39. Suppose A is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the definition of linear independence to explain why the columns of A must be linearly independent.

40. Suppose an $m \times n$ matrix A has n pivot columns. Explain why for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. [*Hint:* Explain why $A\mathbf{x} = \mathbf{b}$ cannot have infinitely many solutions.]

[**M**] In Exercises 41 and 42, use as many columns of A as possible to construct a matrix B with the property that the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Solve $B\mathbf{x} = \mathbf{0}$ to verify your work.

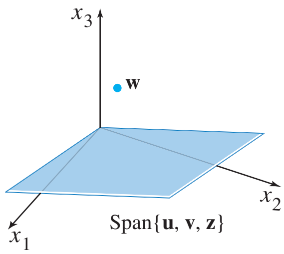
41. $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix}$

42. $A = \begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$

43. [**M**] With A and B as in Exercise 41, select a column $\mathbf{v}$ of A that was not used in the construction of B and determine if $\mathbf{v}$ is in the set spanned by the columns of B. (Describe your calculations.)

44. [**M**] Repeat Exercise 43 with the matrices A and B from Exercise 42. Then give an explanation for what you discover, assuming that B was constructed as specified.

SOLUTIONS TO PRACTICE PROBLEMS



1. a. Yes. In each case, neither vector is a multiple of the other. Thus each set is linearly independent.

b. No. The observation in Part (a), by itself, says nothing about the linear independence of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$.

c. No. When testing for linear independence, it is usually a poor idea to check if one selected vector is a linear combination of the others. It may happen that the selected vector is not a linear combination of the others and yet the whole set of vectors is linearly dependent. In this practice problem, $\mathbf{w}$ is not a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$.

d. Yes, by Theorem 8. There are more vectors (four) than entries (three) in them.

2. Applying the definition of linearly dependent to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ implies that there exist scalars c_1, c_2, and c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Adding $0\,\mathbf{v}_4 = \mathbf{0}$ to both sides of this equation results in

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + 0\,\mathbf{v}_4 = \mathbf{0}.$$

Since c_1, c_2, c_3 and 0 are not *all* zero, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ satisfies the definition of a linearly dependent set.

1.8 INTRODUCTION TO LINEAR TRANSFORMATIONS

The difference between a matrix equation $A\mathbf{x} = \mathbf{b}$ and the associated vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ is merely a matter of notation. However, a matrix equation $A\mathbf{x} = \mathbf{b}$ can arise in linear algebra (and in applications such as computer graphics and signal processing) in a way that is not directly connected with linear combinations of vectors. This happens when we think of the matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

For instance, the equations

$$\underset{A}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}} \quad \text{and} \quad \underset{A}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\mathbf{u}}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}} = \underset{\mathbf{0}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

say that multiplication by A transforms $\mathbf{x}$ into $\mathbf{b}$ and transforms $\mathbf{u}$ into the zero vector. See Figure 1.

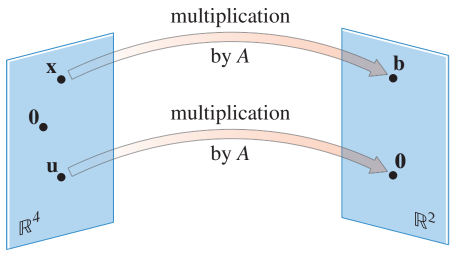


FIGURE 1 Transforming vectors via matrix multiplication.

From this new point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^2$ under the "action" of multiplication by A.

The correspondence from $\mathbf{x}$ to $A\mathbf{x}$ is a *function* from one set of vectors to another. This concept generalizes the common notion of a function as a rule that transforms one real number into another.

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the **range** of T. See Figure 2.

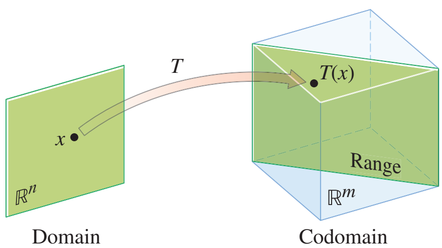


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

The new terminology in this section is important because a dynamic view of matrix–vector multiplication is the key to understanding several ideas in linear algebra and to building mathematical models of physical systems that evolve over time. Such *dynamical systems* will be discussed in Sections 1.10, 4.8, and 4.9 and throughout Chapter 5.

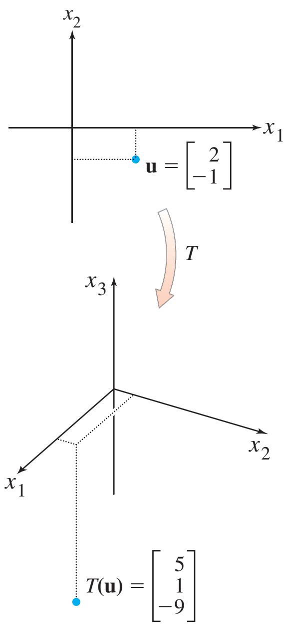
Matrix Transformations

The rest of this section focuses on mappings associated with matrix multiplication. For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix. For simplicity, we sometimes denote such a *matrix transformation* by $\mathbf{x} \mapsto A\mathbf{x}$. Observe that the domain of

T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

EXAMPLE 1 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$



a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

SOLUTION

a. Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

b. Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \tag{1}$$

Using the method discussed in Section 1.4, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$. The image of this $\mathbf{x}$ under T is the given vector $\mathbf{b}$.

c. Any $\mathbf{x}$ whose image under T is $\mathbf{b}$ must satisfy equation (1). From (2), it is clear that equation (1) has a unique solution. So there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$.

d. The vector $\mathbf{c}$ is in the range of T if $\mathbf{c}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some $\mathbf{x}$. This is just another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, $0 = -35$, shows that the system is inconsistent. So $\mathbf{c}$ is *not* in the range of T. ■

The question in Example 1(c) is a *uniqueness* problem for a system of linear equations, translated here into the language of matrix transformations: Is $\mathbf{b}$ the image of a *unique* $\mathbf{x}$ in $\mathbb{R}^n$? Similarly, Example 1(d) is an *existence* problem: Does there *exist* an $\mathbf{x}$ whose image is $\mathbf{c}$?

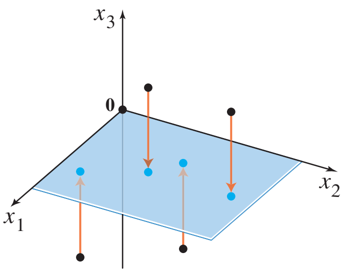


FIGURE 3
A projection transformation.

The next two matrix transformations can be viewed geometrically. They reinforce the dynamic view of a matrix as something that transforms vectors into other vectors. Section 2.7 contains other interesting examples connected with computer graphics.

EXAMPLE 2 If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ *projects* points in $\mathbb{R}^3$ onto the x_1x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

See Figure 3. ■

sheep

sheared sheep

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**. It can be shown that if T acts on each point in the 2×2 square shown in Figure 4, then the set of images forms the shaded parallelogram. The key idea is to show that T maps line segments onto line segments (as shown in Exercise 27) and then to check that the corners of the square map onto the vertices of the parallelogram. For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is $T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. T deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and crystallography. ■

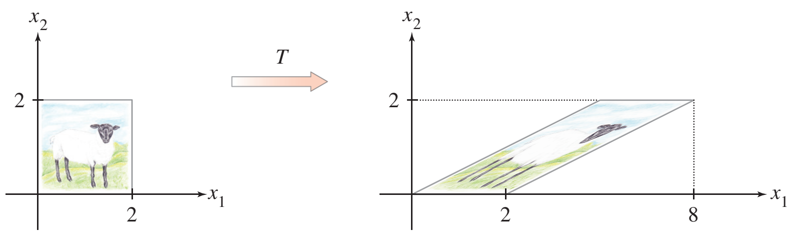


FIGURE 4 A shear transformation.

Linear Transformations

Theorem 5 in Section 1.4 shows that if A is $m \times n$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} \quad \text{and} \quad A(c\mathbf{u}) = cA\mathbf{u}$$

for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ and all scalars c. These properties, written in function notation, identify the most important class of transformations in linear algebra.

DEFINITION

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Every matrix transformation is a linear transformation. Important examples of linear transformations that are not matrix transformations will be discussed in Chapters 4 and 5.

Linear transformations *preserve the operations of vector addition and scalar multiplication*. Property (i) says that the result $T(\mathbf{u}+\mathbf{v})$ of first adding $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and then applying T is the same as first applying T to $\mathbf{u}$ and to $\mathbf{v}$ and then adding $T(\mathbf{u})$ and $T(\mathbf{v})$ in $\mathbb{R}^m$. These two properties lead easily to the following useful facts.

