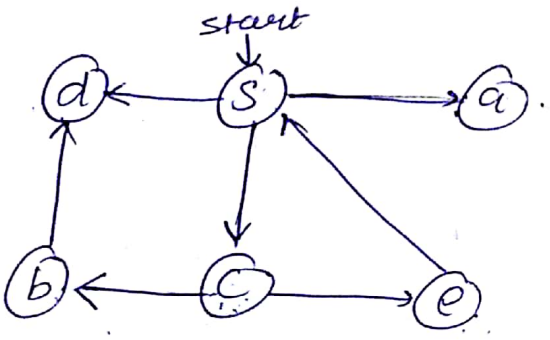


min cost = 7.

Graph Traversing :

1) BFS 2) DFS.

4e.



d e b c a s

1) BFS.

~~s~~	~~a~~	c	d	b	e	

s, a, c, d, b, e

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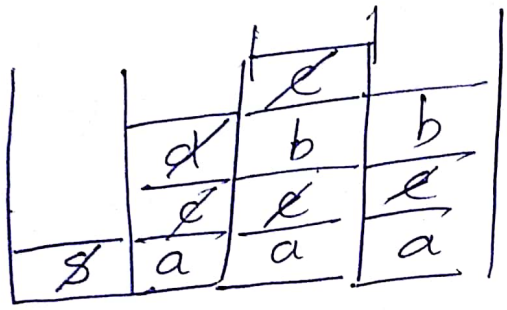
2) DFS

s, a,

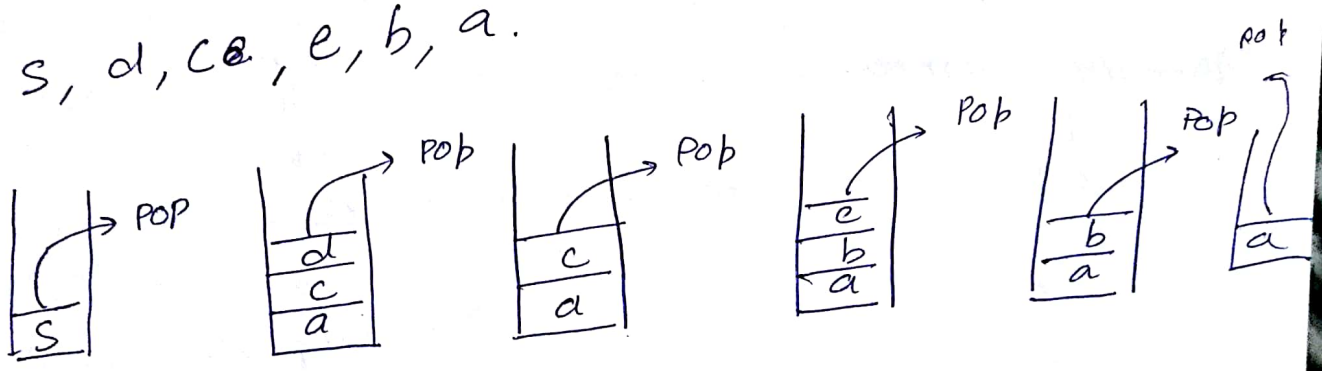
V.O. : s, a, c, b, d, e.

DFS : a, d, b, e, c, s

Ans



s, d, c, e, b, a.



s, d, c, e, b, a.

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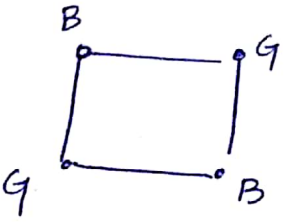
GRAPH COLORING:

* To identify the min. no. of colour in such a way that no two adjacent vertex can are same colour.

* Chromatic Number: least no. of colors (K) required for proper colouring of graph.

* Chromatic number for a null graph is 1.

Ans. C_4



$K = 2$.

For a cycling graph.

Chromatic number.	Graph.
1	NULL
1	C_1
2	C_2
3	C_3
2	C_4
3	C_5
2	C_6
3	C_7
2	C_8
3	C_9
⋮	⋮
1	1

$\chi(C_{even}) = 2$

$\chi(C_{odd}) = 3$

$\chi(K_n) = n$.

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$\chi(W_{odd}) = 4$
$\chi(W_{even}) = 3.$ } wheel (W_n).

