The troublesome reflection rule

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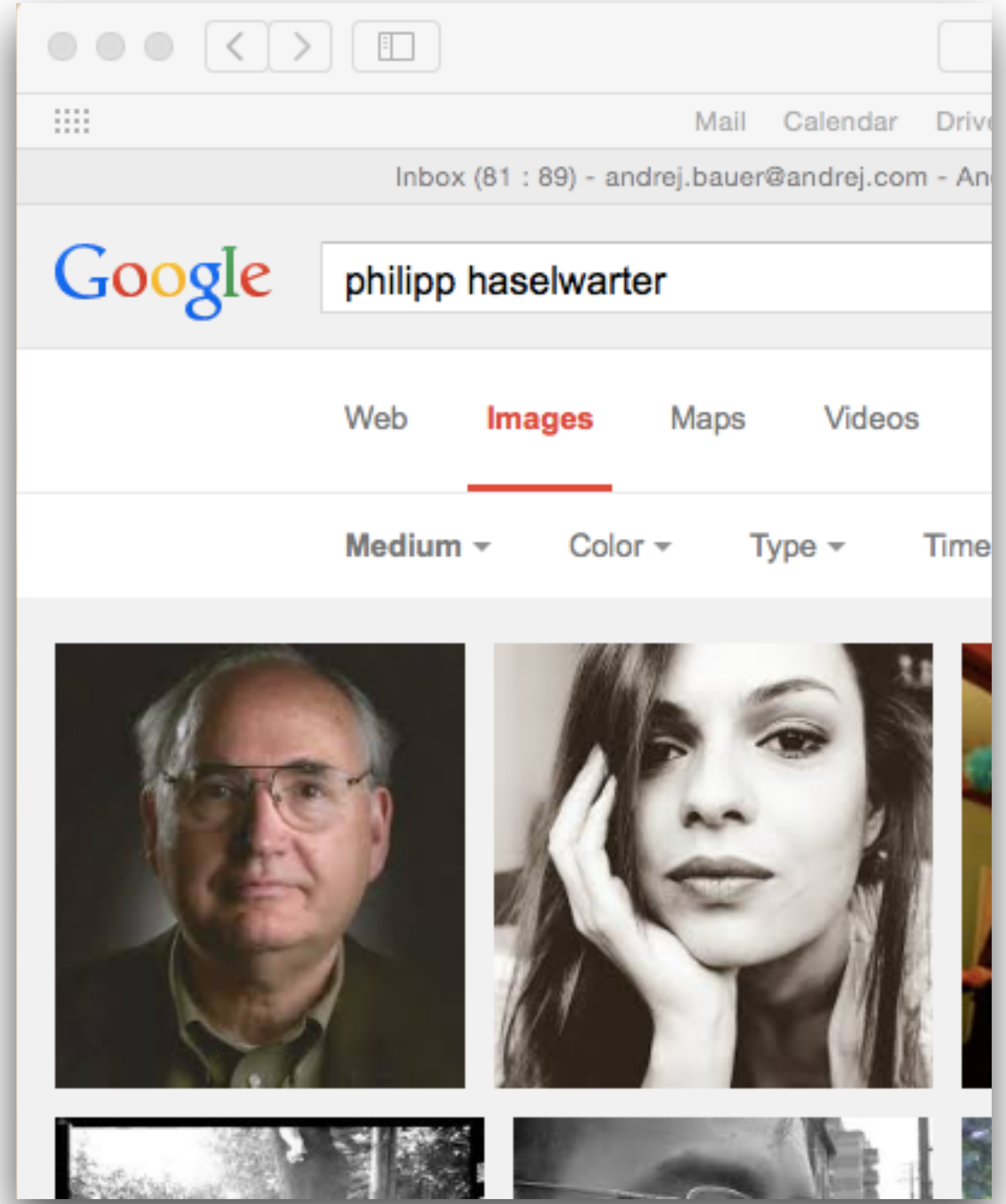


Philipp Haselwarter?