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \tag{3}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \tag{4}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ in the domain of T and all scalars c, d.

Property (3) follows from condition (ii) in the definition, because $T(\mathbf{0}) = T(0\mathbf{u}) = 0T(\mathbf{u}) = \mathbf{0}$. Property (4) requires both (i) and (ii):

$$T(c\mathbf{u} + d\mathbf{v}) = T(c\mathbf{u}) + T(d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

Observe that *if a transformation satisfies* (4) *for all* $\mathbf{u}$, $\mathbf{v}$ *and* c, d, *it must be linear*. (Set $c = d = 1$ for preservation of addition, and set $d = 0$ for preservation of scalar multiplication.) Repeated application of (4) produces a useful generalization:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) \tag{5}$$

In engineering and physics, (5) is referred to as a *superposition principle*. Think of $\mathbf{v}_1, \dots, \mathbf{v}_p$ as signals that go into a system and $T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)$ as the responses of that system to the signals. The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is *the same* linear combination of the responses to the individual signals. We will return to this idea in Chapter 4.

EXAMPLE 4 Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a **contraction** when $0 \le r \le 1$ and a **dilation** when $r > 1$. Let $r = 3$, and show that T is a linear transformation.

SOLUTION Let $\mathbf{u}$, $\mathbf{v}$ be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) && \text{Definition of } T \\ &= 3c\mathbf{u} + 3d\mathbf{v} && \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) && \Big\}\ \text{Vector arithmetic} \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

Thus T is a linear transformation because it satisfies (4). See Figure 5. ■

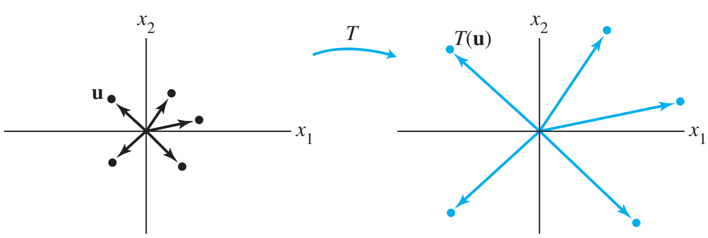


FIGURE 5 A dilation transformation.

EXAMPLE 5 Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images under T of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

SOLUTION

$$T(\mathbf{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \qquad T(\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix},$$

$$T(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Note that $T(\mathbf{u}+\mathbf{v})$ is obviously equal to $T(\mathbf{u}) + T(\mathbf{v})$. It appears from Figure 6 that T rotates $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ counterclockwise about the origin through 90°. In fact, T transforms the entire parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ into the one determined by $T(\mathbf{u})$ and $T(\mathbf{v})$. (See Exercise 28.) ∎

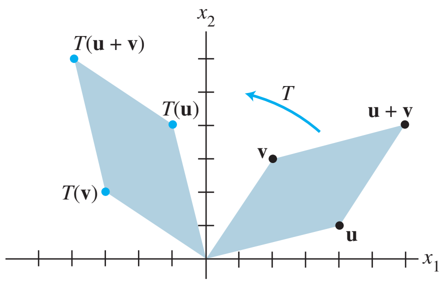


FIGURE 6 A rotation transformation.

The final example is not geometrical; instead, it shows how a linear mapping can transform one type of data into another.

EXAMPLE 6 A company manufactures two products, B and C. Using data from Example 7 in Section 1.3, we construct a "unit cost" matrix, $U = [\,\mathbf{b} \quad \mathbf{c}\,]$, whose columns describe the "costs per dollar of output" for the products:

$$\begin{array}{c c l} & \text{Product} & \\ & \begin{array}{cc} \text{B} & \text{C} \end{array} & \\ U = & \begin{bmatrix} .45 & .40 \\ .25 & .30 \\ .15 & .15 \end{bmatrix} & \begin{array}{l} \text{Materials} \\ \text{Labor} \\ \text{Overhead} \end{array} \end{array}$$

Let $\mathbf{x} = (x_1, x_2)$ be a "production" vector, corresponding to x_1 dollars of product B and x_2 dollars of product C, and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T(\mathbf{x}) = U\mathbf{x} = x_1\begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} + x_2\begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix} = \begin{bmatrix} \text{Total cost of materials} \\ \text{Total cost of labor} \\ \text{Total cost of overhead} \end{bmatrix}$$

The mapping T transforms a list of production quantities (measured in dollars) into a list of total costs. The linearity of this mapping is reflected in two ways:

1. If production is increased by a factor of, say, 4, from $\mathbf{x}$ to $4\mathbf{x}$, then the costs will increase by the same factor, from $T(\mathbf{x})$ to $4T(\mathbf{x})$.

2. If $\mathbf{x}$ and $\mathbf{y}$ are production vectors, then the total cost vector associated with the combined production $\mathbf{x}+\mathbf{y}$ is precisely the sum of the cost vectors $T(\mathbf{x})$ and $T(\mathbf{y})$. ■

PRACTICE PROBLEMS

1. Suppose $T: \mathbb{R}^5 \to \mathbb{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A and for each $\mathbf{x}$ in $\mathbb{R}^5$. How many rows and columns does A have?
2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.
3. The line segment from $\mathbf{0}$ to a vector $\mathbf{u}$ is the set of points of the form $t\mathbf{u}$, where $0 \le t \le 1$. Show that a linear transformation T maps this segment into the segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.8 EXERCISES

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

In Exercises 3–6, with T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

7. Let A be a 6×5 matrix. What must a and b be in order to define $T: \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ into $\mathbb{R}^5$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

For Exercises 9 and 10, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

11. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

12. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbb{R}^2$.

13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

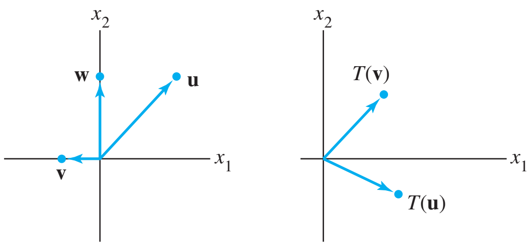
14. $T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u} + 2\mathbf{v}$.

18. The figure shows vectors **u**, **v**, and **w**, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. [*Hint:* First, write **w** as a linear combination of **u** and **v**.]



19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps **x** into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each **x**.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A linear transformation is a special type of function.

b. If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

c. If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

d. Every linear transformation is a matrix transformation.

e. A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

22. a. Every matrix transformation is a linear transformation.

b. The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.

c. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if **c** is in $\mathbb{R}^m$, then a uniqueness question is "Is **c** in the range of T?"

d. A linear transformation preserves the operations of vector addition and scalar multiplication.

e. The superposition principle is a physical description of a linear transformation.

23. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the x_1-axis. (See Practice Problem 2.) Make two sketches similar to Figure 6 that illustrate properties (i) and (ii) of a linear transformation.

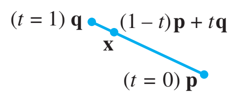
24. Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is the zero transformation. That is, show that if **x** is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

25. Given $\mathbf{v} \neq \mathbf{0}$ and **p** in $\mathbb{R}^n$, the line through **p** in the direction of **v** has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).

26. Let **u** and **v** be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through **u**, **v**, and **0**. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through **0**, or onto a line through **0**, or onto just the origin in $\mathbb{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?

27. a. Show that the line through vectors **p** and **q** in $\mathbb{R}^n$ may be written in the parametric form $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$. (Refer to the figure with Exercises 21 and 22 in Section 1.5.)

b. The line segment from **p** to **q** is the set of points of the form $(1 - t)\mathbf{p} + t\mathbf{q}$ for $0 \leq t \leq 1$ (as shown in the figure below). Show that a linear transformation T maps this line segment onto a line segment or onto a single point.



28. Let **u** and **v** be vectors in $\mathbb{R}^n$. It can be shown that the set P of all points in the parallelogram determined by **u** and **v** has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \leq a \leq 1, 0 \leq b \leq 1$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain why the image of a point in P under the transformation T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

29. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.

a. Show that f is a linear transformation when $b = 0$.

b. Find a property of a linear transformation that is violated when $b \neq 0$.

c. Why is f called a linear function?

30. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(x) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and **b** in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in computer graphics.)

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

In Exercises 32–36, column vectors are written as rows, such as $\mathbf{x} = (x_1, x_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$.

32. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

33. Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

34. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors onto a linearly dependent set, then the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. [*Hint:* Suppose $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are linearly independent and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. Then $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ for some weights c_1 and c_2, not both zero. Use this equation.]

35. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation. [See Example 4 for ideas.]

36. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.

