PMPC Tutorial Sheet 4

1. On average I arrive at the office at a certain time in the morning. However, there is some variability due to traffic. I believe that I'm never more than w minutes late and I never show up earlier than w minutes before my average time of arrival. Let's call X the difference between my actual time of arrival and my average time of arrival. Here's my belief about X:

$$p(X = x) \quad = \quad \begin{cases} \frac{x}{w^2} + \frac{1}{w} & \text{if } -w \le x \le 0 \\ -\frac{x}{w^2} + \frac{1}{w} & \text{if } 0 < x \le w \\ 0 & \text{otherwise} \end{cases}$$

   Convince yourself that this probability density function is normalized to 1. What is the value of the density function at 0? Why can this be greater than 1? I have an appointment at my office at $w/2$ minutes after my average time of arrival. What's the probability that I'll be late for the appointment? What is the probability that I arrive exactly at $w/2$? For which time point am I 95% certain that I'll be there by then [4, 7-2, 7-3].

2. X has the probability density function (pdf) $p(X = t) = f(t)$ and the cumulative distribution function (cdf) $P(X \le t) = F(t)$. Define a new random variable $Y = G^{-1}(X)$ for a strictly increasing function G with an inverse G^{-1} and derivative g. What is the cdf for Y? What is the pdf for Y?

3. Sampling from a distribution with pdf g, cdf G, and inverse cdf G^{-1}. X has density

$$p(X = t) = f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases},$$

   i.e. X is distributed uniformly between 0 and 1. Show that $Y = G^{-1}(X)$ has the pdf $p(Y = t) = g(t)$. Use this fact to write a function that generates random samples from the triangular distribution from above.

4. I don't like multiple choice tests. But I would like them better if there wasn't a lot of guessing involved. Let's try out a method for academic testing that asks you for your belief in a statement, instead of just stating whether it's true or false. I want to try out with you whether such a test can be used for an exam. So we'll simulate an exam by you taking a general knowledge test. The idea is not so much that we really want test your general knowledge here, the idea is to see how well you can quantify your uncertainty about what you know and how well you are calibrated. So don't google the answers until you're done with the test. Nevertheless, the test is simulating an academic testing situation after all, and you do want to score as well as possible. You will be scored with the quadratic loss function that we discussed in class. This means you should be honest and try to be well calibrated. Taking the test should take between 30 and 45 minutes. Please take the test at https://ikw.uni-osnabrueck.de/limesurvey

5. A statement X in the test can be true ($X = 1$) or false ($X = 0$). Say, you answered that your probability is q for $X = 1$. You will be scored using the quadratic loss function

$$L_1(X, q) = (X - q)^2 .$$

This is, however, not the only loss function one could use. One could also use

$$L_2(X, q) = -X \cdot \log(q) - (1 - X) \cdot \log(1 - q) ,$$

i.e. the negative log likelihood. Your true belief in the statement is p. Convince yourself that L_2 is a proper scoring rule, i.e. your expected loss will be minimal if you are honest and $p = q$. Find the minimum of the expected loss with respect to q by taking first and second derivatives. Compare the expected loss for L_1 and L_2 for an honest and well-calibrated test-taker. How does the expected loss vary as a function of p? Make a plot for both loss functions. Which loss function is better? What happens when you are not well calibrated?

6. Bibliographical comments: A good book on how experts' beliefs can be elicited is [3]. This book also has useful background information on subjective probability. Often experts have different subjective beliefs and a decision maker has to come up with a decision based on conflicting expert opinions. If you just want to get a flavour of how experts' forecasts and opinions are scored and combined in practice, you can also look at [2, 1].

References

[1] W. Aspinall. A route to more tractable expert advice. *Nature*, 463(7279):294–295, Jan 2010.

[2] G. W. Brier. Verification of forecasts expressed in terms of probabilities. *Monthly Weather Review*, 78(1):1–3, 1950.

[3] R. M. Cooke. *Experts in Uncertainty.* Oxford University Press, Oxford, 1991.

[4] F. Mosteller, R. E. Rourke, and G. B. Thomas. *Probability with statistical applications.* Addison-Wesley, 1970.