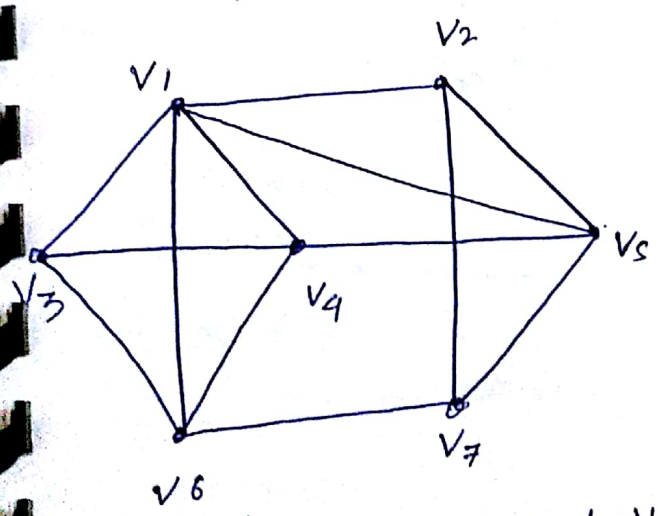
- Every tree with two or more vertices is 2 chromatic.
- Every bipartite graph is 2 chromatic

For any graph G, $\chi(G) \leq 1 + \Delta(G)$

where, $\Delta(G)$ is the maximum degree of a vertex in G.

Welch and Powell graph colouring algorithm:
used to find (k) for proper colouring of graph
→ Greedy approch (doesn't ~~guaru~~ guranty optimal solu^n)

1. order the vertices of G according to decreasing degrees.
   Assign the first color C_1 to the 1st vertex



$V_1 = 5$
$V_2 = 3$
$V_3 = 3$
$V_4 = 4$
$V_5 = 4$
$V_6 = 4$
$V_7 = 3$

Vertices	V_1	V_4	V_5	V_6	V_2	V_3	V_7
degree	5	4	4	4	3	3	3.
colour	a	~~b~~ b	~~c~~ c	c.	b	d	a

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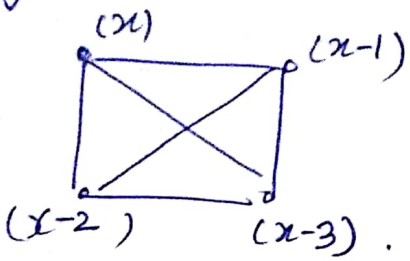
Chromatic Polynomial :

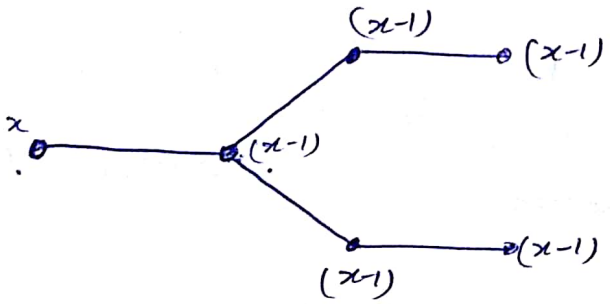
K_4 =



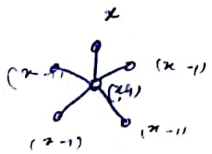
$P\,K_4(x) = x(x-1)(x-2)(x-3)$

for K_n. $\quad P\,K_n(x) = x(x-1)(x-2)(x-3)\ldots(x-(n-1))$.

4: $P_G(x)$ for the following graph.



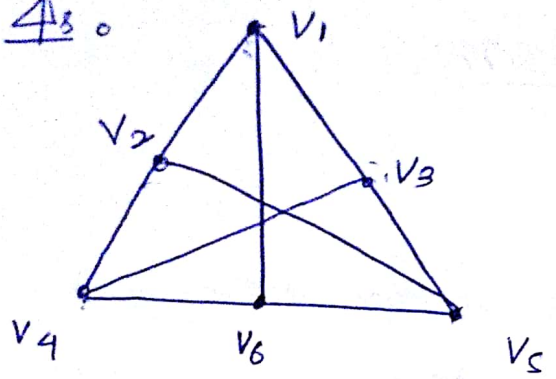
$P_G(x) = x(x-1)^5$.



$x(x-1)^5$.

$x(x-1)^5$.

4s.



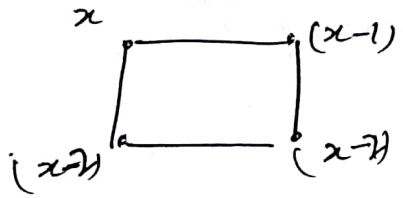
$V_1 = 3$
$V_2 = 3$
$V_3 = 3$
$V_4 = 3$
$V_5 = 3$
$V_6 = 3$

Vertices	V_1	V_2	V_3	V_4	V_5	V_6
degree	3	3	3	3	3	3
colour	a	b	b	a	a	b

$\boxed{k = 2}$

case 1

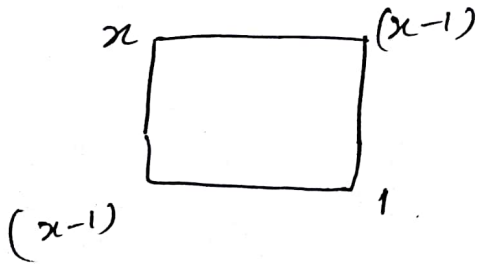
As



$P_2 C_4 = x(x-1)(x-2)^2$

OR

case 2



$P_1 C_4 = x(x-1)^2$

PC_4

total no. of ways $= x(x-1)^2(x-2) + x(x-1)^2$

$= x(x-1)^2[(x-2)+1]$

$= x(x-1)^2(x-1)$

$= x(x-1)^3$

$= x(x^3 - 1 + 3x^2 - 3x)$

$= x^4 - x + 3x^3 - 3x^2$

$= x^4 + 3x^3 - 3x^2 - x$.