[**M**] In Exercises 37 and 38, the given matrix determines a linear transformation T. Find all $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

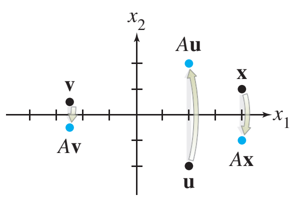
37. $\begin{bmatrix} 4 & -2 & 5 & -5 \\ -9 & 7 & -8 & 0 \\ -6 & 4 & 5 & 3 \\ 5 & -3 & 8 & -4 \end{bmatrix}$ **38.** $\begin{bmatrix} -9 & -4 & -9 & 4 \\ 5 & -8 & -7 & 6 \\ 7 & 11 & 16 & -9 \\ 9 & -7 & -4 & 5 \end{bmatrix}$

39. [**M**] Let $\mathbf{b} = \begin{bmatrix} 7 \\ 5 \\ 9 \\ 7 \end{bmatrix}$ and let A be the matrix in Exercise 37. Is $\mathbf{b}$ in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an $\mathbf{x}$ whose image under the transformation is $\mathbf{b}$.

40. [**M**] Let $\mathbf{b} = \begin{bmatrix} -7 \\ -7 \\ 13 \\ -5 \end{bmatrix}$ and let A be the matrix in Exercise 38. Is $\mathbf{b}$ in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$? If so, find an $\mathbf{x}$ whose image under the transformation is $\mathbf{b}$.

SG Mastering: Linear Transformations 1–34

SOLUTIONS TO PRACTICE PROBLEMS

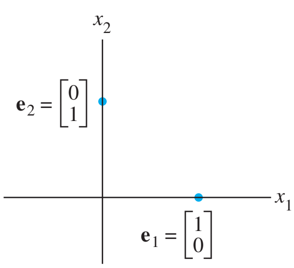


The transformation $\mathbf{x} \mapsto A\mathbf{x}$.

1. A must have five columns for $A\mathbf{x}$ to be defined. A must have two rows for the codomain of T to be $\mathbb{R}^2$.
2. Plot some random points (vectors) on graph paper to see what happens. A point such as $(4, 1)$ maps into $(4, -1)$. The transformation $\mathbf{x} \mapsto A\mathbf{x}$ reflects points through the x-axis (or x_1-axis).
3. Let $\mathbf{x} = t\mathbf{u}$ for some t such that $0 \le t \le 1$. Since T is linear, $T(t\mathbf{u}) = t\,T(\mathbf{u})$, which is a point on the line segment between $\mathbf{0}$ and $T(\mathbf{u})$.

1.9 THE MATRIX OF A LINEAR TRANSFORMATION

Whenever a linear transformation T arises geometrically or is described in words, we usually want a "formula" for $T(\mathbf{x})$. The discussion that follows shows that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ and that important properties of T are intimately related to familiar properties of A. The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n.



EXAMPLE 1 The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

SOLUTION Write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \tag{1}$$

Since T is a *linear* transformation,

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \qquad (2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix} \end{aligned}$$

■

The step from equation (1) to equation (2) explains why knowledge of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ is sufficient to determine $T(\mathbf{x})$ for any $\mathbf{x}$. Moreover, since (2) expresses $T(\mathbf{x})$ as a linear combination of vectors, we can put these vectors into the columns of a matrix A and write (2) as

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

THEOREM 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \tag{3}$$

PROOF Write $\mathbf{x} = I_n\mathbf{x} = [\,\mathbf{e}_1 \ \cdots \ \mathbf{e}_n\,]\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, and use the linearity of T to compute

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) \\ &= \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

The uniqueness of A is treated in Exercise 33. ■

The matrix A in (3) is called the **standard matrix for the linear transformation T**.

We know now that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa. The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented, as Examples 2 and 3 illustrate.

EXAMPLE 2 Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.

SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

■

EXAMPLE 3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. (See Figure 6 in Section 1.8.) Find the standard matrix A of this transformation.

SOLUTION $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotates into $\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotates into $\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$. See Figure 1. By Theorem 10,

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Example 5 in Section 1.8 is a special case of this transformation, with $\varphi = \pi/2$. ■

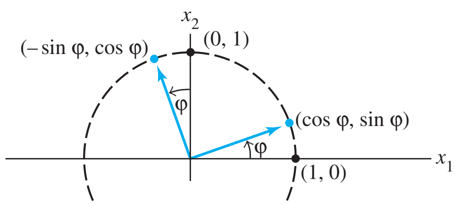


FIGURE 1 A rotation transformation.

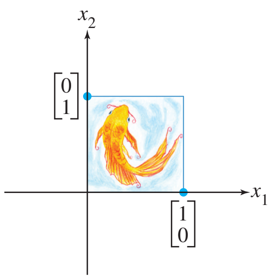


FIGURE 2
The unit square.

Geometric Linear Transformations of $\mathbb{R}^2$

Examples 2 and 3 illustrate linear transformations that are described geometrically. Tables 1–4 illustrate other common geometric linear transformations of the plane. Because the transformations are linear, they are determined completely by what they do to the columns of I_2. Instead of showing only the images of $\mathbf{e}_1$ and $\mathbf{e}_2$, the tables show what a transformation does to the unit square (Figure 2).

Other transformations can be constructed from those listed in Tables 1–4 by applying one transformation after another. For instance, a horizontal shear could be followed by a reflection in the x_2-axis. Section 2.1 will show that such a *composition* of linear transformations is linear. (Also, see Exercise 36.)

Existence and Uniqueness Questions

The concept of a linear transformation provides a new way to understand the existence and uniqueness questions asked earlier. The two definitions following Tables 1–4 give the appropriate terminology for transformations.

TABLE 1 Reflections

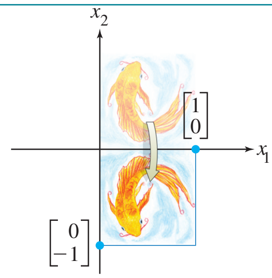
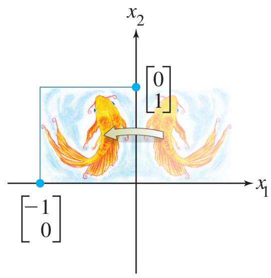
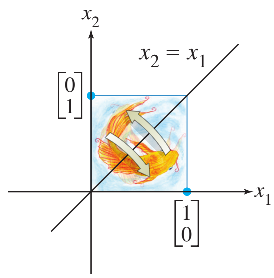
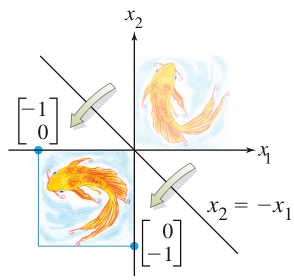
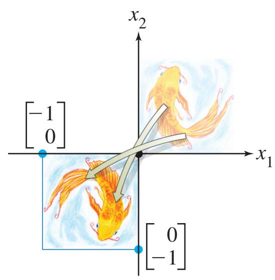
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

TABLE 2 Contractions and Expansions

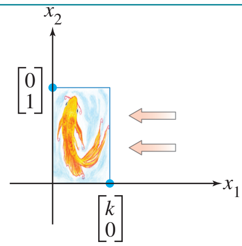
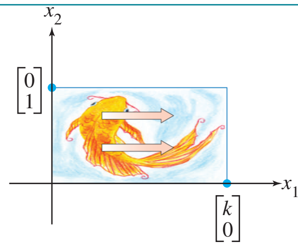
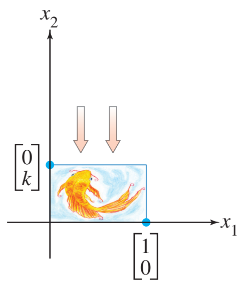
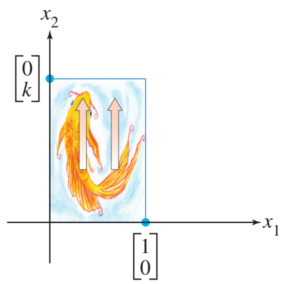
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

