(1)

(a) $\begin{cases} x+y-3z=6 \\ 3x-y+2z=3 \\ -x+2y-z=1 \end{cases}$ corresponding augmented system

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 3 & -1 & 2 & 3 \\ -1 & 2 & -1 & 1 \end{array}\right]$$

we will find equivalent system (and equivalent augmented system with same rank as the original one)

$$\begin{bmatrix} 1 & 1 & -3 & 6 \\ 3 & -1 & 2 & 3 \\ -1 & 2 & -1 & 1 \end{bmatrix} \boxed{\begin{array}{l} R_2 \to R_2-3R_1 \\ R_3 \to R_3+R_1 \end{array}} \Rightarrow \begin{bmatrix} 1 & 1 & -3 & 6 \\ 0 & -4 & 11 & -15 \\ 0 & 3 & -4 & 7 \end{bmatrix} \boxed{R_3 \to R_3+\frac{3}{4}R_2}$$

$$\begin{bmatrix} 1 & 1 & -3 & 6 \\ 0 & -4 & 11 & -15 \\ 0 & 0 & 17/4 & -17/4 \end{bmatrix} \Rightarrow \begin{cases} x+y-3z=6 \\ -4y+11z=-15 \\ \boxed{\frac{17}{4}z=-\frac{17}{4}} \end{cases} \Rightarrow z=-1$$

$\boxed{\text{II eq.}}$ $-4y-11=-15$

$\Rightarrow -4y=-15+11=-4 \Rightarrow \boxed{y=1}$

$\boxed{\text{I eq.}}$ $x+1+3=6 \Rightarrow \boxed{x=2}$

$\Rightarrow \exists!$ solution $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

(b) $\begin{cases} 2x-y+2z=2 \\ x+3y-z=8 \\ -x+4y-3z=6 \end{cases}$ as system (a), augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 2 \\ 1 & 3 & -1 & 8 \\ -1 & 4 & -3 & 6 \end{array}\right] \boxed{\begin{array}{l} R_2 \to R_2-\frac{1}{2}R_1 \\ R_3 \to R_3+\frac{1}{2}R_1 \end{array}} \Rightarrow \begin{bmatrix} 2 & -1 & 2 & 2 \\ 0 & 7/2 & -2 & 7 \\ 0 & 7/2 & -2 & 7 \end{bmatrix}$$

the last two rows are the same $\Rightarrow$ $\mathrm{rank}(A)=\mathrm{rank}(A|b)=2$ $\Rightarrow \infty^1$ solutions. If we proceed with the elimination step $R_3 \to R_3-R_2$

$$\begin{bmatrix} 2 & -1 & 2 & 2 \\ 0 & 7/2 & -2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

last equation $0 \cdot z = 0 \Rightarrow \forall z \in \mathbb{R}$.
We take "z" as a parameter

$2x - y = 2 - 2z$ (I equation) $\Rightarrow$

$\frac{7}{2} y = 7 + 2z$ (reduced II equation)

$\hookrightarrow y = (7+2z) \cdot \frac{2}{7} \Rightarrow$ (I equation) $2x - \frac{2}{7}(7+2z) = 2(1-z)$

$\Rightarrow x = \frac{7+2z}{7} + (1-z) = 2 - \frac{5}{7} z$

(division by 2)

$$\text{Solutions} = \begin{pmatrix} 2 - 5/7\, z \\ (7+2z)2/7 \\ z \end{pmatrix} \quad z \in \mathbb{R}.$$

---

(2) We have the system
$$\begin{cases} x - y - 3w = -1 \\ 4x + 3y + 2z + w = 1 \\ 5x + \alpha y + z + 2w = 0 \\ 2x + z + 2w = 0 \end{cases} \qquad \alpha \in \mathbb{R}$$

we rearrange the rows so that the ~~form~~ "shape" of the matrix associated with the system is more suitable for calculations.

$R_4 \longrightarrow R_1$
$R_3 \longrightarrow R_4$
$R_1 \longrightarrow R_2$
$R_2 \longrightarrow R_3$

$$\begin{cases} 2x + \quad + z + 2w = 0 \\ x - y \quad - 3w = -1 \\ 4x + 3y + 2z + w = 1 \\ 5x + \alpha y + z + 2w = 0 \end{cases} \Rightarrow \text{augmented system}$$

$$\left[\begin{array}{ccccc} 2 & 0 & 1 & 2 & 0 \\ 1 & -1 & 0 & -3 & -1 \\ 4 & 3 & 2 & 1 & 1 \\ 5 & \alpha & 1 & 2 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - \frac{1}{2} R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - \frac{5}{2} R_1 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1/2 & -4 & -1 \\ 0 & 3 & 0 & -3 & 1 \\ 0 & \alpha & -3/2 & -3 & 0 \end{bmatrix} \begin{array}{l} R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 + \alpha R_2 \end{array} \Rightarrow \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1/2 & -4 & -1 \\ 0 & 0 & -3/2 & -15 & -2 \\ 0 & 0 & \frac{-3-\alpha}{2} & -3-4\alpha & -\alpha \end{bmatrix}$$

$$R_4 \to R_4 - \frac{(\alpha+3)}{3} R_3 \Rightarrow \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1/2 & -4 & -1 \\ 0 & 0 & -3/2 & -15 & -2 \\ 0 & 0 & 0 & (\alpha+12) & 2-\alpha/3 \end{bmatrix}$$