$x^3 - 1 + 3a^2b + 3ab^2$

$x^3 - 1 + 3(x^2)(1) + 3(x)$

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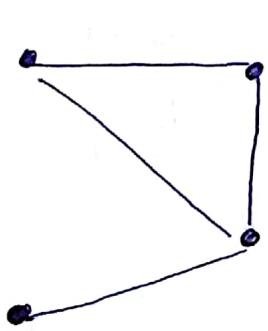
Fundamental Reduction Theorem:
(Deletion- contraction Theorem).

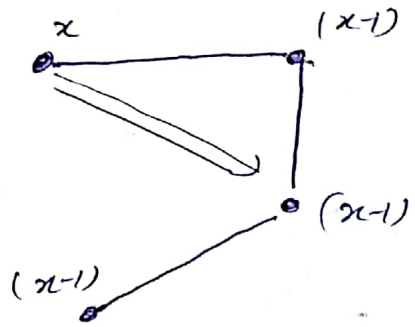
$$P_G(x) = P_{G'}(x) - P_{G''}(x).$$

where $G' \to$ a graph after deletion of an edge from G &
G'' is a graph after contraction (joining) of endpoints of e.

Example: G



G'



$P_{G'} = x(x-1)^3$

G''

$P_{G''} = x(x-1)^2$

$$\begin{aligned}
P_G(x) &= P_{G'}(x) - P_{G''}(x) \\
&= x(x-1)^3 - x(x-1)^2 \\
&= x(x-1)^2[(x-1)-1] \\
&= x(x-1)^2(x-2) \\
&= (x^2-2x)(x^2-2x+1) \\
&= x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x \\
&= x^4 - 4x^3 + 5x^2 - 2x.
\end{aligned}$$

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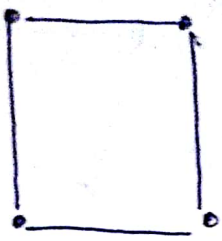
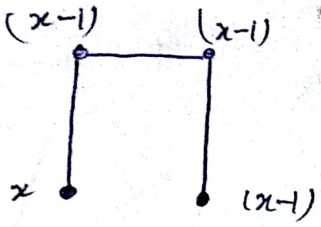
4s C_4

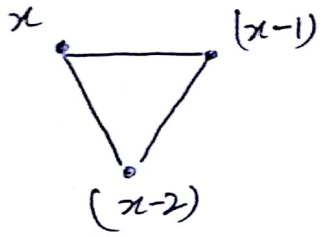
G



G'

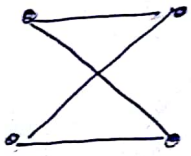


G''

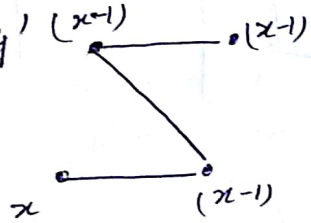


$$P_G(x) = P_{G'}(x) - P_{G''}(x)$$
$$= x(x-1)^3 - x(x-1)(x-2)$$
$$= x(x-1)\left[(x-1)^2 - (x-2)\right]$$
$$= x(x-1)\left[x^2+1-2x-x+2\right]$$
$$= (x^2-x)(x^2-3x+3)$$
$$= x^4 - 3x^3 + 3x^2 - x^3 + 3x^2 - 3x$$
$$= x^4 - 4x^3 + 6x^2 - 3x$$

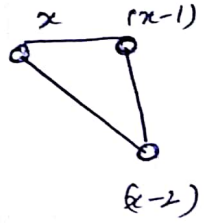
4s. G



G'



G''



$$P_G = P_{G'} - P_{G''}$$
$$= x(x-1)^3 - x(x-1)(x-2).$$
$$= x^4 - 4x^3 + 6x^2 - 3x.$$

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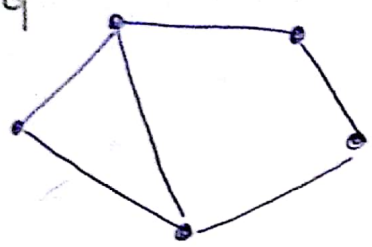
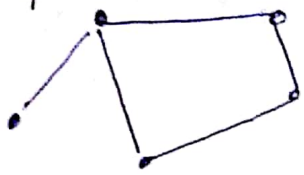
Ans.