TABLE 3 Shears

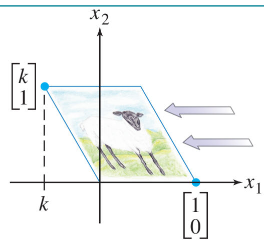
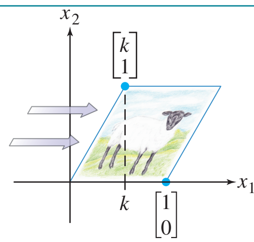
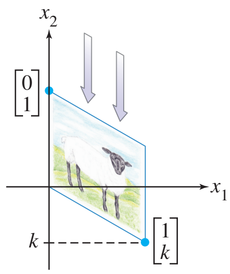
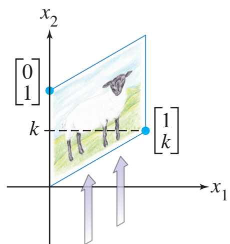
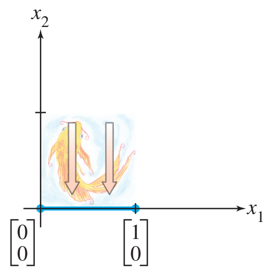
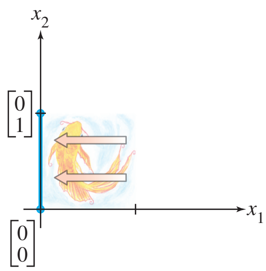
Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

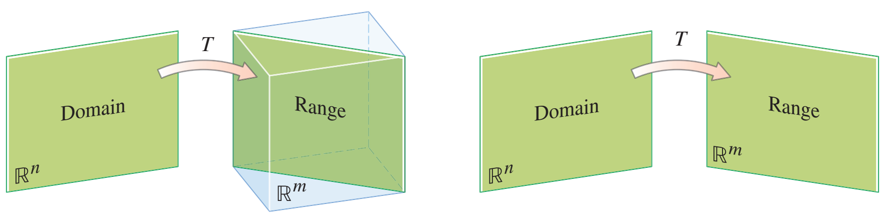
TABLE 4 Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. That is, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if, for each $\mathbf{b}$ in the codomain $\mathbb{R}^m$, there exists at least one solution of $T(\mathbf{x}) = \mathbf{b}$. "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping T is *not* onto when there is some $\mathbf{b}$ in $\mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution. See Figure 3.



T is *not* onto $\mathbb{R}^m$ $\qquad$ T is onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

DEFINITION

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, T is one-to-one if, for each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is T one-to-one?" is a uniqueness question. The mapping T is *not* one-to-one when some $\mathbf{b}$ in $\mathbb{R}^m$ is the image of more than one vector in $\mathbb{R}^n$. If there is no such $\mathbf{b}$, then T is one-to-one. See Figure 4.

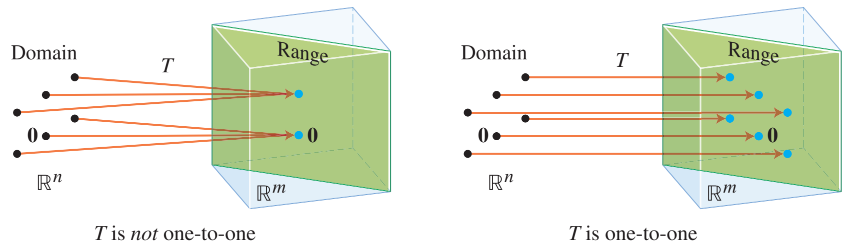


FIGURE 4 Is every $\mathbf{b}$ the image of at most one vector?

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The projection transformations shown in Table 4 are *not* one-to-one and do *not* map $\mathbb{R}^2$ onto $\mathbb{R}^2$. The transformations in Tables 1, 2, and 3 are one-to-one *and* do map $\mathbb{R}^2$ onto $\mathbb{R}^2$. Other possibilities are shown in the two examples below.

Example 4 and the theorems that follow show how the function properties of being one-to-one and mapping onto are related to important concepts studied earlier in this chapter.

EXAMPLE 4 Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

SOLUTION Since A happens to be in echelon form, we can see at once that A has a pivot position in each row. By Theorem 4 in Section 1.4, for each $\mathbf{b}$ in $\mathbb{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent. In other words, the linear transformation T maps $\mathbb{R}^4$ (its domain) onto $\mathbb{R}^3$. However, since the equation $A\mathbf{x} = \mathbf{b}$ has a free variable (because there are four variables and only three basic variables), each $\mathbf{b}$ is the image of more than one $\mathbf{x}$. That is, T is *not* one-to-one. ■

THEOREM 11

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Remark: To prove a theorem that says "statement P is true if and only if statement Q is true," one must establish two things: (1) If P is true, then Q is true and (2) If Q is true, then P is true. The second requirement can also be established by showing (2a): If P is false, then Q is false. (This is called contrapositive reasoning.) This proof uses (1) and (2a) to show that P and Q are either both true or both false.

PROOF Since T is linear, $T(\mathbf{0}) = \mathbf{0}$. If T is one-to-one, then the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution and hence only the trivial solution. If T is not one-to-one, then there is a $\mathbf{b}$ that is the image of at least two different vectors in $\mathbb{R}^n$—say, $\mathbf{u}$ and $\mathbf{v}$. That is, $T(\mathbf{u}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{b}$. But then, since T is linear,

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

The vector $\mathbf{u} - \mathbf{v}$ is not zero, since $\mathbf{u} \neq \mathbf{v}$. Hence the equation $T(\mathbf{x}) = \mathbf{0}$ has more than one solution. So, either the two conditions in the theorem are both true or they are both false. ■

THEOREM 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

Remark: "If and only if" statements can be linked together. For example if "P if and only if Q" is known and "Q if and only if R" is known, then one can conclude "P if and only if R." This strategy is used repeatedly in this proof.

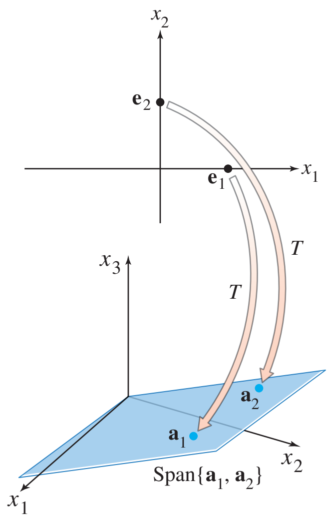
PROOF

a. By Theorem 4 in Section 1.4, the columns of A span $\mathbb{R}^m$ if and only if for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ is consistent—in other words, if and only if for every $\mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has at least one solution. This is true if and only if T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

b. The equations $T(\mathbf{x}) = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ are the same except for notation. So, by Theorem 11, T is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if the columns of A are linearly independent, as was already noted in the boxed statement (3) in Section 1.7. ■

Statement (a) in Theorem 12 is equivalent to the statement "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if every vector in $\mathbb{R}^m$ is a linear combination of the columns of A." See Theorem 4 in Section 1.4.

In the next example and in some exercises that follow, column vectors are written in rows, such as $\mathbf{x} = (x_1, x_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$ instead of the more formal $T((x_1, x_2))$.



The transformation T is not onto $\mathbb{R}^3$.

EXAMPLE 5 Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

SOLUTION When $\mathbf{x}$ and $T(\mathbf{x})$ are written as column vectors, you can determine the standard matrix of T by inspection, visualizing the row–vector computation of each entry in $A\mathbf{x}$.

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

So T is indeed a linear transformation, with its standard matrix A shown in (4). The columns of A are linearly independent because they are not multiples. By Theorem 12(b), T is one-to-one. To decide if T is onto $\mathbb{R}^3$, examine the span of the columns of A. Since A is 3×2, the columns of A span $\mathbb{R}^3$ if and only if A has 3 pivot positions, by Theorem 4. This is impossible, since A has only 2 columns. So the columns of A do not span $\mathbb{R}^3$, and the associated linear transformation is not onto $\mathbb{R}^3$. ■

PRACTICE PROBLEMS

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that first performs a horizontal shear that maps $\mathbf{e}_2$ into $\mathbf{e}_2 - .5\mathbf{e}_1$ (but leaves $\mathbf{e}_1$ unchanged) and then reflects the result through the x_2-axis. Assuming that T is linear, find its standard matrix. [*Hint:* Determine the final location of the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.]

2. Suppose A is a 7×5 matrix with 5 pivots. Let $T(\mathbf{x}) = A\mathbf{x}$ be a linear transformation from $\mathbb{R}^5$ into $\mathbb{R}^7$. Is T a one-to-one linear transformation? Is T onto $\mathbb{R}^7$?

1.9 EXERCISES

In Exercises 1–10, assume that T is a linear transformation. Find the standard matrix of T.

1. $T : \mathbb{R}^2 \to \mathbb{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$, where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (counterclockwise).

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $-\pi/4$ radians (clockwise). [*Hint:* $T(\mathbf{e}_1) = (1/\sqrt{2}, -1/\sqrt{2})$.]

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear transformation that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - 2\mathbf{e}_2$ but leaves the vector $\mathbf{e}_2$ unchanged.

6. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a horizontal shear transformation that leaves $\mathbf{e}_1$ unchanged and maps $\mathbf{e}_2$ into $\mathbf{e}_2 + 3\mathbf{e}_1$.

7. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal x_1-axis. [*Hint:* $T(\mathbf{e}_1) = (-1/\sqrt{2}, 1/\sqrt{2})$.]

8. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

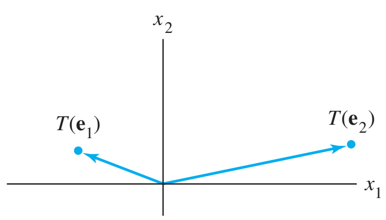
9. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

10. $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the vertical x_2-axis and then rotates points $\pi/2$ radians.

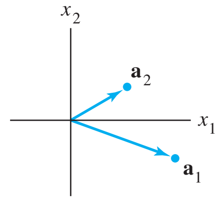
11. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axis and then reflects points through the x_2-axis. Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?

12. Show that the transformation in Exercise 8 is merely a rotation about the origin. What is the angle of the rotation?

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are the vectors shown in the figure. Using the figure, sketch the vector $T(2, 1)$.



14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = [\mathbf{a}_1 \;\; \mathbf{a}_2]$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are shown in the figure. Using the figure, draw the image of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ under the transformation T.



In Exercises 15 and 16, fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

15. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$

16. $\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -2x_1 + x_2 \\ x_1 \end{bmatrix}$

In Exercises 17–20, show that T is a linear transformation by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

17. $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$

18. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

19. $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

20. $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_3 - 4x_4 \qquad (T : \mathbb{R}^4 \to \mathbb{R})$

21. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (3, 8)$.

22. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 - 2x_2, -x_1 + 3x_2, 3x_1 - 2x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (-1, 4, 9)$.

In Exercises 23 and 24, mark each statement True or False. Justify each answer.

23. a. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.

b. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle φ, then T is a linear transformation.

c. When two linear transformations are performed one after another, the combined effect may not always be a linear transformation.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $\mathbf{x}$ in $\mathbb{R}^n$ maps onto some vector in $\mathbb{R}^m$.

e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

24. a. Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

b. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix.

c. The standard matrix of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects points through the horizontal axis, the vertical axis, or the origin has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where a and d are ± 1.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

e. If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

In Exercises 25–28, determine if the specified linear transformation is (a) one-to-one and (b) onto. Justify each answer.

25. The transformation in Exercise 17

26. The transformation in Exercise 2

27. The transformation in Exercise 19

28. The transformation in Exercise 14

In Exercises 29 and 30, describe the possible echelon forms of the standard matrix for a linear transformation T. Use the notation of Example 1 in Section 1.2.

29. $T : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

30. $T : \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

31. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T is one-to-one if and only if A has ____ pivot columns." Explain why the statement is true. [*Hint:* Look in the exercises for Section 1.7.]

32. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with A its standard matrix. Complete the following statement to make it true: "T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if A has ____ pivot columns." Find some theorems that explain why the statement is true.

33. Verify the uniqueness of A in Theorem 10. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B. Show that if A is the standard matrix for T, then $A = B$. [*Hint:* Show that A and B have the same columns.]

34. Why is the question "Is the linear transformation T onto?" an existence question?

35. If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$, can you give a relation between m and n? If T is one-to-one, what can you say about m and n?

36. Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Show that the mapping $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation (from $\mathbb{R}^p$ to $\mathbb{R}^m$). [*Hint:* Compute $T(S(c\mathbf{u} + d\mathbf{v}))$ for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^p$ and scalars c and d. Justify each step of the computation, and explain why this computation gives the desired conclusion.]

[**M**] In Exercises 37–40, let T be the linear transformation whose standard matrix is given. In Exercises 37 and 38, decide if T is a one-to-one mapping. In Exercises 39 and 40, decide if T maps $\mathbb{R}^5$ onto $\mathbb{R}^5$. Justify your answers.

37. $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$

38. $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$

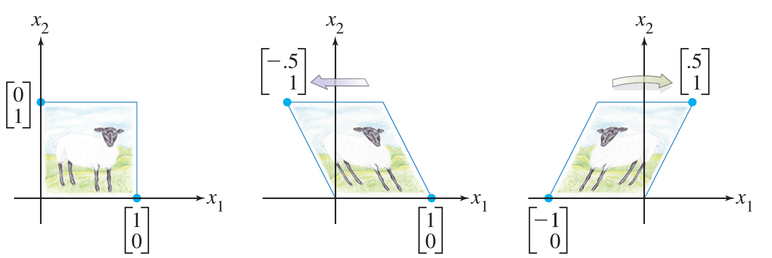
39. $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$

40. $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$

SOLUTION TO PRACTICE PROBLEMS

WEB

1. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. See Figure 5. First, $\mathbf{e}_1$ is unaffected by the shear and then is reflected into $-\mathbf{e}_1$. So $T(\mathbf{e}_1) = -\mathbf{e}_1$. Second, $\mathbf{e}_2$ goes to $\mathbf{e}_2 - .5\mathbf{e}_1$ by the shear transformation. Since reflection through the x_2-axis changes $\mathbf{e}_1$ into $-\mathbf{e}_1$ and



Shear transformation

Reflection through the x_2-axis

FIGURE 5 The composition of two transformations.

leaves $\mathbf{e}_2$ unchanged, the vector $\mathbf{e}_2 - .5\mathbf{e}_1$ goes to $\mathbf{e}_2 + .5\mathbf{e}_1$. So $T(\mathbf{e}_2) = \mathbf{e}_2 + .5\mathbf{e}_1$. Thus the standard matrix of T is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_2 + .5\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} -1 & .5 \\ 0 & 1 \end{bmatrix}$$

2. The standard matrix representation of T is the matrix A. Since A has 5 columns and 5 pivots, there is a pivot in every column so the columns are linearly independent. By Theorem 12, T is one-to-one. Since A has 7 rows and only 5 pivots, there is not a pivot in every row and hence the columns of A do not span $\mathbb{R}^7$. By Theorem 12, and T is not onto.

1.10 LINEAR MODELS IN BUSINESS, SCIENCE, AND ENGINEERING

The mathematical models in this section are all *linear*; that is, each describes a problem by means of a linear equation, usually in vector or matrix form. The first model concerns nutrition but actually is representative of a general technique in linear programming problems. The second model comes from electrical engineering. The third model introduces the concept of a *linear difference equation*, a powerful mathematical tool for studying dynamic processes in a wide variety of fields such as engineering, ecology, economics, telecommunications, and the management sciences. Linear models are important because natural phenomena are often linear or nearly linear when the variables involved are held within reasonable bounds. Also, linear models are more easily adapted for computer calculation than are complex nonlinear models.

As you read about each model, pay attention to how its linearity reflects some property of the system being modeled.

Constructing a Nutritious Weight-Loss Diet

WEB

The formula for the Cambridge Diet, a popular diet in the 1980s, was based on years of research. A team of scientists headed by Dr. Alan H. Howard developed this diet at Cambridge University after more than eight years of clinical work with obese patients.[1] The very low-calorie powdered formula diet combines a precise balance of carbohydrate, high-quality protein, and fat, together with vitamins, minerals, trace elements, and electrolytes. Millions of persons have used the diet to achieve rapid and substantial weight loss.

To achieve the desired amounts and proportions of nutrients, Dr. Howard had to incorporate a large variety of foodstuffs in the diet. Each foodstuff supplied several of the required ingredients, but not in the correct proportions. For instance, nonfat milk was a major source of protein but contained too much calcium. So soy flour was used for part of the protein because soy flour contains little calcium. However, soy flour contains proportionally too much fat, so whey was added since it supplies less fat in relation to calcium. Unfortunately, whey contains too much carbohydrate. . . .

The following example illustrates the problem on a small scale. Listed in Table 1 are three of the ingredients in the diet, together with the amounts of certain nutrients supplied by 100 grams (g) of each ingredient.[2]

---

[1] The first announcement of this rapid weight-loss regimen was given in the *International Journal of Obesity* (1978) **2**, 321–332.

[2] Ingredients in the diet as of 1984; nutrient data for ingredients adapted from USDA Agricultural Handbooks No. 8-1 and 8-6, 1976.

TABLE 1

Amounts (g) Supplied per 100 g of Ingredient				Amounts (g) Supplied by Cambridge Diet in One Day
Nutrient	**Nonfat milk**	**Soy flour**	**Whey**	
Protein	36	51	13	33
Carbohydrate	52	34	74	45
Fat	0	7	1.1	3

EXAMPLE 1 If possible, find some combination of nonfat milk, soy flour, and whey to provide the exact amounts of protein, carbohydrate, and fat supplied by the diet in one day (Table 1).

