Name/Surname: Version A

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Student ID Number: Solution

Graph Theory, Discrete Mathematics, and Optimization
Graph Theory and Discrete Mathematics module

1	2	3	4	5	6
d	**a**	**d**	**d**	**a**	**d**

1. (2 point) The following system

$$\begin{cases} 2x_1 & -2x_2 & & = 0 \\ x_1 & +2x_2 & +2x_3 & = 1 \\ \alpha x_1 & -\alpha x_2 & +x_3 & = 1 \end{cases}$$

is consistent for

a) $\alpha = 0$ c) $\alpha > 0$
b) $\alpha < 0$ **d)** for all real values α

We reduce the augmented matrix of the linear system to the equivalent echelon form (the first swap of rows only for simplicity in the calculations),

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ \alpha & -\alpha & 1 & 1 \end{bmatrix} (R_2 \longleftrightarrow R_1) \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & -2 & 0 & 0 \\ \alpha & -\alpha & 1 & 1 \end{bmatrix} \begin{pmatrix} \\ R_2 - 2R_1 \longrightarrow R_2 \\ R_3 - \alpha R_1 \longrightarrow R_3 \end{pmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -6 & -4 & -2 \\ 0 & -3\alpha & (1-2\alpha) & (1-\alpha) \end{bmatrix}$$

$$\begin{pmatrix} \\ \\ R_3 - \frac{\alpha}{2} R_2 \longrightarrow R_3 \end{pmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -6 & -4 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and we have only one solution regardless of the α value.

2. (2 point) Let

$$A = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \right\}; \; B = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\};$$

a) A and B are linearly independent c) A is linearly independent and B is linearly dependent

b) A and B are linearly dependent d) B is linearly independent and A is linearly dependent

For the set of vectors A (in the following c_i are scalars),

$$c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3c_1 = 0 \Rightarrow c_1 = 0 \\ 2c_2 = 0 \Rightarrow c_2 = 0 \\ c_1 + 2c_2 = 0 \end{cases}$$

For the set of vectors B,

$$c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left\{ \begin{array}{r} c_3 = 0 \\ c_1 + 3c_2 = 0 \\ c_1 + 2c_2 = 0 \end{array} \right.$$

Then $c_1 = c_2 = c_3 = 0$ (just subtract the last equation from the second one to deduce $c_2 = 0$, so $c_1 = 0$).

3. (2 point) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

and $N(A) = \{\mathbf{x} \ : \ A\mathbf{x} = \mathbf{0}\ \}$, $dim(N(A)) =$

a) 0 c) 1
b) 3 **d)** 2

We consider the equivalent echelon form for the matrix A,

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \left(\begin{array}{c} R_2 - 2R_1 \longrightarrow R_2 \\ R_3 - R_1 \longrightarrow R_3 \end{array} \right) \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \left(\begin{array}{c} \\ \\ R_3 + 2R_2 \longrightarrow R_3 \end{array} \right) \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the homogeneous system we obtain two free variables, then $dim(N(A)) = 2$. It is possible also to consider:

$$rank(A) + dim(N(A)) = \ number \ \ of \ \ columns \ \ of \ \ A.$$

4. (2 point) Define the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 - x_3 \\ x_1 + x_3 \end{bmatrix},$$

the corresponding matrix A is

$$\text{(a)} \ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ \text{(b)} \ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \ \text{(c)} \ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ \textbf{(d)} \ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

The columns of the matrix A are the vectors $T(\mathbf{e}_i)$ where $\mathbf{e}_i$ are the the standard basis vectors of $\mathbb{R}^3$,

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

5. (2 point) Let $\mathbf{x} = [1 \ 1]^T$, $\mathbf{y} = [-2 \ 0]^T$, $\mathbf{v}$ the vector projection of $\mathbf{x}$ onto $\mathbf{y}$, and $\mathbf{w}$ the vector projection of $\mathbf{y}$ onto $\mathbf{x}$, then

a) $\mathbf{v} = [1 \ 0]^T$, $\mathbf{w} = [-1 \ -1]^T$ c) $\mathbf{v} = [0 \ 1]^T$, $\mathbf{w} = [-1 \ -1]^T$

b) $\mathbf{v} = [1 \ 0]^T$, $\mathbf{w} = [1 \ 1]^T$ d) $\mathbf{v} = [-1 \ 0]^T$, $\mathbf{w} = [1 \ 1]^T$

From the definition of the vector projection,

$$\mathbf{v} = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \frac{-2}{4}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{w} = \frac{\mathbf{y}\cdot\mathbf{x}}{\|\mathbf{x}\|^2}\mathbf{x} = \frac{-2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

6. (2 point) The following vector is one eigenvector for the given matrix corresponding to the given eigenvalue,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 4 & 1 & 3 \end{bmatrix}, \ \lambda = 3,$$

a) $[0\ 0\ 0]^T$ c) $[1\ 1\ 1]^T$

b) $[1\ 1\ 0]^T$ **d)** $[0\ 0\ 2]^T$

One method consists to verify the vector equality $A\mathbf{V} = 3\mathbf{V}$, where $\mathbf{V}$ is one of the proposed vectors. Alternatively we can calculate the eigenvectors corresponding to the eigenvalue $\lambda = 3$, $\mathbf{V} = (v_1\ v_2\ v_3)^T$,

$$(A - 3I)\mathbf{V} = \mathbf{0} \ \Leftrightarrow \ \begin{cases} -v_1 = 0 \\ v_2 = 0 \\ 4v_1 + v_2 = 0 \end{cases}$$

Then

$$\mathbf{V} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha \in \mathbb{R}.$$

Exercise.(4 points) Find the eigenvalues and eigenvectors of the following matrix A

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 4$$

then

$$(3-\lambda)^2 - 4 = 0 \Longleftrightarrow (3-\lambda)^2 = 4 \Longleftrightarrow \lambda = 3 \pm 2.$$

The matrix A has two distinct eigenvalues: $\lambda_1 = 5$, and $\lambda_2 = 1$.
Eigenvector $\mathbf{V}_1 = (x_1, x_2)^T$ corresponding to λ_1:

$$(A - 5I)\,\mathbf{V}_1 = \mathbf{0} \Longrightarrow \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = -x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

Eigenvector $\mathbf{V}_2 = (x_1, x_2)^T$ corresponding to λ_2:

$$(A - I)\,\mathbf{V}_2 = \mathbf{0} \Longrightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_2 = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$