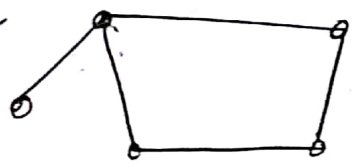
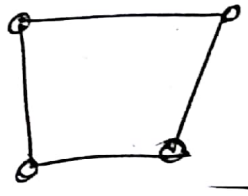
G 

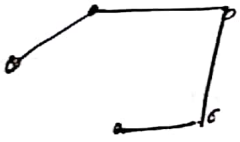
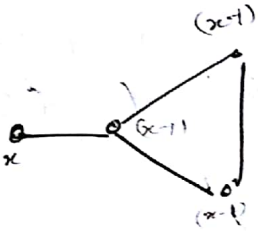
G' 

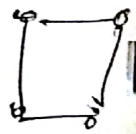
$(G'')'$ =

G'''

$P_G = P_{G'} - P_{G''}$

$= P\Big[$  $\Big] - P\Big[$  $\Big]$

$= P\Big[$  $\Big] - P\Big[$ 

$\Big] - P\Big[$  $\Big]$

$= x(x-1)^4 - \Big[\, x(x-1)(x-2)^4 + x(x-1)(x-2)^2 \,\Big] - \Big[$

$= x(x-1)^4 - \left[x(x-1)^2 + x(x-1)(x-2)^2\right] - \left[x^4 - 4x^3 + 6x^2 - 3x\right]$

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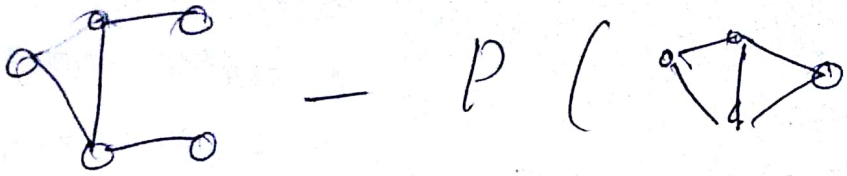

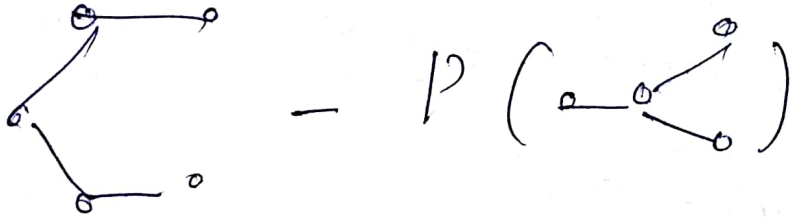

Network flows:

- flow network is a connected, directed graph $G=(V,E)$
- source node : (starting)
- sink node / destination node

Assumption:

capacity are int.

Every nodes has one adjacent node.

constraints on flow: max capacity of edge

$0 \le f(e) \le c_e$ for each edge e. [capacity constraint]

$$\sum_{e \text{ int } v} f(e) = \sum_{e \text{ leaving } v} f(e).$$

maximum flow problem. ?

- Greedy approch.

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Ford Fulkerson Algo: To find optimal soln of network flow.

Forward edge → remaining or avail. capacity (residual capacity)

backward edge → amount of load already sent

Residual Graph.

$G_f(G, s, t)$.

for each edge $(u,v) \in E[G]$

do $f[u,v] \leftarrow 0$

$f[v,u] \leftarrow 0$

while ($\exists$ a path p from s to t in the residual netwf G_f)

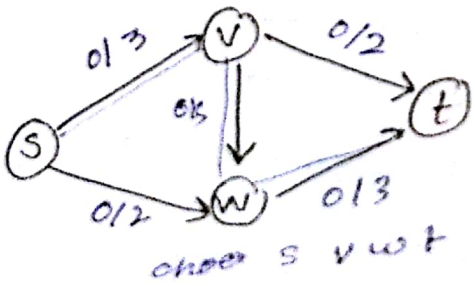
do $c_f(p) \leftarrow \min\{c_f(u,v) : (u,v) \text{ in } p\}$.

for each edge (u,v) in p.

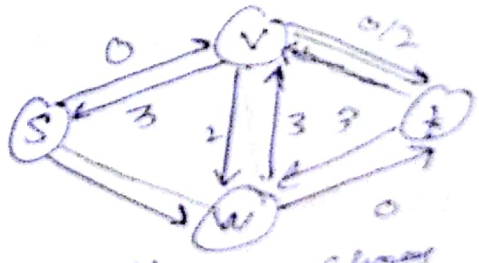
do $f[u,v] \leftarrow f[u,v] + c_f(p)$.

$f[v,u] \leftarrow -f[u,v]$.

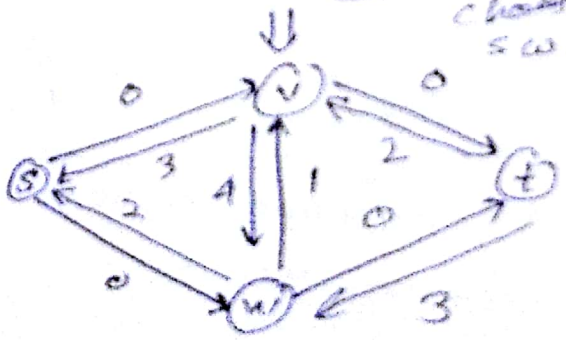
4*



⇒



⇓



max flow = 5.