SOLUTION Let x_1, x_2, and x_3, respectively, denote the number of units (100 g) of these foodstuffs. One approach to the problem is to derive equations for each nutrient separately. For instance, the product

$$\left\{\begin{matrix} x_1 \text{ units of} \\ \text{nonfat milk} \end{matrix}\right\} \cdot \left\{\begin{matrix} \text{protein per unit} \\ \text{of nonfat milk} \end{matrix}\right\}$$

gives the amount of protein supplied by x_1 units of nonfat milk. To this amount, we would then add similar products for soy flour and whey and set the resulting sum equal to the amount of protein we need. Analogous calculations would have to be made for each nutrient.

A more efficient method, and one that is conceptually simpler, is to consider a "nutrient vector" for each foodstuff and build just one vector equation. The amount of nutrients supplied by x_1 units of nonfat milk is the scalar multiple

$$\underset{\text{Scalar}}{\left\{\begin{matrix} x_1 \text{ units of} \\ \text{nonfat milk} \end{matrix}\right\}} \cdot \underset{\text{Vector}}{\left\{\begin{matrix} \text{nutrients per unit} \\ \text{of nonfat milk} \end{matrix}\right\}} = x_1\mathbf{a}_1 \tag{1}$$

where $\mathbf{a}_1$ is the first column in Table 1. Let $\mathbf{a}_2$ and $\mathbf{a}_3$ be the corresponding vectors for soy flour and whey, respectively, and let $\mathbf{b}$ be the vector that lists the total nutrients required (the last column of the table). Then $x_2\mathbf{a}_2$ and $x_3\mathbf{a}_3$ give the nutrients supplied by x_2 units of soy flour and x_3 units of whey, respectively. So the relevant equation is

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b} \tag{2}$$

Row reduction of the augmented matrix for the corresponding system of equations shows that

$$\begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & .277 \\ 0 & 1 & 0 & .392 \\ 0 & 0 & 1 & .233 \end{bmatrix}$$

To three significant digits, the diet requires .277 units of nonfat milk, .392 units of soy flour, and .233 units of whey in order to provide the desired amounts of protein, carbohydrate, and fat. ■

It is important that the values of x_1, x_2, and x_3 found above are nonnegative. This is necessary for the solution to be physically feasible. (How could you use $-.233$ units of whey, for instance?) With a large number of nutrient requirements, it may be necessary to use a larger number of foodstuffs in order to produce a system of equations with a "nonnegative" solution. Thus many, many different combinations of foodstuffs may need to be examined in order to find a system of equations with such a solution. In fact, the manufacturer of the Cambridge Diet was able to supply 31 nutrients in precise amounts using only 33 ingredients.

The diet construction problem leads to the *linear* equation (2) because the amount of nutrients supplied by each foodstuff can be written as a scalar multiple of a vector, as in (1). That is, the nutrients supplied by a foodstuff are *proportional* to the amount of the foodstuff added to the diet mixture. Also, each nutrient in the mixture is the *sum* of the amounts from the various foodstuffs.

Problems of formulating specialized diets for humans and livestock occur frequently. Usually they are treated by linear programming techniques. Our method of constructing vector equations often simplifies the task of formulating such problems.

Linear Equations and Electrical Networks

WEB

Current flow in a simple electrical network can be described by a system of linear equations. A voltage source such as a battery forces a current of electrons to flow through the network. When the current passes through a resistor (such as a lightbulb or motor), some of the voltage is "used up"; by Ohm's law, this "voltage drop" across a resistor is given by

$$V = RI$$

where the voltage V is measured in *volts*, the resistance R in *ohms* (denoted by Ω), and the current flow I in *amperes* (*amps*, for short).

The network in Figure 1 contains three closed loops. The currents flowing in loops 1, 2, and 3 are denoted by I_1, I_2, and I_3, respectively. The designated directions of such *loop currents* are arbitrary. If a current turns out to be negative, then the actual direction of current flow is opposite to that chosen in the figure. If the current direction shown is away from the positive (longer) side of a battery (-|⊢) around to the negative (shorter) side, the voltage is positive; otherwise, the voltage is negative.

Current flow in a loop is governed by the following rule.

> **KIRCHHOFF'S VOLTAGE LAW**
>
> The algebraic sum of the RI voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.

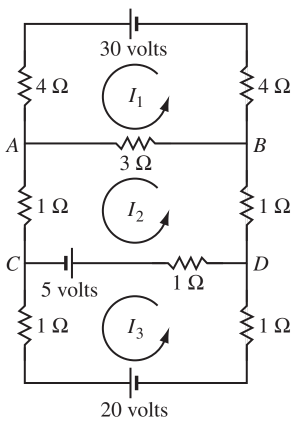


FIGURE 1

EXAMPLE 2 Determine the loop currents in the network in Figure 1.

SOLUTION For loop 1, the current I_1 flows through three resistors, and the sum of the RI voltage drops is

$$4I_1 + 4I_1 + 3I_1 = (4 + 4 + 3)I_1 = 11I_1$$

Current from loop 2 also flows in part of loop 1, through the short *branch* between A and B. The associated RI drop there is $3I_2$ volts. However, the current direction for the branch AB in loop 1 is opposite to that chosen for the flow in loop 2, so the algebraic sum of all RI drops for loop 1 is $11I_1 - 3I_2$. Since the voltage in loop 1 is $+30$ volts, Kirchhoff's voltage law implies that

$$11I_1 - 3I_2 = 30$$

The equation for loop 2 is

$$-3I_1 + 6I_2 - I_3 = 5$$

The term $-3I_1$ comes from the flow of the loop 1 current through the branch AB (with a negative voltage drop because the current flow there is opposite to the flow in loop 2). The term $6I_2$ is the sum of all resistances in loop 2, multiplied by the loop current. The term $-I_3 = -1 \cdot I_3$ comes from the loop 3 current flowing through the 1-ohm resistor in branch CD, in the direction opposite to the flow in loop 2. The loop 3 equation is

$$-I_2 + 3I_3 = -25$$

Note that the 5-volt battery in branch CD is counted as part of both loop 2 and loop 3, but it is -5 volts for loop 3 because of the direction chosen for the current in loop 3. The 20-volt battery is negative for the same reason.

The loop currents are found by solving the system

$$\begin{aligned} 11I_1 - 3I_2 &= 30 \\ -3I_1 + 6I_2 - I_3 &= 5 \\ - I_2 + 3I_3 &= -25 \end{aligned} \tag{3}$$

Row operations on the augmented matrix lead to the solution: $I_1 = 3$ amps, $I_2 = 1$ amp, and $I_3 = -8$ amps. The negative value of I_3 indicates that the actual current in loop 3 flows in the direction opposite to that shown in Figure 1. ∎

It is instructive to look at system (3) as a vector equation:

$$I_1 \underset{\substack{\uparrow \\ \mathbf{r}_1}}{\begin{bmatrix} 11 \\ -3 \\ 0 \end{bmatrix}} + I_2 \underset{\substack{\uparrow \\ \mathbf{r}_2}}{\begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix}} + I_3 \underset{\substack{\uparrow \\ \mathbf{r}_3}}{\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}} = \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 30 \\ 5 \\ -25 \end{bmatrix}} \tag{4}$$

The first entry of each vector concerns the first loop, and similarly for the second and third entries. The first resistor vector $\mathbf{r}_1$ lists the resistance in the various loops through which current I_1 flows. A resistance is written negatively when I_1 flows against the flow direction in another loop. Examine Figure 1 and see how to compute the entries in $\mathbf{r}_1$; then do the same for $\mathbf{r}_2$ and $\mathbf{r}_3$. The matrix form of equation (4),

$$R\mathbf{i} = \mathbf{v}, \quad \text{where} \quad R = [\,\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3\,] \quad \text{and} \quad \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

provides a matrix version of Ohm's law. If all loop currents are chosen in the same direction (say, counterclockwise), then all entries off the main diagonal of R will be negative.

The matrix equation $R\mathbf{i} = \mathbf{v}$ makes the linearity of this model easy to see at a glance. For instance, if the voltage vector is doubled, then the current vector must double. Also, a *superposition principle* holds. That is, the solution of equation (4) is the sum of the solutions of the equations

$$R\mathbf{i} = \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}, \qquad R\mathbf{i} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \quad \text{and} \quad R\mathbf{i} = \begin{bmatrix} 0 \\ 0 \\ -25 \end{bmatrix}$$

Each equation here corresponds to the circuit with only one voltage source (the other sources being replaced by wires that close each loop). The model for current flow is *linear* precisely because Ohm's law and Kirchhoff's law are linear: The voltage drop across a resistor is *proportional* to the current flowing through it (Ohm), and the *sum* of the voltage drops in a loop equals the sum of the voltage sources in the loop (Kirchhoff).

