Name/Surname: Version B

November ..., 2019

Student ID Number:

Graph Theory, Discrete Mathematics, and Optimization
Graph Theory and Discrete Mathematics module

1	2	3	4	5	6
c	**b**	**b**	**c**	**b**	**a**

Exam duration: one hour. The test is composed by 6 multiple-choice questions (one point each), and one open question (up to two points). For the multiple-choice questions please insert the correct answer into the above grid. No cheating or collaboration is allowed. Good luck!

1. (2 point) The rank of the following matrix

$$\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & -9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$$

is

a) 1 **c)** 3
b) 2 d) 4

2. (2 point) The following system (t is a real parameter)

$$\begin{cases} -tx_1 & +(t-1)x_2 & +x_3 & =1 \\ & +(t-1)x_2 & +tx_3 & =1 \\ 2x_1 & & +x_3 & =5 \end{cases}$$

is consistent for

a) $t < 0$ c) $t > 0$
b) $t \neq 1, 2$ d) for all real values t

3. (2 point) The dimension of the subspace S of $\mathbb{R}^3$ spanned by the vector

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -7 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -7 \end{bmatrix},$$

is

a) $= 1$ c) $= 4$

b) $= 2$ d) $= 3$

4. (2 point) Let $\mathbf{x} = [1\ 2\ 3]^T$, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We have $\mathbf{x} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3$,

a) only for $c_1 = 1,\ c_2 = 2,\ c_3 = 3$ **c)** only for $c_1 = 2,\ c_2 = -1,\ c_3 = 2$
b) only for $c_1 = 3,\ c_2 = 2,\ c_3 = 1$ d) for no possible value of c_1, c_2 and c_3

5. (2 point) The following vector is one eigenvector for the given matrix corresponding to the given eigenvalue,

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \ \lambda = 2,$$

a) $[1\ 1\ 1]^T$ c) $[1\ 2\ 2]^T$

b) $[-5\ 2\ 2]^T$ d) $[0\ 0\ 0]^T$

6. (2 point) The following real matrix (α, and β are real parameters)

$$A = \begin{bmatrix} 0 & \beta & 0 \\ \alpha & -\alpha & 1 \\ \beta & 0 & 0 \end{bmatrix}$$

is orthogonal

a) only for $\alpha = 0$, $\beta = 1$ c) for any value of β, and $\alpha = 0$

b) for any value of α, and $\beta = 0$ d) for no possible value of α and β

Exercise.(4 points) Find the eigenvalues and eigenvectors of the following matrix A

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 15$$

then

$$\lambda^2 - 8\lambda + 15 = 0 \iff \lambda = 4 \pm 1.$$

The matrix A has two distinct eigenvalues: $\lambda_1 = 5$, and $\lambda_2 = 3$.
Eigenvector $\mathbf{V}_1 = (x_1, x_2)^T$ corresponding to λ_1:

$$(A - 5I)\,\mathbf{V}_1 = \mathbf{0} \Longrightarrow \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = -x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

Eigenvector $\mathbf{V}_2 = (x_1, x_2)^T$ corresponding to λ_2:

$$(A - 3I)\,\mathbf{V}_2 = \mathbf{0} \Longrightarrow \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_2 = -2x_1$ and (we have chosen x_1 as a free variable),

$$\mathbf{V}_2 = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad x_1 \in \mathbb{R}.$$