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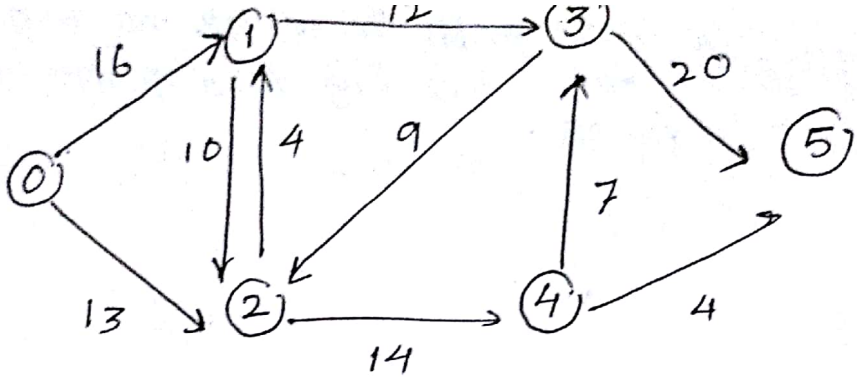
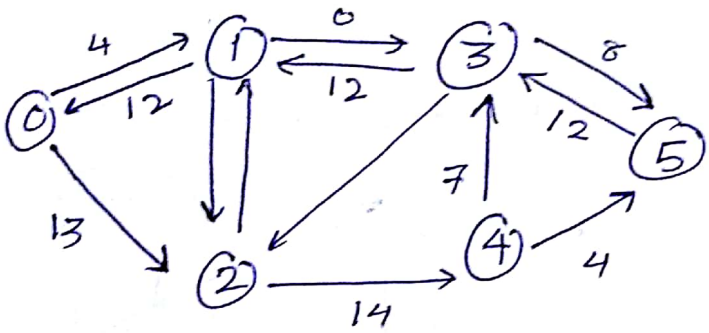
4.



0 · 1 3 5

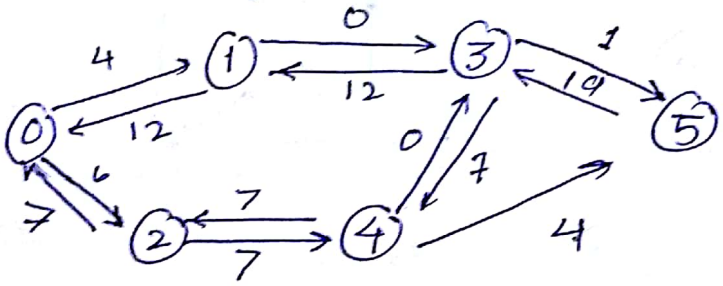


12
7
4

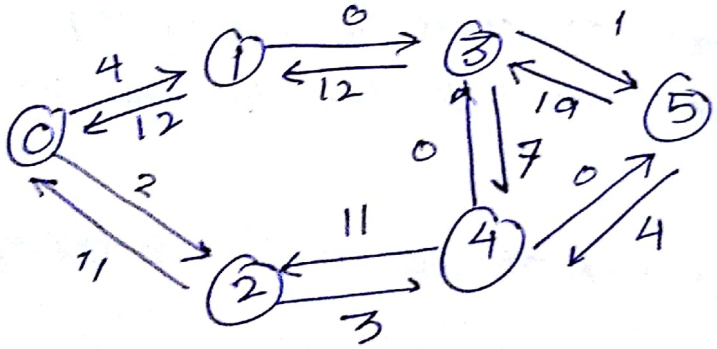
12

0, 2, 4, 3, 5. — 7.

0, 2, 4, 5.



7



4

maximum network flow = 23.

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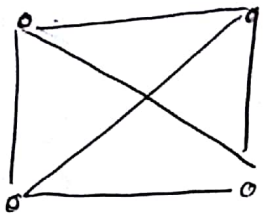
Planar Graphs :

Those graphs in which no two edges are crossing in a plane then it is planar.

K_3

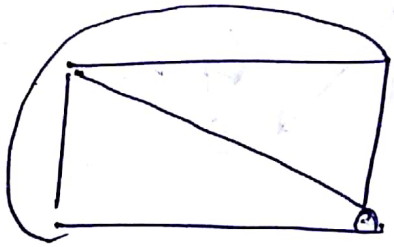
K_4



Here two edges are crossing

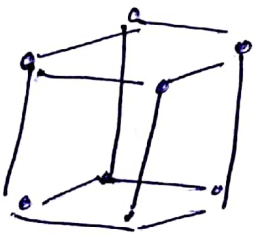
$\Downarrow$

K_4 can also be made as

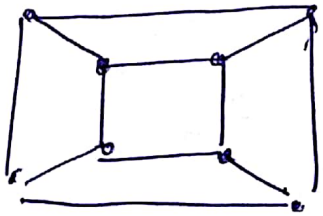


here no edges are crossing

Ex.