Loop currents in a network can be used to determine the current in any branch of the network. If only one loop current passes through a branch, such as from B to D in Figure 1, the branch current equals the loop current. If more than one loop current passes through a branch, such as from A to B, the branch current is the algebraic sum of the loop currents in the branch (*Kirchhoff's current law*). For instance, the current in branch AB is $I_1 - I_2 = 3 - 1 = 2$ amps, in the direction of I_1. The current in branch CD is $I_2 - I_3 = 9$ amps.

Difference Equations

In many fields such as ecology, economics, and engineering, a need arises to model mathematically a dynamic system that changes over time. Several features of the system are each measured at discrete time intervals, producing a sequence of vectors $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2, \ldots$. The entries in $\mathbf{x}_k$ provide information about the *state* of the system at the time of the kth measurement.

If there is a matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1$, and, in general,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{5}$$

then (5) is called a **linear difference equation** (or **recurrence relation**). Given such an equation, one can compute $\mathbf{x}_1$, $\mathbf{x}_2$, and so on, provided $\mathbf{x}_0$ is known. Sections 4.8 and 4.9, and several sections in Chapter 5, will develop formulas for $\mathbf{x}_k$ and describe what can happen to $\mathbf{x}_k$ as k increases indefinitely. The discussion below illustrates how a difference equation might arise.

A subject of interest to demographers is the movement of populations or groups of people from one region to another. The simple model here considers the changes in the population of a certain city and its surrounding suburbs over a period of years.

Fix an initial year—say, 2014—and denote the populations of the city and suburbs that year by r_0 and s_0, respectively. Let $\mathbf{x}_0$ be the population vector

$$\mathbf{x}_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} \begin{array}{l} \text{City population, 2014} \\ \text{Suburban population, 2014} \end{array}$$

For 2015 and subsequent years, denote the populations of the city and suburbs by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} r_3 \\ s_3 \end{bmatrix}, \ldots$$

Our goal is to describe mathematically how these vectors might be related.

Suppose demographic studies show that each year about 5% of the city's population moves to the suburbs (and 95% remains in the city), while 3% of the suburban population moves to the city (and 97% remains in the suburbs). See Figure 2.

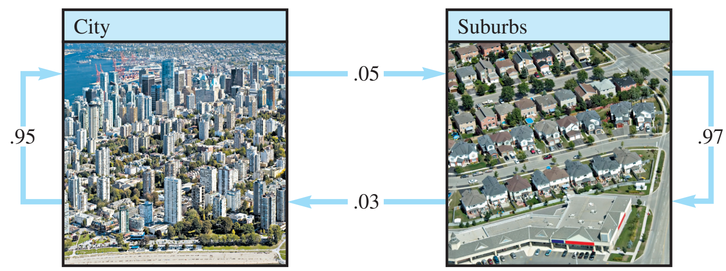


FIGURE 2 Annual percentage migration between city and suburbs.

After 1 year, the original r_0 persons in the city are now distributed between city and suburbs as

$$\begin{bmatrix} .95r_0 \\ .05r_0 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} \begin{array}{l} \text{Remain in city} \\ \text{Move to suburbs} \end{array} \tag{6}$$

The s_0 persons in the suburbs in 2014 are distributed 1 year later as

$$s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} \begin{array}{l} \text{Move to city} \\ \text{Remain in suburbs} \end{array} \tag{7}$$

The vectors in (6) and (7) account for all of the population in 2015.[3] Thus

$$\begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = r_0 \begin{bmatrix} .95 \\ .05 \end{bmatrix} + s_0 \begin{bmatrix} .03 \\ .97 \end{bmatrix} = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

That is,

$$\mathbf{x}_1 = M\mathbf{x}_0 \tag{8}$$

where M is the **migration matrix** determined by the following table:

$$\begin{array}{cl} \text{From:} & \\ \begin{array}{cc} \text{City} & \text{Suburbs} \end{array} & \text{To:} \\ \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} & \begin{array}{l} \text{City} \\ \text{Suburbs} \end{array} \end{array}$$

Equation (8) describes how the population changes from 2014 to 2015. If the migration percentages remain constant, then the change from 2015 to 2016 is given by

$$\mathbf{x}_2 = M\mathbf{x}_1$$

and similarly for 2016 to 2017 and subsequent years. In general,

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{9}$$

The sequence of vectors $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots\}$ describes the population of the city/suburban region over a period of years.

EXAMPLE 3 Compute the population of the region just described for the years 2015 and 2016, given that the population in 2014 was 600,000 in the city and 400,000 in the suburbs.

SOLUTION The initial population in 2014 is $\mathbf{x}_0 = \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$. For 2015,

$$\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix}$$

For 2016,

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix} = \begin{bmatrix} 565{,}440 \\ 434{,}560 \end{bmatrix} \qquad \blacksquare$$

The model for population movement in (9) is *linear* because the correspondence $\mathbf{x}_k \mapsto \mathbf{x}_{k+1}$ is a linear transformation. The linearity depends on two facts: the number of people who chose to move from one area to another is *proportional* to the number of people in that area, as shown in (6) and (7), and the cumulative effect of these choices is found by *adding* the movement of people from the different areas.

PRACTICE PROBLEM

Find a matrix A and vectors $\mathbf{x}$ and $\mathbf{b}$ such that the problem in Example 1 amounts to solving the equation $A\mathbf{x} = \mathbf{b}$.

[3] For simplicity, we ignore other influences on the population such as births, deaths, and migration into and out of the city/suburban region.

1.10 EXERCISES

1. The container of a breakfast cereal usually lists the number of calories and the amounts of protein, carbohydrate, and fat contained in one serving of the cereal. The amounts for two common cereals are given below. Suppose a mixture of these two cereals is to be prepared that contains exactly 295 calories, 9 g of protein, 48 g of carbohydrate, and 8 g of fat.
   a. Set up a vector equation for this problem. Include a statement of what the variables in your equation represent.
   b. Write an equivalent matrix equation, and then determine if the desired mixture of the two cereals can be prepared.

Nutrition Information per Serving

Nutrient	General Mills Cheerios®	Quaker® 100% Natural Cereal
Calories	110	130
Protein (g)	4	3
Carbohydrate (g)	20	18
Fat (g)	2	5

2. One serving of Post Shredded Wheat® supplies 160 calories, 5 g of protein, 6 g of fiber, and 1 g of fat. One serving of Crispix® supplies 110 calories, 2 g of protein, .1 g of fiber, and .4 g of fat.
   a. Set up a matrix B and a vector $\mathbf{u}$ such that $B\mathbf{u}$ gives the amounts of calories, protein, fiber, and fat contained in a mixture of three servings of Shredded Wheat and two servings of Crispix.
   b. **[M]** Suppose that you want a cereal with more fiber than Crispix but fewer calories than Shredded Wheat. Is it possible for a mixture of the two cereals to supply 130 calories, 3.20 g of protein, 2.46 g of fiber, and .64 g of fat? If so, what is the mixture?

3. After taking a nutrition class, a big Annie's® Mac and Cheese fan decides to improve the levels of protein and fiber in her favorite lunch by adding broccoli and canned chicken. The nutritional information for the foods referred to in this exercise are given in the table below.

Nutrition Information per Serving

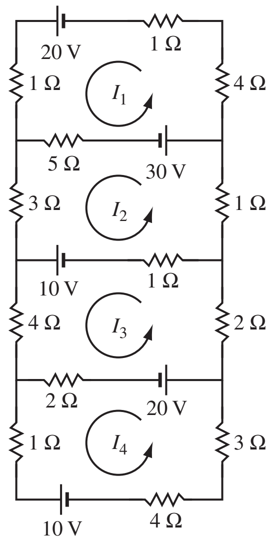
Nutrient	Mac and Cheese	Broccoli	Chicken	Shells
Calories	270	51	70	260
Protein (g)	10	5.4	15	9
Fiber (g)	2	5.2	0	5

   a. **[M]** If she wants to limit her lunch to 400 calories but get 30 g of protein and 10 g of fiber, what proportions of servings of Mac and Cheese, broccoli, and chicken should she use?
   b. **[M]** She found that there was too much broccoli in the proportions from part (a), so she decided to switch from classical Mac and Cheese to Annie's® Whole Wheat Shells and White Cheddar. What proportions of servings of each food should she use to meet the same goals as in part (a)?