Talk outline

- Equality reflection – bad & good
- Current development
- Future possibilities

$$\Gamma \vdash A \ : \ \mathsf{Type}$$

$$\Gamma \vdash e \ : \ A$$

$$\Gamma \vdash e_1 \equiv_A e_2$$

$$\Gamma \vdash A \equiv_{\mathsf{Type}} B$$

$$\Gamma \vdash \mathsf{Type} \quad : \quad \mathsf{Type}$$

$$\Gamma \vdash e \; : \; A$$

$$\Gamma \vdash e_1 \equiv_A e_2$$

Dependent product

$$\frac{\Gamma, x{:}A \vdash B \ : \ \mathsf{Type}}{\Gamma \vdash \prod_{x:A} B \ : \ \mathsf{Type}} \qquad\qquad \frac{\Gamma, x{:}A \vdash e \ : \ B}{\Gamma \vdash (\lambda x{:}A \,.\, e) \ : \ \prod_{x:A} B}$$

$$\frac{\Gamma \vdash e_1 \ : \ \prod_{x:A} B \qquad \Gamma \vdash e_2 \ : \ A}{\Gamma \vdash e_1\, e_2 \ : \ B[e_2/x]}$$

$$(\lambda x{:}A \,.\, e_1)\, e_2 \equiv_{B[e_2/x]} e_1[e_2/x]$$

$$(\lambda x{:}A \,.\, e\, x) \equiv_{\prod_{x:A} B} e$$

Equality

$$\frac{\Gamma \vdash A \ : \ \mathsf{Type} \qquad \Gamma \vdash a \ : \ A \qquad \Gamma \vdash b \ : \ A}{\Gamma \vdash \mathsf{Eq}_A(a, b) \ : \ \mathsf{Type}}$$

$$\frac{\Gamma \vdash a \ : \ A}{\Gamma \vdash \mathsf{refl}_A(a) \ : \ \mathsf{Eq}_A(a, a)} \qquad \frac{\Gamma \vdash p \ : \ \mathsf{Eq}_A(a, b)}{\Gamma \vdash a \equiv_A b}$$

$$p \equiv_{\mathsf{Eq}_A(a,b)} \mathsf{refl}_A(a)$$

J eliminator

$$\frac{\begin{array}{c}\Gamma \vdash a, b \;:\; A \\ \Gamma \vdash p \;:\; \mathsf{Eq}_A(a, b) \\ \Gamma, x{:}A \vdash c \;:\; C(x, x, \mathsf{refl}(x))\end{array}}{\Gamma \vdash \mathsf{J}(\ldots) \;:\; C(a, b, p)}$$

just use c[a/x] for J(..)

$$
\begin{aligned}
A &: \mathsf{Type}, \\
a &: A, \\
B &: \mathsf{Type}, \\
b &: B, \\
p &: \mathsf{Eq}_{\mathsf{Type}}(A, B), \\
q &: \mathsf{Eq}_{A}(a, b)
\end{aligned}
$$

cannot strengthen

cannot exchange

$$p : \mathsf{Eq}_{\mathsf{Type}}(\mathsf{nat} \longrightarrow \mathsf{nat}, \mathsf{nat} \longrightarrow \mathsf{bool})$$

$$\vdash$$

$$0 : \mathsf{bool}$$

$$
\begin{aligned}
\mathsf{nat} &: \mathsf{Type} \\
\mathsf{Z} &: \mathsf{nat} \\
\mathsf{S} &: \mathsf{nat} \longrightarrow \mathsf{nat} \\
\mathsf{ind} &: \textstyle\prod_{P:\mathsf{nat}\longrightarrow\mathsf{Type}} P\mathsf{Z} \longrightarrow \left(\prod_{n:\mathsf{nat}} P\,n \longrightarrow P\,(\mathsf{S}\,n)\right) \longrightarrow \prod_{m:\mathsf{nat}} P\,m
\end{aligned}
$$