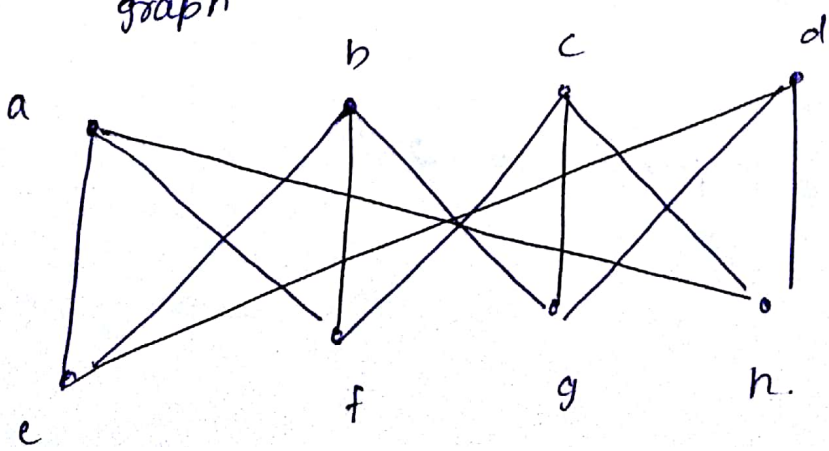
$\Longrightarrow$



Two ways to determine wether graph is planar or not.

(1) Circle chord method : It consist of step by step process to draw an every edge one by one s.t they don't intersect

Step 1 : Find a circuit that contains all the vertices of the graph



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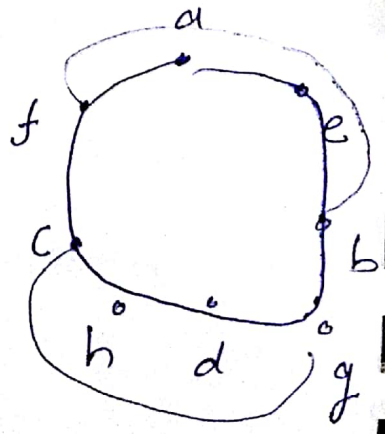

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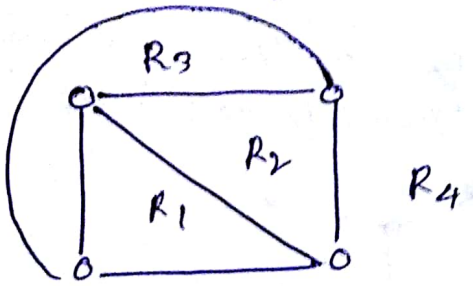
Step 2 : draw in form of circle.

Step 3 : Now connect that edges which are not in circuit such that they don't intersect.

EULER'S PLANER FORMULA

A planar representation of a graph split the plane into regions, where one of them has finite area & is called the finite Region/face



$R_4 \rightarrow \infty$ region

$R_1, R_2 R_3 \rightarrow$ finite region.

In a planar graph $\exists$ one & only one ∞ region.

Degree of region : No. of edges enclosing a ~~finite~~ region.

Let G be a connected planar graph, and consider a planar subgraph of G let

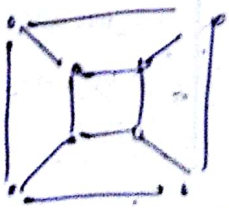
$V \rightarrow$ vertices, $E \rightarrow$ edges, $F \rightarrow$ regions.

Theorem $V+F = E+2$.

$V = 4; F = 4; E = 6$

$V+F = E+2$

$4+4 = 6+2$ ✓



$V = 8; F = 6; E = 12$.

$8+6 = 12+2$

✓

Proof by induction on edges.

Base step:

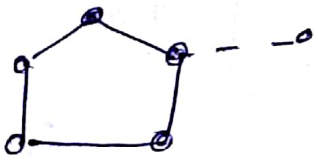
We proved by induction on the no. of edges for $e=1$, we can only have.

$$V+F = E+2$$
$$2+1 = 1+2 \checkmark.$$

Hypothesis step: True for $(e-1)$ edges.

Inductive step: add an edge to get our planar graph there are only two possibilities to consider

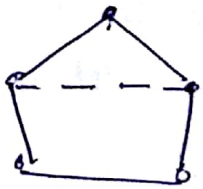
case 1: we add a pendent vertex & edge in this case e & V both increase by one & equality still holds



i,e $$(V+1)+F = (E+1)+2$$

case 2: we add an edge across two existing vertices (s.t. they doesn't cross any other edge). Now, F & e both increase by one. equality still holds.

i,e $$V+(F+1) = (E+1)+2.$$



1. $V=8$; $E=12$

$8+x = 12+2$

$x = 6$

hand shaking

$2e = \sum d(V)$.

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Theorem : (Gives Necessary condition for Planar graph)
For a simple planar graph with $v \geq 3$ and $e \leq 3v-6$.

