PMPC Tutorial Sheet 7

1. The Poisson process is frequently used to model temporal events, e.g. the occurence of earthquakes, phone calls or action potentials. The Poisson process has one free parameter: the rate r at which events occur. The probabity that N events happen during an interval of duration t is given by[1]
$$P(N = n|r, t) = \frac{(rt)^n}{n!} \exp(-rt).$$
Derive the maximum likelihood estimator for the rate parameter r in the Poisson model. Assume the number of events and the time interval to be fixed by an experiment. For which value of the rate parameter is the likelihood function at a maximum? Bonus question: Show that the expected number of events and the variance are both rt.[2]

2. Stimulus decoding [1, Chapters 1 and 3]. In the lobular plate of the fly (*calliphora vicina*) brain there are only two spiking neurons. These neurons are very easy to record from which has made them a popular model system for theoretical studies [2]. These neurons are selective for horizontal visual motion and they are called H1 neurons. One of them is selective for leftward motion, the other one for rightward motion. You record from one of them and find that it has a firing rate of 20 Hz for leftward motion and 80 Hz for rightward motion. In a follow-up experiment either a leftward or a rightward motion (each with probability $\frac{1}{2}$) is presented to the fly in each trial while you are still recording from the same neuron. Assume that the firing of the neuron is a Poisson process for leftward and rightward motion that differs only in the firing rate. Make a plot of the two distributions for the number of spikes in a 100 ms time interval. Where do the two distributions intersect? Calculate the log likelihood ratio for leftward and rightward motion when you observe n spikes in the 100 ms after the stimulus onset. Make a plot of the log likelihood ratio as a function of n.[3]

3. Imagine a homunculus downstream from the the above neuron. The homunculus counts the neurons for 100 ms after the onset of each trial. The homunculus isn't able to see the input to the fly's eye. For simplicity we will also assume that he has no access to the other H1 neuron. His only access to the visual world is through this one neuron. He needs to make a decision as to whether the motion is leftward or rightward because he

[1] Often one finds the Poisson distribution parametrized with only one parameter $\lambda := rt$

[2] Hint: Since the distribution is normalized to one you know that $\sum_{n=0}^{\infty} \frac{(rt)^n}{n!} = \exp(rt)$.

[3] The Poisson distribution is called `poisspdf` in Matlab and Octave. It has one parameter $\lambda := rt$, the expected number of events.

needs to steer the fly away from a wall, for example. How well is he able to do this? Plot the Receiver Operating Characteristic, i.e. the hit-rate vs the false-alarm rate for varying decision criteria.

4. Check that the characteristic function of a Gaussian random variable X with mean μ and variance σ^2 is

$$\chi_X(\omega) = e^{i\mu\omega - \frac{1}{2}\sigma^2\omega^2}.$$

Reminder: The Gaussian distribution is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and the characteristic function is defined as

$$\begin{aligned} \chi_X(\omega) &= E\left(e^{i\omega X}\right) \\ &= \int_{-\infty}^{\infty} f_X(x) e^{i\omega x} dx \end{aligned}$$

where $i^2 = -1$.

5. Show that the n'th derivative of the characteristic function evaluated at zero is related to the n'th moment of the distribution in the following way:

$$\chi_X^{(n)}(0) = i^n E\left(X^n\right).$$

Use this relationship to derive the first two moments of the Gaussian distribution.

References

[1] P. Dayan and L. F. Abbott. *Theoretical Neuroscience.* MIT Press, Cambridge, MA, 2001.

[2] F. Rieke, D. Warland, R. de Ruyter van Steveninck, and W. Bialek. *Spikes: Exploring the Neural Code.* MIT Press, Cambridge, MA, 1997.