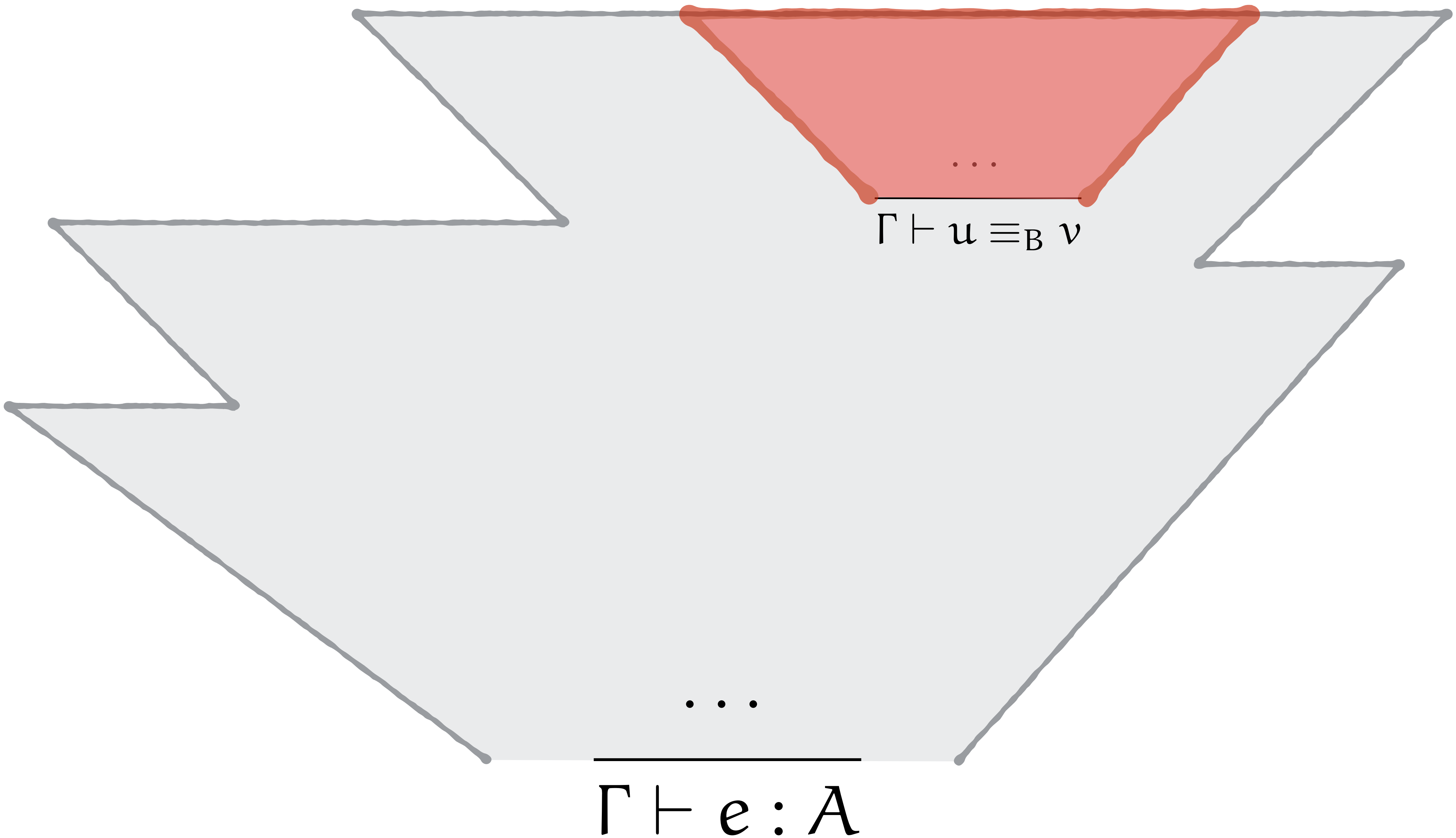
$$
\begin{aligned}
\beta_{\mathsf{Z}} :\ & \textstyle\prod_{P:\mathsf{nat}\longrightarrow\mathsf{Type}} \prod_{x:P\mathsf{Z}} \prod_{f:(\Pi_{n:\mathsf{nat}} P\,n \longrightarrow P\,(\mathsf{S}\,n))} \\
& \qquad \mathsf{Eq}_{P\mathsf{Z}}(\mathsf{ind}\,P\,x\,f\,\mathsf{Z}, x) \\
\beta_{\mathsf{S}} :\ & \textstyle\prod_{P:\mathsf{nat}\longrightarrow\mathsf{Type}} \prod_{x:P\mathsf{Z}} \prod_{f:(\Pi_{n:\mathsf{nat}} P\,n \longrightarrow P\,(\mathsf{S}\,n))} \prod_{n:\mathsf{nat}} \\
& \qquad \mathsf{Eq}_{P\mathsf{Z}}(\mathsf{ind}\,P\,x\,f\,(\mathsf{S}\,n), f\,n\,(\mathsf{ind}\,P\,x\,f\,n))
\end{aligned}
$$