Let r be the number of regions in a planar representation of graph, & for a region R, let $\deg(R)$ be the no. of edges that are adjacent to the region

$\because$ each edge is adjacent to two regions

$$2e = \sum \deg(R) \geq 3r.$$

we can use this with euler's formula ($r = e - v + 2$)
[$v + r = e + 2$].

$$3r \leq 2e$$
$$3(e-v+2) \leq 2e$$
$$e \leq 3v-6.$$

This theorem give necessary condition.

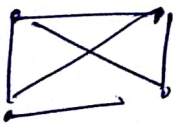
Ans. Show that K_3 is planar or not.

here

$e = 3$; $v = 3$

$$e \leq 3v-6$$
$$3 \leq 3(3)-6$$
$$3 \leq 9-6 \quad \checkmark \text{ true}$$

hence, it is planar.

ii) K_4 

$e = 6$; $v = 4$

$$6 \leq 3(4)-6$$
$$6 \leq 12-6$$
$$6 \leq 6 \quad \checkmark$$

hence, it is planar.

* This theorem can ~~help~~ prove ~~a~~ we the ~~it~~ ~~a~~ graph ~~is~~ is not planar.

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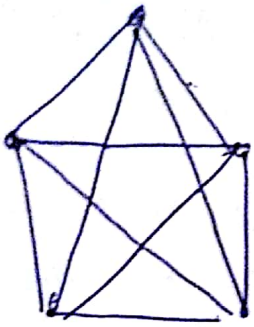
for K_5.



$e = 10$; $v = 5$.

$10 \leq 3(5) - 6$

$10 \leq 15 - 6$

$10 \leq 9$ ✗

K_5 is not a planar graph.

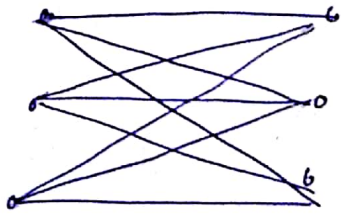
(Necessary condition for ~~non~~ planar).

Theorem: For a simple connected planar graph with $v \geq 3$

$e \leq 2v - 4$.

Q/ Show that K_{33} is non planar.

$v = 6$ $e = 9$. & no triangle, it follows the theorem $e \leq 2v - 4$



$9 \leq 2(6) - 4$

$9 \leq 8$.

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Kuratowski's Theorem :

If any graph have $K_{3,3}$ and K_5 as sub graph. Then that graph will not be planar.

∵ individually $K_{3,3}$ & K_5 are not planar.

UNIT - 2 VECTOR

Vector space :

A VS is a set V on which two operations + & * are defined, called vector addition and scalar multiplication.

The operation + must satisfy the following condition

1. Closure : If U & V are any vector $\in V$ then sum $U+V \in V$
2. Commutative law : $\forall\, u, v \in V$; $u+v = v+u$.
3. Associative law : $\forall\, u, v, w \in V$; $u+(v+w) = (v+u)+w$
4. Additive Identity :
   The set V contains an additive identity element, denoted by 0, such that for any vector v in V,
   $0 + v = v$ and $v + 0 = v$.
5. Additive inverse : For each vector $v \in V$ the equation $v + x = 0$ and $x + v = 0$ have a solution x in V. called as additive inverse of v, and denoted by $-v$.

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The operation must satisfied.

1) Closure: if V is any vector V, and c is any real number, then the product $c \cdot v \in V$.

2) Distributive law: $\forall$ real no. c and all vector $u, v \in V$,

$c \cdot (u+v) = c \cdot u + c \cdot v$.

3) dis law: $\forall$ R c, d and all Vector $v \in V$

$(c+d) \cdot v = c \cdot v + d \cdot v$

4) Associative law: $\forall$ R c, d & all vector $v \in V$

$c \cdot (d \cdot v) = (c \cdot d) \cdot v$

5) Unitary law: $\forall$ vector $v \in V$ $v \cdot 1 = v$.

Real vector space

~~[illegible]~~

Vector in R^n

A vector in n-space is represented by an order n-tuple.

The set of al order n-tuple is called n-space.

$R^1 = a, b, c$

$R^2 = (a_1, a_2)\ (a_3, a_4)\ (b_1, b_2)$

$R^3 = (x, y, z)$

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Q1 the set V of all 2×2 matrices with real entries is a vector space if addition is defined to be matrix addition & scalar multiplication is defined as matrix multiplication.

for + (matrix addition)

1) closure : let $U = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ & $V = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

$$U+V = \begin{bmatrix} x_1+a_1 & x_2+a_2 \\ y_1+b_1 & y_2+b_2 \end{bmatrix} \in V.$$

2) commutative $U+V = V+U$.

3) associative $U+(V+Z) = (U+V)+Z$

4) Additive identity : $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$0+U = U$.

5) Additive inverse : $U+x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$x = -\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$\boxed{x = -U}$$

for * (matrix multiplication).

1) closure : let $U \in V$

$$K \cdot U = K\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} Kx_1 & Kx_2 \\ Ky_1 & Ky_2 \end{bmatrix} \in V.$$

Hence it is closed

2) Distributive I : $K \cdot (U+V) = KU + KV \in V$

3) Distributive II $(K_1+K_2) \cdot U = K_1U + K_2U \in V$

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4) Associative : let $c, d \in R$ $\quad u \in V$

$(c \cdot d) \cdot u$.