Last equation of the equivalent (reduced) system is

$(\alpha+12)\, w = \left(2 - \frac{\alpha}{3}\right)$ if $\alpha + 12 = 0 \Rightarrow \alpha = -12$

$$2 - \frac{\alpha}{3} = 2 + \frac{\cancel{12}^{\,4}}{\cancel{3}_{\,1}} = 6 \neq 0$$

$\Rightarrow$ the system is impossible $0 \cdot w = 6$!

if $(\alpha+12) \neq 0 \Rightarrow$ we have one and only one solution.

---

③ We have the system
$$\begin{cases} x + 2y + \alpha z = 1 \\ 2x + \alpha y + 8z = -1 \\ 4x + 7y + z = \beta \end{cases} \qquad \alpha, \beta \in \mathbb{R}$$

$$\Rightarrow (A \,\vdots\, b) = \left[\begin{array}{ccc|c} 1 & 2 & \alpha & 1 \\ 2 & \alpha & 8 & -1 \\ 4 & 7 & 1 & \beta \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array} \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & \alpha & 1 \\ 0 & (\alpha-4) & (8-2\alpha) & -3 \\ 0 & -1 & (1-4\alpha) & \beta-4 \end{bmatrix}$$

if $\alpha - 4 = 0 \Rightarrow$ pivot is equal zero
moreover the II equatione becomes
$0 \quad 0 \quad 0 \quad -3 \Rightarrow$
the system is impossible.

So $\alpha \neq 4$

$$\begin{bmatrix} 1 & 2 & \alpha & 1 \\ 0 & (\alpha-4) & (8-2\alpha) & -3 \\ 0 & -1 & 1-4\alpha & (\beta-4) \end{bmatrix} \quad R_3 \to R_3 + \frac{1}{(\alpha-4)} R_2 \Rightarrow \begin{bmatrix} 1 & 2 & \alpha & 1 \\ 0 & (\alpha-4) & 2(4-\alpha) & -3 \\ 0 & 0 & -4\alpha-1 & \beta-4-\frac{3}{\alpha-4} \end{bmatrix}$$

if $-4\alpha-1=0$ $(\Rightarrow \alpha = -1/4)$ last equation becomes

$$0 \cdot z = \beta-4-\frac{3}{(\alpha-4)} = \beta-4-\frac{3}{-1/4-4} = \beta-\frac{56}{17}$$

if $\beta-\frac{56}{17} \neq 0$ the system is impossible,

otherwise if $\beta-\frac{56}{17}=0 \Rightarrow \infty^1$ solutions.

---

(4) $\alpha_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (we check linear independence)

$$\begin{cases} 2\alpha_2 = 0 \\ \alpha_1 = 0 \\ -\alpha_1+\alpha_2 = 0 \end{cases} \Rightarrow \alpha_1=\alpha_2=0 \quad \text{(only solution)}$$

the vectors are linearly independent but not a basis for $\mathbb{R}^3$ because dimension $(\mathbb{R}^3)=3$ and we have two vectors.

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} + \alpha_3 \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} \alpha_1 + \quad + 6\alpha_3 = 0 \\ 2\alpha_1 + 4\alpha_2 + 7\alpha_3 = 0 \\ 3\alpha_1 + 5\alpha_2 + 8\alpha_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 6 \\ 2 & 4 & 7 \\ 3 & 5 & 8 \end{pmatrix}$$ the associated matrix $\Rightarrow$

$$\begin{pmatrix} 1 & 0 & 6 \\ 2 & 4 & 7 \\ 3 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 6 \\ 0 & 4 & -5 \\ 0 & 5 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & -\frac{15}{4} \end{pmatrix}$$

All nonzero rows $\Rightarrow$ linear independ.

the set is a basis (3 lin. ind. vectors in $\mathbb{R}^3$)

$$\begin{pmatrix} 1 & -1 & 0 \\ 4 & 2 & 6 \\ 3 & 1 & 8 \end{pmatrix}$$

we consider the reduction to an equivalent matrix

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 4 & 8 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

two nonzero rows $\Rightarrow$ the vectors are linearly dependent

the set is not a basis for $\mathbb{R}^3$

$$\begin{pmatrix} 1 & 1 & 1 \\ t & (t-1) & (t+1) \\ t^2 & (t-1)^2 & (t+1)^2 \end{pmatrix}$$

we consider the equivalent matrix

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & (1-2t) & (1+2t) \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

the vectors are linearly independent $\forall t \in \mathbb{R}$ and they are a basis for $\mathbb{R}^3$: 3 linearly independent vectors.