Extensional type theories

- Nuprl: based on a PER model over an untyped calculus with strong normalization

- HOL: simply typed and classical

- We would like a dependently typed system which is not bound to a single model



. . .
$\Gamma \vdash u \equiv_B v$
. . .
$\Gamma \vdash e : A$

Type theory as a programming language

- **Input:** a program that derives a judgment
- **Evaluation:** construction of the derivation
- **Equality checking:** computational effect
- **Output:** the derived judgment

Input computations

`x`	variable
`Type`	universe
$\prod x{:}c_1.c_2$	product
$\lambda x{:}c_1.c_2$	abstraction
$c_1\ c_2$	application
$\mathtt{Eq}(c_1,c_2)$	equality type
`refl(c)`	reflexivity
$c_1::c_2$	ascription
$\mathtt{hint}\ c_1\ \mathtt{in}\ c_2$	general hint
$\mathtt{beta}\ c_1\ \mathtt{in}\ c_2$	β-hint
$\mathtt{eta}\ c_1\ \mathtt{in}\ c_2$	η-hint

handlers directing equality checks through reflection

Output terms & types

x	variable
$\mathtt{Type}$	universe
$\prod x{:}A.B$	product
$\lambda x{:}A.(e{:}B)$	abstraction
$e_1 \; ^{@(x:A.B)} \; e_2$	application
$\mathtt{Eq}_A(e_1, e_2)$	equality type
$\mathtt{refl}_A(e)$	reflexivity

Output terms & types

x	variable
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$\mathrm{Eq}_A(e_1,e_2)$	equality type
$\mathrm{refl}_A(e)$	reflexivity

β-rule

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad\qquad \Gamma, x{:}A_1 \vdash B_1 \equiv B_2}{\Gamma \vdash ((\lambda x{:}A_1.e_1{:}B_1)\ ^{@(x:A_2.B_2)}\ e_2) \equiv e_1[e_2/x]}$$

$$(\lambda x{:}nat.x{:}nat)\ ^{@(nat\to bool)}\ 0 \not\equiv 0$$

Operational semantics

$$\Gamma;\mathscr{E} \vdash \mathtt{c} \rightsquigarrow (\mathtt{e},\mathtt{A})$$

"In context Γ using hints $\mathscr{E}$
computation c evaluates to (e,A)"

Soundness:

If $\Gamma;\mathscr{E} \vdash \mathtt{c} \rightsquigarrow (\mathtt{e},\mathtt{A})$ then $\Gamma \vdash \mathtt{e}:\mathtt{A}$.

$$\frac{(x : A) \in \Gamma}{\Gamma ; \mathscr{E} \vdash x \rightsquigarrow (x, A)}$$

$$\frac{}{\Gamma ; \mathscr{E} \vdash \mathrm{Type} \rightsquigarrow (\mathrm{Type}, \mathrm{Type})}$$

$$\frac{\begin{array}{l} \Gamma ; \mathscr{E} \vdash c_1 \rightsquigarrow (A, T_1) \\ \Gamma ; \mathscr{E} \vdash T_1 \equiv_{\mathrm{Type}} \mathrm{Type} \\ \Gamma , x : A ; \mathscr{E} \vdash c_1 \rightsquigarrow (B, T_2) \\ \Gamma , x : A ; \mathscr{E} \vdash T_2 \equiv_{\mathrm{Type}} \mathrm{Type} \end{array}}{\Gamma ; \mathscr{E} \vdash \prod x : c_1 . c_2 \rightsquigarrow \prod x : A . B}$$