$(c \cdot v) d = c(v \cdot d) \in V$

Unitary Law : let $v \in V$

$v \cdot x = v \cdot 1 = v \in V$

hence $x = 1$ is the unitary element.

Q. Let $V = \{x, x/2 : x \in R\}$ with standard operation. It is a vector space ?
Justify.

Q. The set V of vector $[2x, x^2]$ with $x \in R^2$: Addition and scale multiplication

Solu^n. for $(+)$.

(1) closure : let $u, v \in V$

$u = [2x_1, x_1^2]$; $v = [2x_2, x_2^2]$

$u + v = [2(x_1 + x_2), 2(x_1)^2 + 2x_2^2]$

$= [2(x_1 + x_2), x_1^2 + x_2^2]$

$\because x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2(x_1 + x_2) \neq (x_1 + x_2)^2$.

Hence, it is not closed under vector addition.

$\therefore$ it is not a vector space.

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Ans. The set of all triples of R (x, y, z) with the operation.

1. $(x, y, z) + (x', y', z') = (x+x', y+y', z+z')$

$K \cdot (x, y, z) = (Kx, y, z)$

2. The set of all pair of real no. (x, y)

$(x, y) + (x', y') = [(x+x'), (y+y')]$ & $K \cdot (x, y) = (2Kx, 2Ky)$

Ans not a vector space.
(no. unitary element)

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Vector Subspace :

Subset of vector set is called vector subspace VS' of vector space VS if VS' follow all 10 properties.

$H \subseteq V$ (Necessary & sufficient condition for H to be a vector subspace).

1. The zero vector of V is in H.
2. For each u & v are in H, u+v is in H. (In this case we say H is closed under vector addition)
3. For each u in H & each scalar c, c.u is in H. (In this case we say H is closed under scale multiplication)

Ex. Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of R^3.

1. zero vector of V is $(0,0,0)$ which $\in H$.
$\Rightarrow$ zero vector of V is present in H.

2. Let $u = \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix}$ and $v = \begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix}$ $\quad u+v = \begin{bmatrix} a_1+a_2 \\ 0 \\ b_1+b_2 \end{bmatrix}$

$u+v \in H$

$\Rightarrow$ It is closed under vector addition.

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3. let $c.u = \begin{bmatrix} a.c \\ 0 \\ b.c \end{bmatrix} \in H$

therefore, it is closed under scalar multiplication.

Ans 2. $S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \;\middle|\; x_1 \geq 0 \right\}$ in the vector space

It is not a vector sub space.

4 3. property. $c.u \in H$.

let say $c = -1$. $\begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}$

here $-x_1 \ngeq 0$

hence, it is not a vector subspace.

Ans 3. $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3 \;\middle|\; x_1 - 4x_2 + 5x_3 = 2 \right\}$

It is not a vector subspace

because.

identity element of $R^3 \notin S_2$.

I of $R^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ but $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S_2$

$\because 0 - 4(0) + 5(0) \neq 2$.

Ans 4. Is $V = \{(a+2b, 2a+3b) : a \text{ and } b \text{ are real}\}$ a sub space of R^2? why or why not.

let $v = \begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix}$ where $a, b \in R$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$ | $v = \begin{bmatrix} a.0+2.0 \\ 2.0-3.0 \end{bmatrix} \in V$ $[a, b = 0 \in R]$

R^2 $I = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — (1)

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$V + I = V$

$\Rightarrow$ identity of R^2 exist in V

2. Vector addition closure:

let V and $U \in V$

$$\Rightarrow \begin{bmatrix} a_1+2b_1 \\ 2a_1+3b_1 \end{bmatrix} + \begin{bmatrix} a_2'+2b_1' \\ 2a_2'+3b_2' \end{bmatrix} = \begin{bmatrix} a_1+a_2'+2(b_1+b_1') \\ 2a_1+2a_2'+3b_1+3b_2' \end{bmatrix}$$

$$= \begin{bmatrix} (a_1+a_1')+2(b_1+b_1') \\ 2(a_2+a_2')+3(b_2+b_2') \end{bmatrix}$$

$\in V$

3. ~~let~~ Scalar multiplication closure.

let $V \in V$

$$c \cdot V = \begin{bmatrix} ca_1+2cb \\ 2ac+3bc \end{bmatrix} \in V$$

where ~~c~~ $c.a + 2cb \in R$
and $2ac+3bc \in R$.

Hence it is closed under scalar multiplication.

$\therefore$ it is a vector subspace

Ans 5. $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : a \,\&\, b \in R \right\}$ a subspace of R^3?

Not a subspace bcoz it doesn't have identity element. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Ans 6. $H = \begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times2}$.

$a, b \in R$. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

✓ existance of zero element
✓ closed under V.A
✓ closed under S.M.

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