4. The Cambridge Diet supplies .8 g of calcium per day, in addition to the nutrients listed in Table 1 for Example 1. The amounts of calcium per unit (100 g) supplied by the three ingredients in the Cambridge Diet are as follows: 1.26 g from nonfat milk, .19 g from soy flour, and .8 g from whey. Another ingredient in the diet mixture is isolated soy protein, which provides the following nutrients in each unit: 80 g of protein, 0 g of carbohydrate, 3.4 g of fat, and .18 g of calcium.
   a. Set up a matrix equation whose solution determines the amounts of nonfat milk, soy flour, whey, and isolated soy protein necessary to supply the precise amounts of protein, carbohydrate, fat, and calcium in the Cambridge Diet. State what the variables in the equation represent.
   b. **[M]** Solve the equation in (a) and discuss your answer.

In Exercises 5–8, write a matrix equation that determines the loop currents. **[M]** If MATLAB or another matrix program is available, solve the system for the loop currents.

5.



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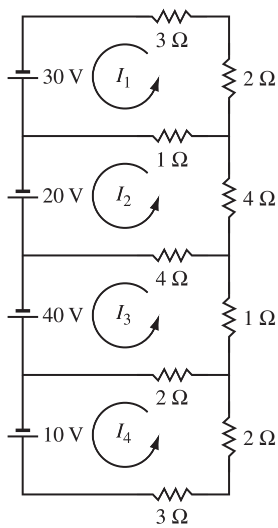

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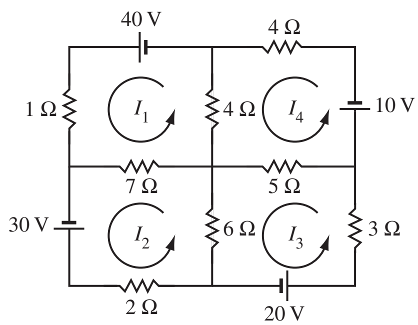

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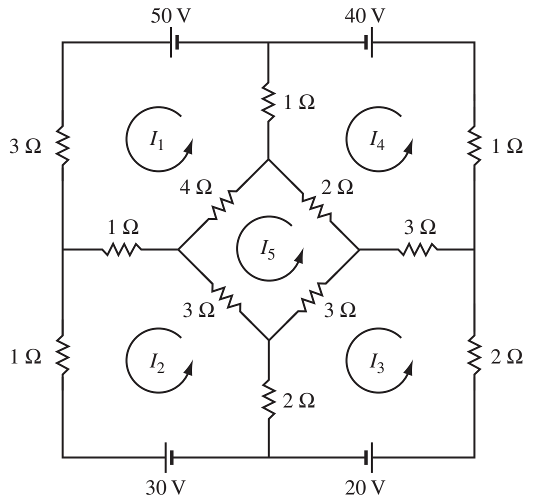


9. In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2015, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2015. Then estimate the populations in the city and in the suburbs two years later, in 2017. (Ignore other factors that might influence the population sizes.)

10. In a certain region, about 6% of a city's population moves to the surrounding suburbs each year, and about 4% of the suburban population moves into the city. In 2015, there were 10,000,000 residents in the city and 800,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2015. Then estimate the populations in the city and in the suburbs two years later, in 2017.

11. In 2012 the population of California was 38,041,430, and the population living in the United States but *outside* California was 275,872,610. During the year, it is estimated that 748,252 persons moved from California to elsewhere in the United States, while 493,641 persons moved to California from elsewhere in the United States.[4]

a. Set up the migration matrix for this situation, using five decimal places for the migration rates into and out of California. Let your work show how you produced the migration matrix.

b. **[M]** Compute the projected populations in the year 2022 for California and elsewhere in the United States, assuming that the migration rates did not change during the 10-year period. (These calculations do not take into account births, deaths, or the substantial migration of persons into California and elsewhere in the United States from other countries.)

[4] Migration data retrieved from http://www.governing.com/

12. **[M]** Budget® Rent A Car in Wichita, Kansas, has a fleet of about 500 cars, at three locations. A car rented at one location may be returned to any of the three locations. The various fractions of cars returned to the three locations are shown in the matrix below. Suppose that on Monday there are 295 cars at the airport (or rented from there), 55 cars at the east side office, and 150 cars at the west side office. What will be the approximate distribution of cars on Wednesday?

Cars Rented From:

$$\begin{array}{c l}
\begin{matrix} \text{Airport} & \text{East} & \text{West} \end{matrix} & \text{Returned To:} \\
\begin{bmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{bmatrix} & \begin{matrix} \text{Airport} \\ \text{East} \\ \text{West} \end{matrix}
\end{array}$$

13. **[M]** Let M and $\mathbf{x}_0$ be as in Example 3.

a. Compute the population vectors $\mathbf{x}_k$ for $k = 1, \dots, 20$. Discuss what you find.

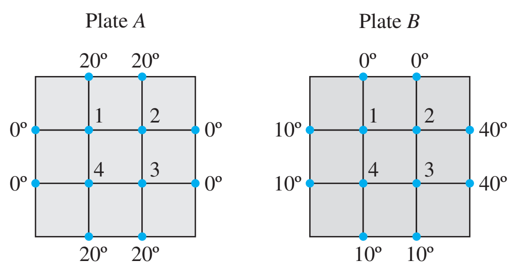
b. Repeat part (a) with an initial population of 350,000 in the city and 650,000 in the suburbs. What do you find?

14. **[M]** Study how changes in boundary temperatures on a steel plate affect the temperatures at interior points on the plate.

a. Begin by estimating the temperatures T_1, T_2, T_3, T_4 at each of the sets of four points on the steel plate shown in the figure. In each case, the value of T_k is approximated by the average of the temperatures at the four closest points. See Exercises 33 and 34 in Section 1.1, where the values (in degrees) turn out to be (20, 27.5, 30, 22.5). How is this list of values related to your results for the points in set (a) and set (b)?

b. Without making any computations, guess the interior temperatures in (a) when the boundary temperatures are all multiplied by 3. Check your guess.

c. Finally, make a general conjecture about the correspondence from the list of eight boundary temperatures to the list of four interior temperatures.



(a) (b)

SOLUTION TO PRACTICE PROBLEM

$$A = \begin{bmatrix} 36 & 51 & 13 \\ 52 & 34 & 74 \\ 0 & 7 & 1.1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 33 \\ 45 \\ 3 \end{bmatrix}$$

CHAPTER 1 SUPPLEMENTARY EXERCISES

1. Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

a. Every matrix is row equivalent to a unique matrix in echelon form.

b. Any system of n linear equations in n variables has at most n solutions.

c. If a system of linear equations has two different solutions, it must have infinitely many solutions.

d. If a system of linear equations has no free variables, then it has a unique solution.

e. If an augmented matrix $[\,A \quad \mathbf{b}\,]$ is transformed into $[\,C \quad \mathbf{d}\,]$ by elementary row operations, then the equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ have exactly the same solution sets.

f. If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x} = \mathbf{0}$.

g. If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of A span $\mathbb{R}^m$.

h. If an augmented matrix $[\,A \quad \mathbf{b}\,]$ can be transformed by elementary row operations into reduced echelon form, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

i. If matrices A and B are row equivalent, they have the same reduced echelon form.

j. The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if there are no free variables.

k. If A is an $m \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, then A has m pivot columns.

l. If an $m \times n$ matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

m. If an $n \times n$ matrix A has n pivot positions, then the reduced echelon form of A is the $n \times n$ identity matrix.

n. If 3×3 matrices A and B each have three pivot positions, then A can be transformed into B by elementary row operations.

o. If A is an $m \times n$ matrix, if the equation $A\mathbf{x} = \mathbf{b}$ has at least two different solutions, and if the equation $A\mathbf{x} = \mathbf{c}$ is consistent, then the equation $A\mathbf{x} = \mathbf{c}$ has many solutions.

p. If A and B are row equivalent $m \times n$ matrices and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

q. If none of the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$ is a multiple of one of the other vectors, then S is linearly independent.

r. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, then $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not in $\mathbb{R}^2$.

s. In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

t. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^m$, then $-\mathbf{u}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$.

u. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^2$, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

v. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, then $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

w. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are in $\mathbb{R}^5$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

x. A linear transformation is a function.

y. If A is a 6×5 matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^5$ onto $\mathbb{R}^6$.

z. If A is an $m \times n$ matrix with m pivot columns, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a one-to-one mapping.

2. Let a and b represent real numbers. Describe the possible solution sets of the (linear) equation $ax = b$. [*Hint:* The number of solutions depends upon a and b.]

3. The solutions (x, y, z) of a single linear equation

$$ax + by + cz = d$$

form a plane in $\mathbb{R}^3$ when a, b, and c are not all zero. Construct sets of three linear equations whose graphs (a) intersect in a single line, (b) intersect in a single point, and (c) have no