“normalization”

$$\frac{\begin{array}{l}\Gamma;\mathscr{E} \vdash c_1 \rightsquigarrow (e_1, A_1) \\ \Gamma;\mathscr{E} \vdash A_1 \mapsto^{\mathrm{whnf}} \Pi x{:}C.D \\ \Gamma;\mathscr{E} \vdash c_2 \rightsquigarrow (e_2, A_2) \\ \Gamma;\mathscr{E} \vdash C \equiv_{\mathrm{Type}} A_2\end{array}}{\Gamma;\mathscr{E} \vdash c_1\ c_2 \rightsquigarrow (e_1\ {}^{@(x:A.B)}\ e_2,\ D[e_2/x])}$$

$$\frac{\begin{array}{l}\Gamma;\mathcal{E} \vdash c_1 \rightsquigarrow (e_1,A_1) \\ \Gamma;\mathcal{E} \vdash c_2 \rightsquigarrow (e_2,A_2) \\ \Gamma;\mathcal{E} \vdash A_2 \equiv_{\mathrm{Type}} \mathrm{Type} \\ \Gamma;\mathcal{E} \vdash A_1 \equiv_{\mathrm{Type}} e_2\end{array}}{\Gamma;\mathcal{E} \vdash c_1{::}c_2 \rightsquigarrow (e_1,e_2)}$$

Equality hints

$$\mathcal{E} \;=\; \mathcal{E}_{\equiv},\ \mathcal{E}_{\beta},\ \mathcal{E}_{\eta}$$

general hints

β-hints

η-hints

A hint is a universally quantified equation:

$$\prod x_1 : A_1 \ldots x_n : A_n .\, \mathrm{Eq}_B(e_1, e_2)$$

β-hints

```
pair_fst:
  ∏A,B:Type. ∏x:A. ∏y:B.
    Eq_A(fst A B (pair A B x y),x)
```

η-hints

```
pair_eta:
  ∏A,B:Type. ∏u,v:A×B.
    Eq_A(fst A B u, fst A B v) →
    Eq_B(snd A B u, snd A B v) →
    Eq_A×B(u,v)
```

$$\frac{\begin{array}{l}\Gamma;\mathscr{E}_{\equiv},\mathscr{E}_{\beta},\mathscr{E}_{\eta} \vdash \mathtt{c}_1 \rightsquigarrow (\mathtt{e}_1,\mathtt{A}_1) \\ \Gamma;\mathscr{E}_{\equiv},\mathscr{E}_{\beta},\mathscr{E}_{\eta} \vdash \mathtt{A}_1 \mapsto \prod \mathtt{x}_1{:}\mathtt{A}_1 \ldots \mathtt{x}_n{:}\mathtt{A}_n.\mathtt{Eq}_{\mathtt{B}}(\mathtt{e}_1,\mathtt{e}_2) \\ \Gamma;\mathscr{E}_{\equiv},\mathscr{E}_{\beta} \cup \{\prod \mathtt{x}_1{:}\mathtt{A}_1 \ldots \mathtt{x}_n{:}\mathtt{A}_n.\mathtt{Eq}_{\mathtt{B}}(\mathtt{e}_1,\mathtt{e}_2)\},\mathscr{E}_{\eta} \vdash \mathtt{c}_2 \rightsquigarrow (\mathtt{e}_2,\mathtt{A}_2)\end{array}}{\Gamma;\mathscr{E}_{\equiv},\mathscr{E}_{\beta},\mathscr{E}_{\eta} \vdash \mathtt{beta}\ \mathtt{c}_1\ \mathtt{in}\ \mathtt{c}_2 \rightsquigarrow (\mathtt{e}_2,\mathtt{A}_2)}$$

Checking $e_1 \equiv_A e_2$

1. Decompose $e_1 \equiv_A e_2$ into subgoals that have smaller types, e.g.

   η-rules

   $$e_1 \equiv_{A \times B} e_2$$

   reduces to

   $$\texttt{fst}\ e_1 \equiv_A \texttt{fst}\ e_2 \quad \text{and} \quad \texttt{snd}\ e_1 \equiv_B \texttt{snd}\ e_2$$

2. When the type cannot be decomposed further, check that e_1 and e_2 are equal by normalization.

   β-rules

β-hints as definitions

```
a : A
a_def : Eq(a,e)

beta a_def in …
```

β-hints as definitions

```
a : A
a_def1 : Eq(a,e₁)
a_def2 : Eq(a,e₂)

beta a_def1 in …
beta a_def2 in …
```

What else?

- Voevodsky’s Homotopy Type System
- Enrich the input language with other computational effects and handlers
- Implement standard proof assistant techniques (implicit arguments, proof search, type clases, …) in the language

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